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Article

On the Finite Orthogonality of q-Pseudo-Jacobi Polynomials

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Abstract: Using the Sturm–Liouville theory in q-difference spaces, we prove the finite orthogonality of q-Pseudo Jacobi polynomials. Their norm square values are then explicitly computed by means of the Favard theorem.

Keywords: q-Pseudo Jacobi Polynomials; Sturm–Liouville problems; q-difference equations; finite sequences of q-orthogonal polynomials

1. Introduction

For $\alpha, \beta > -1$, the Jacobi polynomials are defined as [1]

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left\{(1-x)^{\alpha}(1+x)^{\beta}(1-x^2)^n\right\}. \quad (1)$$

Another representation of Jacobi polynomials is as [2,3]

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array}\middle|\, \frac{1-x}{2}\right) \\ &= \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(\alpha+\beta+1)_k}{(\alpha+1)_k}\frac{(1-x)^k}{2^k k!}, \end{aligned} \quad (2)$$

where

$$(a)_k := \prod_{j=0}^{k-1}(a+j), \quad (a)_0 := 1, \quad (3)$$

and

$${}_rF_s\left(\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_s \end{array}\middle|\, z\right) := \sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_r)_k}{(b_1)_k\dots(b_s)_k}\frac{z^k}{k!}, \quad (4)$$

in which $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_s, z \in \mathbb{C}$ and $b_1, \dots, b_s \neq 0, -1, -2, \cdots, -(k-1)$.

The weight function corresponding to Jacobi polynomials is known in statistics as the shifted beta distribution

$$w(x;\alpha,\beta) = (1-x)^{\alpha}(1+x)^{\beta}, \quad x \in [-1,1].$$

An interesting subclass of Jacobi polynomials is when $\alpha = -u + iv$ and $\beta = -u - iv$ for $i^2 = -1$ in (2), so that the real polynomials

$$J_n^{(u,v)}(x) = (-i)^n P_n^{(-u+iv,-u-iv)}(ix), \quad (5)$$

satisfy the equation

$$(1+x^2)J_n''(x)+2((1-u)x+v)J_n'(x)-n(n-2u+1)J_n(x)=0. \tag{6}$$

It is proved in [4] that $\{J_n^{(u,v)}(x)\}$ are finitely orthogonal with respect to the weight function

$$w(x;u,v)=(1+x^2)^{-u}\exp\left(2v\arctan x\right),$$

on $(-\infty,\infty)$ and can be explicitly represented in form of hypergeometric functions as

$$J_n^{(u,v)}(x)=\frac{(-2i)^n(1-u+iv)_n}{(n-2u+1)_n}\,{}_2F_1\left(\begin{array}{c}-n,n-2u+1\\1-u+iv\end{array}\middle|\,\frac{1-ix}{2}\right).$$

The so-called q-polynomials have found many applications in Eulerian series and continued fractions [3], q-algebras and quantum groups [5–7], and q-oscillators [8–10]. See also [11,12] in this regard.

It has been acknowledged that the theory of q-special functions is essentially based on the relation

$$\lim_{q\to 1}\frac{1-q^\alpha}{1-q}=\alpha.$$

Hence, a basic number in q-calculus is defined as

$$[\alpha]_q=\frac{1-q^\alpha}{1-q}.$$

There is a q-analogue of the Pochhammer symbol (3) (called q-shifted factorial) as

$$(a;\,q)_k:=\prod_{j=0}^{k-1}(1-aq^j),\quad (a;\,q)_0:=1.$$

Moreover we have

$$(a;\,q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)\quad\text{for}\quad 0<|q|<1,$$

and

$$(a_1,a_2,...,a_m;q)_\infty=(a_1;q)_\infty(a_2;q)_\infty...(a_m;q)_\infty. \tag{7}$$

There exist several q-analogues of classical hypergeometric orthogonal polynomials that are known as basic hypergeometric orthogonal polynomials [3].

In the present work, using the Sturm–Liouville theory in q-difference spaces, we prove that a special case of big q-Jacobi polynomials is finitely orthogonal on $(-\infty,\infty)$. The big q-Jacobi polynomials are defined as

$$P_n(x;a,b,c;q)\;=\;{}_3\phi_2\left(\begin{array}{c}q^{-n},a\,b\,q^{n+1},x\\a\,q,c\,q\end{array}\middle|\,q;q\right), \tag{8}$$

where

$${}_r\phi_s\left(\begin{array}{c}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{array}\middle|\,q;z\right):=\sum_{k=0}^{\infty}\frac{(a_1;q)_k\ldots(a_r;q)_k}{(b_1;q)_k\ldots(b_s;q)_k}\frac{z^k}{(q;q)_k}\left((-1)^kq^{\frac{k(k-1)}{2}}\right)^{1+s-r}, \tag{9}$$

is known as the basic hypergeometric series.

Again, $a_1,a_2,\ldots,a_r,b_1,b_2,\ldots,b_s,z\in\mathbb{C}$ and $b_1,b_2,\ldots,b_s\neq 1,q^{-1},q^{-2},\cdots,q^{1-k}$.

Notice that [3] (p. 15)

$$\lim_{q\to 1}{}_r\phi_s\left(\begin{array}{c}q^{a_1},\ldots,q^{a_r}\\q^{b_1},\ldots,q^{b_s}\end{array}\middle|\,q;(q-1)^{1+s-r}z\right)={}_rF_s\left(\begin{array}{c}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{array}\middle|\,z\right). \tag{10}$$

On the other side, if we set $c=0$, $a=q^\alpha$ and $b=q^\beta$ in (8) and then let $q\to 1$, we find the Jacobi polynomials (2) as

$$\lim_{q\to 1}P_n(x;q^\alpha,q^\beta,0;q)=\frac{P_n^{(\alpha,\beta)}(2x-1)}{P_n^{(\alpha,\beta)}(1)}.$$

Moreover, by referring to (8), one can define another family of big q-Jacobi polynomials [13] with four free parameters as

$$P_n^*(x;a,b,c,d;q) = P_n(qac^{-1}x;a,b,-ac^{-1}d;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, qac^{-1}x \\ aq, -qac^{-1}d \end{matrix} \,\Big|\, q;q\right),$$

which yields

$$P_n(x;a,b,c;q) = P_n^*(-q^{-1}c^{-1}x;a,b,-ac^{-1},1;q).$$

Because a particular case of Jacobi polynomials (5) are called the pseudo Jacobi polynomials, it is reasonable to similarly consider a special case of big q-Jacobi polynomials preserving the limit relation as $q \to 1$. This means that the q-pseudo Jacobi polynomials will be derived by substituting

$$a = iq^{\frac{1}{2}(u-iv)}, \quad b = -iq^{\frac{1}{2}(u+iv)}, \quad c = iq^{\frac{1}{2}(-u+iv)} \quad \text{and} \quad d = -iq^{\frac{1}{2}(-u-iv)}$$

in a special case of the polynomials (8) as

$$P_n(cx;c/b,d/a,c/a;q) \quad \text{where} \quad a,b,c,d \in \mathbb{C} \quad \text{and} \quad (ab)/(qcd) > 0,$$

so that

$$\lim_{q\to 1} P_n(iq^{\frac{1}{2}(-u+iv)}x; -q^{-u}, -q^{-u}, q^{-u+iv}; q) = \frac{J_n^{(u,v)}(x)}{J_n^{(u,v)}(i)}.$$

Therefore, the q-pseudo Jacobi polynomials are defined as

$$J_n^{(u,v)}(x;q) = P_n(iq^{\frac{1}{2}(-u+iv)}x; -q^{-u}, -q^{-u}, q^{-u+iv}; q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{u+n+1}, -q^{1+u-iv}x \\ -q^{1+\frac{1}{2}(u-iv)}, iq^{1+\frac{1}{2}(u-3iv)} \end{matrix} \,\Big|\, q;q\right). \quad (11)$$

The main aim of this paper is to apply a q-Sturm–Liouville theorem in order to obtain a finite orthogonality for the real polynomials (11) on $(-\infty,\infty)$, which is a new contribution in the literature.

A regular Sturm–Liouville problem of continuous type is a boundary value problem of the form

$$\frac{d}{dx}\left(K(x)\frac{dy_n(x)}{dx}\right) + \lambda_n w(x) y_n(x) = 0, \quad (K(x) > 0,\ w(x) > 0), \quad (12)$$

which is defined on an open interval, say (γ_1, γ_2) with the boundary conditions

$$\alpha_1 y(\gamma_1) + \beta_1 y'(\gamma_1) = 0 \quad \text{and} \quad \alpha_2 y(\gamma_2) + \beta_2 y'(\gamma_2) = 0, \quad (13)$$

where α_1, α_2 and β_1, β_2 are constant numbers and $K(x)$, and $w(x)$ in (12) are to be assumed continuous functions for $x \in [\gamma_1, \gamma_2]$. The function $w(x)$ is called the weight or density function.

Let y_n and y_m be two eigenfunctions of Equation (12). According to the Sturm–Liouville theory [14], they have an orthogonality property with respect to the weight function $w(x)$ under the given condition (13), so that we have

$$\int_{\gamma_1}^{\gamma_2} w(x) y_n(x) y_m(x) dx = \left(\int_{\gamma_1}^{\gamma_2} w(x) y_n^2(x) dx\right) \delta_{m,n}, \quad (14)$$

in which

$$\delta_{m,n} = \begin{cases} 0 & (n \neq m), \\ 1 & (n = m). \end{cases}$$

There are generally two types of orthogonality for relation (14), i.e. infinitely orthogonality and finitely orthogonality. In the finite case, one has to impose some constraints on n, while in the infinite case, n is free up to infinity [4].

By referring to the differential Equation (6), it is proved in [4] that

$$\int_{-\infty}^{\infty}(1+x^2)^{-u}\exp\left(2v\arctan x\right)J_n^{(u,v)}(x)J_m^{(u,v)}(x)dx = \frac{2\pi\, n!\, 2^{2n+1-2u}\,\Gamma(2u-n)}{(2u-2n-1)\Gamma(u-n+iv)\Gamma(u-n-iv)}\delta_{m,n}$$
$$\Leftrightarrow \quad m,n=0,1,2,\ldots, \quad N=\max\{m,n\}<u-\frac{1}{2} \quad \text{and} \quad v\in\mathbb{R},$$

where $\Gamma(.)$ is the well-known gamma function.

Similarly, q-orthogonal functions can be solutions of a q-Sturm-Liouville problem in the form [15]

$$D_q\left(K(x;q)D_q y_n(x;q)\right)+\lambda_{n,q}w(x;q)y_n(x;q)=0, \quad (K(x;q)>0,\ w(x;q)>0), \tag{15}$$

where

$$D_q f(x)=\frac{f(qx)-f(x)}{(q-1)x} \qquad (x\neq 0,\ q\neq 1),$$

and (15) satisfies a set of boundary conditions like (13). This means that if $y_n(x;q)$ and $y_m(x;q)$ are two eigenfunctions of the q-difference Equation (15), they are orthogonal with respect to a weight function $w(x;q)$ on a discrete set [16].

Let $\varphi(x)$ and $\psi(x)$ be two polynomials of degree at most 2 and 1, respectively, as

$$\varphi(x)=ax^2+bx+c \quad \text{and} \quad \psi(x)=dx+e \quad (a,b,c,d,e\in\mathbb{C},\ \ d\neq 0).$$

If $\{y_n(x;q)\}_n$ is a sequence of polynomials that satisfies the q-difference equation [3]

$$\varphi(x)D_q^2 y_n(x;q)+\psi(x)D_q y_n(x;q)+\lambda_{n,q}y_n(qx;q)=0, \tag{16}$$

where

$$D_q^2(f(x))=\frac{f(q^2x)-(1+q)f(qx)+qf(x)}{q(q-1)^2x^2},$$

$\lambda_{n,q}\in\mathbb{C}$ and $q\in\mathbb{R}\setminus\{-1,0,1\}$, then the following orthogonality relation holds

$$\int_{\rho_1}^{\rho_2} w(x;q)y_n(x;q)y_m(x;q)d_qx=\left(\int_{\rho_1}^{\rho_2} w(x;q)y_n^2(x;q)d_qx\right)\delta_{n,m},$$

in which

$$\int_{\rho_1}^{\rho_2} f(t)d_qt=(1-q)\sum_{j=0}^{\infty}q^j\left(\rho_2 f(q^j\rho_2)-\rho_1 f(q^j\rho_1)\right),$$

and $w(x;q)$ is a solution of the Pearson q-difference equation

$$D_q\left(w(x;q)\varphi(q^{-1}x)\right)=w(qx;q)\psi(x). \tag{17}$$

Note that $w(x;q)$ is assumed to be positive and $w(q^{-1}x;q)\varphi(q^{-2}x)x^k$ for $k\in\mathbb{N}$ must vanish at $x=\rho_1,\rho_2$.

If $\bar{P}_n(x)=x^n+\cdots$ is a monic solution of Equation (16), the eigenvalue $\lambda_{n,q}$ is explicitly derived as

$$\lambda_{n,q}=-\frac{[n]_q}{q^n}(a[n-1]_q+d).$$

The q-integral as an inverse of the q-difference operator [3,17,18] is defined as

$$\int_0^x f(t)d_qt=(1-q)x\sum_{j=0}^{\infty}q^j f(q^jx) \quad (x\in\mathbb{R})$$

provided that the series converges absolutely. Furthermore, we have

$$\int_0^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n),$$

and

$$\int_{-\infty}^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} q^n \left(f(q^n)+f(-q^n)\right).$$

2. Finite Orthogonality of q-Pseudo Jacobi Polynomials

Let us consider the following q-difference equation

$$\begin{aligned}&(q^{2-u}x^2+2\sin\left(\frac{v}{2}\ln q\right)x+1)D_q^2y_n(x;q)\\&\quad+\left(\frac{q^u-q^{2-u}}{1-q}x-2\sin(\frac{v}{2}\ln q)(q^{1-\frac{u}{2}}-q^{\frac{u}{2}})\right)D_qy_n(x;q)+\lambda^*_{n,q}y_n(qx;q)=0,\end{aligned} \tag{18}$$

with

$$\lambda^*_{n,q} = -\frac{[n]_q}{q^n}\left(q^{2-u}[n-1]_q+\frac{q^u-q^{2-u}}{1-q}\right),$$

for $n = 0, 1, 2, \dots$ and $q \in \mathbb{R} \setminus \{-1, 0, 1\}$.

It is clear that

$$\lim_{q\to 1}\lambda^*_{n,q} = -n(n-2u+1),$$

gives the same eigenvalues as in the continuous case (6).

Theorem 1. *Let $\{J_n^{(u,v)}(x;q)\}_n$ defined in (11) be a sequence of polynomials that satisfies the q-difference Equation (18). Subsequently, we have*

$$\int_{-\infty}^\infty w^{(u,v)}(x;q)J_n^{(u,v)}(x;q)J_m^{(u,v)}(x;q)d_qx = \left(\int_{-\infty}^\infty w^{(u,v)}(x;q)\left(J_n^{(u,v)}(x;q)\right)^2 d_qx\right)\delta_{n,m},$$

where $N < u - \frac{1}{2}$ for $N = \max\{m, n\}$ and the positive function $w^{(u,v)}(x;q)$ is a solution of the Pearson-type q-difference equation

$$\begin{aligned}&D_q\left(w^{(u,v)}(x;q)\left(q^{2-u}x^2+2\sin(\tfrac{v}{2}\ln q)x+1\right)\right)\\&=\left(\tfrac{q^u-q^{2-u}}{1-q}x-2\sin(\tfrac{v}{2}\ln q)(q^{1-\frac{u}{2}}-q^{\frac{u}{2}})\right)w^{(u,v)}(qx;q),\end{aligned}$$

which is equivalent to

$$\frac{w^{(u,v)}(x;q)}{w^{(u,v)}(qx;q)} = \frac{q^ux^2-2q^{\frac{u}{2}}\sin(\frac{v}{2}\ln q)x+1}{q^{-u}x^2+2q^{-\frac{u}{2}}\sin(\frac{v}{2}\ln q)x+1}. \tag{19}$$

Proof. First, according to [3] and referring to (7) it is not difficult to verify that

$$\begin{aligned}w^{(u,v)}(x;q) &= \frac{(iq^{(u-iv)/2}x, -iq^{(u+iv)/2}x;q)_\infty}{(iq^{(-u+iv)/2}x, -iq^{(-u-iv)/2}x;q)_\infty}\\&= x^{-2u}\frac{(-iq^{(-u+iv)/2}x^{-1}, iq^{(-u-iv)/2}x^{-1};q^{-1})_\infty}{(-iq^{(u-iv)/2}x^{-1}, iq^{(u+iv)/2}x^{-1};q^{-1})_\infty},\end{aligned} \tag{20}$$

is a solution of Equation (19).

Now, if Equation (18) is written in the self-adjoint form

$$D_q\left(w^{(u,v)}(x;q)\left(q^{2-u}x^2+2\sin(\tfrac{v}{2}\ln q)x+1\right)D_qJ_n^{(u,v)}(x;q)\right)+\lambda^*_{n,q}w^{(u,v)}(qx;q)J_n^{(u,v)}(qx;q)=0, \tag{21}$$

and for m as

$$D_q\left(w^{(u,v)}(x;q)\,(q^{2-u}x^2+2\sin(\tfrac{v}{2}\ln q)x+1)\,D_qJ_m^{(u,v)}(x;q)\right)+\lambda^*_{m,q}w^{(u,v)}(qx;q)J_m^{(u,v)}(qx;q)=0, \tag{22}$$

by multiplying (21) by $J_m^{(u,v)}(qx;q)$ and (22) by $J_n^{(u,v)}(qx;q)$ and subtracting each other we get

$$\begin{aligned}&(\lambda^*_{m,q}-\lambda^*_{n,q})w^{(u,v)}(x;q)J_m^{(u,v)}(x;q)J_n^{(u,v)}(x;q)\\ &\quad=q^2D_q\left(w^{(u,v)}(q^{-1}x;q)\left(q^{2-u}x^2+2\sin(\frac{v}{2}\ln q)x+1\right)D_qJ_n^{(u,v)}(q^{-1}x;q)\right)J_m^{(u,v)}(x;q)\\ &\quad-q^2D_q\left(w^{(u,v)}(q^{-1}x;q)\left(q^{2-u}x^2+2\sin(\frac{v}{2}\ln q)x+1\right)D_qJ_m^{(u,v)}(q^{-1}x;q)\right)J_n^{(u,v)}(x;q).\end{aligned} \tag{23}$$

Hence, q-integration by parts on both sides of (23) over $(-\infty,\infty)$ yields

$$\begin{aligned}&(\lambda^*_{m,q}-\lambda^*_{n,q})\int_{-\infty}^{\infty}w^{(u,v)}(x;q)J_m^{(u,v)}(x;q)J_n^{(u,v)}(x;q)d_qx\\ &=q^2\int_{-\infty}^{\infty}\{D_q\left(w^{(u,v)}(q^{-1}x;q)\left(q^{2-u}x^2+2\sin(\frac{v}{2}\ln q)x+1\right)D_qJ_n^{(u,v)}(q^{-1}x;q)\right)J_m^{(u,v)}(x;q)\\ &\quad-D_q\left(w^{(u,v)}(q^{-1}x;q)\left(q^{2-u}x^2+2\sin(\frac{v}{2}\ln q)x+1\right)D_qJ_m^{(u,v)}(q^{-1}x;q)\right)J_n^{(u,v)}(x;q)\,\}d_qx\\ &\qquad=q^2\Big[w^{(u,v)}(q^{-1}x;q)\left(q^{2-u}x^2+2\sin(\frac{v}{2}\ln q)x+1\right)\\ &\qquad\times\left(D_qJ_n^{(u,v)}(q^{-1}x;q)J_m^{(u,v)}(x;q)-D_qJ_m^{(u,v)}(q^{-1}x;q)J_n^{(u,v)}(x;q)\right)\Big]_{-\infty}^{\infty}.\end{aligned} \tag{24}$$

Because

$$\max\ \deg\{D_qJ_n^{(u,v)}(q^{-1}x;q)J_m^{(u,v)}(x;q)-D_qJ_m^{(u,v)}(q^{-1}x;q)J_n^{(u,v)}(x;q)\}=m+n-1,$$

the left-hand side of (24) is zero if

$$\lim_{x\to\infty}w^{(u,v)}(q^{-1}x;q)\left(q^{2-u}x^2+2\sin(\frac{v}{2}\ln q)x+1\right)x^{m+n-1}=0. \tag{25}$$

By taking $\max\{m,n\}=N$, relation (25) would be equivalent to

$$\lim_{x\to\infty}\frac{(-iq^{(-u+iv)/2}x^{-1},iq^{(-u-iv)/2}x^{-1};q^{-1})_\infty}{(-iq^{(u-iv)/2}x^{-1},iq^{(u+iv)/2}x^{-1};q^{-1})_\infty}\quad x^{2N-2u+1}=0. \tag{26}$$

Note that (26) is valid if and only if

$$2N+1-2u<0\quad\text{or}\quad N<u-\frac{1}{2}.$$

Therefore, the right-hand side of (24) tends to zero and

$$\int_{-\infty}^{\infty}w^{(u,v)}(x;q)J_m^{(u,v)}(x;q)J_n^{(u,v)}(x;q)d_qx=0,$$

if and only if $m\neq n$ and $N<u-\frac{1}{2}$ for $N=\max\{m,n\}$. □

Corollary 1. *The finite polynomial set* $\{J_n^{(u,v)}(x;q)\}_{n=0}^{N<u-\frac{1}{2}}$ *is orthogonal with respect to the weight function* (20) *on* $(-\infty,\infty)$.

2.1. Computing the Norm Square Value

According to (17), because $J_n^{(u,v)}(x;q)$ is a particular case of the big q-Jacobi polynomials, it satisfies the recurrence relation [3]

$$\bar{J}_{n+1}^{(u,v)}(x;q)=(x-c_n(u,v;q))\,\bar{J}_n^{(u,v)}(x;q)-d_n(u,v;q)\bar{J}_{n-1}^{(u,v)}(x;q),$$

with the initial terms

$$\bar{J}_0^{(u,v)}(x;q)=1,\quad \bar{J}_1^{(u,v)}(x;q)=x+\frac{2\sin(\frac{v}{2}\ln q)(1-q)(q^{2-u/2}+q^{1+u/2})}{(q^u-q^{2-u})},$$

where

$$c_n(u,v;q)=\frac{2\sin(\frac{v}{2}\ln q)q^n}{(q^u-q^{2n-2})(q^u-q^{2n})}$$
$$\times\{(q^u-q^{n-1})\left(q^{-u/2}[n]_q(1+q)+(q^{2-u/2}+q^{1+u/2})\right)-q^{n+1-u}(1-q^{n+1})(q^{1-u/2}+q^{u/2})\},$$

and

$$d_n(u,v;q)=\frac{(q^{n+1}-q^{2n+1})(q^u-q^{n-u})}{(1-q)^2(q^u-q^{2n-u-1})(q^u-q^{2n-u})^2(q^u-q^{2n-u+1})}$$
$$\times\{4\sin^2(\frac{v}{2}\ln q)q^{n-1-u/2}(1-q)\left(1+q-q^2+q^u-q^{1+u}-q^{n-1}\right)\left(1-q^{n-u+1}(1+q-q^2)-q^{n+1}(1-q)\right)$$
$$-(q^{4n-2u}+2q^{2n}+q^{2u})\}.$$

Now, by applying the Favard theorem [19] for the monic type of polynomials (11), we get

$$\int_{-\infty}^{\infty}w^{(u,v)}(x;q)\bar{J}_m^{(u,v)}(x;q)\bar{J}_n^{(u,v)}(x;q)d_qx=\left(\mu_0\prod_{k=1}^{n}d_k(u,v;q)\right)\delta_{n,m},$$

where

$$\mu_0=\int_{-\infty}^{\infty}\frac{(iq^{(u-iv)/2}x,-iq^{(u+iv)/2}x;q)_\infty}{(iq^{(-u+iv)/2}x,-iq^{(-u-iv)/2}x;q)_\infty}d_qx.$$

Hence, it remains to explicitly compute the above μ_0. For this purpose, we can refer to the general formula ([13] Formula 128)

$$\int_{z_-q^{\mathbb{Z}}\cup z_+q^{\mathbb{Z}}}\frac{(ax,bx;q)_\infty}{(cx,dx;q)_\infty}d_qx=\frac{(q,a/c,a/d,b/c,b/d;q)_\infty}{(a\,b/(q\,c\,d);q)_\infty}\frac{\theta(z_-/z_+;q)\theta(c\,d\,z_-z_+;q)}{\theta(c\,z_-;q)\theta(d\,z_-;q)\theta(c\,z_+;q)\theta(d\,z_+;q)},\quad(27)$$

in which

$$\theta(x;q)=(x,q/x;q)_\infty.$$

Therefore, it is enough to replace $z_-=-1,z_+=1$ in (27) to finally obtain

$$\mu_0=\frac{(q,q^{u-iv},-q^u,-q^u,q^{u+iv};q)_\infty}{(q^{2u-1};q)_\infty}\times$$
$$\frac{(-1,-q,-q^u,-q^{u+1};q)_\infty}{(-iq^{\frac{-u+iv}{2}},iq^{\frac{-u+iv}{2}},-iq^{\frac{-u-iv}{2}},iq^{\frac{-u-iv}{2}},-iq^{1-\frac{-u+iv}{2}},iq^{1-\frac{-u+iv}{2}},-iq^{1-\frac{-u-iv}{2}},iq^{1-\frac{-u-iv}{2}};q)_\infty}.$$

For example, the set $\{J_n^{(21,1)}(x;q)\}_{n=0}^{20}$ is a finite sequence of q-orthogonal polynomials that satisfies the orthogonality relation

$$\int_{-\infty}^{\infty}\frac{(iq^{(21-i)/2}x,-iq^{(21+i)/2}x;q)_\infty}{(iq^{-(21-i)/2}x,-iq^{-(21+i)/2}x;q)_\infty}\bar{J}_m^{(21,1)}(x;q)\bar{J}_n^{(21,1)}(x;q)d_qx=$$
$$\left(\frac{(q,q^{21-i},-q^{21},-q^{21},q^{21+i},-1,-q,-q^{21},-q^{22};q)_\infty}{(q^{41},-iq^{\frac{-21+i}{2}},iq^{\frac{-21+i}{2}},-iq^{\frac{-21-i}{2}},iq^{\frac{-21-i}{2}},-iq^{\frac{23-i}{2}},iq^{\frac{23-i}{2}},-iq^{\frac{23+i}{2}},iq^{\frac{23+i}{2}};q)_\infty}\prod_{k=1}^{n}d_k(21,1;q)\right)\delta_{m,n}$$
$$\iff m,n<20,$$

where

$$d_k(21,1;q) = \frac{(q^{k+1}-q^{2k+1})(q^{21}-q^{k-21})}{(1-q)^2(q^{21}-q^{2k-22})(q^{21}-q^{2k-21})^2(q^{21}-q^{2k-20})}$$
$$\times \{4\sin^2(\frac{1}{2}\ln q)q^{k-23/2}(1-q)\left(1+q-q^2+q^{21}-q^{22}-q^{k-1}\right)\left(1-q^{k-20}(1+q-q^2)-q^{k+1}(1-q)\right)$$
$$-(q^{4k-42}+2q^{2k}+q^{42})\}. \tag{28}$$

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