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On A Hybrid Concept for Approximating Self-Excited Periodic Oscillations of Large-Scaled Dynamical Systems

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Concerning the approximation of self-excited periodic oscillations in large-scaled mechanical systems involving strong nonlinearities, this contribution suggests a concept for an efficient treatment. The presented *Hybrid FD-HB* method takes the advantages of both schemes *Harmonic Balance* and *Finite Difference* to enhance the ratio of computational cost and accuracy for mechanical systems with many degrees of freedom. Within this contribution the residual equations, required when applying a NEWTON-RAPHSON-scheme, are derived and the method is applied to a stiff nonlinear mechanical system.

1 Introduction

The analysis of self-excited periodic oscillations in large-scaled systems is a current field of research. For the direct approximation of periodic vibrations in nonlinear systems numerous numerical schemes have been developed. Besides *Shooting* or *Collocation* methods *Harmonic Balance Method* (HB) and *Finite Difference* (FD) schemes may be employed. In contrast to numerical time integration these methods formulate an algebraic equation system that can be solved by NEWTON-RAPHSON schemes. Also, all these methods may perform different, particularly when dealing with systems involving strong nonlinearities (*stiff systems*) [1].

Besides various enhancements for these classical methods, hybrid concepts were developed combining advantages of two methods, e.g. in [2] a *Mixed Shooting-HB* method is proposed. Here, a *Hybrid FD-HB* (HFH) method is suggested taking advantage of both: sophisticated resolution of nonlinear domains (FD) and approximating the linear domains via HB with a few harmonics for achieving sufficient accuracy. For systems with locally acting nonlinearities and many degrees of freedom this will enhance the balance of computational cost and accuracy.

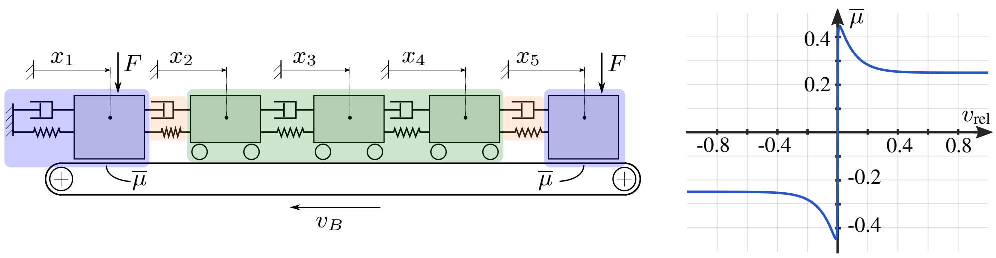

Fig. 1: Minimal model with 5 degrees of freedom and (regularized) friction curve with negative gradient. Here, the system is graphically divided into a linear (green) domain Γ_{lin} and a nonlinear (blue) domain Γ_{nl}. The transition zone (orange) is denoted with $\partial\Gamma$.

Within this contribution, the HFH is applied to a chain of oscillators showing stick-slip vibrations caused by the regularised friction force $\overline{\mu}F$ with negative slope, see Fig. (1). To this end, the algebraic residual equations for the NEWTON-RAPHSON scheme including phase condition are deduced, and special attention is given to the transition zone (orange) $\partial\Gamma$ between linear and nonlinear domain. Finally, a comparison of FD and HFH is shown.

2 Method

This method focusses on large-scaled autonomous systems with locally acting nonlinearities, where only a few degrees of freedom (DoF) are affected by nonlinear forces $\mathrm{vol}(\Gamma_{\mathrm{nl}}) \ll \mathrm{vol}(\Gamma_{\mathrm{lin}})$. That allows the separation of the nonlinear and linear domain, so the governing equations of a mechanical problem can be transformed to

$$\boldsymbol{M}_{11}\ddot{\boldsymbol{x}}_{\mathrm{nl}} + \boldsymbol{P}_{11}\dot{\boldsymbol{x}}_{\mathrm{nl}} + \boldsymbol{C}_{11}\boldsymbol{x}_{\mathrm{nl}} + \boldsymbol{f}_{\mathrm{nl}}(\boldsymbol{x}_{\mathrm{nl}}, \dot{\boldsymbol{x}}_{\mathrm{nl}}) = -\boldsymbol{M}_{12}\ddot{\boldsymbol{x}}_{\mathrm{lin}} + \boldsymbol{P}_{12}\dot{\boldsymbol{x}}_{\mathrm{lin}} + \boldsymbol{C}_{12}\boldsymbol{x}_{\mathrm{lin}} \tag{1a}$$

$$\boldsymbol{M}_{22}\ddot{\boldsymbol{x}}_{\mathrm{lin}} + \boldsymbol{P}_{22}\dot{\boldsymbol{x}}_{\mathrm{lin}} + \boldsymbol{C}_{22}\boldsymbol{x}_{\mathrm{lin}} = -\boldsymbol{M}_{21}\ddot{\boldsymbol{x}}_{\mathrm{nl}} + \boldsymbol{P}_{21}\dot{\boldsymbol{x}}_{\mathrm{nl}} + \boldsymbol{C}_{21}\boldsymbol{x}_{\mathrm{nl}} \tag{1b}$$

where $\boldsymbol{x}_{\mathrm{nl}}$ denotes the nonlinear DoF lying in Γ_{nl} and $\boldsymbol{x}_{\mathrm{lin}}$ the linear DoF lying in Γ_{lin} as it is done in [2]. Here, it is assumed that the mechanism of self-excitation is evoked by local mechanisms included in the eq. (1a). Thus for stationary solutions,

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the linear substructure shows forced vibrations excited by the neighbouring nonlinear DoF, see eq. (1). The basic idea is to express the nonlinear DoF adjoining $\partial\Gamma$ as a FOURIER series $\boldsymbol{x}_{\text{nl}} \approx \Re\{\sum_{k=0}^{H} \boldsymbol{X}_{k,\text{nl}}\, \mathrm{e}^{\,\mathrm{j}k\omega t}\}$ with base frequency ω, complex FOURIER coefficients $\boldsymbol{X}_{k,\text{nl}}$ and a defined number of harmonics H. Since the nonlinear DoF $\boldsymbol{x}_{\text{nl}}$ will be given at N equidistant grid points in time domain, the coefficients $\boldsymbol{X}_{k,\text{nl}}$ are evaluated using the Discrete FOURIER Transformation.

Assuming that the linear DoF $\boldsymbol{x}_{\text{lin}}$ are representable as a FOURIER series and inserting the approximation for $\boldsymbol{x}_{\text{nl}}$ into eq. (1b) enables an analytical evaluation of the complex FOURIER coefficients of the linear DoF with

$$\boldsymbol{X}_{k,\text{lin}} = -\boldsymbol{H}_{22}^{-1}\boldsymbol{H}_{21}\boldsymbol{X}_{k,\text{nl}}, \quad \text{whith} \quad \boldsymbol{H}_{ij} = -(k\omega)^2\boldsymbol{M}_{ij} + \mathrm{j}k\omega\boldsymbol{P}_{ij} + \boldsymbol{C}_{ij}, \quad i,j \in \{1,2\} \tag{2}$$

via HB, where ω is the base frequency of the periodic solution and $k = 0, 1, \ldots, H$ holds. Inserting the time domain expression of the linear DoF $\boldsymbol{x}_{\text{lin}} = \Re\{\sum_{k=0}^{H} \boldsymbol{X}_{k,\text{lin}}\, \mathrm{e}^{\,\mathrm{j}k\omega t}\}$ and its derivatives into the right hand side of eq. (1a) gives the feedback

$$\boldsymbol{f}_{\text{lin}} = \Re\left\{\sum_{k=0}^{H} \boldsymbol{F}_{k,\text{lin}}\, \mathrm{e}^{\,\mathrm{j}k\omega t}\right\} = -\Re\left\{\sum_{k=0}^{H} \boldsymbol{H}_{12}\boldsymbol{H}_{22}^{-1}\boldsymbol{H}_{21}\boldsymbol{X}_{k,\text{nl}}\, \mathrm{e}^{\,\mathrm{j}k\omega t}\right\} \tag{3}$$

acting on the nonlinear structure. Next, the nonlinear DoF $\boldsymbol{x}_{\text{nl}}$ are evaluated at N grid points in time domain. Transforming eq. (1a) into state space and inserting the feedback forces $\boldsymbol{f}_{\text{lin}}$, the resulting algebraic equations at any time grid point t_i read

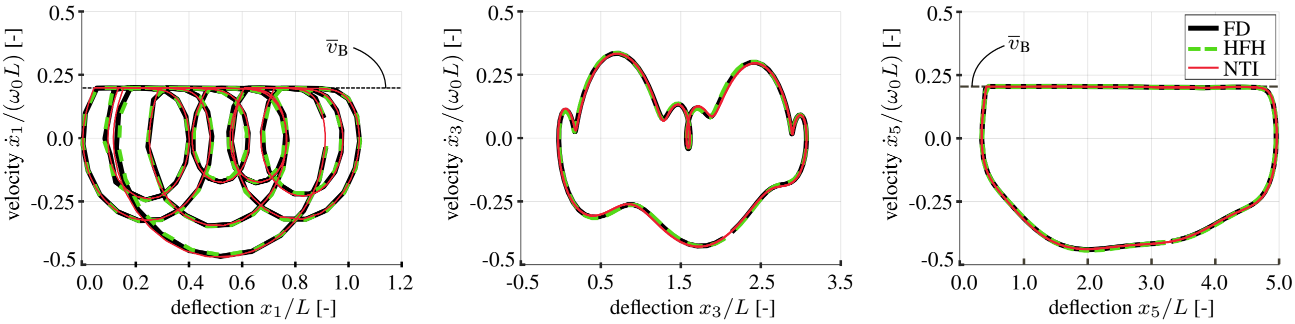
$$\boldsymbol{R}_{\text{res}} = \left\{ \begin{matrix} \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M}_{11} \end{pmatrix} \dot{\boldsymbol{z}}_i - \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{C}_{11} & -\boldsymbol{P}_{11} \end{pmatrix} \boldsymbol{z}_i + \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f}_{\text{nl}}(\boldsymbol{z}_i) \end{pmatrix} + \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f}_{\text{lin}}(\boldsymbol{z}_i) \end{pmatrix} \\ p_c(\boldsymbol{z},\omega) \end{matrix} \right\} \overset{!}{=} \boldsymbol{0}, \qquad i = 1, \ldots, N \tag{4}$$

where $\boldsymbol{z}_i = (\boldsymbol{x}_{\text{nl}}, \dot{\boldsymbol{x}}_{\text{nl}})_i^\top$ denotes the nonlinear DoF in state space notation at the i-th time step. Since self-excited oscillations are calculated, a path condition $p_c(\boldsymbol{z},\omega)$ is added. Finally, the derivative is estimated using FD $\dot{\boldsymbol{z}} \approx \sum_{j\in\mathcal{M}} w_j \boldsymbol{z}_{i+j}$. The weights w_j can be evaluated using a general formula for arbitrary degree and order of accuracy, see for example [3].

Accounting periodicity of $\boldsymbol{z}_i$ on the time grid, $\sum_{j\in\mathcal{M}} w_j \boldsymbol{z}_{i+j}$ can be written as matrix multiplication. Inserting that into eq. (4) gives an algebraic equation system that can be solved by NEWTON-RAPHSON schemes, where the unknowns are the nonlinear DoF $\boldsymbol{z}_i = (\boldsymbol{x}_{\text{nl}}, \dot{\boldsymbol{x}}_{\text{nl}})_i^\top$ at N points in time domain and the base frequency ω.

3 Application

As a first application, periodic limit cycles of the system, shown in Fig. 1 are calculated. Therefore, $N_{\text{FD}} = 150$ time samples were taken and $H = 10$ harmonics were considered, while the derivative $\dot{\boldsymbol{z}}_i$ is approximated using a third order upwind scheme. Although the calculation time of FD takes two times longer, both methods achieve nearly same accuracy, see Fig. 2.



(a) Limit cycle of the first DoF **(b)** Limit cycle of the third DoF **(c)** Limit cycle of the fifth DoF

Fig. 2: Periodic limit cycle of the system shown in Fig. 1 at $\bar{v}_B = 0.2$: comparison of *Finite Difference* (FD) and *Hybrid FD-HB* (HFH) versus the *numerical time integration* (NTI) as reference solution.

Here, HFH shows better performance mainly caused by a smaller algebraic equation system to solve. So, future research will be addressed to efficiency and applicability to large-scaled dynamical systems involving a higher amount of linear DoF.

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References

[1] S. Karkar, B. Cochelin, and C. Vergez, A comparative study of the harmonic balance method and the orthogonal collocation method on stiff nonlinear systems, Journal of Sound and Vibration **333**, p. 2554–2567 (2014).
[2] F. Schreyer, and R. Leine A mixed shooting-harmonic balance method for unilaterally constrained mechanical systems, Archive of mechanical engineering, De Gruyter Open VOL. **LXIII** (2016).
[3] H.Z. Hassan, A.A. Mohamad, and G.E. Atteia, An algorithm for the finite difference approximation of derivatives with arbitrary degree and order of accuracy, Journal of Computational and Applied Mathematics **236**, p. 2622–2631 (2012).

