The principle of JOURDAIN for Dynamic Ideal Elastoplasticity

The principle of JOURDAIN for dissipative continua is based on the stationarity of the balance of energy

\[ \text{stat sup} \inf_{\mathbf{u}, \mathbf{e}_p, \sigma, \gamma} P(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{e}_p, \sigma, \gamma) \] \[ \text{with} \quad P = \dot{K} + \dot{E} + P^* + D. \] (1)

Equation (1) consists of the rate of kinetic energy \( K \), the rate of internal energy \( E \), the power of external loads \( P^* \) and the pseudopotential \( D \) characterizing plastic effects, cf. [2, 4]. It furthermore depends on the stress tensor \( \sigma \), the plastic multiplier \( \gamma \), the displacement vector \( \mathbf{u} \), the plastic strain tensor \( \mathbf{e}_p \) as well as on corresponding first and second time derivatives. Evaluating Equation (1), performing a spatial discretization, and applying a semi-smooth NEWTON method yields the linearized semidiscrete form

\[ M \dot{\mathbf{w}} + D \mathbf{w} + K_1 \mathbf{w} = \mathbf{R}_1^t - \mathbf{R}_1, \quad K_2 \mathbf{w} = -\mathbf{R}_2, \quad \gamma^{i,k+1} = 0 \quad \forall i \in \mathcal{I}^k, \] (2)

\[ \mathcal{A}^{k+1} := \{ i | \gamma^{i,k+1} + d r_{2,i}^{k+1} > 0 \}, \quad \mathcal{T}^{k+1} := \{ i | \gamma^{i,k+1} + d r_{2,i}^{k+1} \leq 0 \}. \]

Therein the vector of unknowns \( \mathbf{w} = [\mathbf{u}, \mathbf{e}_p, \sigma, \gamma] \), the generalized mass matrix \( M \), the generalized damping matrix \( D \), the generalized stiffness matrices \( K_1, K_2 \), the generalized residual load vectors \( \mathbf{R}_1, \mathbf{R}_2 \), the external load vector \( \mathbf{R}_1^t \) and the inactive as well as active set \( \mathcal{I}, \mathcal{A} \) are included. Moreover, \( r_{2,i}^{k+1} \) represents the nodal evaluation of the VON MISES yield function at iteration \( k + 1 \) and \( d \geq 0 \) an arbitrary constant, cf. [3, 4].

3 RUNGE-KUTTA Schemes in Multifield Plasticity

For a temporal solution of Equation (2) higher order stiffly accurate fully implicit RUNGE-KUTTA integrators can be applied. Therefore, the time interval of interest \([0,T]\) is split into time steps \( \Delta t = t_{n+1} - t_n \) and the stages \( t_{ni} = t_n + c_i \Delta t \) with \( c_i \in [0,1] \) and \( i = 1, \ldots, s \) are introduced. At the latter points in time Equation (2) is solved simultaneously. Therefore, the link

\[ \mathbf{w}_t = \mathbf{w}_{t,n} + \Delta t \mathbf{A}_t \mathbf{w}_t \] with \( \mathbf{A}_t = \begin{bmatrix} a_{11} & \cdots & a_{11} & \cdots & a_{12} & \cdots & a_{1s} & \cdots & a_{1s} \\ a_{21} & \cdots & a_{21} & \cdots & a_{22} & \cdots & a_{2s} & \cdots & a_{2s} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{s1} & \cdots & a_{s2} & \cdots & a_{ss} & \cdots & a_{ss} \end{bmatrix} \] (3)

is established, whereby the matrix \( \mathbf{A}_t \) represents an extension of the RUNGE-KUTTA coefficients \( a_{ij} \), cf. [1]. Furthermore, the stage values are aggregated in the vector \( \mathbf{w}_t = [\mathbf{w}_{t,1}^1, \ldots, \mathbf{w}_{t,s}^s] \), the corresponding time derivatives in \( \dot{\mathbf{w}}_t \) and the vector \( \mathbf{w}_{t,n} \) consists of \( s \) copies of \( \mathbf{w}_{t,n} \). Inserting the linearization of Relation (3) into Equation (2), yields the linear system of equations

\[ \left[ \frac{1}{\Delta t^2} M \mathbf{A}_t^{-1} \mathbf{A}_t^{-1} + \frac{1}{\Delta t} D \mathbf{A}_t^{-1} + K_{1,t} \right] \Delta \mathbf{w}_t = \mathbf{R}_{1,t}^1 - \mathbf{R}_{1,t}, \quad K_{2,t} \Delta \mathbf{w}_t = -\mathbf{R}_{2,t}, \quad \gamma^{i,k+1}_t = 0 \quad \forall i \in \mathcal{I}^k. \]
\[ A_i^{k+1} := \{ i | \gamma_i^{k+1} + d \mathbf{r}_2^{k+1} > 0 \}, \quad I_i^{k+1} := \{ i | \gamma_i^{k+1} + d \mathbf{r}_2^{k+1} \leq 0 \}, \quad d > 0, \]

where the incorporated matrices are extended properly. For a final solution a solver for linear systems of equations is applied.

## 4 Time Discretization Error Analysis for an Axisymmetric Model Problem

With the preceding procedure at hand, a time discretization error analysis for the axisymmetric benchmark problem of Fig. 1, where a time dependent sinusoidal load is applied to a cylinder with LAME constants \( \lambda, \mu \), density \( \rho_0 \) and yield stress \( \sigma_y \), is performed, cf. [4]. Hence, at the end of each time interval the global \( h \)-error of a quantity \( X \) is calculated by subtracting the numerically determined values \( X_{ns} \) from those obtained with a very small time step size \( X_{ns}^{\Delta t/16} \), while identical initial values are taken into account. The local \( h \)-error estimator, however, is obtained, by performing two calculations with two distinct time step sizes simultaneously and comparing the results after each time step. Additionally, the results from the calculation using the bigger time step size are passed to the other one as initial condition for the next time step.

\[
e_{ns}^{glob} = ||X_{ns}^{\Delta t/16}(t_0) - X_{ns}(t_0)||, \quad e_{ns} = ||X_{ns}^{\Delta t/2} - X_{ns}||, \quad q^{glob}/q^{loc} = \text{mean}(\text{linear fit}(\log(\Delta t), \log(e_{ns}^{glob})/(e_{ns}^{loc}))). (5)
\]

If these errors are determined for distinct time step sizes and collocated in vectors, expression (5) can be used to determine the order of convergence of the time discretization scheme. A more detailed explanation is given in [4,6] and references therein. For distinct RUNGE-KUTTA schemes the orders of convergence for different field variables, obtained from the local and the global error, are shown in Table 1. Considering the local estimates, it becomes apparent that the Lobatto IIIC(2) method is quite far away from its theoretical order of two. In contrast, the Lobatto IIIC(3) scheme almost reaches its theoretical order of four for all considered field variables. A similar behavior can be recognized within the Radau IIA schemes. Analyzing the corresponding global measurements, however, yields a deviating picture. Apart from the Lobatto IIIC(2) scheme, all methods are estimated to reach orders well above two but not more.

<table>
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<tr>
<th>Scheme</th>
<th>( q^{loc}(u) )</th>
<th>( q^{loc}(\varepsilon_p) )</th>
<th>( q^{loc}(\sigma) )</th>
<th>( q^{glob}(u) )</th>
<th>( q^{glob}(\varepsilon_p) )</th>
<th>( q^{glob}(\sigma) )</th>
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<td>2.19</td>
<td>2.48</td>
</tr>
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Fig. 1: Axisymmetric benchmark problem

Table 1: Estimation of the order of convergence

## 5 Conclusion and Outlook

In the present paper the applicability of higher order accurate fully implicit RUNGE-KUTTA schemes to ideal elastoplasticity using a multifield approach is demonstrated. With the help of a model problem time discretization errors and the linked orders of convergence for distinct time integrators are compared. Orders of convergence of about five are obtained if the local \( h \)-error estimator is considered. But for the global \( h \)-error estimator an order of convergence greater than three could not be determined. An explanation for this order reduction phenomenon is still missing and hence an open task for further investigations.

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**References**

