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An Investigation on Limiting Potentials for Damage Prediction of Viscoelastic Adhesives

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In his recent work VOLOKH introduced the energy limiting method as an alternative to continuum damage mechanics developed and envolved by KACHANOV, RABOTNOV, LEMAITRE and many more. In contrast to continuum damage mechanics, the energy limiting method is physically grounded by the process of atomistic debonding and, therefore, motivated on an energy basis. In order to use the energy limiting method within the framework of continuum mechanics, a limiting potential needs to be defined, which can be seen as a macroscopic manifestation of the LENNARD-JONES bonding potential. The first limiting potential was proposed by VOLOKH and applies an exponential function with one parameter describing the maximum energy that can be absorbed by a material. Consequently, this potential can only capture the onset of damage of different materials, but not the evolution of damage. To overcome this restriction, VOLOKH proposed a generalization applying a Gamma function with two parameters describing the maximum energy and the shape of the damage evolution. Besides this improvement, this generalization has the disadvantages of violating physical plausibility conditions, needing an integration for the evaluation and having coupled parameters, which affect each other. The aim of this contribution is to propose and investigate a collection of analytical limiting potentials satisfying all physical plausibility conditions while having decoupled parameters. Finally, these potentials are used for the damage prediction of thick, viscoelastic adhesives.

1 Energy limiting method and existing limiting potentials

The energy limiting method, proposed by VOLOKH, offers a convenient way to enhance elastic material models to strain softening and, therefore, the prediction of damage, see [1]. The modeling of damage is introduced by wrapping a limiting potential Ψ^{d} around an elastic approach for the HELMHOLTZ free energy Ψ^{eq}, where the limiting potential needs to satisfy four physical plausibility conditions:

$$\lim_{\Psi^{\mathrm{eq}}\to 0^+} \Psi^{\mathrm{d}} = 0 \qquad \lim_{\Psi^{\mathrm{eq}}\to \infty} \Psi^{\mathrm{d}} = \Phi_0 \qquad \lim_{\Psi^{\mathrm{eq}}\to 0^+} \frac{\mathrm{d}\Psi^{\mathrm{d}}}{\mathrm{d}\Psi^{\mathrm{eq}}} = 1 \qquad 1 \geq \frac{\mathrm{d}\Psi^{\mathrm{d}}}{\mathrm{d}\Psi^{\mathrm{eq}}} \geq 0 \tag{1}$$

The first two conditions on a limiting potential ensure that the energy has a value of zero in the unloaded state on the one hand and reaches a finite saturation value Φ_0, where $\Phi_0 \in \mathbb{R}^+_{\neq 0}$, in the limit of large elastic energy on the other hand. The other two conditions concern the derivative of the limiting potential with respect to the elastic energy, which arises in the stress calculation as a factor in front of the elastic stresses due to the additional application of the chain rule while calculating the time derivative of the limiting potential:

$$\Psi^{\mathrm{d}} = \Psi^{\mathrm{d}}\left(\Psi^{\mathrm{eq}}\right) \qquad \Rightarrow \qquad \dot{\Psi}^{\mathrm{d}} = \frac{\mathrm{d}\Psi^{\mathrm{d}}}{\mathrm{d}\Psi^{\mathrm{eq}}}\dot{\Psi}^{\mathrm{eq}} \qquad \Rightarrow \qquad \boldsymbol{\sigma}^{\mathrm{d}} = \frac{\mathrm{d}\Psi^{\mathrm{d}}}{\mathrm{d}\Psi^{\mathrm{eq}}}\boldsymbol{\sigma}^{\mathrm{eq}} \tag{2}$$

In order to be physicially reasonable, this factor needs to have a value of one in the unloaded state on the one hand and must lie between one and zero for every possible energy obtained on the other hand. Concerning the multiplier, it can be noted that it provides a simple link to continuum damage mechanics and its damage variable D, by means of Eq. (3). Thus, the concept of the energy limiting method can also be used to develop evolution equations for the damage variable and its usage within continuum damage mechanics. However, this connection to the continuum damage mechanics will not be pursued further here:

$$1 - D = \frac{\mathrm{d}\Psi^{\mathrm{d}}}{\mathrm{d}\Psi^{\mathrm{eq}}} \qquad \Rightarrow \qquad \dot{D} = -\frac{\mathrm{D}}{\mathrm{D}t}\frac{\mathrm{d}\Psi^{\mathrm{d}}}{\mathrm{d}\Psi^{\mathrm{eq}}} \tag{3}$$

At the present time, only two limiting potentials exist in the literature, whereby both potentials have been proposed by VOLOKH, see [1] and [2]. The first limiting potential applies an exponential function with one parameter describing the maximum energy Φ_0 that can be absorbed by a material. Since this potential can only capture the onset of damage for different materials, but not the evolution of damage, VOLOKH introduced a generalization of the first limiting potential, which employs a lower Gamma function Γ_1 with an additional parameter m, where $m \in \mathbb{R}^+_{\neq 0}$, in oder to modify the evolution of damage. Increasing values of the parameter m can be used to model more brittle material failure, see Fig. 1 a) and b):

$$\Psi^{\mathrm{d}}_{\mathrm{VOLOKH,m}} = \frac{\Phi_0}{m}\Gamma_1\left(\frac{1}{m}, \left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^m\right) \qquad \frac{\mathrm{d}\Psi^{\mathrm{d}}_{\mathrm{VOLOKH,m}}}{\mathrm{d}\Psi^{\mathrm{eq}}} = \exp\left(-\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^m\right) \tag{4}$$

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The generalized limiting potential by VOLOKH, see Eq. (4), has the advantage of containing two parameters, which enable a flexible fitting of numerical simulations to the experimental data. However, the potential also has two drawbacks: First it is defined in terms of the Gamma function, so an integration is required for the evaluation. Assuming the usage of the finite element method, this integration has to be carried out in each time step and at each GAUSSian point, which causes an unnecessarily high numerical effort. The second drawback results from a flawed definition of the potential, since the physical plausibility conditions are not satisfied for all parameter combinations, but only for $m = 1$ and $m \to \infty$:

$$\lim_{\Psi^{\mathrm{eq}} \to \infty} \Psi^{\mathrm{d}}_{\mathrm{VOLOKH,m}} = \Gamma\left(1 + \frac{1}{m}\right) \Phi_0 \leq \Phi_0 \tag{5}$$

By means of the limit in Eq. (5), it is observable that the limiting potential indeed saturates for every combination of parameters, but it does not converge to the specified value of the parameter Φ_0, but instead to a value of Φ_0 multiplied by the Gamma function in terms of the parameter m, see Fig. 1 a). This results in a coupling of the parameters Φ_0 and m, which is not pleasing with respect to parameter identification.

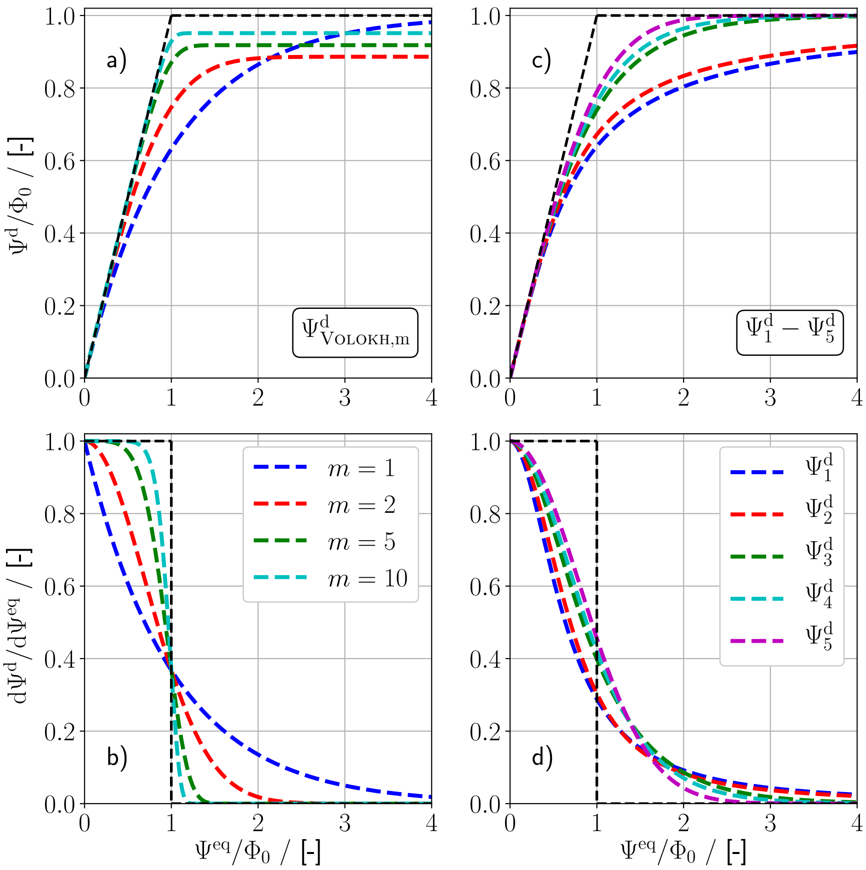

Fig. 1: Limiting potentials $\Psi^{\mathrm{d}}_{\mathrm{VOLOKH,m}}$, $\Psi^{\mathrm{d}}_1 - \Psi^{\mathrm{d}}_5$ and $\Psi^{\mathrm{d}}_{\mathrm{IFM,n}}$ related to the saturation value, as well as their derivatives with respect to the elastic energy in dependence of the elastic energy devided by the saturation value

2 Proposal of new limiting potentials

Considering the drawbacks of the generalized limiting potential by VOLOKH, see Eq. (4), the goal is to develop limiting potentials that are defined by analytical functions and also contain at least two, decoupled material parameters. The first considerations concern the well-known saturation functions, namely the invers tangent function Ψ^{d}_1, the LANGEVIN function Ψ^{d}_2, the GUDERMANN function Ψ^{d}_3, the logistic function Ψ^{d}_4 and the error function Ψ^{d}_5 of GAUSS:

$$\Psi^{\mathrm{d}}_1 = \Phi_0 \frac{2}{\pi} \arctan\left(\frac{\pi}{2}\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right) \qquad \frac{\mathrm{d}\Psi^{\mathrm{d}}_1}{\mathrm{d}\Psi^{\mathrm{eq}}} = \left(\left(\frac{\pi}{2}\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^2 + 1\right)^{-1} \tag{6}$$

$$\Psi^{\mathrm{d}}_2 = \Phi_0 \left(\coth\left(3\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right) - \frac{1}{3}\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^{-1}\right) \qquad \frac{\mathrm{d}\Psi^{\mathrm{d}}_2}{\mathrm{d}\Psi^{\mathrm{eq}}} = \frac{1}{3}\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^{-2} - 3\left(\sinh\left(3\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)\right)^{-2} \tag{7}$$

$$\Psi_3^{\mathrm{d}} = \Phi_0 \left(1 - \frac{4}{\pi} \arctan\left(\exp\left(-\frac{\pi}{2}\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)\right)\right) \qquad \frac{\mathrm{d}\Psi_3^{\mathrm{d}}}{\mathrm{d}\Psi^{\mathrm{eq}}} = \left(\cosh\left(\frac{\pi}{2}\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)\right)^{-1} \tag{8}$$

$$\Psi_4^{\mathrm{d}} = \Phi_0 \tanh\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right) \qquad \frac{\mathrm{d}\Psi_4^{\mathrm{d}}}{\mathrm{d}\Psi^{\mathrm{eq}}} = \left(\cosh\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)\right)^{-2} \tag{9}$$

$$\Psi_5^{\mathrm{d}} = \Phi_0 \mathrm{erf}\left(\frac{\sqrt{\pi}}{2}\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right) \qquad \frac{\mathrm{d}\Psi_5^{\mathrm{d}}}{\mathrm{d}\Psi^{\mathrm{eq}}} = \exp\left(-\frac{\pi}{4}\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^2\right) \tag{10}$$

All these saturation functions have been modified to satisfy all physical plausibility conditions, but, as observable, each contains only one parameter Φ_0 describing the maximum energy that can be absorbed by a material, see Fig. 1 c) and d). Every further modification, e.g. including an additional material parameter in Eqn. (6) to (10), would lead to the violation of the physical plausibility conditions. In regard of the unflexibility, these limiting potentials are neglected.

Further investigation lead into a new, analytical saturation function $\Psi^{\mathrm{d}}_{\mathrm{IFM,n}}$, whose basic idea is motivated by relating the identity function $\Psi^{\mathrm{eq}}/\Phi_0$ at the $\mathcal{L}^n$-norm of the point $(1, \Psi^{\mathrm{eq}}/\Phi_0)$. This idea results in an analytical limiting potential satisfying all physical plausibility conditions and also having an additional parameter n, where $n \in \mathbb{R}^+_{\neq 0}$, in order to modify the evolution of damage. Analogous to the generalized model of VOLOKH, more brittle material failure can be modeled with increasing values of the parameter n, see Fig. 1 e) and f):

$$\Psi^{\mathrm{d}}_{\mathrm{IFM,n}} = \Psi^{\mathrm{eq}} \left(1 + \left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^n\right)^{-\frac{1}{n}} \qquad \frac{\mathrm{d}\Psi^{\mathrm{d}}_{\mathrm{IFM,n}}}{\mathrm{d}\Psi^{\mathrm{eq}}} = \left(1 + \left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^n\right)^{-\frac{1}{n}-1} \tag{11}$$

Besides being an analytical function and containing two material parameters, it can also be shown that the newly developed limiting potential, see Eq. (11), is able to capture the behaviour of the generalized potential by VOLOKH, see Eq. (4), and the saturation functions, see. Eqn. (6) to (10), with a deviation of less than 5 % by means of optimizing a value for the parameter n in order to fit to the other limiting potentials. Another advantage of the new limiting potential is given by the possibility of its further generalization to $\Psi^{\mathrm{d}}_{\mathrm{IFM,n,f}}$ by relating a function $f(\Psi^{\mathrm{eq}}/\Phi_0)$ at the $\mathcal{L}^n$-norm of the point $(1, f(\Psi^{\mathrm{eq}}/\Phi_0))$:

$$\Psi^{\mathrm{d}}_{\mathrm{IFM,n,f}} = \Phi_0 f\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)\left(1 + f\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^n\right)^{-\frac{1}{n}} \qquad \frac{\mathrm{d}\Psi^{\mathrm{d}}_{\mathrm{IFM,n,f}}}{\mathrm{d}\Psi^{\mathrm{eq}}} = \Phi_0 \frac{\mathrm{d}f}{\mathrm{d}\Psi^{\mathrm{eq}}}\left(1 + f\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^n\right)^{-\frac{1}{n}-1} \tag{12}$$

This generalization offers the flexibility to choose a function f and to introduce further material parameters within this function on the one hand. But the limiting potential still needs to satisfy all the physical plausibility conditions on the other hand. However, reaching the saturation value is guaranteed by the structure of the generalized potential itself. The other three conditions can be simplified by inserting the generalization according to Eq. (12) into Eq. (1):

$$\lim_{\Psi^{\mathrm{eq}} \to 0^+} f = 0 \qquad \lim_{\Psi^{\mathrm{eq}} \to 0^+} \Phi_0 \frac{\mathrm{d}f}{\mathrm{d}\Psi^{\mathrm{eq}}} = 1 \qquad (1 + f^n)^{\frac{1}{n}+1} \geq \Phi_0 \frac{\mathrm{d}f}{\mathrm{d}\Psi^{\mathrm{eq}}} \geq 0 \tag{13}$$

Possible approaches, satisfying these plausibility conditions, are given by the functions $f_{\mathrm{k},1}$ and $f_{\mathrm{k},2}$, each of which has another material parameter k_1, where $k_1 \in [0, 1]$, and k_2, where $k_2 \in \mathbb{R}^+_{\neq 0}$, respectively:

$$f_{\mathrm{k},1}\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right) = \ln^{k_1}\left(1 + \frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^{1-k_1} \qquad f_{\mathrm{k},2}\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right) = (k_2 + 1)\left(\left(1 + \frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^{\frac{1}{k_2+1}} - 1\right) \tag{14}$$

The most general approach is achieved by employing a weighted average, because it enables an infinite amount of material parameters: The identity function, the functions $f_{\mathrm{k},1}$, $f_{\mathrm{k},2}$ and possible further functions, which satisfy the physical plausibility conditions according to Eq. (13), can be used for averaging by a weighting α_i, where $\alpha_i \in [0, 1]$:

$$f\left(\frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right) = (1 - \sum_{i=1}^{n} \alpha_i)\frac{\Psi^{\mathrm{eq}}}{\Phi_0} + \sum_{i=1}^{n} \alpha_i f_{\mathrm{k},i} \tag{15}$$

Within the parameter identification, each parameter must be determined by means of experimental data. Thus, limiting potentials with two or three parameters seem to be desirable as a compromise between the flexibility in modeling and the effort required for parameter determination. As a limiting potential with two parameters, the potential $\Psi^{\mathrm{d}}_{\mathrm{IFM,n}}$ is recommended for this purpose. The potential $\Psi^{\mathrm{d}}_{\mathrm{IFM,n,ln}}$, which is built by the weighted average of the indentity function and the function $f_{\mathrm{k},1}$ with $k_1 = 1$ and $\alpha \in [0, 1]$, is again recommended as the limiting potential with three parameters:

$$\begin{aligned} \Psi^{\mathrm{d}}_{\mathrm{IFM,n,ln}} = \quad & \Phi_0\left((1-\alpha)\frac{\Psi^{\mathrm{eq}}}{\Phi_0} + \alpha \ln\left(1 + \frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)\right) \cdot \left(1 + \left((1-\alpha)\frac{\Psi^{\mathrm{eq}}}{\Phi_0} + \alpha \ln\left(1 + \frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)\right)^n\right)^{-\frac{1}{n}} \\ \frac{\mathrm{d}\Psi^{\mathrm{d}}_{\mathrm{IFM,n,ln}}}{\mathrm{d}\Psi^{\mathrm{eq}}} = \quad & \left((1-\alpha)\frac{\Psi^{\mathrm{eq}}}{\Phi_0} + 1\right)\left(1 + \frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)^{-1} \cdot \left(1 + \left((1-\alpha)\frac{\Psi^{\mathrm{eq}}}{\Phi_0} + \alpha \ln\left(1 + \frac{\Psi^{\mathrm{eq}}}{\Phi_0}\right)\right)^n\right)^{-\frac{1}{n}-1} \end{aligned} \tag{16}$$

In regard to employing a weighted average according to Eq. (15), the function $f_{\mathrm{k},1}$ in general seems to be more suitable to be used in contrast to $f_{\mathrm{k},2}$, since it offers a higher concavity. Therefore, a broader range of possible models for the evolution of damage is established.

3 Introduction of irreversibility

The basic conception of the energy limiting method by VOLOKH, see [1], models reversible damage, which heals during unloading of a material, although it introduces the loss of convexity by means of reaching a saturation value, see Fig. 2 a) and b). The reversibility of the damage is already observable in Eq. (2) since it lacks dissipative terms while calculating the time derivative of the HELMHOLTZ free energy. The absence of dissipative terms can be further traced back while deriving a formulation for the stress calculation by inserting the time derivative into the CLAUSIUS-PLANCK inequality, see e.g. [3] and [4], and undergoing the COLEMAN-NOLL procedure, see [5].

In order to transform the physically implausible modeling approach from reversible to irreversible damage, an additional internal variable must be introduced, see [6]. In this work, the all-time maximum elastic energy $\Psi^{\mathrm{eq,max}}$ is proposed as a new variable:

$$\Psi^{\mathrm{eq,max}} = \max_{-\infty<s\leq t} \Psi^{\mathrm{eq}} \tag{17}$$

The all-time maximum elastic energy is then used for the calculation of the multiplier in front of the elastic stress variable by means of Eq. (2), instead of the elastic energy. Therefore, during unloading of a material, the multiplier remains constant at the highest damage value reached within the deformation process of the material, see Fig. 2 c) and d).

This approach can be generalized by introducing two additional material parameters Ψ^{el} and Ψ^{rev}, where $\Psi^{\mathrm{el}} \in \mathbb{R}^+_{\neq 0}$ and also $\Psi^{\mathrm{rev}} \in \mathbb{R}^+_{\neq 0}$, denoting the upper bound of the elastic material behaviour and the upper bound of the range of reversible damage, respectivly:

$$\Psi^{\mathrm{eq,max}} = \begin{cases} \left\langle \Psi^{\mathrm{eq}} - \Psi^{\mathrm{el}} \right\rangle, & \Psi^{\mathrm{eq}} < \Psi^{\mathrm{rev}} \\ \max\limits_{-\infty<s\leq t} \left\langle \Psi^{\mathrm{eq}} - \Psi^{\mathrm{el}} \right\rangle, & \Psi^{\mathrm{eq}} \geq \Psi^{\mathrm{rev}} \end{cases} \tag{18}$$

Due to the case distinction, three ranges of material behaviour are defined: For the case $0 \leq \Psi^{\mathrm{eq}} < \Psi^{\mathrm{el}}$, purely elastic material behaviour is modeled, since the multiplier in front of the elastic stress variable always has a value of one due to the MACAULAY bracket. For the case $\Psi^{\mathrm{el}} \leq \Psi^{\mathrm{eq}} < \Psi^{\mathrm{rev}}$, the MACAULAY bracket has no impact and, as a result, reversible damage to the material is modeled following the original energy limiting method. In the latter case, for $\Psi^{\mathrm{rev}} \leq \Psi^{\mathrm{eq}}$, an irreversible damage model is obtained by means of Eq. (17).

4 Damage prediction of viscoelastic adhesives

In order to illustrate the damage prediction of the energy limiting method for thick, viscoelastic adhesives, a material model for finite deformations is employed: This model includes assumptions concerning the isothermy, homogeneity and isotropy, see [7]. Further it contains a volumetric-isochoric split of the deformation gradient allowing a seperate description of the volumetric part $\Psi^{\mathrm{eq}}_{\mathrm{vol}}$ and the isochoric part $\Psi^{\mathrm{eq}}_{\mathrm{iso}}$ of the HELMHOLTZ free energy, see [8]. In addition, the isochoric part of the HELMHOLTZ free energy is also split into elastic and viscous components.

The volumetric-elastic part of the HELMHOLTZ free energy is described by the volumetric model by OGDEN, containing two material parameters: The bulk modulus K, where $K \in \mathbb{R}^+_{\neq 0}$, and a phenomenological parameter β, where $\beta \in \mathbb{R}^- \setminus [-1, 0]$, see [9]. For the description of the isochoric-elastic part of the HELMHOLTZ free energy, the isochoric model by ARRUDA-BOYCE with two material parameters is employed: The shear modulus G, where $G \in \mathbb{R}^+_{\neq 0}$, and a parameter N, where $N \in \mathbb{R}^+_{\neq 0}$, describing the polymer chain length, see [10]. To describe the isochoric-viscous part, the generalized MAXWELL-model is chosen, which has two material parameters for each MAXWELL-chain: The factors of involvement γ_i, where $\gamma_i \in [0, 1]$, and the relaxation times τ_i, where $\tau_i \in \mathbb{R}^+_{\neq 0}$, see [11]. Also the energy limiting method is used by means of the two parameter limiting potential, stated in Eq. (11). The volumetric and isochoric part of the HELMHOLTZ free energy are extended by seperate limiting potentials $\Psi^{\mathrm{d}}_{\mathrm{vol,IFM}}$ and $\Psi^{\mathrm{d}}_{\mathrm{iso,IFM}}$ with possibly various values for the material parameters $\Phi_{\mathrm{vol},0}$, $\Phi_{\mathrm{iso},0}$, n_{vol} and n_{iso} in order to enable a different damage evolution in both parts, see [12].

The following formula (19) for the calculation of the CAUCHY stress tensor $\boldsymbol{\sigma}^{\mathrm{d}}$ is obtained by taking the assumptions on isothermality, homogeneity and isotropy into account. The models by OGDEN and ARRUDA-BOYCE have not yet been used in order to keep the formula as general as possible. The already used generalized MAXWELL-model could be switched off by $\gamma_\infty = 1$ and $\gamma_i = 0$. Analogously, the already included energy limiting method could also be switched off by neglecting the derivatives of the limiting potentials with respect to the volumetric-elastic and the isochor-elastic energy:

$$\boldsymbol{\sigma}^{\mathrm{d}} = \frac{\mathrm{d}\Psi^{\mathrm{d}}_{\mathrm{vol,IFM}}}{\mathrm{d}\Psi^{\mathrm{eq}}_{\mathrm{vol}}}\frac{\partial\Psi^{\mathrm{eq}}_{\mathrm{vol}}}{\partial III_{\mathrm{F}}}\mathbf{I} + \frac{\mathrm{d}\Psi^{\mathrm{d}}_{\mathrm{iso,IFM}}}{\mathrm{d}\Psi^{\mathrm{eq}}_{\mathrm{iso}}}\left(2\gamma_\infty III_{\mathrm{F}}^{-1}\left(\left(\frac{\partial\Psi^{\mathrm{eq}}_{\mathrm{iso}}}{\partial I_{\mathrm{b,iso}}} + \frac{\partial\Psi^{\mathrm{eq}}_{\mathrm{iso}}}{\partial II_{\mathrm{b,iso}}}I_{\mathrm{b,iso}}\right)\mathbf{b}_{\mathrm{iso}} - \frac{\partial\Psi^{\mathrm{eq}}_{\mathrm{iso}}}{\partial II_{\mathrm{b,iso}}}\mathbf{b}^2_{\mathrm{iso}} \right.\right.$$
$$\left.\left. -\frac{1}{3}\left(\frac{\partial\Psi^{\mathrm{eq}}_{\mathrm{iso}}}{\partial I_{\mathrm{b,iso}}}I_{\mathrm{b,iso}} + 2\frac{\partial\Psi^{\mathrm{eq}}_{\mathrm{iso}}}{\partial II_{\mathrm{b,iso}}}II_{\mathrm{b,iso}}\right)\mathbf{I}\right) + \sum_{i=1}^{n}\gamma_i\int_{-\infty}^{t}\exp\left(-\frac{t-s}{\tau_i}\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\sigma}^{\mathrm{eq}}_{\mathrm{iso}}\right)\mathrm{d}s\right) \quad (19)$$

For the first load case, the simple shear test is employed, which gets loaded in two cycles from $k = 0.00$ to $k = 2.50$ and $k = 0.00$ to $k = 5.00$, where k denotes the shear strain. The shear strain rate is set to be $\dot{k} = \pm 0.10\,\mathrm{s}^{-1}$.

As the kinematics of the simple shear is concerned, the material parameters for the model by ARRUDA-BOYCE and the generalized MAXWELL-model need to be specified: The shear modulus is left arbitrary, since the resulting stresses and parameters concerning the energy will be scaled by the shear modulus in order to obtain unitless quantities. The polymer chain length will be fixed to $N = 15$, where higher values would lead to a softer, elastic behaviour. For the generalized MAXWELL-model, two MAXWELL-chains are employed, where the first chain is left purely elastic with an involvement of $\gamma_\infty = 0.90$ and the second chain has a factor of involvement of $\gamma_1 = 0.10$ and a relaxation time of $\tau_1 = 10\,\mathrm{s}$. The isochoric damage parameters $\Phi_{\mathrm{iso},0}$ and n_{iso} are chosen in ranges of $\Phi_{\mathrm{iso},0}/G = \{10, 12, 14, 16\}$ and $n_{\mathrm{iso}} = \{1, 2, 5, 10\}$, respectively in order to show their impact on the damage prediction within a parametric study.

The load case of simple shear will be calculated twice: On the one hand, the original energy limiting method with reversible damage is used, see. Fig. 2 a) and b). On the other hand, the irreversibility is introduced by means of Eq. (17), see Fig. 2 c) and d).

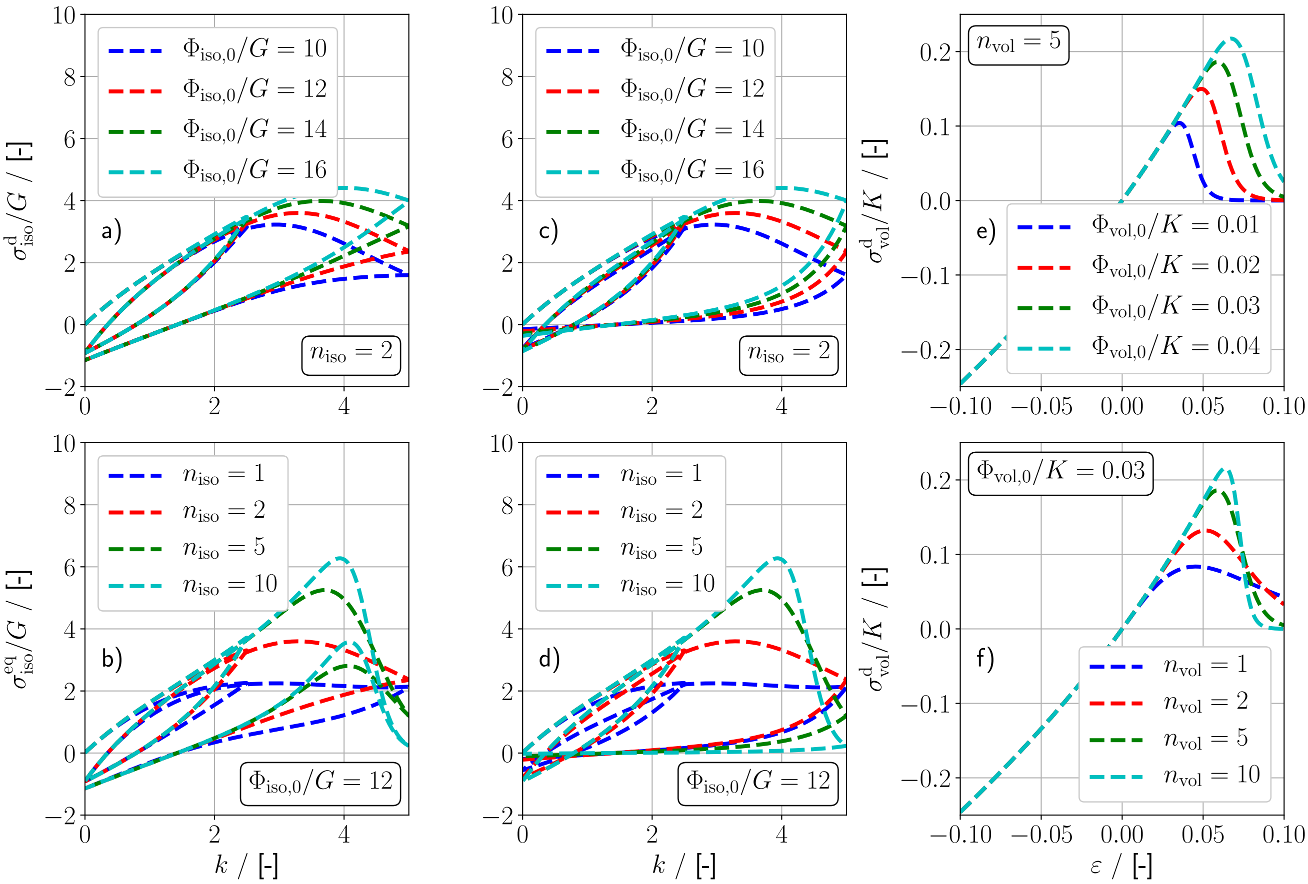

Fig. 2: Parametric study on the damage parameters for the stress response using the simple shear test with a reversible damage model in a) and b) and an irreversible damage model in c) and d) and using the kinematics of the hydrostatic pressure with an irreversible damage model in e) and f), where the compression remains undamaged

For the first load case in Fig. 2 a) and c), it is observable that constant values for the parameter n_{iso} and increasing values for the maximum energy that can be absorbed by a material $\Phi_{\mathrm{iso},0}$ result in the prediction of increasing stress maxima at augmenting shear strain. Furthermore, it can be observed that the evolution of damage remains almost unaffected by the parameter $\Phi_{\mathrm{iso},0}$, since the slope of strain softening is modeled nearly parallel for different values of $\Phi_{\mathrm{iso},0}$, which can be seen best in the second cycle of loading. Concerning constant values for the parameter $\Phi_{\mathrm{iso},0}$ and increasing values for the parameter n_{iso} in Fig. 2 b) and d), it can been seen that increasing stress maxima at rising shear strain are modeled. However, it is observable that increasing values for the parameter n_{iso} lead to more brittle material failure, which also can be seen best in the second cycle of loading.

By comparing the models of reversible and irreversible damage in Fig. 2 a) and b), respectively c) and d), it can be observed that the reversible damage leads to a strain hardening while unloading on the one hand, which is not reasonable in a physical sense. In this case the path dependence is a pure outcome of the generalized MAXWELL-model. On the other hand, the introduction of the irreversibility by means of Eq. (17) leads to a, physicially speaking, reasonable model to predict the material behaviour, since the modeling does not allow the healing from damage of a material during unloading. Additionally, it is observed that the combination of the generalized MAXWELL-model and the irreversible energy limiting method enables the modeling of the MULLINS effect, which describes strain softening due to cyclic loading, see [13].

For the second load case, the kinematics of the hydrostatic pressure is used, which is loaded in the range of $\varepsilon = [-0.10, 0.10]$, where ε denotes the strain. A strain rate does not need to be defined, because the volumetric model is not time dependent.

In regard to the kinematics of the hydrostatic pressure, the material parameters for the model by OGDEN need to be specified: The bulk modulus is left arbitrary, because the resulting stresses and parameters concerning the energy will be scaled by the bulk modulus in order to obtain unitless quantities. The additional, phenomenological parameter will be fixed to $\beta = -4$, where smaller values would lead to softer, elastic behaviour in the range of compression and stiffer, elastic behaviour in the range of dilatation. The impact of the volumetric damage parameters $\Phi_{\mathrm{vol},0}$ and n_{vol} is shown within a parametric study, so that the parameters are considered on the ranges of $\Phi_{\mathrm{vol},0}/K = \{0.01, 0.02, 0.03, 0.04\}$ and $n_{\mathrm{vol}} = \{1, 2, 5, 10\}$, respectively.

Since the evolution of damage in the range of compression is neglectable compared to the range of dilatation, the volumetric damage is turned off in the range of compression:

$$\Psi_{\mathrm{vol}}^{\mathrm{eq,max}} = \begin{cases} \max\limits_{-\infty < s \leq t} \Psi_{\mathrm{vol}}^{\mathrm{eq}}, & \mathit{III}_{\mathrm{F}} \in \mathbb{R}^{+} \\ 0, & \mathit{III}_{\mathrm{F}} \in \mathbb{R}_{\neq 0}^{-} \end{cases} \tag{20}$$

By means of Eq. (20), the the energy limiting method is subdivided with respect to positive and negative values of the third invariant of the deformation gradient to a compression and dilatation part. Within the range of dilatation, the irreversible damage model according to Eq. (17) is used, while in the range of compression the internal variable is kept at a value of zero in order to obtain a multiplier for the elastic stresses of one resulting in pure elastic material behaviour, see Fig. 2 e) and f).

In regard to the second load case in Fig. 2 e) and f), a comparable impact of the damage parameters $\Phi_{\mathrm{vol},0}$ and n_{vol} can be observed as in the first load case for the simple shear test. Thus, increasing values for the maximum energy that can be absorbed by a material $\Phi_{\mathrm{vol},0}$ lead to the prediction of increasing stress maxima at augmenting strain. Also, increasing values for the parameter n_{vol} result in the modeling of more brittle material failure.

Finally, in regard to the kinematics shown, it can be observed that the combination of the framework of continuum mechanics by means of the finite viscoelasticity and the energy limiting method leads to a material model, which produces a reasonable material behaviour in a physical sense. In addition, the proposed new limiting potentials satisfy all physical plausibility conditions and contain at least two or three material parameters, thus, ensuring a flexible fit of the material model to the experimental data until material failure.

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