Investigation and Elimination of Substructures in Formal Concept Analysis focusing on Boolean Suborders and Subcontexts

Dissertation

zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften

(Dr. rer. nat.)

vorgelegt im

Fachbereich Elektrotechnik/Informatik

der

Universität Kassel

von

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Die Dissertation wurde in der Zeit von April 2018 bis Februar 2023 am Fachgebiet Knowledge & Data Engineering angefertigt.

Eingereicht am 14.02.2023

Verteidigt am 12.07.2023

Gutachter

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Acknowledgements

First and foremost, I want to thank my supervisor Prof. Dr. Gerd Stumme for his support and understanding during the past years as well as for his help with finding interesting research questions and encouraging my ideas. Further, I would like to thank Prof. Dr. Bernhard Ganter for his interesting ideas and comments on my research works. I could not have taken this journey without my colleagues at the Knowledge and Data Engineering group at the university of Kassel, from whom I have learned so much. Therefore, I thank Dominik Dürschnabel, Tobias Hille, Johannes Hirth, Mark Kibanov, Bastian Schäfermeier, Andreas Schmidt and Maximilian Stubbemann for many helpful discussions and cherished time spent together in the office. In particular, I thank Maximilian Felde, for his willingness to listen to and discuss all my research problems when entering his office spontaneously, Dr. Tom Hanika for introducing me to the world of writing research papers, Björn Fries for always helping with technical problems and Monika Vopicka for organizing everything that had to be organized in the office.

Last I want to thank my friends and family for their unwavering support and belief in me and, in particular, the unconditional love and encouragement of my mother.

Abstract

In the field of Formal Concept Analysis, data is mainly presented in so-called *formal contexts*, which assign to a set of objects their respective attributes. From those *concept lattices* can be generated, where the objects are grouped with respect to their common attributes to represent the relationships in the data in a way that enhances the understandability for humans. However, since a concept lattice can be of exponential size compared to its associated formal context, the presented relationships often become hard to grasp, even for data sets of moderate size. Therefore, the question arises of finding ways to reduce the size of a dataset to make it more understandable to the user while retaining the original information and structures as best as possible.

Since *Boolean substructures* are significantly responsible for the exponential size of concept lattices, whereas the objects in these structures just slightly differ concerning their attribute set, we consider them first in the present work. We give the possibility to infer from a *Boolean subcontext* in the formal context directly to a corresponding *Boolean suborder* in the associated concept lattice and vice versa. Next, we deal with reducing the size of the concept lattice. To this end, we consider two different types of feature selection in the formal context. Finally, we consider changes directly in the lattice. First, we give a way to collapse intervals (and thus Boolean suborders) by factorization while preserving (as many as possible of) the remaining elements. Second, we investigate under which conditions an interval can be entirely removed without changing anything in the rest of the structure.

Overview of Authors' Contribution

This work is based on papers that were created during collaborations with other researchers and previously published in peer-reviewed conferences and journals. This section gives for each paper the detailed authors' contributions and references to the respective chapters in this work. The remaining parts are the work of the author. The following publications have been incorporated into this Ph.D thesis:

Boolean Substructures in Formal Concept Analysis [42]

> M. Koyda and G. Stumme. "Boolean Substructures in Formal Concept Analysis." In: *International Conference on Formal Concept Analysis*. Springer. 2021, pp. 38–53. DOI: 10.1007/978-3-030-77867-5_3

Paper Contents. The paper "Boolean Substructures in Formal Concept Analysis" by Maren Koyda and Gerd Stumme connects Boolean subcontexts in a formal context with Boolean suborders and Boolean sub(semi)lattices in the corresponding concept lattice. The authors give a direct connection between closed-subcontexts and sublattices. Moreover, mappings from the subcontexts to the suborders and vice versa are introduced, and their interplay is investigated.

Author Contributions. Both authors did the conceptualization. Maren Koyda did the investigation, the writing of the original draft, and the visualization-related tasks such as the creation of figures. Maren Koyda and Gerd Stumme performed the review and editing. Gerd Stumme supervised the writing of this paper.

Quoted in Thesis. Most of this paper is quoted verbatim or paraphrased in Chapter 3, Chapter 4, and Chapter 5.

Relevant Attributes in Formal Contexts [31]

T. Hanika, M. Koyda, and G. Stumme. "Relevant Attributes in Formal Contexts." In: *International Conference on Conceptual Structures.* Vol. 11530. Lecture Notes in Computer Science. Springer, 2019, pp. 102–116. DOI: `10.1007/978-3-030-23182-8_8`

Paper Contents. In the paper "Relevant Attributes in Formal Contexts" Tom Hanika, Maren Koyda, and Gerd Stumme present an approach for attribute selection in formal contexts based on the structural impact of the selected attributes, more precisely on the lattice size and the object distribution on the concepts. They give an approximation of this approach based on entropy functions to enable the selection without requiring the computation of the whole concept lattice. The suitability of this approximation is evaluated in several presented experiments.

Author Contributions. The conceptualization of the research ideas, the initial writing, and the discussion of the results, as well as the review and editing, was a joint work between Tom Hanika and Maren Koyda. Tom Hanika ran the experiments and plotted Figure 6.3, Figure 6.4, and Figure 6.5. Maren Koyda gave the idea for measuring the change of the extent size, developed the first outlines of the presented proofs and did the required research for this work. The writing of this paper was supervised by Gerd Stumme.

Quoted in Thesis. Most of this paper is quoted verbatim or paraphrased in Chapter 4, and Chapter 6.

Attribute Selection Using Contranominal Scales [21]

D. Dürrschnabel, M. Koyda, and G. Stumme. "Attribute Selection Using Contranominal Scales." In: *Graph-Based Representation and Reasoning.* Springer International Publishing, 2021, pp. 127–141. ISBN: 978-3-030-86982-3. DOI: `10.1007/978-3-030-86982-3_10`

Paper Contents. The paper "Attribute Selection Using Contranominal Scales" by Dominik Dürrschnabel, Maren Koyda and Gerd Stumme provides the algorithm `ContraFinder` to generate the set of all contranominal scales of a formal context. This enables the selection of attributes in a formal context based on their appearance in contranominal scales. The runtime of `ContraFinder` is evaluated compared to other related approaches. The suitability of the presented attribute selection for decreasing the size of the corresponding concept lattice and the canonical base is illustrated by several experiments. Moreover, the encapsulated information in the selection is measured.

Author Contributions. The idea of the paper was developed and conceptualized by Dominik Dürrschnabel and Maren Koyda. Both discussed the results and wrote and edited the paper, while Gerd Stumme supervised the writing of this paper. Dominik Dürrschnabel developed and programmed the algorithm `ContraFinder`, executed the experiments, and created the tables. Maren Koyda did the investigation in the realm of attribute selection and investigated the impact of reducing a formal context before δ-adjusting and the properties of the implications. All authors reviewed the paper.

Quoted in Thesis. Parts of this paper are quoted verbatim or paraphrased in Chapter 3, Chapter 4, and Chapter 7.

Factorizing Lattices by Interval Relations [43]

> Maren Koyda and Gerd Stumme. "Factorizing Lattices by Interval Relations." In: *International Journal of Approximate Reasoning* 157 (2023), pp. 70–87. DOI: `10.1016/j.ijar.2023.03.003`

Paper Contents. In the paper "Factorizing Lattices by Interval Relations" Maren Koyda and Gerd Stumme investigate different possibilities to implode selected intervals of a lattice by factorization. To this end, they introduce interval relations on a lattice to enable the implosion of precisely the chosen intervals. They investigate the properties of the generated factor set as well as the possibility of imploding an interval directly in the corresponding formal context.

Author Contributions. The conceptualization of the research ideas was done by Maren Koyda and Gerd Stumme. Maren Koyda did the investigation and the writing of the original draft as well as the editing and also created the figures. Gerd Stumme reviewed and supervised the writing of this paper.

Quoted in Thesis. Most of this paper is quoted verbatim or paraphrased in Chapter 3, Chapter 4, and Chapter 8.

Interval-Dismantling for Lattices [25]

M. Felde and M. Koyda. "Interval-Dismantling for Lattices." In: *International Journal of Approximate Reasoning* 159 (2023). DOI: 10.1016/j.ijar.2023.108931

Paper Contents. In the paper "Interval-Dismantling for Lattices" Maximilian Felde and Maren Koyda expand the notion of dismantling single elements to the notion of dismantling and quasi-dismantling for intervals in a lattice. The authors demonstrate the connection of those structures to the complete subrelations and complete-subcontexts in the corresponding formal context and prove the uniqueness of the DI-core. Moreover, they present an approach to generate all dismantling intervals in the corresponding formal context.

Author Contributions. The conceptualization and writing of the initial paper were done by Maximilian Felde and Maren Koyda. Several ideas are outcomes of the discussion with Prof. Dr. Bernhard Ganter. Maren Koyda provided the proofs of the statements and the extension from dismantling to quasi-dismantling intervals. The algorithm to generate all dismantling intervals for a given lattice was implemented by Maximilian Felde. He also generated the smallest non-trivial lattice without dismantling interval. Both discussed the results and contributed to the final manuscript.

Quoted in Thesis. Most of this paper is quoted verbatim or paraphrased in Chapter 3, Chapter 4, and Chapter 9.

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CHAPTER 1

Motivation and Structure of the Work

In 1982 Rudolf Wille introduced Formal Concept Analysis (FCA) in his article "Restructuring Lattice Theory" [61] as an answer to the search for the real-world meaning of mathematical order theory and thereby presenting datasets in an understandable way for the human operator. Therefore, the connections in the data are illustrated by a hierarchical presentation (a *concept lattice*) of so-called *formal concepts*. Such a formal concept consists of a object set together with the common attributes. Those object sets are maximal with regard of the common attributes.

One frequently – in real-world data as well as in randomly generated datasets [24] – occurring substructure in a *formal context*, the main data structure in FCA being roughly a data table where every row represents an object associated with attributes described through columns, are *contranominal scales*, i. e., contexts of type $(\{1,\ldots,k\},\{1,\ldots,k\},\neq)$. This means, in particular, the existence of k objects that differ only slightly on k attributes. However, despite of the only slight difference, these subcontexts are responsible for an exponential growth of the corresponding concept lattice, by generating a k-dimensional *Boolean lattice*, that consists of 2^k concepts. More precisely, a concept lattice contains a Boolean lattice of dimension k as a substructure if and only if its corresponding formal context contains a contranominal scale of the same dimension [2]. Since humans tend to observe connections in smaller parts of data, the understandability is decreased by this exponential nature, as theconcept lattices become large and hard to grasp. Consequently, reducing large and complex data to meaningful substructures enhances the application of FCA. This motivates us to examine Boolean structures in more detail. To this end, we state the following research questions:

(RQ1) How are Boolean substructures in formal contexts and in concept lattices connected, and in particular, how can such a substructure in the formal context be identified with a corresponding substructure in the associated concept lattice and vice versa?

(RQ2) How can the size of a concept lattice be decreased by alterations of the data while preserving the underlying structure to enable a human observer to grasp the presented information of the data? In particular, what changes can be made in the corresponding formal context (RQ2a) or directly in the lattice (RQ2b)?

After revisiting the required foundations in Part I, we start in Part II by expanding the notion of contranominal scales to cover (RQ1). We introduce *Boolean subcontexts of dimension k* to cover all substructures of a formal context that generate a concept lattice isomorphic to the one corresponding to a contranominal scale of the same dimension. On the side of the concept lattice *Boolean sublattices*, as well as *Boolean suborders*, are examined. An approach for the connection of substructures in the formal context and the corresponding concept lattice is introduced by Wille [58]. He proposed the one-to-one correspondence between *closed subrelations* of a formal context and *complete sublattices* of the associated concept lattice. In the realm of our investigation, we adapt Willes approach to the set of all sublattices and, in particular, Boolean sublattices in Chapter 5. Moreover, we define mappings between the Boolean subcontexts in a formal context and the Boolean sublattices and suborders in its corresponding concept lattice based on the maps introduced by Ganter and Wille [29, Prop. 32], and introduce *associated subcontexts* to map Boolean suborders to Boolean subcontexts. All those maps are investigated with regard to their structural properties as well as the interplay between them.

We address (RQ2) in Part III and IV, in that we aim to generate a smaller concept lattice while preserving as much of the underlying structure as possible. We start in Part III focusing on (RQ2a) with a restriction of the data (more precisely, the attribute set) in formal contexts by attribute selection. By this, we generate sub-$\wedge$-semilattices of the original concept lattice and preserve underlying knowledge by not generating false implications.

In Chapter 6, we investigate the method of attribute selection, as done in machine learning. To this end, we adapt the notion of attribute relevance formalized by Blum and Langley in [10] to the field of Formal Concept Analysis by introducing *relevant attributes* in formal contexts. We utilize this idea to select attributes based on their impact on the lattice size and the distribution of the objects on the concepts and use different entropy functions to approximate the relevance to overcome computational limitations. The suitability of these approximations is shown in several experiments.

Since the size of a concept lattice is heavily influenced by the number of contranominal scales of large dimension in the corresponding formal context, the elimination of those scales is a reasonable approach to decrease the lattice size. Therefore, we focus on the removal of attributes based on their *contranominal influence*, measuring the appearance in contranominal scales, in Chapter 7. Based on this measure, we generate a *δ-adjusted* subcontext that includes the attributes with the lowest contranominal influence. We evaluate the reduction of the lattice size by δ-adjustment in comparison to the selection of relevant attributes and measure the encapsulated information based on a decision tree experiment that illustrates the suitability of both approaches to generate substructures to enhance the understandablility for the observer.

We deal with (RQ2b) in Part IV by eliminating selected parts directly in the concept lattice. Since we aim to reduce the size of a lattice, we turn to the question of imploding a selected interval – for example, a Boolean sublattice – by factorization while imploding as less not selected elements as possible and thus preserving the remaining order structure in Chapter 8. We study the factorization of a selected interval by *congruence relations* and t*olerance relations* which always generates an order-preserving factor lattice and preserves the original meet- and join-operations. However, this approach often results in the implosion of more elements than selected. Inspired by the connection of tolerance relations and *block relations* [59], we add the missing incidences of contranominal scales and Boolean subcontexts to the corresponding formal context. However, this approach can generate new contranominal-scales and possibly result in a concept lattice even larger than the original one. Therefore we turn away from this approach and introduce *interval relations*, a new kind of equivalence relations on lattices. They provide a factorization that implodes a given interval while preserving all other lattice elements. While this implosion is order-preserving, neither the meet- nor the join-operation is generally preserved in the factor set. To generate a factor set with a lattice structure, we introduce *lattice-generating* interval relations as a requirement. We conclude this chapter with a context construction for the factorization by interval relations, the *enrichments of formal contexts by intervals*. In the case of imploding *pure* intervals, those enrichments are in one-to-one correspondence to the interval relations.

Instead of imploding a selected interval and thus generating a new representative, the elimination of selected elements is also possible. Based on the dismantling of irreducible elements [19] – where a sublattice is generated through an elimination of doubly irreducible concepts – we introduce the idea of dismantling (and quasi-dismantling) for intervals in Chapter 9. After stating the requirements for an interval to be dismantling (quasi-dismantling) in a lattice, we prove the existence of a unique DI-kernel. We connect dismantling intervals with closed subrelations and quasi-dismantling intervals with closed-subcontexts and propose an algorithm to compute all dismantling intervals of a lattice directly in the associated formal context.

Note that our work is triggered by complexity issues in data analysis where only finite sets are considered. Therefore **all statements in this work are about finite sets and structures only** unless explicitly stated otherwise.

We conclude our work in Chapter 10 and give an outlook on possible further work.

Part I

Foundations and State of the Art

CHAPTER 2

Order Theory

Order theory is an area of mathematics introduced during the 1930s by Garrett Birkhoff and others [9]. It formalizes the intuitive notion of order, such as comparisons like “greater than” or “lesser than” between different objects while also providing the possibility of incomparable elements so that the natural idea of ordered numbers is extended to general orders, e.g., alphabetical orders or subset relations.

Since Formal Concept Analysis can be interpreted as an application of order theory, this chapter builds the foundation for understanding the essentials of Formal Concept Analysis by introducing the field and providing basic notions. For a more detailed introduction to order theory (and lattice theory), we refer the reader to the works of Ganter [27] or Davey and Priestley [16]. The basic definitions this chapter relies on can also be revisited in [29].

2.1 Ordered Sets

The basic structure investigated in order theory is the *ordered set* as presented in Definition 2.2. To conceptualize statements of comparison like “lesser than” the notion of an *order relation* on a set is introduced:

Definition 2.1 (Order Relation)
A binary relation R on a set P is called *order relation* if the following statements hold for all elements $x, y, z \in P$:

i) $(x,x) \in R$ (reflexivity)

ii) $(x,y) \in R,\ x \neq y \Rightarrow (y,x) \notin R$ (antisymmetry)

iii) $(x,y),\ (y,z) \in R \Rightarrow (x,z) \in R$ (transitivity)

In the following, we also use the phrase *order*, meaning an order relation. Also, we often write xRy instead of $(x,y) \in R$.

We primarily use the notion $\leq$ for an order relation R. In this case we write $x < y$ to denote $x \leq y$ with $x \neq y$.

Definition 2.2 (Ordered Set)
Let P be a set and $\leq$ an order relation on P. We call $\underline{P} \coloneqq (P, \leq)$ an *ordered set*.

Example 2.1
The set $P = \{1,2,3,5,6,10,15\}$ together with the order relation $x \leq y :\Leftrightarrow x \mid y$ is an ordered set.

Definition 2.3 (Lower and Upper Neighbor)
Let $\underline{P} = (P, \leq)$ be an ordered set and $x, y \in P$. We call x *lower neighbor* of y if $x < y$ and it exists no $z \in P$ with $x < z < y$. In this case, analogous y is called *upper neighbor* of x. We denote this by $x \prec y$.

Every finite ordered set $\underline{P} = (P, \leq)$ can be represented graphically by a *line diagram*. Here every element on P is represented by a circle. For two elements $x, y \in P$ with $x < y$, the circle representing y is positioned above the one representing x. Two circles representing the elements x and y are connected by a line if and only if $x \prec y$ holds. Thus, the order relation can be read in the line diagram as follows: For two elements, $x, y \in P$ holds $x \leq y$ if and only if the circle representing y can be reached from the one representing x by an ascending path. In Figure 2.1, a line diagram for the ordered set in Example 2.1 is given.

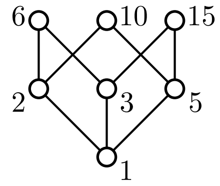


Figure 2.1 Example of a line diagram for the ordered set $(P, \leq)$ in Example 2.1. The elements of P are written next to the circles that representing them.

Note that for an order $\leq$, the inverse relation $\geq$ is an order as well. We call it the *dual order* of $\leq$. The line diagram for the *dual ordered set* $(P, \leq)^d \coloneqq (P, \geq)$ can be generated by horizontal reflection of the line diagram for the ordered set $(P, \leq)$. For a given order-theoretical statement of $(P, \leq)$, the *dual statement* arises by replacing $\leq$ by $\geq$. This principle of duality is used in several proofs.

To investigate different properties of an ordered set, often specific parts of the order are considered.

Definition 2.4 (Suborder)
Let $\underline{P} = (P, \leq)$ be an ordered set and $S \subseteq P$. We call $\underline{S} \coloneqq (S, S^2 \cap \leq)$ *suborder* of $\underline{P}$ and denote this by $\underline{S} \leq \underline{P}$. We define the suborder $\underline{P} \setminus \underline{S} \coloneqq (P \setminus S, (P \setminus S)^2 \cap \leq)$. The set of all suborders of $\underline{P}$ is denoted by $\mathcal{SO}(\underline{P})$.

Note that a suborder of an ordered set is an ordered set itself since all order conditions are passed from the original order.

A special kind of suborder are the *crowns* of order $k \geq 3$ as introduced by Baker, Fishburn, and, Roberts [53] to investigate partial orders of dimension 2. The crown of order 5 is presented in Figure 2.2.

Definition 2.5 (Crown)
A *crown* of order k is a set $\{x_1, y_1, \ldots, x_k, y_k\}$ together with the following order: $x_i \leq y_i$ for $i \in \{1, \ldots, k\}$, $x_i \leq y_{i+1}$ for $i \in \{1, \ldots, k-1\}$, and $x_1 \leq y_k$.

Some other specific suborders can be generated by single elements of an ordered set as follows:

Definition 2.6 (Ideal, Filter, Interval)
Let $\underline{P} = (P, \leq)$ be an ordered set and $x, y \in P$ with $x \leq y$. The *ideal* of x is defined as $(x] \coloneqq \{p \in P \mid p \leq x\}$, the *filter* of x is defined as $[x) \coloneqq \{p \in P \mid x \leq p\}$ and the *interval* of x and y is given by $[x, y] \coloneqq \{p \in P \mid x \leq p \leq y\}$.

Note that an interval $[x, y]$ can always be interpreted as the intersection of the ideal $[x)$ and the filter $(y]$. This is illustrated in the following example:

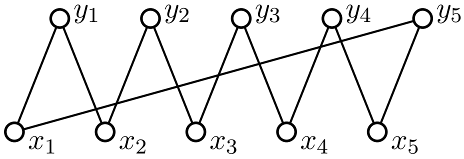


Figure 2.2 The crown of order 5.

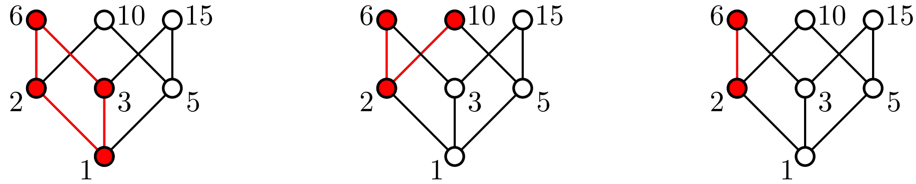


Figure 2.3 Visualization of the ideal $(6]$ (left), the filter $[2)$ (middle) and ideal $[2,6]$ (right) in the ordered set $\underline{P}$ as mentioned in Example 2.2. The suborders are each highlighted red.

Example 2.2

Let $\underline{P}$ be the ordered presented in Example 2.1. The ideal $(6]$ is the set $\{1,2,3,6\}$ together with the original order relation $x \leq y :\Leftrightarrow x \mid y$ restricted to this set, i.e., $(6] = (\{1,2,3,6\}, \{(1,2),(1,3),(1,6),(2,6),(3,6)\})$. An example for a filter in $\underline{P}$ is $(2] = (\{2,6,10\}, \{(2,6),(2,10)\})$. The intersection of both this suborders is the interval $[2,6] = (\{2,6\}, \{(2,6)\})$. Those three suborders are visualized in Figure 2.3.

2.2 Lattices

A special kind of an ordered set is a lattice. Besides the order properties, a lattice has additional structural characteristics as presented in Definition 2.8.

Definition 2.7 (Infimum, Supremum)

Let $\underline{P}$ be an ordered set and $S \subseteq P$. An element $x \in P$ is called *lower bound* of S if $x \leq s$ holds for all $s \in S$. If there is no $y \in P$ so that $x < y$ and y is a lower bound of S, x is a *greatest lower bound* of S. We call x the *infimum* or *meet* of S, if x is the only greatest lower bound of S. Analogously x is called *upper bound* of S if $s \leq x$ holds for all $s \in S$. If there is no $y \in P$ so that $y < x$ and y is a upper bound of S, x is a *smallest upper bound* of S. If x is the only smallest upper bound of S, we call x the *supremum* or *join* of S. The infimum of S is denoted by $\bigwedge S$, and the supremum of S is denoted by $\bigvee S$. If $S = \{a,b\}$ we also write $a \vee b$ or $a \wedge b$, respectively.

Definition 2.8 ((Complete) Lattice, Coatoms, Atoms)

An ordered set $\underline{L} = (L, \leq)$ is called *lattice*, if the supremum $x \vee y$ and infimum $x \wedge y$ exists for every two elements $x, y \in L$. $\underline{L}$ is a *complete lattice* if $\bigvee S$ and $\bigwedge S$ exists for every subset $S \subseteq L$. The smallest element $\bigwedge L$ of a complete lattice $\underline{L}$ is called *zero element* and denoted by $0_{\underline{L}}$ or $\perp$. The upper neighbors of $0_{\underline{L}}$ are called *atoms* of $\underline{L}$.

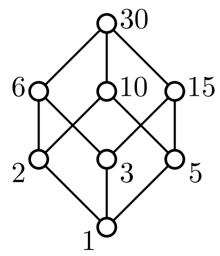


Figure 2.4 Example of a line diagram for the lattice $\underline{L}$ in Example 2.3. The elements of L are written next to the circles representing them.

The largest element $\bigvee L$ of a complete lattice $\underline{L}$ is called *unit element* and denoted by $1_{\underline{L}}$ or $\top$. The lower neighbors of $1_{\underline{L}}$ are called *coatoms* of $\underline{L}$. We denote by $At(\underline{L})$ and $CoAt(\underline{L})$, respectively, the set of all atoms and coatoms of $\underline{L}$.

Note that every finite lattice is complete, since the requirements are always fulfilled.

For a lattice $\underline{L} = (L, \leq)$ the *dual lattice* is given by $\underline{L}^d = (L, \geq)$. Here the duality principle for ordered sets expands to lattices as follows: To obtain a dual order-theoretic statement, in addition to the exchange of $\leq$ and $\geq$, the symbols $\vee$ and $\wedge$ as well as $0_{\underline{L}}$ and $1_{\underline{L}}$, etc. have to be replaced by each other.

Example 2.3
The ordered set $\underline{P}$ presented in Example 2.1 is not a lattice since the elements 6 and 10 have no supremum in P. By expanding P by an additional element the complete lattice $\underline{L} = (\{1, 2, 3, 5, 6, 10, 15, 30\}, \leq)$ with $x \leq y :\Leftrightarrow x \mid y$ arises. A line diagram for this lattice can be seen in Figure 2.4.

The elements of a lattice can be characterized with regard to their interaction with other elements. Those structural information can be used to identify special substructures like *dismantling intervals* (see Chapter 9).

Definition 2.9 (Irreducible Elements)
An element x of a lattice $\underline{L}$ is called *supremum-irreducible* if x has exactly one lower neighbor, meaning $\bigvee\{y \in \underline{L} \mid y < x\} =: x_\star < x$. An element $x \in \underline{L}$ is called *infimum-irreducible* if x has exactly one upper neighbor, meaning $x < x^\star := \bigwedge\{y \in \underline{L} \mid x < y\}$. We call x *doubly-irreducible* if it is both, supremum-irreducible and infimum-irreducible. The set of all supremum-irreducible elements of $\underline{L}$ is denoted by $J(\underline{L})$. The set of all infimum-irreducible elements of $\underline{L}$ is denoted by $M(\underline{L})$.

Definition 2.10 (Supremum-prime, Infimum-prime)
Let $\underline{L}$ be a lattice. An element $x \in \underline{L}$ is called *supremum-prime* if for all $a, b \in \underline{L}$ holds: $x \leq a \vee b \Rightarrow x \leq a$ or $x \leq b$. Analogously, an element $x \in \underline{L}$ is called *infimum-prime* if for all $a, b \in \underline{L}$ holds: $a \wedge b \leq x \Rightarrow a \leq x$ or $b \leq x$.

As in to general ordered sets, in lattices also specific parts are of structural interest. While every part of a lattice can be selected as a suborder, special substructures are generated considering the meet- or join-operators:

Definition 2.11 (Sub(semi)lattice)
Let $\underline{L}$ be a lattice and $S \subset L$. If $(a, b \in S \Rightarrow (a \vee b) \in S)$ holds we call $\underline{S}$ *sub-*$\vee$*-semilattice* of $\underline{L}$. If $(a, b \in S \Rightarrow (a \wedge b) \in S)$ holds we call $\underline{S}$ *sub-*$\wedge$*-semilattice* of $\underline{L}$. If a $\underline{S}$ is both, a sub-$\vee$-semilattice and a sub-$\wedge$-semilattice, it is called *sublattice* of $\underline{L}$. The set of all sublattices of $\underline{L}$ is denoted by $\mathcal{SL}(\underline{L})$. If $(T \subseteq S \Rightarrow (\bigvee T), (\bigwedge T) \in S)$ holds for all $T \subseteq S$ we call $\underline{S}$ *complete sublattice* of $\underline{L}$.

In the case of a finite lattice $\underline{L}$, the requirement for completeness can be translated into $1_{\underline{L}}$ and $0_{\underline{L}}$ being included in the sublattice $\underline{S}$.

Note that every ideal, filter, or interval of a lattice is a sublattice of the original one.

CHAPTER 3

Formal Concept Analysis

This chapter provides the basic ideas of Formal Concept Analysis (FCA) and the notations required in the further work. If not stated differently, the definitions and statements are from [29], to that we refer the reader for a deeper introduction.

Formal Concept Analysis deals with the investigation of (binary) data. Therefore, the main ways to represent data are the *formal context* and the *concept lattice*. Their definitions and connections, as well as some structural properties, are represented in this section. Note that the word "formal" should underline that we deal with a mathematical definition and separate it from the use of the word "context" in the standard language. However, for reasons of readability, we will often leave out the additional adjective and go with *context*, meaning the defined *formal context*. The same holds for the usage of the words *formal concept* and *concept*.

3.1 Formal Contexts and Concept Lattices

One of the primary data structures in Formal Concept Analysis is the *formal context* as defined in Definition 3.1. It can be understood as a binary data structure.

Definition 3.1 (Formal Context)
A *formal context* is a triple consisting of an *object set* G, an *attribute set* M, and a binary *incidence relation* on those sets $I \subseteq (G \times M)$. It is denoted by $\mathbb{K} = (G, M, I)$.

If $(g, m) \in I$ holds for an object $g \in G$ and an attribute $m \in M$, we also write gIm and say *object g has attribute m*. In this work, G and M (and therefore I) are assumed to be finite. The roles of objects and attributes are exchangeable. By this the *dual context* $\mathbb{K}^d = (G, M, I^{-1})$ of a formal context $\mathbb{K}$ arises.

A formal context can be visualized by a *cross-table*, in which each object is represented by a row and each attribute is represented by a column, respectively. Crosses in the table stand for the elements of the incidence relation, meaning a cross in row g and column m represents that the object g has the attribute m.

Example 3.1
An example of a formal context is $\mathbb{K}_{Aladdin} = (G, M, I)$. It contains some characters of the movie *Aladdin* [1] as object set and some of their properties as attribute set. The incidence relation represents that a character in G has a property in M. $\mathbb{K}_{Aladdin}$ is visualized by the cross-table in Figure 3.1.

To determine for a given set of objects which attributes they have in common, and the other way around, the following two *derivations* are defined on the powersets of object set and attribute set:

Definition 3.2 (Derivation)
Let $\mathbb{K} = (G, M, I)$ be a formal context. The *object derivation* on $\mathbb{K}$ is the map

$$\cdot' \colon \mathcal{P}(G) \to \mathcal{P}(M),\ A \mapsto A' \coloneqq \{m \in M \mid \forall\, g \in A \colon (g, m) \in I\}.$$

The *attribute derivation* on $\mathbb{K}$ is the map

$$\cdot' \colon \mathcal{P}(M) \to \mathcal{P}(G),\ B \mapsto B' \coloneqq \{g \in G \mid \forall\, m \in B \colon (g, m) \in I\}.$$

We call A' the set of attributes *common* to objects in A and B' the set of objects that *share* all attributes in B.

[1] `https://de.wikipedia.org/wiki/Aladdin_(2019)`

$\mathbb{K}_{Aladdin}$	human	animal	has magic	can speak	can fly	villain
Aladdin	×			×		
Jasmin	×			×		
Genie			×	×	×	
Jafar	×		×	×		×
Abu		×				
Jago		×		×	×	
Magic Carpet			×		×	

Figure 3.1 Visualization of the formal context $\mathbb{K}_{Aladdin}$ from Example 3.1.

Since it is clear whether the derivation of a set of objects or a set of attributes is addressed, we use the symbol $\cdot'$ for both operators. If it is unclear which incidence relation is used when investigating different contexts or parts of a context, we also write A^I instead of A' to clarify the use of incidence I. Further, we write g' instead of $\{g\}'$ for a single object and A'' instead of $(A')'$ for readability reasons.. The same holds for attribute subsets.

Example 3.2
Considering the object set $A = \{\text{Aladdin}\}$ from the formal context in Example 3.1, we have $A' = \{\text{human, can speak}\}$ as the common attributes and $A'' = \{\text{Aladdin, Jasmin, Jafar}\}$ as the objects that share all those attributes.

The following properties hold as proposed in [29, Prop. 10], where the proof can also be found. Based on them *formal concepts*, a fundamental structure in Formal Concept Analysis, are defined in Definition 3.3.

Proposition 3.1 ([29, Prop. 10])
Let $\mathbb{K} = (G, M, I)$ be a formal context with $A, \widetilde{A} \subseteq G$ and $B, \widetilde{B} \subseteq M$. The following properties hold:

i) $\widetilde{A} \subseteq A \Rightarrow A' \subseteq \widetilde{A}'$

i') $\widetilde{B} \subseteq A \Rightarrow A' \subseteq \widetilde{B}'$

ii) $A \subseteq A''$

ii') $B \subseteq B''$

iii) $A = A'''$

iii') $B = B'''$

v) $A \subseteq B' \Longleftrightarrow B \subseteq A' \Longleftrightarrow A \times B \in I$

Definition 3.3 (Formal Concept, Extent, Intent)
Let $\mathbb{K} = (G, M, I)$ be a formal context, $A \subseteq G$, and $B \subseteq M$. A pair $c = (A, B)$ is called a *formal concept* of $\mathbb{K}$ if and only if $A' = B$ and $B' = A$. A is called the *extent* of c and is denoted by $ext(c)$. B is called the *intent* of c and is denoted by $int(c)$. The set of all formal concepts of a formal context $\mathbb{K}$ is denoted by $\mathfrak{B}(\mathbb{K})$.

It can be seen, that for every object set $A \subseteq G$ the derivation A' is the intent of some concept, to be specific of (A'', A'). Further, A'' is the smallest extent containing A. Therefore we also call (A'', A') the context *generated* by A. Since it is possible that two object sets $A, \widetilde{A}$ have the same set of common attributes, a concept can be generated by more than just one object set. The same holds for attribute sets. However, the *minimal generators* of an object set (an attribute set) can be specified.

Definition 3.4 (Minimal Generator)
An object set $O \subseteq G$ is called *minimal object generator* of a concept (A, B) if $O'' = A$ and $P'' \neq A$ for every subset $P \subsetneq O$. Analogous, the *minimal attribute generator* of a concept (A, B) is defined. The set of all minimal object generators (all minimal attribute generators) of (A, B) is denoted by $minG_{obj}(A, B)$ ($minG_{att}(A, B)$).

It is possible that a single object or attribute can generate a concept. In particular, this is a requirement for a concept to be infimum-irreducible or supremum-irreducible in the concept lattice.

Definition 3.5 (Object Concept, Attribute Concept)
Let $\mathbb{K} = (G, M, I)$ be a formal context with $g \in G$ and $m \in M$. The concept $\gamma g := (g'', g')$ is called *object concept* of g. Analogous, the concept $\mu m := (m', m'')$ is called *attribute concept* of m.

Example 3.3
As for the context $\mathbb{K}_{Aladdin}$ from Example 3.1 the formal concepts are the following:

- ($\varnothing$,{human, animal, has magic, can speak, can fly, villain})
- ({Aladdin, Jasmin, Jafar},{human, can speak})
- ({Genie},{has magic, can speak, can fly})
- ({Jafar},{human, has magic, can speak,villain})
- ({Abu, Jago},{animal})
- ({Jago},{animal, can speak, can fly})
- ({Genie, Magic Carpet},{has magic, can fly})
- ({Genie, Jafar},{has magic, can speak})
- ({Genie, Jafar, Magic Carpet},{has magic})
- ({Aladdin, Jasmin, Genie, Jafar, Jago},{can speak})
- ({Genie, Jago, Magic Carpet},{can fly})

- ({Genie, Jago},{can speak, can fly})
- ({Aladdin, Jasmin, Genie, Jafar, Abu, Jago, Magic Carpet},∅)

The concept c =({Aladdin, Jasmin, Jafar},{human, can speak}) has the minimal object generators {Aladdin} and {Jasmin}. Thus, c is the object concept of both of those sets.

The concepts of a formal context $\mathbb{K}$ can be compared with regard to their extents (or intents). Based on this, an order arises as follows:

Definition 3.6 (Subconcept, Superconcept, Hierarchical Order)
Let $\mathbb{K} = (G, M, I)$ be a formal context and $(A, B), (\widetilde{A}, \widetilde{B}) \in \mathfrak{B}(\mathbb{K})$ two formal concepts. We call (A, B) a *subconcept* of $(\widetilde{A}, \widetilde{B})$ if $A \subseteq \widetilde{A}$ (or equivalent $\widetilde{B} \subseteq B$). In this case $(\widetilde{A}, \widetilde{B})$ is called *superconcept* of (A, B). We also write $(A, B) \leq (\widetilde{A}, \widetilde{B})$ and call this relation the *hierarchical order* of the concepts.

Definition 3.7 (Concept Lattice)
Let $\mathbb{K} = (G, M, I)$ be a formal context. The set of all formal concepts of $\mathbb{K}$ together with the hierarchical order forms the *concept lattice* $\underline{\mathfrak{B}}(\mathbb{K}) \coloneqq (\mathfrak{B}(\mathbb{K}), \leq)$.

Indeed, the concept lattice of a formal context $\mathbb{K}$ is a complete lattice. This is shown in the first part of the Basic Theorem of Formal Concept Analysis:

Theorem 3.1 (Basic Theorem, [29, Thm. 3])
Let $\mathbb{K} = (G, M, I)$ be a formal context. The concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ is a complete lattice with the following infima and suprema:

$$\bigwedge_{t \in T} (A_t, B_t) = \left(\bigcap_{t \in T} A_t, \left(\bigcup_{t \in T} B_t \right)'' \right)$$
$$\bigvee_{t \in T} (A_t, B_t) = \left(\left(\bigcup_{t \in T} A_t \right)'', \bigcap_{t \in T} B_t, \right)$$

The second part of this theorem is omitted in this work since we do not make use of it. However, we want to emphasize that every complete lattice $\underline{L}$ is isomorphic to the concept lattice of the formal context $(L, L, \leq)$. Therefore, the statements in this work on concept lattices can be translated to lattices in general and the other way around. Also, the duality principle does extend to concept lattices.

A concept lattice can be visualized by a line diagram, like every lattice. Instead of labeling each circle with the concept it represents, a simplification can be made in the following way: Every object and attribute is written down only once. The

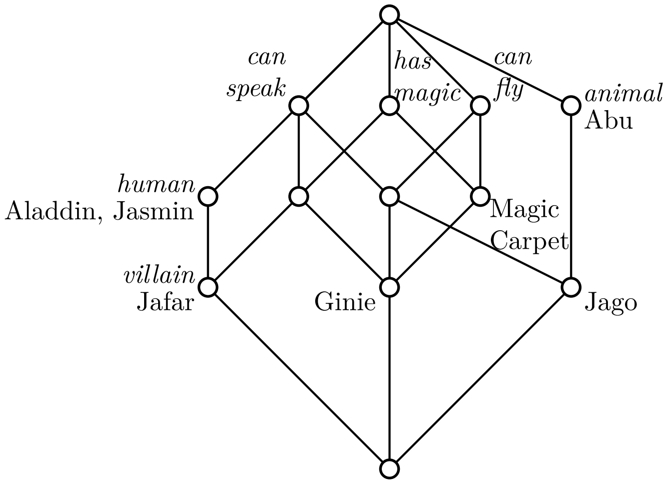


Figure 3.2 Concept lattice of the context $\mathbb{K}_{Aladdin}$. The line diagram has reduced labeling.

object g is written next to the circle, which represents the object concept γg. Analog the attribute is written next to the circle, which represents the attribute concept μm. To clarify whether an object or an attribute generates the concept the objects are placed slightly below the circles while the attributes are placed slightly above them. To read the extent of a concept from the line diagram, one has to look at the corresponding circle and unite the objects from this circle and all circles that can be reached by descending line paths. Dually, the intent of this concept is the union of the attributes on this circle and those on all circles that can be reached by ascending line paths. The line diagram of $\underline{\mathfrak{B}}(\mathbb{K}_{Aladdin})$ is visualized in Figure 3.2.

There are two basic structural approaches for simplifying a formal context, namely *clarifying* and *reduction*. They generate the following contexts:

Definition 3.8 (Clarified Context)
Let $\mathbb{K} = (G, M, I)$ be a formal context. $\mathbb{K}$ is called *object clarified* if for every two objects $g, h \in G$ with $g' = h'$ follows that $h = g$. Analogous, $\mathbb{K}$ is called *attribute clarified* if for every two attributes $m, n, \in M$ from $m' = n'$ follows that $h = g$. If $\mathbb{K}$ is both, object clarified and attribute clarified, we call it *clarified*.

Definition 3.9 (Reduced Context)
Let $\mathbb{K} = (G, M, I)$ be a formal context. An object $g \in G$ is called *reducible*, if there is an object set $X \subseteq G$ with $g \notin X$ and $g' = X'$. Otherwise g is called *irreducible* in $\mathbb{K}$. The same holds for the attributes. If all objects and attributes in $\mathbb{K}$ are irreducible, we call the context *reduced*.

Note that objects and attributes corresponding to full or columns, respectively, are always reducible, i.e. the objects g with $g' = M$ and the attributes m with $m' = G$. In this case, the empty set has the same derivation as these objects or attributes.

Since every element of a finite lattice is the join of supremum-irreducible elements and meet of infimum-irreducible elements, all reducible objects and attributes can be eliminated at once in a finite formal context:

Proposition 3.2
Let $\mathbb{K} = (G, M, I)$ *be a finite clarified formal context,* $G_{irr} \subseteq G$ *the irreducible objects, and* $M_{irr} \subseteq M$ *the irreducible attributes of* $\mathbb{K}$*. Let* $\mathbb{K}_{irr} = (G_{irr}, M_{irr}, I \cap (G_{irr} \times M_{irr}))$*. Then:*

$$\underline{\mathfrak{B}}(\mathbb{K}) \cong \underline{\mathfrak{B}}(\mathbb{K}_{irr})$$

This means numerous formal contexts correspond to (concept) lattices with identical structures. However, only one of them is reduced (up to isomorphism).

The denotations of irreducible elements in a (concept) lattice $\underline{L}$ (Definition 2.9) and in a formal context $\mathbb{K}$ are chosen in this way because they are strongly connected. An element $x \in \underline{L}$ is supremum-irreducible if and only if $x = \gamma g$ for an irreducible object g in the corresponding formal context. Dual, the infimum-irreducible elements of a (concept) lattice are the ones generated by an irreducible attribute.

Proposition 3.3 ([29, Prop. 12])
Let $\underline{L}$ *be a finite lattice. There is (up to isomorphism) one reduced context* $\mathbb{K}$ *with* $\underline{\mathfrak{B}}(\mathbb{K}) \cong \underline{L}$*. This context is* $\mathbb{K} = (J(\underline{L}), M(\underline{L}), \leq)$*.*

Definition 3.10 (Standard Context, Generic Context)
Let $\underline{L}$ be a finite lattice. The context $\mathbb{K} = (J(\underline{L}), M(\underline{L}), \leq)$ is called the *standard context* of $\underline{L}$. The context $\mathbb{K} = (L, L, \leq)$ is called the *generic context* of $\underline{L}$.

Example 3.4
As for the formal context $\mathbb{K}_{Aladdin}$ in Example 3.1, the objects “Aladdin” and “Jasmin” have identical derivations, i.e. Aladdin$'$ = Jasmin$'$. Hence, $\mathbb{K}_{Aladdin}$ is not clarified. To clarify the context both objects have to be merged. To obtain the standard context, also the attribute “villain” has to be removed since villain$'$ = {human, has magic}$'$ holds. The generic context of the lattice $\underline{L} \cong \underline{\mathfrak{B}}(\mathbb{K}_{Aladdin})$ is presented in Figure 3.3. The Numbers denote the elements of the lattice for reasons of readability.

Another method to characterize the irreducible elements of a formal context can be obtained using arrow relations.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	×	×	×	×	×	×	×	×	×	×	×	×	×
2		×			×			×		×	×		×
3			×		×	×	×			×	×	×	×
4				×		×			×	×		×	×
5					×					×	×		×
6						×				×		×	×
7							×				×	×	×
8								×		×			×
9									×				×
10										×			×
11											×		×
12												×	×
13													×

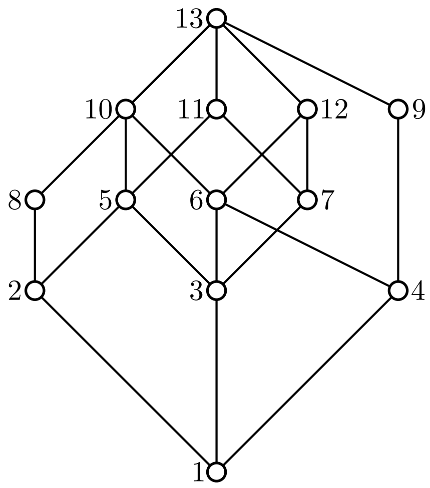


Figure 3.3 A generic formal context $\mathbb{K} = (L, L \leq)$ (left) and its corresponding concept lattice $\underline{L}$ (right). $\underline{L}$ is isomorphic to $\underline{\mathfrak{B}}(\mathbb{K}_{Aladdin})$.

Definition 3.11 (Arrow Relations)

For an object $g \in G$ and an attribute $m \in M$ of a formal context (G, M, I) we write

$$g \swarrow m \ :\Longleftrightarrow \ \begin{cases} (g,m) \notin I \text{ and} \\ \text{if there exists an } h \in G \text{ with } g' \subseteq h' \text{ and } g' \neq h', \text{ then } hIm, \end{cases}$$

$$g \nearrow m \ :\Longleftrightarrow \ \begin{cases} (g,m) \notin I \text{ and} \\ \text{if there exists an } n \in M \text{ with } m' \subseteq n' \text{ and } m' \neq n', \text{ then } gIn, \end{cases}$$

$$g \swarrow\!\!\!\nearrow m \ :\Longleftrightarrow \ g \swarrow m \text{ and } g \nearrow m.$$

We also write $\swarrow^I$, $\nearrow^I$, and $\swarrow\!\!\!\nearrow^I$ to clarify the use of incidence relation I.

We now adapt the definitions of $(\cdot)^\star, (\cdot)_\star$ to object and attribute concepts,

$$(\gamma g)_\star \coloneqq \bigvee\{c \in \mathfrak{B}(\mathbb{K}) \mid c \leq \gamma g\}, \text{ and}$$
$$(\mu m)^\star \coloneqq \bigwedge\{c \in \mathfrak{B}(\mathbb{K}) \mid \mu m \leq c\},$$

in order to characterize the arrow relations in a formal context as follows:

Proposition 3.4

Let $\mathbb{K} = (G, M, I)$ be a formal context with $g \in G$ and $m \in M$. We have

$$g \swarrow m \iff \gamma g \wedge \mu m = (\gamma g)_\star \neq \gamma g$$
$$g \nearrow m \iff \gamma g \vee \mu m = (\mu m)^\star \neq \mu m.$$

$\mathbb{K}_{Aladdin}$	human	animal	has magic	can speak	can fly	villain
Aladdin	$\times$	$\nearrow$	$\swarrow\mkern-15mu\nearrow$	$\times$	$\nearrow$	$\swarrow$
Jasmin	$\times$	$\nearrow$	$\swarrow\mkern-15mu\nearrow$	$\times$	$\nearrow$	$\swarrow$
Genie	$\swarrow\mkern-15mu\nearrow$	$\swarrow\mkern-15mu\nearrow$	$\times$	$\times$	$\times$	$\swarrow$
Jafar	$\times$	$\nearrow$	$\times$	$\times$	$\nearrow$	$\times$
Abu		$\times$	$\nearrow$	$\swarrow\mkern-15mu\nearrow$	$\swarrow\mkern-15mu\nearrow$	
Jago	$\swarrow\mkern-15mu\nearrow$	$\times$	$\swarrow\mkern-15mu\nearrow$	$\times$	$\times$	$\swarrow$
Magic Carpet		$\nearrow$	$\times$	$\swarrow\mkern-15mu\nearrow$	$\times$	

Figure 3.4 $\mathbb{K}_{Aladdin}$ with its arrow relations.

If $(\gamma g)_\star \neq \gamma g$, the object g is irreducible in the formal context, and γg is supremum-irreducible in the corresponding concept lattice. If dually $(\mu m)^\star \neq \mu m$, the attribute m is irreducible in the formal context and μm is infimum-irreducible in the concept lattice. Therefore, the arrow relations can be utilized to identify the irreducible concepts of a lattice as follows:

Proposition 3.5 ([29, Prop. 13])
Let $\mathbb{K} = (G, M, I)$ be a finite formal context with $g \in G$ and $m \in M$. It holds:

i) γg is supremum-irreducible $\Longleftrightarrow$ There is an $n \in M$ with $g \swarrow\mkern-15mu\nearrow n$.

ii) μm is infimum-irreducible $\Longleftrightarrow$ There is an $h \in G$ with $h \swarrow\mkern-15mu\nearrow m$.

Example 3.5
In Figure 3.4 $\mathbb{K}_{Aladdin}$ is visualized with its arrow relations. There is no object $g \in G$ with $g \nearrow$ villain. Thus, the attribute "villain" is reducible.

3.2 Subcontexts and Subrelations

Besides reducing and clarifying a context, arbitrary objects and attributes can be chosen to reduce the size of a formal context. This selection of such a *subcontext* helps to study particular parts of a formal context.

Definition 3.12 (Subcontext)
Let $\mathbb{K} = (G, M, I)$ be a formal context with the subsets $H \subseteq G$ and $N \subseteq M$. We define $\mathbb{S} = [H, N] := (H, N, I \cup (H \times N))$ as a *subcontext* of $\mathbb{K}$ and denote this with $\mathbb{S} \leq \mathbb{K}$. The set of all subcontexts of a formal context $\mathbb{K}$ is denoted by $\mathcal{S}(\mathbb{K})$.

Note that every subcontext is a formal context itself. One kind of subcontext that is often found in formal contexts is the family of *contranominal scales*, denoted by $\mathbb{N}^c(k) := (\{1, 2, \ldots, k\}, \{1, 2, \ldots, k\}, \neq)$. The objects (and dually attributes) in this structure just slightly differ.

	has magic	can speak	can fly
Jafar	×	×	
Jago		×	×
Magic Carpet	×		×

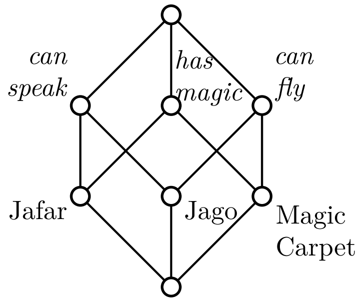


Figure 3.5 A subcontext of $\mathbb{K}_{Aladdin}$ that is a contranominal scale of dimension 3 (left) and the corresponding concept lattice (right).

Definition 3.13 (Contranominal Scale)
The formal context $\mathbb{N}^c(k) \coloneqq (\{1,\ldots,k\},\{1,\ldots,k\},\neq)$ is called *contranominal scale* of dimension k. The concept lattice of $\mathbb{N}^c(k)$ is called *Boolean lattice of dimension k* and is denoted by $\mathfrak{B}(k) \coloneqq \underline{\mathfrak{B}}(\mathbb{N}^c(k))$.

In this work, we call every subcontext $\mathbb{S} \leq \mathbb{K}$ of a formal context $\mathbb{K}$ with $\mathbb{S} \cong \mathbb{N}^c(k)$ contranominal scale as well.

Albano and Chornomaz [2, Prop. 1] proposed that every formal context $\mathbb{K}$ contains a contranominal scale of dimension k if $\mathfrak{B}(\mathbb{K})$ contains a suborder isomorphic to a Boolean lattice of dimension k. We will investigate this connection and the related structures in more detailed in Chapter 5.

Example 3.6
$\mathbb{K}_{Aladdin}$ contains the 3-dimensional contranominal scale $\mathbb{S}$ =[{Jafar, Jago, Magic Carpet}, {has magic, can speak, can fly}] as a subcontext. $\mathbb{S}$ is visualized in Figure 3.5 together with its corresponding concept lattice, a Boolean lattice of dimension 3.

Two other frequently occurring scales in formal contexts are the *nominal scale*, $\mathbb{N}(k) \coloneqq (\{1,\ldots,1\},\{1,\ldots,k\},=)$, in which every object can be assigned to one attribute and vice versa, and the *ordinal scale*, $\mathbb{O}(k) \coloneqq (\{1,\ldots,1\},\{1,\ldots,k\},\leq)$. A small example of both scales can be found in $\mathbb{K}_{Aladdin}$. The respective subcontexts are presented together with their corresponding concept lattices in Figure 3.6.

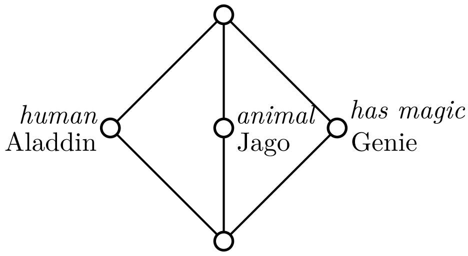
A connection of the concept lattices of a formal context $\mathbb{K} = (G,M,I)$ and its subcontext $\mathbb{S} = [H,N]$ is given by Ganter and Wille by the following maps:

Proposition 3.6 ([29, Prop. 31 and Prop. 32])
Let (G,M,I) be a formal context with $H \subseteq G$ and $N \subseteq M$.
The map

$$\underline{\mathfrak{B}}([G,N]) \to \underline{\mathfrak{B}}((G,M,I)),\ (A,B) \mapsto (A,A')$$

is a meet-preserving order embedding.

	human	animal	has magic
Aladdin	×		
Genie			×
Jago		×	



	has magic	can speak	can fly
Aladdin		×	
Genie	×	×	×
Jago		×	×

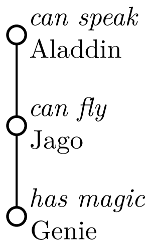


Figure 3.6 A subcontext of $\mathbb{K}_{Aladdin}$ that is a nominal scale of dimension 3 and the corresponding concept lattice (top). A subcontext of $\mathbb{K}_{Aladdin}$ that is an ordinal scale of dimension 3 and the corresponding concept lattice (bottom).

The map

$$\underline{\mathfrak{B}}([H,M]) \to \underline{\mathfrak{B}}((G,M,I)),\ (A,B) \mapsto (B',B)$$

is a join-preserving order embedding.
The two maps

$$\varphi_1 : \underline{\mathfrak{B}}([H,N]) \to \underline{\mathfrak{B}}((G,M,I)),\ (A,B) \mapsto (A'',A'),\ \textit{and}$$
$$\varphi_2 : \underline{\mathfrak{B}}([H,N]) \to \underline{\mathfrak{B}}((G,M,I)),\ (A,B) \mapsto (B',B'')$$

are order embeddings*.*

This means for all $(A_1,B_1),(A_2,B_2) \in \underline{\mathfrak{B}}([H,N])$ that $(A_1,B_1) \leq (A_2,B_2)$ in $\underline{\mathfrak{B}}([H,N])$ if and only if $\varphi_i(A_1,B_1) \leq \varphi_i(A_2,B_2)$ in $\underline{\mathfrak{B}}((G,M,I))$ for both $i \in \{1,2\}$. Therefore, the concept lattice corresponding to a subcontext $\mathbb{S} \leq \mathbb{K}$ can always be found as a suborder of the original concept lattice $\mathfrak{B}(\mathbb{K})$. Thus, the concept lattice $\underline{\mathfrak{B}}(\mathbb{K}_{Aladdin})$ contains a 3-dimensional Boolean lattice as suborder since $\mathbb{K}_{Aladdin}$ contains the 3-dimensional contranominal scale from Example 3.6.

Subcontexts can not only be characterized based on their internal structure but also based on their properties in interaction with the original context like *compatible subcontexts* that were introduced by Rudolf Wille [62] as follows:

Definition 3.14 (Compatible Subcontext)
Let $\mathbb{K} = (G,M,I)$ be a formal context. A subcontext $\mathbb{S} = [H,N] \leq \mathbb{K}$ is called *compatible* if $(A \cap H, B \cap N)$ is a concept of $\mathbb{S}$ for every concept $(A,B) \in \underline{\mathfrak{B}}(\mathbb{K})$.

$\mathbb{K}_{Aladdin}$	human	animal	has magic	can speak	can fly	villain
Aladdin	×			×		
Jasmin	×			×		
Genie			×	×	×	
Jafar	×		×	×		×
Abu		×				
Jago		×		×	×	
Magic Carpet			×		×	

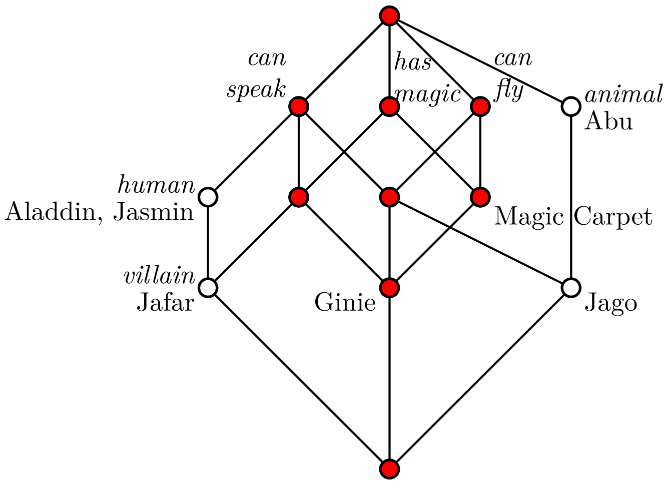


Figure 3.7 A closed subrelation of $\mathbb{K}_{Aladdin}$ (top) and the corresponding complete sublattice in $\underline{\mathfrak{B}}(\mathbb{K}_{Aladdin})$ (bottom). Both substructures are highlighted.

Instead of selecting objects and attributes of a context (G, M, I) also the selection of specific incidences is possible. This generates a *subrelation* of I. A special kind of subrelation, the *closed subrelation*, was introduced by Rudolf Wille [58] to characterize the complete sublattices of its concept lattice as presented in Theorem 3.2.

Definition 3.15 (Closed Subrelation)
Let $\mathbb{K} = (G, M, I)$ be formal context. A relation $J \subseteq I$ is called *closed subrelation* of $\mathbb{K}$ if every concept of the context (G, M, J) is a concept of $\mathbb{K}$ as well.

Theorem 3.2 ([29, Thm. 13])
Let $\mathbb{K}$ be a formal context. The bijection

$$C(\underline{S}) \coloneqq \bigcup\{A \times B | (A, B) \in \underline{S}\}$$

maps the complete sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$ to the closed relations of $\mathbb{K}$.

This means that the set of all closed subrelations of $\mathbb{K}$ and all complete sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$ have a one-to-one correspondence. An example of a closed subrelation of $\mathbb{K}_{Aladdin}$ and the corresponding complete sublattice are presented in Figure 3.7.

A closed subrelation may differ just slightly from the original incidence relation. In the extreme case, only one incidence vanishes. This case corresponds to the approach of *dismantling of doubly-irreducible elements* by Duffus and Rival [19] where doubly-irreducible elements are eliminated from an ordered set (see Proposition 3.7). They state that the dismantling of an element x from a lattice $\underline{L}$ results in a complete sublattice $\underline{L} \setminus \{x\}$ if and only if the element was doubly-irreducible. We expand this statement in Chapter 9 from single elements to suitable intervals.

Proposition 3.7 ([29, Prop. 53])
Let $\mathbb{K} = (G, M, I)$ be a clarified formal context and $c = \gamma g = \mu m$ $(g \in G, m \in M)$ a doubly-irreducible concept of $\underline{\mathfrak{B}}(\mathbb{K})$, then

$$\underline{\mathfrak{B}}((G, M, I)) \setminus \{c\} = \underline{\mathfrak{B}}((G, M, I \setminus \{(g, m)\})).$$

Note that the dismantling of a doubly-irreducible element can result in another element becoming doubly-irreducible. Thus, a successive elimination of all doubly-irreducible elements is possible. The remaining structure, the *DI-kernel*, is unique and therefore does not depend on the order of dismantling elements [19].

Example 3.7
In the case of the lattice $\underline{\mathfrak{B}}(\mathbb{K}_{Aladdin})$ the only doubly-irreducible concepts are ({Abu, Jago},{animal}) and ({Aladdin, Jasimin, Jafar},{can speak, human}). After dismantling both of them the concepts ({Jago},{animal, can fly, can speak}) and ({Jafar},{human, has magic, can speak}) become doubly-irreducible. The dismantling of them results in the DI-kernel.

Instead of selecting a subset of incidences, generating a new incidence relation via additional incidences is possible. If this new relation generates no new concepts in a formal context, it is called *block relation*.

Definition 3.16 (Block Relation)
Let $\mathbb{K} = (G, M, I)$ be a formal context. A relation $J \subseteq G \times M$ is called *block relation* of $\mathbb{K}$ if it satisfies the following conditions:

i) $I \subseteq J$

ii) For all $g \in G$ holds that g^J is an intent of $\mathbb{K}$.

iii) For all $m \in M$ holds that m^J is an extent of $\mathbb{K}$.

Since the intersection of intents is always an intent and the dual holds for extents, the block relations of a formal context (G, M, I) form a closure system.

3.3 Congruence and Tolerance Relations

An equivalence relation on an algebraic structure is called *congruence relation* (or short congruence) if the algebraic operations of the structure are compatible with this equivalence relation [33]. The equivalence classes of the quotient structure are called *congruence classes*. In the realm of lattice theory, the commonly investigated congruence is the *lattice congruence* as defined in [8], where the requirement is compatibility with suprema and infima for finite sets. In contrast to this the *complete congruence relation* as seen in Definition 3.17 requires this characteristic for infinite sets as well. Consequently, every complete lattice is isomorphic to the lattice of the complete congruence lattice of a suitable lattice as seen in [30]. For an overview of congruences on concept lattices, we refer to Reuter and Wille [51], who show the one-to-one correspondence presented in Theorem 3.3.

Definition 3.17 (Complete Congruence Relation)
A *complete congruence relation* of a complete lattice $\underline{L}$ is a equivalence relation θ on $\underline{L}$ that satisfies the following condition:

$$x_t \theta y_t \text{ for all } t \in T \Rightarrow (\bigvee_{t\in T} x_t)\theta(\bigvee_{t\in T} y_t) \text{ and } (\bigwedge_{t\in T} x_t)\theta(\bigwedge_{t\in T} y_t)$$

Thus, congruence relations preserve the meet- and join-operators of $\underline{L}$ in $\underline{L}/\theta$.

In this work, we always refer to complete congruence relations when mentioning a congruence relation (or short congruence).

Theorem 3.3
Let $\mathbb{K} = (G, M, I)$ be a formal context and $\underline{S} = [H, N] \leq \mathbb{K}$ a compatible subcontext. Then exists a unique congruence relation $\theta_{H,N}$ with

$$\underline{\mathfrak{B}}([H, N]) \cong \underline{\mathfrak{B}}(\mathbb{K})/\theta_{H,N}$$

so that

$$(A_1, B_1)\theta_{H,N}(A_2, B_2) \Longleftrightarrow A_1 \cap H = A_2 \cap H \Longleftrightarrow B_1 \cap N = B_2 \cap N.$$

If $\mathbb{K}$ is finite and reduced, for every congruence relation θ exists a unique compatible subcontext $[H, N] \leq \mathbb{K}$ with $\theta = \theta_{H,N}$.

Since we only consider finite lattices in this work, the only requirement for a context for applying this theorem is to be reduced.

A generalization of congruence relations are (not necessarily transitive) *tolerance relations*. Czedli [14] and Bandelt [6] showed that they also generate a factor lattice.

Definition 3.18 (Tolerance Relation)
A *complete tolerance relation* of a complete lattice $\underline{L}$ is a reflexive and symmetric relation $\theta \subseteq \underline{L} \times \underline{L}$ on $\underline{L}$ that satisfies the following condition:

$$x_t \theta y_t \text{ for } t \in T \Rightarrow (\bigvee_{t \in T} x_t)\theta(\bigvee_{t \in T} y_t) \text{ and } (\bigwedge_{t \in T} x_t)\theta(\bigwedge_{t \in T} y_t)$$

We also shortly say tolerance when referring to a (complete) tolerance relation.

It is also possible to describe the tolerance relations of a lattice in the associated formal context. They correspond to the block relations, as investigated by Wille in [59] in the following way:

Theorem 3.4 ([29, Thm. 15])
Let $\mathbb{K} = (G, M, I)$ be a formal context. The lattice of all block relations of $\mathbb{K}$ is isomorphic to the lattice of all tolerance relations of $\underline{\mathfrak{B}}(\mathbb{K})$. For every tolerance relation θ the isomorphism β maps to the block relation defined by

$$g\beta(\theta)m :\Longleftrightarrow \gamma g \theta(\gamma g \wedge \mu m) \Longleftrightarrow (\gamma g \vee \mu m)\theta \mu m.$$

Conversely,

$$(A, B)\beta^{-1}(J)(C, D) \Longleftrightarrow A \times D \cup C \times B \subseteq J$$

yields the tolerance relation corresponding to the block relation J.

Proposition 3.8 ([29, Prop. 40])
If a complete congruence θ is induced by a compatible subcontext $[H, N] \leq \mathbb{K}$ then:

$$H = \{g \in G \mid \gamma g \textit{ is the smallest element of a } \theta\textit{-class}\} \textit{ and}$$

$$N = \{m \in M \mid \mu m \textit{ is the greatest element of a } \theta\textit{-class}\}.$$

Example 3.8
Considering $\underline{\mathfrak{B}}(\mathbb{K}_{Aladdin})$, we find no congruence relation but the trivial ones – namely, the one with just one equivalence relation and the one with a different equivalence relation for each element of the lattice. In contrast, a non-trivial tolerance relation can be found as pictured in Figure 3.8. This relation consists of two relation classes: one is highlighted in red, and one is highlighted with dotted boxes. The corresponding block relation of $\mathbb{K}_{Aladdin}$ is presented in Figure 3.8 as well.

$\mathbb{K}_{Aladdin}$	human	animal	has magic	can speak	can fly	villain
Aladdin	×	•	•	×	•	•
Jasmin	×	•	•	×	•	•
Genie	•	•	×	×	×	•
Jafar	×	•	×	×	•	×
Abu		×	•	•	•	
Jago	•	×	•	×	×	•
Magic Carpet		•	×	•	×	

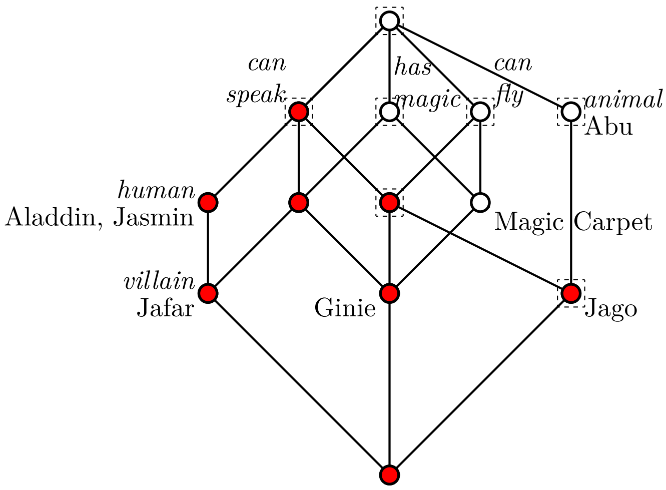


Figure 3.8 A not transitive tolerance relation θ on $\underline{\mathfrak{B}}(\mathbb{K}_{Aladdin})$ (bottom). The two different relation-classes of θ are highlighted, one by color and one by dotted boxes. Adding the •-marked incidences to $\mathbb{K}_{Aladdin}$ results in the block relation corresponding to θ (top).

3.4 Implications

The relationships between the attributes of a context are represented as *implications*. By this, a *valid implication* represents a characteristic underlying the data.

Definition 3.19 (Implication, Premise, Conclusion)
Let $\mathbb{K} = (G, M, I)$ be a formal context. An *implication* (in M) is a pair of attribute subsets $X, Y \subseteq M$, denoted by $X \to Y$, with *premise* X and *conclusion* Y. It is called *valid* in $\mathbb{K}$ if and only if $X' \subseteq Y'$. In this case, we call $X \to Y$ an *implication of* $\mathbb{K}$. The set of all implications of a formal context $\mathbb{K}$ is denoted by $Imp(\mathbb{K})$.

Instead of stating all implications of a formal context, the declaration of a *implication base* from that all implications follow is sufficient.

Definition 3.20 (Canonical Base)
Let $\mathbb{K} = (G, M, I)$be a formal context. A minimal set $\mathfrak{L} \subseteq Imp(\mathbb{K})$ defines an *implication base* if every implication of $\mathbb{K}$ follows from $\mathfrak{L}$ by composition. An implication base of minimal size is called *canonical base* of $\mathbb{K}$ and is denoted by $\mathfrak{C}(\mathbb{K})$

Example 3.9
One valid implication of the context $\mathbb{K}_{Aladdin}$ is {human}→{can speak}. The reverse implication {can speak}→{human} is not valid in the context since the object Genie is a counterexample.

CHAPTER 4

Related Work

In the field of Formal Concept Analysis, several approaches aim to analyze (small) parts of a formal context or a concept lattice, as well as to investigate the connection between the two data structures. E.g., Albano [1] explores local changes to a formal context and their effects on the corresponding concept lattice, namely the number of concepts. In [3], in particular, the impact of contranominal scales in a formal context on the size of the corresponding concept lattice is studied by giving an upper bound for $\mathfrak{B}(k)$-free lattices. We extend the notion of contranominal scales to Boolean subcontexts in Chapter 5. Based on this, we present direct connections between Boolean subcontexts of a formal context and Boolean suborders of the corresponding concept lattice. Focussing on Boolean sublattices, we build up on Willes work [58] where he presents a one-to-one correspondence between closed subrelations of a formal context and complete sublattices of the associated concept lattice, which is also the basis for the work of Kauer and Krupke [36]. They investigate the problem of constructing the closed subrelation referring to a complete sublattice generated by a given subset of elements while not computing the whole concept lattice.

Since one of the central aspects of Formal Concept Analysis is the representation and analysis of (binary) data in a way suitable for the human mind, an approach to improve the readability of concept lattices is made by optimizing their presentation. Here nested line diagrams [60] and drawing algorithms [20] can be used. However, neither of them compresses the size of the datasets, and thus grasping relationships in large concept lattices remains hard. Therefore, many approaches aim to compress the size of a data set by selecting special substructures or altering the data.

One way is the selection of a suitable attribute set like done in [4] by a procedure based on random projection. In the case of many-valued contexts, Ganter and Kuznetzov [28] select features based on their scaling. We introduce the *contranominal influence* as a measure for selection attributes based on their appearance in contranominal scales to enable the elimination of structures that support the exponential growth of the concept lattice directly in the corresponding formal context.

Also, in the field of supervised machine learning there are numerous approaches for feature set selection. The authors from [34] introduce a beneficial categorization for those in two categories: wrapper models – A a representative of this model type is the class of selective Bayesian classifiers in Langley and Sage [46] – and filters – like RELIEF [38]. The wrapper models evaluate feature subsets using the underlying learning algorithm which allows responding to redundant or correlated features while the filter models work independently from the underlying learning algorithm. Instead these methods make use of general characteristics like the attribute distribution with respect to the labels in order to weigh an attribute's importance [63]. An entropy-based approach of a filter model is introduced by Koller et al. [41] where the authors select features based on the Kullback-Leibler-distance. All these methods incorporate an underlying notion of attribute relevance as captured and formalized in the seminal work by Blum and Langley in [10], on which we base the notion of *relevant attributes* in formal contexts, we present in Chapter 6.

Besides meaningful reduction, altering the dataset is a standard method in FCA, which is motivated by an attempt to reduce the complexity of the dataset or deal with noise. In this realm, Dias and Vieira investigate the replacement of *similar* objects by a single representative [18]. They evaluate this strategy by measuring the appearance of false implications on the new object set. In the attribute case, a similar approach is explored by Kuitche et al. [44]. Approximate frequent item sets are investigated to handle noisy data in [47], where the authors state an additional threshold for both rows and columns of the dataset. More related works originate from granular computing with FCA, e.g. [50]. A basic idea here is to find information granules based on entropy. To this end, the authors of [48] introduce an (object) entropy function for formal contexts, which we utilize in Chapter 6 as well. Their approach used the principles of granulation as in [64], which is based on merging attributes to reduce the data set.

Another approach is the direct selection of entire concepts. This can be done by random sampling [11] or concept selection [39]. For such a selection various measures are applicated. A natural idea is the consideration of extent and intent size of the concepts. Based on this, Kuznetsov [45] proposes a stability measure for formal concepts, measuring the ratio of extent subsets generating the same intent. Another measure, the support measure of association rule mining, is borrowed by Stumme et al. [56] to generate iceberg concept lattices, special subsemilattices of the original concept lattice. However, a concept selection does not always result in a sub(semi)lattice of the original one. To this end, a structural approach is given in [19] through dismantling where a sublattice is generated by the iterative elimination of all doubly irreducible concepts. The dismantling of elements in ordered sets, in particular, that of irreducible elements, is examined in several works like [52, 37, 5, 57]. We build up on this approach in Chapter 9 by expanding the notion of dismantling single elements to dismantling intervals for a lattice and show the uniqueness of the core in this case inspired by the proof given by Farley for irreducible elements in [23].

Instead of selecting concepts, a factorization of them to generate some representatives is also possible. For $\bigwedge$-sublattices, $\bigvee$-sublattices and lattices – and in general for algebras (i.e., a set with operations defined on its elements) – homomorphism, congruence relation, and factor algebra are defined explicitly. In the field of lattice theory, lattice congruences, as defined in [8], where the requirement is compatibility with suprema and infima for finite sets, are examined. In the realm of ordered sets, no such operations can be utilized. Thus, there are different approaches to expand these theoretical aspects on ordered sets. Moorth and Karpagavalli introduce a congruence relation on partially ordered sets that is not a lattice congruence [26]. In particular, the congruence classes of this relation do not have to be intervals so that an intense change of the original structure is valid. Other approaches, as given by Snasel and Jukl [55] or Kolibiar [40], aim to define congruence relations on ordered sets, that are precisely the lattice congruence if applied to lattices. Those approaches are related to our approach in Chapter 8 as we introduce an equivalence relation on lattices that is no lattice congruence in general.

Part II

Boolean Substructures

CHAPTER 5

Boolean Substructures in Formal Concept Analysis

One type of substructure (more precisely: suborder or sub(semi)lattice) that is frequently occurring in a concept lattice is that of Boolean algebras. In a formal context, they correspond to subcontexts isomorphic to a contranominal scale. This chapter examines the connection and interplay between Boolean substructures in a formal context and its corresponding concept lattice.

A (concept) lattice contains an k-dimensional Boolean suborder if and only if the context contains an k-dimensional contranominal scale as subcontext. In the following, we investigate more closely the interplay between the Boolean subcontexts of a given finite context and the Boolean suborders of its concept lattice. To this end, we define mappings from the set of subcontexts of a context to the set of suborders of its concept lattice and vice versa and study their structural properties. In addition, we introduce closed-subcontexts as an extension of closed subrelations to investigate the set of all (Boolean) sublattices of a given lattice.

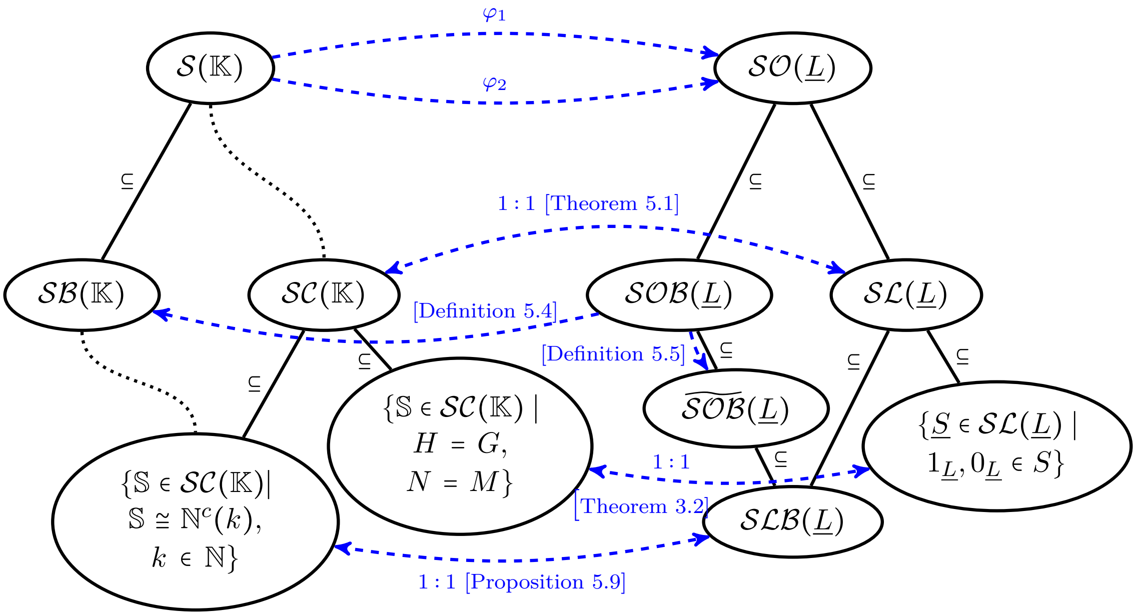


Figure 5.1 Connections between the subcontexts of a formal context $\mathbb{K}$ and the suborders of the corresponding concept lattice $\underline{L} := \underline{\mathfrak{B}}(\mathbb{K})$. The set of all subsemilattices of $\underline{L}$ is denoted by $\widetilde{\mathcal{SOB}}(\underline{L})$.

5.1 Introduction

The substructures in a formal context $\mathbb{K}$ correspond to the substructures in the associated concept lattice. To this end, Wille [58] presents closed subrelations to characterize complete sublattices of a concept lattice. Building on this, we introduce *closed-subcontexts* and present a one-to-one correspondence to all sublattices. Through this, we merge the obvious two-step approach of limiting the lattice to an interval and determining its complete sublattices in one structure. Since this construction is an – almost arbitrary and difficult to handle – mixture of subcontext and subrelation and in addition is not directly specific to the field of Boolean substructures, we investigate the connection between Boolean subcontexts and Boolean sublattices and suborders, respectively, in Section 5.4 in a direct way without having to manipulate the incidence relation. Therefore, we lift the embeddings φ_1 and φ_2 (see Proposition 3.6) to the level of subcontexts and suborders to find the Boolean suborders corresponding to a Boolean subcontext. In addition, we introduce a construction to generate the Boolean subcontext associated to a given Boolean suborder. We combine these methods to investigate to which degree the join- and meet-operators of a lattice are respected by those maps.

5.2 Boolean Subcontexts and Sublattices

In this chapter, we investigate Boolean substructures in formal contexts as well as in the corresponding concept lattices. Therefore, as illustrated in Figure 5.1, we link the different substructures of a formal context with the substructures of the corresponding concept lattice. To begin with, we introduce the concrete definitions that serve as a foundation to analyze those connections.

Definition 5.1 (Boolean Subcontext)
Let $\mathbb{K}$ be a formal context and $\mathbb{S} \leq \mathbb{K}$ a subcontext. $\mathbb{S}$ is called *Boolean subcontext of dimension* k of $\mathbb{K}$, if $\underline{\mathfrak{B}}(\mathbb{S}) \cong \mathfrak{B}(k)$. $\mathbb{S}$ is called *reduced* if $\mathbb{S}$ is a reduced context. We call $\mathbb{S}$ *maximum* Boolean subcontext of dimension k if there is no subcontext $\mathbb{T} \leq \mathbb{K}$ that is a Boolean subcontext of dimension k with $\mathbb{S} \leq \mathbb{T}$ and $\mathbb{S} \neq \mathbb{T}$. The set of all Boolean subcontexts of dimension k of $\mathbb{K}$ and the set of all reduced Boolean subcontexts of dimension k of $\mathbb{K}$ are denoted by $\mathcal{SB}_k(\mathbb{K})$ and $\mathcal{SRB}_k(\mathbb{K})$, respectively.

Note that every reduced Boolean subcontext of dimension k is isomorphic to the contranominal scale $\mathbb{N}^c(k)$.

Definition 5.2 (Boolean Suborder, Boolean Sublattice)
Let $\underline{L}$ be a lattice and $\underline{S} \leq \underline{L}$ a suborder. $\underline{S}$ is called *Boolean suborder of dimension* k if $\underline{S} \cong \mathfrak{B}(k)$. If $\underline{S}$ is a sublattice of $\underline{L}$, $\underline{S}$ is called *Boolean sublattice of dimension* k. The set of all Boolean suborders , or all Boolean sublattices, of dimension k of a lattice $\underline{L}$ is denoted by $\mathcal{SOB}_k(\underline{L})$ and $\mathcal{SLB}_k(\ \underline{\mathrm{L}}))$, respectively.

If all dimensions are considered, the number k is left out in the following.

Note that $\mathcal{SLB}_k(\underline{\mathrm{L}})$ is a subset of $\mathcal{SOB}_k(\underline{\mathrm{L}})$ and the standard context of a Boolean lattice $\underline{L}$ of dimension k consists of a formal context $\mathbb{K} \cong \mathbb{N}^c(k)$ due to Proposition 3.3. Conversely, a formal context $\mathbb{K}$ consisting of a reduced Boolean subcontext of dimension k and an arbitrary number of additional reducible attributes and objects has a corresponding concept lattice $\underline{\mathfrak{B}}(\mathbb{K}) \cong \mathfrak{B}(k)$.

For a better understanding of these structures, we introduce the formal context and its corresponding concept lattice given in Figure 5.2. We will refer back to this illustration throughout this chapter.

Example 5.1
$\mathbb{S} = (\{4,5,6\}, \{b,c,d,e\}, J)$ with $J = I \cap (\{4,5,6\} \times \{b,c,d,e\})$ is a Boolean subcontext of dimension 3 of the context $\mathbb{K}$ in Figure 5.2 (left). $\mathbb{S}$ is not reduced since $d^J = e^J$ holds. However, $\mathbb{S}$ includes two reduced Boolean subcontexts: $\mathbb{S}_1 = [\{4,5,6\}, \{b,c,d\}]$ and $\mathbb{S}_2 = [\{4,5,6\}, \{b,c,e\}]$. The third reduced Boolean subcontext of dimension 3 in $\mathbb{K}$

	a	b	c	d	e
1	×	×			
2	×		×		
3		×	×	×	×
4	×	×	×		
5		×		×	×
6			×	×	×
7				×	
8					×

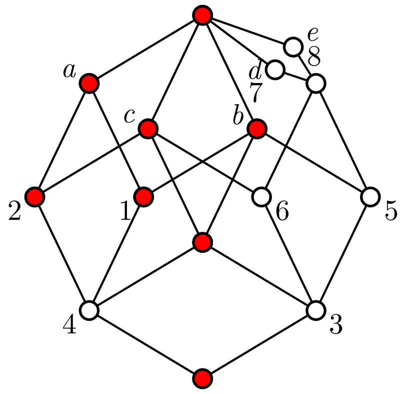


Figure 5.2 A formal context $\mathbb{K} = (G, M, I)$ containing three reduced Boolean subcontexts (left) and its corresponding concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ (right).

is $\mathbb{S}_3 = [\{1,2,3\}, \{a,b,c\}]$. In total, $\mathbb{K}$ contains 118 Boolean subcontexts of dimension 3, of which 10 are maximum. The reduced and the maximum Boolean subcontexts are listed in Table 5.1. The concept lattice of $\mathbb{K}$ (see Figure 5.2 right) contains 12 Boolean suborders of dimension 3, two of which are also Boolean sublattices. All of them are visualized in Figure 5.3.

Example 5.1 illustrates that a context can include numerous Boolean subcontexts of dimension k compared to its context size and to the number of Boolean suborders (or sublattices) of the same dimension in its corresponding concept lattice. The problem of finding all reduced Boolean subcontexts of dimension k is investigated in detail in [21], where an algorithm to compute the set of all contranominal scales

Table 5.1 List of all reduced (top) and maximum (bottom) Boolean subcontexts of dimension 3 of the context $\mathbb{K}$ from Figure 5.2 together with the suborders of $\underline{\mathfrak{B}}(\mathbb{K})$ they are mapped to by φ_1 and φ_2, respectively.

Subcontext $\mathbb{S}$	$\varphi_1(\mathbb{S})$	$\varphi_2(\mathbb{S})$
$[\{1,2,3\}, \{a,b,c\}]$	No. 3	No. 2
$[\{4,5,6\}, \{b,c,d\}]$	No. 12	No. 5
$[\{4,5,6\}, \{b,c,e\}]$	No. 12	No. 7
$[\{1,2,3\}, M]$	No. 3	No. 3
$[G, \{a,b,c\}]$	No. 2	No. 2
$[\{1,2,3,7\}, \{a,b,c,e\}]$	No. 3	No. 3
$[\{1,2,3,8\}, \{a,b,c,d\}]$	No. 3	No. 3
$[\{4,5,6\}, M]$	No. 12	No. 12
$[G, \{b,c,d\}]$	No. 5	No. 5
$[\{4,5,6,7,8\}, \{a,b,c,d\}]$	No. 10	No. 10
$[\{1,2,3,4,5,6\}, \{b,c,d,e\}]$	No. 9	No. 9
$[G, \{b,c,e\}]$	No. 7	No. 7
$[\{4,5,6,7,8\}, \{a,b,c,e\}]$	No. 11	No. 11

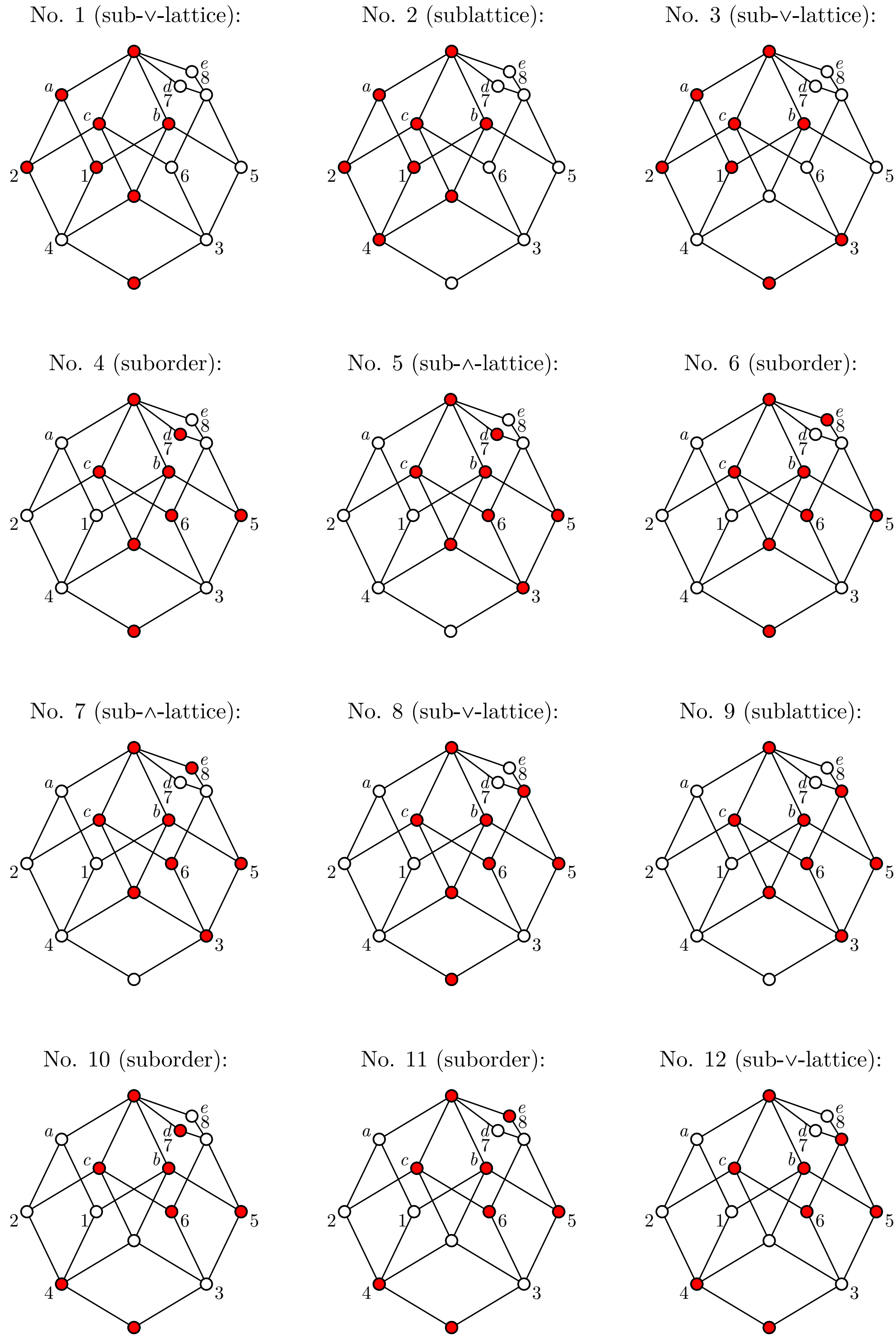


Figure 5.3 The context from Figure 5.2 with all Boolean suborders of dimension 3 highlighted red. The suborders are numbered above the respective lattice.

Algorithm 1: Generation of $\mathcal{SB}_k(\mathbb{K})$ from $\mathcal{RSB}_k(\mathbb{K})$

Input: $\mathbb{K} = (G, M, I)$, k, $\mathcal{RSB}_k(\mathbb{K})$
Output: $\mathcal{SB}_k(\mathbb{K})$

$Y \coloneqq M \cup G$
$n \coloneqq |Y|$
$X_0 \coloneqq \mathcal{RSB}_k(\mathbb{K})$
$X_1 = X_2 = \cdots = X_{n-2k} = \varnothing$
for $i \in \{0, 1, .., (n - 2k - 1)\}$ **do**
    for $\mathbb{S} = (H, N, J) \in X_i$ **do**
        for $y \in Y \setminus (H \cup N)$ **do**
            if $y \in M$ **then**
                if $\exists R \subseteq N : y^J = R^J$ **then**
                    $X_{i+1} \coloneqq X_{i+1} \cup (H, (N \cup y), I \cap (H \times (N \cup y)))$
            if $y \in G$ **then**
                if $\exists R \subseteq H : y^J = R^J$ **then**
                    $X_{i+1} \coloneqq X_{i+1} \cup ((H \cup y), N, I \cap ((H \cup y) \times N))$
return $\bigcup_{i=0}^{n-2k} X_i$

of a given formal context is presented. Building on that, we reduce the problem of finding all Boolean subcontexts of dimension k to the problem of finding all reduced Boolean subcontexts of the same dimension as follows: In a formal context $\mathbb{K}$ every Boolean subcontext of dimension k contains a reduced Boolean subcontext of the same dimension. Therefore, we start with a reduced Boolean subcontext $\mathbb{S} \cong \mathbb{N}^c(k)$ and successively add a reducible attribute or object. This procedure constructs a new Boolean subcontext of dimension k in every step. To obtain all possible attribute and object combinations that belong to a Boolean subcontext every additional attribute and object combination has to be investigated for each subcontext $\mathbb{S} \in \mathcal{SRB}_k(\mathbb{K})$ as starting point. In the case of $M \cap G \neq \varnothing$ the attributes and objects have to be renamed before computing the set $\mathcal{RSB}_k(\mathbb{K})$ and starting the algorithm in the following way: Let $\mathbb{K}^* = (G^*, M^*, I^*)$ with $M^* = \{(1, x) | x \in M\}$, $G^* = \{(2, x) | x \in G\}$ and $I^* = \{(x, y) | x = (1, g), y = (2, m) \text{ for } (g, m) \in I\}$. Then we generate $\mathcal{SB}_k(\mathbb{K}^*)$ through the following strategy and subsequently extract $\mathcal{SB}_k(\mathbb{K})$ by renaming the attributes and objects in the reverse way. Since we consider $\mathbb{K}$ to be finite, only a finite number of reduced Boolean subcontexts, attributes, and objects exist. Therefore the procedure presented in Algorithm 1 ends after a finite number of steps. Note that we turn away from optimizing the algorithm since this naive strategy already answers the structural question of finding all Boolean subcontexts.

Since the reduction of a finite clarified formal context is not affected by the order in which attributes and objects are reduced the following statement yields:

Corollary 5.1
Let $\mathbb{K} = (G, M, I)$ *be a formal context with* $\mathbb{R} = [H_R, N_R] \leq \mathbb{S} = [H_S, N_S] \leq \mathbb{K}$ *so that* $\mathbb{S} \in \mathcal{SB}_k(\mathbb{K})$ *and* $\mathbb{R} \in \mathcal{RSB}_k(\mathbb{K})$. *If* $\mathbb{S}$ *is clarified, the reduction of* $\mathbb{S}$ *results in* $\mathbb{R}$.

If $\mathbb{S}$ is not clarified, $\mathbb{S}$ may contain more than just one reduced Boolean subcontext of dimension k. Depending on which clarifiable objects and attributes are removed, a reduction can lead to different outcomes, namely the reduced Boolean subcontexts of dimension k contained in $\mathbb{S}$. Those are all isomorphic.

Reversing the former observation, we can conclude that Algorithm 1 generates all Boolean subcontexts of dimension k given the set $\mathcal{SRB}_k(\mathbb{K})$.

Proposition 5.1
Given $\mathcal{RSB}_k(\mathbb{K})$, *every Boolean subcontext of dimension* k *of a formal context* $\mathbb{K}$ *is computed by the execution of Algorithm 1.*

Proof Let $\mathbb{S} = [H_S, N_S] \leq \mathbb{K}$ be a Boolean subcontext of dimension k. Hence, there is a subcontext $\mathbb{R} = [H_R, N_R] \in \mathcal{RSB}_k(\mathbb{K})$ with $\mathbb{R} \leq \mathbb{S}$. Let $\{g_1, g_2, \ldots, g_u\} = H_S \setminus H_R$ and $\{m_1, m_2, \ldots, m_v\} = N_S \setminus H_R$ be the sets of all objects and attributes included in $\mathbb{S}$ but not in $\mathbb{R}$. Since the order of removing those reducible elements does not matter, inverse the order of the addition of reducible elements does not matter by the generation of a Boolean subcontext from the reduced Boolean subcontext. Following the set X_{u+v} with $u = |H_S \setminus H_R|$ and $v = |N_S \setminus N_R|$ includes $\mathbb{S}$. □

For a formal context $\mathbb{K}$ the subcontext-relation is a natural order on $\mathcal{SB}_k(\mathbb{K})$. With this order, the subcontexts can be represented by a line diagram. In general, this line diagram is no lattice since multiple minima (the elements of $\mathcal{RSB}_k(\mathbb{K})$) and maxima (the maximum Boolean subcontexts of dimension k) can exist. Additionally, it is not necessarily connected. The line diagram represents a part of the lattice on the powerset of attributes and objects of $\mathbb{K}$. Since Algorithm 1 successively adds elements to the reduced subcontexts, it is possible to generate the order and, therefore, the line diagram while computing the set of all Boolean subcontexts of a given dimension.

As for the formal context $\mathbb{K}$ in Figure 5.2 there are 3 reduced Boolean subcontexts of dimension 3 as mentioned in Example 5.1. Those are the minimal elements referring to the order on $\mathcal{SB}_3(\mathbb{K})$. By successive adding of an additional attribute or object, all subcontexts $\mathbb{S} \in \mathcal{SB}_3(\mathbb{K})$ are built. We illustrate this procedure by a cutout of the extension of $\mathbb{S}_1 = [\{4, 5, 6\}, \{b, c, d\}]$ in Figure 5.4.

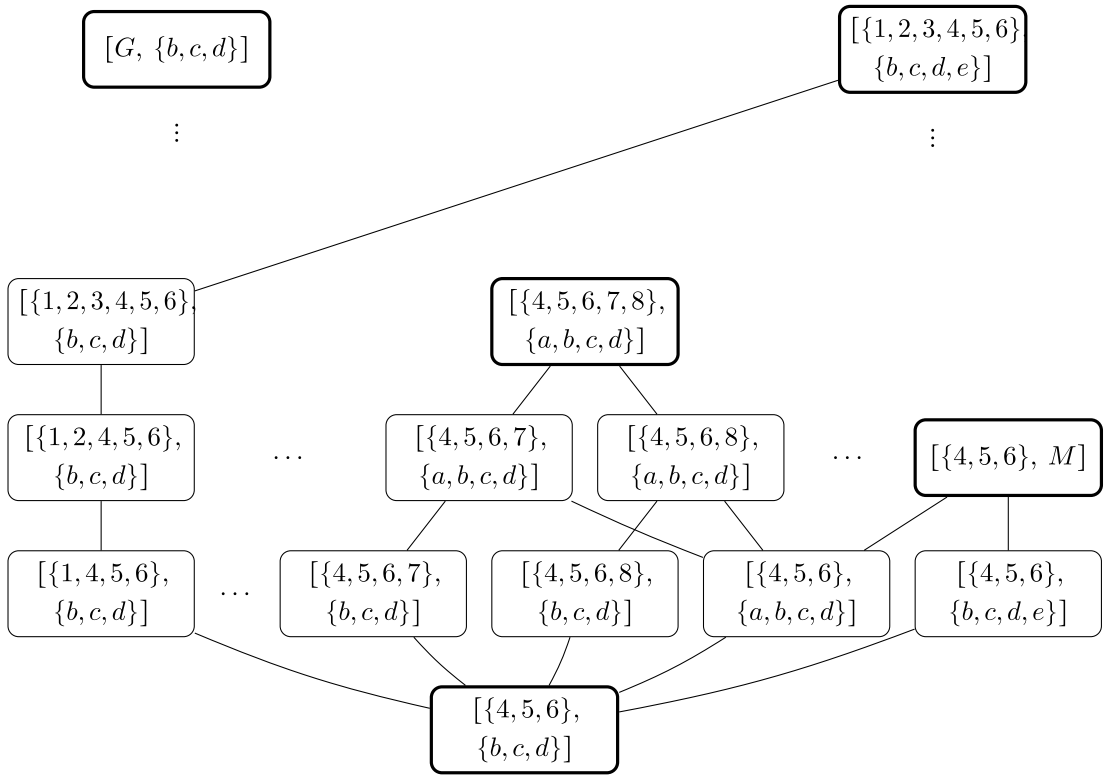


Figure 5.4 A cutout of $\mathcal{SB}_3(\mathbb{K})$ with the subcontext-order. The maxima and minima of this part are marked with an intense border.

5.3 Closed-Subcontexts

At first, we leave the field of (Boolean) suborders and narrow our focus on (Boolean) sublattices. On the context side, we introduce so-called *closed-subcontexts* and show their one-to-one relationship to the sublattices of the concept lattice.

In [58], Wille introduced closed relations of a context that can be utilized to characterize the complete sublattices of its concept lattice. In finite lattices, complete sublattices differ from (non-complete) sublattices in that they always include the unit element and the zero element of the lattice. We adopt Wille's construction to match with (non-necessarily complete) sublattices.

Definition 5.3 (Closed-Subcontext)
Let $\mathbb{K} = (G, M, I)$ and $\mathbb{S} = (H, N, J)$ be two formal contexts. We call $\mathbb{S}$ *closed-subcontext of* $\mathbb{K}$ if $H \subseteq G$, $N \subseteq M$, $J \subseteq I \cap (H \times N)$ and every concept of $\mathbb{S}$ is a concept of $\mathbb{K}$ as well. The set of all closed-subcontexts of $\mathbb{K}$ is denoted by $\mathcal{SC}(\mathbb{K})$.

The sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$ are in a one-to-one correspondence with the closed-subcontexts of $\mathbb{K}$ as follows:

Theorem 5.1
Let $\mathbb{K}$ be a formal context and $\underline{S}$ be a sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$. Then

$$\mathbb{K}_{\underline{S}} := \Big(\bigcup_{(A,B)\in\underline{S}} A, \bigcup_{(A,B)\in\underline{S}} B, \bigcup_{(A,B)\in\underline{S}} A \times B \Big)$$

is a closed-subcontext of $\mathbb{K}$. Conversely, for every closed-subcontext $\mathbb{S}$ of $\mathbb{K}$, $\underline{\mathfrak{B}}(\mathbb{S})$ is a sublattice of $\mathfrak{B}(\mathbb{K})$.
Furthermore, the map $f(\underline{S}) := \mathbb{K}_{\underline{S}}$ maps the set of sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$ bijectively onto the set of closed-subcontexts of $\mathbb{K}$.

Proof For a formal concept $(A,B) \in \underline{S}$ the concept $(A,B) \in \mathfrak{B}(\mathbb{K}_{\underline{S}})$ is a concept in $\mathbb{K}$ due to construction. On the other side, let $\mathbb{S} = (H,N,J)$ be a closed-subcontext of $\mathbb{K}$. We have $\mathfrak{B}(\mathbb{S}) \subseteq \mathfrak{B}(\mathbb{K})$ and therefore $\underline{\mathfrak{B}}(\mathbb{S})$ is a suborder of $\underline{\mathfrak{B}}(\mathbb{K})$. Let $(A_1,B_1),(A_2,B_2) \in \underline{\mathfrak{B}}(\mathbb{S})$. Let (A_S,B_S) be the infimum of both concepts in $\mathbb{S}$ and (A_K,B_K) the infimum of both concepts in $\mathbb{K}$. So $A_S = A_1 \cap A_2 = A_K$, which implies $(A_S,B_S) = (A_K,B_K)$ since (A_S,B_S) is by definition a concept in $\mathbb{K}$. The dual argument shows that $\mathbb{S}$ is closed under suprema. So $\underline{\mathfrak{B}}(\mathbb{S})$ is a sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$. □

An example of a small formal context with all closed-subcontexts and their corresponding sublattices is presented in Figure 5.5.

Note that the closed-subsets of a formal context do not form a closure system since the intersection of two closed-subcontexts, in general, is not a closed-subcontext, even though the sublattices of the concept lattice do so.

In the construction of $\mathbb{K}_{\underline{S}}$, $\bigcup_{(A,B)\in\underline{S}} A$ is the concept extent of the unit element of the sublattice and $\bigcup_{(A,B)\in\underline{S}} B$ is the concept intent of its zero element. Since the extents and intents of a closed-subcontext have to be extents and intents in the original context, the following statement arises:

Proposition 5.2
Let $\mathbb{K} = (G,M,I)$ be a formal context and $\mathbb{S} = (H,N,J)$ a closed-subcontext of $\mathbb{K}$. Then $H = G$ or an attribute $m \in N$ with $m' = H$ exists. And $N = M$ or an object $g \in H$ with $g' = N$ exists.

Proof Due to Definition 5.3, every concept of $\mathbb{S}$ is a concept of $\mathbb{K}$ as well. In particular, this has to hold for the concepts $(\varnothing'',\varnothing')$ and (H'',H') of $\mathbb{S}$. □

We provide next some basic statements about closed-subcontexts. Since the following propositions are based on the work of Wille [58] and lifted to our approach, the proofs are similar to the ones from in [29, Sec. 3.3] and we omit them.

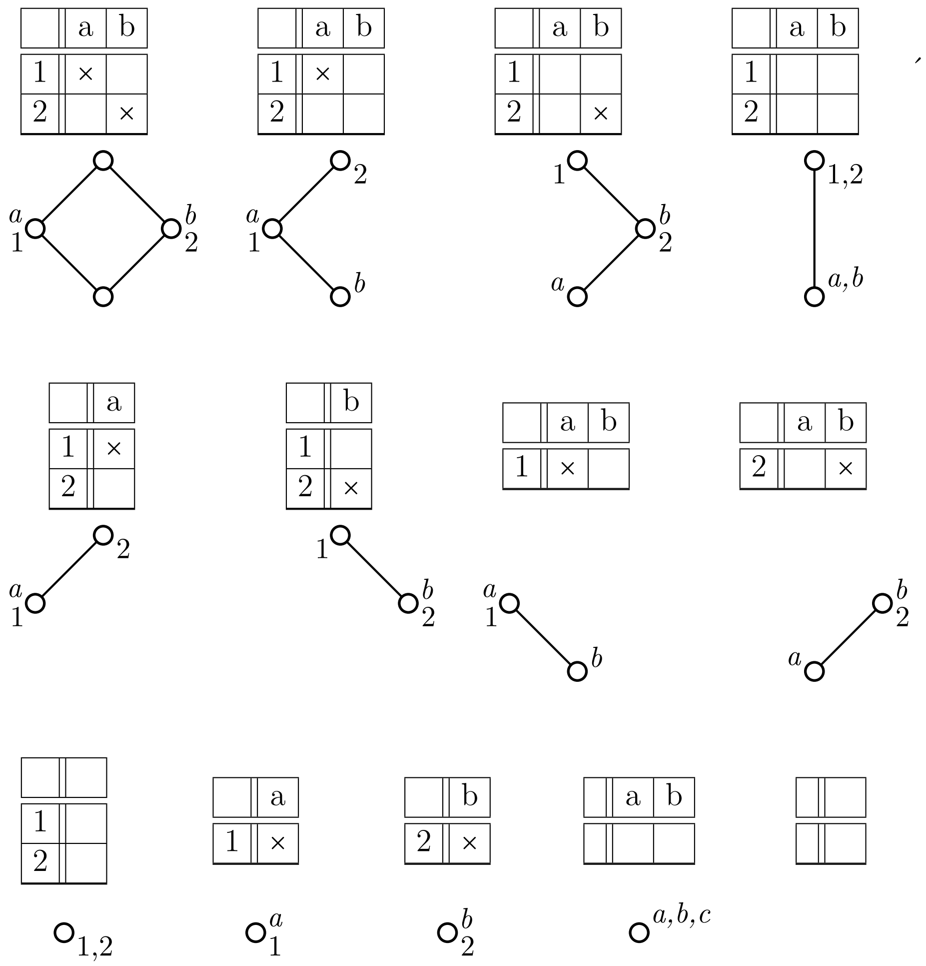


Figure 5.5 A formal context $\mathbb{K} = (G, M, I)$ together with its corresponding concept lattice (top left). All closed-subcontexts of $\mathbb{K}$ are given. Their corresponding sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$ are each presented beneath the particular context. The contexts in the first row include G and M. Therefore, their corresponding sublattices are complete sublattices.

Proposition 5.3
For every set $T \subseteq \mathfrak{B}(G, M, I)$ there is a smallest closed-subcontext $\mathbb{S}$ of $\mathbb{K}$, so that all incidences $(A \times B)$ for $(A, B) \in T$ are contained in $\mathbb{S}$. $\underline{\mathfrak{B}}(\mathbb{S})$ is the sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$ generated by *T.*

Proof The proof follows the structure of the proof of Proposition 45 in [29]. □

Proposition 5.4
$\mathbb{S} = (H, N, J)$ is a closed-subcontext of the formal context $\mathbb{K} = (G, M, I)$ if and only if $X^{JJ} \supseteq X^{JI}$ holds for each $X \subseteq H$ and for each $X \subseteq N$.

Proof The proof follows the structure of the proof of Proposition 46 in [29]. □

Proposition 5.5
The closed-subcontexts (H,N,J) *of* (G,M,I) *are exactly the subcontexts that satisfy the following condition:*
If $(g,m) \in (H \times N)$ *and* $(g,m) \in I \setminus J$ *then* $(h,m) \notin I$ *for* $h \in H$ *with* $g^J \subseteq h^J$ *and* $(g,n) \notin I$ *for* $n \in N$ *with* $m^J \subseteq n^J$.

Proof The proof follows the structure of the proof of Proposition 47 in [29]. □

Proposition 5.6
Let $\mathbb{K} = (G,M,I)$ *be a formal context,* $H \subseteq G$, $N \subseteq M$, *and* $\mathbb{S} = (H,N,J)$ *a clarified formal context. Then* $\mathbb{S} = (H,N,J)$ *is a closed-subcontext of* $\mathbb{K}$ *if and only if* $J \subseteq I \cap (H \times N) \subseteq H \times N \setminus (\nearrow^J \cup \swarrow^J)$.

Proof The proof follows the structure of the proof of Proposition 49 in [29]. □

Proposition 5.7
Let $\mathbb{K} = (G,M,I)$ *be a formal context and* $(A,B),(C,D)$ *concepts of* $\mathbb{K}$. *Then* $(A,B,A{\times}B)$, $(A,M,I{\cap}(A{\times}M))$, *and* $(G,B,I{\cap}(G{\times}B))$ *are closed-subcontexts. The associated lattices are* $\underline{\mathfrak{B}}(A,B,A\times B) = \{(A,B)\}$, $\underline{\mathfrak{B}}(A,M,I\cap(A\times M)) = ((A,B)]$, *and* $\underline{\mathfrak{B}}(G,B,I\cap(G\times B)) = [(A,B))$.
If $(A,B) \leq (C,D)$ *also* $(C,B,(A\times B \cup C\times D))$ *and* $(C,B,I\cap(C\times B))$ *are closed-subcontexts with the concept lattices* $\underline{\mathfrak{B}}(C,B,(A\times B\cup C\times D)) = \{(A,B),(C,D)\}$, *and* $\underline{\mathfrak{B}}(C,B,I\cap(C\times B)) = [(A,B),(C,D)]$.

Proof The proof follows the structure of the proof of Proposition 50 in [29]. □

Also, the set of the arrow relations of a closed-subcontext $\mathbb{S}$ is a subset of the set of the arrow relations of the original context $\mathbb{K}$.

Proposition 5.8
Let $\mathbb{K} = (G,M,I)$ *be a formal context and* $\mathbb{S} = (H,N,J)$ *a closed-subcontext. Then* $\nearrow^J \subseteq \nearrow^I$ *and* $\swarrow^J \subseteq \swarrow^I$ *hold.*

Proof Let $g \in H, m \in N$ and $g \swarrow^J m$. Assumed that $g \not\swarrow^I m$, then there exists $h \in G$ with $g^I \subseteq h^I$ and $(h,m) \notin I$. It follows $g^J \subseteq g^{I\cap(G\times H)} \subseteq h^{I\cap(G\times H)}$ and therefore $h \in h^{I\cap(G\times H)} \subseteq g^{JI} = g^{JJ} \subseteq H \Rightarrow g^J \subseteq h^J$. This is a conflict to $g \swarrow^J m$. □

Now we transfer our approach to the field of Boolean substructures. To find all Boolean sublattices (of dimension k) in a lattice $\mathfrak{B}(\mathbb{K})$, the closed-subcontexts of $\mathbb{K}$ that are also Boolean subcontexts have to be found. Hence, Theorem 5.1 can be restricted in the following way:

Proposition 5.9
Let $\mathbb{K}$ be a formal context with $\underline{S} \subseteq \underline{\mathfrak{B}}(\mathbb{K})$. Then $\underline{S} \in \mathcal{SLB}_k(\underline{\mathfrak{B}}(\mathbb{K}))$ if and only if $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}}) \cong \mathfrak{B}(k)$ for the context $\mathbb{K}_{\underline{S}} = (\bigcup_{(A,B)\in\underline{S}} A, \bigcup_{(A,B)\in\underline{S}} B, \bigcup_{(A,B)\in\underline{S}} A \times B)$.

The properties of closed-subcontexts can be utilized to identify the Boolean closed-subcontexts in a formal context $\mathbb{K}$ in a direct way. Since every concept in $\mathbb{K}$ is either retained or erased but not altered in a closed-subcontext $\mathbb{S}$, its Boolean structure has to be preserved from $\mathbb{K}$. Every subcontext $\mathbb{T} = (H, N, J) \in \mathcal{SRB}(\mathbb{K})$ provides the Boolean structure. By Lifting each concept $(A_{\mathbb{T}}, B_{\mathbb{T}}) \in \underline{\mathfrak{B}}(\mathbb{T})$ to a concept $(A_{\mathbb{K}}, B_{\mathbb{K}}) \in \underline{\mathfrak{B}}(\mathbb{K})$ with $A_{\mathbb{T}} \subseteq A_{\mathbb{K}}$ and $B_{\mathbb{T}} \subseteq B_{\mathbb{K}}$, an extension of the sets H, N and J that provides a Boolean closed-subcontext $\mathbb{S} = (\widetilde{H}, \widetilde{N}, \widetilde{J})$ of $\mathbb{K}$ is generated as follows: $\widetilde{H} \coloneqq H \cup \bigcup_{(A_{\mathbb{T}},B_{\mathbb{T}})\in\underline{\mathfrak{B}}(\mathbb{T})} A_{\mathbb{K}}$, $\widetilde{N} \coloneqq H \cup \bigcup_{(A_{\mathbb{T}},B_{\mathbb{T}})\in\underline{\mathfrak{B}}(\mathbb{T})} B_{\mathbb{K}}$ and $\widetilde{J} \coloneqq \bigcup_{(A_{\mathbb{T}},B_{\mathbb{T}})\in\underline{\mathfrak{B}}(\mathbb{T})} (A_{\mathbb{K}} \times B_{\mathbb{K}})$. This approach is represented through the dotted lines in Figure 5.1.

5.4 Connecting Suborders and Subcontexts

In this section, we investigate the relationship between Boolean subcontexts and Boolean suborders. For this purpose, we use the embeddings φ_1 and φ_2 and expand them to the set of Boolean subcontexts. Further, we present a construction to get from a Boolean suborder to a corresponding Boolean subcontext. Both approaches are analyzed with focus on the structural information they transfer and their interplay.

5.4.1 Embeddings of Boolean Substructures

To investigate the connection between Boolean subcontexts $\mathbb{S}$ of a formal context $\mathbb{K}$ and Boolean suborders of $\mathfrak{B}(\mathbb{K})$ we consider embeddings of $\mathfrak{B}(\mathbb{S})$ in $\mathfrak{B}(\mathbb{K})$. Therefore we lift the embeddings φ_1 and φ_2 (see Proposition 3.6) to the level of subcontexts and suborders:

$$\varphi_1 : \mathcal{S}(\mathbb{K}) \to \mathcal{SO}(\underline{\mathfrak{B}}(\mathbb{K})),\ \mathbb{S} \mapsto (\{\varphi_1(c) \mid c \in \mathfrak{B}(\mathbb{S})\}, \leq) \text{ and}$$
$$\varphi_2 : \mathcal{S}(\mathbb{K}) \to \mathcal{SO}(\underline{\mathfrak{B}}(\mathbb{K})),\ \mathbb{S} \mapsto (\{\varphi_2(c) \mid c \in \mathfrak{B}(\mathbb{S})\}, \leq).$$

From the input (concept or context), it is clear whether the original or the lifted versions of the embeddings φ_1 and φ_2 are used in the following.

Example 5.2
All reduced and all maximum Boolean subcontexts of dimension 3 of the context $\mathbb{K}$ from Figure 5.2 are listed in Table 5.1. For each of them the suborders of $\underline{\mathfrak{B}}(\mathbb{K})$ they are mapped to by φ_1 and φ_2 are given.

We will, in particular, study these mappings for Boolean subcontexts. In this case, an additional structural benefit arises: The images of reduced Boolean subcontexts are sub-$\vee$-semilattice and sub-$\wedge$-semilattices of the original concept lattice:

Proposition 5.10
Let $\mathbb{K}$ be a formal context and $\mathbb{S} = [H, N] \in \mathcal{SRB}_k(\mathbb{K})$. Then $\varphi_1(\underline{\mathfrak{B}}(\mathbb{S}))$ is a sub-$\vee$-semilattice of $\underline{\mathfrak{B}}(\mathbb{K})$ and $\varphi_2(\underline{\mathfrak{B}}(\mathbb{S}))$ is a sub-$\wedge$-semilattice of $\underline{\mathfrak{B}}(\mathbb{K})$.

Proof Consider φ_1: Let $J \coloneqq I \cap (H \times N)$ and $(A, B), (C, D)$ be two concepts of $\underline{\mathfrak{B}}(\mathbb{S})$. Then $\varphi_1(A, B) \vee \varphi_1(C, D) = (A'', A') \vee (C'', C') = ((A'' \cup C'')'', (A' \cap C')) = ((A' \cap C')', (A \cup C)') = ((A \cup C)'', (A \cup C)')$ and in addition $((A \cup C)'', (A \cup C)') = \varphi_1((A \cup C), (B \cap D)) = \varphi_1((A, B) \vee (C, D))$. Since $\mathbb{S}$ is a reduced Boolean context, it includes all possible object combinations as extents so that $E = E^{JJ}$ holds for every $E \subseteq H$. Thus, $(A, B) \vee (c; D) = ((A \cup C)^{JJ}, B \cap D) = (A \cup C, B \cap D)$ holds in $\mathfrak{B}(\mathbb{S})$. The procedure for φ_2 is analogous. $\square$

Note that this conclusion does not hold for Boolean reducible subcontexts, e.g., the formal context given in Figure 5.2 and its subcontext $\mathbb{S} = [\{1, 2, 3, 7\}, \{a, b, c, e\}]$ where $\varphi_1(\mathbb{S}) = \varphi_2(\mathbb{S})$. The corresponding suborder is suborder No. 1 (see Figure 5.3). Therefore it is a sub-$\vee$-semilattice but no sub-$\wedge$-semilattice.

The images of the two maps of a reduced Boolean context are in general just a sub-$\vee$-semilattice and a sub-$\wedge$-semilattice, respectively. Hence, the images of φ_1 and φ_2 have to be identical for $\mathbb{S} \in \mathcal{SRB}_k(\mathbb{K})$ to certainly generate a lattice. This means $\varphi_1(A, B) = (A'', A) = (B', B'') = \varphi_2(A, B)$ has to hold for all $(A, B) \in \mathfrak{B}(\mathbb{S})$.

For every concept (A, B) of a subcontext $\mathbb{S} = (H, N, J) \leq \mathbb{K}$ we can differ between the four cases:

1) $A' = A^J = B$, $B' = B^J = A$,
2) $A' = A^J = B$, $A = B^J \subset B'$,
3) $B = A^J \subset A'$, $B' = B^J = A$, and
4) $B = A^J \subset A'$, $A = B^J \subset B'$.

The condition under which $\varphi_1(A, B) = \varphi_2(A, B)$ holds is the following:

Proposition 5.11
Let $\mathbb{K} = (G, M, I)$ be a formal context and $\mathbb{S} \leq \mathbb{K}$. Then $\varphi_1(\mathbb{S}) = \varphi_2(\mathbb{S})$ holds if and only if $(A' \setminus B) \times (B' \setminus A) \subseteq I$ holds for all $(A, B) \in \underline{\mathfrak{B}}(\mathbb{S})$. If case 1, 2 or 3 holds for all $(A, B) \in \underline{\mathfrak{B}}(\mathbb{S})$, then $\varphi_1(\mathbb{S}) = \varphi_2(\mathbb{S})$ holds directly.

	a	b	c	d	e
1	×	×		×	×
2	×		×	×	×
3		×	×		×
4	×			×	
5	×		×		

Figure 5.6 A formal context $\mathbb{K}$ with a highlighted subcontext $\mathbb{S}$ that is a maximum Boolean subcontext of dimension 3 with $\varphi_1(\mathbb{S}) \neq \varphi_2(\mathbb{S})$.

Proof For a concept $(A, B) \in \mathfrak{B}(\mathbb{S})$ the identity of both embeddings leads to $\varphi_1(A, B) = \varphi_2(A, B) \Leftrightarrow (A'', A') = (B', B'') = (B', A') \Leftrightarrow (B' \times A') \subseteq I$. This set can be written as $B' \times A' = A \times B \,\cup\, (B' \setminus A) \times B \,\cup\, A \times (A' \setminus B) \,\cup\, (B' \setminus A) \times (A' \setminus B)$. We know $A \times B \subseteq I$ since $(A, B) \in \mathfrak{B}(\mathbb{S})$ and $A \times A' \subseteq I$ and $B' \times B \subseteq I$ by definition of the $\cdot'$ operator. The remaining part equals $(A' \setminus B) \times (B' \setminus A)$. In cases 1 to 3 $(A'', A') = (B', B'')$ holds by construction. □

If a subcontext $\mathbb{S} \leq \mathbb{K}$ contains the whole object set G (or the whole attribute set M), for every concept in $\mathbb{S}$ either case 1 or 2 (or either case 1 or 3, respectively) hold. Therefore, $\varphi_1 = \varphi_2$ holds in all those subcontexts. Note that this is not the case for maximum Boolean subcontexts in general, e.g., considering the formal context in Figure 5.6. Its highlighted subcontext is a maximum Boolean subcontext of dimension 3. However, its concept $c = (\{2, 3\}, \{c\})$ is mapped to the two different concepts $\varphi_1(c) = (\{2, 3\}, \{c, e\})$ and $\varphi_2(c) = (\{2, 3, 5\}, \{c\})$. Therefore $\varphi_1 = \varphi_2$ does not hold in the subcontext.

Proposition 5.12
Let $\mathbb{K} = (G, M, I)$ be a formal context and $\mathbb{S} = [H, N] \in \mathcal{SB}_k(\mathbb{K})$. If $H = G$ or $N = M$, then $\varphi_1(\mathbb{S}) = \varphi_2(\mathbb{S})$ holds.

However, the relationship between the images of both mappings φ_1 and φ_2 of a specific concept is always (not only in the Boolean case) the same, namely:

Proposition 5.13
Let $\mathbb{K}$ be a formal context and $\mathbb{S} \leq \mathbb{K}$. Then $\varphi_1(A, B) \leq \varphi_2(A, B)$ for all concepts $(A, B) \in \mathfrak{B}(\mathbb{S})$.

In particular, an interval containing exactly the concepts $(C, D) \in \underline{\mathfrak{B}}(\mathbb{K})$ with $A \subseteq C$ and $B \subseteq D$ exists between $\varphi_1(A, B)$ and $\varphi_2(A, B)$ with $\varphi_1(A, B)$ as its zero element and $\varphi_2(A, B)$ as its unit element. In the extreme case, this interval can comprise all of $\underline{\mathfrak{B}}(\mathbb{K})$, as the following example shows:

	a	b	c	d
1	×	×	×	×
2	×	×	×	×
3	×	×	×	
4	×	×		

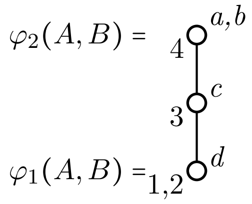


Figure 5.7 A formal context $\mathbb{K}$ (left) containing the subcontext $\mathbb{S} = [\{1,2\},\{a,b\}] = [A,B]$ with $[\varphi_1(A,B),\varphi_2(A,B)] = \underline{\mathfrak{B}}(\mathbb{K})$ (right).

Example 5.3

Let $\mathbb{K}$ be the formal context in Figure 5.7 and $\mathbb{S} = [\{1,2\},\{a,b\}] \leq \mathbb{K}$. Then for the concept $(A,B) = (\{1,2\},\{a,b\})$ of $\mathbb{S}$, $\varphi_1(A,B) = (\{1,2\},\{a,b,c,d\})$ and $\varphi_2(A,B) = (\{1,2,3,4\},\{a,b\})$ hold. These are the zero and the unit element of the whole concept lattice of $\mathbb{K}$.

This raises the question of whether there is a concept lattice where a Boolean suborder exists that can not be obtained by embedding. This is indeed the case, e.g., in Figure 5.3 the Boolean order No. 1.

An approach to make any Boolean suborder of a (concept) lattice reachable is to expand $\mathbb{K}$ by additional objects and attributes so that every concept $c \in \underline{\mathfrak{B}}(\mathbb{K})$ can be generated by one object and by one attribute. For a (concept) lattice $\underline{L}$, this is the case with the generic context $\mathbb{K} = (L, L, \leq)$. Here $\underline{S} \in \mathcal{SOB}_k(\underline{L})$ is the image of both $\varphi_1(\mathbb{S})$ and $\varphi_2(\mathbb{S})$ for the Boolean subcontext $\mathbb{S} = (S, S, \leq)$.

Since we are interested in the connections between the existence of Boolean subcontexts on the one hand and the existence of Boolean suborders on the other, we observe a first relationship between these sets.

Proposition 5.14

Let $\mathbb{K}$ be a formal context and $\mathcal{SB}_k(\mathbb{K}) \neq \varnothing$. Then $\mathcal{SOB}_k(\underline{\mathfrak{B}}(\mathbb{K})) \neq \varnothing$.

Proof Let $\mathbb{S} \in \mathcal{SB}_k(\mathbb{K})$. By definition $\underline{\mathfrak{B}}(\mathbb{S}) \cong \mathfrak{B}(k)$. Since $\varphi_1 : \underline{\mathfrak{B}}(\mathbb{S}) \to \underline{\mathfrak{B}}(\mathbb{K})$ is an order embedding $\varphi_1(\underline{\mathfrak{B}}(\mathbb{S}))$ is a Boolean suborder of dimension k in $\underline{\mathfrak{B}}(\mathbb{K})$. □

In general the images of $\varphi_1(\mathbb{S})$ and $\varphi_2(\mathbb{S})$ are neither lattices nor semilattices. However, we know from Proposition 5.10 that if $\mathbb{S}$ is a reduced Boolean subcontext and $\varphi_1(\mathfrak{B}(\mathbb{S})) = \varphi_2(\mathfrak{B}(\mathbb{S}))$ holds, there exists a Boolean sublattice $\underline{S}$ of the same dimension in $\underline{\mathfrak{B}}(\mathbb{K})$. We can generalize the previous statement as follows:

Proposition 5.15

Let $\mathbb{K}$ be a clarified context and $\mathbb{S}_1 = [H_1, N_1], \mathbb{S}_2 = [H_2, N_2] \in \mathcal{SRB}_k(\mathbb{K})$ with $\mathbb{S}_1 \neq \mathbb{S}_2$. If $H_1 \neq H_2$, then $\varphi_1(\mathbb{S}_1) \neq \varphi_1(\mathbb{S}_2)$ holds. If $N_1 \neq N_2$, then $\varphi_2(\mathbb{S}_1) \neq \varphi_2(\mathbb{S}_2)$ holds.

	a	b	c	d	e
1	×	×		×	
2	×		×	×	
3		×	×		
4	×	×		×	×
5				×	

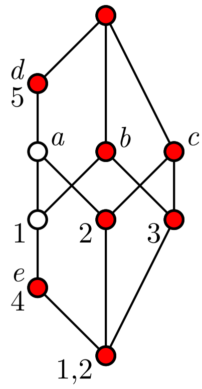


Figure 5.8 A formal context $\mathbb{K}$ with $|\mathcal{SRB}_3(\mathbb{K})| = |\mathcal{SOB}_3(\underline{\mathfrak{B}}(\mathbb{K}))| = 4$.

Proof Since $\mathbb{S}_1, \mathbb{S}_2 \in \mathcal{SRB}_k(\mathbb{K})$, $|H_1| = |H_2|$ holds. If $H_1 \neq H_2$ holds, $g_1 \in H_1$ with $g_1 \notin H_2$ and $g_2 \in H_2$ with $g_2 \notin H_1$ exist. Since $\mathbb{S}_1$ and $\mathbb{S}_2$ are reduced and Boolean there is a concept $c_1 = (g_1, g_1'') \in \mathfrak{B}(\mathbb{S}_1)$ and a concept $c_2 = (g_2, g_2'') \in \mathfrak{B}(\mathbb{S}_2)$. Hence $\mathbb{K}$ is clarified, $\varphi_1(c_1) = (g_1'', g_1') \neq (g_2'', g_2') = \varphi_1(c_2)$. If $N_1 \neq N_2$ holds, the analogous procedure can be executed using φ_2. □

Based on this, we can assume that the number of reduced Boolean subcontexts of a context $\mathbb{K}$ is a lower bound for the number of Boolean suborders of $\underline{\mathfrak{B}}(\mathbb{K})$:

Conjecture
Let $\mathbb{K}$ be a clarified context with $|\mathcal{SRB}_k(\mathbb{K})| = n$. Then $|\mathcal{SOB}_k(\underline{\mathfrak{B}}(\mathbb{K}))| \geq n$ holds.

This conjecture can not be proved as straight forward as Proposition 5.15 since φ_1 and φ_2 can be identical for some $\mathbb{S} \in \mathcal{SRB}_k(\mathbb{K})$. In addition not every Boolean suborder is the image of $\varphi_1(\mathbb{S})$ or $\varphi_2(\mathbb{S})$ for a $\mathbb{S} \in \mathcal{SRB}_k(\mathbb{K})$. Both phenomena occur in the example given in Figure 5.8, where the marked Boolean suborder is not the image of the embedding by φ_1 or φ_2 of any Boolean subcontext contained in the given formal context, although in this case the number of Boolean subcontexts of dimension 3 and Boolean suborders of dimension 3 is identical.

5.4.2 Subconcepts associated to Suborders

After investigating mappings of Boolean subcontexts to Boolean suborders, we now analyze the connection between those substructures the other way around. As presented by Albano and Chornomaz [2, Prop. 1] every formal context $\mathbb{K}$ contains a Boolean subcontext $\mathbb{S} \in \mathcal{SB}_k(\mathbb{K})$ if $\mathfrak{B}(\mathbb{K})$ contains a Boolean suborder $\underline{S} \in \mathcal{SOB}_k(\mathfrak{B}(\mathbb{K}))$. Based on this statement, we introduce a construction to generate a (not necessarily reduced) Boolean subcontext of a formal context based on a Boolean suborder of the corresponding concept lattice.

Definition 5.4 (Associated Subcontext)
Let $\mathbb{K}$ be a formal context and $\underline{S} \in \mathcal{SOB}_k(\mathfrak{B}(\mathbb{K}))$. We define $\psi(\underline{S}) \coloneqq [H, N]$ with $H \coloneqq \bigcup_{c \in At(\underline{S})} minG_{obj}(c)$ and $N \coloneqq \bigcup_{c \in CoAt(\underline{S})} minG_{att}(c)$ as the *subcontext of* $\mathbb{K}$ *associated to* $\underline{S}$.

Indeed the structure arising from the construction given in Definition 5.4 is a Boolean subcontext of the same dimension as $\underline{S}$:

Proposition 5.16
Let $\mathbb{K}$ be a formal context, $\underline{S} \in \mathcal{SOB}_k(\mathfrak{B}(\mathbb{K}))$, and $\mathbb{S} = [H, N] \coloneqq \psi(\underline{S})$ the subcontext of $\mathbb{K}$ associated to $\underline{S}$. Then $\mathbb{S} \in \mathcal{SB}_k(\mathbb{K})$.

Proof Let $At(\underline{S}) = \{a_1, a_2, \ldots, a_k\}$ and $CoAt(\underline{S}) = \{c_1, c_2, \ldots, c_k\}$ be the sets of atoms and coatoms of $\underline{S}$. Due to the Boolean structure of $\underline{S}$ the atoms can be ordered holding the following condition: a_i is a lower bound for the set $CoAt(\underline{S}) \setminus c_i$ for all $1 \leq i \leq k$, and analogous c_i is an upper bound for the set $At(\underline{S}) \setminus a_i$ for all $1 \leq i \leq k$. It follows gIm for all $g \in minG_{obj}(a_i)$, $m \in N \setminus minG_{att}(c_i)$ and $(g, m) \notin I$ else. Therefore $\mathbb{S} \cong \mathbb{N}^c(k)$ holds. □

Example 5.4
All Boolean suborders of dimension 3 from Figure 5.3 are listed in Table 5.2 together with their associated subcontexts in the formal context $\mathbb{K}$ from Figure 5.2.

In the following, we study the interplay of the mapping ψ from suborders to subcontexts with the mappings φ_1 and φ_2 from subcontexts to suborders.

Table 5.2 List of all Boolean suborders $\underline{S}$ presented in Figure 5.3 together with their associated subcontexts $\psi(\underline{S})$ in the formal context $\mathbb{K}$ from Figure 5.2.

Suborder $\underline{S}$	$\psi(\underline{S})$
No. 1	$[\{1,2,3,4\},\{a,b,c\}]$
No. 2	$[\{1,2,3,4\},\{a,b,c\}]$
No. 3	$[\{1,2,3\},\{a,b,c\}]$
No. 4	$[\{3,4,5,6\},\{b,c,d\}]$
No. 5	$[\{3,4,5,6\},\{b,c,d\}]$
No. 6	$[\{3,4,5,6\},\{b,c,e\}]$
No. 7	$[\{3,4,5,6\},\{b,c,e\}]$
No. 8	$[\{3,4,5,6\},\{b,c,d,e\}]$
No. 9	$[\{3,4,5,6\},\{b,c,d,e\}]$
No. 10	$[\{4,5,6\},\{b,c,d\}]$
No. 11	$[\{4,5,6\},\{b,c,e\}]$
No. 12	$[\{4,5,6\},\{b,c,d,e\}]$

Proposition 5.17
Let $\mathbb{K}$ be a formal context and $\mathbb{S} = [H, N] \in \mathcal{SRB}_k(\mathbb{K})$. Then $\mathbb{S} = \psi(\varphi_1(\mathbb{S}))$ if and only if $(n', n'') \in CoAt(\varphi_1(\mathbb{S}))$ holds for all $n \in N$. Dually, $\mathbb{S} = \psi(\varphi_2(\mathbb{S}))$ if and only if $(h'', h') \in At(\varphi_2(\mathbb{S}))$ holds for all $h \in H$.

Proof Consider φ_1: Let $H = \{h_1, h_2, \ldots, h_k\}$, $N = \{n_1, n_2, \ldots, n_k\}$, and $\psi(\varphi_1(\mathbb{S})) = [\widetilde{H}, \widetilde{N}]$. Due to the construction of φ_1, we have $At(\varphi_1(\mathbb{S})) = \{a_1, a_2, \ldots, a_k\}$ with $a_i = (h_i'', h_i')$. Since every h_i is a minimal object generator of an atom of $\varphi_1(\mathbb{S})$, $\widetilde{H} = H$ holds. Let $CoAt(\varphi_1(\mathbb{S})) = \{c_1, c_2, \ldots, c_k\}$. $\widetilde{N}$ consists of the minimal attribute generators of the coatoms of $\varphi_1(\mathbb{S})$. Following, $\widetilde{N} = N$ if and only if a renumbering of the coatoms exists so that $c_i = (n_i', n_i'')$ for all $i \in \{1, 2, \ldots, k\}$. The procedure for φ_2 is analogous. □

In Example 5.5 all possibilities for the interplay of the maps φ_1, φ_2 and ψ are presented. In particular, a Boolean subcontext $\mathbb{S}$ can be the associated subcontext to one of the embeddings $\varphi_1(\mathbb{S})$ or $\varphi_2(\mathbb{S})$, to both embeddings or to none of the embeddings.

Example 5.5
The reduced Boolean subcontexts of dimension 3 of the formal context in Figure 5.8 are $\mathbb{S}_1 = [\{1,2,3\}, \{a,b,c\}]$, $\mathbb{S}_2 = [\{2,3,4\}, \{a,b,c\}]$, $\mathbb{S}_3 = [\{1,2,3\}, \{b,c,d\}]$ and $\mathbb{S}_4 = [\{2,3,4\}, \{b,c,d\}]$). We have $\mathbb{S}_1 = \psi(\varphi_1(\mathbb{S}_1)) = \psi(\varphi_2(\mathbb{S}_1))$, $\mathbb{S}_2 = \psi(\varphi_2(\mathbb{S}_2))$ and $\mathbb{S}_3 = \psi(\varphi_1(\mathbb{S}_3))$. For $\mathbb{S}_4$ we have $\mathbb{S}_4 \neq \psi(\varphi_1(\mathbb{S}_4))$ and $\mathbb{S}_4 \neq \psi(\varphi_2(\mathbb{S}_4))$.

Proposition 5.18
Let $\mathbb{K}$ be a formal context, $\underline{S} \in \mathcal{SOB}_k(\mathfrak{B}(\mathbb{K}))$, and $\mathbb{S} := \psi(\underline{S})$. Let $c \in \underline{S} \setminus \{0_{\underline{S}}, 1_{\underline{S}}\}$ with either c not being the supremum (in $\mathfrak{B}(\mathbb{K})$) of a subset of $At(\underline{S})$ or c not being the infimum (in $\mathfrak{B}(\mathbb{K})$) of a subset of $CoAt(\underline{S})$. Then (A, B) with $A = \bigcup\{minG_{obj}(X) \mid X \in At(\underline{S}), X \leq c\}$ and $B = \bigcup\{minG_{att}(X) \mid X \in CoAt(\underline{S}), X \geq c\}$ is a concept of $\mathbb{S}$ with $\varphi_1(A, B) \neq \varphi_2(A, B)$.

Proof According to the construction of $\mathbb{S}$, there is a concept $(A, B) \in \mathfrak{B}(\mathbb{S})$ as stated. If c is not the supremum of a subset of $At(\underline{S})$, A does not generate c. Therefore $\varphi_1(A, B) = (A'', A') < c$, due to the construction of A. Also $\varphi_2(A, B) = (B', B'') \geq c$ and consequently $\varphi_1(A, B) < \varphi_2(A, B)$. Similarly, if c is not the infimum of a subset of $CoAt(\underline{S})$, we have $\varphi_1(A, B) = (A'', A') \leq c$, $\varphi_2(A, B) = (B', B'') > c$ and $\varphi_1(A, B) < \varphi_2(A, B)$. □

Utilizing this interplay of the mappings φ_1, φ_2 and ψ, we are able to generate a related subsemilattice for every Boolean suborder in a concept lattice as presented in Proposition 5.19. This approach is formalized in Definition 5.5.

Proposition 5.19
Let $\mathbb{K}$ be a formal context, $\underline{S} \in \mathcal{SOB}(\underline{\mathfrak{B}}(\mathbb{K}))$. Then $\varphi_1(\psi(\underline{S}))$ is a sub-$\vee$-semilattice and $\varphi_2(\psi(\underline{S}))$ is a sub-$\wedge$-semilattice of $\underline{\mathfrak{B}}(\mathbb{K})$.

Proof Let $\mathbb{S} = [H, N] \coloneqq \psi(\underline{S})$. H is the set of all minimal generators of the atoms of $\underline{S}$. Due to the Boolean structure, all concepts in $\mathbb{K}$ that are generated by a subset of H are exactly the supremum of a subset of $At(\mathbb{S})$. Since this generation corresponds to mapping the concepts $c \in \mathfrak{B}(\mathbb{S})$ with φ_1, $\varphi_1(\mathbb{S})$ is a sub-$\vee$-semilattice. The second part of the statement is proved similarly. □

Definition 5.5 (Associated Subsemilattice)
Let $\mathbb{K}$ be a formal context, $\underline{S} \in \mathcal{SOB}_k(\underline{\mathfrak{B}}(\mathbb{K}))$. We call $\varphi_1(\psi(\underline{S}))$ the *sub-$\vee$-sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$ associated to $\underline{S}$* and $\varphi_2(\psi(\underline{S}))$ the *sub-$\wedge$-sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$ associated to $\underline{S}$*.

The statement in Proposition 5.19 holds especcially for a $\underline{S}$ being a Boolean subsemilattice or a Boolean sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$. In this case, it provides $\varphi_1(\psi(\underline{S})) = \underline{S}$ and $\varphi_2(\psi(\underline{S})) = \underline{S}$, respectively, as follows:

Proposition 5.20
Let $\mathbb{K}$ be a formal context and $\underline{S} \in \mathcal{SOB}_k(\underline{\mathfrak{B}}(\mathbb{K}))$. If $\underline{S}$ is a sub-$\vee$-semilattice, $\varphi_1(\psi(\underline{S})) = \underline{S}$ holds. If $\underline{S}$ is a sub-$\wedge$-semilattice, $\varphi_2(\psi(\underline{S})) = \underline{S}$ holds.

Proof Let $\underline{S}$ be a sub-$\vee$-semilattice and $\mathbb{S} = [H, N] \coloneqq \psi(\underline{S})$. H is the set of minimal generators of the atoms of $\underline{S}$. Due to the Boolean structure all concepts in $\underline{\mathfrak{B}}(\mathbb{K})$ that are generated by a subset of H are exactly the supremums of a subset of the atoms of $\underline{S}$. Since this generation corresponds to mapping the concepts $c \in \mathfrak{B}(\mathbb{S})$ with φ_1, every image of $\varphi_1(c)$ is contained in $\underline{S}$. The second part follows dually. □

Proposition 5.21
Let $\mathbb{K}$ be a formal context and $\underline{S} \in \mathcal{SLB}_k(\underline{\mathfrak{B}}(\mathbb{K}))$. Then $\varphi_1(\psi(\underline{S})) = \varphi_2(\psi(\underline{S})) = \underline{S}$.

Our research can be concluded in the following theorems. They give an insight into the interplay of φ_1, φ_2 and ψ and the structural properties they transfer.

Theorem 5.2
Let $\mathbb{K}$ be a formal context and $\mathbb{S} \in \mathcal{SB}(\mathbb{K})$. Then:

i) $\psi(\varphi_1(\mathbb{S})) = \mathbb{S}$ *iff a sub-$\vee$-semilattice $\underline{S} \in \mathcal{SOB}(\underline{\mathfrak{B}}(\mathbb{K}))$ exists with $\psi(\underline{S}) = \mathbb{S}$.*

ii) $\psi(\varphi_2(\mathbb{S})) = \mathbb{S}$ *iff a sub-$\wedge$-semilattice $\underline{S} \in \mathcal{SOB}(\underline{\mathfrak{B}}(\mathbb{K}))$ exists with $\psi(\underline{S}) = \mathbb{S}$.*

iii) $\psi(\varphi_1(\mathbb{S})) = \psi(\varphi_2(\mathbb{S})) = \mathbb{S}$ *iff a $\underline{S} \in \mathcal{SLB}(\underline{\mathfrak{B}}(\mathbb{K}))$ exists with $\psi(\underline{S}) = \mathbb{S}$.*

Furthermore, if $\mathbb{S}$ is reduced, $\varphi_1(\mathbb{S}) = \varphi_1(\psi(\varphi_1(\mathbb{S})))$ and $\varphi_2(\mathbb{S}) = \varphi_2(\psi(\varphi_2(\mathbb{S})))$ hold.

	a	b	c	d	e	f
1	×	×				
2	×		×			
3		×	×	×		
4		×	×		×	
5		×	×			×
6		×	×	×	×	×

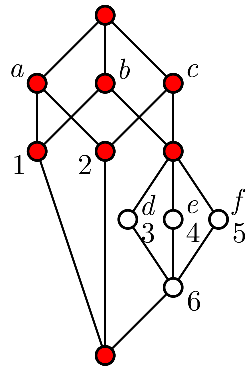


Figure 5.9 A formal context (left) and its corresponding concept lattice (right) that show that neither φ_1 and ψ nor φ_2 and ψ are (dually) adjoint mappings.

Proof Consider i): "⇒:" follows directly from Proposition 5.19 since $\mathbb{S}$ is the subcontext corresponding to the suborder $\varphi_1(\mathbb{S})$.
"⇐:" is presented in Proposition 5.20.
ii) is proved similarly and iii) follows from the combination of i) and ii). The last statement follows from the combination of Proposition 5.10 and Proposition 5.18. □

Theorem 5.3
Let $\mathbb{K}$ be a formal context and $\underline{S} \in \mathcal{SOB}(\underline{\mathfrak{B}}(\mathbb{K}))$. Then:

i) $\varphi_1(\psi(\underline{S})) = \underline{S}$ *iff* $\underline{S}$ *is a sub-*$\vee$*-semilattice.*

ii) $\varphi_2(\psi(\underline{S})) = \underline{S}$ *iff* $\underline{S}$ *is a sub-*$\wedge$*-semilattice.*

iii) $\varphi_1(\psi(\underline{S})) = \varphi_2(\psi(\underline{S})) = \underline{S}$ *iff* $\underline{S}$ *is a sublattice.*

Proof Consider i): "⇒:" follows directly from Proposition 5.19.
"⇐" is presented in Proposition 5.20.
ii) is proved similarly, iii) follows from combining i) and ii). □

Although φ_1 and ψ (or φ_2 and ψ) seem to be (dually) adjoint mappings, they are not. E.g., in Figure 5.9 consider the subcontexts $\mathbb{S}_1 = [\{1,2,3,4\},\{a,b,c\}]$, $\mathbb{S} = [\{1,2,3,4,5\},\{a,b,c\}]$, and $\mathbb{S}_2 = [\{1,2,3,4,5,6\},\{a,b,c\}]$. Then it holds that $\varphi_1(\mathbb{S}_1) = \varphi_1(\mathbb{S}_2) = \varphi_1(\mathbb{S}) = \varphi_2(\mathbb{S}_2) = \varphi_2(\mathbb{S}_1)$ – the image is highlighted in the line diagram, and its associated context is $\mathbb{S}$. This shows that $\psi \circ \varphi_1$ is neither monotonic nor anti-monotonic, and the same holds for $\psi \circ \varphi_2$.

5.5 Interplay of both Approaches

In the previous sections, two approaches to relate Boolean substructures of a formal context $\mathbb{K}$ with those of the corresponding concept lattice $\mathfrak{B}(\mathbb{K})$ were introduced. In this section, we set both of them in relation.

In Section 5.3 a one-to-one correspondence between the closed-subcontexts of a formal context $\mathbb{K}$ and the sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$ is presented. However, subsemilattices and suborders are not addressed. In addition, the closed-subcontexts restrict not only the object set and the attribute set of a formal context but also its incidence relation, whereby they could be understood as a more substantial altering of $\mathbb{K}$ compared to the approach presented in Section 5.4. It provides different maps to associate specific Boolean suborders on the one side with Boolean subcontexts on the other side while transferring some structural information.

The intersection of both approaches is localized in the Boolean subcontexts that are closed-subcontexts as well, and in general, the subcontexts $\mathbb{S} \leq \mathbb{K}$ with $c \in \mathfrak{B}(\mathbb{K})$ for all $c \in \mathfrak{B}(\mathbb{S})$.

Proposition 5.22
Let $\mathbb{K}$ be a formal context and $\mathbb{S} \leq \mathbb{K}$. $\mathbb{S}$ is a closed-subcontext of $\mathbb{K}$ if and only if $\varphi_1(c) = \varphi_2(c) = c$ for all $c \in \mathfrak{B}(\mathbb{S})$.

This statement can be restricted to Boolean subcontexts. E.g., the Boolean subcontext $\mathbb{S} = [G, \{a, b, c\}]$ in Figure 5.2 fulfills the requirement. In general, the set of the Boolean subcontexts of $\mathbb{K}$ that are closed-subcontexts is smaller than the set of all Boolean sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$. So not every Boolean sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$ can be reached by an embedding of a subcontext of such a structure. Refering to those structures we expand the statement of Proposition 5.14 as follows:

Proposition 5.23
Let $\mathbb{K}$ be a formal context and $\mathbb{S} \in \mathcal{SB}_k(\mathbb{K})$ with $\mathbb{S}$ a closed-subcontext of $\mathbb{K}$. Then $\underline{S} \coloneqq \varphi_1(\mathbb{S}) = \varphi_2(\mathbb{S}) \in \mathcal{SLB}_k(\underline{\mathfrak{B}}(\mathbb{K}))$.

However, in general the subcontext $\widetilde{\mathbb{S}}$ associated to $\underline{S}$ is not equal to $\mathbb{S}$. E.g. in Figure 5.2 the subcontext $\mathbb{S} = [G, \{a, b, c\}]$ is embedded to a Boolean sublattice $\underline{S}$ but the sublattice, that is associated to $\underline{S}$ is $\widetilde{\mathbb{S}} = [\{1, 2, 3, 4\}, \{a, b, c\}]$.

5.6 Conclusion

This chapter relates Boolean substructures in a formal context $\mathbb{K}$ with those in its concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$. The notion of closed-subcontexts of $\mathbb{K}$ was presented to generalize closed relations and provide a one-to-one correspondence to the set of all sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$ using a direct construction. In particular, this relationship can be restricted to the set of all Boolean closed-subcontexts of $\mathbb{K}$, that can be generated based on the set of all reduced Boolean subcontexts of $\mathbb{K}$, and all Boolean sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$. Moreover, we investigated two embeddings of Boolean subcontexts of $\mathbb{K}$ into $\underline{\mathfrak{B}}(\mathbb{K})$. The images of those embeddings are, in general, not sub(semi)lattices but only Boolean suborders and do not cover $\mathcal{SOB}(\mathbb{K})$ completely. Through the introduction of the subcontext $\mathbb{S}$ associated to a Boolean suborder $\underline{S}$ of $\underline{\mathfrak{B}}(\mathbb{K})$, the connection between Boolean subcontexts and Boolean suborders is investigated the other way around. The combination of both approaches give an insight of their interplay and the structural information they transfer. Through this every subsemilattice $\underline{S}$ can be associated with a concrete subcontext, that can be mapped to $\underline{S}$ by one of the two embeddings.

We conclude this chapter with two open questions. First, we are curious to which amount the presented findings can be transferred to general substructures of (not necessarily finite) formal contexts and their corresponding concept lattices. Secondly, we are interested in consideration of other special substructures, e.g., the subcontexts of a concept lattice isomorphic to a nominal scale, as those scales also contain nearly identical objects that differ only in one attribute.

Part III

Selecting Attributes

CHAPTER 6

Relevant Attributes in Formal Contexts

In the age of massive data sets, understanding the data is a challenging task that is addressed by various approaches, e.g., random sampling, parallelization, or attribute extraction. A so far not investigated method in the realm of Formal Concept Analysis is attribute selection, as done in machine learning. Building up on this, in this chapter, we turn away from Boolean subcontexts and suborders and introduce a method for attribute selection in formal contexts whereby we generate a sub-$\wedge$-semilattice and avoid the generation of false implications (see Section 7.3). To this end, we propose the notion of relevant attributes, which enables us to define a relative relevance function, reflecting both the order structure of the concept lattice as well as the distribution of objects on it. Finally, we overcome computational challenges for computing the relative relevance through an approximation approach based on information entropy.

6.1 Introduction

Contemporary formal contexts consist of thousands of objects and attributes, which results in even larger concept lattices. To overcome the lack of clarity for the human observer one is required to select a subcontext resembling the original data set most accurately. This can be done by selecting attributes or objects. In this chapter, we will focus on the identification of relevant attributes. This is due to the duality of formal contexts, similar to the problem of selecting relevant objects. There are several comparable works related to FCA, e.g., to concept sampling [11] and concept selection [39], that, however, either need to compute the whole concept lattice or sample from it, which is at best possible with polynomial delay [35].

In this chapter, we overcome this limitation and present a feasible approach for selecting relevant attributes from a formal context using information entropy. To this end, we introduce the notion of *attribute relevance* to the realm of FCA, based on a seminal work by Blum and Langley [10]. In that work, the authors address a comprehensible theory for selecting the most relevant features in supervised machine learning settings. Building up on this, we formalize a *relative relevance* measure in formal contexts in order to identify the most relevant attributes. However, this measure is still prone to the limitation of computing the concept lattice. Finally, we tackle this disadvantage by approximating the relative relevance measure through an information entropy approach. Choosing attributes based on this approximation leads to significantly more relevant selections than random sampling does, which we demonstrate in an empirical experiment.

6.2 Relevant Attributes

A severe computational problem in FCA is to compute the set of all formal concepts, which resembles the CLIQUE problem [35]. Furthermore, the number of formal concepts in a proper sized real-world data set tends to be very large, e.g., 238710 in the (small) mushroom data set, see Section 6.3.1. Hence, concept lattices for contemporary-sized data sets are hard to grasp and hard to cope with through consecutive measures and metrics. Thus, a need for selecting subcontexts from data sets or sublattices is self-evident. This selection can be conducted in the formal context as well as in the concept lattice. However, the computational feasible choice is to do this in the formal context. Considering a subcontext of a formal context (G, M, I) can be done in general in three different ways: One may consider only a subset $H \subseteq G$, a subset $N \subseteq M$, or a combination of those. Our goal for the rest of this work is to identify relevant attributes in a formal context. The notion of (attribute) relevance shall cover two aspects: the lattice structure and the distribution of objects on it. The task at hand is to choose the most relevant attributes which do both reflect a large part of the lattice structure as well as the distribution of the objects on the concepts. For this, in the next section, we introduce a notion of *relevant attributes* in a formal context. Due to the duality in FCA, this can easily be translated to object relevance.

6.2.1 Choosing Attributes

There is a plenitude of conceptions for describing the relevance of an attribute in a data set. Apparently, the relevance should depend on the particular machine learning or knowledge discovery procedure. One very influential work in this direction was done by Blum and Langley in [10], where the authors defined the *(weak/strong) relevance* of an attribute in the realm of labeled data. In particular, for some data set of examples D, described using features from some feature set F, where every $d \in D$ has the label (distribution) $\ell(d)$, the authors stated: A feature $x \in F$ is *relevant* to a target concept-label if there exists a pair of examples $a, b \in D$ such that a and b only differ in their assignment of x and $\ell(a) \neq \ell(b)$. They further expanded their notion by calling some attribute x *weakly relevant* if and only if it is possible to remove a subset of the features (from a and b) such that x becomes relevant.

Since data is commonly unlabeled in the realm of Formal Concept Analysis, we may not directly adapt the above notion to formal contexts. However, we may motivate the following approach with it. We cope with the lack of a label function in the following way. First, we identify the data set D with a formal context (G, M, I), where the elements of G are the examples and M are the features describing the examples. Secondly, a concept lattice exhibits essentially two almost independent properties, the order structure and the distribution of objects (attributes) on it, cf. Example 6.1. Thus, a *conceptual label function* then shall reflect both the order structure as well as the distribution of objects in this structure. To achieve this, we propose the following:

Definition 6.1 (Extent Label Function)
Let $\mathbb{K} = (G, M, I)$ be a formal context with its concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$. The map

$$\ell_{\mathbb{K}} : G \to \mathbb{N}, g \mapsto |\{c \in \mathfrak{B}(\mathbb{K}) \mid g \in \operatorname{ext}(c)\}|$$

is called *extent label function*.

	a	b	c	d	e	f	g	h	i
Leach	×	×					×		
Bream	×	×					×	×	
Frog	×	×	×				×	×	
Dog	×		×				×	×	×
Spike-weed	×	×		×		×			
Bean	×		×	×	×				

	a	b	c	d
Bream	×	×		
Frog	×	×	×	
Dog	×		×	
Spike-weed	×	×		×

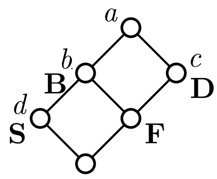


Figure 6.1 Two subcontexts of "Living Beings and Water" [29]. The attributes are: a: needs water to live, b: lives in water, c: lives on land, d: needs chlorophyll to produce food, e: two seed leaves, f: one seed leaf, g: can move around, h: has limbs, i: suckles its offspring. The concept lattice corresponds to the formal context at the bottom.

One may define an *intent label function* analogously. Utilizing the just introduced function, we may now define the notion of relevant attributes in formal contexts.

Definition 6.2 (Relevance)
Let $\mathbb{K} = (G, M, I)$ be a formal context. We call an attribute $m \in M$ *relevant to* $g \in G$ if and only if $\ell_{\mathbb{K}_{\{m\}}}(g) < \ell_{\mathbb{K}}(g)$, where $\mathbb{K}_{\{m\}} \coloneqq [G, M \setminus \{m\}]$. Furthermore, m is *relevant to a subset* $A \subseteq G$ if and only if there is a $g \in A$ such that m is relevant to g. And, we say m is *relevant to the context* $\mathbb{K}$ if and only if m is relevant to G.

For a better understanding of the relevance of an attribute, we introduce the formal context $\mathbb{K}$ illustrated in Figure 6.1 (bottom) that will be revisited during this chapter.

Example 6.1
Figure 6.1 (bottom) shows a formal context $\mathbb{K}$ and its corresponding concept lattice. The objects from there are abbreviated by their first letter in the following. The extent label function of the objects can easily be read from the lattice and is given by $\ell_{\mathbb{K}}(B) = 2$, $\ell_{\mathbb{K}}(F) = 4$, $\ell_{\mathbb{K}}(D) = 2$, $\ell_{\mathbb{K}}(S) = 3$. Additionally, one can deduct the relevant attributes. E.g., for attribute b the equality $\ell_{\mathbb{K}_{\{b\}}}(D) = \ell_{\mathbb{K}}(D)$ holds. In contrast $\ell_{\mathbb{K}_{\{b\}}}(S) < \ell_{\mathbb{K}}(S)$, cf. Figure 6.2. Hence, attribute b is not relevant to “Dog” but relevant to “Spike-weed”. Thus, b is relevant to $\mathbb{K}$.

The two structural approaches in FCA to identify admissible attributes are attribute clarifying and reducibility. Since they are based purely on the lattice structure, the notion of relevant attributes is directly related to reducibility as follows:

Proposition 6.1 (Irreducible)
Let $\mathbb{K} = (G, M, I)$ be a formal context. For an attribute $m \in M$ holds

$$m \text{ is relevant to } \mathbb{K} \Longleftrightarrow m \text{ is irreducible.}$$

Proof "$\Rightarrow$": We use a contraposition. We have to show that the following inequality holds: $|\{c \in \mathfrak{B}(\mathbb{K}) \mid g \in \text{ext}(c)\}| \leq |\{c \in \mathfrak{B}(\mathbb{K}_{\{m\}}) \mid g \in \text{ext}(c)\}|$ (assumed the attribute m to be reducible). Since $g \in \text{ext}(c)$ holds and for any $c \in \mathfrak{B}(\mathbb{K})$ exists a unique concept $\widetilde{c} \in \mathfrak{B}(\mathbb{K}_{\{m\}})$ with $\text{int}(\widetilde{c}) \cup \{m\} = \text{int}(c)$, cf. Proposition 3.7, it follows that $g \in (\text{int}(\widetilde{c}) \cup \{m\})' \subseteq \text{int}(\widetilde{c})'$. We omitted the trivial case when $\text{int}(\widetilde{c}) = \text{int}(c)$.

"$\Leftarrow$": There is a join-preserving order embedding $[G, M \setminus m] \to (G, M, I)$ with $(A, B) \mapsto (A, A')$ (Proposition 3.6). Hence, every extent in $\mathfrak{B}(\mathbb{K}_{\{m\}})$ is also an extent in $\mathfrak{B}(\mathbb{K})$ which implies for all $g \in G$ that $\ell_{\mathbb{K}_{\{m\}}}(g) \leq \ell_{\mathbb{K}}(g)$ holds. Since m is irreducible, there exist less concepts in $\mathfrak{B}(\mathbb{K}_{\{m\}})$ than in $\mathfrak{B}(\mathbb{K})$ so that $\ell_{\mathbb{K}_{\{m\}}}(g) < \ell_{\mathbb{K}}(g)$. □

The last proposition implies that no clarifiable attributes would be considered relevant, even if the removal of all attributes that have identical closure would have a massive impact on the structure of the concept lattice. Therefore a meaningful identification of relevant attributes is equivalent to the identification of meaningful equivalence classes $[x]_{\mathbb{K}} \coloneqq \{y \in M \mid x' = y'\}$ for all $y \in M$. Accordingly, we consider in the following only clarified contexts. Transferring the relevance of an attribute $m \in M$ to its equivalence class is an easy task which can be executed if necessary.

So far, we are only able to decide the relevance of an attribute but not discriminate attributes with respect to their relevance to the concept lattice. To overcome this limitation, we introduce in the following a measure that is able to compare the relevancy of two given attributes in a clarified formal context. We consider the change in the object label distribution $\{(g, \ell_{\mathbb{K}}(g)) \mid g \in G\}$ going from $\mathbb{K}$ to $\mathbb{K}_{\{m\}}$ as characteristic to the relevance of a relevant attribute m. To examine this characteristic in more detail and to make it graspable via a numeric value, we propose the following inequality:

$$\sum_{g \in G} \ell_{\mathbb{K}_{\{m\}}}(g) < \sum_{g \in G} \ell_{\mathbb{K}}(g).$$

This approach does not only offer the possibility to verify the existence of a change in the object label distribution but also to measure the extent of this change. We may quantify this via

$$\frac{\sum_{g \in G} \ell_{\mathbb{K}_{\{m\}}}(g)}{\sum_{g \in G} \ell_{\mathbb{K}}(g)} \eqqcolon t(m)$$

whence $t(m) < 1$ for all attributes $m \in M$.

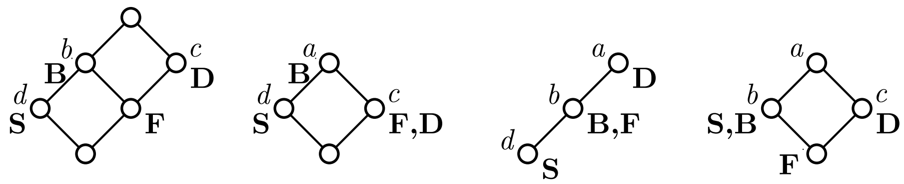


Figure 6.2 Sublattices created through the removal of an attribute from the lattice in Figure 6.1. From left to right: removing a,b,c, or d.

This results in the following relevance measure for attributes:

Definition 6.3 (Relative Relevance)
Let $\mathbb{K} = (G, M, I)$ be a clarified formal context. The attribute $m \in M$ is *relative relevant* to $\mathbb{K}$ with

$$r(m) \coloneqq 1 - \frac{\sum_{g \in G} |\{c \in \mathfrak{B}(\mathbb{K}_{\{m\}}) \mid g \in \operatorname{ext}(c)\}|}{\sum_{g \in G} |\{c \in \mathfrak{B}(\mathbb{K}) \mid g \in \operatorname{ext}(c)\}|} = 1 - t(m).$$

The values of $r(m)$ for an attribute $m \in M$ are in $[0, 1)$. We say $m \in M$ is *more relevant* to the context $\mathbb{K}$ than $n \in M$ if and only if $r(n) < r(m)$. Double counting leads to the following proposition.

Proposition 6.2
Let $\mathbb{K} = (G, M, I)$ be a formal context. For all $m \in M$ holds

$$r(m) = 1 - \frac{\sum_{c \in \mathfrak{B}(\mathbb{K}_{\{m\}})} |ext(c)|}{\sum_{c \in \mathfrak{B}} |ext(c)|}$$

with $\mathfrak{B}(\mathbb{K}_{\{m\}}) = \{c \in \mathfrak{B} | (int(c) \setminus \{m\})' = ext(c)\}$.

This statement reveals an interesting property of the just defined relative relevance. In fact, an attribute $m \in M$ is more relevant to a formal context $\mathbb{K}$ if the sub-$\wedge$-semilattice, which one does obtain by removing m from $\mathbb{K}$, does exhibit a smaller sum of all extent sizes. This will enable us to find proper approximations to the relative relevance in Section 6.2.2.

Example 6.2
Excluding one attribute from the running example in Figure 6.1 (bottom) results in the sublattices in Figure 6.2. The relative relevance of the attributes to the original context is given by $r(a) = 0$, $r(b) = 4/11$, $r(c) = 3/11$, and $r(d) = 1/11$.

By means of $r(\cdot)$ it is also possible to measure the relative relevance of a set $N \subseteq M$. We simply lift Proposition 6.2 by

$$r(N) = 1 - \frac{\sum_{c \in \mathfrak{B}(\mathbb{K}_N)} |\mathrm{ext}(c)|}{\sum_{c \in \mathfrak{B}(\mathbb{K})} |\mathrm{ext}(c)|}$$

with $\mathfrak{B}(\mathbb{K}_N) = \{c \in \mathfrak{B}(K) \mid (\mathrm{int}(c) \setminus N)' = \mathrm{ext}(c)\}$.

For the relevance of two attribute subsets the following statements hold:

Proposition 6.3
Let $\mathbb{K} = (G, M, I)$ be a formal context and $N, O \subseteq M$ attribute sets. Then

i) $N \subseteq O \Rightarrow r(N) \leq r(O)$, *and*

ii) $r(N \cup O) \leq r(O) + r(N)$.

Proof We prove i) by showing $\sum_{c \in \mathfrak{B}(\mathbb{K}_N)} |\mathrm{ext}(c)| > \sum_{c \in \mathfrak{B}(\mathbb{K}_O)} |\mathrm{ext}(c)|$. Since for all $c \in \mathfrak{B}(\mathbb{K})$ we have $(\mathrm{int}(c) \setminus O)' \supseteq (\mathrm{int}(c) \setminus N)' \supseteq \mathrm{ext}(c)$ we obtain $\mathfrak{B}(\mathbb{K}_N) \supseteq \mathfrak{B}(\mathbb{K}_O)$, as required.

For ii) we will use the identity $(\star)$: $\mathfrak{B}(\mathbb{K}_N) \cap \mathfrak{B}(\mathbb{K}_O) = \mathfrak{B}(\mathbb{K}_{N \cup O})$, which follows from $(\mathrm{int}(c) \setminus N)' = \mathrm{ext}(c) \wedge (\mathrm{int}(c) \setminus O)' = \mathrm{ext}(c) \Leftrightarrow (\mathrm{int}(c) \setminus (N \cup O))' = \mathrm{ext}(c)$ for all $c \in \mathfrak{B}(\mathbb{K})$. This equivalence is true since ("$\Rightarrow$"):

$$\begin{aligned}(\mathrm{int}(c) \setminus (N \cup O))' &= ((\mathrm{int}(c) \setminus N) \cap (\mathrm{int}(c) \setminus O))' \\ &= (\mathrm{int}(c) \setminus N)' \cup (\mathrm{int}(c) \setminus O)' = \mathrm{ext}(c) \cup \mathrm{ext}(c) = \mathrm{ext}(c)\end{aligned}$$

("$\Leftarrow$"): From $(\mathrm{int}(c) \setminus (N \cup O))' \supseteq (\mathrm{int}(c) \setminus N)'$ and $(\mathrm{int}(c) \setminus (N \cup O))' \supseteq (\mathrm{int}(c) \setminus O)'$ we obtain with i) that $(\mathrm{int}(c) \setminus N)' = (\mathrm{int}(c) \setminus O)' = \mathrm{ext}(c)$. We now show ii) by proving the inequality $\sum_{\mathfrak{B}(\mathbb{K}_N)} |\mathrm{ext}(c)| + \sum_{\mathfrak{B}(\mathbb{K}_O)} |\mathrm{ext}(c)| \leq \sum_{\mathfrak{B}(\mathbb{K})} |\mathrm{ext}(c)| + \sum_{\mathfrak{B}(\mathbb{K}_{N \cup O})} |\mathrm{ext}(c)|$. In the following, we use the equations $\mathfrak{B}(\mathbb{K}_N) \setminus \mathfrak{B}(\mathbb{K}_{N \cup O}) \cup \mathfrak{B}(\mathbb{K}_{N \cup O}) = \mathfrak{B}(\mathbb{K}_N)$ and $\mathfrak{B}(\mathbb{K}_N) \setminus \mathfrak{B}(\mathbb{K}_{N \cup O}) \cap \mathfrak{B}(\mathbb{K}_{N \cup O}) = \varnothing$ to find an equivalent equation employing $(\star)$:

$$\begin{aligned}\sum_{\mathfrak{B}_N \setminus \mathfrak{B}_{N \cup O}} |\mathrm{ext}(c)| + \sum_{\mathfrak{B}_O \setminus \mathfrak{B}_{N \cup O}} |\mathrm{ext}(c)| + 2 \cdot \sum_{\mathfrak{B}_{N \cup O}} |\mathrm{ext}(c)| \quad &\leq \quad \sum_{\mathfrak{B}_N \setminus \mathfrak{B}_{N \cup O}} |\mathrm{ext}(c)| + \sum_{\mathfrak{B}_O \setminus \mathfrak{B}_{N \cup O}} |\mathrm{ext}(c)| + \\ &\quad\quad \sum_{\mathfrak{B} \setminus (\mathfrak{B}_N \cup \mathfrak{B}_O)} |\mathrm{ext}(c)| + 2 \cdot \sum_{\mathfrak{B}_{N \cup O}} |\mathrm{ext}(c)| \\ 0 \quad &\leq \quad \sum_{\mathfrak{B} \setminus (\mathfrak{B}_N \cup \mathfrak{B}_O)} |\mathrm{ext}(c)|\end{aligned}$$

where $\mathfrak{B}_X$ is short for $\mathfrak{B}(\mathbb{K}_X)$. □

Equipped with the notion of relative relevance and some basic observations, we are ready to state the associated computational problem. We imagine that in real-world applications attribute selection is a task to identify a set $N \subsetneq M$ of the most relevant attributes for a given cardinality $n \in \mathbb{N}$. , i.e., an element from $\{N \subseteq M \mid |N| = n \wedge r(N) \text{ maximal}\}$. We call such a set N a *maximal relevant set.*

Problem 6.1 (Relative Relevance Problem (RRP))
Let $\mathbb{K} = (G, M, I)$ be a formal context and $n \in \mathbb{N}$ with $n < |M|$. Find a subset $N \subseteq M$ with $|N| = n$ such that $r(N) \geq r(X)$ for all $X \subseteq M$ where $|X| = n$.

Aiming to solve Problem 6.1 in a straightforward manner evolves two difficulties. First, as n increases, so does the number of possible subset combinations. The determination of a maximal relevant set requires the computation and comparison of $\binom{|M|}{|N|}$ different relative relevances, which presents itself as infeasible. Secondly, the computation of the relative relevance does presume that the set of formal concepts is computed. This also states an intractable problem for large formal contexts, which are the focus of applications of the proposed relevance selection method. To overcome the first limitation, we suggest an iterative approach. Instead of testings every subset of size n, we construct $N \subseteq M$ by first considering all singleton sets $\{m\} \subseteq M$. Consecutively, in every step i where X is the so far constructed set we find $x \in M$ such that $r(X \cup \{x\}) \geq r(X \cup \{m\})$ for all $m \in M$. This approach requires the computation of only $\sum_{i=|M|-|n|+1}^{|M|} i$ different relative relevances and their comparisons, which is simplified $n \cdot |M| - (n-1) \cdot n/2$. We call a set obtained through this approach an *iterative maximal relevant set* IMRS. In fact, the IMRS does not always correspond to the maximal relevant set. E.g., consider a formal context (G, M, I) with $G = \{1, 2, 3, 4\}$, $M = \{a, b, c, d\}$ and $I = \{(1, a), (1, c), (1, d), (2, a), (2, b), (3, b), (3, c), (4, d)\}$. Then b is the most relevant attribute, i.e., $r(b) > r(x)$ for all $x \in M \setminus \{b\}$. However, we find $r(\{a, c\}) > r(\{b, x\})$ for all $x \in M \setminus \{b\}$. Hence, the relative relevance of an IMRS indicates a lower bound for the relative relevance of the maximal relevant set.

6.2.2 Approximating RRP

Motivated by the computational infeasibility of Problem 6.1 we investigate in this section the possibility of approximating RRP, more specifically the IMRS. Approaches for this approximation have to incorporate both aspects of the relative relevance: the structure of the concept lattice and the distribution of the objects. Considering the former is not complicated since for any context (G, M, I) the lattice $\mathfrak{B}([G, N])$ is join-preserving order embeddable into $\mathfrak{B}((G, M, I))$ for any $N \subseteq M$. Thus, this aspect can be represented through a quotient $|\mathfrak{B}(\mathbb{K}_{M \setminus N}))|/|\mathfrak{B}(\mathbb{K})|$, which is a special case of the maximal common subgraph distance, see [13]. Hence, whenever searching for the largest $\mathfrak{B}([G,N])$ the obvious choice is to optimize for large contranominal scales in subcontexts of (G,M,I). For example, when selecting three attributes in Figure 6.1 (top), the largest meet-preserving order embeddable lattice would be generated by the set $\{b,c,d\}$. However, the relative relevance of $\{b,c,g\}$ is significantly larger, in particular, $r(\{b,c,d\}) = 17/33$ and $r(\{b,c,g\}) = 19/33$, since we have a second requirement. Considering the distribution of the objects on the concept lattice, the sizes of the concept extents have to be incorporated. Since they are unknown, unless we compute the concept lattice, we need a proxy for estimating the influence of those. Accordingly, we want to reflect this with the quotient $E(\mathbb{K}_{M\setminus N})/E(\mathbb{K})$, which estimates the change of the object distribution on the concept lattices when selecting a set $N \subseteq M$. This quotient does employ a mapping $E : \mathcal{K} \to \mathbb{R}, \mathbb{K} \mapsto E(K)$, which is to be found. A natural candidate for this mapping would be information entropy, as introduced by Shannon in [54]. He defined the entropy of a discrete set of probabilities $p_1,\ldots,p_n$ as $H = -\sum_{i\in I} p_i \log\ p_i$. We adapt this formula to the realm of formal contexts as follows.

Definition 6.4 (Shannon Object Information Entropy)
Let $\mathbb{K} = (G,M,I)$ be a formal context. Then the *Shannon object information entropy of* $\mathbb{K}$ is given as follows:

$$E_{SE}(\mathbb{K}) = \sum_{g\in G} -\frac{|g''|}{|G|}\ log_2\left(\frac{|g''|}{|G|}\right)$$

For this entropy function we employ the quotient $|g''|/|G|$, which does reflect the extent sizes of the object concepts of $\mathbb{K}$. Obviously, this choice does not consider all concept extents. However, since every extent in a concept lattice is either the extent of an object concept or the intersection of finitely many extents of object concepts, we see that the Shannon object information entropy does relate to all extents to some degree. We found another candidate for E in the literature [48]. The authors introduced an entropy function which is, roughly speaking, the mean distance of the extents of object concepts to the complete set of objects.

Definition 6.5 (Object Information Entropy)
Let $\mathbb{K} = (G,M,I)$ be a formal context. Then the *object information entropy of* $\mathbb{K}$ is given as follows:

$$E_{OE}(\mathbb{K}) = \frac{1}{|G|}\sum_{g\in G}\left(1 - \frac{|g''|}{|G|}\right)$$

We observe that this entropy decreases as the number of objects having similar attribute sets increases. Furthermore, we recognize an essential difference for E_{OE} compared to E_{SE}. The Shannon object information entropy reflects on the number of necessary bits to encode the formal context. In contrary the object information entropy reflects on the average number of bits to encode an object from the formal context. To enhance the first grasp of the just introduced functions as well as the relative relevance defined in Definition 6.3 we want to investigate them on well-known contextual scales. In particular, the ordinal scale $\mathbb{O}(k) \coloneqq ([k],[k],\leq)$, the nominal scale $\mathbb{N}(k) \coloneqq ([k],[k],=)$, and the contranominal scale $\mathbb{N}^c(k) \coloneqq ([k],[k],\neq)$, where $[k] \coloneqq \{1,\dots,k\}$. Since there is a bijection between the set $\{1,\dots,k\}$ to the extent sizes $|g''|$ in an ordinal scale we obtain that $E_{SE}(\mathbb{O}(k)) = -\sum_{i=1}^{k} \frac{i}{k} log_2\left(\frac{i}{k}\right)$ and $E_{OE}(\mathbb{O}(k)) = \frac{1}{k}\sum_{i=1}^{k}\left(1-\frac{i}{k}\right) = \frac{1}{k}\frac{k(k+1)}{2k} = \frac{k+1}{2n}$. The former diverges to ∞ whereas the latter converges to $1/2$. Based on the linear structure of $\underline{\mathfrak{B}}(\mathbb{O}(k))$ we conclude that the set $\mathfrak{B}(\mathbb{K}) \setminus \mathfrak{B}(\mathbb{K})_{\{m\}} = \{(m', m'')\}$ for all $m \in M$. So the relative relevance amounts to $r(m) = 1 - (\sum_{i=1}^{k} i - |m''|) \;/\; \sum_{i=1}^{k} i = 2|m''|/(k\cdot(k+1))$. Both the nominal scale as well as the contranominal scale satisfy $g'' = g$ for all $g \in G$ for different reasons. We conclude that E_{SE} and E_{OE} evaluate respectively equally for both scales. In detail, we have $E_{SE}(\mathbb{N}(k)) = E_{SE}(\mathbb{N}^c(k)) = -\sum_{g\in G}\frac{1}{k} log_2\left(\frac{1}{k}\right) = \log_2(k)$ and $E_{OE}(\mathbb{N}(k)) = E_{OE}(\mathbb{N}^c(k)) = \frac{1}{k}\sum_{g\in G}\left(1-\frac{1}{k}\right) = \frac{k-1}{k}$. For the relative relevance we observe that $r(m) = r(n)$ for all $m, n \in M$ in the case of the nominal and contranominal scale. This is due to the fact that every attribute is part of the same number of concepts. For the nominal scale holds $r(m) = 1 - \frac{2k-1}{2k}$ for all $m \in M$. Hence, as the number of attributes increases, does the relevance of a single attribute converge to zero. The relative relevance in the case of the contranominal scale is $r(m) = 1 - \frac{\sum_{i=0}^{k}\binom{k}{i}(k-i) - \sum_{i=0}^{k-1}\binom{k-1}{i}(k-1-i)}{\sum_{i=0}^{k}\binom{k}{i}(k-i)}$ for all $m \in M$.

Example 6.3

Revisiting our running example in Figure 6.1 (bottom), this context has four objects with $\{B\}'' = \{B, F, S\}$, $\{F\}'' = \{F\}$, $\{D\}'' = \{F, D\}$ and $\{S\}'' = \{S\}$. Its entropies are given by $E_{OE}(\mathbb{K}) = \frac{1}{4}\sum_{g\in G}\left(1 - \frac{|g''|}{4}\right) \approx 0.56$ and $E_{SE}(\mathbb{K}) \approx 0.45$.

Considering both aspects discussed in this section, we now want to introduce a function that shall be capable of approximating RRP.

Definition 6.6 (Entropic Relevance Approximation (ERA))

Let $\mathbb{K} = (G, M, I)$ be a formal context with $N \subseteq M$. The *entropic relevance approximation (ERA) of N* is defined as

$$ERA(N) \coloneqq \frac{|\mathfrak{B}([G,N])|}{|\mathfrak{B}(\mathbb{K})|} \cdot \frac{E([G,N])}{E(\mathbb{K})}.$$

First, the ERA compares the number of concepts in a given formal context to the number of concepts in a subcontext on the attribute set $N \subseteq M$. This reflects the structural impact when restricting the attribute set. Secondly, a quotient is evaluated where the entropy of $[G, N]$ is compared to the entropy of $\mathbb{K}$. When using Definition 6.6 for finding a subset $N \subseteq M$ with maximal (entropic) relevance it suffices to compute N such that $\mathfrak{B}([G,N]) \cdot E([G,N])$ is minimal since $\mathfrak{B}(\mathbb{K}) \cdot E(\mathbb{K})$ does not change for different attribute selections. This task is essentially less complicated since we only have to compute $\mathfrak{B}([G,N])$ and $E([G,N])$ for some comparable small formal context $[G,N]$.

6.3 Evaluation and Discussion

To assess the ability for approximating relative relevance through Definition 6.6, we carried out several experiments in the following fashion. For all data sets we computed the iterative maximal relevant subsets of M of sizes one to seven (or ten) in the obvious manner. We decided for those fixed numbers for two reasons. First, using a relative number, e.g., 10% of all attributes, would still lead to an infeasible computation when the initial formal context is very large. Secondly, formal contexts with up to ten attributes permit a plenitude of research methods that are impracticable for larger contexts, in particular, human evaluation.

Then we computed subsets of M using ERA, for which we used both introduced entropy functions and their relative relevance. Finally, we sampled subsets of M randomly at least $|M| \cdot 10$ many times and computed their average relative relevance as well as the standard deviation in relative relevance.

Additional evaluations regarding the size of the generated lattice and the size of the canonical base of its implications can be found in Section 7.3.

6.3.1 Data Set Description

A total of 2678 formal contexts were considered in this experimental study. From those 2674 contexts were excerpts from the BibSonomy platform[1] as described in [7]. All those contexts are equipped with an attribute set of twelve elements and a varying number of objects. The particular extraction method is described in detail in [12]. For the rest we revisited three data sets well known in the realm of Formal Concept Analysis, i.e., *mushroom, zoo, water* [17, 29], and additionally a data set

[1] `https://www.kde.cs.uni-kassel.de/wp-content/uploads/bibsonomy/`

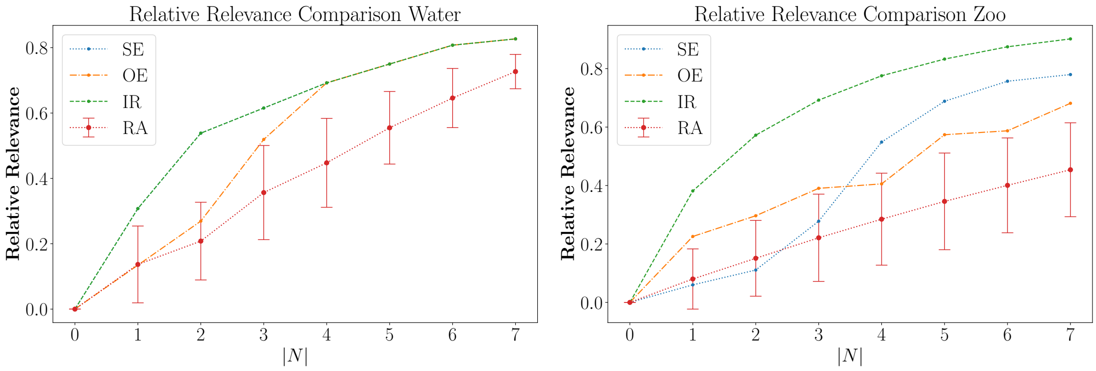


Figure 6.3 Relevance of attribute selections through entropy (SE,OE), IMRS (IR), and random selection (RA) for the "Living beings in water" (left) and the zoo context (right).

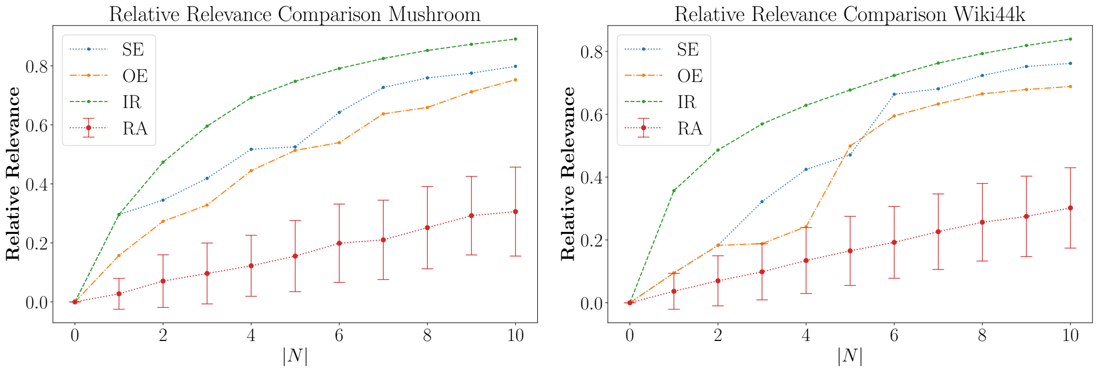


Figure 6.4 Relevance of attribute selections through entropy (SE, OE), IMRS (IR), and random selection (RA) for the mushroom (left) and the wiki44k context (right).

wiki44k introduced in [32], which is based on a 2014 Wikidata[2] database dump. The well-known *mushroom* data set is a collection of 8124 mushrooms described by 119 (scaled) attributes and exhibits 238710 formal concepts. The *zoo* data set possesses 101 animal descriptions using 43 (scaled) attributes and exhibits 4579 formal concepts. The *water* data set, more formally "Living beings and water", has eight objects and nine attributes and exhibits 19 formal concepts. Finally, the *wiki44k* has 45021 objects and 101 attributes exhibiting 21923 formal concepts.

6.3.2 Results

In Figures 6.3 to 6.5, we depicted the results of our computations. We observe in all experiments that the relative relevances of the subsets found through the iterative approach are an upper bound for the relative relevance of all subsets computed through entropic relevance approximation or random selection, with respect to the same size of subset. In particular, we find IMRS of cardinality seven and above have

[2] `https://www.wikidata.org`

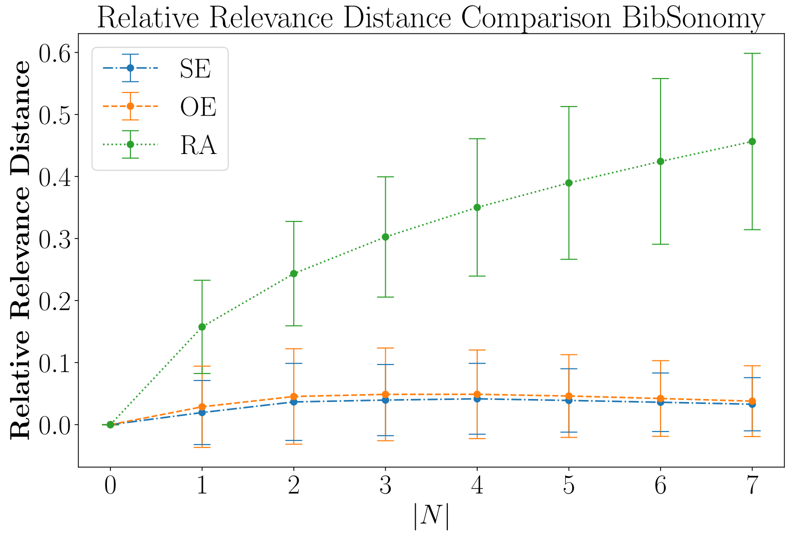


Figure 6.5 Average distance and standard deviation to IMRS for entropy and random based selections of $|N|$ attributes for 2674 formal contexts from BibSonomy.

a relative relevance of at least 0.8. Moreover, the relative relevance of the attribute subsets selected by both ERA versions (SE or OE) exceed the relative relevance of the randomly selected subsets except for the Shannon object information entropy for |N|=1 and |N|=2 in the zoo context. Principally we find for contexts containing a small number of attributes (Figure 6.3) a large increase of the distance between the relative relevance of the randomly selected attributes and the attribute sets selected through the entropy approach. This characteristic manifests in the relative relevance of both ERA selections excelling not only the mean relative relevance of randomly chosen attribute sets but also the standard deviation for subset sizes of $|N| = 4$ and above. In the case of contexts containing a huge number of attributes, this observation can be made for selections with $|N| = 1$ already. Furthermore, the interval between the relative relevance of the attribute subsets selected by both ERA versions and the relative relevance of the randomly selected subsets is significantly larger than in the case of contexts with small attribute set sizes. In general, we may point out that neither of the entropies seems preferable over the other in terms of performance. In Figure 6.5, we show the results for the experiment with the 2674 formal contexts from BibSonomy. We plotted for all three methods, ERA-OE/SE and random, the mean distance in relative relevance to the IMRS of the same size together with the standard deviation. We detect a significant difference between randomly chosen and ERA chosen sets with respect to their relative relevance. The deviation for both ERA is bound by 0 and 0.12. In contrast, the relative relevance for randomly selected sets is bound by 0.09 and 0.6.

6.3.3 Discussion

We found in our investigation that attribute sets obtained through the iterative approach for relative relevance do have a high relevance value. Even though their relative relevance is only a lower bound compared to the maximal relevant set they exhibit a relative relevance of 0.8 for attribute set sizes seven and above. We conclude from this that the iterative approach is a sufficient solution to the relative relevance problem. Based on this we may deduct that entropic relative approximation is also a good approximation for a solution to the RRP. In particular, in large formal contexts investigated in this work the approximation was even better than in the smaller ones.

6.4 Conclusion

By defining the relative relevance of attribute sets in formal contexts, we introduced a novel notion of attribute selection. This notion respects both the structure of the concept lattice and the distribution of the objects on it. To overcome computational limitations, which arose from the notion of relative relevance, we introduced an approximation based on two different entropy functions adapted to formal contexts. For this, we used a combination of two factors: the change in the number of concepts and the change in entropy that arise from the selection of an attribute subset. The experimental evaluation for relative relevance as well as the entropic approximation seems to comply with the theoretical modeling.

We conclude this chapter with two open questions. First, even though IMRS seems a good choice for relevant attributes, we suspect that computing the maximal relevant set, with respect to RRP, can be achieved more feasible. Secondly, so far our justification for RRP is based on theoretical assumptions and a basic experimental study. We imagine, and are curious, if maximal relevant attribute sets are also employable in supervised machine learning setups. For example, one may perceive the task of adding a new object to a given formal context as instance of such a setup. The question is, how capable is the context to add this object to an already existing concept.

CHAPTER 7

Selecting Attributes using Contranominal Scales

Since the size of a lattice depends on the number of contranominal scales of high dimension in the corresponding formal context, we follow up on the previous idea of attribute selection by introducing *δ-adjusting*, a novel approach to decrease the number of contranominal scales in a formal context by selecting an appropriate attribute subset. Considering that one of the main goals of Formal Concept Analysis is to enable humans to comprehend the information that is encapsulated in the data, the large size of concept lattices is a limiting factor for the feasibility of understanding the underlying structural properties. Therefore, we demonstrate that *δ-adjusting* a context – as well as selecting relative relevant attributes as done in the Chapter 6 – reduces the size of the hereby emerging subsemilattice and that the original implication set is restricted to meaningful implications. This is evaluated with respect to its associated knowledge by means of a classification task. Hence, our proposed techniques strongly improve the understandability while preserving important conceptual structures.

7.1 Introduction

Since the size of the concept lattice is heavily influenced by the number of its Boolean suborders their elimination, and thus the elimination of contranominal scales in the corresponding formal context, is a reasonable approach to reduce the lattice size. Therefore, we focus on the removal of attributes based on their appearance in contranominal scales in this chapter.

We present our novel approach *δ-adjusting*, which focuses on the selection of an appropriate attribute subset of a formal context. To this end, we measure the influence of each attribute with respect to the number of contranominal scales. By restricting the attribute set, a subsemilattice is computed that preserves the meet-operation. This provides the advantage of not only maintaining all implications between the selected attributes but also does not produce false implications and thus retains underlying structure. We conduct experiments to demonstrate that the subcontexts that arise by *δ-adjusting* decrease the size of the concept lattice and the implication set while preserving underlying knowledge and show that this is also the case for the approach of selecting relative relevant attributes. We evaluate the remaining knowledge by training a classification task. This results in a more understandable depiction of the encapsulated data for the human mind.

7.2 Attribute Selection

In this section, we propose δ-`adjusting`, a method to select attributes based on measuring their influence for contranominal scales. For this, we first introduce a method to rate the attributes as follows:

Definition 7.1 (Contranominal-Influence)
Let $\mathbb{K} = (G, M, I)$ be a formal context and $k \in \mathbb{N}$. We call the attribute set $N \subseteq M$ *k-cubic* if there is a set $H \subseteq G$ with $[H, N]$ being a contranominal scale of dimension k and $\nexists \widetilde{N} \subseteq M$ with $\widetilde{N} \supseteq N$ such that $\widetilde{N}$ is $(k+1)$-cubic. We define the *contranominal-influence* of $m \in M$ in $\mathbb{K}$ as

$$\zeta(m) \coloneqq \sum_{k=1}^{\infty} \left(|\{N \subseteq M \mid m \in N, N \text{is } k\text{-cubic}\}| \cdot \frac{2^k}{k} \right).$$

Subcontexts that are k-cubic are directly influencing the concept lattice, as those dominate the structure, as the following shows.

Proposition 7.1
Let $\mathbb{K} = (G, M, I)$ be a formal context. An attribute set $N \subseteq M$ is k-cubic if and only if $\underline{\mathfrak{B}}([G,N]) \cong \mathfrak{B}(k)$ and it has no Boolean superlattice in $\underline{\mathfrak{B}}(\mathbb{K})$.

The contranominal influence thus measures the impact of an attribute on the lattice structure. In this, only the maximal contranominal scales are considered since the smaller non-maximal-ones have no additional structural impact. As each contranominal scale of dimension k corresponds to 2^k concepts, we scale the number of attribute combinations with this factor. To distribute the impact of a contranominal scale evenly over all involved attributes, the measure is scaled by $\frac{1}{k}$. With this measure, we now define the notions of δ-`adjusting` a formal context and its concept lattice.

Definition 7.2 (δ-adjusted Subcontext, δ-adjusted Sublattice)
Let $\mathbb{K} = (G, M, I)$ be a formal context and $\delta \in [0,1]$. Let $N \subseteq M$ be minimal such that $\frac{|N|}{|M|} \geq \delta$ and $\zeta(n) < \zeta(m)$ for all $n \in N, m \in M \setminus N$. We call $\mathbb{A}_\delta(\mathbb{K}) \coloneqq [G,N]$ the δ-*adjusted subcontext* of $\mathbb{K}$ and $\underline{\mathfrak{B}}(\mathbb{A}_\delta(\mathbb{K}))$ the δ-*adjusted sublattice* of $\underline{\mathfrak{B}}(\mathbb{K})$.

Note that δ-`adjusting` always results in unique contexts since all attributes with identical contranominal-influence are simultaneously added to N. Moreover, every δ-`adjusted` sublattice is a sub-$\wedge$-semilattice of the original one and $\mathbb{A}_1 = \mathbb{K}$ and $\mathbb{A}_0 = [G, \varnothing]$ for every context $\mathbb{K} = (G, M, I)$. A context from a medical diagnosis dataset with measured contranominal influence and its $\frac{1}{2}$-`adjusted` subcontext is presented in Figure 7.1. The associated concept lattices are visualized in Figure 7.2.

For a context $\mathbb{K}$ and its reduced context $\mathbb{K}_{irr}$ a different attribute set can remain if they are δ-`adjusted` (e.g., see Figure 7.3). Thus, the resulting concept lattices for $\mathbb{K}$ and $\mathbb{K}_{irr}$ can differ. To preserve structural integrity between δ-`adjusted` formal contexts and their concept lattices, we thus recommend only to consider clarified and reduced contexts and perform these steps prior to δ-`adjusting` in the rest of this chapter. Note that since no attributes are generated and the incidence relation is not altered, no new contranominal scales can arise by δ-`adjusting`. Furthermore, removing attributes can not turn another attribute from irreducible to reducible. On the other hand, however, objects can become reducible as can be seen again in Figure 7.3. While 6 is irreducible in the original context, it is reducible in $\mathbb{A}_{\frac{3}{5}}(\mathbb{K})$.

7.2.1 Properties of Implications

In this section, we investigate δ-`adjusting` with respect to the influence on implications. However, the statements generally hold for contexts generated via attribute selection and, therefore, especially for the ones that utilized relative relevance.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
111			×	×	×		×	×		×					×
119	×				×	×	×	×				×	×		×
31		×	×	×	×				×	×	×		×		
32	×		×		×			×	×	×					×
17		×		×	×	×			×	×	×		×		
27		×	×	×	×			×	×	×			×		
105	×	×				×	×				×	×	×		
58		×		×	×	×				×	×		×	×	
65	×				×	×		×				×	×	×	×
103	×		×				×				×	×			×
56		×	×	×	×					×	×		×	×	
98	×	×	×				×				×	×	×		
43		×	×	×	×			×		×			×	×	
50	×		×		×			×		×				×	×

	Attribute Name	2	3	4	ζ
a:	Lumbar pain y	1	22	6	84.7
b:	Bladder inflammation y	1	29	0	79.3
c:	Burning n	1	31	9	120.7
d:	Lumbar pain n	2	19	0	54.7
e:	Nausea n	0	16	3	54.7
f:	Burning y	1	31	0	84.7
g:	Temp. $\in [40.0,\ 42.0]$	2	24	5	88.0
h:	Micturition pains n	1	18	5	70.0
i:	Temp. $\in [35.0,\ 37.5]$	3	16	0	48.7
j:	Pelvis nephritis n	1	19	1	56.7
k:	Micturition pains y	1	33	0	90.0
l:	Pelvis nephritis y	3	17	0	51.3
m:	Urine pushing y	0	21	7	84.0
n:	Temp. $\in [37.5,\ 40.0]$	2	23	3	77.3
o:	Bladder inflammation n	1	26	1	75.3

Figure 7.1 Top: Reduced and clarified medical diagnosis dataset [15]. The $\frac{1}{2}$-`adjusted` subcontext is highlighted. The objects are patient numbers.
Bottom: The attributes are described in the figure together with the count of k-cubic subcontexts and their contranominal influence ζ.

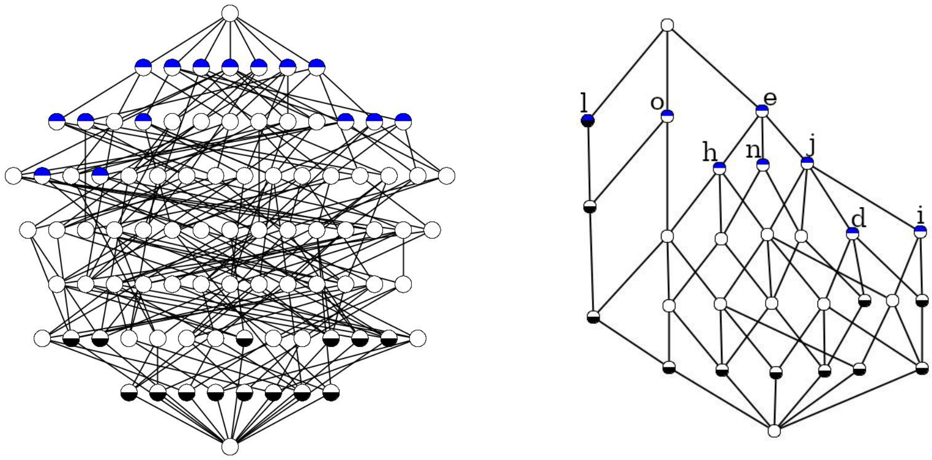
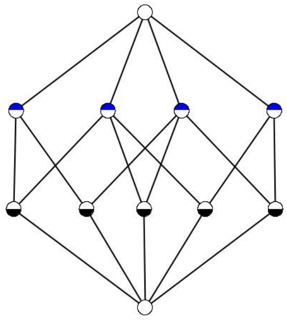


Figure 7.2 Lattice of the original (left) and the $\frac{1}{2}$-`adjusted` (right) dataset shown in Figure 7.1.

Let $\mathbb{K} = (G, M, I)$ be a formal context, $m \in M$, and $X \to Y$ an implication in $\mathbb{K}$. If m is part of the implication, i.e., $m \in X$ or $m \in Y$, this implication vanishes when removing m from the context. Therefore the removal of m in an implication $X \to Y$ of some implication base $\mathcal{C}(\mathbb{K})$ is of interest. If m is neither part of a premise nor a conclusion of an implication $X \to Y \in \mathcal{C}(\mathbb{K})$, its removal has no impact on this implication base. In the case of $m \in Y$, its elimination changes all implications $X \to Y$ to $X \to Y \setminus \{m\}$. Note that, even though all implications can still be deduced from $\mathcal{C}' = \{X \to Y : X \to Y \cup \{m\} \in \mathcal{C}(\mathbb{K})\} \cup \{X \to Y \in \mathcal{C}(\mathbb{K}) : m \notin Y\}$ this set is not necessarily minimal and in this case is not a base. Especially if $\{m\} = Y$, the resulting implication $X \to \varnothing$ is never part of an implication base. In the case of $m \in X$, every $Z \to X$ in the base is changed to $Z \to X \setminus \{m\} \cup Y$ while $X \to Y$ is removed. Similarly to the conclusion case, the resulting set of implications can be used to deduce all implications but is not necessarily an implication base. Moreover, as the following shows, no new implications can emerge from the removal of attributes.

Proposition 7.2
Let $\mathbb{K} = (G, M, I)$ be a formal context, $N \subseteq M$ and $X, Y \subseteq N$ with $X \to Y$ being a non-valid implication in $\mathbb{K}$. Then $X \to Y$ is also non-valid in $[G, N]$.

Proof Since $X \to Y$ is not valid in $\mathbb{K}$, there exists an object $g \in G$ with $X \subseteq g'$ and $Y \nsubseteq g'$. As the objects in $\mathbb{K}$ and $[G, N]$ are identical on N (especially if $X, Y \subseteq N$), g is a counterexample for $X \to Y$ in $[G, N]$. □



	a	b	c	d	e
1	×	×			
2	×		×		
3		×	×		
4			×	×	×
5				×	
6		×		×	

	a	b	c	d
1	×	×		
2	×		×	
3		×	×	
4			×	×
5				×
6		×		×

Figure 7.3 A concept lattice (left) together with two of its contexts $\mathbb{K}$ (middle) and $\mathbb{K}_{irr}$ (right) whereby $\mathbb{K}_{irr}$ is attribute reduced while $\mathbb{K}$ contains the reducible element e. In both contexts the $\frac{3}{5}$-`adjusted` subcontext is highlighted. Their lattices (right to each context) differ.

Thus, the relationship between the implications of a subcontext with all objects and the original context is as follows:

Corollary 7.1
Let $\mathbb{K} = (G, M, I)$ be a formal context and $\mathbb{S} = [G, N] \leq \mathbb{K}$ a subcontext. Then $Imp(\mathbb{S}) \subseteq Imp(\mathbb{K})$ holds.

In particular, the minimal implication base cannot grow by selecting subcontexts based on attributes. This influences the size of the implication base of a subcontext generated by attribute selection as follows:

Proposition 7.3
Let $\mathbb{K} = (G, M, I)$ a formal context and $\mathbb{S} = [G, N] \leq \mathbb{K}$. Then $|\mathcal{C}(\mathbb{S})| \leq |\mathcal{C}(\mathbb{K})|$ holds.

Proof Assume not; i.e., $|\mathcal{C}(\mathbb{S})| > |\mathcal{C}(\mathbb{K})|$. Let J be the set of implications containing $m \in M \setminus N$ in $\mathcal{C}(\mathbb{K})$. A set of implications that can generate the whole implication set with size $|\mathcal{C}(\mathbb{K})|$ or less is given by altering the implications $X \to Y$ $X, Y \in M$ in $|\mathcal{C}(\mathbb{K})|$ as follows. If $m \in Y$, $X \to Y$ is replaced by $X \to Y \setminus m$. If $m \in X$ and $Z \to X \in \mathcal{C}(\mathbb{K})$, $X \to Y$ is replaced by $Z \to Y$. This yields a contradiction. □

Example 7.1
Revisiting the context in Figure 7.1 together with its $\frac{1}{2}$-`adjusted` subcontext the selection of nearly 50% of the attributes (8 out of 15) results in a sub-$\wedge$-semilattice containing only 33% of the concepts (29 out of 88). Moreover, the implication base of the original context includes 40 implications. After the alteration, its size is decreased to 11 implications.

7.3 Evaluation and Discussion

In this section, we evaluate the process of δ-`adjusting` using real-world datasets. Hereby, the selection of relative relevant attributes as presented in Chapter 6 is used as a baseline. To compute all contranominal scales of the data sets, we utilize the algorithm `Contrafinder` as proposed by Dürrschnabel et al. [21].

7.3.1 Datasets

The datasets used in this chapter are published in [22]. Table 7.1 provides descriptive properties of the different datasets. The zoo and mushroom datasets and the Wiki44k dataset have already been used in Chapter 6. The Wikipedia dataset depicts the edit relation between authors and articles. Finally, the students dataset depicts the grades of students together with properties such as parental level of education. All experiments are conducted on the reduced and clarified versions of the contexts.

7.3.2 Structural Effects of δ-Adjusting

We measure the number of formal concepts generated by the formal context as well as the size of the canonical base. To demonstrate the effects of δ-adjusting we focus on $\delta = \frac{1}{2}$. Our two baselines are selecting the same number of attributes using random sampling and choosing the attributes of highest the relative relevance. It can be observed that in all three cases the number of concepts heavily decrease (see Table 7.2). However, this effect is considerably stronger for $\frac{1}{2}$-`adjusting` and the relative relevance approach compared to random sampling. Hereby, $\frac{1}{2}$-`adjusting` yields smaller concept lattices on four datasets. A similar effect can be observed for the sizes of the canonical bases where this method yields three times in the smallest cardinality. The selection based on the relative relevance results in the smallest concept lattice in one case and the smallest canonical base in two cases.

Table 7.1 Datasets used for the evaluation of δ-`adjusting`.

	Zoo	Students	Wikipedia	Wiki44k	Mushroom
Objects:	101	1000	11273	45021	8124
Attributes:	43	32	102	101	119
Density:	0.40	0.28	0.015	0.045	0.19
Number of concepts:	4579	17603	14171	21923	238710
Mean objects per concept:	18.48	16.73	20.06	109.47	91.89
Mean attributes per concept:	7.32	5.97	5.88	7.013	16.69
Size of canonical base:	401	2826	4575	7040	2323

7.3.3 Knowledge in the δ-Adjusted Context

To measure the degree of encapsulated knowledge in δ-`adjusted` formal contexts, we conduct the following experiment using random sampling and the relative relevant attributes as baselines again. In order to measure if the remaining subcontexts still encapsulate knowledge, we train a decision tree classifier on them, predicting an attribute that is removed beforehand. This attribute is sampled randomly in each step. To prevent a random outlier from distorting the result, we repeat this same experiment 1000 times for each context and method and report the mean value and the standard-deviation in Table 7.2. The experiment is conducted using a 0.5-split on the train and test data. For all five datasets, the results of the decision tree on the $\frac{1}{2}$-`adjusted` context are consistently high, however $\frac{1}{2}$-`adjusting` and the relative relevance approach outperform the sampling approach. Both these methods achieve the highest score on four contexts; in two cases, the highest result is shared. The single highest score of the sampling approach is just slightly above the score of the other two approaches.

7.3.4 Discussion

To evaluate the impact on the understandability of the δ-adjusted formal contexts, we conduct experiments measuring the sizes of the concept lattices and the canonical bases. All three evaluated methods heavily decrease the size of the concept lattice

Table 7.2 Evaluation of k-adjusted contexts. The standard deviation is given in parenthesis. "Acc of DT" is the abbreviation for "Accuracy of the Decision Tree".

		Zoo	Students	Wikipedia	Wiki44k	Mushroom
$\lvert\mathfrak{B}(\mathbb{K})\rvert$:	$\frac{1}{2}$-adjusted:	**90**	**312**	**65**	323	**426**
	Sampling:	496	1036	833	1397	8563
		(205)	(327)	(517)	(627)	(4532)
	rel. relevance:	95	341	67	**254**	561
$\lvert\mathcal{C}(\mathbb{K})\rvert$:	$\frac{1}{2}$-adjusted:	98	**105**	626	**1003**	**339**
	Sampling:	**95**	156	758	1360	574
		(17)	(35)	(101)	(135)	(93)
	rel. relevance:	100	**105**	**553**	1091	490
Acc of DT:	$\frac{1}{2}$-adjusted:	0.88	0.88	**0.99**	**0.98**	**0.98**
		(0.08)	(0.06)	(0.01)	(0.03)	(0.02)
	Sampling:	**0.89**	0.81	0.9	0.95	0.92
		(0.15)	(0.15)	(0.14)	(0.06)	(0.13)
	rel. relevance:	0.88	**0.89**	**0.99**	**0.98**	0.97
		(0.09)	(0.06)	(0.01)	(0.16)	(0.03)

as well as the canonical base. Compared to the random sampling $\frac{1}{2}$-adjusting and selecting the relative relevant attributes influence the size of these structural components much stronger. Among those two, $\frac{1}{2}$-adjusting seems to outperform the method of relative relevant attribute selection slightly.

To evaluate to what extent knowledge is encapsulated in the selected subcontexts, we conduct the decision tree experiment. It demonstrates that the selected subcontexts can be used in order to deduce relationships of the remaining attributes in the context. While meaningful implications are preserved, and the implication set is downsized, $\frac{1}{2}$-adjusted lattices seem suitable to preserve large amounts of data from the original dataset. Similar good results can be achieved by selecting relative relevant attributes. The approach of $\frac{1}{2}$-adjusting combines this with producing smaller concept lattices and canonical bases and is thus more suitable for preparing data for a human analyst by only reducing sizes of structural constructs.

We conclude from these experiments that δ-adjusting is a solution to the problem of making information more feasible for manual analysis while retaining essential parts of the data. In particular, if large formal contexts are investigated, this method provides a way to extract relevant subcontexts.

7.4 Conclusion

In this chapter, we defined the contranominal-influence of an attribute. This measure allows us to select a subset of attributes in order to reduce a formal context to its δ-adjusted subcontext. The size of its lattice is significantly reduced compared to the original lattice and thus enables researchers to analyze and understand much larger datasets using Formal Concept Analysis. Furthermore, the size of the canonical base, which can be used to derive relationships of the remaining attributes, shrinks significantly. Still, the remaining data can be used to deduce relationships between attributes, as our classification experiment shows. This approach, therefore, identifies subcontexts whose sub-$\wedge$-semilattice is a restriction of the original lattice of a formal context to a small meaningful part.

Part IV

Eliminating Intervals

CHAPTER 8

Interval Factorization

This chapter investigates the factorization of finite lattices to implode selected intervals while preserving the remaining order structure. We examine how complete congruence relations and complete tolerance relations can be utilized for this purpose and answer the question of finding the finest of those relations to implode a given interval in the generated factor lattice. To overcome the limitations of the factorization based on those relations, we introduce a new lattice factorization that enables the imploding of selected disjoint intervals of a finite lattice. To this end, we propose an *interval relation* that generates this factorization. To obtain lattices rather than arbitrary ordered sets, we restrict this approach to so-called *pure intervals*. Further, we also provide a new FCA construction by introducing the *enrichment* of an incidence relation by a set of intervals in a formal context to investigate the approach for lattice-generating interval relations on the context side.

8.1 Introduction

A complete lattice $\underline{L}$ consists of several intervals. Such an interval is often considered as a unity. Therefore, a transformation of the lattice that implodes such an interval (or a set of intervals) is a suitable way to obtain a more compact representation of the original structure. In this chapter, we investigate different ways to factorize a lattice to "implode" selected intervals, meaning to condense an interval into a single element in the following manner:

Definition 8.1 (Implosion)
An *implosion* of a set of disjoint intervals $\underline{S}_1, \ldots, \underline{S}_k$ in an ordered set $(P, \leq_P)$ is an surjective, order-preserving mapping $f\colon (P, \leq_P) \to (Q \leq_Q)$ on an ordered set Q so that $|f(\underline{S}_i)| = 1$ for all $i \in \{1, \ldots, k\}$.

We pursue two – partially conflict – goals for such an implosion. The first goal is that the function f is as compatible as possible with the lattice structure – in the best case, f is a lattice homomorphism. The second goal is that the remaining part of the lattice remains intact as much as possible – in the best case, f is injective in $\underline{L} \setminus \underline{S}$. As we will discuss in the sequel, there are different ways to realize an implosion of one or more intervals with different trade-offs regarding to our goals.

We examine how lattice factorizations based on complete congruence relations and complete tolerance relations can be utilized for imploding intervals in a lattice. In particular, we answer the question of finding the finest of those relations that implodes a given interval in the generated factor lattice while preserving as many of the other elements of the lattice as possible. While both of these approaches result in an infimum- and supremum-preserving factor lattice, the imploded intervals are often larger than just the selected interval. It is even possible for the factor lattice to implode the whole lattice and thus only contain one element.

To directly address intervals generated by Boolean suborders, we add the incidences missing in contranominal scales and in the subcontexts associated to the chosen suborders. However, this approach is not suitable for imploding a Boolean suborder if other parts of the context support the concepts of the associated Boolean subcontext. In general, other incidences have to be added beyond those in the associated Boolean subcontext.

To preserve all elements of the lattice except those in selected intervals, we investigate then a new kind of factorization based on *interval relations* and study the different types of implosion. We introduce *enrichments of the incidence relation by intervals* and show their one-to-one correspondence to interval relations. They can be utilized to implode selected intervals in the lattice while preserving the original order relation. By restricting the approach to *pure intervals*, we also ensure the lattice properties in the generated structure. Since every finite lattice is isomorphic to a concept lattice, all statements can be translated to finite lattices in general.

8.2 Imploding with Congruences and Tolerances

This section presents two methods for lattice factorization to implode selected intervals while preserving certain structural properties of the original lattice. Since we utilize approaches of FCA, we phrase the statements mostly for concept lattices. However, all statements can be translated to finite lattices in general.

8.2.1 Complete Congruence Relations

Due to its definition, a congruence relation θ preserves the meet- and join-operators of a lattice $\underline{L}$ in $\underline{L}/\theta$. Further, for every lattice $\underline{L}$ and every interval $\underline{S} \leq \underline{L}$ at least one congruence relation θ on $\underline{L}$ exists that implodes $\underline{S}$, meaning that $f\colon \underline{L} \to \underline{L}/\theta$ with $f(x) = [x]\theta$ (the equivalence class of θ including x) is an implosion of $\underline{S}$ in $\underline{L}$. This is always the case for the trivial congruence relation θ that has a single θ-class $[x]\theta = L$, so that $|f(\underline{L})| = 1$. To utilize this method to our aim of imploding specific intervals while preserving as much of the remaining structure as possible, the following question arises: Given a lattice $\underline{L}$ and an interval $\underline{S} \leq \underline{L}$, which congruence relation θ on $\underline{L}$ is the finest (meaning that $|\underline{L}/\theta|$ is as large as possible) so that $f\colon \underline{L} \to \underline{L}/\theta$ is an implosion of $\underline{S}$ in $\underline{L}$ and how can we determine this θ?

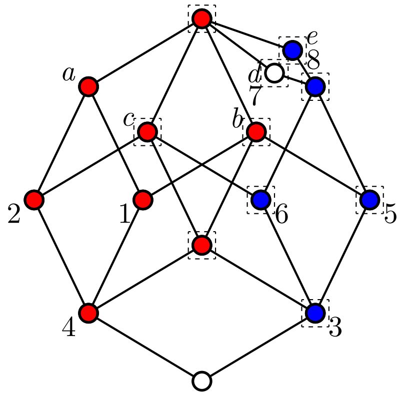
The congruence relations on a given lattice $\underline{L}$ are a closure system. So a unique finest congruence with the required property exists. Also, in the finite case, the congruence relations and the compatible subcontexts of the reduced formal context $\mathbb{K}$ with $\mathfrak{B}(\mathbb{K}) \cong \underline{L}$ have a one-to-one correspondence. We adapt this statement to our question setting as follows:

Proposition 8.1
Let $\mathbb{K} = (G, M, I)$ *be a reduced formal context and* $\underline{S} = [(A, B), (C, D)] \leq \underline{\mathfrak{B}}(\mathbb{K})$ *an interval. Let* $H = \{A \cup \{g \in G \mid g \notin C\}\}$ *and* $N = \{D \cup \{m \in M \mid m \notin B\}\}$*. The set of all compatible subcontexts* $[O, P] \leq \mathbb{K}$ *with* $O \subseteq N$ *and* $P \subseteq H$ *corresponds to the set of all congruence relations* θ *on* $\underline{\mathfrak{B}}(\mathbb{K})$ *with* $f\colon \underline{L} \to \underline{L}/\theta$ *is an implosion of* $\underline{S}$ *in* $\underline{L}$*. The largest compatible subcontext* $[O, P] \leq \mathbb{K}$ *with* $O \subseteq N$ *and* $P \subseteq H$ *corresponds to the finest of those congruence relations.*

Proof The compatible subcontexts of $\mathbb{K}$ correspond to the complete congruences of $\mathfrak{B}(\mathbb{K})$ so that they induce the complete congruences, and the ordered set of the compatible subcontexts is dually isomorphic to the congruence lattice. Due to the order of the congruence lattice, for a given interval $\underline{S}$, the set of all congruence relations θ on $\underline{L}$ with $f\colon \underline{L} \to \underline{L}/\theta$ is an implosion of $\underline{S}$ in $\underline{L}$ are an order filter, meaning $\exists [x]\theta$ with $\underline{S} \subseteq [x]\theta$. This filter is generated by the unique finest complete congruence with this property. Analogously, there is a corresponding compatible subcontext that is the unique greatest one generating an order ideal of all compatible subcontexts corresponding to the formerly mentioned congruences. For every object g of a compatible subcontext, the concept (g'', g') is the smallest element of a θ-class of the corresponding congruence θ. Analogously for every attribute m of a compatible subcontext, the concept (m', m'') is the greatest element of a θ-class of the corresponding congruence θ (see Proposition 3.8). Thus, the compatible subcontexts that correspond to the congruences with a θ-class containing $\underline{S}$ must not contain the object set $\{\{g \in G \mid (g'', g') \in \underline{S}\} \setminus A\}$ and the attribute set $\{\{m \in M \mid (m', m'') \in \underline{S}\} \setminus D\}$. So the greatest compatible subcontext fulfilling this requirement corresponds to the finest congruence with a θ-class containing $\underline{S}$. However, the possible selection of objects and attributes can be reduced as follows: Let $g \notin H$. Then $(g'', g') \leq (C, D)$ and $(g'', g') \nleq (A, B)$ and therefore $(g'', g') \wedge (C, D) = (g'', g')$ and $(g'', g') \wedge (A, B) < (g'', g')$ hold. So (g'', g') is not the smallest element of a θ-class, and therefore g is not contained in the compatible subcontext. $\square$

Note that the statement of Proposition 8.1 holds especially for intervals that are Boolean sublattices or generated by Boolean suborders, meaning the smallest intervals containing all elements of a given Boolean suborder.

Example 8.1
Considering the red highlighted interval $\underline{S}_1$ in $\underline{\mathfrak{B}}(\mathbb{K})$ in Figure 8.1 (left), the application of Proposition 8.1 results in the sets $H = \{4\}$ and $N = \{d, e\}$. The compatible subcontexts of the corresponding reduced formal context $\mathbb{K}$ (right) are $[\varnothing, \varnothing]$, $[3, a]$, and $[G, M]$. Thus, the compatible subcontext corresponding to the finest (and the only) congruence relation θ, that implodes $\underline{S}_1$ is $[\varnothing, \varnothing]$. So, the congruence relation we looked for is the trivial one that contains every concept in the same equivalence class. For $\underline{S}_2$ the sets $H = \{1, 2, 3, 4, 7\}$ and $N = \{a, e\}$ arise. Thus, the compatible subcontext $[3, a]$ corresponds to the finest congruence relation θ that implodes $\underline{S}_2$. θ partitions the concepts in two intervals, the one highlighted with dotted boxes and the remaining one.



	a	b	c	d	e
1	×	×	•	•	•
2	×	•	×	•	•
3		×	×	×	×
4	×	×	×	•	•
5		×	•	×	×
6		•	×	×	×
7		•	•	×	•
8		•	•	•	×

Figure 8.1 A (concept) lattice $\underline{\mathfrak{B}}(\mathbb{K})$(left) and its corresponding reduced formal context $\mathbb{K} = (G, M, I)$ (right). The intervals $\underline{S}_1$(red) and $\underline{S}_2$(blue) are highlighted in $\underline{\mathfrak{B}}(\mathbb{K})$. The finest congruence relation that implodes $\underline{S}_2$ partitions the concepts in two intervals, the one highlighted with dotted boxes and the remaining one. Adding the •-marked incidences to I results in the block relation that corresponds to the finest tolerance relation that implodes $\underline{S}_2$.

8.2.2 Complete Tolerance Relations

Another tool for lattice factorization are tolerance relations. They also generate a factor lattice $\underline{L}/\theta$, in which, similar to the congruence relations, the meet- and join-operators of $\underline{L}$ are preserved. However, since a tolerance relation does not have to be transitive, we can not expect to find a lattice homomorphism between $\underline{L}$ and $\underline{L}/\theta$. Instead, we have to arrange with a sublattice homomorphism. The two possible maps are given by Ganter and Wille in [29, Prop. 56]. This entails to consider those maps as implosions in this section, meaning there is a θ-class $[x]\theta$ with $\underline{S} \subseteq [x]\theta$.

Since every congruence relation is a tolerance relation, the trivial tolerance relation to implode a given interval exists on every lattice. Thus, the question of finding the finest tolerance relation with this property also arises. The tolerance relations of a lattice are in a one-to-one correspondence with the block relations of its corresponding formal context. Hence, we propose the following statement to search for the finest block relation that implodes a chosen interval.

Proposition 8.2

Let $\mathbb{K}$ be a reduced formal context and $\underline{S} = [(A, B), (C, D)] \leq \underline{\mathfrak{B}}(\mathbb{K})$ an interval. Let $\widetilde{J} = I \cup (C \times B)$. The block relations $J \supseteq I$ with $J \supseteq \widetilde{J}$ corresponds to the tolerance relations θ on $\underline{\mathfrak{B}}(\mathbb{K})$ with with $\underline{\mathfrak{B}}(\mathbb{K}) \to \underline{\mathfrak{B}}(\mathbb{K})/\theta$ being an implosion of $\underline{S}$. The finest block relation J with $J \supseteq \widetilde{J}$ corresponds to the finest of those congruence relations.

Proof Since the set of all block relations is a closure system, there is a unique finest block relation J, which includes $\widetilde{J}$. The tolerance relation θ corresponding to J has a θ-class containing $\underline{S}$ due to the initial inclusion of $(C \times B)$ in J. $\square$

Algorithm 2: Generation of J, the finest block relation to implode $\underline{S}$

Input: $\mathbb{K} = (G, M, I)$, $\underline{S} = [(A, B), (C, D)]$
Output: J

$J \coloneqq I \cup (C \times B)$
$ext \coloneqq \{H | (H, N) \in \mathfrak{B}(\mathbb{K})\}$
$int \coloneqq \{N | (H, N) \in \mathfrak{B}(\mathbb{K})\}$
$check \coloneqq C \cup B$
while $|check| > 0$ **do**
  $x \coloneqq first(check)$
  if $x \in G$ **then**
    if $x^J \notin int$ **then**
      $candidates \coloneqq \{y | y \in int, x \subset y\}$
      for $y \in candidates$ **do**
        $m_y \coloneqq y \setminus x^J$
      $add \coloneqq min_{|m_y|}\{m_y\}$
      $J \coloneqq J \cup \{(x, m) | m \in add\}$
      $check \coloneqq check \cup add$
  if $x \in M$ **then**
    if $x^J \notin ext$ **then**
      $candidates \coloneqq \{y | y \in ext, x \subset y\}$
      for $y \in candidates$ **do**
        $g_y \coloneqq i \setminus x^J$
      $add \coloneqq min_{|g_i|}\{g_i\}$
      $J \coloneqq J \cup \{(g, x) | g \in add\}$
      $check \coloneqq check \cup add$
  $check \coloneqq check \setminus x$
return J

In Algorithm 2, we give a strategy to find those relations: Given a complete lattice $\underline{L}$ and an interval $\underline{S} = [(A, B), (C, D)] \leq \underline{L}$, in the first step $(C \times D)$ is added to the incidence relation to ensure, that (A, B) and (C, D) are in the same equivalence-class and therefore are mapped to the same element by the factorization. Then for every object $g \in C$ and every attribute $m \in B$, it is checked whether it satisfies the conditions for block relations with the new incidence relation $J = I \cup (C \times B)$. If this is not the case for an object g, for the smallest intent $N \subseteq M$ in $\underline{\mathfrak{B}}(\mathbb{K})$ with $g^J \subset N$ all incidences (g, n) with $n \in N \setminus g^J$ are added to J. The method for attributes is analogous. This process is repeated iteratively until the conditions for block relations hold for every object and attribute. Note that since the intersection of two intents is an intent itself, the smallest intent selected in every step is unique. The same holds for extents.

Proposition 8.3
Algorithm 2 results in J, the finest block relation that implodes $\underline{S}$.

Proof J is a block relation, since for every $g \in G$ and every $m \in M$ with $(g,m) \in J$ and $(g,m) \notin I$ holds that g^J is an intent in (G,M,I) and m^J is an extent in (G,M,I). Further, the tolerance relation θ corresponding to J has a θ-class containing $\underline{S}$ due to the initial inclusion of $(C \times B)$ in J.

Since the set of all block relations is a closure system, there is a unique finest block relation θ with the requested properties. Assume $\widetilde{J}$ with $\widetilde{J} \subset J$ to be this finest block relation so that $I \subset \widetilde{J} \subset J$. Let $(g,m) \in J$ with $(g,m) \notin \widetilde{J}$ be the first incidence that is added to J while $\widetilde{J}$ does not contain it. In each iteration of the algorithm, the unique minimal intent containing the current derivation of g is selected as the new derivation of g. Following $J \subseteq \widetilde{J}$ and therefore $\widetilde{J} = J$ holds. □

Note that in some cases, the addition of the incidences $(C \times B)$ already results in the wanted outcome. In general, this is not the case, e.g., see Example 8.2.

Example 8.2
As for the lattice $\underline{\mathfrak{B}}(\mathbb{K})$ given in Figure 8.1 (left) and the red highlighted interval $\underline{S}_1$ the corresponding formal context $\mathbb{K} = (G,M,I)$ (right) is examined to find the finest tolerance relation θ imploding $\underline{S}_1$. Since $\underline{S}_1 = [(\{a\},\{a,b,c\}),(G,\varnothing)]$ the incidence relation $\widetilde{J} = I \cup G \times \{a,b,c\}$ is generated. After this step, the conditions for block relations have to be checked iteratively. As for the attribute set, the condition holds for every attribute since $a^{\widetilde{J}} = b^{\widetilde{J}} = c^{\widetilde{J}} = \varnothing^I$, $d^J = d^I$ and $e^{\widetilde{J}} = e^I$. For the objects $1,2,\ldots,6$ also the condition holds. This is not the case for the objects 7 and 8 and the incidences $(7,e)$ and $(8,d)$ have to be added to $\widetilde{J}$. After this step, the attributes e and d have to be considered again. Thus the finest block relation J with $\widetilde{J} \subseteq J$ is $J = G \times M$. This block relation corresponds to the trivial tolerance relation.
For the blue highlighted interval $\underline{S}_1 = [(\{3\},\{b,c,d,e\}),(\{3,5,6,8\},\{e\})]$ we have $\widetilde{J} = I \cup (\{3,5,6,8\} \times \{b,c,d,e\})$. The finest block relation J with $\widetilde{J} \subseteq J$ is depicted in Figure 8.1 (right) by the additional •-incidences.

Note that in the case of imploding Boolean sublattices (or intervals generated by a Boolean suborder) the approach of filling the incidences in $(C \times B)$ results in filling at least the associated Boolean subcontext (and, therefore, a contranominal scale). We investigate the approach of directly filling contranominal scales and associated subcontexts of Boolean suborders in the following subsection.

	a	b	c	d	e	f
1	×	×	•			
2	×	•	×			
3	•	×	×			
4				×		
5					×	
6						×

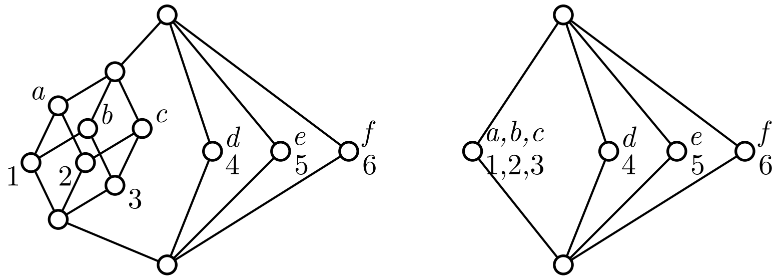
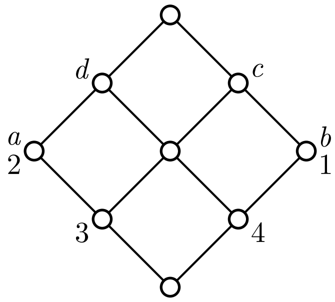


Figure 8.2 A formal context $\mathbb{K}$ (left) and its corresponding concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ (middle). The concept lattice on the right corresponds to the context that arises by adding the •-marked incidences to $\mathbb{K}$ and filling a 3-dimensional Boolean subcontext of $\mathbb{K}$. In this case a 3-dimensional Boolean suborder in $\underline{\mathfrak{B}}(\mathbb{K})$ collapses.

8.2.3 Adding Incidences to Boolean Subcontexts

In the previous section, we investigate lattice factorization using congruence relations and tolerance relations to implode given intervals (and, therefore, possibly Boolean suborders). By this, we maintain the meet- and join-operators of the original lattice. However, often the factor lattice implodes not only the chosen interval but also significant parts of the former lattice and, worst case, the whole lattice as seen in Example 8.2. Now we want to turn to another approach to preserve more elements of the original lattice while collapsing a Boolean suborder. To this end, we utilize the strong connection between Boolean substructures in a (concept) lattice and the corresponding formal context. Therefore as a first step, we consider changes in the contranominal scales as those are directly related to the Boolean substructures of the corresponding lattice. Since the objects in a contranominal scale $[H,N] \leq \mathbb{K}$ are nearly identically on the attributes in this scale, the adding of the missing incidences (meaning uniting the original incidence relation of the context $\mathbb{K}$ with $H \times N$) could be used to collapse a Boolean suborder in the corresponding lattice. If the concepts of the contranominal scale are independent of other concepts, this approach does work like pictured in Figure 8.2. However, a Boolean suborder generally corresponds not only to a single contranominal scale. On the one hand, a clarifyable object $g \in G$ may be contained in the contranominal scale. In this case, all objects with identical derivation as g must be considered. This case can be avoided by considering only clarified formal contexts. Therefore it is useful only to consider standard contexts for this approach. On the other hand, a different case independent of the clarification and reduction of the context can occur as follows. An object not contained in the considered contranominal scale $[H,N]$ may have the same derivation as an object $g \in H$ restricted to the attribute set N. This is the case if an atom of the Boolean suborder in the concept lattice is not irreducible, meaning a single irreducible object

	a	b	c	d
1	•	×	×	
2	×	•		×
3	×		×	×
4		×	×	×



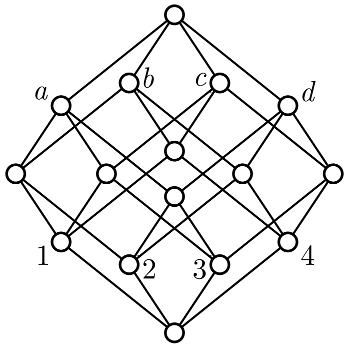


Figure 8.3 A formal context $\mathbb{K}$ (left) and its corresponding concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ (middle). The concept lattice on the right corresponds to the context that arises by adding the •-marked incidences to $\mathbb{K}$ and filling a 2-dimensional Boolean subcontext of $\mathbb{K}$. In this case, the size of the concept lattice expands by filling a contranominal scale.

can not generate it. Both cases do also hold for attributes dually. To avoid this problem, it is necessary not only to consider a single contranominal scale but a complete Boolean subcontext $\underline{S} = [H, N] \leq \mathbb{K}$ that is associated to the previously selected Boolean suborder $\underline{S}$ in the corresponding concept lattice. Then the incidence relation of the original formal context $\mathbb{K}$ can be united with $H \times N$.

However, by only considering a random contranominal scale or a single associated Boolean subcontext, additional structures in the concept lattice can arise, as seen in Figure 8.3. In this case, the filling of a 2-dimensional contranominal scale (that is also the associated Boolean subcontext of a two-dimensional Boolean suborder) constructs a new 4-dimensional contranominal scale, and therefore the size of the concept lattices increases.

To collapse a Boolean substructure in a lattice by adding incidences, those must be chosen beyond the associated Boolean subcontext. Therefore, in the following, we again focus on the lattice side and investigate an appropriate relation to transfer this approach afterwords to the context side.

8.3 Interval Factorization of Lattices

Our goal in this section is to generate, from a set of intervals $\underline{S}_1, \ldots, \underline{S}_k \leq \underline{L}$, a factorization $\underline{L}/\theta$ that can be obtained by an implosion f of the intervals such that f is injective on $\underline{L} \setminus \bigcup_{i=1}^{k} \underline{S}_i$ (i.e., $|f(\underline{L} \setminus \bigcup_{i=1}^{k} \underline{S}_i)| = |\underline{L} \setminus \bigcup_{i=1}^{k} \underline{S}_i|$ holds) and $f(S_i) \neq f(S_j)$ for $i, j \in \{1, \ldots, k\}$ with $i \neq j$. To this end, we present the *interval relation* θ on $\underline{L}$, which enables us to generate factor sets that implode exactly the chosen intervals. By restricting the interval relations to *pure intervals* (see Section 8.3.3) the generation of lattices is guaranteed.

8.3.1 Interval Relations

To overcome the problem of imploding more than the selected interval, we now introduce a new equivalence relation.

Definition 8.2 (Interval Relation)
Let $\underline{L}$ be anordered set and $\{\underline{S}_1, \underline{S}_2, \ldots, \underline{S}_k\}$ a set of pairwise disjoint intervals of $\underline{L}$. We call the equivalence relation

$$\theta_{\underline{S}_1, \underline{S}_2, \ldots, \underline{S}_k} \coloneqq \bigcup_{i=1}^{k} S_i \times S_i \cup \{(x,x) \mid x \in \underline{L}\}$$

an *interval relation on* $\underline{L}$. If $k = 1$ holds, we call $\theta = \theta_{\underline{S}_1}$ *1-generated.* For an interval relation $\theta \coloneqq \theta_{\underline{S}_1, \underline{S}_2, \ldots, \underline{S}_k}$ we denote the factor set with $\underline{L}/\theta \coloneqq \{[x]\theta \mid x \in \underline{L}\}$ and the equivalence classes of the interval relation by $[x]\theta \coloneqq \{y \in \underline{L} \mid x\theta y\}$.

Note that θ truly is an equivalence relation since the reflexivity is provided by $\{(x,x) \mid x \in \underline{L}\}$, the symmetry is given by using only the $\times$ operator, and the transitivity is based on $\underline{S}_1, \underline{S}_2, \ldots, \underline{S}_k$ being disjoint intervals.

Since each equivalence class is an interval, it includes its supremum and its infimum. We denote them by $x^\theta \coloneqq \bigvee [x]\theta$ and $x_\theta \coloneqq \bigwedge [x]\theta$, respectively. We also use the notations $[\underline{S}]_\theta$ and $[\underline{S}]^\theta$ for the infimum and supremum of the equivalence class $\{x \in \underline{L} \mid x \in \underline{S}\}$ that is generated by $\underline{S}$.

Note that every congruence relation on a complete lattice $\underline{L}$ is an interval relation as well. More precisely, it is a special case of the *lattice-generating interval relations* that are defined in Definition 8.8.

In the case of $\underline{L}$ being a lattice, the incidence relations can be characterized in an additional way:

Proposition 8.4
Let θ be an equivalence relation on lattice $\underline{L}$. The following statements are equivalent:

a) θ is an interval relation.

b) The two following conditions hold for all $x_1, x_2, y_1, y_2 \in \underline{L}$:

i) $x_1 \theta x_2 \Rightarrow (x_1 \vee x_2)\theta x_1$ and $(x_1 \wedge x_2)\theta x_1$

ii) $x_1 \theta x_2$, $y_1 \theta y_2$, $(x_1, y_1) \notin \theta$, and $x_1 > y_1 \Rightarrow x_2 \nless y_2$

Proof $a) \Rightarrow b)$: Let $\theta = \theta_{\underline{S}_1, \underline{S}_2, \ldots, \underline{S}_k}$ be an interval relation on $\underline{L}$. If $x_1 \theta x_2$ then $x_1, x_2 \in \underline{S}_i$ for some i and therefore $(x_1 \vee x_2), (x_1 \wedge x_2) \in \underline{S}_i$ hold. Therefore i) holds.

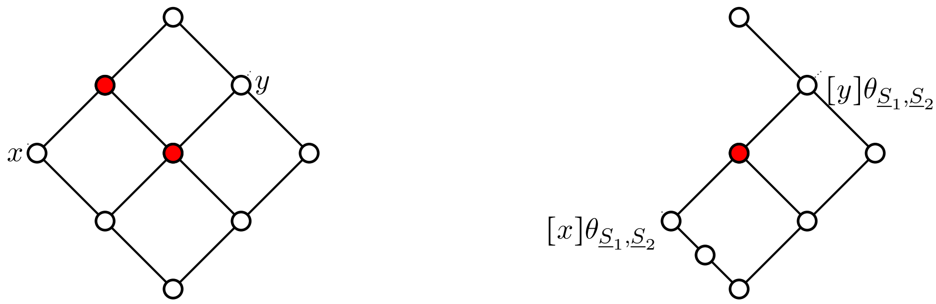


Figure 8.4 Example of a lattice $\underline{L}$ (left) with a red highlighted interval $\underline{S} \leq \underline{L}$ and two elements x, y that are not comparable. In the factor set $\underline{L}/\theta_{\underline{S}}$ (right) the equivalence class corresponding to $\underline{S}$ is highlighted in red. Now the elements $[x]\theta_{\underline{S}}$ and $[y]\theta_{\underline{S}}$ are comparable.

Let $x_1, x_2 \in \underline{S}_i$ and $y_1, y_2 \in \underline{S}_j$ with $\underline{S}_i \neq \underline{S}_j$ and $x_1 > y_1$. Assumed $x_2 < y_2$. If $x_1 \leq x_2$ then $y_1 < x_1 \leq x_2 < y_2$ and therefore $\underline{S}_i = \underline{S}_j$ holds. $\lightning$

In the case of $x_1 > x_2$ and $y_1 \geq y_2$ we have $x_2 < y_2 \leq y_1 < x_1$ also $\underline{S}_i = \underline{S}_j$. $\lightning$

If $x_1 > x_2$ and $y_1 < y_2$ then y_2 and x_1 are both upper bounds of y_1 and x_2. Since $\underline{L}$ is a lattice we have $y_1 \vee x_2 \leq y_2$ and $y_1 \vee x_2 \leq x_1$ and therefore $y_1 \vee x_2 \in \underline{S}_i$ and $y_1 \vee x_2 \in \underline{S}_j$. $\lightning$

In all cases, this is a contradiction to the assumptions. Hence ii) holds.
b) $\Rightarrow$ *a)*: Let θ be an equivalence relation on $\underline{L}$ as in b). For an arbitrary equivalence class $[x]\theta$ the supremum x^θ and the infimum x_θ exist in $[x]\theta$ because of i). Now let $y \in [x_\theta, x^\theta]$ with $y \notin [x]\theta$. Then $y \in [x_\theta, x^\theta] \Rightarrow y \leq x^\theta, y \geq x_\theta$. This is a contradiction to ii). So the equivalence classes of θ are intervals. □

Defining $\leq_\theta$ for 1-generated interval relations

Since implosions are defined as order-preserving maps, on the factor set $\underline{L}/\theta$ the question remains which order the factorization should impose. To answer this we now define the relation $\leq_\theta$ on the factor set $\underline{L}/\theta$. We start with the case of a 1-generated interval relation $\theta = \theta_{\underline{S}}$ as presented in Definition 8.4. Our construction is motivated by the aim to preserve all comparabilities of $\underline{L}$ in $\underline{L}/\theta$. So, for an element $x \in \underline{L}$ that is smaller than at least one element of $\underline{S}$, it should also $[x]\theta \leq_\theta [\underline{S}]\theta$ hold. Dually, for an element $y \in \underline{L}$ that is larger than any element of $\underline{S}$ should also $[\underline{S}]\theta \leq_\theta [y]\theta$ hold. Therefore also the elements $[x]\theta$ and $[y]\theta$ should become comparable in $\underline{L}/\theta$. This is illustrated in Figure 8.4.

To define the order $\leq_\theta$ with the required properties, we first have to define some areas of an ordered set generated by an interval.

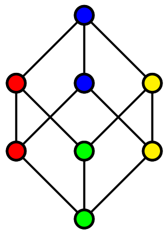


Figure 8.5 For the interval $\underline{S}$ (red), the sets $S^{\mapsup}$ (blue), $S^{\mapsdown}$ (green) and $S^{\parallel}$ (yellow) are highlighted. In this example, all four sets are intervals. They are not pure but build up a lattice-generating interval relation.

Definition 8.3 ($S^{\mapsup}$, $S^{\mapsdown}$, $S^{\parallel}$)
Let $\underline{L}$ be an ordered set and $\underline{S}$ an interval of $\underline{L}$. We define the sets

$$\begin{aligned} S^{\mapsup} &:= \{x \in \underline{L} \setminus \underline{S} \mid \exists y \in \underline{S} : y < x\}, \\ S^{\mapsdown} &:= \{x \in \underline{L} \setminus \underline{S} \mid \exists y \in \underline{S} : x < y\} \text{ and} \\ S^{\parallel} &:= \{x \in \underline{L} \setminus \underline{S} \mid \nexists y \in \underline{S} : y < x \text{ or } x < y\}. \end{aligned}$$

Proposition 8.5
Let $\underline{L}$ be an ordered set and let $\underline{S}$ be an interval of $\underline{L}$. Then S, $S^{\mapsup}$, $S^{\mapsdown}$ and $S^{\parallel}$ are pairwise disjoint and together cover $\underline{L}$. In other words, $\{S, S^{\mapsup}, S^{\mapsdown}, S^{\parallel}\}$ is a partition of the elements in $\underline{L}$, with possibly empty classes.

Proof Let $\underline{S} = [u, v]$. Then $S^{\mapsup} = [u) \setminus S$ and $S^{\mapsdown} = (v] \setminus S$. Since $[u, v]$ is an interval $S^{\mapsup} \cap S^{\mapsdown} = \varnothing$. Also $S^{\parallel} = L \setminus (S \cup S^{\mapsup} \cup S^{\mapsdown})$ holds. □

An example for this division of the lattice elements can be seen in Figure 8.5.

Definition 8.4 ($\leq_\theta$ for 1-generated interval relations)
Let $\underline{L}$ be an ordered set and $\underline{S} \leq \underline{L}$ an interval. On $\underline{L}/\theta$ we define the relation $[x]\theta \leq_\theta [y]\theta :\Leftrightarrow (x_\theta \leq y^\theta$ or $x \in S^{\mapsdown}, y \in S^{\mapsup})$.

Example 8.3
Utilizing congruence or tolerance relations to implode the red highlighted interval in Figure 8.1, the trivial factor lattice, consisting of a single element, arose. With the new construction, we preserve everything outside of the interval: In Figure 8.6 the original lattice $\underline{\mathfrak{B}}(\mathbb{K})$ and the factor set $\underline{\mathfrak{B}}(\mathbb{K})/\theta_{\underline{S}}$ for the interval relation $\theta_{\underline{S}}$ are depicted. In $\underline{\mathfrak{B}}(\mathbb{K})/\theta_{\underline{S}}$ the interval $\underline{S}$ implodes, but the rest of the lattice is preserved. On the other hand, not all joins and meets are preserved.

Following, we take a look into the order properties of the factor set with the previously defined relation. For a lattice $\underline{L}$, in general, $\underline{L}/\theta$ is neither a lattice (see Figure 8.7) nor even an ordered set (see Figure 8.8). However, considering only one interval with size larger one, $\underline{L}/\theta$ is always an ordered set:

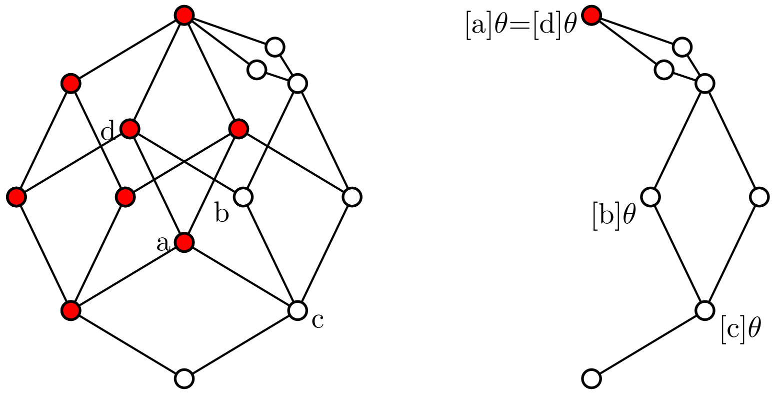


Figure 8.6 A (concept) lattice $\underline{\mathfrak{B}}(\mathbb{K})$ with a pure interval $\underline{S}$ (highlighted red) on the left. The lattice $\underline{\mathfrak{B}}(\mathbb{K})/\theta_{\underline{S}}$ is pictured on the right. Some elements of the lattices are labeled.

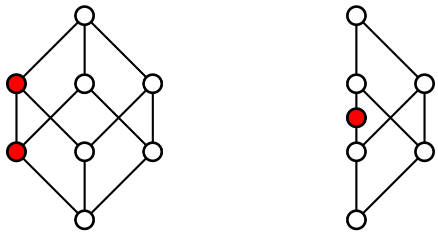


Figure 8.7 A lattice $\underline{L}$ with a red highlighted interval $\underline{S} \leq \underline{L}$ (left). The factor set $\underline{L}/\theta_{\underline{S}}$ (right) is no lattice but still an ordered set. The equivalence class $[\underline{S}]\theta$ is highlighted in red.

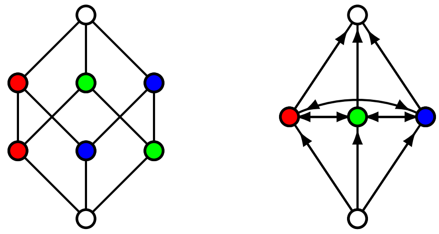


Figure 8.8 "Penrose crown": a lattice $\underline{L}$ with three pairwise comparable intervals $\underline{S}_1$(red), $\underline{S}_2$(green), $\underline{S}_3$(blue)$\leq \underline{L}$ (left). The factor set $\underline{L}/\theta_{\underline{S}_1,\underline{S}_2,\underline{S}_3}$ (right) is not even an ordered set. It holds $[\underline{S}_1]\theta \leq [\underline{S}_2]\theta \leq [\underline{S}_3]\theta \leq [\underline{S}_1]\theta$. Thus $\leq_\theta$ is not anti-symmetric and, therefore no order relation.

Proposition 8.6

Let $\theta = \theta_{\underline{S}}$ be an interval relation on the ordered set $\underline{L}$. Then $\leq_\theta$ is an order on $\underline{L}/\theta$.

Proof Reflexivity: $\leq$ is an order on $\underline{L}$, meaning $x \leq x$ for all $x \in \underline{L}$ and especially $x_\theta \leq x \leq x^\theta$. Therefore $[x]\theta \leq_\theta [x]\theta$ holds in $\underline{L}/\theta$.

Transitivity: Let $[x]\theta \leq_\theta [y]\theta$ and $[y]\theta \leq_\theta [z]\theta$ in $\underline{L}/\theta$. In the case that $x_\theta \leq y^\theta$ and $y_\theta \leq z^\theta$ in $\underline{L}$ either $y_\theta = y^\theta$ and therefore $x_\theta \leq z^\theta$. Otherwise, in the case of $y_\theta \neq y^\theta$ we have $y \in \underline{S}$, $x \in S^{\downarrow} \cup S$ and $z \in S^{\uparrow} \cup S$ and then $[x]\theta \leq_\theta [z]\theta$. In the case $x \in S^{\downarrow}, y \in S^{\uparrow}$ and $y_\theta \leq z^\theta$ holds $z \in S^{\uparrow}$ and therefore $[x]\theta \leq_\theta [z]\theta$. The case $y \in S^{\downarrow}, z \in S^{\uparrow}$ and $x_\theta \leq y^\theta$ is analogous.

Anti-symmetry: Let $[x]\theta \leq_\theta [y]\theta$ and $[y]\theta \leq_\theta [x]\theta$. If $x_\theta \leq y^\theta$ and $y_\theta \leq x^\theta$, follows $[x]\theta = [y]\theta$ directly since either $x_\theta = x^\theta$ or $x_\theta = x^\theta$ or $x_\theta = y_\theta$ and $x^\theta = y^\theta$. The cases $x \in S^{\downarrow}, y \in S^{\uparrow}$, $y_\theta \leq x^\theta$ and $y \in S^{\downarrow}, x \in S^{\uparrow}$, $x_\theta \leq y^\theta$ can not occur as well as the case of $x \in S^{\downarrow}, y \in S^{\uparrow}$, $y \in S^{\downarrow}, x \in S^{\uparrow}$. □

Proposition 8.7
Let $\underline{L}$ be an ordered set and $\theta = \theta_{\underline{S}}$ an interval relation on $\underline{L}$. Then $\leq_\theta$ is the smallest order on $\underline{L}/\theta$ so that the map $\varphi: \underline{L} \to \underline{L}/\theta,\ x \mapsto [x]\theta$ is surjective and order-preserving.

Proof The surjectivity follows directly from the fact that for every $[x]\theta \in \underline{L}/\theta$ we have $x \in \underline{L}$ as representative. The order is preserved due to the definition of $\leq_\theta$: If $x \leq y$ in $\underline{L}$ holds, we have $x_\theta \leq y^\theta$ and therefore $[x]\theta \leq_\theta [y]\theta$ in $\underline{L}/\theta$.

We show that $\leq_\theta$ is the smallest of those relations by contraposition: Let R be an order on $\underline{L}/\theta$ so that φ is surjective and order-preserving. Assumed there are $x, y \in \underline{L}$ with $[x]\theta \leq_\theta [y]\theta$ and $[x]\theta \not R [y]\theta$. Since φ is order-preserving we know that $x \nleq y$ in $\underline{L}$ and moreover no element of the θ-class $[x]\theta$ is less or equal an element of the θ-class $[y]\theta$. In particular, $x_\theta \nleq y^\theta$ holds in $\underline{L}$. Thus, $x \in S^{\updownarrow}$ and $y \in S^{\updownarrow}$ due to the definition of $\leq_\theta$. Therefore, we have $x_\theta \leq [\underline{S}]^\theta$ and $[\underline{S}]_\theta \leq y^\theta$ and due to φ being order-preserving also $[x]\theta R [\underline{S}]\theta$ and $[\underline{S}]\theta R [y]\theta$. Consequently, $[x]\theta R [y]\theta$ follows from the transitivity of the order R. ↯ □

Defining $\leq_\theta$ in general

Considering more than one interval with size larger one, we define the relation $\leq_\theta$ on $\underline{L}/\theta$, by generalizing Definition 8.4 as follows:

Definition 8.5 ($\leq_\theta$ for general interval relations)
Let $\underline{L}$ be an ordered set and $\theta \coloneqq \theta_{\underline{S}_1,\ldots,\underline{S}_k}$ an interval relation on $\underline{L}$. We define the relation $[x]\theta \leq_\theta [y]\theta :\Leftrightarrow (x_\theta \leq y^\theta$ or $\exists i_1, \ldots, i_l \in \{1, \ldots, k\}$ with $x_\theta \in S_{i_1}^{\updownarrow}$, $[\underline{S}_{i_1}]_\theta \in S_{i_2}^{\updownarrow}$, $\ldots$, $[\underline{S}_{i_{l-1}}]_\theta \in S_{i_l}^{\updownarrow}$, $y^\theta \in S_{i_l}^{\updownarrow}$).

The relation is illustrated in Figure 8.9.

Note that $\leq_\theta$ for one interval in Definition 8.4 is a special case of $\leq_\theta$ in Definition 8.5. Therefore we use the notion $\leq_\theta$ in the following both for imploding one as well as multiple intervals.

We can show that considering several intervals, $\leq_\theta$ is always a preorder on $\underline{L}/\theta$, i.e., it is always reflexive and symmetric but ot necessarily anti-symmetric:

Proposition 8.8
Let $\underline{L}$ be an ordered set and $\theta = \theta_{\underline{S}_1,\ldots,\underline{S}_k}$ an interval relation on $\underline{L}$. Then $\leq_\theta$ is a preorder on $\underline{L}/\theta$.

Proof Reflexivity: $\leq$ is an order on $\underline{L}$, meaning $x \leq x$ for all $x \in \underline{L}$ and especially $x_\theta \leq x \leq x^\theta$. Therefore $[x]\theta \leq_\theta [x]\theta$ holds in $\underline{L}/\theta$.

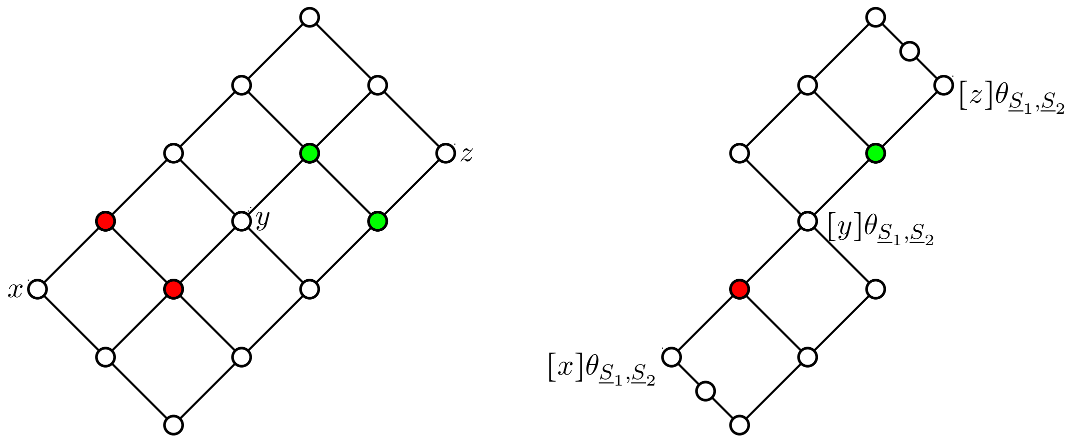


Figure 8.9 In the lattice $\underline{L}$ (left) – with intervals $\underline{S}_1, \underline{S}_2 \leq \underline{L}$ highlighted in red and green, respectively – the three elements x, y, z that are not comparable. In the factor set $\underline{L}/\theta_{\underline{S}_1,\underline{S}_2}$ (right) the equivalence classes corresponding to $\underline{S}_1$ and $\underline{S}_2$ are highlighted red and green, respectively. Now the elements $[x]\theta_{\underline{S}_1,\underline{S}_2}, [y]\theta_{\underline{S}_1,\underline{S}_2}$ and $[z]\theta_{\underline{S}_1,\underline{S}_2}$ are comparable.

Transitivity: Let $[x]\theta \leq_\theta [y]\theta$ and $[y]\theta \leq_\theta [z]\theta$ in $\underline{L}/\theta$. If $[x]\theta = [y]\theta$ or $[y]\theta = [z]\theta$ the statement is similar to the proof of Proposition 8.6. Assume $[x]\theta \neq [y]\theta \neq [z]\theta$. If $x_\theta \leq y^\theta$ and $y_\theta \leq z^\theta$ in $\underline{L}$ then either $y^\theta = y_\theta$ and therefore $x_\theta \leq z^\theta$ or there is an interval $\underline{S}_i$ with $y \in \underline{S}_i$. In this case $x_\theta \in \underline{S}_i^{\downarrow}$ and $z^\theta \in \underline{S}_i^{\uparrow}$ hold. In both cases follows $[x]\theta \leq_\theta [z]\theta$. If $x_\theta \leq y^\theta$ and $\exists \underline{S}_{i_1}, \dots, \underline{S}_{i_l}$ as described with $y_\theta \in \underline{S}_{i_1}^{\downarrow}$ and $z^\theta \in \underline{S}_{i_l}^{\uparrow}$ then $x_\theta \in \underline{S}^{\downarrow}$ and $[\underline{S}]_\theta \in \underline{S}_{i_1}^{\downarrow}$ for the interval $\underline{S} = ([y]\theta, \leq)$. Then $[x]\theta \leq_\theta [z]\theta$. The case of $y_\theta \leq z^\theta$ and $\exists \underline{S}_{i_1}, \dots, \underline{S}_{i_l}$ as described with $x_\theta \in \underline{S}_{i_1}^{\downarrow}$ and $y^\theta \in \underline{S}_{i_l}^{\uparrow}$) follows analogously. If $\exists \underline{S}_{i_1}, \dots, \underline{S}_{i_l}$ and $\underline{S}_{j_1}, \dots, \underline{S}_{j_l}$ as described with $x_\theta \in \underline{S}_{i_1}^{\downarrow}$, $y^\theta \in \underline{S}_{i_l}^{\uparrow}$), $y_\theta \in \underline{S}_{j_1}^{\downarrow}$ and $z^\theta \in \underline{S}_{j_l}^{\uparrow}$) there is an interval $\underline{S} = ([y]\theta, \leq)$ so that $[\underline{S}_{i_l}]_\theta \in \underline{S}^{\downarrow}$ and $[\underline{S}]_\theta \in \underline{S}_{j_1}^{\downarrow}$. Then $[x]\theta \leq_\theta [z]\theta$. □

Note that $\leq_\theta$ as given in Definition 8.5 is the same relation as $[x]\theta \leq_\theta [y]\theta :\Leftrightarrow (x_\theta \leq y^\theta$ or $\exists i_1, \dots, i_l \in \{1, \dots, k\}$ with $x_\theta \in \underline{S}_{i_1}^{\downarrow}$, $[\underline{S}_{i_2}]_\theta \in \underline{S}_{i_1}^{\uparrow}$, ... , $[\underline{S}_{i_l}]_\theta \in \underline{S}_{i_l-1}^{\uparrow}$, $y^\theta \in \underline{S}_{i_l}^{\uparrow}$).

We observe that $\leq_\theta$ is the transitive closure of a simpler relation as follows:

Proposition 8.9

Let $\underline{L}$ be an ordered set and θ an interval relation on $\underline{L}$. Let $[x]\theta \leq^* [y]\theta :\Leftrightarrow x_\theta \leq y^\theta$ be a relation on $\underline{L}/\theta$. Then $\leq_\theta$ is the transitive closure of $\leq^*$.

Proof Let $x, y \in \underline{L}$ and $\underline{S}_i$ an interval of θ. Then $x_\theta \in S_i^{\downarrow}$ is equivalent to $x_\theta \leq [S_i]^\theta$. Further, $y^\theta \in S_i^{\uparrow}$ is equivalent to $y^\theta \leq [S_i]_\theta$. Since $[S_i]_\theta \leq [S_i]^\theta$, the statement follows directly. □

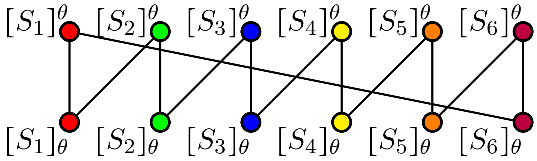


Figure 8.10 Penrose crown of order 6.

8.3.2 Order-preserving Interval Relations

Up to now, we have only shown (in Proposition 8.8) that $\leq_\theta$ is a preorder and Figure 8.8 showed that it will not always be anti-symmetric. We now investigate the order properties of $\leq_\theta$ in more detail:

Definition 8.6 (Order-preserving interval Relation)
Let $\underline{L}$ be an ordered set and θ an interval relation on $\underline{L}$. We call θ *order-preserving* on $\underline{L}$ if $(\underline{L}/\theta, \leq_\theta)$ is an ordered set.

Considering a 1-generated interval relation, from Proposition 8.8 follows directly:

Corollary 8.1
Let $\underline{L}$ be an ordered set and θ an interval relation on $\underline{L}$. If θ is 1-generated, θ is order-preserving.

For more than one interval, we can provide a necessary and sufficient condition for an interval relation θ to be order-preserving in Theorem 8.1.

Definition 8.7 (Penrose crown)
Let $\underline{L}$ be an ordered set and $2 \leq k$. A set $\{\underline{S}_1, \dots, \underline{S}_k\}$ of intervals in $\underline{L}$ are called *Penrose crown of order k* in $\underline{L}$ if they are pairwise disjoint and if $[\underline{S}_1]_\theta \in \underline{S}_2^\curlyvee$, $[\underline{S}_2]_\theta \in \underline{S}_3^\curlyvee$, $\dots$, $[\underline{S}_{k-1}]_\theta \in \underline{S}_k^\curlyvee$, $[\underline{S}_k]_\theta \in \underline{S}_1^\curlyvee$.

We call such a constellation of intervals a Penrose crown, named after the "impossible staircase" created by L. Penrose and R. Penrose in 1958 (and previously by O. Reutersvärd in 1937) [49]. The construction became popular by M.C. Escher's lithograph "Ascending ans Descending". The intervals in the lattice in Figure 8.8 form a Penrose crown of order 3. Another example is illustrated in Figure 8.10.

Theorem 8.1
Let $\underline{L}$ be a finite lattice and $\theta = \theta_{\underline{S}_1,\dots,\underline{S}_k}$ an interval relation on $\underline{L}$. θ is order-preserving if and only if there exists no Penrose crown $\{\underline{S}_{i_1}, \dots, \underline{S}_{i_l}\}$ in $\underline{L}$ with $i_1, \dots, i_l \in \{1, \dots, k\}$.

Proof "$\Rightarrow$": Assumed some intervals $\underline{S}_1, \dots, \underline{S}_l \leq \underline{L}$ exist as described. Then $[\underline{S}_1]\theta \leq_\theta [\underline{S}_l]\theta$ and $[\underline{S}_l]\theta \leq_\theta [\underline{S}_1]\theta$ by definition of $\leq_\theta$. Since $[\underline{S}_1] \neq [\underline{S}_l]$ the preorder $\leq_\theta$ is not anti-symmetric and therefore not an order.

"⇐": Assume the relation $\leq_\theta$ is not an order. Then there are two equivalence classes $[S_i]\theta$ and $[S_j]\theta$ with $[S_i]\theta \leq_\theta [S_j]\theta$, $[S_j]\theta \leq_\theta [S_i]\theta$ and $[S_i]\theta \neq [S_j]\theta$ in $\underline{L}/\theta$. For two intervals $\underline{S}, \underline{T}$ with $[S]\theta \neq [T]\theta$ it holds that $[S]_\theta \leq [T]^\theta \Rightarrow [S]_\theta \in \underline{T}^\curlyvee$. Due to the definition of $\leq_\theta$, one of the following cases has to occur. In the case of $[\underline{S}_i]_\theta \leq [\underline{S}_j]^\theta$ and $[\underline{S}_j]_\theta \leq [\underline{S}_i]^\theta$ $\{\underline{S}_i, \underline{S}_j\}$ is a Penrose crown of order 2. In the case of $[S_i]_\theta \leq [S_j]^\theta$ and $[S_j]_\theta \nleq [S_i]^\theta$ we have $[S_j]_\theta \in \underline{S}_1^\curlyvee$, $[\underline{S}_1]_\theta \in \underline{S}_2^\curlyvee$, $\ldots$, $[\underline{S}_l]_\theta \in [S_i]\theta^\curlyvee$ because of $[S_j]\theta \leq_\theta [S_i]\theta$. The case of $[S_j]_\theta \leq [S_i]^\theta$ and $[S_i]_\theta \nleq [S_j]^\theta$ follow analogously. In the case of $[S_j]_\theta \nleq [S_i]^\theta$ and $[S_i]_\theta \nleq [S_j]^\theta$ we have $[S_j]_\theta \in \underline{S}_1^\curlyvee$, $[\underline{S}_1]_\theta \in \underline{S}_2^\curlyvee$, $\ldots$, $[\underline{S}_l]_\theta \in [S_i]\theta^\curlyvee$ and $[S_i]_\theta \in \underline{S}_m^\curlyvee$, $[\underline{S}_m]_\theta \in \underline{S}_{m+1}^\curlyvee$, $\ldots$, $[\underline{S}_s]_\theta \in [S_j]\theta^\curlyvee$. □

Since a Penrose crown consists of at least two intervals, an interval relation is always order-preserving if at most one of its intervals consists of more than one element:

Proposition 8.10
Let $\underline{L}$ be an ordered set and $\theta = \theta_{\underline{S}_1,\ldots,\underline{S}_k}$ an interval relation on $\underline{L}$. If $|[\underline{S}_i]\theta| \geq 2$ for at most one $i \in \{1,\ldots,k\}$, θ is an order-preserving interval relation.

Proof If θ includes no interval of size 2 or larger, $\underline{L}/\theta = \underline{L}$. If $\theta = \theta_{\underline{S}_1}$ with $|\underline{S}_1| \geq 2$ the statement follows from Proposition 8.6. □

In the case of $\underline{L}$ being a lattice, such a constellation can not occur if at most two intervals of the interval relation θ include more than a single element of $\underline{L}$:

Proposition 8.11
Let $\underline{L}$ be a finite lattice and $\theta = \theta_{\underline{S}_1,\ldots,\underline{S}_k}$ an interval relation on $\underline{L}$. If $|[\underline{S}_i]\theta| \geq 2$ for at most two $i \in \{1,\ldots,k\}$, θ is an order-preserving interval relation.

Proof If θ includes no or one interval of size 2 or larger, the proof is the same as in Proposition 8.10. Let $\theta = \theta_{\underline{S}_1,\underline{S}_2}$ with $|\underline{S}_1|, |\underline{S}_2| \geq 2$. Assume that $[\underline{S}_1]_\theta \in S_2^\curlyvee$ and $[\underline{S}_2]_\theta \in S_1^\curlyvee$. Then $[\underline{S}_1]_\theta \vee [\underline{S}_2]_\theta \in \underline{S}_1$ and $[\underline{S}_1]_\theta \vee [\underline{S}_2]_\theta \in \underline{S}_2$. Hence, the intervals $\underline{S}_1$ $\underline{S}_2$ are not disjoint. This is a contradiction to θ being an interval relation. □

Moreover, the case mentioned in Theorem 8.1 can only occur if $\underline{L}$ contains a Penrose crown of order $l \geq 2$. If $\underline{L}$ is a lattice, it has to contain a Penrose crown of order $l \geq 3$ and therefore a crown of the same order as suborder.

Corollary 8.2
Let $\underline{L}$ be an ordered set and θ an interval relation on $\underline{L}$. If $\underline{L}$ does not contain a Penrose crown of order $l \geq 2$, then θ is order-preserving.

Corollary 8.3
Let $\underline{L}$ be a lattice and θ an interval relation on $\underline{L}$. If $\underline{L}$ does not contain a crown of order $l \geq 3$ as a suborder, then θ is order-preserving.

Since a dismantlable lattice $\underline{L}$ – meaning the iterative elimination of all doubly irreducible elements results in the elimination of the whole lattice – never contains a crown [37], every interval relation on such a lattice is order-preserving.

Corollary 8.4
Let $\underline{L}$ be an ordered set and θ an interval relation on $\underline{L}$. If $\underline{L}$ is planar than θ is order-preserving.

As shown, for an order-preserving interval relation $\theta_{\underline{S}_1,\dots,\underline{S}_k}$, we have an implosion of the intervals $\underline{S}_1,\dots,\underline{S}_k$ as defined in Definition 8.1. In the following section we investigate the preservation of the lattice properties.

8.3.3 Lattice-generating Interval Relations

So far, we have been interested in interval relations with factor sets that are ordered sets. Now we focus on interval relations where the resulting factor set is even a lattice. Therefore we restrict us to the case where $\underline{L}$ is a (finite) lattice.

Definition 8.8 (Lattice-generating Interval Relation)
Let $\underline{L}$ be a lattice and θ an interval relation on $\underline{L}$. We call θ a *lattice-generating interval relation* on $\underline{L}$ if $(\underline{L}/\theta, \leq_\theta)$ is a lattice.

Note that every congruence relation is a lattice-generating interval relation since the equivalence classes form pairwise disjoint intervals and the order (denoted by $\leq_c$ in the following lemma) which is defined on the factor lattice $\underline{L}/\theta$ for a congruence relations θ is equal to $\leq_\theta$:

Proposition 8.12
Let $\underline{L}$ be a complete lattice and θ a complete congruence relation on $\underline{L}$. Then the orders $\leq_\theta$ and $[x]\theta \leq_c [y]\theta :\Leftrightarrow x\theta(x \wedge y)$ are identical on $\underline{L}/\theta$.

Proof "$\Leftarrow$": Let $[x]\theta \leq_c [y]\theta$ and therefore $(x \wedge y) \in [x]\theta$. Since $x \wedge y \leq y$ holds, we have $x_\theta \leq (x \wedge y) \leq y \leq y^\theta$ and consequently $[x]\theta \leq_\theta [y]\theta$.

"$\Rightarrow$": Let $[x]\theta \leq_\theta [y]\theta$. In the case of $x_\theta \leq y^\theta$ we have $x\theta x_\theta$ and $y\theta y^\theta$ and therefore (due to the definition of congruence relations) $(x \wedge y)\theta(x_\theta \wedge y^\theta) = x_\theta$. Then $[x]\theta \leq_c [y]\theta$ holds.

If $x_\theta \nleq y^\theta$ then there are intervals $\underline{S}_1,\dots,\underline{S}_k$ in θ with $x_\theta \leq [\underline{S}_1]^\theta$, $[\underline{S}_1]_\theta \leq [\underline{S}_2]^\theta$, $\dots$, $[\underline{S}_k]_\theta \leq y^\theta$. Then we have $[x]\theta \leq_c [\underline{S}_1]\theta \leq_c \cdots \leq_c [y]\theta$. □

We will now provide a characterization of lattice-generating interval relations.

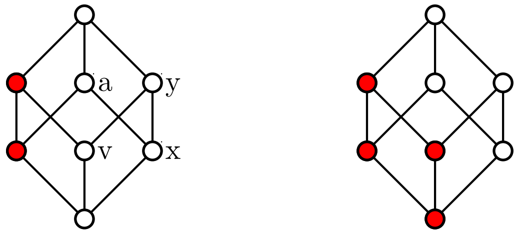


Figure 8.11 On the left, a nested interval $\underline{S}$ of the lattice $\underline{L}$ is highlighted in red. $\underline{S}$ is part of a Penrose crows of order 3 with the Intervals $[x, a]$ and $[v, y]$. As presented in Figure 8.7 $\underline{L}/\theta_{\underline{S}}$ is no lattice. On the right, a pure interval is highlighted in red.

Definition 8.9 (Nested Interval, Pure Interval)
Let $\underline{L}$ be a lattice and $\underline{S} \leq \underline{L}$ an interval. We call $\underline{S}$ a *nested interval* of $\underline{L}$ if there are two intervals $\underline{T}, \underline{U} \leq \underline{L}$ so that $\underline{S}, \underline{T}, \underline{U}$ are a Penrose crown of order 3 in $\underline{L}$. We call $\underline{S}$ a *pure interval* of $\underline{L}$ if it is not nested.

Corollary 8.5
Let $\underline{L}$ be a lattice and $\underline{S} \leq \underline{L}$ an interval. $\underline{S}$ is nested if and only if there are $x, y \in S^{\parallel}, a \in S^{\updownarrow}$ and $v \in S^{\downarrow}$ with $y = x \vee v, x = y \wedge a, y \nleq a$ and $v \nleq x$.

An example of a lattice with a nested and a pure interval is given in Figure 8.11. Also, both highlighted intervals in the lattice in Figure 8.1 (and therefore the one in Figure 8.6) are pure. In the 1-generated case, the pure intervals are exactly the lattice-generating intervals:

Proposition 8.13
Let $\theta = \theta_{\underline{S}}$ be a 1-generated interval relation on lattice $\underline{L}$. Then θ is lattice-generating if and only if $\underline{S}$ is pure.

Proof "$\Rightarrow$": We show the contraposition: Let $\underline{S}$ be a nested interval, meaning $\exists x, y \in S^{\parallel}, a \in S^{\uparrow}$ and $v \in S^{\downarrow}$ with $y = x \vee v, x = y \wedge a, y \nleq a$ and $v \nleq x$. For all elements $d \in S^{\downarrow}, e \in S^{\uparrow}$ holds $[d]\theta <_{\theta} [e]\theta$ in $\underline{L}/\theta_{\underline{S}}$. It follows that $[v]\theta <_{\theta} [a]\theta$ in $\underline{L}/\theta_{\underline{S}}$ and $\exists [c]\theta \in S^{\uparrow}$ with $[v]\theta <_{\theta} [c]\theta \leq_{\theta} [a]\theta$ and $[x]\theta <_{\theta} [c]\theta$. So $[y]\theta$ and $[c]\theta$ are two different minimal upper bounds of $[v]\theta$ and $[x]\theta$. Thus $\underline{L}/\theta_{\underline{S}}$ is not a lattice.

"$\Leftarrow$": We show the contraposition: $\underline{L}/\theta$ is an ordered set by Proposition 8.6. Suppose $\underline{L}\theta$ is not a lattice. Then exist $[x]\theta, [v]\theta$ in $\underline{L}/\theta$ with two smallest upper bounds or two greatest lower bounds. Due to the duality of lattices, we only examine the case of two incomparable smallest upper bounds $[a]\theta, [y]\theta$. We have $[x]\theta \neq [v]\theta$ in $\underline{L}/\theta$ and thus $x \neq v$ in $\underline{L}$. Since $\underline{L}$ is a lattice and the factorization does not affect the order of the elements in $S^{\downarrow}, S^{\uparrow}$ and $S^{\parallel}$ we have that v, x are not both in the same of those sets. Otherwise $[x]\theta \vee [v]\theta = [x \vee v]\theta$ holds.

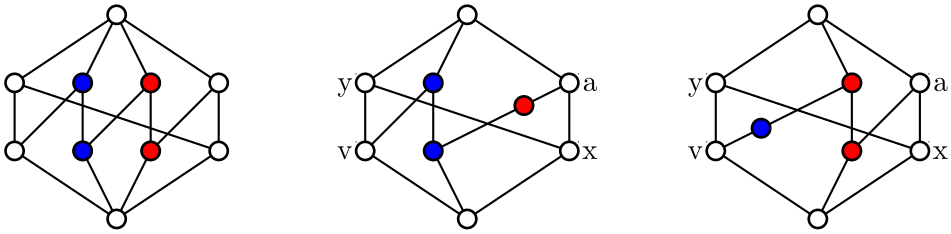


Figure 8.12 A lattice $\underline{L}$ with two pure intervals that are red and blue highlighted (left). If the red interval is factorized, the blue interval becomes nested (middle). If the blue interval is factorized, the red interval becomes nested (right).

In addition, we show that $x \notin S$: If $x \in S$ and $v \in S^{\updownarrow}$, we have $[x]\theta \vee [v]\theta = [v]\theta$. If $x \in S$ and $v \in S^{\updownarrow}$, we have $[x]\theta \vee [v]\theta = [x]\theta$. If $x \in S$ and $v \in S$, we have $[x]\theta \vee [v]\theta = [v]\theta = [x]\theta$. If $x \in S$ and $v \in S^{\parallel}$, we have $[x]\theta \vee [v]\theta = [x_\theta \vee v]\theta$ or otherwise $\underline{L}$ would not be a lattice. Analogous, one can show that $v \notin S$.

In case of x or v in $S^{\updownarrow}$ we have: W.l.o.g. let $v \in S^{\updownarrow}$. If $x \in S^{\updownarrow}$ we have $[x]\theta \vee [v]\theta = [v]\theta$. If $x \in S^{\parallel}$ we have $[x]\theta \vee [v]\theta = [x \vee v]\theta$ Therefore the only possibility for $[x]\theta$ and $[v]\theta$ having two minimal upper bounds is $x \in S^{\parallel}$ and $v \in S^{\updownarrow}$ with $v \nleq x$ (or the other way around).

W.l.o.g. let $y = v \vee x$ in $\underline{L}$. Since $x \in S^{\parallel}$ and $v \in S^{\updownarrow}$ we have $y \in S^{\updownarrow} \cup S^{\parallel}$. If $y \in S^{\updownarrow}$ we have $[x]\theta \vee [v]\theta = [x_\theta \vee v]\theta \leq_\theta [y]\theta$ as the supremum of $[x]\theta$ and $[v]\theta$. So let $y \in S^{\parallel}$. Then $[y]\theta$ is a smallest upper bound of $[x]\theta$ and $[v]\theta$. Let $[a]\theta \neq [y]\theta$ be another smallest upper bound of $[x]\theta$ and $[v]\theta$. Then either $x \nleq a$ or $v \nleq a$ in $\underline{L}$. Due to the definition of the order in $\underline{L}/\theta$ we have $v \nleq a$, $x \leq a$ and $a \in S^{\updownarrow}$ in $\underline{L}$. Then S is a nested interval. □

The example shown in Figure 8.7 illustrates the implosion of a nested interval $\underline{S}$ in a lattice $\underline{L}$. In this case, $\underline{L}/\theta_{\underline{S}}$ is an ordered set but no lattice.

Let $\underline{L}$ be a finite lattice and $\underline{S}_1, \underline{S}_2$ two disjoint pure intervals of $\underline{L}$. Note, that in general $\underline{S}_2$ is not a pure interval in $\underline{L}/\theta_{\underline{S}_1}$. Consequently, an interval relation $\theta = \theta_{\underline{S}_1,\ldots,\underline{S}_k}$ it not necessarily lattice-generating just because all considered intervals are pure - Figure 8.12 shows a counterexample. Also, not every lattice-generating interval relation consists of only pure intervals as can be seen in Figure 8.5.

This means that, having an interval relation $\theta_{\underline{S}}$ with a nested interval $\underline{S}$, it is possible to alter θ in a way that *purifies* it by adding additional intervals. In this case, it is necessary to make the elements a and y or the elements x and v comparable in the lattice. Some possibilities to purify a nested interval are illustrated in Figure 8.13. Note that the purification of an interval can also necessarily require several new intervals since there is possibly more than just one set of elements a, v, x, y that

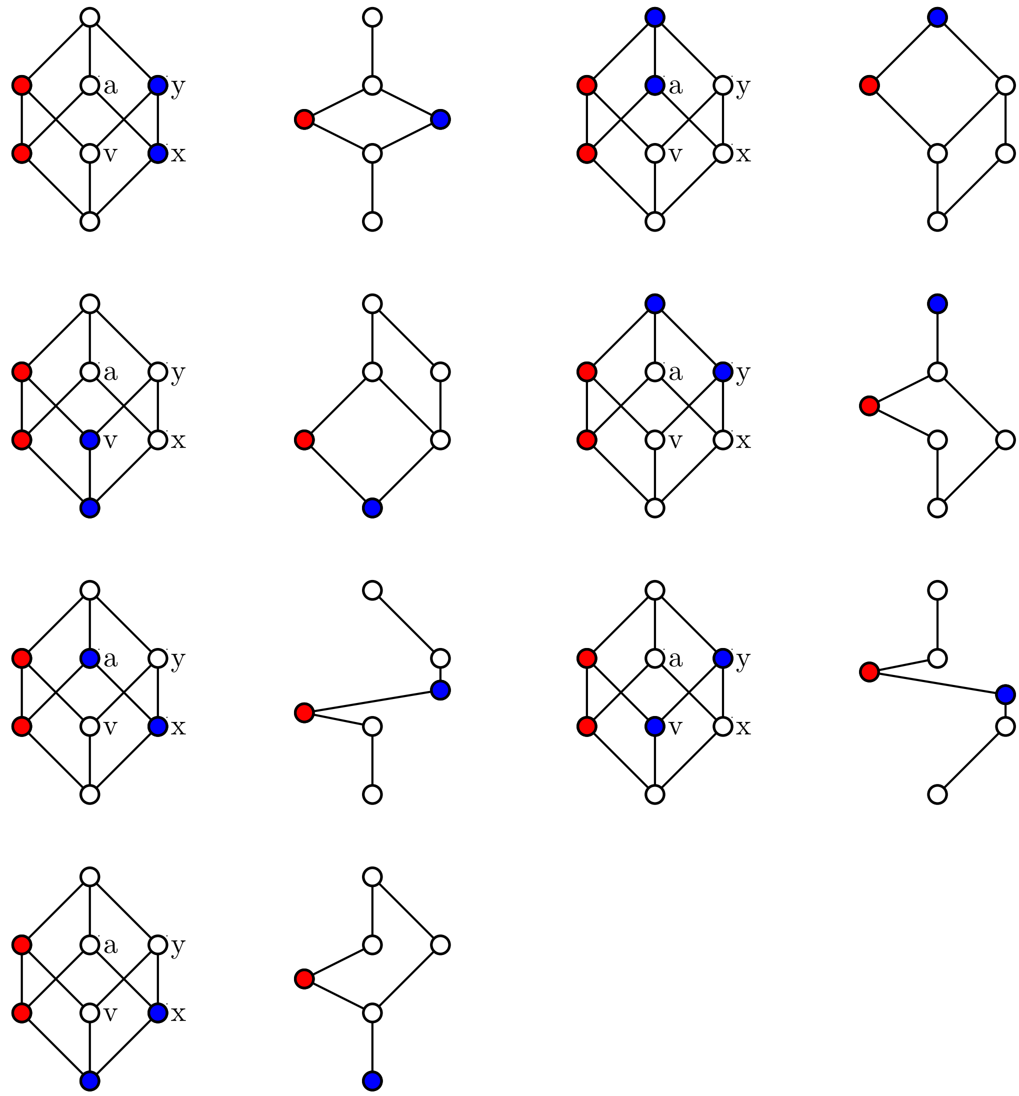


Figure 8.13 For the lattice $\underline{L}$ with a nested interval $\underline{S}$ (red), the diagrams show all different possibilities to purify the interval relation $\theta_{\underline{S}}$ by adding an additional interval $\underline{S}_{new}$(blue) with $|\underline{S}| = 2$.

makes $\underline{S}$ nested. Also, each additional interval may interact with the other added intervals as well as with $\underline{S}$ so that new problematic elements can arise.

Since our goal was to find a factorization to generate a lattice that can be obtained by a surjective, order-preserving mapping, the lattice-generating interval relation fulfills this purpose. However, the meet- and join-operations of the lattice are not generally preserved by φ, i.e., φ is, in general, not a lattice homomorphism. For example consider the two concepts a and b in Figure 8.6 (left). Their infimum in the original lattice is c, but in the factor lattice $[a]\theta \wedge [b]\theta = [b]\theta \neq [c]\theta$ holds.

However, it is possible to determine where the lattice operations are not preserved after a factorization using an interval relation:

Proposition 8.14
Let $\underline{L}$ be a lattice and $\theta = \theta_{\underline{S}}$ a lattice-generating interval relation on $\underline{L}$. Let $u, v, w, x, y, z \in \underline{L}$ with $u \wedge v = w$ and $x \vee y = z$. Then:

i) $u \in \underline{S} \cup S^{\updownarrow}, v, w \in S^{\updownarrow}, v \neq w \Rightarrow [u]\theta \wedge [v]\theta \neq [w]\theta$

ii) $x \in \underline{S} \cup S^{\updownarrow}, y, z \in S^{\updownarrow}, y \neq z \Rightarrow [x]\theta \vee [y]\theta \neq [z]\theta$

iii) $[u]\theta \wedge [v]\theta \neq [w]\theta \Rightarrow u \in \underline{S} \cup S^{\updownarrow}, v \in S^{\updownarrow} \cup S^{\parallel}, v \neq w$

iv) $[x]\theta \vee [y]\theta \neq [z]\theta \Rightarrow x \in \underline{S} \cup S^{\updownarrow}, y \in S^{\updownarrow} \cup S^{\parallel}, y \neq z$

Proof We show i): Because $u \in \underline{S} \cup S^{\updownarrow}$ and $v \in S^{\updownarrow}$ we have $[v]\theta \leq_\theta [u]\theta$. Since $w < v$ we have $[w]\theta <_\theta [v]\theta$. This means $[u]\theta \wedge [v]\theta = [v]\theta \neq [w]\theta$.
ii) can be shown analogously.

We show iii): We show the contraposition: Assumed $v = w$, we have $v \leq u$ and therefore $[u]\theta \wedge [v]\theta = [v]\theta = [w]\theta$. Thus, let $v \neq w$. In case of $u, v \in \underline{S} \cup S^{\updownarrow}$ we have $w \in \underline{S}$ or $w \in S^{\updownarrow}$. If $w \in \underline{S}$, we have $[u]\theta \wedge [v]\theta = [S]\theta = [w]\theta$, if $w \in S^{\updownarrow}$ the order between the three elements is not affected by the factorization and $[u]\theta \wedge [v]\theta = [w]\theta$ as well. In case of $u, v \in S^{\parallel} \cup S^{\updownarrow}$ we have $w \in S^{\updownarrow}$ or $w \in S^{\parallel}$. The order between the three elements is not affected by the factorization and $[u]\theta \wedge [v]\theta = [w]\theta$.
iv) can be shown analogously. □

As seen in the previous section, crowns play an essential role in determining whether an interval relation is ordered. Those substructures can also be used to determine the pureness of an interval (relation) as follows:

Proposition 8.15
Let $\underline{L}$ be a lattice and $\underline{S} = [S_\perp, S_\top]$ an interval on $\underline{L}$. Then the following equivalence holds:

$$\underline{S} \text{ is nested in } \underline{L} \Leftrightarrow S_\perp \text{ and } S_\top \text{ are elements of a crown of order } 3 \text{ in } \underline{L}$$

Proof "$\Leftarrow$": Let $A_3 \leq \underline{L}$ be a crown consisting of $x_1 = S_\perp, x_2, x_3, y_1 = S_\top, y_2$ and y_3. By definition of a crown we have $y_2 \in S^{\updownarrow}, x_3 \in S^{\updownarrow}$ and $x_2, y_3 \in S^{\parallel}$. Let $x_2 = y_2 \wedge y_3$, $y_3 = x_2 \vee x_3$, $y_3 \nleq y_2$ and $x_3 \nleq x_2$. Thus, $\underline{S}$ is nested in $\underline{L}$.
"$\Rightarrow$": Let $\underline{S}$ be nested with the elements a, v, x, y. Then we have the relations $x \leq a$, $S_\perp \leq a$, $v \leq y$, $v \leq S_\top$, $x \leq y$ and $S_\perp \leq S_\top$ as the only relations between those elements. Thus the set $s, v, x, y, S_\perp, S_\top$ is a crown of order 3 in $\underline{L}$. □

Using this, we can now generalize Corollary 8.3 to lattice-generating interval relations:

Proposition 8.16
Let $\underline{L}$ be a lattice and θ an interval relation on $\underline{L}$. If $\underline{L}$ does not contain a crown of order 3 as a suborder, then θ is lattice-generating.

Corollary 8.6
Let $\underline{L}$ be a lattice and θ an interval relation on $\underline{L}$. If $\underline{L}$ is planar θ is an order-preserving interval relation on $\underline{L}$.

Using a lattice-generating interval relation, a lattice arises by factorization so that exactly the chosen intervals of the original lattice implode. In the following, we investigate this approach on the context side.

8.3.4 Context Construction for Interval Factorization

We aim to investigate the structure of finite lattices. Since every finite lattice is isomorphic to a concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ of a formal context $\mathbb{K}$ and formal contexts tend to be smaller than their corresponding concept lattices, we discuss the corresponding context constructions of our approach in the following section.

Definition 8.10 (Enrichment of an Incidence Relation)
Let $\mathbb{K} = (G, M, I)$ be a formal context and $\{\underline{S}_1, \underline{S}_2, \ldots, \underline{S}_k\}$ a set of pairwise disjoint intervals of $\underline{\mathfrak{B}}(\mathbb{K})$ with $\underline{S}_i = [(A_i, B_i), (C_i, D_i)]$. The incidence relation

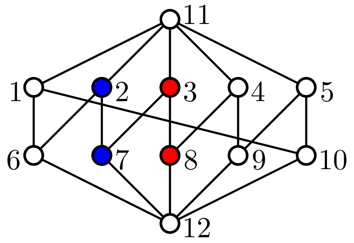
$$I_{\underline{S}_1,\ldots,\underline{S}_k} \coloneqq I \cup \bigcup_{i=1}^{k} (C_i \times B_i)$$

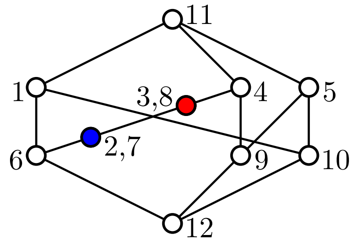
is the *enrichment of relation I by the intervals $\underline{S}_1, \ldots, \underline{S}_k$*. We call the context $\mathbb{K}_{\underline{S}} \coloneqq (G, M, I_{\underline{S}})$ the *enrichment* of context $\mathbb{K}$ by the interval $\underline{S}$.

Note that the simultaneously and the iterative enrichment of a relation by a set of intervals generally does not end in the same context, e.g., Figure 8.14.

Therefore, we present the following statements just for single intervals. We present a one-to-one correspondence between the set of the enrichments of the incidence relation by an interval for a generic formal context $\mathbb{K}$ and the interval relations $\theta_{\underline{S}}$ on $\underline{\mathfrak{B}}(\mathbb{K})$ in the following lemma. Note that the statement does not hold for reduced formal concepts in general. This fact is discussed in Lemma 8.3 in more detail.

Lemma 8.1
Let $\underline{L}$ be a lattice and $\mathbb{K} = (G, M, I)$ its generic formal context. If $\theta_{\underline{S}}$ is an interval relation on $\underline{L}$, then $I_{\underline{S}} = I \cup (C \times B)$ is an enrichment of I by the interval $\underline{S} = [(A, B), (C, D)]$. Conversely, for every enrichment $I_{\underline{S}}$ of I by an interval $\underline{S}$ the relation $\theta_{\underline{S}}$ is an interval relation on $\underline{\mathfrak{B}}(\mathbb{K})$.





	1	2	3	4	5	6	7	8	9	10	11	12
1	×										×	
2		×	•				•				×	
3			×	•				•			×	
4				×							×	
5					×						×	
6	×	×	•			×	•				×	
7		×	×	•			×	•			×	
8			×	×				×			×	
9				×	×				×		×	
10	×				×					×	×	
11											×	
12	×	×	×	×	×	×	×	×	×	×	×	×

	1	2	3	4	5	6	7	8	9	10	11	12
1	×										×	
2		×	•	•			•	•			×	
3			×	•				•			×	
4				×							×	
5					×						×	
6	×	×	•	•		×	•	•			×	
7		×	×	•			×	•			×	
8			×	×				×			×	
9				×	×				×		×	
10	×				×					×	×	
11											×	
12	×	×	×	×	×	×	×	×	×	×	×	×

Figure 8.14 A lattice $\underline{L}$ with two pure intervals $\underline{S}_1$, $\underline{S}_2$(red and blue highlighted) (top left) and the lattice $\underline{L}/\theta_{\underline{S}_1,\underline{S}_2}$(top right). For $\mathbb{K} = (G, M, I)$, the generic formal context of $\underline{L}$, the enrichments $(G, M, I_{\underline{S}_1,\underline{S}_2})$ (bottom left) and $(G, M, (I_{\underline{S}_1})_{\underline{S}_2}) = (G, M, (I_{\underline{S}_2})_{\underline{S}_1})$ (bottom right) are given. The incidences that are added by the enrichment are depicted by •.

Proof Since $\underline{S}$ is a single interval, the statement follows directly from the definitions of enrichments and interval relations. □

In a generic formal context we can also determine weather an interval is pure or nested in the corresponding concept lattice.

Lemma 8.2
Let $\underline{L}$ be a finite lattice and $\mathbb{K} = (G, M, I)$ its generic formal context. Let $\theta_{\underline{S}}$ be an interval relation on $\underline{L}$. $\underline{S}$ is nested interval in $\underline{L}$ if and only if there exists $\mathbb{S} = [H, N] \leq \mathbb{K}$ with $\mathbb{S}$ being a Boolean subcontext of dimension 3, $[\underline{S}]^\theta \in N$, $[\underline{S}]_\theta \in H$ and $[\underline{S}]_\theta I [\underline{S}]^\theta$.

Proof Follows directly from Proposition 8.15: A lattice contains a Boolean suborder of dimension 3 if and only if it contains a crown of order 3 as a suborder. Due to the definition of a generic formal context, there is a Boolean subcontext $[\{a, b, c\}, \{x, y, z\}]$ of dimension 3 in $\mathbb{K}$ precisely if there is a Boolean suborder of dimension 3 and thus a crown of order 3 as suborder in the corresponding lattice so that a, b, c are the lower elements of the crown and x, y, z are the upper elements of the crown. □

In Lemma 8.1 we considered $\mathbb{K}$ to be generic. Otherwise, additional reducible concepts may vanish even if they are not in the chosen interval, as presented in the following example.

Example 8.4

In Figure 8.15 two contexts that generate (up to isomorphism) the same concept lattice are represented. Both have enrichments by the interval $\underline{S} = [(4'', 4'), (G, G')]$. Consider context $\widetilde{\mathbb{K}} = (\widetilde{G}, \widetilde{M}, \widetilde{I})$ presented in Figure 8.15 (bottom left) and its corresponding formal context $\underline{\mathfrak{B}}(\widetilde{\mathbb{K}}) = \underline{\mathfrak{B}}(\mathbb{K})$ (top right). The enrichment of $\widetilde{I}$ by the red highlighted interval $\underline{S} = [(4'', 4'), (13'', 13')]$ is given by adding the $\bullet$-marked incidences to $\widetilde{I}$. $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}})$ is presented in the figure (bottom right). It consists of the new generated interval (red) and the remaining concepts in the original order, i.e. $\underline{\mathfrak{B}}(\widetilde{\mathbb{K}}_{\underline{S}}) \cong \underline{\mathfrak{B}}(\widetilde{\mathbb{K}})/\theta_{\underline{S}}$. If $\mathbb{K}$ (top middle), the standard context of $\underline{\mathfrak{B}}(\mathbb{K})$, is considered, the enrichment of the incidence relation by the same interval results in the smaller lattice (top right). Since in $\underline{\mathfrak{B}}(\mathbb{K})$, e.g., the concepts $(5'', 5')$ and $(6'', 6')$ only differ in an attribute set that is totally included in $\underline{S}$, their difference vanishes by the enrichment if no attribute o or l persists to differ them.

This illustrates that the lattice, based on the enrichment of an incidence by an interval of a corresponding context, depends on the selection of the context. It is clear that using the generic formal context leads to an upper bound for the size of the arising lattice, since all concepts are generated by a single object and a single attribute. In the following, we determine the objects and attributes necessary for generating a lattice isomorphic to the one obtained using the generic formal context.

Definition 8.11

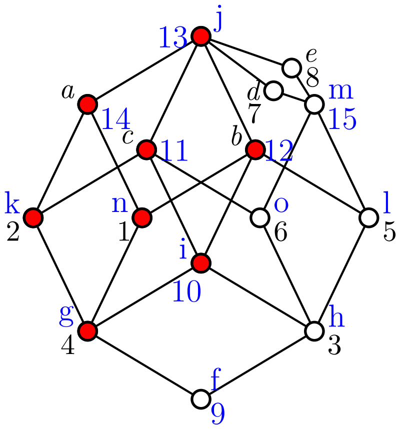
Let $\theta_{\underline{S}}$ be an interval relation on the lattice $\underline{L}$ with $\underline{S} \leq \underline{L}$ an interval. We call $x \in \underline{L}$ θ-$\bigvee$-*irreducible* if either $x \in J(\underline{L})$ or for $x \notin \underline{S}$ if $|\{y \in \underline{L} \setminus \underline{S} \mid y \text{ is an lower neighbour of } x\}| \leq 1$ holds. Analogous we call an element $x \in \underline{L}$ θ-$\bigwedge$-*irreducible* if either $x \in M(\underline{L})$ or for $x \notin \underline{S}$ if $|\{y \in \underline{L} \setminus \underline{S} \mid y \text{ is an upper neighbour of } x\}| \leq 1$ holds.

Definition 8.12

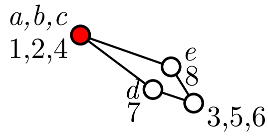
Let $\theta_{\underline{S}}$ be an interval relation on the lattice $\underline{L}$ with $\underline{S} \leq \underline{L}$ an interval. Let $U = \{x \in \underline{L} \mid x \; is \; \theta - \bigwedge -\text{irreducible}\}$ and $V = \{x \in \underline{L} \mid x \; is \; \theta - \bigvee -irreducible\}$. We call a context $\mathbb{K} = (H, N, \leq)$ with $V \subseteq H \subseteq L$ and $U \subseteq N \subseteq L$ a θ-*irreducible* context of $\underline{L}$.

Lemma 8.3

Let $\underline{L}$ be a lattice, $\theta_{\underline{S}}$ an interval relation on $\underline{L}$, $\mathbb{K} = (G, M, I)$ the generic context of $\underline{L}$, and $(H, N, \leq)$ a θ-irreducible context of $\underline{L}$. Then $\underline{\mathfrak{B}}(H, N, \leq) \cong \underline{\mathfrak{B}}(G, M, I_{\underline{S}})$ holds.



	a	b	c	d	e
1	×	×	•		
2	×	•	×		
3	•	×	×	×	×
4	×	×	×		
5	•	×	•	×	×
6	•	•	×	×	×
7	•	•	•	×	
8	•	•	•		×



	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
1	×	×	•				•		•	×	•			×	
2	×	•	×				•		•	×	×			•	
3	•	×	×	×	×		•	×	×	×	•	×	×	•	×
4	×	×	×				×		×	×	×			×	
5	•	×	•	×	×		•		•	×	•	×	×	•	
6	•	•	×	×	×		•		•	×	•		×	•	×
7	•	•	•	×			•		•	×	•			•	
8	•	•	•		×		•		•	×	•			•	
9	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
10	•	×	×				•		×	×	•			•	
11	•	•	×				•		•	×	•			•	
12	•	×	•				•		•	×	•			•	
13	•	•	•				•		•	×	•			•	
14	×	•	•				•		•	×	•			•	
15	•	•	•	×	×		•		•	×	•		×	•	

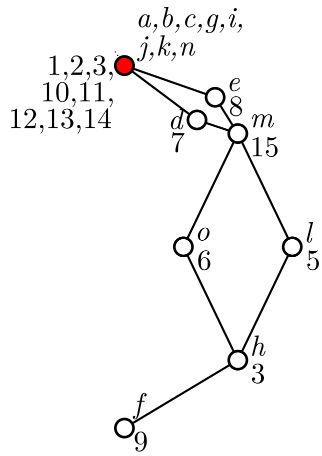


Figure 8.15 A (concept) lattice $\underline{\mathfrak{B}}(\mathbb{K}) = \underline{\mathfrak{B}}(\widetilde{\mathbb{K}})$ with a pure interval $\underline{S}$ highlighted red (top left). The objects and attributes with blue highlighted labels are reducible. The corresponding reduced formal context $\mathbb{K} = (G, M, I)$ (top middle) and an corresponding generic formal context $\widetilde{\mathbb{K}} = (\widetilde{G}, \widetilde{M}, \widetilde{I})$ (bottom left) have additional incidences marked by $\bullet$, that represent the enrichments of the contexts by $\underline{S}$. $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}})$ is displayed on the top right, and $\underline{\mathfrak{B}}(\widetilde{\mathbb{K}}_{\underline{S}})$ on the bottom right.

Proof We show that every object $g \in G$ with $g \notin H$ is reducible in $(G, M, I_{\underline{S}})$. If $g \notin H$, we have $c = (g'', g')$ is not θ-$\bigvee$-irreducible in $\underline{L} \cong \underline{\mathfrak{B}}(\mathbb{K})$. Let $c_1, \ldots, c_l \notin \underline{S}$ with $l \geq 2$ be the lower neighbors of c in $\underline{\mathfrak{B}}(\mathbb{K})$. Since the original order relation is preserved by the factorization, $[c_1]\theta, \ldots, [c_l]\theta$ are lower neighbors of $[c]\theta$ in $\underline{\mathfrak{B}}(\mathbb{K})/\theta$. Therefore g is reducible in $(G, M, I_{\underline{S}})$. Analogous θ-$\bigwedge$-reducible elements are unnecessary for the attribute set. □

It follows that not the whole generic context has to be considered in the following but only the context containing all θ-$\bigvee$-irreducible elements as the object set and all θ-$\bigwedge$-irreducible elements as the attribute set. E.g. the concept $(15'', 15') = (m', m'')$ in Figure 8.15 is neither θ-$\bigvee$-irreducible nor θ-$\bigwedge$-irreducible. Therefore, object 15 and attribute m have no impact on the factor set.

Lemma 8.4
Let $\underline{L}$ be a lattice, $\mathbb{K} = (G, M, I)$ a θ-irreducible context of $\underline{L}$, and $\underline{S} \leq \underline{L}$ an interval. Then:

i) $\underline{S}$ is pure $\Leftrightarrow$ $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}}) \cong \underline{L}/\theta_{\underline{S}}$

ii) $\underline{S}$ is nested $\Leftrightarrow$ $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}}) \ncong \underline{L}/\theta_{\underline{S}}$ and $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}})$ is the Dedekind–MacNeille completion of $\underline{L}/\theta_{\underline{S}}$.

Proof i): "⇒:" We assume $\mathbb{K} = (\underline{L}, \underline{L}, \leq)$ to be the generic context of $\underline{L}$ and thus $\mathbb{K}_S = (\underline{L}, \underline{L}, \leq_{\underline{S}})$. The factor lattice $\underline{L}/\theta_{\underline{S}}$ is isomorphic to the concept lattice of its generic context $(\underline{L}/\theta_{\underline{S}}, \underline{S}/\theta_{\underline{S}}, \leq_\theta)$. Via definition of the order $\leq_\theta$, for two elements $[g]\theta, [m]\theta \in \underline{L}/\theta$ we have $[x]\theta \leq_\theta [y]\theta$ if and only if $x \leq y$ or $x \leq [\underline{S}]^\theta$ and $y \geq [\underline{S}]_\theta$ in $\underline{L}$. Considering $\mathbb{K}_S$, for two elements $x, y \in$ Ł we have $x \leq_{\underline{S}} y$ if and only if $x \leq y$ or $x \in \{c \in \underline{L} \mid c \leq [\underline{S}]^\theta\}$ and $y \in \{c \in \underline{L} \mid c \geq [\underline{S}]_\theta\}$. By identifying each element $x \in \underline{L}$ with the equivalence class $[x]\theta \in \underline{L}/\theta$, the isomorphism between $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}})$ and $\underline{L}/\theta$ follows.

"⇐:" If $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}}) \cong \underline{L}/\theta_{\underline{S}}$ holds, $\underline{L}/\theta_{\underline{S}}$ is a lattice and therefore $\underline{S}$ is pure.

ii): "⇒:" If $\underline{S}$ is nested, $\underline{L}/\theta_S$ is an ordered set but no lattice and thus $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}}) \ncong \underline{L}/\theta_S$ holds. With [29, Theorem 4] follows, that the formal context $(\underline{L}/\theta_S, \underline{L}/\theta_S, \leq_\theta)$ corresponds to the Dedekind–MacNeille completion of $\underline{L}/\theta_S$. Further, as seen before, the concept lattices corresponding to $\mathbb{K}_{\underline{S}}$ and $(\underline{L}/\theta_S, \underline{L}/\theta_S, \leq_\theta)$ are isomorphic.

"⇐:" If $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}})$ is the Dedekind–MacNeille completion of $\underline{L}/\theta_S$, $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}})$ is isomorphic to the concept lattice $\underline{\mathfrak{B}}((\underline{L}/\theta_{\underline{S}}, \underline{L}/\theta_{\underline{S}}, \leq))$. Since $\underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}}) \ncong \underline{L}/\theta_{\underline{S}}$ holds, $\underline{L}/\theta_{\underline{S}}$ is not a lattice and therefore $\underline{S}$ is nested in $\underline{L}$. □

Therefore we can transfer the statement of $\varphi : \underline{L} \to \underline{L}/\theta,\ x \mapsto [x]\theta$ being surjective and order-preserving to formal contexts:

Lemma 8.5

Let $\underline{L}$ be a lattice, $\underline{S} = [(A_S, B_S), (C_S, D_S)] \leq \underline{L}$ a pure interval, and $\mathbb{K} = (G, M, I)$ a θ-irreducible context of $\underline{L}$. Then the map

$$\varphi : \underline{\mathfrak{B}}(\mathbb{K}) \to \underline{\mathfrak{B}}((G, M, I_{\underline{S}}))$$
$$(A, B) \mapsto \begin{cases} (C_S, B_S) & , (A, B) \in \underline{S} \\ (A, B \cup B_S) & , (A, B) \in S^{\updownarrow} \\ (A \cup C_S, B) & , (A, B) \in S^{\uparrow} \\ (A, B) & , (A, B) \in S^{\parallel} \end{cases}$$

is surjective and order preserving.

Proof follows directly from Lemma 8.4. □

Note that the approach in Section 8.3.4 does relate to the one presented in Section 8.2.3. For a selected Boolean sublattice $\underline{L} \leq \underline{S}$ in the corresponding induced concept lattice besides $\psi(\underline{S}) = [N, H]$ additional incidences are added. This is done to ensure that an element $\underline{L}$ smaller than a specific part of $\underline{S}$ is also smaller than the newly generated element. Therefore, for every object $g \in G$ with $x \subseteq g'$ the incidences in the subcontext $[g, N]$ are added where $x = int(\bigvee CoAt(\underline{S}))$. The analogous approach is made with attributes and the extent of the infimum of all atoms of $\underline{S}$.

8.4 Conclusion

In this chapter, we presented methods to factorize a lattice so that selected intervals implode. We started with the investigation of factor lattices generated by complete congruence relations in Section 8.2.1 and presented an approach to find the finest congruence relation, i.e., the one with as many different congruence classes as possible, to implode a selected interval. Since every congruence relation is an equivalence relation, the elements of the original lattice can be mapped to the elements of the factor lattice in a unique way. This property does not hold when using complete tolerance relations, a generalization of the complete congruence relations, instead. In both cases, the generated factor lattice preserves the original meet- and join-operators. However, both approaches can result in an overaggressive reduction of the lattice size, imploding not only the selected interval since their construction results in bigger classes. To overcome this problem, we introduced a kind of factorization based on newly introduced interval relations in Section 8.3. The equivalence classes of which include precisely the selected intervals so that it is possibility to implode selected disjunct intervals while preserving all other elements of the original lattice and their order. As a trade-off, the original $\bigwedge$- and $\bigvee$-relations are no longer preserved in this case. To ensure that a lattice arises as the factor set, we restricted the approach to single pure intervals. In this case, by taking advantage of the one-to-one correspondence between interval relations and enrichments of incidence relations by intervals in the corresponding context, we get the corresponding context of the factor set directly.

CHAPTER 9

Dismantling for Intervals

The research question of this chapter is a natural follow-up to the one of Chapter 8, expanding the idea of factorizing an interval to a single representative to eliminating it completely from a lattice. Dismantling allows for the removal of elements of a set, or in our case lattice, without disturbing the remaining structure. In this chapter, we extend the notion of dismantling by single elements to dismantling by intervals in a lattice. We show that lattices dismantled by intervals correspond to closed subrelations in the respective formal context and that there exists a unique kernel with respect to dismantling by intervals. Furthermore, we show that dismantling intervals can be identified directly in the formal context utilizing a characterization via arrow relations and provide an algorithm to compute all dismantling intervals.

9.1 Introduction

In Formal Concept Analysis, the removal of a doubly irreducible element in a concept lattice corresponds to the removal of a single incidence from its (clarified) formal context. This approach results in a complete sublattice of the original concept lattice. In particular, this sublattice contains all but one of the original concepts.

In this chapter, we extend the notion of dismantling from single elements to intervals in order to remove multiple (not necessarily irreducible) concepts at once while preserving the remaining concept lattice. To this end, we make use of the one-to-one correspondence between closed subrelations of a formal context and the complete sublattices of its concept lattice [58], and, more generally, of the one-to-one correspondence between closed-subcontexts of a context and the sublattices of its concept lattice as presented in Section 5.3.

Extending dismantling to intervals, we call an interval $[u, v] =: \underline{S}$ *dismantling for a lattice* $\underline{L}$ if v is infimum-prime in the filter of u, u is supremum prime in the ideal of v and $u, v \notin \{\bot, \top\}$. Because infimum-prime (supremum-prime) implies infimum-irreducible (supremum-irreducible), dismantling intervals that consist of a single element are precisely the doubly irreducible elements. We show that an interval $\underline{S}$ is dismantling for $\underline{L}$ if and only if the incidences of all concepts not in $\underline{S}$ form a closed subrelation. Furthermore, we show that the core obtained by the iterative removal of dismantling intervals for a lattice is unique. Where possible, we use the more general notion of an interval $\underline{S}$ being *quasi-dismantling for a lattice*, which allows for $u, v \in \{\bot, \top\}$ and show that an interval $\underline{S}$ is quasi-dismantling for $\underline{L}$ if and only if the objects, attributes and incidences of all concepts not in $\underline{S}$ form a closed-subcontext.

Finally, we give a characterization of dismantling intervals via arrow relations on the context side and provide an algorithm to determine whether an interval is dismantling using this characterization. Furthermore, the arrow relations provide a way to compute all dismantling intervals for a given context $\mathbb{K}$ without having to compute the concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ itself.

9.2 Dismantling Intervals for a Lattice

To identify the intervals that can be removed from a lattice without disturbing the remaining structure, we introduce the notions of *dismantling* and *quasi-dismantling intervals for a lattice*, by extending the notion of dismantling single elements, in Definition 9.1. These notions build up the basis for our further investigation.

$\mathbb{K}$	a	b	c	d	e
1	×				
2	×				×
3		×	×	×	
4		×	×		×
5		×		×	
6		×			×

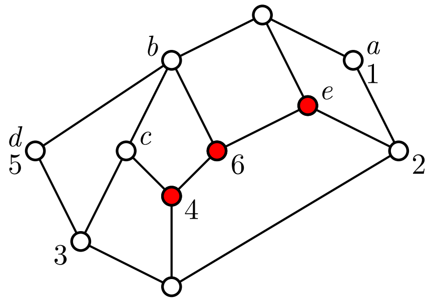


Figure 9.1 Example for a context $\mathbb{K}$ (left) and its concept lattice $\underline{L}$ (right) where the colored elements in the lattice are an example for an interval $\underline{S}$ dismantling for $\underline{L}$, and the highlighted incidences in $\mathbb{K}$ correspond to the S-removed incidences. The interval $\underline{S} = [\gamma 4, \mu e]$ is interval dismantling for the lattice $\underline{L}$ since the object concept $\gamma 4$ is supremum-prime in $(\mu e]$, and the attribute concept μe is infimum-prime in $[\gamma 4)$.

Definition 9.1 ((Quasi-)Dismantling Intervals)
Let $\underline{L}$ be a lattice and $[u, v] = \underline{S} \leq \underline{L}$ an interval of $\underline{L}$. We call $\underline{S}$ *quasi-dismantling* for $\underline{L}$ if u is supremum-prime in $(v]$ and v is infimum-prime in $[u)$. If $u \neq \bot$ and $v \neq \top$ hold, we call $\underline{S}$ *dismantling* for $\underline{L}$.

On the context side, the removal of a set of concepts S corresponds to the removal of all incidences that only belong to concepts in the respective interval. We call the remaining context (objects, attributes, incidences) the *S-removed* context (objects, attributes, incidences).

Definition 9.2 (S-removed Context)
Let $\mathbb{K} = (G, M, I)$ be a formal context and $S \subseteq \mathfrak{B}(\mathbb{K})$. We call

- $I_S := I \setminus \left(\bigcup_{(A,B)\in S} A \times B \setminus \bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})\setminus S} A \times B\right)$ *S-removed incidences,*
- $G_S := G \setminus \left(\bigcup_{(A,B)\in S} A \setminus \bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})\setminus S} A\right)$ *S-removed objects,*
- $M_S := M \setminus \left(\bigcup_{(A,B)\in S} B \setminus \bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})\setminus S} B\right)$ *S-removed attributes.*

In the following we let $\mathbb{K}_S := (G_S, M_S, I_S)$ be the *S-removed context* for $\mathbb{K}$.

Next, in Proposition 9.1, we show that instead of removing incidences (objects, attributes) to obtain the S-removed sets, they can be generated as the union of all incidences (objects, attributes) that are not part of a concept in S. This leads to a simpler representation of the S-removed sets.

Proposition 9.1
Let $\mathbb{K} = (G, M, I)$ be a formal context and $S \subseteq \mathfrak{B}(\mathbb{K})$ a set of formal concepts, then $I_S = \bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})\setminus S} A \times B$, $G_S = \bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})\setminus S} A$ and $M_S = \bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})\setminus S} B$.

Proof We show the proof for the S-removed incidences. The proofs for the other two sets are analogous.

$$
\begin{aligned}
&I \setminus \left(\bigcup_{(A,B)\in S} A\times B \setminus \bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})\setminus S} A\times B\right)\\
&= \bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})} A\times B \setminus \left(\bigcup_{(A,B)\in S} A\times B \setminus \bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})\setminus S} A\times B\right)\\
&= \left(\bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})} A\times B \cap \left(\bigcup_{(A,B)\in S} A\times B\right)^c\right)\\
&\quad \cup \left(\bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})} A\times B \cap \bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})\setminus S} A\times B\right)\\
&= \bigcup_{(A,B)\in\mathfrak{B}(\mathbb{K})\setminus S} A\times B
\end{aligned}
$$

□

As an example consider the formal context $\mathbb{K}$ in Figure 9.1 and the set of concepts $S = \{(\{4\},\{b,c,e\}),(\{4,6\},\{b,e\}),(\{2,4,6\},\{e\})\}$ colored in the lattice. Then the highlighted S-removed incidences in $\mathbb{K}$ can be obtained both, by the removal of incidences (Definition 9.2), or as the union of incidences of concepts not in S (Proposition 9.1).

Further, if we consider an interval $\underline{S}$, we see that $\underline{S}$ being quasi-dismantling corresponds to obtaining a closed-subcontext on S-removal. More precisely, the S-removed context $\mathbb{K}_S$ for a formal context $\mathbb{K}$ is a closed-subcontext if and only if the interval $\underline{S} \leq \underline{\mathfrak{B}}(\mathbb{K})$ is quasi-dismantling for $\underline{\mathfrak{B}}(\mathbb{K})$.

Proposition 9.2
Let $\mathbb{K} = (G,M,I)$ be a formal context and $\underline{\mathfrak{B}}(\mathbb{K})$ its corresponding concept lattice. Let $\underline{S} = [u,v] \leq \underline{\mathfrak{B}}(\mathbb{K})$ be an interval. Then, $\underline{S}$ is quasi-dismantling for $\underline{\mathfrak{B}}(\mathbb{K})$ if and only if $\mathbb{K}_S = (G_S,M_S,I_S)$ is a closed-subcontext of $\mathbb{K}$.

Proof "⇒": We show the contraposition: Assume that (G_S,M_S,I_S) is no closed-subcontext. By definition holds $G_S \subseteq G$, $M_S \subseteq M$ and $I_S \subseteq I$. Then there exists some $c \in \underline{\mathfrak{B}}(\mathbb{K}_S)$ such that $c \notin \underline{\mathfrak{B}}(\mathbb{K})$. Since $\underline{\mathfrak{B}}(\mathbb{K}_S)$ is a lattice generated from $\mathfrak{B}(\mathbb{K}) \setminus S$ there exist $x,y \in \mathfrak{B}(\mathbb{K}) \setminus S$ such that $x \vee y = c$ or $x \wedge y = c$ in $\underline{\mathfrak{B}}(\mathbb{K}_S)$. In case $x \vee y = c$: Since $\underline{\mathfrak{B}}(\mathbb{K})$ is a lattice it follows that there exists some $z \in \mathfrak{B}(\mathbb{K})$ with $z \notin \mathfrak{B}(\mathbb{K}_S)$ and $z = x \vee y$. Thus, $z \in S = [u,v]$ and therefore $z \geq u$ in $\underline{\mathfrak{B}}(\mathbb{K})$. Because $x,y \notin S$ we have $x,y \ngeq u$. Hence, u is not supremum-prime in $(v]$ and S is not quasi-dismantling. The case $x \wedge y = c$ is analogous.

"⇐": We show the contraposition: Assume $\underline{S}$ is not quasi-dismantling. Then u is not supremum-prime in $(v]$ or v is not infimum-prime in $[u)$. In case that u is not supremum-prime in $(v]$: There exist $x,y \in (v]$ such that $z := x \vee y \geq u$, $x \ngeq u$ and $y \ngeq u$. Thus, $x,y \notin \underline{S}$ and $z \in \underline{S}$. Therefore, $z \notin \mathfrak{B}(\mathbb{K}_S)$. There is some supremum $c = x \vee y$ in $\mathfrak{B}(\mathbb{K}_S)$. Because the intent of c is the intent of z by [29, Thm. 3] we have $c \notin \mathfrak{B}(\mathbb{K})$. Thus, (G_S,M_S,I_S) is no closed-subcontext. The case that v is not infimum-prime in $[u)$ is analogous. □

In particular, for a dismantling interval $\underline{S}$ we have $G_S = G$ and $M_S = M$ (because $\top, \bot \in \mathfrak{B}(\mathbb{K}) \setminus S$) and therefore, the correspondence, in this case, is to closed subrelations.

The removal of a quasi-dismantling interval leaves the remaining lattice intact with respect to supremum and infimum:

Proposition 9.3
Let $\underline{L}$ be a lattice and $\underline{S} \leq \underline{L}$ an interval. If $\underline{S}$ is quasi-dismantling for $\underline{L}$, then $\underline{L} \setminus \underline{S}$ is a lattice. In particular, $\underline{L} \setminus \underline{S}$ is a sublattice of $\underline{L}$.

Proof Let $x, y, z \in \underline{L}$ with $z = x \vee y$. Because $\underline{S}$ is quasi-dismantling if $x, y \notin \underline{S}$ then $z \notin \underline{S}$. Analogously for $z = x \wedge y$. □

Note that $\underline{L} \setminus \underline{S}$ has a unique unit and zero element even if the original ones were part of the quasi-dismantlable interval $\underline{S}$. If $\underline{S}$ does not include $\top$ or $\bot$ of $\underline{L}$, i.e. $\underline{S}$ is a dismantling for $\underline{L}$, both of those elements are preserved in $\underline{L} \setminus \underline{S}$.

Corollary 9.1
If $\underline{S}$ is dismantling for $\underline{L}$ then $\underline{L} \setminus \underline{S}$ is a complete sublattice of $\underline{L}$.

Combining the previous propositions, it follows for an interval $\underline{S}$ which is quasi-dismantling in a lattice $\underline{\mathfrak{B}}(\mathbb{K})$ that the removal of $\underline{S}$ from $\underline{\mathfrak{B}}(\mathbb{K})$ is isomorphic to the concept lattice of the S-removed context $\mathbb{K}_S$.

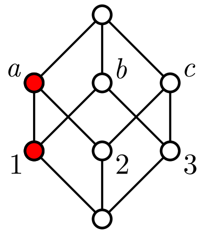
Theorem 9.1
Let $\mathbb{K} = (G, M, I)$ be a formal context and $\underline{S} \leq \underline{\mathfrak{B}}(\mathbb{K})$ an interval. If $\underline{S}$ is quasi-dismantling for $\underline{L}$, then

$$\underline{\mathfrak{B}}(\mathbb{K}) \setminus \underline{S} = \underline{\mathfrak{B}}(\mathbb{K}_S).$$

Proof We know from Proposition 9.3 that $\underline{\mathfrak{B}}(\mathbb{K}) \setminus \underline{S}$ is a sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$. Further, from Proposition 9.2 follows that $\underline{\mathfrak{B}}(\mathbb{K}_S)$ is a sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$. Both contain exactly the concepts of $\mathfrak{B}(\mathbb{K})$ that are not included in $\underline{S}$. □

A lattice that has no doubly irreducible elements but contains a dismantling interval $\underline{S}$ is given in Figure 9.2 (left lattice). The concept lattice of $\mathbb{K}_S$, in this case a closed subcontext of $\mathbb{K}$, is the concept lattice of $\mathbb{K}$ without the interval $\underline{S}$. Note that Theorem 9.1 does not hold for intervals that are not dismantling for the lattice. See Figure 9.3 for a counterexample.

Theorem 9.1 is a generalization of the statement in Proposition 3.7 concerning the dismantling of doubly irreducible lattice elements. In the following Propositions 9.4 and 9.5 clarify how Proposition 3.7 and Theorem 9.1 are connected. The intervals $\underline{S} \leq \underline{\mathfrak{B}}(\mathbb{K})$ consisting of a single element and being dismantling for a lattice are



	a	b	c
1	×	×	
2	×		×
3		×	×

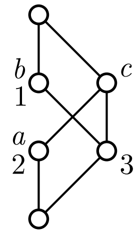
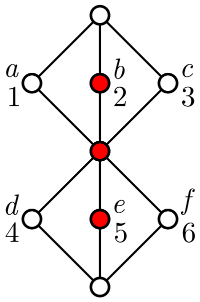


Figure 9.2 A lattice $\underline{L}$ (left) and the corresponding context $\mathbb{K} = (G, M, I)$ (middle). The highlighted interval $\underline{S}$ is dismantling for $\underline{L}$. Therefore, the highlighted S-removed incidences of $\mathbb{K}$ are a closed subrelation of I. The lattice on the right corresponds to the context $\mathbb{K}_S$.



	a	b	c	d	e	f
1	×					
2		×				
3			×			
4	×	×	×	×		
5	×	×	×		×	
6	×	×	×			×

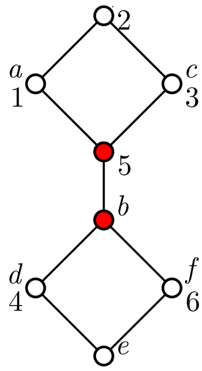


Figure 9.3 A lattice (left) and the corresponding formal context $\mathbb{K}$ (middle). The highlighted interval $\underline{S}$ is not dismantling for the lattice. Therefore, the highlighted S-removed incidences of $\mathbb{K}$ are no closed subrelation of I. The lattice on the right corresponds to the context $\mathbb{K}_S$. The highlighted concepts $(\{4,6\},\{a,b,c\})$ and $(\{4,5,6\},\{a,c\})$ do not exist in the original lattice.

exactly the doubly irreducible concepts of $\underline{\mathfrak{B}}(\mathbb{K})$. If $\underline{S}$ is quasi-dismantling, the cases of $\top \in \underline{S}$ and $\bot \in \underline{S}$ have to be considered additionally.

Proposition 9.4
Let $\mathbb{K} = (G, M, I)$ be a formal context and $\underline{S} \leq \underline{\mathfrak{B}}(\mathbb{K})$ an interval with $|\underline{S}| = 1$. $\underline{S}$ is dismantling for $\underline{\mathfrak{B}}(\mathbb{K})$ if and only if $\underline{S}$ is doubly irreducible.

Proposition 9.5
Let $\mathbb{K} = (G, M, I)$ be a formal context and $\underline{S} \leq \underline{\mathfrak{B}}(\mathbb{K})$ an interval with $|\underline{S}| = 1$. $\underline{S}$ is quasi-dismantling if and only if

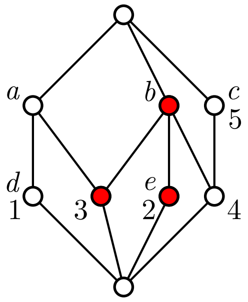
i) $\underline{S}$ is doubly irreducible or

ii) $\underline{S} = \top$ and $\underline{S}$ is supremum-irreducible or

iii) $\underline{S} = \bot$ and $\underline{S}$ is infimum-irreducible or

iv) $\underline{S} = \top = \bot$.

If we consider multiple intervals at once, the previous statements do not hold in general. However, one direction of Proposition 9.2 still holds:



	a	b	c	d	e
1	×			×	
2		×			×
3	×	×			
4		×	×		
5			×		

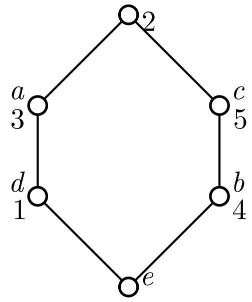


Figure 9.4 A formal context with a highlighted closed subrelation (middle) and its corresponding concept lattice (left). The highlighted concepts are the ones, that vanish by the closed subrelation. The arising lattice is pictured on the right. This closed subrelation can not be obtained via dismantling intervals.

Proposition 9.6
Let $\mathbb{K} = (G, M, I)$ be a formal context and $\underline{S}_1, \ldots, \underline{S}_k$ be intervals in $\underline{\mathfrak{B}}(\mathbb{K})$. If $\underline{S}_1, \ldots, \underline{S}_k$ are quasi-dismantling, then $(G_{S_1\cup\ldots\cup S_k}, M_{S_1\cup\ldots\cup S_k}, I_{S_1\cup\ldots\cup S_k})$ is a closed-subcontext of $\mathbb{K}$.

Proof We show the contraposition. Assume $(G_{S_1\cup\ldots\cup S_k}, M_{S_1\cup\ldots\cup S_k}, I_{S_1\cup\ldots\cup S_k})$ is no closed-subcontext. By definition, we have $G_{S_1\cup\ldots\cup S_k} \subseteq G$, $M_{S_1\cup\ldots\cup S_k} \subseteq M$ and $I_{S_1\cup\ldots\cup S_k} \subseteq I$. Then there exists $c \in \mathfrak{B}(\mathbb{K}_{S_1\cup\ldots\cup S_k})$ with $c \notin \mathfrak{B}(\mathbb{K})$. Hence, there exist $x, y \in \mathfrak{B}(\mathbb{K})$ such that $c = x \vee y$ or $c = x \wedge y$ in $\mathfrak{B}(\mathbb{K}_{S_1\cup\ldots\cup S_k})$. In case $c = x \vee y$ in $\mathfrak{B}(\mathbb{K}_{S_1\cup\ldots\cup S_k})$ there exists $z \in \mathfrak{B}(\mathbb{K})$ such that $z = x \vee y$. Since $z \in S_1 \cup \ldots \cup S_k$ we have $z \in S_i$ for some i. The rest follows analogous to the proof of Proposition 9.2. □

However, not all closed-subcontexts (and therefore sublattices of the corresponding concept lattice) can be obtained via a quasi-dismantling interval or a set of quasi-dismantling intervals for the corresponding lattice, see e.g. Figure 9.4. The interval $[(\{3\}, \{a, b\}), (\{2, 3, 4\}, \{b\})]$ is not dismantling, and neither $(\{3\}, \{a, b\})$ nor $(\{2, 3, 4\}, \{b\})$ are doubly irreducible.

Further note that not every lattice contains a quasi-dismantling interval besides the trivial one (the complete lattice) or any dismantling interval at all. Figure 9.5 shows the smallest (non-trivial) lattice that has no dismantling interval.

However, there is always a unique smallest lattice that can be obtained by iteratively removing all dismantling intervals, as shown in Theorem 9.2. To this end, we make use of the following proposition concerning the dismantlability of intervals upon removing one of them.

Proposition 9.7
Let $\underline{L}$ be a lattice and $\underline{S}_1, \underline{S}_2$ dismantling intervals for $\underline{L}$ such that $\underline{S}_2 \notin \underline{S}_1$. Then, $\underline{S}_2 \setminus \underline{S}_1$ is a dismantling interval for $\underline{L} \setminus \underline{S}_1$.

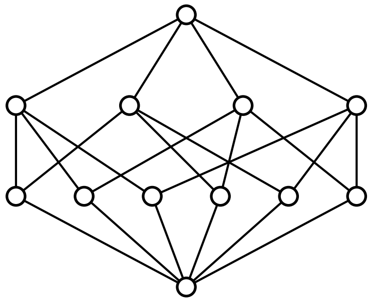


Figure 9.5 A lattice that has no dismantling interval. This lattice and its dual are the smallest (non-trivial) lattices for which this is the case.

Proof We first show that $\underline{S}_2 \setminus \underline{S}_1$ is an interval: Assume $\underline{S}_2 \setminus \underline{S}_1$ is no interval in $\underline{L} \setminus \underline{S}_1$. Then, without loss of generality, there exist $x, y \in \underline{S}_2 \setminus \underline{S}_1$ such that $z := x \vee y \notin \underline{S}_2 \setminus \underline{S}_1$ in $\underline{L} \setminus \underline{S}_1$. Either $z = x \vee y$ in $\underline{L}$, thus $z \in \underline{S}_2$ and therefore $z \in \underline{S}_2 \setminus \underline{S}_1(\lightning)$; or $z \neq x \vee y$ in $\underline{L}$, hence $w = x \vee y$ in $\underline{L}$ with $w \in \underline{S}_1$ and since $\underline{S}_1$ is dismantling, $x \in \underline{S}_1$ or $y \in \underline{S}_1(\lightning)$. Thus, $\underline{S}_2 \setminus \underline{S}_1$ is an interval in $\underline{L} \setminus \underline{S}_1$.

It remains to show that $\underline{S}_2 \setminus \underline{S}_1$ is dismantling for $\underline{L} \setminus \underline{S}_1$. Let $[u, v] = \underline{S}_2 \setminus \underline{S}_1$ and assume $\underline{S}_2 \setminus \underline{S}_1$ is not dismantling. Then, u is not supremum-prime in $(v]$ or v is not infimum-prime in $[u)$. Without loss of generality, assume u is not supremum-prime in $(v]$. Then, there exist $x, y \in \underline{L} \setminus \underline{S}_1$ such that $x, y \notin [u, v]$ and $x \vee y \in [u, v]$. Hence, $x \vee y \in \underline{S}_2$ in $\underline{L}$. Because $\underline{S}_2$ is dismantling for $\underline{L}$ it follows that $x \in \underline{S}_2$ or $y \in \underline{S}_2$ and therefore u is supremum-prime($\lightning$). Thus, $\underline{S}_2 \setminus \underline{S}_1$ is dismantling in $\underline{L} \setminus \underline{S}_1$. □

Let $DInt(\underline{L})$ be the family of all subsets of $\underline{L}$ that can be obtained by iterated dismantling by intervals from $\underline{L}$, i. e., by iteratively removing dismantling intervals starting from $\underline{L}$. A smallest element of $DInt(\underline{L})$ is called a *DInt-core* of $\underline{L}$.

Theorem 9.2
Let $\underline{L}$ be a lattice. There exists a unique DInt-core*.*

Proof Let $\underline{U}, \underline{V} \in DInt(\underline{L})$ be two minimal elements in $DInt(\underline{L})$. Then, there is a minimal upper bound $\underline{T} \in DInt(\underline{L})$ of $\underline{U}$ and $\underline{V}$, i. e., both $\underline{U}$ and $\underline{V}$ are obtained by removing dismantling intervals from $\underline{T}$. Hence, there are two sequences of intervals $\underline{S}_1, \ldots, \underline{S}_k, \underline{R}_1, \ldots, \underline{R}_l$ such that $\underline{U} = \underline{T} \setminus (\underline{S}_1 \cup \cdots \cup \underline{S}_k) = (\underline{T} \setminus \underline{S}_1) \ldots \setminus \underline{S}_k$ and $\underline{V} = \underline{T} \setminus (\underline{R}_1 \cup \cdots \cup \underline{R}_l) = (\underline{T} \setminus \underline{R}_1) \ldots \setminus \underline{R}_l$. By iterative application of Proposition 9.7 we have that $\underline{T} \setminus (\underline{S}_1 \cup \cdots \cup \underline{S}_k \cup \underline{R}_1 \cup \cdots \cup \underline{R}_l) \in DInt(\underline{L})$. □

In particular, we call a lattice *interval-dismantlable* if the DInt-Core of the lattice is trivial, i. e., a lattice of two elements. For example, the lattices in Figures 9.1 to 9.4 are interval-dismantlable whereas the lattice in Figure 9.5 is not. Note that the DInt-Core of a lattice does not contain any doubly irreducible elements.

9.3 Dismantling in the Formal Context

In this section, we show that dismantling intervals can be identified directly in the formal context. Based on this, we propose an algorithm to find all dismantling intervals for a given formal context. Thus, we can omit the (expensive) computation of the concept lattice. To this end, we make use of the arrow relations to identify the irreducible concepts of a lattice as seen in Proposition 3.5. Thus, the statement of Proposition 3.4 can be adapted to sublattices and their corresponding subcontexts, in particular to filters and ideals.

Proposition 9.8
Let $\mathbb{K} = (G, M, I)$ with $g \in G$ and $m \in M$. Then

i) γg *is supremum-irreducible in* $(\mu m]$ *if and only if an attribute* $n \in M$ *with* $g \swarrow\!\!\!\nearrow n$ *in* $[m', M] \leq \mathbb{K}$ *(clarified) exists.*

ii) μm *is supremum-irreducible in* $[\gamma g)$ *if and only if an object* $h \in G$ *with* $h \swarrow\!\!\!\nearrow m$ *in* $[G, g'] \leq \mathbb{K}$ *(clarified) exists.*

Proof The proof follows directly from Proposition 3.5 with $\underline{\mathfrak{B}}([m', M]) = (\mu m]$ and $\underline{\mathfrak{B}}([G, g']) = [\gamma g)$. □

Based on this equivalence, we propose a characterization for supremum-prime and infimum-prime concepts in the formal context as follows:

Proposition 9.9
Let $\mathbb{K} = (G, M, I)$ be a formal context with $g \in G$ and $m \in M$. Then

1. γg *is supremum-prime in* $(\mu m]$ *if and only if*

   i) $\exists n \in M : g \swarrow\!\!\!\nearrow n$ *in* $[m', M]$ *(attribute-clarified) and*

   ii) $\nexists n \neq k \in M : g \nearrow k$ *in* $[m', M]$ *(attribute-clarified).*

2. μm *is infimum-prime in* $[\gamma g)$ *if and only if*

   i) $\exists h \in G : h \swarrow\!\!\!\nearrow m$ *in* $[G, g']$ *(object-clarified) and*

   ii) $\nexists h \neq o \in M : o \nearrow m$ *in* $[G, g']$ *(object-clarified).*

Proof We show the first part of the statement:
"$\Leftarrow$": We show this by contraposition. Assume γg is not supremum-prime in $(\mu m]$ but is supremum-irreducible (otherwise, use Proposition 9.8). Hence, $\exists c_1, c_2 \in (\mu m]$ with $c_1 \neq c_2, \gamma g \nleq c_1, c_2, \gamma g \leq c_1 \vee c_2$. Let $c_i = \mu k_i$ with $k_i \in M$ such that there is no

$l \in M$ with $\mu l > \mu k_i$ and $\gamma g \nleq \mu l$, i. e., choose maximal attribute concepts not larger than γg.

We show that $g \nearrow k_i$ using the characterization $\gamma g \vee \mu k_i = (\mu k_i)^\star \neq \mu k_i$. The second part, $(\mu k_i)^\star \neq \mu k_i$, is fulfilled by choice of μk_i. If we assume $\gamma g \vee \mu k_i \neq (\mu k_i)^\star$, then $(\mu k_i)^\star < \gamma g \vee \mu k_i$ and thus there exists some $l \in M$ with $l \in int((\mu k_i)^\star)$, $l \notin int(\gamma g)$ and $\text{!`}l \in int(\mu k_i)(\lightning)$. Hence, $g \nearrow k_i$ in $\mathbb{K}|_{m',M}$.

"$\Rightarrow$": We show this by contraposition. Assume $k, n \in M$, $k \neq n$, $g \nearrow n$, $g \nearrow k$ in $\mathbb{K}|_{m',M}$ (clarified). From $g \nearrow n$ we have $\gamma g \vee \mu n = (\mu n)^\star \neq \mu n$ and thus $\mu n \ngeq \gamma g$. Analogously, $g \nearrow k$ implies $\gamma g \vee \mu k = (\mu k)^\star \neq \mu k$ and thus $\mu k \ngeq \gamma g$. Since $\gamma g \vee \mu k = (\mu k)^\star$ and $\gamma g \vee \mu n = (\mu n)^\star$, we have $\mu k \ngeq \mu n$ and $\mu n \ngeq \mu k$. Thus, we have $\mu k \vee \mu n \geq \gamma g$.

The second part of the statement can be shown analogously. □

Now, two questions arise. First, given that we have a formal context $\mathbb{K}$ and an interval $[\gamma g, \mu m]$ between an object concept and an attribute concept, is this interval dismantling in $\underline{\mathfrak{B}}(\mathbb{K})$? And second, given a formal context $\mathbb{K}$, which are the dismantling intervals in the corresponding concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$?

To answer the first question Proposition 9.9 tells us that it suffices to check the arrow relations of g in the subcontext $[m', M]$ and of m in $[G, g']$: If g only has a single $\swarrow\!\!\!\nearrow$ in $[m', M]$ and no additional $\nearrow$, then γg is supremum-prime in $(\mu m]$. Analogously, if m only has a single $\swarrow\!\!\!\nearrow$ in $[G, g']$ and no additional $\swarrow$, then μm is infimum-prime in $[\gamma g)$. If both conditions hold, then the interval $[\gamma g, \mu m]$ is dismantling in $\underline{\mathfrak{B}}(\mathbb{K})$. Note that, if $\gamma g \nleq \mu m$ then $g \notin [m', M]$ and $m \notin [G, g']$.

For example, consider the context and concept lattice from Figure 9.1. If we want to check in $\mathbb{K}$ if $[\gamma 4, \mu e]$ is a dismantling interval for $\underline{\mathfrak{B}}(\mathbb{K})$ it suffices to check the arrow relations of the object "4" in $[e', M]$ and of the attribute "e" in $[G, 4']$, cf. Figure 9.6. In $[e', M]$ we see that $4 \swarrow\!\!\!\nearrow a$ and that there is no $n \in M, n \neq a$ with $4 \nearrow n$, hence $\gamma 4$ is supremum-prime in $(\mu e]$. Similarly, in $[G, 4']$ we see that $3 \swarrow\!\!\!\nearrow e$ and that there is no $h \in G$, $h \neq 3$ with $h \swarrow e$, hence μe is infimum-prime in $[\gamma 4)$. Therefore, the interval $[\gamma 4, \mu e]$ is dismantling for $\underline{\mathfrak{B}}(\mathbb{K})$.

In order to compute all dismantling intervals for a given formal context, the naive approach is to check all intervals $[\gamma g, \mu m]$ between object concepts and attribute concepts. However, this (essentially iterative) approach results in the repeated computation of the same subcontexts. To prevent this, we instead compute each subcontext only once and for each object concept γg we check which attribute concepts are infimum-prime in $[\gamma g)$, i. e., we check the arrow relations in $[G, g']$, and

$[e', M]$	a	b	c	d	e
2	×	⤢	↙	↙	×
4	⤢	×	×	↙	×
6	↗	×	⤢		×

$[G, 4']$	b	c	e
1	↗		↗
2	⤢		×
3	×	×	⤢
4	×	×	×
5	×	↗	↗
6	×	⤢	×

Figure 9.6 Subcontexts of the context $\mathbb{K}$ from Figure 9.1 where the arrow relations are checked to decide whether the interval $[\gamma 4, \mu e]$ is dismantling for $\underline{\mathfrak{B}}(\mathbb{K})$.

Algorithm 3: Computation of all dismantling intervals for $\mathbb{K}$

Input: $\mathbb{K} = (G, M, I)$
Result: The set of all dismantling intervals for $\mathbb{K}$.

$U = \varnothing$
$O = \varnothing$
for $g \in G$ **do**
  compute $[G, g']$ and clarify objects
  compute $\nearrow([G, g']) = \{(h, m) \mid h \nearrow m \text{ in } [G, g']\}$
  for $m \in g'$ **do**
    $H_m = (G \times \{m\}) \cap \nearrow([G, g'])$
    if $H_m = \{(h, m)\}$ *and* $h \swarrow m$ *in* $[G, g']$ **then**
      $U = U \cup \{(g, m)\}$
for $m \in M$ **do**
  compute $[m', M]$ and clarify attributes
  compute $\swarrow([m', M]) = \{(g, n) \mid g \swarrow n \text{ in } [m', M]\}$
  for $g \in m'$ **do**
    $N_g = (\{g\} \times M) \cap \swarrow([m', M])$
    if $N_g = \{(g, n)\}$ *and* $g \nearrow n$ *in* $[m', M]$ **then**
      $O = O \cup \{(g, m)\}$
return $\{[\gamma g, \mu m] \mid (g, m) \in O \cap U\}$

---

vice versa. More precisely, for each object g we take the attributes m where μm is infimum-prime in $[\gamma g)$ and collect them in the set

$$U = \{(g, m) \mid g \in G, \mu m \text{ infimum-prime in } [\gamma g)\}.$$

Similarly, for each attribute m we take the objects g where γg is supremum-prime in $(\mu m]$ and collect them in the set

$$O = \{(g, m) \mid m \in M, \gamma g \text{ supremum-prime in } (\mu m]\}.$$

If a pair (g, m) is in both U and O, then the respective interval $[\gamma g, \mu m]$ is dismantling for the lattice $\underline{\mathfrak{B}}(\mathbb{K})$. Note that it suffices to consider the reduced formal context since it includes all objects with irreducible object concepts and all attributes with irreducible attribute concepts. In Algorithm 3 we present an implementation in pseudo-code.

If we are interested in the dismantling intervals of a lattice $\underline{L}$, we can simply compute them for its standard context, i. e., for the context $(J(\underline{L}), M(\underline{L}), \leq)$.

9.4 Conclusion

In this chapter, we introduced the notion of dismantling intervals for a lattice in order to transfer the notion of dismantling doubly irreducible elements to a set of elements. In particular, we showed the connection between closed subrelations on the context side and dismantling intervals on the lattice side, and more generally, the connection between closed-subcontexts and quasi-dismantling intervals. While a lattice can always be shrunk to the trivial empty lattice by removing a quasi-dismantling interval, iteratively removing only dismantling intervals for a lattice results in a unique (not necessarily trivial) smallest sublattice. The dismantling intervals can be found directly in the formal context $\mathbb{K}$ with the help of the arrow relations. We showed how to decide in $\mathbb{K}$ if a given interval is dismantling for $\underline{\mathfrak{B}}(\mathbb{K})$. Additionally, given $\mathbb{K}$, we propose an algorithm to compute all dismantling intervals of $\underline{\mathfrak{B}}(\mathbb{K})$ without first computing the concept lattice itself.

Part V

Summary and Outlook

CHAPTER 10

Summary and Outlook

Our goals in this thesis were to investigate substructures that are responsible for concept lattices tending to become large and hard to grasp for the human observer and to condense the visual representation of data without creating information artifacts. Therefor, we at first analyzed the connection of Boolean substructures in formal contexts and in the corresponding concept lattices, motivated by the exponential growth of lattices based on those substructures.

After introducing the required definitions and notations for our work in Part I, we connected Boolean substructures in formal contexts and in the associated concept lattices in Part II. First, we introduced Boolean subcontexts of a formal context as an enlargement of contranominal scales to investigate their interplay with Boolean suborders in the corresponding concept lattice. We further expanded the notion of closed subrelations of a formal context to the notion of closed-subcontexts and showed that those are in a one-to-one correspondence to the set of all sublattices of the referring concept lattice. This connection can be restricted to Boolean sublattices (and Boolean closed-subcontexts). To not be limited to Boolean sublattices but in addition investigate Boolean subsemilattices and Boolean suborders in general, we lifted the embeddings φ_1 und φ_2 of Ganter and Wille to the level of subcontexts and suborders. In addition, we introduced the mapping ψ from Boolean suborders on the lattice side to the newly defined associated Boolean subcontexts on the context side. We examined the structural properties and the interplay of all those maps and detected that φ_1 is the reversion of ψ if and only if the addressed Boolean suborder is join-preserving. Dually, φ_2 is the reversion of ψ if and only if the addressed Boolean suborder is meet-preserving. This means that every subsemilattice of a given lattice can be associated with a concrete subcontext of the corresponding concept lattice.

The second goal of introducing and investigating different approaches to decrease the size of a lattice to improve the readability and understandability for the human observer was focused in Part III and Part IV where the preservation of the underlying structure was measured in different ways. In Part III, we chose subcontexts of the original formal context by attribute selection based on two different approaches.

In Chapter 6, we selected attributes based on their relative relevance to the concept lattice. This measure is based on the impact of an attribute on the distribution of the objects in the concepts and on the preservation of the lattice structure. To overcome computational limitations, we presented an approximation for attribute relevance based on entropy functions adapted to formal contexts. Based on this, we measured the change in the lattice size and entropy by eliminating an attribute set. We concluded from our experiments, that the approximation is a good choice to compute relative relevant attributes.

The second approach, presented in Chapter 7, was based on the connection between contranominal scales in a formal context and Boolean suborders in the corresponding concept lattice, as well as the exponential size of those suborders. To this end, we defined the contranominal-influence of attributes based on the occurrence of an attribute in large contranominal scales and generated the δ-adjusted subcontext by eliminating attributes with high contranominal-influence.

Both presented approaches result in a sub-$\wedge$-semilattice of the original lattice of significantly reduced size. In addition, the size of the canonical base of the underlying implications decreases. However, most of the underlying knowledge is still incorporated in the generated substructures, as shown by an experiment with decision trees. Thus, our approaches enable a human observer to understand large datasets.

In Part IV, another approach for reducing the lattice size was investigated. Here, we focused on eliminating selected intervals of a lattice directly since they represent structural units. In particular, when operating those approaches, it is possible to select intervals that include Boolean sublattices or Boolean suborders.

In Chapter 8, we examined the possibility of imploding intervals in a lattice, meaning mapping each interval to a single representative by a surjective, order-preserving map. Here our goal is to preserve as much of the other elements of the lattice and the lattice structure as possible. We examined factorizations based on congruence relations and tolerance relations and found out that there is always a relation that implodes the given interval and constructed an order-preserving and also join- and meet-preserving factor lattice. To this end, we introduced approaches to find the finest of those relations utilizing the one-to-one correspondence of congruence relations to compatible subcontexts and the one-to-one correspondence of tolerance relations to block relations in the generic formal context of the considered lattice. However, the imploded interval is often significantly larger than the one originally selected. Since our motivation is based on the exponential size of the Boolean suborders and, therefore, on eliminating intervals including Boolean suborders, we tried to implode those structures directly. Inspired by block relations adding new incidences in the formal context, this was done by adding the missing incidences of a contranominal scale, creating a new concept that includes all attributes and objects of the considered subcontext and eliminates the other concepts referring to the contranominal scale. However, that approach can generate new contranominal scales and therefore result in a lattice of larger size than the original one. This can also be the case when considering not only contranominal scales but also associated Boolean subcontexts for Boolean suborders or even for Boolean sublattices. Therefore, we turned away from this approach and focused on the factorization of lattices. To overcome the phenomenon of imploding more than the selected interval, we introduced interval relations. They enable us to implode exactly a given interval. However, as a trade-off, the generated factor set of this approach is, in general, neither join- nor meet-preserving. In addition, the selected interval has to meet the introduced criteria of being pure for the factor set to be a lattice. Otherwise, just an ordered set arises. In the case of imploding more than one interval at a time, the generated factor set is, in general, not even an ordered set. This can be determined by the occurrence of the intervals in Penrose crowns included in the lattice. We concluded this chapter with a context construction for factorizations based on interval relations by introducing enrichments of formal contexts by intervals. In the case of imploding pure intervals, we utilized the one-to-one correspondence between those enrichments and the interval relations to provide the corresponding formal context of the factor set directly.

In Chapter 9, we followed the idea of eliminating an interval of a lattice. However, we do not want to have a (new generated) representative in the generated lattice but eliminate the interval entirely and, at the same time, preserve all of the other elements of the lattice as well as their order and lattice properties. To this end, we introduced dismantling (and quasi-dismantling) intervals of a lattice as a generalization of dismantling an element of a lattice. We connected closed subrelations on the context side and dismantling intervals on the lattice side. This connection can be generalized to closed-subcontexts and quasi-dismantling intervals. Further, we proved the existence of a unique DI-kernel. Utilizing arrow relations of a formal context $\mathbb{K}$ the dismantling intervals of the corresponding concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ can be determined directly. To this end, we proposed an algorithm to compute all dismantling intervals of $\underline{\mathfrak{B}}(\mathbb{K})$ in $\mathbb{K}$.

We conclude this work with an outlook to possible further work. In the realm of the investigations in Chapter 5, we are interested in the proof of the proposed conjecture. It leads to a follow-up question of how many Boolean suborders of a lattice can be reached from the Boolean subcontexts of the generic formal context by utilizing φ_1, φ_2 or a combination of both, meaning a map, that decides for each concept of a Boolean subcontext whether it uses φ_1 or φ_2. Further, the intervals between the images of φ_1 or φ_2 for concepts are disjoint. Therefore, a factorization based on this characteristic could be examined. In addition, investigating non-transitive interval relations as a generalization of tolerance relations is a possible research question.

In the realm of attribute selection a combination of both presented approaches is of interest. Further, the dual application on the object is a possible extension of the approaches. However, in this case, no sub-$\wedge$-semilattice arises, and wrong implications could be generated.

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