Twin Basic Quantum Calculus and its Applications

By

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Contents

Chapter 1

Introduction

The q-analysis (including q-differentiation, q-integration, partial q-differentiation, q-integral transform,...) and q-special functions (including q-hypergeometric series, q-Gamma functions, q-Beta functions, q-Mittag-Leffler functions,...) essentially started in 1748 when Euler considered the infinite product

$$(q;q)_\infty = \prod_{k=0}^{\infty}(1-q^{k+1})^{-1}$$

as a generating function for $p(n)$, the number of partitions of a positive integer into positive integers. But it was not until about hundred years later that the subject acquired an independent status when Heine (see [41]) converted the simple observation that

$$\lim_{q\to 1}\frac{1-q^a}{1-q} = a$$

into a systematic theory of ${}_2\phi_1$ basic hypergeometric series parallel to Gauss' ${}_2F_1$ hypergeometric series. This led to an intensive investigation on q-calculus during this century. Few of the significant results are for example the relation of Heine's ${}_2\phi_1$ with the Ramanujan formula, the relation between Euler's identities and the Jacobi triple product.

The systematic development of q-calculus started with Jackson who reconsidered the Euler-Jackson q-difference operator (see [45]) in 1908, who gave a q-form of Taylor's theorem (see [46]) and introduced the q-definite integral on a finite interval (see [47]). This theory has now played a crucial role in almost every branch of mathematics. It found applications for example in the field of Special functions, differential equations, combinatorics, number theory.

During the same period, new mathematics objects of the theory of symmetries appeared. They are quantum groups and quantum algebras (q-deformations of Lie groups and Lie algebras). Investigations of representations of these groups and algebras showed that these representations of Lie groups are related to special functions of mathematical physics.

The two-parameter quantum algebra, $U_{p,q}(gl(2))$, was first introduced in [25] in order to generalize and unify a series of q-oscillator algebra variants, known in the earlier physics and mathematics literature on the representation theory of single parameter quantum algebra. Then investigations came up in the same direction among which the work of Burban and Klimyk [23] on representations of two-parameter quantum groups and models of two parameter quantum algebra $U_{p,q}(su_{1,1})$ and (p,q)-deformed algebra. In the same paper [23], Burban and Klimyk introduced the (p,q)-hypergeometric functions. The (p,q)-deformation rapidly found applications in physics and mathematical physics as described for example in [37].

In the same vein, after recalling the connection between the Roger-Szegö polynomials and the q-oscillator, Jagannathan and Sridhar [50] have defined (p,q)-Rogers-Szegö polynomials and have shown that they are connected with the (p,q)-deformed oscillator associated with the Jagannathan-Srinavasa (p,q)-numbers [49], and proposed a new realization of this algebra.

This work is divided into eleven chapters.

Chapter 1 is a general introduction of the thesis.

Chapter 2 provides some definitions about the (p,q)-differential and the (p,q)-derivative. The (p,q)-analogues of the binomial coefficients are introduced and the (p,q)-Leibniz rule for the nth (p,q)-derivative of a product of two functions is stated and proved. Results of this chapter are published in [69].

Chapter 3 introduces a polynomial basis called (p,q)-power, that generalizes both the canonical power basis and the classical q-Pochhammer. Several properties of the (p,q)-powers are stated and proved. Those (p,q)-powers are finally used to state and prove the (p,q)-analogues of Taylor's formula for polynomials. These (p,q)-Taylor formulas are used to provide a (p,q)-analogue of the Taylor expansion of $f(x)=\dfrac{1}{(x-1)^n}$ known in the q-theory as Heine's binomial formula. Results of this chapter are published in [69].

In *Chapter 4*, the (p,q)-binomial coefficients are studied in detail. Their recurrence relations are given. A new and generalized orthogonality relation is obtained and the (p,q)-powers are used to state a (p,q)-analogue of the Vandermonde formula. Some results of this are available in [32]. The generalized orthogonality relation and the (p,q)-Vandermonde formula appear here for the first time.

In *Chapter 5*, we introduce two (p,q)-analogues of the exponential function and provide several of their representations based on the Taylor formulas proved in Chapter 3. Next, the (p,q)-trigonometric functions and the hyperbolic (p,q)-trigonometric functions are introduced and their main properties stated. Results of this chapter are published in [69, 68].

In *Chapter 6*, we derive the (p,q)-antiderivative and the (p,q)-integral. Their algebraic properties are studied, the fundamental theorem of (p,q)-calculus is proved and the formula of (p,q)-integration by parts is provided. Results of this chapter are published in [69].

In *Chapter 7*, two (p,q)-analogues of the Gamma function are introduced and their relevant properties are proved. Next, three (p,q)-analogues of the Beta function are given. It is proved that they are related to the (p,q)-Gamma function previously introduced. Results of this chapter are published in [67].

In *Chapter 8*, we discuss the (p,q)-hypergeometric series. They are generalizations of q-hypergeometric series. Note that from Proposition 90, it is seen that any well behaved ϕ-series can be written as a Φ-series. But the converse is not always true. In the general case, when $p \neq 1$, this is possible only for an ${}_r\Phi_{r-1}$. To see this, it is enough to look at the ${}_0\Phi_0$ case. Indeed,

$$\begin{aligned} {}_0\Phi_0\left(\left.\begin{matrix} - \\ (p,q) \end{matrix}\right| (p,q); z\right) &= \sum_{n=0}^{\infty} \frac{(-1)^n (q/p)^{\binom{n}{2}}}{(p\ominus q)_{p,q}^n} z^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (\rho/p)^{\binom{n}{2}}}{(\rho;\rho)_n} (z/\rho)^n, \quad \text{with } \rho = q/p, \end{aligned}$$

which shows that ${}_0\Phi_0$ becomes a ϕ-series if and only if $p = 1$. Similarly, one is easily convinced that a generic ${}_r\Phi_s$-series cannot be identified within the class of ϕ-series unless $p = 1$ or $s = r - 1$. It is now clear that the (p,q)-series is a larger structure in which the q-series gets embedded. Also, whereas in the case of the ϕ-series one will have to resort to the limit process of confluence, namely, replacing z by z/a_r and taking the limit $a_r \to \infty$, in the (p,q)-series, it is sometimes enough to make the choice $a_{ip} = 0$ or $b_{ip} = 0$ for some specific i.

In *Chapter 9*, we introduce a (p,q)-analogue of Sturm-Liouville problems and study their orthogonal solutions. Next, applications are done to find a (p,q)-analogue of the Jacobi, the Laguerre and the Hermite polynomials. Some results of this chapter were published in [84].

In *Chapter 10*, two (p,q)-analogues of the Laplace transform are introduced and their relevant fundamental properties are stated and proved. It is shown how they can be used to solve (p,q)-differential equations. Next, we introduce double (p,q)-Laplace transforms for solving partial (p,q)-differential equations and some functional equations. Some results of this chapter were published in [68].

Finally, *Chapter 11* deals with a special class of (p,q)-polynomials, that are (p,q)-analogues of Appell polynomials. Their main characterizations and their algebraic structure are studied. Next, some examples of such polynomials are given, namely the (p,q)-Bernoulli, the (p,q)-Euler, the (p,q)-Genocchi, a second (p,q)-analogue of the Hermite polynomials and a kind of bivariate (p,q)-Bernoulli polynomials.

The thesis ends with a conclusion and further perspectives.

Chapter 2

The (p, q)-Derivative

2.1 Definition and properties

Definition 1 (Njionou [69]). *Let f be an arbitrary function. Its (p, q)-differential is defined by*

$$d_{p,q}f(x) = f(px) - f(qx). \tag{2.1}$$

In particular, $d_{p,q}x = (p - q)x$.

Proposition 2 (Njionou [69]). *The (p, q)-differential fulfils the following product rule*

$$d_{p,q}(f(x)g(x)) = f(px)d_{p,q}g(x) + g(qx)d_{p,q}f(x). \tag{2.2}$$

Proof. For f and g two arbitrary functions, we have

$$\begin{aligned} d_{p,q}(f(x)g(x)) &= f(px)g(px) - f(qx)g(qx) \\ &= f(px)g(px) - f(px)g(qx) + f(px)g(qx) - f(qx)g(qx) \\ &= f(px)(g(px) - g(qx)) + g(qx)(f(px) - f(qx)) \\ &= f(px)d_{p,q}g(x) + g(qx)d_{p,q}f(x). \end{aligned}$$

□

With the two parameter quantum differential, we can also define the corresponding two-parameter quantum derivative.

Definition 3 (Chakrabarti and Jagannathan [25]). *The following expression*

$$D_{p,q}f(x) = \frac{d_{p,q}f(x)}{d_{p,q}x} = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0 \tag{2.3}$$

is called the (p, q)-derivative of the function $f(x)$.

Note that when $p = 1$, the $D_{p,q}$ reduces to the quantum derivative D_q (see Kac and Cheung [52])

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}.$$

Note also that if $f(x)$ is differentiable, then

$$\lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx}.$$

It is clear that as with the ordinary derivative, the action of the (p, q)-derivative of a function is a linear operator. More precisely, for any constants a and b,

$$D_{p,q}(af(x) + bg(x)) = aD_{p,q}f(x) + bD_{p,q}g(x).$$

Example 4. *Compute the (p,q)-derivative of $f(x) = x^n$, where n is a positive integer. By definition*

$$D_{p,q}x^n = \frac{(px)^n - (qx)^n}{(p-q)x} = \frac{p^n - q^n}{p-q}x^{n-1}. \tag{2.4}$$

Since $\frac{p^n - q^n}{p-q}$ appears quite frequently, let us introduce the following notation

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}, \tag{2.5}$$

for any positive integer n. This is called the (p,q)-bracket, the (p,q)-number, the twin-basic number or the (p,q)-analogue of n. Then, (2.5) becomes

$$D_{p,q}x^n = [n]_{p,q}x^{n-1}. \tag{2.6}$$

The twin-basic number is a natural generalization of the q-number, that is

$$\lim_{p\to 1}[n]_{p,q} = [n]_q. \tag{2.7}$$

Proposition 5 (Bukweli and Hounkonnou, [22]). *If n and m are non-negative integers, then*

$$[n+m]_{p,q} = q^m[n]_{p,q} + p^n[m]_{p,q} = p^m[n]_{p,q} + q^n[m]_{p,q}, \tag{2.8}$$

$$[-m]_{p,q} = -q^{-m}p^{-m}[m]_{p,q}, \tag{2.9}$$

$$[n-m]_{p,q} = q^{-m}\left([n]_{p,q} - p^{n-m}[m]_{p,q}\right) = p^{-m}\left([n]_{p,q} - q^{n-m}[m]_{p,q}\right). \tag{2.10}$$

Proposition 6. *For $n \geq m$, $n, m \in \{0, 1, 2, \ldots\}$, the following equations apply*

$$p^{n-1}([n]_{p,q} - q^{n-m}[m]_{p,q}) = p^{n+m-1}[n-m]_{p,q} \tag{2.11}$$

$$[n]_{p,q}[n-1]_{p,q} - (pq)^{n-m}[m]_{p,q}[m-1]_{p,q} = [n-m]_{p,q}[n+m-1]_{p,q}. \tag{2.12}$$

Remark 7. *Note that* (2.11) *and* (2.12) *reduce to Equations (2.2.3) and (2.2.4) in [53, Page 30]*

Proposition 8 (Njionou [69]). *The (p,q)-derivative fulfils the following product rules*

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x), \tag{2.13}$$

$$D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x) \tag{2.14}$$

Proof. From the definition of the (p,q)-derivative and (2.2), we have

$$D_{p,q}(f(x)g(x)) = \frac{d_{p,q}(f(x)g(x))}{(p-q)x} = \frac{f(px)d_{p,q}g(x) + g(qx)d_{p,q}f(x)}{(p-q)x},$$

hence

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x).$$

This proves (2.13). (2.14) is obtained by symmetry. □

Proposition 9 (Njionou [69]). *The (p,q)-derivative fulfils the following quotient rules*

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)} \tag{2.15}$$

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)} \tag{2.16}$$

Proof. In order to get those quotient rules, we remark that $f(x) = g(x)\dfrac{f(x)}{g(x)}$. Thus, applying (2.13) to this relation we get

$$D_{p,q}f(x) = g(px)D_{p,q}\left(\frac{f(x)}{g(x)}\right) + \frac{f(qx)}{g(qx)}D_{p,q}g(x),$$

and thus (2.15). In the same manner, applying (2.16) provides (2.14). □

2.2 (p,q)-Binomial coefficients

In this section we introduce the (p,q)-factorial and the (p,q)-binomials coefficients. Chapter 4 is devoted to these (p,q)-binomial coefficients since they are very useful in combinatorics.

Definition 10 (**(p,q)-factorial** (R. Jagannathan, R. Sridhar [50], Njionou [68])). *The (p,q)-factorial is defined by*

$$[n]_{p,q}! = \prod_{k=1}^{n}[k]_{p,q}!, \quad n \geq 1, \quad [0]_{p,q}! = 1. \tag{2.17}$$

Let us introduce also the so-called (p,q)-binomial coefficients.

Definition 11 (**(p,q)-Binomial** (R. Jagannathan and R. Sridhar [50], Njionou [68])). *The (p,q)-binomial coefficients are defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n. \tag{2.18}$$

Note that as $p \to 1$, the (p,q)-binomial coefficients reduce to the q-binomial coefficients (see Kac and Cheung [52]).
It is clear by definition that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q}. \tag{2.19}$$

Proposition 12 (Njionou [68]). *Let n be a non-negative integer, then the following formula applies*

$$D^n_{p,q}\left[\frac{1}{x}\right] = (-1)^n\frac{[n]_{p,q}!}{(pq)^{\binom{n+1}{2}}x^{n+1}}. \tag{2.20}$$

Proof. The relation is obvious for $n = 0$. Let $n \geq 1$, assume that (2.20) holds true. Then

$$\begin{aligned}
D^{n+1}_{p,q}\left[\frac{1}{x}\right] &= D_{p,q}\left[(-1)^n\frac{[n]_{p,q}!}{(pq)^{\binom{n+1}{2}}x^{n+1}}\right] \\
&= \frac{(-1)^n[n]_{p,q}!}{(pq)^{\binom{n+1}{2}}} \times \frac{1}{(p-q)x}\left(\frac{1}{(px)^{n+1}} - \frac{1}{(qx)^{n+1}}\right) \\
&= \frac{(-1)^n[n]_{p,q}!}{(pq)^{\binom{n+1}{2}}} \times \frac{-[n+1]_{p,q}}{(pq)^{n+1}x^{n+2}} = [(-1)^{n+1}\frac{[n+1]_{p,q}!}{(pq)^{\binom{n+2}{2}}x^{n+2}}.
\end{aligned}$$

The proof is then complete. □

The next proposition generalizes (2.20) and is proved in the same way.

Proposition 13 (Njionou [68]). *Let a be a non-zero complex number. Then the following equation holds true:*

$$D_{p,q}^{n}\left[\frac{1}{ax+b}\right]=\frac{(-a)^{n}[n]_{p,q}!}{\prod\limits_{k=0}^{n}(ap^{n-k}q^{k}x+b)} \tag{2.21}$$
$$=\frac{(-a)^{n}[n]_{p,q}!}{(ap^{n}x+b)(ap^{n-1}qx+b)\cdots(apq^{n-1}x+b)(aq^{n}x+b)}.$$

Note that for $a=1$ and $b=0$, (2.21) reduces to (2.20).

2.3 (p,q)-Leibniz formula and power derivative

Let n be a nonnegative function. If f and g are two n times differentiable functions, the classical Leibniz formula states that

$$(f(x)g(x))^{(n)}=\sum_{m=0}^{n}\binom{n}{m}f^{(m)}(x)g^{(n-m)}(x), \tag{2.22}$$

where $f^{(n)}(x)$ stands for the n-th derivative of $f(x)$.
The q-analogue of this formula states (Kac and Cheung [53])

$$D_{q}[f(x)g(x)]=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_{q}(D_{q}^{n-k}f)(q^{k}x)(D_{q}^{k}g)(x). \tag{2.23}$$

In the following theorem, we state the (p,q)-generalization of these results.

Theorem 14 (**(p,q)-Leibniz theorem**, (Araci et *al.* [15])). *Let f and g be two (p,q)-differentiable functions. The following (p,q)-derivative rules are valid*

$$D_{p,q}^{n}[f(x)g(x)]=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(D_{p,q}^{k}f)(p^{n-k}x)(D_{p,q}^{n-k}g)(q^{k}x), \tag{2.24}$$

$$D_{p,q}^{n}[f(x)g(x)]=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(D_{p,q}^{n-k}f)(p^{k}x)(D_{p,q}^{k}g)(q^{n-k}x). \tag{2.25}$$

Proof. (2.24) is true for $n=0$. Let n be a nonnegative integer, we have

$$\begin{aligned}
D_{p,q}^{n+1}(f(x)g(x)) &= D_{p,q}D_{p,q}^{n}(f(x)g(x))\\
&= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q} D_{p,q}\left[(D_{p,q}^{k}f)(p^{n-k}x)(D_{p,q}^{n-k}g)(q^{k}x)\right]\\
&= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\Big\{(D_{p,q}^{k}f)(p^{n+1-k}x)D_{p,q}\left[(D_{p,q}^{n-k}g)(q^{k}x)\right]\\
&\quad + D_{p,q}\left[(D_{p,q}^{k}f)(p^{n-k}x)\right](D_{p,q}^{n-k}g)(q^{k+1}x)\Big\}\\
&= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\Big\{q^{k}(D_{p,q}^{k}f)(p^{n+1-k}x)(D_{p,q}^{n+1-k}g)(q^{k}x)\\
&\quad + p^{n-k}(D_{p,q}^{k+1}f)(p^{n-k}x)(D_{p,q}^{n-k}g)(q^{k+1}x)\Big\}\\
&= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q} q^{k}(D_{p,q}^{k}f)(p^{n+1-k}x)(D_{p,q}^{n+1-k}g)(q^{k}x)\\
&\quad + \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q} p^{n-k}(D_{p,q}^{k+1}f)(p^{n-k}x)(D_{p,q}^{n-k}g)(q^{k+1}x)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q} q^{k}(D_{p,q}^{k}f)(p^{n+1-k}x)(D_{p,q}^{n+1-k}g)(q^{k}x)\\
&\quad + \sum_{k=1}^{n+1}\begin{bmatrix}n\\k-1\end{bmatrix}_{p,q} p^{n+1-k}(D_{p,q}^{k}f)(p^{n+1-k}x)(D_{p,q}^{n+1-k}g)(q^{k}x)\\
&= \sum_{k=0}^{n+1}(-1)^{k}\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}}q^{\binom{k}{2}}(D_{p,q}^{k}f)(p^{n+1-k}x)(D_{p,q}^{n+1-k}g)(q^{k}x).
\end{aligned}$$

(2.25) follows from (2.24) and the use of (2.19). □

Next, we state the following power derivative for the operator $D_{p,q}$.

Theorem 15 (**Power of** $D_{p,q}$, (Araci et *al.* [15]) **).** *The following derivative rule applies*

$$D_{p,q}^{n}f(x) = (p-q)^{-n}x^{-n}(pq)^{-\binom{n}{2}}\sum_{k=0}^{n}(-1)^{k}\begin{bmatrix}n\\k\end{bmatrix}_{p,q} p^{\binom{k}{2}}q^{\binom{n-k}{2}}f(p^{n-k}q^{k}x),\quad x\neq 0. \tag{2.26}$$

Proof. The relation is obvious for $n=1$. Let $n\geq 1$, assume (2.26) is valid. Thus:

$$\begin{aligned}
(D^{n+1}_{p,q}f)(x) &= D_{p,q}(D^n_{p,q}f)(x)\\
&= (p-q)^{-n}(pq)^{-\binom{n}{2}}[(p-q)x]^{-1}\Bigg\{(px)^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}q^{\binom{n-k}{2}}f(p^{n+1-k}q^kx)\\
&\qquad -(qx)^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}q^{\binom{n-k}{2}}f(p^{n-k}q^{k+1}x)\Bigg\}\\
&= (p-q)^{-(n+1)}(pq)^{-\binom{n}{2}}x^{-(n+1)}(pq)^{-n}\Bigg\{\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}q^{\binom{n-k}{2}+n}f(p^{n+1-k}q^kx)\\
&\qquad -\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}+n}q^{\binom{n-k}{2}}f(p^{n-k}q^{k+1}x)\Bigg\}\\
&= (p-q)^{-(n+1)}(pq)^{-\binom{n+1}{2}}x^{-(n+1)}\Bigg\{\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}q^{\binom{n+1-k}{2}+k}f(p^{n+1-k}q^kx)\\
&\qquad +\sum_{k=1}^{n+1}(-1)^k\begin{bmatrix}n\\k-1\end{bmatrix}_{p,q}p^{\binom{k}{2}+n+1-k}q^{\binom{n+1-k}{2}}f(p^{n+1-k}q^kx)\Bigg\}\\
&= (p-q)^{-(n+1)}(pq)^{-\binom{n+1}{2}}x^{-(n+1)}\\
&\qquad\times\sum_{k=0}^{n+1}(-1)^k\Bigg\{q^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}+p^{n+1-k}\begin{bmatrix}n\\k-1\end{bmatrix}_{p,q}\Bigg\}p^{\binom{k}{2}}q^{\binom{n+1-k}{2}}f(p^{n+1-k}q^kx)\\
&= (p-q)^{-(n+1)}(pq)^{-\binom{n+1}{2}}x^{-(n+1)}\sum_{k=0}^{n+1}(-1)^k\begin{bmatrix}n+1\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}q^{\binom{n+1-k}{2}}f(p^{n+1-k}q^kx).
\end{aligned}$$

This ends the proof. □

Remark 16. *Using (2.19), the relation (2.26) can be written as*

$$D^n_{p,q}f(x)=(q-p)^{-n}x^{-n}(pq)^{-\binom{n}{2}}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}q^{\binom{k}{2}}f(p^kq^{n-k}x). \tag{2.27}$$

For $p=1$, (2.27) reduces to the power derivative for the operator D_q (Annaby et al.[11] F. Ryde [77]).

Chapter 3

The (p, q)-Power and the Taylor Formulas

3.1 The (p, q)-power basis

Here, we introduce the so-called (p, q)-power and investigate some of its relevant properties. These polynomials are useful to state our Taylor formulas.

Definition 17 (Njionou [69]). *The expression*

$$(x \ominus a)^n_{p,q} = (x - a)(px - aq) \cdots (p^{n-1}x - aq^{n-1}) \tag{3.1}$$

is called the (p, q)-power.

Note that for $p = 1$, the (p, q)-power reduces to the q-power (see Kac and Cheung [52]). It should be noted the following important relation between the (p, q)-power basis and the (p, q)-factorial

$$[n]_{p,q}! = \frac{(p \ominus q)^n_{p,q}}{(p - q)^n}. \tag{3.2}$$

Proposition 18 (Njionou [69]). *The following assertion is valid.*

$$D_{p,q}(x \ominus a)^n_{p,q} = [n]_{p,q}(px \ominus a)^{n-1}_{p,q}, \quad n \geq 1, \tag{3.3}$$

and $D_{p,q}(x \ominus a)^0_{p,q} = 0$.

Proof. The assertion $D_{p,q}(x \ominus a)^0_{p,q} = 0$ is obvious. Let $n \geq 1$, then we have

$$\begin{aligned}
(px \ominus a)^n_{p,q} &= (p^n x - aq^{n-1})(px \ominus a)^{n-1}_{p,q} \\
(qx \ominus a)^n_{p,q} &= (qx - a)\prod_{k=1}^{n-1}(qxp^k - aq^k) \\
&= (qx - a)\prod_{k=0}^{n-2}(qxp^{k+1} - aq^{k+1}) \\
&= (qx - a)q^{n-1}\prod_{k=0}^{n-2}(pxp^k - aq^k) \\
&= (xq^n - aq^{n-1})(px \ominus q)^{n-1}_{pq}.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned} D_{p,q}(x\ominus a)^n_{p,q} &= \frac{\left((p^n x - aq^{n-1}) - (xq^n - aq^{n-1})\right)}{(p-q)x}(px\ominus q)^{n-1}_{pq} \\ &= [n]_{p,q}(px\ominus q)^{n-1}_{pq}. \end{aligned}$$

□

We can also handle the proof by induction with respect to n as follows. The result is already true for $n=1$. Let $n\geq 1$, assume that

$$D_{p,q}(x\ominus a)^n_{p,q} = [n]_{p,q}(px\ominus a)^{n-1}_{p,q},$$

therefore, by using the product rule (2.13) where we take $f(x) = (x\ominus a)^n_{p,q}$ and $g(x) = (xp^n - aq^n)$, it follows that

$$\begin{aligned} D^k_{p,q}(x\ominus a)^{n+1}_{p,q} &= D^k_{p,q}\left[(x\ominus a)^n_{p,q}(xp^n - aq^n)\right] \\ &= (px\ominus a)^n_{p,q}p^n + (xqp^n - aq^n)D_{p,q}(x\ominus a)^n_{p,q} \\ &= (px\ominus a)^n_{p,q}p^n + q[n]_{p,q}(xp^n - aq^{n-1})(px\ominus a)^{n-1}_{p,q} \\ &= (px\ominus a)^n_{p,q}p^n + q[n]_{p,q}(x\ominus a)^n_{p,q} \\ &= (p^n + q[n]_{p,q})(px\ominus a)^n_{p,q} \\ &= [n+1]_{p,q}(px\ominus a)^n_{p,q}. \end{aligned}$$

Proposition 19 (Njionou [69])**.** *Let γ be a complex number and $n\geq 1$ be an integer, then*

$$D_{p,q}(\gamma x\ominus a)^n_{p,q} = \gamma[n]_{p,q}(\gamma px\ominus a)^{n-1}_{p,q}. \tag{3.4}$$

Proof. The proof is done exactly as the proof of (3.3). □

We now generalize this result in the following proposition.

Proposition 20 (Njionou [69])**.** *Let $n\geq 1$ be an integer, and $0\leq k\leq n$, we have the following*

$$D^k_{p,q}(x\ominus a)^n_{p,q} = p^{\binom{k}{2}}\frac{[n]_{p,q}!}{[n-k]_{p,q}!}(p^k x\ominus a)^{n-k}_{p,q}. \tag{3.5}$$

In particular for $a=0$ we get

$$D^k_{p,q}x^n = \frac{[n]_{p,q}!}{[n-k]_{p,q}!}x^{n-k}. \tag{3.6}$$

Proof. The proof is done by induction with respect to k. Let $n\geq 1$, for $k=1$, it is the previous proposition. Assume that for a fixed $k<n$, we have

$$D^k_{p,q}(x\ominus a)^n_{p,q} = p^{\binom{k}{2}}\frac{[n]_{p,q}!}{[n-k]_{p,q}!}(p^k x\ominus a)^{n-k}_{p,q}.$$

Then, we get

$$\begin{aligned} D^{k+1}_{p,q}(x\ominus a)^n_{p,q} &= D_{p,q}\left(D^k_{p,q}(x\ominus a)^n_{p,q}\right) \\ &= p^{\binom{k}{2}}\frac{[n]_{p,q}!}{[n-k]_{p,q}!}D_{p,q}(p^k x\ominus a)^{n-k}_{p,q} \\ &= p^{\binom{k}{2}}\frac{[n]_{p,q}!}{[n-k]_{p,q}!}p^k[n-k]_{p,q}(p^k x\ominus a)^{n-k-1}_{p,q} \\ &= p^{\binom{k+1}{2}}\frac{[n]_{p,q}!}{[n-k-1]_{p,q}!}(p^k x\ominus a)^{n-k-1}_{p,q}. \end{aligned}$$

□

Remark 21. *For the classical derivative, it is known that for any real number α, one has*

$$\frac{d}{dx}x^{\alpha} = \alpha x^{\alpha-1}.$$

In what follows, we would like to state a similar result for the $D_{p,q}$ derivative as done for the D_q derivative in [52]. We follow the same procedure.

Proposition 22 (Njionou [69])**.** *Let m and n be two non-negative integers. Then the following assertion is valid.*

$$(x\ominus a)_{p,q}^{m+n} = (x\ominus a)_{p,q}^{m}(p^m x\ominus q^m a)_{p,q}^{n}. \tag{3.7}$$

Proof. By definition,

$$\begin{aligned}
(x\ominus a)_{p,q}^{m+n} &= \prod_{k=0}^{n+m-1}(xp^k - aq^k)\\
&= \prod_{k=0}^{m-1}(xp^k - aq^k)\prod_{k=m}^{n+m-1}(xp^k - aq^k)\\
&= \prod_{k=0}^{m-1}(xp^k - aq^k)\prod_{k=0}^{n-1}((xp^m)p^k - (aq^m)q^k)\\
&= (x\ominus a)_{p,q}^{m}(p^m x\ominus q^m a)_{p,q}^{n}.
\end{aligned}$$

This is the expected result. □

In Proposition 22, if we take $m = -n$, then we get the following extension of the (p,q)-power basis.

Definition 23 (Njionou [69])**.** *Let n be a non-negative integer, then we set the following definition.*

$$(x\ominus a)_{p,q}^{-n} = \frac{1}{(p^{-n}x\ominus q^{-n}a)_{p,q}^{n}}. \tag{3.8}$$

Proposition 24 (Njionou [69])**.** *For any two integers m and n, (3.7) holds.*

Proof. The case $m > 0$ and $n > 0$ has already been proved, and the case where one of m and n is zero is easy. Let us first consider the case $m = -m' < 0$ and $n > 0$. Then,

$$\begin{aligned}
(x\ominus a)_{p,q}^{m}(p^m x\ominus q^m a)_{p,q}^{n} &= (x\ominus a)_{p,q}^{-m'}(p^{-m'}x\ominus q^{-m'}a)_{p,q}^{n}\\
\text{by (3.8)}\quad &= \frac{(p^{-m'}x\ominus q^{-m'}a)_{p,q}^{n}}{(p^{-m'}x\ominus q^{-m'}a)_{p,q}^{m'}}\\
\text{by (3.7)}\quad &= \begin{cases} (p^m(p^{-m}x)\ominus q^m(q^{-m}a))_{p,q}^{n-m'} & \text{if } \ n\geq m'\\ \frac{1}{\left(q^n(q^{-m'}x)\ominus q^n(q^{-m'}a)\right)_{p,q}^{m'-n}} & \text{if } \ n< m'\end{cases}\\
\text{by (3.8)}\quad &= (x\ominus a)_{p,q}^{n-m'} = (x\ominus a)_{p,q}^{n+m}.
\end{aligned}$$

If $m \geq 0$ and $n = -n' < 0$, then

$$\begin{aligned}
(x \ominus a)^m_{p,q}(p^m x \ominus q^m a)^n_{p,q} &= (x \ominus a)^m_{p,q}(p^m x \ominus q^m a)^{-n'}_{p,q} \\
&= \frac{(x \ominus a)^m_{p,q}}{(p^{m-n'} x \ominus q^{m-n'} a)^{n'}_{p,q}} \\
&= \begin{cases} \frac{(x \ominus a)^{m-n'}_{p,q}(p^{m-n'} x \ominus a q^{m-n'})^{n'}_{p,q}}{(p^{m-n'} x \ominus q^{m-n'} a)^{n'}_{p,q}} & \text{if } \ m > n' \\ \frac{(x \ominus a)^m_{p,q}}{(p^{m-n'} x \ominus q^{m-n'} a)^{n'-m}_{p,q}(p^{n'-m}(p^{m-n'}) x \ominus q^{n'-m}(q^{m-n'} a))^m_{p,q}} & \text{if } \ m < n' \end{cases} \\
&= \begin{cases} (x \ominus a)^{m-n'}_{p,q} & \text{if } \ m > n' \\ \frac{1}{(p^{m-n'} x \ominus q^{m-n'} a)^{n'-m}_{p,q}} & \text{if } \ m < n' \end{cases} \\
&= (x \ominus a)^{m-n'}_{p,q} = (x \ominus a)^{m+n}_{p,q}.
\end{aligned}$$

Lastly, if $m = -m' < 0$ and $n = -n' < 0$,

$$\begin{aligned}
(x \ominus a)^m_{p,q}(p^m x \ominus q^m a)^n_{p,q} &= (x \ominus a)^{-m'}_{p,q}(p^{-m'} x \ominus q^{-m'} a)^{-n'}_{p,q} \\
&= \frac{1}{(p^{-m'} x \ominus q^{-m'} a)^{m'}_{p,q}(p^{-n'-m'} x \ominus q^{-n'-m'} a)^{n'}_{p,q}} \\
&= \frac{1}{(p^{-n'-m'} x \ominus q^{-n'-m'} a)^{n'+m'}_{p,q}} \\
&= (x \ominus a)^{-m'-n'}_{p,q} = (x \ominus a)^{m+n}_{p,q}.
\end{aligned}$$

Therefore, (3.7) is true for all integers m and n. □

It is natural to ask ourselves if (3.3) is valid for any integer as well. But before trying to answer this question, let us generalise the twin-basic number as follows.

Definition 25 (Njionou [69])**.** *Let α be any complex number,*

$$[\alpha]_{p,q} = \frac{p^\alpha - q^\alpha}{p - q}. \tag{3.9}$$

Proposition 26 (Njionou [69])**.** *For any integer n,*

$$D_{p,q}(x \ominus a)^n_{p,q} = [n]_{p,q}(px \ominus a)^{n-1}_{p,q}. \tag{3.10}$$

Proof. Note that $[0]_{p,q} = 0$. The result is already proved for $n \geq 0$. If $n = -n' < 0$, using

(2.15) and (3.8) it follows that

$$\begin{aligned}
D_{p,q}(x\ominus a)^n_{p,q} &= D_{p,q}\left(\frac{1}{(p^{-n'}x\ominus q^{-n'}a)^{n'}_{p,q}}\right)\\
&= -\frac{D_{p,q}(p^{-n'}x\ominus q^{-n'}a)^{n'}_{p,q}}{(q^{-n'}(px)\ominus q^{-n'}a)^{n'}_{p,q}(qp^{-n'}x\ominus q^{-n'}a)^{n'}_{p,q}}\\
&= -\frac{p^{-n'}[n']_{p,q}(p^{-n'}x\ominus q^{-n'}a)^{n'-1}_{p,q}}{(q^{-n'}(px)\ominus q^{-n'}a)^{n'}_{p,q}(qp^{-n'}x\ominus q^{-n'}a)^{n'}_{p,q}}\\
&= -\frac{p^{-n'}[n']_{p,q}}{(x-q^{-1}a)(qp^{-n'}x\ominus q^{-n'}a)^{n'}_{p,q}}\\
&= \frac{-p^{-n'}q^{-n'}[n']_{p,q}}{(p^{-n'-1}(px)\ominus q^{-n'-1}a)^{n'+1}_{p,q}}\\
&= -p^{-n'}q^{-n'}[n']_{p,q}(px\ominus a)^{-n'-1}_{p,q}\\
&= [n]_{p,q}(px\ominus a)^{n-1}_{p,q}.
\end{aligned}$$

This was announced. □

Remark 27. *It should be noted that* $(a\ominus x)^n_{p,q}\neq(-1)^n(x\ominus a)^n_{p,q}$. *Instead, for* $n\geq 1$,

$$\begin{aligned}
(a\ominus x)^n_{p,q} &= (a-x)(pa-xq)\cdots(p^{n-1}a-xq^{n-1})\\
&= (-1)^n(pq)^{\binom{n}{2}}(x-a)(p^{-1}x-aq^{-1})\cdots(p^{-n+1}x-aq^{-n+1})\\
&= (-1)^n(pq)^{\binom{n}{2}}\left(p^{-n+1}x\ominus aq^{-n+1}\right)^n_{p,q}
\end{aligned}$$

Proposition 28 (Njionou [69])**.** *The following relations are valid:*

$$D_{p,q}\frac{1}{(x\ominus a)^n_{p,q}} = \frac{-q[n]_{p,q}}{(qx\ominus a)^{n+1}_{p,q}}, \tag{3.11}$$

$$D_{p,q}(a\ominus x)^n_{p,q} = -[n]_{p,q}(a\ominus qx)^{n-1}_{p,q}, \tag{3.12}$$

$$D_{p,q}\frac{1}{(a\ominus x)^n_{p,q}} = \frac{p[n]_{p,q}}{(a\ominus px)^{n+1}_{p,q}}. \tag{3.13}$$

Proof. For the relation (3.11), we first do the following remark

$$\frac{1}{(x\ominus a)^n_{p,q}} = \frac{1}{(p^{-n}(p^nx)\ominus(q^{-n}(q^na)))^n_{p,q}} = (p^nx\ominus q^na)^{-n}_{p,q}.$$

If follows that

$$\begin{aligned}
D_{p,q}\frac{1}{(x\ominus a)^n_{p,q}} &= D_{p,q}(p^nx\ominus q^na)^{-n}_{p,q}\\
&= [-n]_{p,q}p^n\left(p^n(px)\ominus q^na\right)^{-n-1}_{p,q}\\
&= [-n]_{p,q}p^n\left(p^{n+1}x\ominus q^{n+1}(aq^{-1})\right)^{-(n+1)}_{p,q}\\
&= \frac{[-n]_{p,q}p^n}{(x\ominus aq^{-1})^{n+1}_{p,q}},
\end{aligned}$$

Taking into account that

$$[-n]_{p,q} = -\frac{[n]_{p,q}}{(pq)^n} \quad \text{and} \quad (\alpha x \ominus y)_{p,q} = \alpha^n (x \ominus a\alpha^{-1})^n_{p,q},$$

we finally get

$$D_{p,q}\frac{1}{(x \ominus a)^n_{p,q}} = \frac{-q[n]_{p,q}}{(qx \ominus a)^{n+1}_{p,q}}.$$

For the relation (3.12), we use twice the above remark as follows

$$\begin{aligned}
D_{p,q}(a \ominus x)^n_{p,q} &= (-1)^n (pq)^{\binom{n}{2}} D_{p,q}\left(p^{-n+1}x \ominus q^{-n+1}a\right)^n_{p,q} \\
&= (-1)^n (pq)^{\binom{n}{2}} p^{-n+1}[n]_{p,q}\left(p^{-n+1}(px) \ominus q^{-n+1}a\right)^{n-1}_{p,q} \\
&= -[n]_{p,q}(pq)^{n-1}p^{-n+1}(pq)^{\binom{n-1}{2}}\left(p^{-n+2}x \ominus q^{-n+2}(q^{-1}a)\right)^{n-1}_{p,q} \\
&= -[n]_{p,q}q^{n-1}\left(q^{-1}a \ominus x\right)^{n-1}_{p,q} \\
&= -[n]_{p,q}(a \ominus qx)^{n-1}_{p,q}.
\end{aligned}$$

For the proof of (3.13), we use the quotient rule (2.16) as follows

$$\begin{aligned}
D_{p,q}\frac{1}{(a \ominus x)^n_{p,q}} &= -\frac{D_{p,q}(a \ominus x)^n_{p,q}}{(a \ominus px)^n_{p,q}(a \ominus qx)^n_{p,q}} \\
&= \frac{[n]_{p,q}(a \ominus qx)^{n-1}_{p,q}}{(a \ominus px)^n_{p,q}(a \ominus qx)^n_{p,q}} \\
&= \frac{[n]_{p,q}}{(a \ominus px)^n_{p,q}(ap^{n-1} - q^n x)} \\
&= \frac{[n]_{p,q}}{p^n(ap^{-1} \ominus x)^n_{p,q}((aq^{-1})q^n - q^n x)} \\
&= \frac{[n]_{p,q}}{p^n(ap^{-1} \ominus x)^n_{p,q}} \\
&= \frac{[n]_{p,q}}{p^n(p^{-1})^{n+1}(a \ominus px)^{n+1}_{p,q}} \\
&= \frac{p[n]_{p,q}}{(a \ominus px)^{n+1}_{p,q}}.
\end{aligned}$$

□

Proposition 29 (Njionou [69]). *Let $n \geq 1$ be an integer, and $0 \leq k \leq n$, we have the following identity*

$$D^k_{p,q}(a \ominus x)^n_{p,q} = (-1)^k q^{\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}!}(a \ominus q^k x)^{n-k}_{p,q}. \tag{3.14}$$

Proof. Let $n \geq 1$, for $k = 1$, it is the relation (3.12). Assume that for a fixed $k < n$, we have

$$D^k_{p,q}(a \ominus x)^n_{p,q} = (-1)^k q^{\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}!}(a \ominus q^k x)^{n-k}_{p,q}.$$

Then, we have

$$\begin{aligned}
D_{p,q}^{k+1}(a\ominus x)_{p,q}^{n} &= D_{p,q}\left(D_{p,q}^{k}(a\ominus x)_{p,q}^{n}\right)\\
&=(-1)^k q^{\binom{k}{2}}\frac{[n]_{p,q}!}{[n-k]_{p,q}!}D_{p,q}(a\ominus q^k x)_{p,q}^{n-k}\\
&=(-1)^k q^{\binom{k}{2}}\frac{[n]_{p,q}!}{[n-k]_{p,q}!}\left(-q^k\right)[n-k]_{p,q}(a\ominus q^{k+1})_{p,q}^{n-k-1}\\
&=(-1)^{k+1} q^{\binom{k+1}{2}}\frac{[n]_{p,q}!}{[n-k-1]_{p,q}!}(a\ominus q^{k+1}x)_{p,q}^{n-k-1}.
\end{aligned}$$

Hence, (3.14) is valid for all non-negative integers n. □

3.2 (p,q)-Taylor formulas

In this section, two Taylor formulas for polynomials are given and some of their consequences are investigated.

Theorem 30 (Njionou [69]). *For any polynomial $f(x)$ of degree N, and any number a, we have the following (p,q)-Taylor expansion:*

$$f(x)=\sum_{k=0}^{N}p^{-\binom{k}{2}}\frac{\left(D_{p,q}^{k}f\right)(ap^{-k})}{[k]_{p,q}!}(x\ominus a)_{p,q}^{k}. \tag{3.15}$$

Proof. Let f be a polynomial of degree N, then we have the expansion

$$f(x)=\sum_{j=0}^{N}c_j(x\ominus a)_{p,q}^{j}. \tag{3.16}$$

Let k be an integer such that $0\le k\le N$, then, applying $D_{p,q}^{k}$ on both sides of (3.16) and using (3.5), we get

$$\left(D_{p,q}^{k}f\right)(x)=\sum_{j=k}^{N}c_j\frac{[j]_{p,q}!}{[j-k]_{p,q}!}p^{\binom{k}{2}}(p^k x\ominus q)_{p,q}^{j-k}.$$

Substituting $x=ap^{-k}$, it follows that

$$\left(D_{p,q}^{k}f\right)(ap^{-k})=c_k[k]_{p,q}!p^{\binom{k}{2}},$$

thus we get

$$c_k=p^{-\binom{k}{2}}\frac{\left(D_{p,q}^{k}f\right)(ap^{-k})}{[k]_{p,q}!}.$$

This proves the desired result. □

Corollary 31 (Njionou [69]). *The following connection formula holds*

$$x^n=\sum_{k=0}^{n}p^{-\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(ap^{-k})^{n-k}(x\ominus a)_{p,q}^{k}. \tag{3.17}$$

Proof. Consider $f(x) = x^n$, where n is a positive integer. For $k \leq n$, we have

$$\left(D^k_{p,q}f\right)(x) = \frac{[n]_{p,q}!}{[n-k]_{p,q}!}x^{n-k}.$$

Thus we have

$$\begin{aligned} x^n &= \sum_{k=0}^{n} p^{-\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}(ap^{-k})^{n-k}(x \ominus a)^j_{p,q} \\ &= \sum_{k=0}^{n} p^{-\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (ap^{-k})^{n-k}(x \ominus a)^k_{p,q}, \end{aligned}$$

which proves the result. □

Theorem 32 (Njionou [69]). *For any polynomial $f(x)$ of degree N, and any number a, we have the following (p,q)-Taylor expansion:*

$$f(x) = \sum_{k=0}^{N} (-1)^k q^{-\binom{k}{2}} \frac{\left(D^k_{p,q}f\right)(aq^{-k})}{[k]_{p,q}!}(a \ominus x)^k_{p,q}. \tag{3.18}$$

Proof. Let f be a polynomial of degree N, then we have the expansion

$$f(x) = \sum_{j=0}^{N} c_j (a \ominus x)^j_{p,q}. \tag{3.19}$$

Let k be an integer such that $0 \leq k \leq N$, then, applying $D^k_{p,q}$ to both sides of (3.19) and using (3.14), we get

$$\left(D^k_{p,q}f\right)(x) = \sum_{j=k}^{N} c_j (-1)^j \frac{[j]_{p,q}!}{[j-k]_{p,q}!} q^{-\binom{k}{2}} (a \ominus q^k x)^{j-k}_{p,q}.$$

Substituting $x = aq^{-k}$, it follows that

$$\left(D^k_{p,q}f\right)(aq^{-k}) = c_k(-1)^k [k]_{p,q}! q^{-\binom{k}{2}},$$

thus we get

$$c_k = (-1)^k q^{-\binom{k}{2}} \frac{\left(D^k_{p,q}f\right)(aq^{-k})}{[k]_{p,q}!}.$$

This proves the desired result. □

Corollary 33 (Njionou [69]). *The following connection formula holds*

$$x^n = \sum_{k=0}^{n} (-1)^k q^{-\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (aq^{-k})^{n-k}(a \ominus x)^k_{p,q}. \tag{3.20}$$

Proof. Consider again $f(x) = x^n$, where n is a positive integer. For $k \leq n$, we have

$$\left(D^k_{p,q}f\right)(x) = \frac{[n]_{p,q}!}{[n-k]_{p,q}!}x^{n-k}.$$

Thus we have

$$x^n = \sum_{k=0}^{n} (-1)^k q^{-\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!} (aq^{-k})^{n-k} (a \ominus x)^k_{p,q}$$
$$= \sum_{k=0}^{n} (-1)^k q^{-\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (aq^{-k})^{n-k} (a \ominus x)^k_{p,q}.$$

□

Corollary 34 (Njionou [69]). *The following connection formulas hold:*

$$(x \ominus b)^n_{p,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (a \ominus b)^{n-k}_{p,q} (x \ominus a)^k_{p,q}, \tag{3.21}$$

$$(b \ominus x)^n_{p,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (b \ominus a)^{n-k}_{p,q} (a \ominus x)^k_{p,q}. \tag{3.22}$$

Remark 35. *If one substitutes b by ab in (3.21), then one gets*

$$(x \ominus ab)^n_{p,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} (1 \ominus b)^{n-k}_{p,q} (x \ominus a)^k_{p,q}.$$

Now, take $x = 1$ and $p = 1$, the following well known q-binomial theorem follows

$$(ab; q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} (b; q)_{n-k} (a; q)_k. \tag{3.23}$$

Therefore, (3.21) is an obvious generalization of (3.23).

Corollary 36 (Njionou [69]). *The following expansion holds*

$$\frac{1}{(1 \ominus x)^n_{p,q}} = 1 + \sum_{j=0}^{\infty} \frac{p^{j-\binom{j}{2}} [n]_{p,q} [n+1]_{p,q} \cdots [n+j-1]_{p,q}}{[j]_{p,q}!} x^n$$
$$= 1 + \sum_{j=0}^{\infty} \begin{bmatrix} n+j-1 \\ j \end{bmatrix}_{p,q} p^{j-\binom{j}{2}} x^j, \tag{3.24}$$

Proof. Consider the function $f(x) = \dfrac{1}{(1 \ominus x)^n_{p,q}}$. From (3.13), we have

$$D_{p,q} f(x) = D_{p,q} \frac{1}{(1 \ominus x)^n_{p,q}} = \frac{p[n]_{p,q}}{(1 \ominus x)^{n+1}_{p,q}},$$

and by induction,

$$D^j_{p,q} f(x) = \frac{p^j [n]_{p,q} [n+1]_{p,q} \cdots [n+j-1]_{p,q}}{(1 \ominus x)^{n+j}_{p,q}}.$$

Hence $(D^j_{p,q} f)(0) = p^{j-\binom{j}{2}} [n]_{p,q} [n+1]_{p,q} \cdots [n+j-1]_{p,q}$ for any $j \geq 1$ and hence the formula follows. □

Note that (8.25) is the (p, q)-analogue of Taylor's expansion of $f(x) = \dfrac{1}{(1-x)^n}$ in ordinary calculus. Note also that when $p \to 1$, (8.25) becomes the well known Heine binomial formula.

Chapter 4

(p,q)-Binomial Coefficients and their Properties

The classical binomial coefficients, usually denoted by $\binom{n}{k}$, play a very important role in enumerative combinatorics. These numbers appear as the coefficients in the expansion of the binomial expression $(x+y)^n$. More precisely,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

This identity is known as the Binomial Theorem (see [26, 31, 76, 85]), which for $y = 1$ becomes

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

the horizontal generating function for the binomial coefficients. These coefficients can be interpreted as the number of possible k-subsets out of a set of n distinct elements or the number of ways to choose k elements from the set of n distinct elements. The binomial coefficients are also known as combinatorics or combinatorial numbers.

q-Analogues of binomial coefficients are introduced in [24], extensively studied in [31] and [52]. They are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-i+1}}{q^i - 1} = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1, \quad q \neq 1.$$

In this chapter we introduce the (p,q)-binomial coefficients and establish some properties and identities similar to the ones known in the q-case and the classical case.

4.1 Recurrence relations for the (p,q)-binomial coefficients

Theorem 37 (Corcino [32]). *The (p,q)-binomial coefficients satisfy the following triangular recurrence relations*

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}, \tag{4.1}$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = q^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}. \tag{4.2}$$

Proof. First remark that for any integer k such that $0 \leq k \leq n+1$, we have

$$\begin{aligned} [n+1]_{p,q} &= \frac{p^{n+1} - q^{n+1}}{p-q} \\ &= \frac{p^{n+1} - q^k p^{n+1-k} + q^k p^{n+1-k} - q^{n+1}}{p-q} \\ &= \frac{p^{n+1-k}(p^k - q^k) + q^k(p^{n+1-k} - q^{n+1-k})}{p-q} \\ &= q^k [n+1-k]_{p,q} + p^{n+1-k}[k]_{p,q}. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} &= \frac{[n+1]_{p,q}!}{[k]_{p,q}![n+1-k]_{p,q}!} = \frac{[n+1]_{p,q}[n]_{p,q}!}{[k]_{p,q}![n+1-k]_{p,q}!} \\ &= \frac{(q^k[n+1-k]_{p,q} + p^{n+1-k}[k]_{p,q})[n]_{p,q}!}{[k]_{p,q}![n+1-k]_{p,q}!} \\ &= \frac{q^k[n+1-k]_{p,q}[n]_{p,q}!}{[k]_{p,q}![n+1-k]_{p,q}!} + \frac{p^{n+1-k}[k]_{p,q}[n]_{p,q}!}{[k]_{p,q}![n+1-k]_{p,q}!} \\ &= q^k \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!} + p^{n+1-k} \frac{[n]_{p,q}!}{[k-1]_{p,q}![n+1-k]_{p,q}!} \\ &= p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}. \end{aligned}$$

This proves (4.1), (4.2) follows from the fact that the (p,q)-binomial coefficients are symmetric in p and q. $\square$

Remark 38. *Note that taking $p = 1$ in* (4.1) *yields*

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{q} = \begin{bmatrix} n \\ k \end{bmatrix}_{q} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{q}$$

which is the so-called triangular recurrence relation for the q-binomial coefficients (Comtet [31]).

Note that if we apply (4.1) three times, to $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q}$, we get

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q} &= q^{n-(k+1)-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{k+1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_{p,q} \\
&= q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{k+1} \left(q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} + p^{k+1} \begin{bmatrix} n-1 \\ k+1 \end{bmatrix}_{p,q} \right) \\
&= q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{k+1} q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} + p^{2(k+1)} \left(q^{n-k-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} + p^{k+1} \begin{bmatrix} n-2 \\ k+1 \end{bmatrix}_{p,q} \right) \\
&= q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{k+1} q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} + p^{2(k+1)} q^{n-k-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} + p^{3(k+1)} \begin{bmatrix} n-2 \\ k+1 \end{bmatrix}_{p,q}
\end{aligned}$$

Continuing this process until the $(n-k)$th application of (4.1), we get

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q} = q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{k+1} q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} + p^{2(k+1)} q^{n-k-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} \\
+ p^{3(k+1)} q^{n-k-3} \begin{bmatrix} n-2 \\ k+1 \end{bmatrix}_{p,q} + \cdots + p^{(n-k)(k+1)} \begin{bmatrix} k \\ k \end{bmatrix}_{p,q}.
\end{aligned}$$

This is known as the vertical recurrence relation for the (p,q)-binomial coefficients. Next, rewriting (4.2) in the form

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{-(n-k)} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q} - p^{-(n-k)} q^{k+1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_{p,q},$$

and iterating this recurrence relation, it follows the horizontal recurrence relation. These results are contained in the following theorem.

Theorem 39 (Corcino [32])**.** *The (p,q) binomial coefficients satisfy the following vertical recurrence relation*

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q} = \sum_{j=k}^{n} p^{(n-j)(k+1)} q^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_{p,q}$$

and the horizontal recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \sum_{j=0}^{n-k} (-1)^j p^{-(j+1)(n-k)+\binom{j+1}{2}} q^{jk+\binom{j+1}{2}} \begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix}_{p,q}.$$

The horizontal and the vertical recurrence relations (39) may be regarded as the (p,q)-analogues of the Hockey Stick identities (Hilton [42]). They are also known as Chu Shi-Chieh's identities (Chuan-Chong and Khee-Meng [26]). Indeed, if $p=1$ we get

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{q} = \sum_{j=k}^{n} q^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_{q} \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_{q} = \sum_{j=0}^{n-k} (-1)^j q^{jk+\binom{j+1}{2}} \begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix}_{q}$$

which are the vertical and the horizontal recurrence relations for the q-binomial coefficients, respectively. Moreover, as q tends to 1, the former relations reduce to

$$\binom{n+1}{k+1} = \sum_{j=k}^{n} \binom{j}{k} \quad \text{and} \quad \binom{n}{k} = \sum_{j=0}^{n-k} (-1)^j \binom{n+1}{k+j+1},$$

which are the classical Chu Shi-Chieh's identities (Chuan-Chong and Khee-Meng [26]).

Proposition 40 (**Generating function**, Corcino [32] or Njionou [69])**.** *The (p,q)-binomial coefficients are generated by the (p,q)-power*

$$(x \ominus b)^n_{p,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} (-b)^{n-k} x^k. \tag{4.3}$$

Proof. Taking $b = 0$ in (3.21), and using the fact that

$$(x \ominus 0)^n_{p,q} = p^{\binom{n}{2}} x^n$$
$$(0 \ominus x)^n_{p,q} = (-1)^n q^{\binom{n}{2}} x^n$$

we get the result. □

Corollary 41 (Corcino [32])**.** *For $n \geq 1$, we have*

$$\sum_{k\,\text{even}} p^{\binom{k}{2}} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \sum_{k\,\text{odd}} p^{\binom{k}{2}} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q}. \tag{4.4}$$

Proof. For $x = b = 1$ in (4.3) it follows that

$$(1 \ominus 1)^n_{p,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^{n-k} = 0,$$

therefore the result follows. □

4.2 Orthogonality relations

Here, using twice the connection formula (3.21), we prove a general orthogonality relation for (p,q)-binomial coefficients.

Theorem 42. *The following orthogonality relation hold true for all complex numbers a and b*

$$\sum_{k=j}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \begin{bmatrix} k \\ j \end{bmatrix}_{p,q} (a \ominus b)^{n-k}_{p,q} (b \ominus a)^{k-j}_{p,q} = \delta_{n,j}, \tag{4.5}$$

where $\delta_{n,j} = \begin{cases} 1 & n = j \\ 0 & n \neq 0 \end{cases}$ *is the Kronecker delta.*

Proof. Using (3.21) we get

$$(x\ominus b)^n_{p,q}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(a\ominus b)^{n-k}_{p,q}(x\ominus a)^k_{p,q}.$$

In this relation, commute the position of a and b in the (p,q)-power in x appearing on the right hand side, apply again (3.21) combined with the summation formula

$$\sum_{k=0}^{n}\sum_{j=0}^{k}A(k,j)=\sum_{j=0}^{n}\sum_{k=j}^{n}A(k,j),$$

it follows that

$$\begin{aligned}(x\ominus b)^n_{p,q}&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(a\ominus b)^{n-k}_{p,q}(x\ominus a)^k_{p,q}\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(a\ominus b)^{n-k}_{p,q}\left(\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_{p,q}(b\ominus a)^{k-j}_{p,q}(x\ominus b)^j_{p,q}\right)\\&=\sum_{k=0}^{n}\sum_{j=0}^{k}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\begin{bmatrix}k\\j\end{bmatrix}_{p,q}(a\ominus b)^{n-k}_{p,q}(b\ominus a)^{k-j}_{p,q}(x\ominus b)^j_{p,q}\\&=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\begin{bmatrix}k\\j\end{bmatrix}_{p,q}(a\ominus b)^{n-k}_{p,q}(b\ominus a)^{k-j}_{p,q}\right)(x\ominus b)^j_{p,q}.\end{aligned}$$

The proof follows by equating the coefficients of $(x\ominus a)^j_{p,q}$ on both sides. □

The special cases of (4.5) where $a=0$ and $b=1$ or $b=0$ and $a=1$ appeared in [32, Theorem 4]. The proof of this result is done there by a very long induction process.

Corollary 43 (Corcino [32])**.** *The following orthogonality relations for the (p,q)-binomial coefficients holds*

$$\sum_{k=j}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\begin{bmatrix}k\\j\end{bmatrix}_{p,q}(-1)^{k-j}p^{\binom{n-k}{2}}q^{\binom{k-j}{2}}=\delta_{n,j},$$

$$\sum_{k=j}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\begin{bmatrix}k\\j\end{bmatrix}_{p,q}(-1)^{n-k}q^{\binom{n-k}{2}}p^{\binom{k-j}{2}}=\delta_{n,j}.$$

4.3 (p,q)-Vandermonde's identity

Vandermonde's identity states the following

$$\binom{n+m}{k}=\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j}. \tag{4.6}$$

The corresponding q-analogue of Vandermonde's identity is given by [38]

$$\begin{bmatrix}m+n\\k\end{bmatrix}_q=\sum_{j=0}^{k}q^{j(m-k+j)}\begin{bmatrix}n\\j\end{bmatrix}_q\begin{bmatrix}m\\k-j\end{bmatrix}_q. \tag{4.7}$$

In this section, we derive a (p,q)-analogue of Vandermonde's identity.

Theorem 44. *The following identity for the (p,q)-binomial coefficients holds*

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_{p,q} = \sum_{j=0}^{k} p^{j(j-m-k)+mk} q^{j(n-k+j)} \begin{bmatrix} m \\ j \end{bmatrix}_{p,q} \begin{bmatrix} n \\ k-j \end{bmatrix}_{p,q}. \tag{4.8}$$

Proof. First let us introduce the following notation

$$(a \oplus b)^n_{p,q} = (a \ominus (-b))^n_{p,q} = \prod_{k=0}^{n-1} (ap^k + bq^k).$$

From (3.7) we can write

$$(x \ominus b)^{m+n}_{p,q} = (x \ominus b)^m_{p,q} (p^m x \ominus q^m b)^n_{p,q}.$$

Using (4.3), respectively, with $(x \ominus b)^m_{p,q}$ and $(p^m x \ominus q^m b)^n_{p,q}$ combined with the Cauchy product, it follows that

$$\begin{aligned}
(x \oplus b)^{m+n}_{p,q} &= (x \oplus b)^m_{p,q} (p^m x \oplus q^m b)^n_{p,q} \\
&= \left(\sum_{k=0}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{m-k}{2}} b^{m-k} x^k \right) \left(\sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} (q^m b)^{n-k} (p^m x)^k \right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \begin{bmatrix} m \\ j \end{bmatrix}_{p,q} p^{\binom{j}{2}} q^{\binom{m-j}{2}} b^{m-j} x^j \begin{bmatrix} n \\ k-j \end{bmatrix}_{p,q} p^{\binom{k-j}{2}} q^{\binom{n+j-k}{2}} (q^m b)^{n+j-k} (p^m x)^{k-j} \right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \begin{bmatrix} m \\ j \end{bmatrix}_{p,q} \begin{bmatrix} n \\ k-j \end{bmatrix}_{p,q} p^{\binom{j}{2}+\binom{k-j}{2}+m(k-j)} q^{\binom{m-j}{2}+\binom{n+j-k}{2}+m(n+j-k)} b^{m-j} x^j b^{n+j-k} x^{k-j} \right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \begin{bmatrix} m \\ j \end{bmatrix}_{p,q} \begin{bmatrix} n \\ k-j \end{bmatrix}_{p,q} p^{\binom{j}{2}+\binom{k-j}{2}+m(k-j)} q^{\binom{m-j}{2}+\binom{n+j-k}{2}+m(n+j-k)} \right) b^{m+n-k} x^k \\
&= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \begin{bmatrix} m \\ j \end{bmatrix}_{p,q} \begin{bmatrix} n \\ k-j \end{bmatrix}_{p,q} p^{j(j-m-k)+mk} q^{j(n-k+j)} \right) p^{\binom{k}{2}} q^{\binom{n+m-k}{2}} b^{m+n-k} x^k.
\end{aligned}$$

Also, by (4.3), we can write

$$(x \oplus b)^{n+m}_{p,q} = \sum_{k=0}^{\infty} \begin{bmatrix} n+m \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} b^{n+m-k} x^k.$$

Identifying the coefficients of x^k on both sides, the result follows. □

Remark 45. *For $p = 1$ in* (4.8) *and using the symmetric role of m and n, we recover* (4.7).

Chapter 5

(p,q)-Exponential and (p,q)-Trigonometric Functions

In this chapter we introduce three (p,q)-analogues of the exponential function and their associated (p,q)-trigonometric functions.

5.1 The usual (p,q)-exponential functions

Definition 46 (Jagannathan et *al.* [49], Njionou [68, 84, 70])**.** *The small (p,q)-exponential function $e_{p,q}(z)$ and the big (p,q)-exponential function $E_{p,q}(z)$ are defined, respectively, by*

$$e_{p,q}(z) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} z^n, \tag{5.1}$$

$$E_{p,q}(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} z^n. \tag{5.2}$$

Remark 47. *It is worth noting that $e_{p,q}(x) = E_{q,p}(x)$.*

Proposition 48 (Jagannathan et *al.* [49])**.** *The following equation applies:*

$$e_{p,q}(x) E_{p,q}(-x) = 1. \tag{5.3}$$

Proof. The result is proved in [49] using (p,q)-hypergeometric series. We provide here a direct proof. From (5.1) and (5.2), and the general identity (which is a direct consequence of the Cauchy product)

$$\left(\sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty} b_n \frac{t^n}{[n]_{p,q}!}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_k b_{n-k}\right) \frac{t^n}{[n]_{p,q}!}, \tag{5.4}$$

it follows that

$$\begin{aligned} e_{p,q}(x) E_{p,q}(-x) &= \left(\sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} x^n\right)\left(\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} (-x)^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^k q^{\binom{k}{2}} p^{\binom{n-k}{2}}\right) \frac{x^n}{[n]_{p,q}!} \end{aligned}$$

It remains to prove that

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^k q^{\binom{k}{2}} p^{\binom{n-k}{2}} = \delta_{n,0}.$$

Taking $a=0$ in (3.15) it follows that

$$f_n(x) = \sum_{k=0}^{n} \frac{(D_{p,q}^k f)(0)}{[k]_{p,q}!} x^n$$

for any polynomial $f_n(x)$ of degree n. Applying this formula to $f_n(x) = (a \ominus x)_{p,q}^n$, it follows that

$$(a \ominus x)_{p,q}^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{\binom{k}{2}} p^{\binom{n-k}{2}} (-x)^k a^{n-k}.$$

Taking finally $x = a = 1$, the result follows. □

Proposition 49 (Njionou [69]). *Let λ be a complex number, then the following relations hold*

$$\begin{aligned} D_{p,q} e_{p,q}(\lambda x) &= \lambda e_{p,q}(\lambda p x), \\ D_{p,q} E_{p,q}(\lambda x) &= \lambda E_{p,q}(\lambda q x). \end{aligned}$$

Proof. From the definitions of the (p,q)-derivative and the (p,q)-exponential function $e_{p,q}(x)$, it follows that

$$\begin{aligned} D_{p,q} e_{p,q}(\lambda x) &= \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}} \lambda^n}{[n]_{p,q}!} [n]_{p,q} z^{n-1} = \lambda \sum_{n=1}^{\infty} \frac{p^{\binom{n}{2}}}{[n-1]_{p,q}!} (\lambda z)^{n-1} \\ &= \lambda \sum_{n=0}^{\infty} \frac{p^{\binom{n+1}{2}}}{[n]_{p,q}!} (\lambda z)^n = \lambda \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}+n}}{[n]_{p,q}!} (\lambda z)^n \\ &= \lambda \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} (\lambda p z)^n = e_{p,q}(\lambda p z). \end{aligned}$$

The proof of the second equation follows in the same way. □

Proposition 50 (Njionou [69]). *Let n be a nonnegative integer, then the following equations hold*

$$D_{p,q}^n e_{p,q}(\lambda x) = \lambda^n p^{\binom{n}{2}} e_{p,q}(\lambda p^n x), \tag{5.5}$$

$$D_{p,q}^n E_{p,q}(\lambda x) = \lambda^n q^{\binom{n}{2}} \lambda E_{p,q}(\lambda q^n x). \tag{5.6}$$

Proof. The proof follows by induction from the definitions of the (p,q)-exponentials and the (p,q)-derivative. □

Theorem 51 (Njionou [69]). *Let a be a complex number. The following expansions hold:*

$$\begin{aligned} e_{p,q}(\lambda x) &= e_{p,q}(\lambda a) \sum_{n=0}^{\infty} \frac{((p-q)\lambda)^n}{(p \ominus q)_{p,q}^n} (x \ominus a)_{p,q}^n, \\ E_{p,q}(\lambda x) &= \sum_{n=0}^{\infty} \left(\frac{q}{p}\right)^{\binom{n}{2}} \frac{\lambda^n E_{p,q}(\lambda a (q/p)^n)}{[n]_{p,q}!} (x \ominus a)_{p,q}^n. \end{aligned}$$

Proof. Note that in (3.15), N can be taken to be ∞ with the condition that the infinite series obtained is convergent. The formula becomes

$$f(x) = \sum_{n=0}^{\infty} p^{-\binom{n}{2}} \frac{\left(D_{p,q}^n f\right)(ap^{-n})}{[n]_{p,q}!}(x\ominus a)_{p,q}^n.$$

For $f(x) = e_{p,q}(x)$, using the relations (3.2) and (5.5), it follows that

$$\begin{aligned}
e_{p,q}(\lambda x) &= \sum_{n=0}^{\infty} p^{-\binom{n}{2}} \frac{\lambda^n p^{\binom{n}{2}} e_{p,q}(\lambda a)}{[n]_{p,q}!}(x\ominus a)_{p,q}^n \\
&= e_{p,q}(\lambda a)\sum_{n=0}^{\infty} \frac{\lambda^n}{[n]_{p,q}!}(x\ominus a)_{p,q}^n \\
&= e_{p,q}(\lambda a)\sum_{n=0}^{\infty} \frac{(x\ominus a)_{p,q}^n}{(p\ominus q)_{p,q}^n}((p-q)\lambda)^n.
\end{aligned}$$

For $f(x) = E_{p,q}(x)$, using the relations (3.2) and (5.6), it follows that

$$\begin{aligned}
E_{p,q}(\lambda x) &= \sum_{n=0}^{\infty} p^{-\binom{n}{2}} \frac{\lambda^n q^{\binom{n}{2}} E_{p,q}(\lambda a(q/p)^n)}{[n]_{p,q}!}(x\ominus a)_{p,q}^n \\
&= \sum_{n=0}^{\infty} \left(\frac{q}{p}\right)^{\binom{n}{2}} \frac{\lambda^n E_{p,q}(\lambda a(q/p)^n)}{[n]_{p,q}!}(x\ominus a)_{p,q}^n.
\end{aligned}$$

□

Theorem 52 (Njionou [69]). *Let a be a complex number. The following expansions hold:*

$$\begin{aligned}
e_{p,q}(x) &= \sum_{n=0}^{\infty} \left(-\frac{p}{q}\right)^{\binom{n}{2}} \frac{\lambda^n e_{p,q}(\lambda a(p/q)^n)}{[n]_{p,q}!}(a\ominus x)_{p,q}^n, \\
E_{p,q}(x) &= E_{p,q}(\lambda a)\sum_{n=0}^{\infty} \frac{((q-p)\lambda)^n}{(p\ominus q)_{p,q}^n}(a\ominus x)_{p,q}^n.
\end{aligned}$$

Proof. Note that in (3.18), N can be taken to be ∞ with the condition that the infinite series obtained is convergent. The formula becomes

$$f(x) = \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} \frac{\left(D_{p,q}^n f\right)(aq^{-n})}{[n]_{p,q}!}(a\ominus x)_{p,q}^n.$$

For $f(x) = e_{p,q}(x)$, using the relations (3.2) and (5.5), it follows that

$$\begin{aligned}
e_{p,q}(\lambda x) &= \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} \frac{\lambda^n p^{\binom{n}{2}} e_{p,q}(\lambda a(p/q)^n)}{[n]_{p,q}!}(a\ominus x)_{p,q}^n \\
&= \sum_{n=0}^{\infty} \left(-\frac{p}{q}\right)^{\binom{n}{2}} \frac{\lambda^n e_{p,q}(\lambda a(p/q)^n)}{[n]_{p,q}!}(a\ominus x)_{p,q}^n.
\end{aligned}$$

For $f(x) = E_{p,q}(x)$, using the relations (3.2) and (5.6), it follows that

$$\begin{aligned}
E_{p,q}(\lambda x) &= \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} \frac{\lambda^n q^{\binom{n}{2}} E_{p,q}(\lambda a)}{[n]_{p,q}!}(a\ominus x)_{p,q}^n \\
&= E_{p,q}(\lambda a)\sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{[n]_{p,q}!}(a\ominus x)_{p,q}^n \\
&= E_{p,q}(\lambda a)\sum_{n=0}^{\infty} \frac{(a\ominus x)_{p,q}^n}{(p\ominus q)_{p,q}^n}((q-p)\lambda)^n.
\end{aligned}$$

$\square$

5.2 (p,q)-trigonometric functions

From (5.1) we can derive

$$e_{p,q}(iz) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!}(iz)^n = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}. \tag{5.7}$$

By (5.7), we define the (p,q)-cosine and the (p,q)-sine functions as follows:

$$cos_{p,q}(z) = \frac{e_{p,q}(ix) + e_{p,q}(-ix)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n}, \tag{5.8}$$

$$sin_{p,q}(z) = \frac{e_{p,q}(ix) - e_{p,q}(-ix)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}. \tag{5.9}$$

Analogously, from (5.2) we can derive

$$E_{p,q}(iz) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!}(iz)^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}. \tag{5.10}$$

And by (5.7), we define the big (p,q)-cosine and the big (p,q)-sine functions as follows:

$$Cos_{p,q}(z) = \frac{E_{p,q}(ix) + E_{p,q}(-ix)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n}, \tag{5.11}$$

$$Sin_{p,q}(z) = \frac{E_{p,q}(ix) - E_{p,q}(-ix)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}. \tag{5.12}$$

It is easy to see that

$$cos_{p,q}(z) = Cos_{q,p}(z) \quad \text{and} \quad sin_{p,q}(z) = Sin_{q,p}(z).$$

Proposition 53 (Njionou [69])**.** *The following equations hold true:*

$$\begin{aligned} D_{p,q}cos_{p,q}(z) &= -sin_{p,q}(pz), \\ D_{p,q}sin_{p,q}(z) &= cos_{p,q}(pz), \\ D_{p,q}Cos_{p,q}(z) &= -Sin_{p,q}(qz), \\ D_{p,q}Sin_{p,q}(z) &= Cos_{p,q}(qz). \end{aligned}$$

Proof. From the definition of $cos_{p,q}(z)$ and the derivative property (Proposition 49) of the (p,q)-exponential function $e_{p,q}(z)$, it follows that

$$\begin{aligned} D_{p,q}cos_{p,q}(x) &= \frac{1}{2}\left(D_{p,q}cos_{p,q}(iz) + D_{p,q}cos_{p,q}(-iz)\right) \\ &= \frac{1}{2}\left(isin_{p,q}(ipz) - isin_{p,q}(-ipz)\right) = -sin_{p,q}(pz). \end{aligned}$$

The three other equations are established in the same way. $\square$

Remark 54. *Note that, using the fact relation*

$$D_{p,q}\left[f(\gamma z)\right] = \gamma (D_{p,q} f)(\gamma z)$$

we can see that both the (p,q)-cosine and the (p,q)-sine functions are solutions of the second-order (p,q)-difference equation

$$D^2_{p,q} y(z) + p^2 y(p^2 z) = 0.$$

Proposition 55 (Njionou [69])**.** *The following equations hold*

$$\begin{aligned} cos_{p,q}(x)Cos_{p,q}(x) + sin_{p,q}(x)Sin_{p,q}(x) &= 1, \\ sin_{p,q}(x)Cos_{p,q}(x) - cos_{p,q}(x)Sin_{p,q}(x) &= 0. \end{aligned}$$

Proof. Using (5.3), it follows that

$$\begin{aligned} cos_{p,q}(x)Cos_{p,q}(x) &= \left(\frac{e_{p,q}(ix) + e_{p,q}(-ix)}{2}\right)\left(\frac{E_{p,q}(ix) + E_{p,q}(-ix)}{2}\right) \\ &= \frac{1}{4}\left(e_{p,q}(ix)E_{p,q}(ix) + e_{p,q}(-ix)E_{p,q}(ix) + 2\right) \\ sin_{p,q}(x)Sin_{p,q}(x) &= \left(\frac{e_{p,q}(ix) - e_{p,q}(-ix)}{2i}\right)\left(\frac{E_{p,q}(ix) - E_{p,q}(-ix)}{2i}\right) \\ &= -\frac{1}{4}\left(e_{p,q}(ix)E_{p,q}(ix) + e_{p,q}(-ix)E_{p,q}(ix) - 2\right) \end{aligned}$$

Hence

$$cos_{p,q}(x)Cos_{p,q}(x) + sin_{p,q}(x)Sin_{p,q}(x) = 1.$$

The second equation follows in the same way. □

5.3 Hyperbolic (p,q)-trigonometric functions

Let us now define the hyperbolic (p,q)-cosine and the hyperbolic (p,q)-sine functions as follows

$$cosh_{p,q}(z) = \frac{e_{p,q}(z) + e_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n}, \tag{5.13}$$

$$sinh_{p,q}(z) = \frac{e_{p,q}(z) - e_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}, \tag{5.14}$$

$$Cosh_{p,q}(z) = \frac{E_{p,q}(z) + E_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{q^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n}, \tag{5.15}$$

$$Sinh_{p,q}(z) = \frac{E_{p,q}(z) - E_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{q^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}. \tag{5.16}$$

Proposition 56 (Njionou [69])**.** *The following equations hold*

$$\begin{aligned} cosh_{p,q}(z)Cosh_{p,q}(z) + sinh_{p,q}(z)Sinh_{p,q}(z) &= 1, \\ cosh_{p,q}(z)Sinh_{p,q}(z) - sinh_{p,q}(z)Cosh_{p,q}(z) &= 0. \end{aligned}$$

Proof. The proof is similar to the proof of Proposition (55). □

Chapter 6

(p, q)-Antiderivative and (p, q)-Integral

6.1 The (p, q)-antiderivative

The function $F(x)$ is a (p, q)-antiderivative of $f(x)$ if $D_{p,q}F(x) = f(x)$. It is denoted by

$$\int f(x) d_{p,q}x. \tag{6.1}$$

Note that we say "a" (p, q)-antiderivative instead of "the" (p, q)-antiderivative, because, as in ordinary calculus, an antiderivative is not unique. In ordinary calculus, the uniqueness is up to a constant since the derivative of a function vanishes if and only if it is a constant. The situation in the twin basic quantum calculus is more subtle. $D_{p,q}\varphi(x) = 0$ if and only if $\varphi(px) = \varphi(qx)$, which does not necessarily imply φ a constant. If we require φ to be a formal power series, the condition $\varphi(px) = \varphi(qx)$ implies $p^n c_n = q^n c_n$ for each n, where c_n is the coefficient of x^n. It is possible only when $c_n = 0$ for any $n \geq 1$, that is, φ is constant. Therefore, if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a formal power series, then among formal power series, $f(x)$ has a unique (p, q)-antiderivative up to a constant term, which is

$$\int f(x) d_{p,q}x = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{[n+1]_{p,q}} + C. \tag{6.2}$$

6.2 The (p, q)-integral

We define the inverse of the (p, q)-differentiation called the (p, q)-integration. Let $f(x)$ be an arbitrary function and $F(x)$ be a function such that $D_{p,q}F(x) = f(x)$, then

$$\frac{F(px) - F(qx)}{(p-q)x} = f(x).$$

Therefore, $F(px) - F(qx) = \varepsilon x f(x)$ where $\varepsilon = (p - q)$. This relation leads to the formula

$$\begin{aligned}
F\left(p^1q^{-1}x\right) - F\left(p^0q^{-0}x\right) &= \varepsilon p^0q^{-1}xf\left(p^0q^{-1}x\right) \\
F\left(p^2q^{-2}x\right) - F\left(p^1q^{-1}x\right) &= \varepsilon p^1q^{-2}xf\left(p^1q^{-2}x\right) \\
F\left(p^3q^{-3}x\right) - F\left(p^2q^{-2}x\right) &= \varepsilon p^2q^{-3}xf\left(p^2q^{-3}x\right) \\
&\vdots \\
F\left(p^{n+1}q^{-(n+1)}x\right) - F\left(p^nq^{-n}x\right) &= \varepsilon p^nq^{-(n+1)}xf\left(p^nq^{-(n+1)}x\right)
\end{aligned}$$

By adding these formulas term by term, we obtain

$$F\left(p^{n+1}q^{-(n+1)}x\right) - F(x) = (p-q)x\sum_{k=0}^{n} f\left(p^kq^{-(k+1)}x\right).$$

Assuming $\left|\frac{p}{q}\right| < 1$ and letting $n \to \infty$, we have

$$F(x) - F(0) = (q-p)x\sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}x\right).$$

Similarly, for $\left|\frac{p}{q}\right| > 1$, we have

$$F(x) - F(0) = (p-q)x\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}x\right).$$

Therefore, we give the following definition.

Definition 57 (Njionou [69]). *Let f be an arbitrary function. We define the (p, q)-integral of f as follows:*

$$\int f(x)d_{p,q}x = (p-q)x\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}x\right). \tag{6.3}$$

Remark 58. *Note that this is a formal definition since we do not care about the convergence of the right-hand side of (6.3).*

From this definition, one easily derives a more general formula

$$\begin{aligned}
\int f(x)D_{p,q}g(x)d_{p,q}x &= (p-q)x\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}x\right) D_{p,q}g\left(\frac{q^k}{p^{k+1}}x\right) \\
&= (p-q)x\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}x\right) \frac{g\left(\frac{q^k}{p^k}x\right) - g\left(\frac{q^{k+1}}{p^{k+1}}x\right)}{(p-q)\frac{q^k}{p^{k+1}}x} \\
&= \sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}}x\right)\left(g\left(\frac{q^k}{p^k}x\right) - g\left(\frac{q^{k+1}}{p^{k+1}}x\right)\right),
\end{aligned}$$

or otherwise stated

$$\int f(x)d_{p,q}g(x) = \sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}}x\right)\left(g\left(\frac{q^k}{p^k}x\right) - g\left(\frac{q^{k+1}}{p^{k+1}}x\right)\right). \tag{6.4}$$

We have derived (6.3) merely formally and have yet to examine under what conditions it really converges to a (p, q)-antiderivative. The theorem below gives a sufficient condition for this.

Theorem 59 (Njionou [69]). *Suppose $0 < \dfrac{q}{p} < 1$. If $|f(x)x^{\alpha}|$ is bounded on the interval $(0,A]$ for some $0 \le \alpha < 1$, then the (p,q)-integral (6.3) converges to a function $F(x)$ on $(0,A]$, which is a (p,q)-antiderivative of $f(x)$. Moreover, $F(x)$ is continuous at $x=0$ with $F(0)=0$.*

Proof. Let us assume that $|f(x)x^{\alpha}| < M$ on $(0,A]$. For any $0 < x < A$, $j \ge 0$,

$$\left| f\left(\frac{q^j}{p^{j+1}}x\right)\right| < M\left(\frac{q^j}{p^{j+1}}x\right)^{-\alpha}.$$

Thus, for $0 < x \le A$, we have

$$\left|\frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}x\right)\right| < M\frac{q^j}{p^{j+1}}\left(\frac{q^j}{p^{j+1}}x\right)^{-\alpha} = Mp^{\alpha-1}x^{-\alpha}\left[\left(\frac{q}{p}\right)^{1-\alpha}\right]^j. \tag{6.5}$$

Since, $1-\alpha > 0$ and $0 < \dfrac{q}{p} < 1$, we see that our series is bounded above by a convergent geometric series. Hence, the right-hand size of (6.3) converges point-wise to some function $F(x)$. It follows directly from (6.3) that $F(0)=0$. The fact that $F(x)$ is continuous at $x=0$, that is $F(x)$ tends to zero as $x \to 0$, is clear if we consider, using (6.5)

$$\left|(p-q)x\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}x\right)\right| < \frac{M(p-q)x^{1-\alpha}}{p^{1-\alpha}-q^{1-\alpha}}, \quad 0 < x \le A.$$

In order to check that $F(x)$ is a (p,q)-antiderivative we (p,q)-differentiate it:

$$\begin{aligned}
D_{p,q}F(x) &= \frac{1}{(p-q)x}\left((p-q)px\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}px\right) - (p-q)qx\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}qx\right)\right)\\
&= \sum_{k=0}^{\infty}\frac{q^k}{p^k} f\left(\frac{q^k}{p^k}x\right) - \sum_{k=0}^{\infty}\frac{q^{k+1}}{p^{k+1}} f\left(\frac{q^{k+1}}{p^{k+1}}x\right)\\
&= \sum_{k=0}^{\infty}\frac{q^k}{p^k} f\left(\frac{q^k}{p^k}x\right) - \sum_{k=1}^{\infty}\frac{q^k}{p^k} f\left(\frac{q^k}{p^k}x\right)\\
&= f(x).
\end{aligned}$$

Note that if $x \in (0,A]$ and $0 < \dfrac{q}{p} < 1$, then $\dfrac{q}{p}x \in (0,A]$, and the (p,q)-differentiation is valid. $\square$

Remark 60. *Note that if the assumption of Theorem 59 is satisfied, the (p,q)-integral gives the unique (p,q)-antiderivative that is continuous at $x=0$, up to a constant. On the other hand, if we know that $F(x)$ is a (p,q)-antiderivative of $f(x)$ and $F(x)$ is continuous at $x=0$, $F(x)$ must be given, up to a constant, by (6.3), since a partial sum of the (p,q)-integral is*

$$\begin{aligned}
(p-q)x\sum_{j=0}^{N}\frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}x\right) &= (p-q)x\sum_{j=0}^{N}\frac{q^j}{p^{j+1}} D_{p,q}\, F(t)\big|_{t=\frac{q^j}{p^{j+1}}x}\\
&= (p-q)x\sum_{j=0}^{N}\frac{q^j}{p^{j+1}}\left(\frac{F\left(\frac{q^j}{p^j}x\right) - F\left(\frac{q^{j+1}}{p^{j+1}}x\right)}{(p-q)\frac{q^j}{p^{j+1}}x}\right)\\
&= \sum_{j=0}^{N}\left(F\left(\frac{q^j}{p^j}x\right) - F\left(\frac{q^{j+1}}{p^{j+1}}x\right)\right)\\
&= F(x) - F\left(\frac{q^{N+1}}{p^{N+1}}x\right)
\end{aligned}$$

which tends to $F(x) - F(0)$ as N tends to ∞, by the continuity of $F(0)$ at $x = 0$.

Let us emphasize on an example where the (p,q)-derivative fails. Consider $f(x) = \dfrac{1}{x}$. Since

$$D_{p,q} \ln x = \frac{\ln px - \ln qx}{(p-q)x} = \frac{\ln p - \ln q}{p-q} \frac{1}{x}, \tag{6.6}$$

we have

$$\int \frac{1}{x} d_{p,q}x = \frac{p-q}{\ln p - \ln q} \ln x. \tag{6.7}$$

However, the formula (6.3) gives

$$\int \frac{1}{x} d_{p,q}x = (p-q) \sum_{j=0}^{\infty} 1 = \infty.$$

The formula fails because $f(x)x^{\alpha}$ is not bounded for any $0 \leq \alpha < 1$. Note that $\ln x$ is not continuous at $x = 0$.
We now apply formula (6.3) to define the definite (p,q)-integral.

Definition 61 (Njionou [69]). *Let f be an arbitrary function and a be a real number, we set*

$$\int_0^a f(x) d_{p,q}x = (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}} a\right) \quad \text{if} \quad \left|\frac{p}{q}\right| < 1 \tag{6.8}$$

$$\int_0^a f(x) d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}} a\right) \quad \text{if} \quad \left|\frac{p}{q}\right| > 1. \tag{6.9}$$

Example 62. *Let us compute the (p,q)-integral of the function $f(x) = x^n$. We take the case where $\left|\dfrac{p}{q}\right| < 1$, the other case being similar:*

$$\begin{aligned}
\int_0^a x^n d_{p,q}x &= (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} \left(\frac{p^k}{q^{k+1}} a\right)^n \\
&= \frac{q-p}{q^n} \frac{1}{1 - \frac{p^{n+1}}{q^{n+1}}} a^{n+1} \\
&= \frac{q-p}{q^{n+1} - p^{n+1}} a^{n+1} \\
&= \frac{a^{n+1}}{[n+1]_{p,q}}.
\end{aligned}$$

Example 63. *For $g(x) = \mathrm{e}_{p,q}(x) = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{[n]_{p,q}!}$.*

$$\begin{aligned}
\int_0^a \mathrm{e}_{p,q}(x) d_{p,q}x &= \sum_{n=0}^{\infty} \int_0^a \frac{x^n}{[n]_{pq}!} d_{p,q}x \\
&= \sum_{n=0}^{\infty} \frac{a^{n+1}}{[n]_{p,q}![n+1]_{p,q}} = \sum_{n=0}^{\infty} \frac{a^{n+1}}{[n+1]_{p,q}!} \\
&= \mathrm{e}_{p,q}(a) - 1.
\end{aligned}$$

Remark 64. *Note that for $p = 1$, the definition (6.9) reduces to the well known Jackson integral (see [52, P. 67])*

$$\int f(x)d_qx = (1-q)x\sum_{k=0}^{\infty} q^k f(q^k x).$$

For $p = r^{1/2}$, $q = s^{-1/2}$,

$$\left|\frac{p}{q}\right| < 1 \iff |rs| < 1,$$

the formula (6.8) reads

$$\int_0^a f(x)d_{p,q}x = (s^{-1/2} - r^{1/2})a\sum_{k=0}^{\infty} r^{k/2}s^{(k+1)/2} f\left(r^{k/2}s^{(k+1)/2}a\right),$$

which is the formula (11) given in [23]. Once more, for $p = r^{1/2}$, $q = s^{-1/2}$,

$$\left|\frac{p}{q}\right| > 1 \iff |rs| > 1,$$

the formula (6.9) reads

$$\int_0^a f(x)d_{p,q}x = (r^{1/2} - s^{-1/2})a\sum_{k=0}^{\infty} s^{-k/2}r^{-(k+1)/2} f\left(s^{-k/2}r^{-(k+1)/2}a\right),$$

which is the formula (10) given in [23].

Definition 65 (Njionou [69]). *Let f be an arbitrary function, a and b be two non-negative numbers such that $a < b$, then we set*

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x. \tag{6.10}$$

We cannot obtain a good definition of the improper integral by simply letting $a \to \infty$ in (6.9). Instead, since

$$\begin{aligned}
\int_{q^{j+1}/p^{j+1}}^{q^j/p^j} f(x)d_{p,q}x &= \int_0^{\frac{q^j}{p^j}} f(x)d_{p,q}x - \int_0^{\frac{q^{j+1}}{p^{j+1}}} f(x)d_{p,q}x \\
&= (p-q)\left\{\sum_{k=0}^{\infty}\frac{q^{k+j}}{p^{k+1+j}} f\left(\frac{q^{k+j}}{p^{k+1+j}}\right) - \sum_{k=0}^{\infty}\frac{q^{k+j+1}}{p^{k+j+2}} f\left(\frac{q^{k+j+1}}{p^{k+j+2}}\right)\right\} \\
&= (p-q)\frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right),
\end{aligned}$$

it is natural to define the improper (p,q)-integral as follows.

Definition 66 (Njionou [69]). *The improper (p,q)-integral of $f(x)$ on $[0;\infty)$ is defined to be*

$$\begin{aligned}
\int_0^{\infty} f(x)d_{p,q}x &= \sum_{j=-\infty}^{\infty}\int_{q^{j+1}/p^{j+1}}^{q^j/p^j} f(x)d_{p,q}x \\
&= (p-q)\sum_{j=-\infty}^{\infty}\frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right) \qquad (6.11)
\end{aligned}$$

if $0 < \frac{q}{p} < 1$ *or*

$$\int_0^\infty f(x) d_{p,q}x = \sum_{j=-\infty}^{\infty} \int_{q^j/p^j}^{q^{j+1}/p^{j+1}} f(x) d_{p,q}x \tag{6.12}$$

if $\frac{q}{p} > 1$ *where the formula is used.*

Proposition 67 (Njionou [69])**.** *Suppose that* $0 < \frac{q}{p} < 1$. *The improper* (p,q)*-integral defined above converges if* $x^\alpha f(x)$ *is bounded in a neighbourhood of* $x = 0$ *with* $\alpha < 1$ *and for sufficiently large* x *with some* $\alpha > 1$.

Proof. By (6.11) we have

$$\begin{aligned}\int_0^\infty f(x) d_{p,q}x &= (p-q) \sum_{j=-\infty}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right) \\ &= (p-q) \left\{ \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right) + \sum_{j=1}^{\infty} \frac{q^{-j}}{p^{-j+1}} f\left(\frac{q^{-j}}{p^{-j+1}}\right) \right\}\end{aligned}$$

The convergence of the first sum is proved by Theorem 59. For the second sum, suppose for x large we have $|x^\alpha f(x)| < M$ where $\alpha > 1$ and $M > 0$. Then, we have for sufficiently large j,

$$\begin{aligned}\left| \frac{q^{-j}}{p^{-j+1}} f\left(\frac{q^{-j}}{p^{-j+1}}\right) \right| &= p^{\alpha-1} \left(\frac{q}{p}\right)^{j(\alpha-1)} \left| \left(\frac{q^{-j}}{p^{-j+1}}\right)^\alpha f\left(\frac{q^{-j}}{p^{-j+1}}\right) \right| \\ &< M p^{\alpha-1} \left(\frac{q}{p}\right)^{j(\alpha-1)}.\end{aligned}$$

Therefore, the second sum is also bounded above by a convergent geometric series, and thus converges. □

Note that a similar proposition can be stated when $\frac{q}{p} > 1$.

Definition 68 (Njionou [69])**.** *Let* f *be an arbitrary function and* a *be a nonnegative real number, then we put*

$$\int_a^\infty f(x) d_{p,q}x = (q-p) a \sum_{k=0}^{\infty} \frac{p^{-k}}{q^{-(k+1)}} f\left(\frac{p^{-k}}{q^{-(k+1)}} a\right) \quad \text{if} \quad \left|\frac{p}{q}\right| < 1 \tag{6.13}$$

$$\int_a^\infty f(x) d_{p,q}x = (p-q) a \sum_{k=0}^{\infty} \frac{q^{-k}}{p^{-(k+1)}} f\left(\frac{q^{-k}}{p^{-(k+1)}} a\right) \quad \text{if} \quad \left|\frac{p}{q}\right| > 1. \tag{6.14}$$

Remark 69. *Combining (6.8) with (6.13) and (6.9) with (6.14) we get for* $a = 1$

$$\int_0^\infty f(x) d_{p,q}x = (q-p) \sum_{k=-\infty}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}\right) \quad \text{if} \quad \left|\frac{p}{q}\right| < 1 \tag{6.15}$$

$$\int_0^\infty f(x) d_{p,q}x = (p-q) \sum_{k=-\infty}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}\right) \quad \text{if} \quad \left|\frac{p}{q}\right| > 1. \tag{6.16}$$

6.3 The fundamental theorem of (p,q)-calculus

In ordinary calculus, a derivative is defined as the limit of a ratio, and a definite integral is defined as the limit of an infinite sum. Their subtle and surprising relation is given by the Newton-Leibniz formula, also called the fundamental theorem of calculus. Following the work done in q-calculus, where the introduction of the definite integral (see [52]) has been motivated by an antiderivative, the relation between the (p,q)-derivative and the (p,q)-integral is more obvious. Similarly to the ordinary and the q cases, we have the following fundamental theorem, or (p,q)-Newton-Leibniz formula.

Theorem 70 (**Fundamental theorem of** (p,q)**-calculus** (Njionou [69])). *If $F(x)$ is a (p,q)-antiderivative of $f(x)$ and $F(x)$ is continuous at $x=0$, we have*

$$\int_a^b f(x)d_{p,q}x = F(b) - F(a), \tag{6.17}$$

where $0 \le a < b \le \infty$.

Proof. Since $F(x)$ is continuous at $x=0$, $F(x)$ is given by the formula

$$F(x) = (p-q)x\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}f\left(\frac{q^j}{p^{j+1}}x\right) + F(0).$$

Since by definition,

$$\int_0^a f(x)d_{p,q}x = (p-q)a\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}f\left(\frac{q^j}{p^{j+1}}a\right),$$

we have

$$\int_0^a f(x)d_{p,q}x = F(a) - F(0).$$

Similarly, we have, for finite b,

$$\int_0^b f(x)d_{p,q}x = F(b) - F(0),$$

and thus

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x = F(b) - F(a).$$

Putting $a = \dfrac{q^{j+1}}{p^{j+1}}$ and $b = \dfrac{q^j}{p^j}$ and considering the definition of the improper (p,q)-integral (6.11), we see that (6.17) is true for $b=\infty$. □

Corollary 71. *If $f'(x)$ exists in a neighbourhood of $x=0$ and is continuous at $x=0$, where $f'(x)$ denotes the ordinary derivative of $f(x)$, we have*

$$\int_a^b D_{p,q}f(x)d_{p,q}x = f(b) - f(a). \tag{6.18}$$

Proof. Using L'Hospital's rule, we get

$$\begin{aligned}\lim_{x\to 0} D_{p,q}f(x) &= \lim_{x\to 0}\frac{f(px)-f(qx)}{(p-q)x}\\ &= \lim_{x\to 0}\frac{pf'(px)-qf'(qx)}{p-q} = f'(0).\end{aligned}$$

Hence $D_{p,q}f(x)$ can be made continuous at $x=0$ if we define $(D_{p,q}f)(0) = f'(0)$, and (6.18) follows from the theorem. □

Similarly as the q-integral, an important difference between the (p,q)-integral and the its ordinary counterpart is that even if we are integrating a function on an interval like $[1;2]$, we have to care about the behaviour at $x=0$. This has to do with the definition of the definite (p,q)-integral and the condition for the convergence of the (p,q)-integral.

Now suppose that $f(x)$ and $g(x)$ are two functions whose ordinary derivatives exist in a neighbourhood of $x=0$. Using the product rule (2.13), we have

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x).$$

Since the product of differentiable functions is also differentiable in ordinary calculus, we can apply Corollary 71 to obtain

$$f(b)g(b) - f(a)g(a) = \int_a^b f(px)\left(D_{p,q}g(x)\right) d_{p,q}x + \int_a^b g(qx)\left(D_{p,q}f(x)\right) d_{p,q}x,$$

or

$$\int_a^b f(px)\left(D_{p,q}g(x)\right) d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)\left(D_{p,q}f(x)\right) d_{p,q}x, \tag{6.19}$$

which is the formula of (p,q)-integration by part. Note that $b=\infty$ is allowed.

Chapter 7

(p,q)-Gamma and (p,q)-Beta functions

Being related to solutions of special types of differential equations, many important functions in analysis are defined in terms of definite integrals. The following two functions, introduced by Euler,

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx, \quad t > 0, \tag{7.1}$$

$$B(t,s) = \int_0^1 x^{t-1}(1-x)^{s-1}dx, \quad ,s,t > 0, \tag{7.2}$$

and called the *gamma* and the *beta functions* respectively, are the most important examples. The Euler Gamma function $\Gamma(t)$ first happens in 1729 in a correspondance between Euler and Goldbach. Euler gave an equivalent representation of the Gamma function (see [12, 14, 86])

$$\Gamma(t) = \lim_{n\to\infty} \frac{n!n^t}{t(t+1)\cdots(t+n)}. \tag{7.3}$$

Some of their most important properties are

$$\Gamma(t+1) = t\Gamma(t), \tag{7.4}$$

$$\Gamma(n+1) = n! \quad \text{if } n \text{ is a non-negative integer}, \tag{7.5}$$

$$B(t,s) = \frac{\Gamma(t)\Gamma(s)}{\Gamma(t+s)}. \tag{7.6}$$

Note that equation (7.5) tells us that the gamma function may be regarded as a generalization of the classical factorial. Also, from (7.6), we see that the beta function is symmetric in t and s.

The q-gamma function $\Gamma_q(t)$, a q-analogue of Euler's gamma function, was introduced by Thomae [87] and later by Jackson [44] as the infinite product

$$\Gamma_q(t) = \frac{(1\ominus q)_q^{t-1}}{(1-q)^{t-1}}, \quad t > 0, \tag{7.7}$$

where q is a fixed real number $0 < q < 1$. Notice that, under this assumption on q, the infinite product (7.7) is convergent. Its q-integral representation was given in [52, 83]. Note that for when t is a non-negative integer, (7.7) becomes

$$\Gamma_q(n+1) = \frac{(1\ominus q)^n}{(1-q)^n} = \frac{(q;q)_n}{(1-q)^n} = [n]_q!.$$

Hence, the q-gamma function is regarded as a generalization of the q-factorial. In this chapter we study the (p,q)-analogues of these two functions and their various properties, including the (p,q)-analogues of (7.4)-(7.6). Unless otherwise stated, we shall always assume that $0 < q < p < 1$.

Before we give our definitions of (p,q)-gamma and (p,q)-beta functions, we first give further useful properties for the (p,q)-power.

7.1 Some properties of the (p,q)-power

We prove here some important formulas for the (p,q)-power.

Let us recall the so-called (p,q)-powers

$$\begin{aligned}(x\ominus a)^n_{p,q} &= (x-a)(px-aq)\cdots(xp^{n-1}-aq^{n-1}),\\ (x\oplus a)^n_{p,q} &= (x+a)(px+aq)\cdots(xp^{n-1}+aq^{n-1}).\end{aligned}$$

These definitions are extended to

$$(a\ominus b)^{\infty}_{p,q} = \prod_{k=0}^{\infty}(ap^k - q^k b), \tag{7.8}$$

$$(a\oplus b)^{\infty}_{p,q} = \prod_{k=0}^{\infty}(ap^k + q^k b) \tag{7.9}$$

with the assumption that the infinite products are convergent.

Proposition 72 (Njionou [67]). *The following identities are easily verified*

$$(a\ominus b)^n_{p,q} = \frac{(a\ominus b)^{\infty}_{p,q}}{(ap^n\ominus bq^n)^{\infty}_{p,q}}, \tag{7.10}$$

$$(a\ominus b)^{n+k}_{p,q} = (a\ominus b)^n_{p,q}(ap^n\ominus bq^n)^k_{p,q}, \tag{7.11}$$

$$(ap^n\ominus bq^n)^k_{p,q} = \frac{(a\ominus b)^k_{p,q}(ap^k\ominus bq^k)^n_{p,q}}{(a\ominus b)^n_{p,q}}, \tag{7.12}$$

$$(ap^k\ominus bq^k)^{n-k}_{p,q} = \frac{(a\ominus b)^n_{p,q}}{(a\ominus b)^k_{p,q}}, \tag{7.13}$$

$$(ap^{2k}\ominus bq^{2k})^{n-k}_{p,q} = \frac{(a\ominus b)^n_{p,q}(ap^n\ominus bq^n)^k_{p,q}}{(a\ominus b)^{2k}_{p,q}}, \tag{7.14}$$

$$(a^2\ominus b^2)^n_{p^2,q^2} = (a\ominus b)^n_{p,q}(a\oplus b)^n_{p,q}, \tag{7.15}$$

$$(a\ominus b)^{2n}_{p,q} = (a\ominus b)^n_{p^2,q^2}(ap\ominus bq)^n_{p^2,q^2}, \tag{7.16}$$

$$(a\ominus b)^{3n}_{p,q} = (a\ominus b)^n_{p^3,q^3}(ap\ominus bq)^n_{p^3,q^3}(ap^2\ominus bq^2)^n_{p^3,q^3}, \tag{7.17}$$

$$(a\ominus b)^{\ell n}_{p,q} = \prod_{j=0}^{\ell-1}(ap^j\ominus bq^j)^n_{p^\ell,q^\ell}. \tag{7.18}$$

Proof. From (7.8), we can write

$$\begin{aligned}(a \ominus b)_{p,q}^{\infty} &= \prod_{k=0}^{\infty}(ap^k - q^k b) \\ &= \prod_{k=0}^{n-1}(ap^k - q^k b)\prod_{k=n}^{\infty}(ap^k - q^k b) \\ &= \prod_{k=0}^{n-1}(ap^k - q^k b)\prod_{k=0}^{\infty}(ap^n p^k - q^n q^k b) \\ &= (a \ominus b)_{p,q}^{n}(aq^n \ominus bq^n)_{p,q}^{\infty}.\end{aligned}$$

This proves (7.10). Next,

$$\begin{aligned}(a \ominus b)_{p,q}^{n+k} &= \prod_{k=0}^{n+k}(ap^k - q^k b) \\ &= \prod_{k=0}^{n-1}(ap^k - q^k b)\prod_{k=n}^{n+k}(ap^k - q^k b) \\ &= \prod_{k=0}^{n-1}(ap^k - q^k b)\prod_{k=0}^{n}(ap^n p^k - q^n q^k b) \\ &= (a \ominus b)_{p,q}^{n}(ap^n \ominus bq^n)_{p,q}^{k}.\end{aligned}$$

Hence (7.11) is proved. Commuting the role of n and k in (7.11), we can write

$$\begin{aligned}(a \ominus b)_{p,q}^{n+k} &= (a \ominus b)_{p,q}^{n}(ap^n \ominus bq^n)_{p,q}^{k} \\ &= (a \ominus b)_{p,q}^{k}(ap^k \ominus bq^k)_{p,q}^{n},\end{aligned}$$

so we obtain (7.12). (7.13) is obtained from (7.11) by substituting n by $n-k$. □

From equation (7.10) we can define the (p, q)-power for any complex number α as follows

$$(a \ominus b)_{p,q}^{\alpha} = \frac{(a \ominus b)_{p,q}^{\infty}}{(ap^{\alpha} \ominus bq^{\alpha})_{p,q}^{\infty}}. \tag{7.19}$$

7.2 The (p, q)-Gamma functions

Definition 73 (Njionou [67, 68]). *Let x be a complex number. We define the (p, q)-Gamma function as*

$$\Gamma_{p,q}(x) = \frac{(p \ominus q)_{p,q}^{\infty}}{(p^x \ominus q^x)_{p,q}^{\infty}}(p-q)^{1-x},\ 0 < q < p < 1. \tag{7.20}$$

Remark 74. *Note that in (7.20), if we set $p = 1$, then $\Gamma_{p,q}$ reduces to Γ_q.*

Proposition 75 (Njionou [67, 68]). *The (p, q)-Gamma function fulfils the following fundamental relation*

$$\Gamma_{p,q}(x+1) = [x]_{p,q}\Gamma_{p,q}(x). \tag{7.21}$$

Proof. From definition (7.20), we can write

$$\Gamma_{p,q}(x+1) = \frac{(p \ominus q)_{p,q}^{\infty}}{(p^{x+1} \ominus q^{x+1})_{p,q}^{\infty}}(p-q)^{-x}.$$

Since

$$(p^{x+1} \ominus q^{x+1})^{\infty}_{p,q} = \frac{1}{p^x - q^x}(p^x \ominus q^x)^{\infty}_{p,q},$$

it follows that

$$\begin{aligned}
\Gamma_{p,q}(x+1) &= (p^x - q^x)\frac{(p \ominus q)^{\infty}_{p,q}}{(p^x \ominus q^x)^{\infty}_{p,q}}(p-q)^{-x} \\
&= \frac{(p^x - q^x)}{p-q}\frac{(p \ominus q)^{\infty}_{p,q}}{(p^x \ominus q^x)^{\infty}_{p,q}}(p-q)^{1-x} \\
&= [x]_{p,q}\Gamma(x).
\end{aligned}$$

This is the desired result. □

Remark 76. *If n is a nonnegative integer, it follows from (7.21) that*

$$\Gamma_{p,q}(n+1) = [n]_{p,q}!.$$

It can be also easily seen from the definition that

$$\Gamma_{p,q}(n+1) = \frac{(p \ominus q)^n_{p,q}}{(p-q)^n}.$$

Proposition 77 (((p,q)**-Legendre's multiplication formula**, Njionou [67])**.** *The following multiplication formula applies*

$$\Gamma_{p,q}(2x)\Gamma_{p^2,q^2}\left(\frac{1}{2}\right) = (p+q)^{2x-1}\Gamma_{p^2,q^2}(x)\Gamma_{p^2,q^2}\left(x+\frac{1}{2}\right). \tag{7.22}$$

Proof. From the definition, we have

$$\begin{aligned}
\Gamma_{p^2,q^2}(x) &= \frac{(p^2 \ominus q^2)^{\infty}_{p^2,q^2}}{(p^{2x} \ominus q^{2x})^{\infty}_{p^2,q^2}}(p^2-q^2)^{1-x} \\
\Gamma_{p^2,q^2}\left(x+\frac{1}{2}\right) &= \frac{(p^2 \ominus q^2)^{\infty}_{p^2,q^2}}{(p^{2x+1} \ominus q^{2x+1})^{\infty}_{p^2,q^2}}(p^2-q^2)^{\frac{1}{2}-x} \\
\Gamma_{p^2,q^2}\left(\frac{1}{2}\right) &= \frac{(p^2 \ominus q^2)^{\infty}_{p^2,q^2}}{(p \ominus q)^{\infty}_{p^2,q^2}}(p^2-q^2)^{\frac{1}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\Gamma_{p^2,q^2}(x)\Gamma_{p^2,q^2}\left(x+\frac{1}{2}\right)}{\Gamma_{p^2,q^2}\left(\frac{1}{2}\right)} &= \frac{(p^2 \ominus q^2)^{\infty}_{p^2,q^2}(p \ominus q)^{\infty}_{p^2,q^2}}{(p^{2x} \ominus q^{2x})^{\infty}_{p^2,q^2}(p^{2x+1} \ominus q^{2x+1})^{\infty}_{p^2,q^2}}(p^2-q^2)^{1-2x} \\
&= \frac{(p \ominus q)^{\infty}_{p,q}}{(p^{2x} \ominus q^{2x})^{\infty}_{p,q}}(p-q)^{1-2x}(p+q)^{1-2x} \\
&= (p+q)^{1-2x}\Gamma_{p,q}(2x).
\end{aligned}$$

This proves the proposition. □

The (p,q)-Legendre's multiplication formula is generalized as follows.

Proposition 78 (**(p,q)-Gauss' multiplication formula**, Njionou [67]). *The following multiplication formula applies*

$$\Gamma_{p,q}(nx)\prod_{k=1}^{n-1}\Gamma_{p^n,q^n}\left(\frac{k}{n}\right) = ([n]_{p,q})^{nx-1}\prod_{k=0}^{n-1}\Gamma_{p^n,q^n}\left(x+\frac{k}{n}\right). \tag{7.23}$$

Proof. As for the previous proposition, we start by using the definition as follows

$$\begin{aligned}\Gamma_{p^n,q^n}\left(\frac{k}{n}\right) &= \frac{(p^n\ominus q^n)^\infty_{p^n,q^n}}{(p^k\ominus q^k)^\infty_{p^n,q^n}}(p^n-q^n)^{1-\frac{k}{n}},\\ \Gamma_{p^n,q^n}\left(x+\frac{k}{n}\right) &= \frac{(p^n\ominus q^n)^\infty_{p^n,q^n}}{(p^{nx+k}\ominus q^{nx+k})^\infty_{p^n,q^n}}(p^n-q^n)^{1-\frac{k}{n}-x}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\prod_{k=1}^{n-1}\Gamma_{p^n,q^n}\left(\frac{k}{n}\right) &= \frac{\left[(p^n\ominus q^n)^\infty_{p^n,q^n}\right]^{n-1}}{\prod\limits_{k=1}^{n-1}(p^k\ominus q^k)^\infty_{p^n,q^n}}(p^n-q^n)^{\sum\limits_{k=1}^{n-1}(1-\frac{k}{n})}\\ &= \frac{\left[(p^n\ominus q^n)^\infty_{p^n,q^n}\right]^{n}}{\prod\limits_{k=0}^{n-1}(p.p^k\ominus q.q^k)^\infty_{p^n,q^n}}(p^n-q^n)^{\frac{n-1}{2}}\\ &= \frac{\left[(p^n\ominus q^n)^\infty_{p^n,q^n}\right]^{n}}{(p\ominus q)^\infty_{p,q}}(p^n-q^n)^{\frac{n-1}{2}}\end{aligned}$$

and

$$\begin{aligned}\prod_{k=0}^{n-1}\Gamma_{p^n,q^n}\left(x+\frac{k}{n}\right) &= \frac{\left[(p^n\ominus q^n)^\infty_{p^n,q^n}\right]^{n}}{\prod\limits_{k=0}^{n-1}(p^{nx+k}\ominus q^{nx+k})^\infty_{p^n,q^n}}(p^n-q^n)^{\sum\limits_{k=0}^{n-1}(1-\frac{k}{n}-x)}\\ &= \frac{\left[(p^n\ominus q^n)^\infty_{p^n,q^n}\right]^{n}}{(p^{nx}\ominus q^{nx})^\infty_{p,q}}(p^n-q^n)^{(\frac{n-1}{2}+1-nx)}.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\prod\limits_{k=0}^{n-1}\Gamma_{p^n,q^n}\left(x+\frac{k}{n}\right)}{\prod\limits_{k=1}^{n-1}\Gamma_{p^n,q^n}\left(\frac{k}{n}\right)} &= \frac{(p\ominus q)^\infty_{p,q}}{(p^{nx}\ominus q^{nx})^\infty_{p,q}}(p^n-q^n)^{1-nx}\\ &= \frac{(p\ominus q)^\infty_{p,q}}{(p^{nx}\ominus q^{nx})^\infty_{p,q}}(p-q)^{1-nx}\left(\frac{p^n-q^n}{p-q}\right)^{1-nx}\\ &= ([n]_{p,q})^{1-nx}\Gamma_{p,q}(nx).\end{aligned}$$

The proposition is therefore proved. □

Proposition 79 (Njionou [68]). *The (p,q)-Gamma function* (7.20) *has the (p,q)-integral representation*

$$\Gamma_{p,q}(z) = p^{\frac{z(z-1)}{2}}\int_0^\infty t^{z-1}E_{p,q}(-qt)d_{p,q}t. \tag{7.24}$$

Proof. It is enough to prove that both $\Gamma_{p,q}(z)$ defined by (7.20) and

$$G(z) = p^{\frac{z(z-1)}{2}} \int_0^\infty t^{z-1} E_{p,q}(-qt) d_{p,q}t$$

satisfy the same recurrence relation with the same initial condition. Obviously,

$$G(1) = \int_0^\infty E_{p,q}(-qt) d_{p,q}t = \left[-E_{p,q}(-t)\right]_0^\infty = -E_{p,q}(\infty) + E_{p,q}(0) = 1.$$

Moreover, using equation (7.24) and the (p,q)-integration by part (6.19), we have:

$$\begin{aligned}
\Gamma_{p,q}(z+1) &= p^{\frac{z(z+1)}{2}} \int_0^\infty t^z E_{p,q}(-qt) d_{p,q}t \\
&= -p^{\frac{z(z-1)}{2}} \int_0^\infty (pt)^z D_{p,q} E_{p,q}(-t) d_{p,q}t \\
&= -p^{\frac{z(z-1)}{2}} \left[t^z E_{p,q}(-t)\right]_0^\infty + p^{\frac{z(z-1)}{2}} [z]_{p,q} \int_0^\infty t^{z-1} E_{p,q}(-qt) d_{p,q}t \\
&= [z]_{p,q} \Gamma_{p,q}(z).
\end{aligned}$$

Hence, $G(z) = \Gamma_{p,q}(z)$. □

7.3 The (p,q)-Beta functions

7.3.1 (p,q)-Beta function of the first kind

We introduce the following (p,q)-Beta function of the first kind.

Definition 80. *Let m and n be to non-negative integers. We define the (p,q)-integral of the first kind by*

$$B_{p,q}(m,n) = p^{\binom{m}{2}} \int_0^p x^{m-1} (p \ominus qx)_{p,q}^{n-1} d_{p,q}x. \tag{7.25}$$

Note that for $p = 1$, (7.25) reduces to the q-Beta function of the first kind [83, 52].

Theorem 81. *The following equation is valid:*

$$B_{p,q}(m,n) = p^{mn+\binom{n}{2}} \frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}. \tag{7.26}$$

Proof. Using the definition of the (p,q)-function (7.25) and the formula of integration by parts (6.19) with $f(x) = x^m$ and $g(x) = (p \ominus x)_{p,q}^{n-1}$, it follows that

$$\begin{aligned}
B_{p,q}(m+1,n) &= p^{\binom{m+1}{2}} \int_0^p x^m (p \ominus qx)_{p,q}^{n-1} d_{p,q}x \\
&= p^{\binom{m}{2}} \int_0^p (px)^m (p \ominus qx)_{p,q}^{n-1} d_{p,q}x \\
&= -\frac{p^{\binom{m}{2}}}{[n]_{p,q}} \int_0^p f(px) D_{p,q} g(x) d_{p,q}x \\
&= -\frac{p^{\binom{m}{2}}}{[n]_{p,q}} \left\{ [f(x)g(x)]_0^p - \int_0^p g(qx) D_{p,q} f(x) d_{p,q}x \right\} \\
&= \frac{[m]_{p,q}}{[n]_{p,q}} p^{\binom{m}{2}} \int_0^p x^{m-1} (p \ominus qx)_{p,q}^n d_{p,q}x \\
&= \frac{[m]_{p,q}}{[n]_{p,q}} B(m, n+1).
\end{aligned}$$

Also, we have

$$\begin{aligned}
B_{p,q}(m,n+1) &= p^{\binom{m}{2}}\int_0^p x^{m-1}(p\ominus qx)_{p,q}^{n}d_{p,q}x\\
&= p^{\binom{m}{2}}\int_0^p x^{m-1}(p\ominus qx)_{p,q}^{n-1}(p^n-q^nx)d_{p,q}x\\
&= p^np^{\binom{m}{2}}\int_0^p x^{m-1}(p\ominus qx)_{p,q}^{n-1}d_{p,q}x - q^np^{\binom{m}{2}}\int_0^p x^{m}(p\ominus qx)_{p,q}^{n-1}d_{p,q}x\\
&= p^n\mathcal{B}_{p,q}(m,n) - \frac{q^n}{p^m}p^{\binom{m+1}{2}}\int_0^p x^{m}(p\ominus qx)_{p,q}^{n-1}d_{p,q}x\\
&= p^n\mathcal{B}_{p,q}(m,n) - \frac{q^n}{p^m}B(m+1,n).
\end{aligned}$$

Hence,

$$\begin{aligned}
B_{p,q}(m,n+1) = p^nB(m,n) - \frac{q^n}{p^m}B(m+1,n)&\\
= p^nB(m,n) - \frac{q^n}{p^m}\frac{[m]_{p,q}}{[n]_{p,q}}B(m,n+1)&.
\end{aligned}$$

This gives the following relation

$$B_{p,q}(m,n+1) = p^{n+m}\frac{[n]_{p,q}}{[n+m]_{p,q}}B(m,n).$$

Iterating this relation gives

$$\begin{aligned}
B_{p,q}(m,n) &= p^{n+m-1}\frac{[n-1]_{p,q}}{[n+m-1]_{p,q}}B(m,n-1)\\
&= p^{n+m-1}\frac{[n-1]_{p,q}}{[n+m-1]_{p,q}}p^{n+m-2}\frac{[n-2]_{p,q}}{[n+m-2]_{p,q}}B(m,n-2)\\
&= p^{(n+m-1)+(n+m-2)+\ldots+(m+1)}\frac{[n-1]_{p,q}[n-2]_{p,q}\ldots[1]_{p,q}}{[n+m-1]_{p,q}[n+m-2]_{p,q}\ldots[m+1]_{p,q}}B(m,1).
\end{aligned}$$

Further, by definition of (p,q)-integration, it follows that

$$B_{p,q}(m,1) = \int_0^p x^{m-1}d_{p,q}x = \frac{p^m}{[m]_{p,q}}.$$

Hence,

$$\begin{aligned}
B_{p,q}(m,n) &= p^{(n+m-1)+(n+m-2)+\ldots+(m+1)+m}\frac{[n-1]_{p,q}[n-2]_{p,q}\ldots[1]_{p,q}}{[n+m-1]_{p,q}[n+m-2]_{p,q}\ldots[m+1]_{p,q}[m]_{p,q}}\\
&= p^{mn+\binom{n}{2}}\frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}.
\end{aligned}$$

□

Another (p,q)-Beta function of the first kind was introduced in [62] as follows.

Definition 82 (Milovanović, Gupta and Malik [62]). *Let m and n be two non-negative integers, the (p,q)-Beta function is defined as*

$$\tilde{B}_{p,q}(m,n) = \int_0^1 x^{m-1}(1\ominus qx)_{p,q}^{n-1}d_{p,q}x. \tag{7.27}$$

This (p,q)-Beta of the first kind fulfil the following property.

Proposition 83 (Milovanović, Gupta and Malik [62]). *The (p,q)-Gamma and the (p,q)-Beta functions fulfil the following fundamental relation:*

$$\tilde{B}_{p,q}(m,n) = p^{(n-1)(2m+n-2)/2}\frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}. \tag{7.28}$$

7.3.2 (p,q)-Beta function of the second kind

The (p,q)-Beta of the second kind was defined in [17] as follows.

Definition 84 (Aral and Gupta [17]). *Let m,n be to non-negative integer, the (p,q)-Beta function of the second kind is defined by*

$$\mathcal{B}_{p,q}(m,n) = \int_0^\infty \frac{x^{m-1}}{(1\oplus px)_{p,q}^{m+n}} d_{p,q}x. \tag{7.29}$$

The following theorem provides the link between the (p,q)-Beta function of the second kind and the (p,q)-Gamma function of the first kind.

Theorem 85 (Compare to [17]). *Let m,n be two non-negative integers, the following equation holds*

$$\mathcal{B}_{p,q}(m,n) = p^{-m}q^{-\binom{m}{2}}\frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}. \tag{7.30}$$

Proof. From (3.13), we can write

$$D_{p,q}\frac{1}{(a\oplus x)_{p,q}} = -\frac{p[n]_{p,q}}{(a\oplus px)_{p,q}^{n+1}}.$$

By choosing $f(x) = \dfrac{1}{(1\oplus x)_{p,q}^{n+m}}$ and $g(x) = x^m$ and using the formula of (p,q)-integration by parts (6.19), we get

$$\begin{aligned}
\mathcal{B}_{p,q}(m,n) &= \int_0^\infty \frac{x^{m-1}}{(1\oplus px)_{p,q}^{m+n}} d_{p,q}x \\
&= \frac{1}{[m]_{p,q}}\int_0^\infty f(px)D_{p,q}g(x)d_{p,q}x \\
&= \frac{1}{[m]_{p,q}}\left\{[f(x)g(x)]_0^\infty - \int_0^\infty D_{p,q}f(x)g(qx)d_{p,q}x\right\} \\
&= \frac{1}{[m]_{p,q}}\left\{0 - \int_0^\infty (qx)^m D_{p,q}\frac{1}{(1\oplus x)_{p,q}^{m+n}}d_{p,q}x\right\} \\
&= \frac{pq^m[m+n]_{p,q}}{[m]_{p,q}}\int_0^\infty \frac{x^m}{(1\oplus px)_{p,q}^{m+n+1}}d_{p,q}x \\
&= \frac{pq^m[m+n]_{p,q}}{[m]_{p,q}}\mathcal{B}_{p,q}(m+1,n).
\end{aligned}$$

Hence

$$\mathcal{B}_{p,q}(m+1,n) = \frac{[m]_{p,q}}{pq^m[m+n]_{p,q}}\mathcal{B}_{p,q}(m,n).$$

Iterating this relation leads to

$$\begin{aligned}\mathcal{B}_{p,q}(m,n) &= \frac{[m-1]_{p,q}}{pq^{m-1}[m+n-1]_{p,q}}\mathcal{B}_{p,q}(m-1,n)\\ &= \frac{[m-1]_{p,q}}{pq^{m-1}[m+n-1]_{p,q}}\frac{[m-2]_{p,q}}{pq^{m-2}[m+n-2]_{p,q}}\mathcal{B}_{p,q}(m-2,n)\\ &= \frac{[m-1]_{p,q}}{pq^{m-1}[m+n-1]_{p,q}}\frac{[m-2]_{p,q}}{pq^{m-2}[m+n-2]_{p,q}}\cdots\frac{1}{pq[n+1]_{p,q}}\mathcal{B}_{p,q}(1,n).\end{aligned}$$

Using the fact that

$$\mathcal{B}_{p,q}(1,n) = \int_0^\infty \frac{1}{(1\oplus px)_{p,q}^{n+1}}d_{p,q}x = -\frac{1}{p[n]_{p,q}}\int_0^\infty D_{p,q}\frac{1}{(1\oplus x)_{p,q}^{n}}d_{p,q}x = \frac{1}{p[n]_{p,q}},$$

it follows that

$$\mathcal{B}_{p,q}(m,n) = \frac{1}{p^m q^{\binom{m}{2}}}\frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}.$$

□

Chapter 8

(p,q)-Hypergeometric Series

8.1 (p,q)-Hypergeometric series

We first recall that the q-hypergeometric or basic hypergeometric series [53] is defined by

$$ {}_r\phi_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| q \right) z = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r;\, q)_n}{(b_1, \ldots, b_s;\, q)_n} \frac{z^n}{(q;q)_n} \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{1+s-r}, \tag{8.1} $$

where

$$ (a_1, \ldots, a_r; q)_n = (a_1; q)_1 \cdots (a_r; q)_n, $$

and

$$ (a;q)_n = \begin{cases} 1 & n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & n > 0 \end{cases} $$

is the so-called q-Pochhammer.

It is clear from the definition that

$$ \lim_{q \to 1} {}_r\phi_s \left(\begin{matrix} q^{a_1}, \ldots, q^{a_r} \\ q^{b_1}, \ldots, q^{b_s} \end{matrix} \middle| q, (q-1)^{1+s-r} z \right) = {}_rF_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| z \right), \tag{8.2} $$

where

$$ {}_rF_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| z \right) = \sum_{j=0}^{\infty} \frac{(a_1, \ldots, a_r)_j}{(b_1, \ldots, b_s)_j} \frac{z^j}{j!}, $$

denotes the usual hypergeometric series in which

$$ (a_1, \ldots, a_r)_j = (a_1)_j \cdots (a_r)_j, $$

where

$$ (a)_n = \begin{cases} 1 & n = 0 \\ a(a+1)\cdots(a+n-1) & n > 0 \end{cases} $$

is the Pochhammer symbol.

The q-hypergeometric series is extended to the (p,q)-hypergeometric series in the following way [23, 48, 49, 68, 78].

Definition 86 (Compare [49]). *The (p,q)-hypergeometric series is defined by*

$$
{}_r\Phi_s\left(\begin{matrix}(a_{1p},a_{1q}),\dots(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\middle|(p,q);z\right)
= \sum_{n=0}^{\infty}\frac{(a_{1p}\ominus a_{1q})^n_{p,q}\cdots(a_{rp}\ominus a_{rq})^n_{p,q}}{(b_{1p}\ominus b_{1q})^n_{p,q}\cdots(b_{sp}\ominus b_{sq})^n_{p,q}(p\ominus q)^n_{p,q}}\left[(-1)^n\left(\frac{q}{p}\right)^{\binom{n}{2}}\right]^{1+s-r}z^n. \tag{8.3}
$$

Note that for $s=r-1$, (8.3) reads

$$
{}_r\Phi_{r-1}\left(\begin{matrix}(a_{1p},a_{1q}),\dots(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\middle|(p,q);z\right)
= \sum_{n=0}^{\infty}\frac{(a_{1p}\ominus a_{1q})^n_{p,q}\cdots(a_{rp}\ominus a_{rq})^n_{p,q}}{(b_{1p}\ominus b_{1q})^n_{p,q}\cdots(b_{sp}\ominus b_{sq})^n_{p,q}(p\ominus q)^n_{p,q}}z^n. \tag{8.4}
$$

Also, when $a_{1p}=a_{2p}=\cdots=a_{rp}=b_{1p}=b_{2p}=\cdots=b_{sp}=1$, $a_{1q}=a_1,\dots,a_{rq}=a_r$ and $b_{1q}=b_1,\dots,b_{s,q}=b_s$ we get

$$
\lim_{p\to1}{}_r\Phi_s\left(\begin{matrix}(1,a_1),\dots,(1,a_r)\\(1,b_1),\dots,(1,b_s)\end{matrix}\middle|(p,q);z\right)={}_r\phi_s\left(\begin{matrix}a_1,\dots,a_r\\b_1,\dots,b_s\end{matrix}\middle|q;z\right).
$$

As we will see later, note that any well behaved ϕ-series can be written as a Φ-series but the converse of this proposition is not true in general (see [48, 49]).

8.2 From the ${}_r\phi_s$-series to the ${}_r\Phi_s$-series

In this section we show how to embed the usual ${}_r\phi_s$-series (8.1) from new defined ${}_r\Phi_s$ series (8.3).

Let us start with some links between the (p,q)-power and the q-Pochhammer symbol.

Proposition 87. *Let a and b be two non-zero complex numbers. The following property is valid.*

$$
(a\ominus b)^n_{p,q}=a^np^{\binom{n}{2}}\left(\frac{b}{a};\frac{q}{p}\right)_n. \tag{8.5}
$$

Proof. From the definition of the (p,q)-power basis (3.1), it follows that

$$
\begin{aligned}
(a\ominus b)^n_{p,q}&=(a-b)(ap-bq)\cdots(ap^{n-1}-bq^{n-1})\\
&=a\left(1-\frac{b}{a}\right)\times ap\left(1-\frac{b}{a}\frac{q}{p}\right)\times\cdots\times ap^{n-1}\left(1-\frac{b}{a}\left(\frac{q}{p}\right)^{n-1}\right)\\
&=a^np^{\binom{n}{2}}\left(\frac{b}{a};\frac{q}{p}\right)_n.
\end{aligned}
$$

□

Corollary 88. *Let a, b, c and d be four non-zero complex numbers. The following property is valid.*

$$
\frac{\left(\frac{b}{a};\frac{q}{p}\right)_\infty}{\left(\frac{d}{c};\frac{q}{p}\right)_\infty}=\frac{(a\ominus b)^\infty_{p,q}}{\left(a\ominus\frac{ad}{c}\right)^\infty_{p,q}}=\frac{\left(c\ominus\frac{cb}{a}\right)^\infty_{p,q}}{(c\ominus d)^\infty_{p,q}}. \tag{8.6}
$$

Proof. It follows from (8.5) that

$$\left(\frac{b}{a};\frac{q}{p}\right)_n = a^{-n}p^{-\binom{n}{2}}(a\ominus b)^n_{p,q},$$

hence,

$$\begin{aligned}
\frac{\left(\frac{b}{a};\frac{q}{p}\right)_\infty}{\left(\frac{d}{c};\frac{q}{p}\right)_\infty} &= \lim_{n\to\infty}\frac{\left(\frac{b}{a};\frac{q}{p}\right)_n}{\left(\frac{d}{c};\frac{q}{p}\right)_n}\\
&= \lim_{n\to\infty}\frac{a^{-n}p^{-\binom{n}{2}}(a\ominus b)^n_{p,q}}{c^{-n}p^{-\binom{n}{2}}(c\ominus d)^n_{p,q}}\\
&= \begin{cases}\lim\limits_{n\to\infty}\dfrac{\left(\frac{c}{a}\right)^n(a\ominus b)^n_{p,q}}{(c\ominus d)^n_{p,q}}\\ \lim\limits_{n\to\infty}\dfrac{(a\ominus b)^n_{p,q}}{\left(\frac{a}{c}\right)^n(c\ominus d)^n_{p,q}}\end{cases}\\
&= \begin{cases}\lim\limits_{n\to\infty}\dfrac{(c\ominus\frac{cb}{a})^n_{p,q}}{(c\ominus d)^n_{p,q}} = \dfrac{(c\ominus\frac{cb}{a})^\infty_{p,q}}{(c\ominus d)^\infty_{p,q}}\\ \lim\limits_{n\to\infty}\dfrac{(a\ominus b)^n_{p,q}}{(c\ominus\frac{ad}{c})^n_{p,q}} = \dfrac{(a\ominus b)^\infty_{p,q}}{(a\ominus\frac{ad}{c})^\infty_{p,q}}.\end{cases}
\end{aligned}$$

This proves the corollary. □

Remark 89. *Note that for every complex number λ, the following equation applies*

$$(\lambda a\ominus\lambda b)^n_{p,q} = \lambda^n\,(a\ominus b)^n_{p,q}\,. \tag{8.7}$$

Proposition 90 (Jagannathan and Srinisvasa [49])**.** *The following relation between the ${}_r\phi_s$-series and the ${}_r\Phi_s$-series is valid:*

$$\begin{aligned}
&{}_r\phi_s\left(\left.\begin{matrix}\frac{a_{1q}}{a_{1p}},\frac{a_{2q}}{a_{2p}},\dots,\frac{a_{rq}}{a_{rp}}\\ \frac{b_{1q}}{b_{1p}},\frac{b_{2q}}{b_{2p}},\dots,\frac{b_{sq}}{b_{sp}}\end{matrix}\,\right|\,\frac{q}{p};z\right)\\
&= \begin{cases}{}_r\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots,(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\right|(p,q);\mu z\right), & s=r-1\\
{}_{s+1}\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots,(a_{rp},a_{rq}),(0,1),\dots,(0,1)\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\right|(p,q);\mu z\right), & s>r-1\\
{}_r\Phi_{r-1}\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots,(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq},(0,1),\dots,(0,1))\end{matrix}\right|(p,q);\mu z\right), & s<r-1.\end{cases}
\end{aligned} \tag{8.8}$$

with

$$\mu = \frac{pb_{1p}b_{2p}\dots b_{sp}}{a_{1p}a_{2p}\dots a_{rp}}. \tag{8.9}$$

Proof. We write the proof for $s=r-1$. The cases $s>r-1$ and $s<r-1$ are done in a

similar way. For $s = r-1$, using (8.6) and (8.4), it follows that

$$
\begin{aligned}
{}_r\Phi_s\left(\begin{matrix}(a_{1p},a_{1q}),\ldots,(a_{rp},a_{rq})\\ (b_{1p},b_{1q}),\ldots,(b_{sp},b_{sq})\end{matrix}\middle|\,(p,q);z\right) &= {}_r\Phi_s\left(\begin{matrix}(a_{1p},a_{1q}),\ldots,(a_{rp},a_{rq})\\ (b_{1p},b_{1q}),\ldots,(b_{sq},b_{sq})\end{matrix}\middle|\,(p,q);z\right)\\
&= \sum_{n=0}^{\infty}\frac{(a_{1p}\ominus a_{1q})^n_{p,q}\cdots(a_{rp}\ominus a_{rq})^n_{p,q}}{(b_{1p}\ominus b_{1q})^n_{p,q}\cdots(b_{sp}\ominus b_{sq})^n_{p,q}(p\ominus q)^n_{p,q}}z^n\\
&= \sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r} a_{ip}^n p^{\binom{n}{2}}\left(\frac{a_{iq}}{a_{ip}};\frac{q}{p}\right)_n}{\prod\limits_{i=1}^{s} b_{ip}^n p^{\binom{n}{2}}\left(\frac{b_{iq}}{b_{ip}};\frac{q}{p}\right)_n p^n p^{\binom{n}{2}}\left(\frac{q}{p};\frac{q}{p}\right)}z^n\\
&= \sum_{n=0}^{\infty}\frac{\left(\frac{a_{1q}}{a_{1p}};\frac{q}{p}\right)_n\cdots\left(\frac{a_{rq}}{a_{rp}};\frac{q}{p}\right)_n}{\left(\frac{b_{1q}}{b_{1p}};\frac{q}{p}\right)_n\cdots\left(\frac{b_{sq}}{b_{sp}};\frac{q}{p}\right)_n}\left(\frac{a_{1p}a_{2p}\ldots a_{rp}}{b_{1p}b_{2p}\ldots b_{rp}p}z\right)^n\\
&= {}_r\phi_s\left(\begin{matrix}\frac{a_{1q}}{a_{1p}},\cdots\frac{a_{rq}}{a_{rp}}\\ \frac{b_{1q}}{b_{1p}},\cdots\frac{b_{sq}}{b_{sp}}\end{matrix}\middle|\,\frac{a_{1p}a_{2p}\ldots a_{rp}}{b_{1p}b_{2p}\ldots b_{rp}p}z\right),
\end{aligned}
$$

where it is assumed that the ${}_r\phi_s$-series is convergent or terminating. This proves the proposition for $s = r-1$. □

Remark 91. *From the Proposition 90, it is seen that any well behaved ϕ-series can be written as a Φ-series. But the converse is not true, in general; in the general case, when $p \neq 1$, this is possible only for an ${}_r\Phi_{r-1}$. To see this, it is enough to look at the ${}_0\Phi_0$ case. Indeed,*

$$
{}_0\Phi_0\left(\begin{matrix}-\\ (p,q)\end{matrix}\middle|\,(p,q);z\right) = \sum_{n=0}^{\infty}\frac{(-1)^n(q/p)^{\binom{n}{2}}}{(p\ominus q)^n_{p,q}}z^n \tag{8.10}
$$

$$
= \sum_{n=0}^{\infty}\frac{(-1)^n(\rho/p)^{\binom{n}{2}}}{(\rho;\rho)_n}(z/\rho)^n, \quad \textit{with}\ \ \rho = q/p, \tag{8.11}
$$

which shows that ${}_0\Phi_0$ becomes a ϕ-series if and only if $p = 1$. Similarly, one is easily convinced that a generic ${}_r\Phi_s$-series cannot be identified within the class of ϕ-series unless $p = 1$ or $s = r-1$. It is now clear that the (p,q)-series is larger than structure in which the q-series gets embedded.

Remark 92. *Note that in the usual ϕ-series theory, there is no direct analogue for the choice $a_{ip} = 0$ or $b_{ip} = 0$, for any i, permissible, in general (of course, subject to conditions of convergence and so on), in the (p,q)-series; to obtain a corresponding result in the case of the ϕ-series one will have to resort to the limit process of confluence, namely, replacing z by z/a_r and taking the limit $a_r \to \infty$. For example*

$$
\lim_{a_r\to\infty} {}_r\phi_s\left(\begin{matrix}a_1,\ldots a_r\\ b_1,\ldots b_s\end{matrix}\middle|\,q;\frac{z}{a_r}\right) = {}_{r-1}\phi_s\left(\begin{matrix}a_1,\ldots,a_{r-1}\\ b_1,\ldots b_s\end{matrix}\middle|\,q;z\right). \tag{8.12}
$$

For the (p,q)-hypergeometric series, using the fact that

$$
\lim_{a\to\infty}\frac{(a\ominus b)^n_{p,q}}{a^n} = \lim_{a\to\infty}\left(1\ominus\frac{b}{a}\right)^n_{p,q} = (1\ominus 0)^n_{p,q},
$$

or

$$\lim_{b\to\infty}\frac{(a\ominus b)^n_{p,q}}{b^n}=\lim_{b\to\infty}\left(\frac{a}{b}\ominus 1\right)^n_{p,q}=(0\ominus 1)^n_{p,q},$$

it follows that

$$\lim_{a_{rp}\to\infty}{}_r\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\right|(p,q);\frac{z}{a_{rp}}\right)={}_r\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{(r-1)p},a_{(r-1)q}),(1,0)\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\right|(p,q);z\right)$$

$$\lim_{a_{rq}\to\infty}{}_r\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\right|(p,q);\frac{z}{a_{rq}}\right)={}_r\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{(r-1)p},a_{(r-1)q}),(0,1)\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\right|(p,q);z\right).$$

Also, we can obtain the following formulas

$$\lim_{b_{sp}\to\infty}{}_r\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\right|(p,q);zb_{sp}\right)={}_r\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{(s-1)p},b_{(s-1)q}),(1,0)\end{matrix}\right|(p,q);z\right)$$

$$\lim_{b_{rq}\to\infty}{}_r\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\right|(p,q);zb_{rq}\right)={}_r\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{(s-1)p},b_{(s-1)q}),(0,1)\end{matrix}\right|(p,q);z\right).$$

and

$$\lim_{a_{rp},b_{sp}\to\infty}{}_r\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\right|(p,q);zb_{sp}\right)={}_{r-1}\Phi_{s-1}\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{(r-1)p},a_{(r-1)q})\\(b_{1p},b_{1q}),\dots,(b_{(s-1)p},b_{(s-1)q})\end{matrix}\right|(p,q);z\right)$$

$$\lim_{a_{rq},b_{rq}\to\infty}{}_r\Phi_s\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{rp},a_{rq})\\(b_{1p},b_{1q}),\dots,(b_{sp},b_{sq})\end{matrix}\right|(p,q);zb_{rq}\right)={}_{r-1}\Phi_{s-1}\left(\left.\begin{matrix}(a_{1p},a_{1q}),\dots(a_{(r-1)p},a_{(r-1)q})\\(b_{1p},b_{1q}),\dots,(b_{(s-1)p},b_{(s-1)q})\end{matrix}\right|(p,q);z\right).$$

Now we write in detail some relevant cases of Proposition 90 that we will use to obtain some (p,q)-transformations and (p,q)-summation formulas.

$${}_1\phi_1\left(\left.\begin{matrix}\frac{b}{a}\\\frac{d}{c}\end{matrix}\right|\frac{q}{p};\theta\right)={}_2\Phi_1\left(\left.\begin{matrix}(a,b),(0,1)\\(c,d)\end{matrix}\right|(p,q);\frac{pc\theta}{a}\right); \tag{8.13}$$

$${}_2\phi_1\left(\left.\begin{matrix}\frac{b}{a},\frac{d}{c}\\\frac{f}{e}\end{matrix}\right|\frac{q}{p},\theta\right)={}_2\Phi_1\left(\left.\begin{matrix}(a,b),(c,d)\\(e,f)\end{matrix}\right|(p,q);\frac{pe\theta}{ac}\right); \tag{8.14}$$

$${}_2\phi_2\left(\left.\begin{matrix}\frac{b}{a},\frac{d}{c}\\\frac{f}{e},\frac{h}{g}\end{matrix}\right|\frac{q}{p},\theta\right)={}_3\Phi_2\left(\left.\begin{matrix}(a,b),(c,d),(0,1)\\(e,f),(g,h))\end{matrix}\right|(p,q);\frac{peg\theta}{ac}\right); \tag{8.15}$$

$${}_3\phi_2\left(\left.\begin{matrix}\frac{b}{a},\frac{d}{c},\frac{f}{e}\\\frac{h}{g},\frac{j}{i}\end{matrix}\right|\frac{q}{p},\theta\right)={}_3\Phi_2\left(\left.\begin{matrix}(a,b),(c,d),(e,f)\\(g,h),(i,j))\end{matrix}\right|(p,q);\frac{pgi\theta}{ace}\right). \tag{8.16}$$

8.2.1 The (p,q)-binomial theorem

The q-binomial theorem (Kac and Cheung [52] or Koekoek, Lesky and Swarttouw [53]) states that

$$ {}_1\phi_0\left(\begin{array}{c|c} a & \\ & q;z \\ - & \end{array}\right) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{8.17} $$

The following theorem is a (p,q)-analogue of (8.17).

Theorem 93 (Jagannathan [48]). *Let a, b be two non-zero complex numbers, then we have the following*

$$ {}_1\Phi_0\left(\begin{array}{c|c} (a,b) & \\ & (p,q);z \\ - & \end{array}\right) = \sum_{n=0}^{\infty}\frac{(a\ominus b)^n_{p,q}}{(p\ominus q)^n_{p,q}}z^n = \frac{(p\ominus bz)^\infty_{p,q}}{(p\ominus az)^\infty_{p,q}}. \tag{8.18} $$

Proof. We first note that $\dfrac{(a\ominus b)^n_{p,q}}{(p\ominus q)^n_{p,q}} = \dfrac{\left(\frac{b}{a};\frac{q}{p}\right)_n}{\left(\frac{q}{p};\frac{q}{p}\right)_n}\left(\dfrac{a}{p}\right)^n$. It follows from the q-binomial theorem that

$$ \begin{aligned} \sum_{n=0}^{\infty}\frac{(a\ominus b)^n_{p,q}}{(p\ominus q)^n_{p,q}}z^n &= \sum_{n=0}^{\infty}\frac{\left(\frac{b}{a};\frac{q}{p}\right)_n}{\left(\frac{q}{p};\frac{q}{p}\right)_n}\left(\frac{az}{p}\right)^n \\ &= \frac{\left(\frac{bz}{p};\frac{q}{p}\right)_\infty}{\left(\frac{az}{p};\frac{q}{p}\right)_\infty} = \frac{(p\ominus bz)^\infty_{p,q}}{(p\ominus az)^\infty_{p,q}}. \end{aligned} $$

□

The usual q-binomial theorem (8.17) is obtained when $a = 1$ and $p = 1$. An interesting feature of the (p,q)-binomial theorem (8.18) may be noted here. The product

$$ \prod_{k=1}^{n} {}_1\Phi_0\left(\begin{array}{c|c} (a_{kp},a_{kq}) & \\ & (p,q);z \\ - & \end{array}\right) $$

is seen to be an invariant under the group of independent permutations of parameters of the p-components $(a_{1p},a_{2p},\ldots,a_{np})$ and the q-components $(a_{1q},a_{2q},\ldots,a_{nq})$. This product has value 1 if the n-tuple of p-components $(a_{1p},a_{2p},\ldots,a_{np})$ is related to the n-tuple of components $(a_{1q},a_{2q},\ldots,a_{nq})$ by a mere permutation.

For the case $n = 2$ this result implies that

$$ {}_1\Phi_0\left(\begin{array}{c|c} (a,b) & \\ & (p,q);z \\ - & \end{array}\right) {}_1\Phi_0\left(\begin{array}{c|c} (b,a) & \\ & (p,q);z \\ - & \end{array}\right) = 1. \tag{8.19} $$

A special case of this equation is

$$ {}_1\Phi_0\left(\begin{array}{c|c} (1,0) & \\ & (p,q);z \\ - & \end{array}\right) {}_1\Phi_0\left(\begin{array}{c|c} (0,1) & \\ & (p,q);z \\ - & \end{array}\right) = 1. \tag{8.20} $$

Recognizing that

$$ {}_1\Phi_0\left(\begin{array}{c|c} (1,0) & \\ & (p,q);z \\ - & \end{array}\right) = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}}{(p\ominus q)_{p,q}^{n}}z^n = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}}{[n]_{p,q}!}((p-q)z)^n = \mathrm{e}_{p,q}((p-q)z) \quad (8.21) $$

and

$$ {}_1\Phi_0\left(\begin{array}{c|c} (0,1) & \\ & (p,q);-z \\ - & \end{array}\right) = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{(p\ominus q)_{p,q}^{n}}z^n = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{[n]_{p,q}!}((p-q)z)^n = E_{p,q}((p-q)z) \quad (8.22) $$

we recover the relation (5.3). For $p=1$, $\mathrm{e}_{p,q}(z)$ and $E_{p,q}(z)$ reduces to $e_q(z)$ and $E_q(z)$, respectively.

For $n=3$, the above general result and the relation (8.18) imply

$$ {}_1\Phi_0\left(\begin{array}{c|c} (u,v) & \\ & (p,q);z \\ - & \end{array}\right){}_1\Phi_0\left(\begin{array}{c|c} (v,w) & \\ & (p,q);z \\ - & \end{array}\right) = {}_1\Phi_0\left(\begin{array}{c|c} (u,w) & \\ & (p,q);z \\ - & \end{array}\right). \quad (8.23) $$

Choose $u=1$, $v=a$, $w=ab$ and $p=1$ in (8.23), to obtain

$$ {}_1\Phi_0\left(\begin{array}{c|c} (1,a) & \\ & (1,q);z \\ - & \end{array}\right){}_1\Phi_0\left(\begin{array}{c|c} (a,ab) & \\ & (1,q);z \\ - & \end{array}\right) = {}_1\Phi_0\left(\begin{array}{c|c} (1,ab) & \\ & (1,q);z \\ - & \end{array}\right), $$

which is nothing but the well known product formula for the ${}_1\phi_0$ function, namely

$$ {}_1\phi_0\left(\begin{array}{c|c} a & \\ & q;z \\ - & \end{array}\right){}_1\phi_0\left(\begin{array}{c|c} b & \\ & q;az \\ - & \end{array}\right) = {}_1\phi_0\left(\begin{array}{c|c} ab & \\ & q;z \\ - & \end{array}\right). \quad (8.24) $$

8.2.2 (p,q)-Heine transformation for ${}_2\Phi_1$

The Heine transformation of the ${}_2\phi_1$ series states that [38, (III.1) P. 359]

$$ {}_2\phi_1\left(\begin{array}{c|c} a,b & \\ & q;z \\ c & \end{array}\right) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}{}_2\phi_1\left(\begin{array}{c|c} c/b,z & \\ & q;b \\ az & \end{array}\right). \quad (8.25) $$

Proposition 94 (Jagannathan and Srinisvasa [49]). *The following (p,q)-Heine transformation formula holds for the ${}_2\Phi_1$*

$$ {}_2\Phi_1\left(\begin{array}{c|c} (a,b),(c,d) & \\ & (p,q);z \\ (e,f) & \end{array}\right) = \frac{(ce\ominus de)_{p,q}^{\infty}(pe\ominus bcz)_{p,q}^{\infty}}{(ce\ominus cf)_{p,q}^{\infty}(pe\ominus acz)_{p,q}^{\infty}}{}_2\Phi_1\left(\begin{array}{c|c} (dc,cf),(pe,acz) & \\ & (p,q);\frac{p}{ce} \\ (pe,bcz) & \end{array}\right). \quad (8.26) $$

Proof. From the Heine transformation formula (8.25), we can write

$$ {}_2\phi_1\left(\left.\begin{matrix}\frac{b}{a},\frac{d}{c}\\ \frac{f}{e}\end{matrix}\right|\frac{q}{p},\theta\right)=\frac{\left(\frac{d}{c};\frac{q}{p}\right)_\infty\left(\frac{b\theta}{a};\frac{q}{p}\right)_\infty}{\left(\frac{f}{e};\frac{q}{p}\right)_\infty\left(\theta;\frac{q}{p}\right)_\infty}{}_2\phi_1\left(\left.\begin{matrix}\frac{cf}{de},\theta\\ \frac{b\theta}{a}\end{matrix}\right|\frac{q}{p},\frac{d}{c}\right). $$

Using the first case of (8.8), we can write

$$ {}_2\phi_1\left(\left.\begin{matrix}\frac{b}{a},\frac{d}{c}\\ \frac{f}{e}\end{matrix}\right|\frac{q}{p},\theta\right)={}_2\Phi_1\left(\left.\begin{matrix}(a,b),(c,d)\\ (e,f)\end{matrix}\right|(p,q);\frac{pe\theta}{ac}\right), $$

$$ {}_2\phi_1\left(\left.\begin{matrix}\frac{cf}{de},\theta\\ \frac{b\theta}{a}\end{matrix}\right|\frac{q}{p},\frac{d}{c}\right)={}_2\Phi_1\left(\left.\begin{matrix}(de,cf),(1,\theta)\\ (a,b\theta)\end{matrix}\right|(p,q);\frac{pa}{ce}\right). $$

Next the use of (8.6) produces

$$ \frac{\left(\frac{d}{c};\frac{q}{p}\right)_\infty\left(\frac{b\theta}{a};\frac{q}{p}\right)_\infty}{\left(\frac{f}{e};\frac{q}{p}\right)_\infty\left(\theta;\frac{q}{p}\right)_\infty}=\frac{(c\ominus d)_{p,q}^\infty\,(a\ominus b\theta)_{p,q}^\infty}{\left(c\ominus\frac{cf}{e}\right)_{p,q}^\infty(a\ominus a\theta)_{p,q}^\infty}=\frac{(ce\ominus de)_{p,q}^\infty(pe\ominus bcz)_{p,q}^\infty}{(ce\ominus cf)_{p,q}^\infty(pe\ominus acz)_{p,q}^\infty}. $$

Hence,

$$ {}_2\Phi_1\left(\left.\begin{matrix}(de,cf),(1,\theta)\\ (a,b\theta)\end{matrix}\right|(p,q);\frac{pa}{ce}\right)=\frac{(ce\ominus de)_{p,q}^\infty(pe\ominus bcz)_{p,q}^\infty}{(ce\ominus cf)_{p,q}^\infty(pe\ominus acz)_{p,q}^\infty}{}_2\Phi_1\left(\left.\begin{matrix}(de,cf),(1,\theta)\\ (a,b\theta)\end{matrix}\right|(p,q);\frac{pa}{ce}\right). $$

Finally, taking $\theta=\dfrac{acz}{pe}$ we obtain the announced result. □

Setting $a=0$, $b=c=e=1$ and $p=1$, it follows that

$$ {}_2\Phi_1\left(\left.\begin{matrix}(0,1),(1,d)\\ (1,f)\end{matrix}\right|(1,q);z\right)=\frac{(1\ominus d)_{1,q}^\infty(1\ominus z)_{1,q}^\infty}{(1\ominus f)_{1,q}^\infty(1\ominus 0)_{1,q}^\infty}{}_2\Phi_1\left(\left.\begin{matrix}(d,f),(1,0)\\ (1,z)\end{matrix}\right|(1,q);1\right). $$

Using the facts that

$$ (1\ominus x)_{1,q}^n=(x;q)_n,\quad (d\ominus f)_{p,q}^n=d^n\left(1\ominus\frac{f}{d}\right)_{p,q}^n, $$

we get the following transformation formula for the ϕ-series

$$ {}_1\phi_1\left(\left.\begin{matrix}d\\ f\end{matrix}\right|q;z^n\right)=\frac{(d;q)_\infty(z;q)_\infty}{(f;q)_\infty}{}_2\phi_1\left(\left.\begin{matrix}0,\frac{f}{d}\\ z\end{matrix}\right|q;d\right). $$

By relabelling d as a and f as b, this read

$$ {}_1\phi_1\left(\left.\begin{matrix}a\\ b\end{matrix}\right|q;z\right)=\frac{(a;q)_\infty(z;q)_\infty}{(b;q)_\infty}{}_2\phi_1\left(\left.\begin{matrix}0,\frac{b}{a}\\ z\end{matrix}\right|q;a\right). \tag{8.27} $$

8.2.3 (p,q)-Gauss sum

The q-Gauss sum is (Gasper and Rahman [38, (II.8) P. 354])

$$ {}_2\phi_1\left(\begin{array}{c} a,b \\ c \end{array}\middle| q;\frac{c}{ab}\right) = \frac{\left(\frac{c}{a};q\right)_\infty\left(\frac{c}{b};q\right)_\infty}{(c;q)_\infty\left(\frac{c}{ab};q\right)_\infty}, \quad \left|\frac{c}{ab}\right| < 1. \tag{8.28} $$

Proposition 95 (Jagannathan and Srinisvasa [49])**.** *The following (p,q)-Gauss sum holds:*

$$ {}_2\Phi_1\left(\begin{array}{c} (a,b),(c,d) \\ (e,f) \end{array}\middle| (p,q);\frac{pf}{bd}\right) = \frac{(be\ominus af)^\infty_{p,q}(de\ominus cf)^\infty_{p,q}}{(e\ominus f)^\infty_{p,q}(bde\ominus acf)^\infty_{p,q}}, \quad \left|\frac{acf}{dbe}\right| < 1. \tag{8.29} $$

Proof. Using the q-Gauss sum (8.28), it follows that

$$ {}_2\phi_1\left(\begin{array}{c} \frac{b}{a},\frac{d}{c} \\ \frac{f}{e} \end{array}\middle| \frac{q}{p};\frac{acf}{bde}\right) = \frac{\left(\frac{af}{be};\frac{q}{p}\right)_\infty\left(\frac{cf}{de};\frac{q}{p}\right)_\infty}{\left(\frac{f}{e};\frac{q}{p}\right)_\infty\left(\frac{acf}{bde};\frac{q}{p}\right)_\infty}. $$

Next, from the first case of (8.8), the left-hand side of the previous equation reads

$$ {}_2\phi_1\left(\begin{array}{c} \frac{b}{a},\frac{d}{c} \\ \frac{f}{e} \end{array}\middle| \frac{q}{p};\frac{acf}{bde}\right) = {}_2\Phi_1\left(\begin{array}{c} (a,b),(c,d) \\ (e,f) \end{array}\middle| (p,q);\frac{pf}{bd}\right). $$

The use of (8.6) gives

$$ \frac{\left(\frac{af}{be};\frac{q}{p}\right)_\infty\left(\frac{cf}{de};\frac{q}{p}\right)_\infty}{\left(\frac{f}{e};\frac{q}{p}\right)_\infty\left(\frac{acf}{bde};\frac{q}{p}\right)_\infty} = \frac{(be\ominus af)^\infty_{p,q}(de\ominus cf)^\infty_{p,q}}{(be\ominus bf)^\infty_{p,q}\left(de\ominus\frac{acf}{b}\right)^\infty_{p,q}} = \frac{(be\ominus af)^\infty_{p,q}(de\ominus cf)^\infty_{p,q}}{(e\ominus f)^\infty_{p,q}(bde\ominus acf)^\infty_{p,q}}. $$

This proves the desired result. $\square$

If we set $a=c=0$, $b=d=e=1$, $f=qz$ and $p=1$ in (8.29), it follows that

$$ {}_2\Phi_1\left(\begin{array}{c} (0,1),(0,1) \\ (1,qz) \end{array}\middle| (1,q);qz\right) = \frac{(1\ominus 0)^\infty_{1,q}(1\ominus 0)^\infty_{1,q}}{(1\ominus qz)^\infty_{1,q}(1\ominus 0)^\infty_{1,q}}. $$

which reads

$$ \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q,qz;q)_\infty}z^n = \frac{1}{(qz;q)_\infty}. \tag{8.30} $$

Note that this result is usually obtained from the q-Gauss sum (8.28) by setting $c=qz$ and letting $a\to\infty$ and $b\to\infty$.

Note also if we take $c=0$ and $d=1$, and relabelling e by c and f by d we get

$$ {}_2\Phi_1\left(\begin{array}{c} (a,b),(0,1) \\ (c,d) \end{array}\middle| (p,q);\frac{pd}{b}\right) = \frac{(bc\ominus ad)^\infty_{p,q}}{(bc\ominus bd)^\infty_{p,q}}. $$

The last equation is nothing but the well known summation formula (Gasper and Rahman [38, (II-5), P. 354])

$$ {}_1\phi_1\left(\begin{array}{c} a \\ c \end{array}\middle| q; \frac{c}{a}\right) = \frac{\left(\frac{c}{a};q\right)_\infty}{(c;q)_\infty}. \tag{8.31}$$

Finally remark that (8.31) is usually obtained from the q-Gauss summation formula (8.28) by letting $b \to \infty$.

8.2.4 The (p,q)-Kummer sum

The q-Kummer sum is (Gasper and Rahman [38, (II.9), P. 354])

$$ {}_2\phi_1\left(\begin{array}{c} a, b \\ \frac{aq}{b} \end{array}\middle| q; -\frac{q}{b}\right) = \frac{(-q;q)_\infty \left(aq, \frac{aq^2}{b^2}; q^2\right)_\infty}{\left(-\frac{q}{b}, \frac{aq}{b}; q\right)_\infty}. \tag{8.32}$$

Proposition 96. *The following (p,q)-Kummer summation formula is valid:*

$$ {}_2\Phi_1\left(\begin{array}{c} (a,b),(c,d) \\ (adp, bcq) \end{array}\middle| (p,q); -pq\right) = \frac{(dp\oplus dq)_{p,q}^\infty (ap\ominus bq)_{p^2,q^2}^\infty \left(ad^2p^2\ominus bc^2q^2\right)_{p^2,q^2}^\infty}{(dp\oplus qc)_{p,q}^\infty (adp\ominus bcq)_{p,q}^\infty}. \tag{8.33}$$

Proof. From (8.14) and (8.32), it follows that

$$\begin{aligned}
{}_2\Phi_1\left(\begin{array}{c} (a,b),(c,d) \\ (adp, bcq) \end{array}\middle| (p,q); -pq\right) &= {}_2\phi_1\left(\begin{array}{c} \frac{b}{a}, \frac{d}{c} \\ \frac{bcq}{adp} \end{array}\middle| \frac{q}{p}; -\frac{cq}{dp}\right) \\
&= \frac{\left(-\frac{q}{p};\frac{q}{p}\right)_\infty \left(\frac{bq}{ap};\frac{q^2}{p^2}\right)_\infty \left(\frac{bc^2q^2}{ad^2p^2};\frac{q^2}{p^2}\right)_\infty}{\left(-\frac{cq}{dp};\frac{q}{p}\right)_\infty \left(\frac{bcq}{adq};\frac{q}{p}\right)_\infty} \\
&= \frac{\left(-\frac{q}{p};\frac{q}{p}\right)_\infty \left(\frac{bq}{ap};\frac{q^2}{p^2}\right)_\infty \left(\frac{bc^2q^2}{ad^2p^2};\frac{q^2}{p^2}\right)_\infty}{\left(-\frac{cq}{dp};\frac{q}{p}\right)_\infty \left(\frac{bcq}{adq};\frac{q^2}{p^2}\right)_\infty \left(\frac{bcq^2}{adq^2};\frac{q^2}{p^2}\right)_\infty} \\
&= \frac{(p\oplus q)_{p,q}^\infty (ap\ominus bq)_{p^2,q^2}^\infty \left(ad^2p^2\ominus bc^2q^2\right)_{p^2,q^2}^\infty}{\left(p\oplus\frac{qc}{d}\right)_{p,q}^\infty \left(ap\ominus\frac{bcq}{d}\right)_{p^2,q^2}^\infty \left(ad^2p^2\ominus bcdq^2\right)_{p^2,q^2}^\infty} \\
&= \frac{(dp\oplus dq)_{p,q}^\infty (ap\ominus bq)_{p^2,q^2}^\infty \left(ad^2p^2\ominus bc^2q^2\right)_{p^2,q^2}^\infty}{(dp\oplus qc)_{p,q}^\infty (adp\ominus bcq)_{p^2,q^2}^\infty \left(adp^2\ominus bcq^2\right)_{p^2,q^2}^\infty} \\
&= \frac{(dp\oplus dq)_{p,q}^\infty (ap\ominus bq)_{p^2,q^2}^\infty \left(ad^2p^2\ominus bc^2q^2\right)_{p^2,q^2}^\infty}{(dp\oplus qc)_{p,q}^\infty (adp\ominus bcq)_{p,q}^\infty}.
\end{aligned}$$

This proves (8.33). □

8.2.5 A (p,q)-analogue of Bailey's ${}_2F_1(-1)$ sum

The q-analogue of the Bailey's ${}_2F_1(-1)$ sum is (Gasper and Rahman [38, (II.10), P. 354])

$$ {}_2\phi_2\left(\begin{array}{c} a, \dfrac{q}{a} \\ -q, b \end{array}\middle| q; -b\right) = \frac{\left(ab, \dfrac{bq}{a}; q^2\right)}{(b;q)_\infty} \tag{8.34} $$

Proposition 97. *The following (p,q)-analogue of the Bailey's ${}_2F_1(-1)$ is valid:*

$$ {}_3\Phi_2\left(\begin{array}{c} (a,b),(pb,qa),(0,1) \\ (p,-q),(c,d) \end{array}\middle| (p,q); -\frac{pd}{ab}\right) = \frac{(ac\ominus bd)^{\infty}_{p^2,q^2}\,(bcp\ominus adq)^{\infty}_{p^2,q^2}}{(ac\ominus ad)^{\infty}_{p^2,q^2}\,(bcp\ominus bdq)^{\infty}_{p^2,q^2}}. \tag{8.35} $$

Proof. From (8.15) and (8.34) we get

$$ \begin{aligned} {}_3\Phi_2\left(\begin{array}{c} (a,b),(pb,qa),(0,1) \\ (p,-q),(c,d) \end{array}\middle| (p,q); -\frac{pd}{ab}\right) &= {}_2\phi_2\left(\begin{array}{c} \dfrac{b}{a}, \dfrac{aq}{bp} \\ -\dfrac{q}{p}, \dfrac{d}{c} \end{array}\middle| \frac{q}{p}; -\frac{d}{c}\right) \\ &= \frac{\left(\dfrac{bd}{ac}; \dfrac{q^2}{p^2}\right)_\infty \left(\dfrac{adq}{bcp}; \dfrac{q^2}{p^2}\right)_\infty}{\left(\dfrac{d}{c}; \dfrac{q^2}{p^2}\right)_\infty \left(\dfrac{dq}{cp}; \dfrac{q^2}{p^2}\right)_\infty} \\ &= \frac{(ac\ominus bd)^{\infty}_{p^2,q^2}\,(bcp\ominus adq)^{\infty}_{p^2,q^2}}{(ac\ominus ad)^{\infty}_{p^2,q^2}\,(bcp\ominus bdq)^{\infty}_{p^2,q^2}}. \end{aligned} $$

Equation (8.35) is therefore proved. □

8.2.6 A (p,q)-analogue of Gauss's ${}_2F_1(-1)$ sum

A q-analogue of Gauss's ${}_2F_1(-1)$ sum is (Gasper and Rahman [38, (II.11), P. 355])

$$ {}_2\phi_2\left(\begin{array}{c} a^2, b^2 \\ abq^{1/2}, -abq^{1/2} \end{array}\middle| q; -q\right) = \frac{(a^2q, b^2q; q^2)_\infty}{(q; a^2b^2q; q^2)_\infty}. \tag{8.36} $$

Proposition 98. *The following (p,q)-analogue of the Gauss ${}_2F_1(-1)$ summation formula holds:*

$$ {}_3\Phi_2\left(\begin{array}{c} (a^2,b^2),(c^2,d^2),(0,1) \\ (acp^{1/2},bdq^{1/2}),(acp^{1/2},-bdq^{1/2}) \end{array}\middle| (p,q); -pq\right) = \frac{(a^2p\ominus b^2q)^{\infty}_{p^2,q^2}\,(c^2p\ominus d^2q)^{\infty}_{p^2,q^2}}{(p\ominus q)^{\infty}_{p^2,q^2}\,(a^2c^2p\ominus b^2d^2q)^{\infty}_{p^2,q^2}}. \tag{8.37} $$

Proof. From (8.15) and (8.36) we get

$$
\begin{aligned}
{}_3\Phi_2\left(\begin{matrix}(a^2,b^2),(c^2,d^2),(0,1)\\(acp^{1/2},bdq^{1/2}),(acp^{1/2},-bdq^{1/2})\end{matrix}\middle|(p,q);-pq\right) &= {}_2\phi_2\left(\begin{matrix}\frac{b^2}{a^2},\frac{d^2}{c^2}\\ \frac{bdq^{1/2}}{acp^{1/2}},-\frac{bdq^{1/2}}{acp^{1/2}}\end{matrix}\middle|\frac{q}{p},-\frac{q}{p}\right)\\
&= \frac{\left(\frac{b^2q}{a^2p};\frac{q^2}{p^2}\right)_\infty\left(\frac{d^2q}{c^2p};\frac{q^2}{p^2}\right)_\infty}{\left(\frac{q}{p};\frac{q^2}{p^2}\right)_\infty\left(\frac{b^2d^2q}{a^2c^2p};\frac{q^2}{p^2}\right)_\infty}\\
&= \frac{(a^2p\ominus b^2q)^\infty_{p^2,q^2}(c^2p\ominus d^2q)^\infty_{p^2,q^2}}{(p\ominus q)^\infty_{p^2,q^2}(a^2c^2p\ominus b^2d^2q)^\infty_{p^2,q^2}}.
\end{aligned}
$$

□

8.2.7 The (p,q)-Saalschütz sum

The q-Saalschütz sum (Gasper and Rahman [38, (II.12), P. 355]) is

$$
{}_3\phi_2\left(\begin{matrix}q^{-n},a,b\\ \frac{ab}{cq^{n-1}},c\end{matrix}\middle|q;q\right) = \frac{\left(\frac{c}{a},\frac{c}{b};q\right)_n}{\left(c;\frac{c}{ab};q\right)_n}. \tag{8.38}
$$

Proposition 99. *The following (p,q)-Saalschütz sum holds*

$$
{}_3\Phi_2\left(\begin{matrix}(p^{-n},q^{-n}),(a,b),(c,d)\\(acfp^{1-n},bdeq^{1-n}),(e,f)\end{matrix}\middle|(p,q);pqf\right) = \frac{(be\ominus af)^n_{p,q}(de\ominus cf)^n_{p,q}}{(e\ominus f)^n_{p,q}(bde\ominus acf)^n_{p,q}}. \tag{8.39}
$$

Proof. From the relations (8.16) and (8.38), it follows that

$$
\begin{aligned}
{}_3\Phi_2\left(\begin{matrix}(p^{-n},q^{-n}),(a,b),(c,d)\\(acfp^{1-n},bdeq^{1-n}),(e,f)\end{matrix}\middle|(p,q);pqef\right) &= {}_3\phi_2\left(\begin{matrix}\frac{q^{-n}}{p^{-n}},\frac{b}{a},\frac{d}{c}\\ \frac{bdeq^{1-n}}{acfp^{1-n}},\frac{f}{e}\end{matrix}\middle|\frac{q}{p};\frac{q}{p}\right)\\
&= \frac{\left(\frac{af}{be};\frac{q}{p}\right)_n\left(\frac{cf}{de};\frac{q}{p}\right)_n}{\left(\frac{f}{e};\frac{q}{p}\right)_n\left(\frac{acf}{bde};\frac{q}{p}\right)_n}\\
&= \frac{(be\ominus af)^n_{p,q}(de\ominus cf)^n_{p,q}}{(e\ominus f)^n_{p,q}(bde\ominus acf)^n_{p,q}}.
\end{aligned}
$$

This proves (8.39). □

8.2.8 (p,q)-Jackson's transformations of ${}_2\Phi_1$

Jackson's transformation formula for ${}_2\phi_1$, ${}_2\phi_2$ is (Gasper and Rahman [38, (III.4) P. 359])

$$
{}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|q;z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}{}_2\phi_2\left(\begin{matrix}a,\frac{c}{b}\\c,az\end{matrix}\middle|q;bz\right). \tag{8.40}
$$

Proposition 100. *The following (p,q)-Jackson's transformation is valid:*

$$
{}_2\Phi_1\left(\begin{matrix}(a,b),(c,d)\\(e,f)\end{matrix}\middle|(p,q);z\right)=\frac{(pe\ominus bcz)_{p,q}^{\infty}}{(pe\ominus acz)_{p,q}^{\infty}}\,{}_3\Phi_2\left(\begin{matrix}(a,b),(de,cf),(0,1)\\(e,f),(a,bz)\end{matrix}\middle|(p,q);\frac{az}{c}\right). \tag{8.41}
$$

Proof. From (8.14) and (8.40) it follows that

$$
\begin{aligned}
{}_2\Phi_1\left(\begin{matrix}(a,b),(c,d)\\(e,f)\end{matrix}\middle|(p,q);\frac{pe}{ac}\theta\right)&={}_2\phi_1\left(\begin{matrix}\frac{b}{a},\frac{d}{c}\\\frac{f}{e}\end{matrix}\middle|\frac{q}{p};\theta\right)\\
&=\frac{\left(\frac{b\theta}{a};\frac{q}{p}\right)_\infty}{(\theta;\frac{q}{p})_\infty}\,{}_2\phi_2\left(\begin{matrix}\frac{b}{a},\frac{cf}{de}\\\frac{f}{e},\frac{b\theta}{a}\end{matrix}\middle|\frac{q}{p};\frac{d\theta}{c}\right)\\
&=\frac{(a\ominus b\theta)_{p,q}^{\infty}}{(a\ominus a\theta)_{p,q}^{\infty}}\,{}_3\Phi_2\left(\begin{matrix}(a,b),(de,cf),(0,1)\\(e,f),(a,b\theta)\end{matrix}\middle|(p,q);\frac{p\theta}{c}\right).
\end{aligned}
$$

Taking $\theta=\frac{ac}{pe}z$, (8.41) follows. □

8.2.9 Transformations of ${}_3\Phi_2$ series

The following transformations of ${}_3\phi_2$ series are valid (Gasper and Rahman [38, (III.9) and (III.10), P 359])

$$
{}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix}\middle|q;\frac{de}{abc}\right)=\frac{\left(\frac{e}{a},\frac{de}{bc};q\right)_\infty}{\left(e,\frac{de}{abc};q\right)_\infty}\,{}_3\phi_2\left(\begin{matrix}a,\frac{d}{b},\frac{d}{c}\\d,\frac{de}{bc}\end{matrix}\middle|q,\frac{e}{a}\right) \tag{8.42}
$$

$$
=\frac{\left(b,\frac{de}{ab},\frac{de}{bc}\right)_\infty}{\left(d,e,\frac{de}{abc}\right)_\infty}\,{}_3\phi_2\left(\begin{matrix}\frac{d}{b},\frac{e}{b},\frac{de}{abc}\\\frac{de}{ab},\frac{de}{bc}\end{matrix}\middle|q;b\right). \tag{8.43}
$$

Proposition 101. *The following transformation formula for the ${}_3\Phi_2$-series are valid:*

$$
\begin{aligned}
&{}_3\Phi_2\left(\begin{matrix}(a,b),(c,d),(e,f)\\(g,h),(i,j)\end{matrix}\middle|(p,q);\frac{phj}{bdf}\right)\\
&=\frac{(bi\ominus aj)_{p,q}^{\infty}\,(dfgi\ominus cehj)_{p,q}^{\infty}}{\left(bi\ominus\frac{bi^2}{j}\right)_{p,q}^{\infty}\left(dfgi\ominus\frac{acehj}{b}\right)_{p,q}^{\infty}}\,{}_3\Phi_2\left(\begin{matrix}(a,b),(dg,ch),(fg,eh)\\(g,h),(dfgi,cehj)\end{matrix}\middle|\frac{q}{p};\frac{aj}{bi}\right).
\end{aligned} \tag{8.44}
$$

$$
{}_3\Phi_2\left(\begin{matrix}(a,b),(c,d),(e,f)\\ (g,h),(i,j)\end{matrix}\middle|(p,q);\frac{phj}{bdf}\right)
$$
$$
=\frac{(c\ominus d)_{p,q}^{\infty}(bdgi\ominus achj)_{p,q}^{\infty}(dfgi\ominus cehj)_{p,q}^{\infty}}{(cg\ominus ch)_{p,q}^{\infty}(di\ominus dj)_{p,q}^{\infty}(bdfgi\ominus acehj)_{p,q}^{\infty}}{}_3\Phi_2\left(\begin{matrix}(dg,ch),(di,cj),(bdfgi,acehj)\\ (bdgi,achj),(dfgi,cehj)\end{matrix}\middle|(p,q);\frac{p}{c}\right). \tag{8.45}
$$

Proof. From (8.42) and (8.16) it follows that

$$
{}_3\Phi_2\left(\begin{matrix}(a,b),(c,d),(e,f)\\ (g,h),(i,j)\end{matrix}\middle|(p,q);\frac{phj}{bdf}\right)={}_3\phi_2\left(\begin{matrix}\frac{b}{a},\frac{d}{c},\frac{f}{e}\\ \frac{h}{g},\frac{j}{i}\end{matrix}\middle|\frac{q}{p};\frac{acehj}{bdfgi}\right)
$$
$$
=\frac{\left(\frac{aj}{bi};\frac{q}{p}\right)_\infty\left(\frac{cehj}{dfgi};\frac{q}{p}\right)_\infty}{\left(\frac{j}{i};\frac{q}{p}\right)_\infty\left(\frac{acehj}{bdfgi};\frac{q}{p}\right)_\infty}{}_3\phi_2\left(\begin{matrix}\frac{b}{a},\frac{ch}{dg},\frac{eh}{fg}\\ \frac{h}{g},\frac{cehj}{dfgi}\end{matrix}\middle|\frac{q}{p};\frac{aj}{bi}\right)
$$
$$
=\frac{(bi\ominus aj)_{p,q}^{\infty}(dfgi\ominus cehj)_{p,q}^{\infty}}{\left(bi\ominus\frac{bi^2}{j}\right)_{p,q}^{\infty}\left(dfgi\ominus\frac{acehj}{b}\right)_{p,q}^{\infty}}{}_3\Phi_2\left(\begin{matrix}(a,b),(dg,ch),(fg,eh)\\ (g,h),(dfgi,cehj)\end{matrix}\middle|\frac{q}{p};\frac{aj}{bi}\right).
$$

This proves (8.44). (8.45) is obtained in the same way. Indeed,

$$
{}_3\Phi_2\left(\begin{matrix}(a,b),(c,d),(e,f)\\ (g,h),(i,j)\end{matrix}\middle|(p,q);\frac{phj}{bdf}\right)={}_3\phi_2\left(\begin{matrix}\frac{b}{a},\frac{d}{c},\frac{f}{e}\\ \frac{h}{g},\frac{j}{i}\end{matrix}\middle|\frac{q}{p};\frac{acehj}{bdfgi}\right)
$$
$$
=\frac{\left(\frac{d}{c};\frac{q}{p}\right)_\infty\left(\frac{achj}{bdgi};\frac{q}{p}\right)_\infty\left(\frac{cehj}{dfgi};\frac{q}{p}\right)_\infty}{\left(\frac{h}{g};\frac{q}{p}\right)_\infty\left(\frac{j}{i};\frac{q}{p}\right)_\infty\left(\frac{acehj}{bdfgi};\frac{q}{p}\right)_\infty}{}_3\phi_2\left(\begin{matrix}\frac{ch}{dg},\frac{cj}{di},\frac{acehj}{bdfgi}\\ \frac{achj}{bdgi},\frac{cehj}{dfgi}\end{matrix}\middle|\frac{q}{p};\frac{d}{c}\right)
$$
$$
=\frac{(c\ominus d)_{p,q}^{\infty}(bdgi\ominus achj)_{p,q}^{\infty}(dfgi\ominus cehj)_{p,q}^{\infty}}{(cg\ominus ch)_{p,q}^{\infty}(di\ominus dj)_{p,q}^{\infty}(bdfgi\ominus acehj)_{p,q}^{\infty}}{}_3\Phi_2\left(\begin{matrix}(dg,ch),(di,cj),(bdfgi,acehj)\\ (bdgi,achj),(dfgi,cehj)\end{matrix}\middle|(p,q);\frac{p}{c}\right).
$$

□

8.3 Power representation of terminating (p,q)-series

The following power representations of some terminating q-series are valid:

$$
\frac{\left((-1)^n q^{\binom{n}{2}}\right)^{(s-r)}(a_2,\ldots,a_{r+1};q)_n}{(b_1,b_2,\ldots,b_s;q)_n}x^n=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{k}{2}}{}_{r+1}\phi_s\left(\begin{matrix}q^{-k},a_2,\ldots,a_{r+1}\\ b_1,b_2,\ldots,b_s\end{matrix}\middle|q;qx\right). \tag{8.46}
$$

and

$$\frac{\left((-1)^n q^{\binom{n}{2}}\right)^{(s-r)}(a_3,\ldots,a_{r+1})_n}{(b_1,b_2,\ldots,b_s;q)_n}x^n$$
$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \frac{(-1)^k q^{\binom{k}{2}}}{(a_2q^k,;q)_k(a_2q^{2k+1};q)_{n-k}}\,{}_{r+1}\phi_s\left(\begin{array}{c}q^{-k},a_2q^k,a_3,\ldots,a_{r+1}\\ b_1,b_2,\ldots,b_s\end{array}\middle|\, q;qx\right) \tag{8.47}$$

They can be obtained [66] from Verma's q-extension [88] of Fields and Wimp [36] expansion of

$$ {}_{r+t}\phi_{s+u}\left(\begin{array}{c}(a_r),(c_t)\\ (b_s),(d_u)\end{array}\middle|\, q;y\omega\right)=\sum_{j=0}^{\infty}\frac{((c_t),(e_k);q)_j}{(q,(d_u),\gamma q^j;q)_j}y^j\left[(-1)^j q^{\binom{j}{2}}\right]^{u+3-t-k}$$
$$\cdot{}_{t+k}\phi_{u+1}\left(\begin{array}{c}(c_tq^j),(e_kq^j)\\ \gamma q^{2j+1},(d_uq^j)\end{array}\middle|\, q,yq^{j(u+2-t-k)}\right)$$
$$\cdot{}_{r+2}\phi_{s+k}\left(\begin{array}{c}q^{-j},\gamma q^j,(a_r)\\ (b_s),(e_k)\end{array}\middle|\, q,\omega q\right) \tag{8.48}$$

in powers of $y\omega$ as given in Gasper and Rahman [38, (3.7.9)].

In this section we give (p,q)-analogues of (8.46) and (8.47).

Proposition 102. *The following power representation is valid:*

$$x^n=A_n\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k+1}{2}-nk}q^{\binom{k}{2}}\Phi_k(x), \tag{8.49}$$

where

$$A_n=\left(\frac{a_{2p}\ldots a_{(r+1)p}}{b_{1p}\ldots b_{sp}}\right)^n\frac{\left(b_{1p}\ominus b_{1q}\right)^n_{p,q}\cdots\left(b_{sp}\ominus b_{sq}\right)^n_{p,q}}{\left(a_{2p}\ominus a_{2q}\right)^n_{p,q}\cdots\left(a_{(r+1)p}\ominus a_{(r+1)q}\right)^n_{p,q}}\left((-1)^n q^{\binom{n}{2}}\right)^{r-s}, \tag{8.50}$$

and

$$\Phi_k(x)=\begin{cases}{}_{r+1}\Phi_s\left(\begin{array}{c}(p^{-k},q^{-k}),(a_{2p},a_{2q}),\ldots,(a_{(r+1)p},a_{(r+1)q})\\ (b_{1p,b_{1q}}),(b_{2p},b_{2q}),\ldots(b_{sp},b_{sq})\end{array}\middle|\,(p,q);\mu x\right) & s=r\\ {}_{s+1}\Phi_s\left(\begin{array}{c}(p^{-k},q^{-k}),(a_{2p},a_{2q}),\ldots,(a_{(r+1)p},a_{(r+1)q}),(0,1),\ldots,(0,1)\\ (b_{1p,b_{1q}}),(b_{2p},b_{2q}),\ldots(b_{sp},b_{sq})\end{array}\middle|\,(p,q);\mu x\right) & s>r\\ {}_{r+1}\Phi_r\left(\begin{array}{c}(p^{-k},q^{-k}),(a_{2p},a_{2q}),\ldots,(a_{(r+1)p},a_{(r+1)q})\\ (b_{1p,b_{1q}}),(b_{2p},b_{2q}),\ldots(b_{sp},b_{sq}),(0,1),\ldots,(0,1)\end{array}\middle|\,(p,q);\mu x\right) & s<r\end{cases}. \tag{8.51}$$

with

$$\mu=\frac{p^kb_{1p}b_{2p}\ldots b_{sp}}{a_{2p}\ldots a_{(r+1)p}}. \tag{8.52}$$

Proof. We give the proof for (8.49), that is for the case $s=r$. The other formulas are obtained in the same way.
From (8.46), we can write

$$\frac{\left(\frac{a_{2q}}{a_{2p}},\dots\frac{a_{(r+1)q}}{a_{(r+1)p}};\frac{q}{p}\right)_n}{\left(\frac{b_{1q}}{b_{1p}},\dots\frac{b_{sq}}{b_{sp}};\frac{q}{p}\right)_n}x^n=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{\frac{q}{p}}\left(\frac{q}{p}\right)^{\binom{k}{2}}{}_{r+1}\phi_s\left(\begin{array}{c|c}\frac{q^{-k}}{p^{-k}},\frac{a_{2q}}{a_{2p}},\dots,\frac{a_{(r+1)q}}{a_{(r+1)p}} & \\ \frac{b_{1q}}{b_{1p}},\frac{a_{2q}}{a_{2p}}\dots,\frac{b_{sq}}{b_{sp}} & \frac{q}{p};\frac{qx}{p}\end{array}\right).$$

Next, using the relation

$$\begin{bmatrix}n\\k\end{bmatrix}_{p,q}=p^{k(n-k)}\begin{bmatrix}n\\k\end{bmatrix}_{\frac{q}{p}},$$

together with the first case of (8.8) and proceeding with some simplifications (8.49) follows. □

Proposition 103. *The following power representation holds:*

$$x^n=B_n\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\frac{p^{n(k+1)-\binom{k+1}{2}}q^{\binom{k}{2}}}{\left(a_{2p}p^k\ominus a_{2q}q^k\right)_{p,q}^k\left(a_{2p}p^{2k+1}\ominus a_{2q}q^{2k+1}\right)_{p,q}^{n-k}}\overline{\Phi}_k(x),\tag{8.53}$$

where

$$B_n=\left(\frac{a_{2p}a_{3p}\dots a_{(r+1)p}}{b_{1p}\dots b_{sp}}\right)^n\frac{\left(b_{1p}\ominus b_{1q}\right)_{p,q}^n\dots\left(b_{sp}\ominus b_{sq}\right)_{p,q}^n}{\left(a_{3p}\ominus a_{3q}\right)_{p,q}^n\dots\left(a_{(r+1)p}\ominus a_{(r+1)q}\right)_{p,q}^n}\left((-1)^nq^{\binom{n}{2}}\right)^{r-s}p^{\binom{n}{2}},\tag{8.54}$$

and

$$\overline{\Phi}_k(x)=\begin{cases}{}_{r+1}\Phi_s\left(\begin{array}{c|c}(p^{-k},q^{-k}),(a_{2p}p^k,a_{2q}q^k),\dots,(a_{(r+1)p},a_{(r+1)q}) & \\ (b_{1p,b_{1q}}),(b_{2p},b_{2q}),\dots(b_{sp},b_{sq}) & (p,q);\mu x\end{array}\right) & s=r\\ {}_{s+1}\Phi_s\left(\begin{array}{c|c}(p^{-k},q^{-k}),(a_{2p}p^k,a_{2q}q^k),\dots,(a_{(r+1)p},a_{(r+1)q}),(0,1),\dots,(0,1) & \\ (b_{1p,b_{1q}}),(b_{2p},b_{2q}),\dots(b_{sp},b_{sq}) & (p,q);\mu x\end{array}\right) & s>r\\ {}_{r+1}\Phi_r\left(\begin{array}{c|c}(p^{-k},q^{-k}),(a_{2p}p^k,a_{2q}q^k),\dots,(a_{(r+1)p},a_{(r+1)q}) & \\ (b_{1p,b_{1q}}),(b_{2p},b_{2q}),\dots(b_{sp},b_{sq}),(0,1),\dots,(0,1) & (p,q);\mu x\end{array}\right) & s<r\end{cases}\tag{8.55}$$

with

$$\mu=\frac{qb_{1p}\dots b_{sp}}{a_{2p}\dots a_{(r+1)p}}.\tag{8.56}$$

Proof. The proof is achieved in the same way as for Proposition 102. □

We write (8.49) and (8.53) for $r=s=1$.

$$\frac{c^n\left(a\ominus b\right)_{p,q}^n}{a^n\left(c\ominus d\right)_{p,q}^n}x^n=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k+1}{2}-nk}q^{\binom{k}{2}}{}_2\Phi_1\left(\begin{array}{c|c}(p^{-k},q^{-k}),(a,b) & \\ (c,d) & (p,q);\frac{cqp^k}{a}x\end{array}\right);\tag{8.57}$$

and

$$\frac{c^n p^{\binom{n}{2}}}{(c\ominus d)_{p,q}^n} x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{(-1)^k p^{\binom{k+1}{2}+4k^2-nk} q^{\binom{k}{2}} a^{2k}}{(ap^k \ominus bq^k)_{p,q}^k \, (ap^{2k+1} \ominus bq^{2k+1})_{p,q}^{n-k}} {}_2\Phi_1\left(\begin{matrix} (p^{-k},q^{-k}),(ap^k,bq^k) \\ (c,d) \end{matrix} \middle| (p,q); \frac{cq}{a}x\right). \tag{8.58}$$

Chapter 9

(p,q)-Sturm-Liouville Problems and Their Orthogonal Solutions

In mathematics and its applications, a classical Sturm-Liouville theory, named after Jacques Charles François Sturm (1803-1855) and Joseph Liouville (1809-1882), is the theory of a real second-order linear differential equation of the form

$$\frac{d}{dx}\left[u(x)\frac{dy}{dx}\right]+v(x)y(x)=-\lambda w(x)y, \tag{9.1}$$

where y is a function of the free variable x. Here the functions $u(x)$, $v(x)$, and $w(x)>0$ are specified at the outset. In the simplest of cases all coefficients are continuous on the finite closed interval $[a,b]$, and u has continuous derivative. In this case, this function y is called a solution if it is two times continuously differentiable on (a,b) and satisfies the equation (9.1) at every point in (a,b). In addition, the unknown function y is typically required to satisfy some boundary conditions at a and b. The function $w(x)$, which is sometimes also denoted $\rho(x)$, is called the *weight* or *density* function.

The value of λ is not specified in the equation; finding the values of λ for which there exists a non-trivial solution of (9.1) satisfying the boundary conditions is part of the Sturm-Liouville **(S-L)** problem. Such values of λ, when they exist, are called the eigenvalues of the boundary value problem defined by (9.1) and the prescribed set of boundary conditions. The corresponding solutions (for each such λ) are the *eigenfunctions* of this problem.

The resulting theory of the existence and asymptotic behaviour of the eigenvalues, the corresponding qualitative theory of the eigenfunctions and their completeness in a suitable function space became known as Sturm-Liouville theory. This theory is important in applied mathematics, where **(S-L)** problems occur very commonly, particularly when dealing with linear partial differential equations that are separable.

A regular Sturm-Liouville problem of continuous type is a boundary value problem in the form

$$\frac{d}{dx}\left(r(x)\frac{dy_n(x)}{dx}\right)+\lambda_n w(x)y_n(x)=0 \qquad (r(x)>0,\ w(x)>0), \tag{9.2}$$

which is defined on an open interval, say (a,b), with the boundary conditions

$$\alpha_1 y(a)+\beta_1 y'(a)=0, \qquad \alpha_2 y(b)+\beta_2 y'(b)=0, \tag{9.3}$$

where α_1, α_2 and β_1, β_2 are constant numbers and $r(x)$, $r'(x)$ and $w(x)$ in (9.2) are to be assumed continuous for $x\in[a,b]$. In this sense, if y_n and y_m are two eigenfunctions of equation (9.2), then according to Sturm-Liouville theory [65], they are orthogonal with respect to the weight function $w(x)$ under the given condition (9.3), *i.e.* we have

$$\int_a^b w(x)y_n(x)y_m(x)dx = d_n^2\delta_{m,n},$$

where $d_n^2 = \int_a^b w(x)y_n^2(x)dx$ denotes the norm square of the functions y_n and $\delta_{m,n}$ stands for the Kronecker delta [58, 59].

It is well known that q-orthogonal functions may be solutions of a q-Sturm-Liouville problem [18]. One of the important cases of these functions are the q-classical orthogonal polynomials which are of special interest inside the class of special functions and play an important role in several problems such as Eulerian series and continued fractions [53], q-algebras and quantum groups [56, 57] or q-oscillators [10, 19, 39].

q-Orthogonal functions can also be similarly solutions of a q-Sturm-Liouville problem in the form [51]

$$(D_q(r\,D_q y_n))(x;q) + \lambda_{n,q}w(x;q)y_n(x;q) = 0 \qquad (r(x;q) > 0,\, w(x;q) > 0), \tag{9.4}$$

with $(D_q f)(0) = f'(0)$ (provided $f'(0)$ exists), and (9.4) satisfies a set of boundary conditions like (9.3). This means that if $y_n(x;q)$ and $y_m(x;q)$ are two eigenfunctions of the q-difference equation (9.4), then they are orthogonal with respect to a weight function $w(x;q)$ on a discrete set [64].

In this chapter, we study the extension of q-Sturm-Liouville problems to (p,q)-Sturm-Liouville problems and seek for finding some (p,q)-orthogonal functions that are solutions of them.

9.1 Eigenvalue problems

We consider the eigenvalue problem (see [60] or [84])

$$\varphi(x)(D_{p,q}^2 y_n)(x) + \psi(x)(D_{p,q}y_n)(px) = \lambda_n y_n(pqx) \tag{9.5}$$

for polynomials y_n of degree n, where $D_{p,q}^2 y_n = D_{p,q}(D_{p,q}y_n)$ with $\lambda_n \in \mathbb{C}$ and $n \in \{0,1,2,\ldots\}$, φ is a polynomial of degree at most 2 and ψ is a polynomial of exact degree 1, say

$$\varphi(x) = ax^2 + bx + c, \quad \psi(x) = dx + e, \quad a,b,c,d,e \in \mathbb{C}, \quad d \neq 0. \tag{9.6}$$

Since

$$D_{p,q}(x^n) = \frac{p^n x^n - q^n x^n}{(p-q)x} = [n]_{p,q}x^{n-1},$$

comparing the coefficients of x^n in (9.5), we get

$$\lambda_n = \frac{[n]_{p,q}}{(pq)^n}\left(a[n-1]_{p,q} + dp^{n-1}\right). \tag{9.7}$$

Consequently, the corresponding (p,q)-difference equation (9.5) takes the form

$$\begin{aligned}(ax^2+bx+c)&(D_{p,q}^2 y_n)(x) + (dx+e)(D_{p,q}y_n)(px) \\ &= \frac{[n]_{p,q}}{(pq)^n}\left(a[n-1]_{p,q} + dp^{n-1}\right)y_n(pqx), \quad n = 0,1,2,\ldots.\end{aligned} \tag{9.8}$$

Note that

$$(D^2_{p,q}y_n)(x) = \frac{qy_n(p^2x) - (p+q)y_n(pqx) + py_n(q^2x)}{(p-q)^2pqx^2}. \tag{9.9}$$

Therefore the equation (9.8) can be written in the form

$$\begin{aligned}\left(\frac{ax^2+bx+c}{(p-q)^2px^2} + \frac{dx+e}{(p-q)px}\right) y_n(p^2x) \\ - \left(\frac{(ax^2+bx+c)(p+q)}{(p-q)^2px^2} + \frac{dx+e}{(p-q)px} - \lambda_n\right) y_n(pqx) \\ + \frac{(ax^2+bx+c)}{(p-q)^2qx^2} y(q^2x).\end{aligned} \tag{9.10}$$

If now we replace x by $x/(pq)$ we obtain the so-called symmetric form

$$\begin{aligned}C(x)y_n(p^{-1}qx) - \{C(x)+D(x)\}\, y_n(x) + D(x)y_n(pq^{-1}x) \\ = \frac{(p^n-q^n)\left(aq^np - p^naq - dp^npq + q^2dp^n\right)}{p^{n+1}q^{n+1}\,(p-q)^2} y_n(x),\end{aligned} \tag{9.11}$$

for $n = 0, 1, 2, \ldots$ with

$$C(x) = \frac{ax^2 + bpqx + cp^2q^2}{q\,(p-q)^2\,x^2} \tag{9.12}$$

and

$$D(x) = \frac{(a+(p-q)d)\,x^2 + (b+(p-q)e)\,pqx + cp^2q^2}{p\,(p-q)^2\,x^2}. \tag{9.13}$$

Note that $D(x)$ is related to $C(x)$ by

$$D(x) = \frac{q}{p}C(x) + \frac{dx+epq}{p(p-q)x}. \tag{9.14}$$

9.2 The regularity condition

In this section we will point out in which cases the eigenvalue problem (9.5) has essentially unique polynomial solutions $y_n(x)$ of degrees $n = 0, 1, 2, \ldots, N$ for some positive integer N with possibly $N = \infty$. Solutions are called essentially unique if they are determined up to a factor independent of x. We have

Theorem 104. *Let N denote a positive integer (possibly $N \to \infty$). Then the following statements are equivalent:*

1. *For each $n = 0, 1, 2, \ldots, N$ there exists a solution of the eigenvalue problem* (9.5) *and all eigenspaces are one-dimensional.*
2. *For $m, n \in \{0, 1, 2, \ldots, N\}$ with $m \neq n$ we have $\lambda_m \neq \lambda_n$.*

Proof. Assume that $\lambda_m = \lambda_n$ for $m \neq n$. Then there is either no polynomial solution for one of the degrees m and n or the solutions y_m and y_n belong to the same eigenspace. This shows that the first statement implies the second.
Now we use induction to show that the second statement implies the first. For $n = 0$, we have $\lambda_0 = 0$ and the one-dimensional eigenspace generated by $y_0(x) = 1$. Now we

assume that $n \in \{1,2,3,\ldots\}$. Suppose that the polynomials $y_k(x)$ are solutions of degree k for $k = 0,1,2,\ldots n-1$. Then the (monic) polynomial $y_n(x)$ of degree n given by

$$y_n(x) = x^n + \sum_{k=0}^{n-1} \alpha_k y_k(x) \quad \text{with} \quad \alpha_k \in \mathbb{C} \tag{9.15}$$

is clearly a solution of (9.5) if

$$\begin{aligned}\varphi(x)D_{p,q}^2(x^n) &+ \psi(x)(D_{p,q}\mathcal{L}_p)(x^n)\\ &+ \varphi(x)\left(D_{p,q}^2 \sum_{k=0}^{n-1} \alpha_k y_k\right)(x) + \psi(x)(D_{p,q}\mathcal{L}_p)\left(\sum_{k=0}^{n-1} \alpha_k y_k\right)(x)\\ &= \lambda_n\left((px)^n + \sum_{k=0}^{n-1} \alpha_k y_k(pqx)\right)\end{aligned}$$

with $\mathcal{L}_p(f)(x) = f(px)$, holds. Clearly, the polynomial $\varphi(x)D_{p,q}^2(x^n) + \psi(x)(D_{p,q}\mathcal{L}_p)(x^n)$ has degree at most n. Hence we may write

$$\varphi(x)D_{p,q}^2(x^n) + \psi(x)(D_{p,q}\mathcal{L}_p)(x^n) = \beta_n(px)^n + \sum_{k=0}^{n-1} \beta_k y_k(pqx)$$

with $\beta_j \in \mathbb{C}$ for $j = 0,1,\ldots,n$. Combining the last two equations, we get

$$\beta_n(px)^n + \sum_{k=0}^{n-1}(\beta_k + \lambda_k\alpha_k)y_k(pqx) = \lambda_n\left((px)^n + \sum_{k=0}^{n-1} \beta_k y_k(pqx)\right)$$

and therefore

$$(\beta_n - \lambda_n)(px)^n + \sum_{k=0}^{n-1}(\alpha_k(\lambda_k - \lambda_n) + \beta_k)y_k(pqx) = 0.$$

Since $\lambda_k \neq \lambda_n$, this implies that the numbers α_k are uniquely determined by this equation. Hence the (monic) polynomial solution y_n given by (9.15) is uniquely determined and this implies that the corresponding eigenspace is one-dimensional. □

Now, since $\lambda_n = \frac{[n]_{p,q}}{(pq)^n}\left(a[n-1]_{p,q} + dp^{n-1}\right)$, using the relations (2.11) and (2.12) we get

$$\begin{aligned}(pq)^n(\lambda_n - \lambda_m) &= [n]_{p,q}\left(a[n-1]_{p,q} + dp^{n-1}\right) - (pq)^{n-m}[m]_{p,q}\left(a[m-1]_{p,q} + dp^{m-1}\right)\\ &= a([n]_{p,q}[n-1]_{p,q} - (pq)^{n-m}[m]_{p,q}[m-1]_{p,q}) + dp^{n-1}([n]_{p,q} - q^{n-m}[m]_{p,q})\\ &= a[n-m]_{p,q}[n+m-1]_{p,q} + dp^{n+m-1}[n-m]_{p,q}\\ &= [n-m]_{p,q}\left(a[n+m-1]_{p,q} + dp^{n+m-1}\right), \quad n \geq m, \ \ n,m \in \{0,1,2,\ldots\}.\end{aligned}$$

Hence we have

$$\lambda_n - \lambda_m = \frac{[n-m]_{p,q}}{(pq)^n}\left(a[n+m-1]_{p,q} + dp^{n+m-1}\right), \quad n \geq m, \ \ n,m \in \{0,1,2,\ldots\}. \tag{9.16}$$

For $n \neq m$, it follows that $[n-m]_{p,q} \neq 0$ and so $\lambda_m \neq \lambda_n$ is equivalent to $a[n+m-1]_{p,q} + dp^{n+m-1} \neq 0$. Therefore, Theorem 104 leads to:

Corollary 105. *Let N denote a positive integer (possibly $N \to \infty$). Then the eigenvalue problem* (9.5) *has polynomial solutions y_n of degree n for all $n = 0,1,2,\ldots,N$ with one-dimensional eigenspaces if and only if the regularity condition*

$$a[n]_{p,q} + dp^n \neq 0, \quad n = 0,1,2,\ldots,2N-2 \tag{9.17}$$

holds.

9.3 Orthogonality of the polynomial solutions

In this section, we prove that the polynomial solutions of (9.5) are orthogonal with respect to some measure ρ.

Theorem 106. *If $\lambda_n = -(pq)^{-n}[n]_{p,q}(a[n-1]_{p,q} + dp^{n-1}),\quad n \in \mathbb{N}$, then the (p,q)-differential equation (9.5) has a polynomial solution of degree n, $P_n(x)$. Moreover, if ρ is a nonnegative solution of the Pearson type equation on an interval $(a;b)$ (the latter interval may be finite or infinite)*

$$D_{p,q}[\phi(q^{-1}x)\rho(q^{-1}x)] = \psi(x)\rho(x), \tag{9.18}$$

and if the limiting conditions

$$\lim_{x\to a} x^n\phi(x)\rho(x) = \lim_{x\to b} x^n\phi(x)\rho(x) = 0, \quad \forall n \in \mathbb{N} \tag{9.19}$$

hold. Then the polynomial system $(P_n(x))_n$ satisfies the orthogonality relation

$$\int_a^b \rho(x)P_m(pqx)P_n(pqx)d_{p,q}x = k_n\delta_{n,m}. \tag{9.20}$$

Proof. When $c = 0$, it is clear that for $\lambda_n = -(pq)^{-n}[n]_{p,q}(a[n-1]_{p,q} + dp^{n-1})$, $f_m = 0$ for $m > n$ and $f_n \neq 0$. Next, multipliying (9.5) by $\rho(x)$, we get

$$\phi(x)\rho(x)D^2_{p,q}y(x) + \psi(x)\rho(x)(D_{p,q}y)(px) + \lambda\rho(x)y(pqx) = 0.$$

Using the (p,q)-Pearson equation (9.18), and the product rule (2.13), we get the following self-adjoint form of (9.5)

$$D_{p,q}[\phi(q^{-1}x)\rho(q^{-1}x)D_{p,q}y(x)] = -\lambda\rho(x)y(pqx).$$

We write this self-adjoint form for P_n and P_m, $m \neq n$

$$\begin{aligned} D_{p,q}[\phi(q^{-1}x)\rho(q^{-1}x)D_{p,q}P_m(x)] &= -\lambda_m\rho(x)P_m(pqx) \\ D_{p,q}[\phi(q^{-1}x)\rho(q^{-1}x)D_{p,q}P_n(x)] &= -\lambda_n\rho(x)P_n(pqx). \end{aligned}$$

Multiplying the first equation by $P_n(pqx)$ and the second one by $P_m(pqx)$ and subtract the second equation from the first one, it follows that

$$\begin{aligned} (\lambda_n - \lambda_m)\rho(x)P_m(pqx)P_n(pqx) &= P_m(pqx)D_{p,q}[\phi(q^{-1}x)\rho(q^{-1}x)D_{p,q}P_n(x)] \\ &\quad - P_n(pqx)D_{p,q}[\phi(q^{-1}x)\rho(q^{-1}x)D_{p,q}P_m(x)] \\ &= D_{p,q}\left[\phi(q^{-1}x)\rho(q^{-1}x)\left(P_m(qx)D_{p,q}P_n(x) - P_n(qx)D_{p,q}P_m(x)\right)\right] \end{aligned}$$

Therefore

$$\begin{aligned} &(\lambda_n - \lambda_m)\int_a^b \rho(x)P_m(pqx)P_n(pqx)d_{p,q}x \\ &\qquad = \left[\phi(q^{-1}x)\rho(q^{-1}x)\left(P_m(qx)D_{p,q}P_n(x) - P_n(qx)D_{p,q}P_m(x)\right)\right]_a^b \\ &\qquad = 0 \quad (\text{ limit conditions (9.19)}) \end{aligned}$$

The orthogonality then follows from the regularity condition (9.17). $\square$

Now, note that from the (p,q)-Pearson equation (9.18) we get

$$D_{p,q}[\phi(q^{-1}x)\rho(q^{-1}x)] = \psi(x)\rho(x) \iff \frac{\phi(pq^{-1}x)\rho(pq^{-1}x) - \phi(x)\rho(x)}{(p-q)x} = \psi(x)\rho(x)$$
$$\iff \frac{\rho\left(\frac{p}{q}x\right)}{\rho(x)} = \frac{\phi(x) + (p-q)x\psi(x)}{\phi(pq^{-1}x)}.$$

We state the following theorem.

Theorem 107. *Let $\theta(x)$ be a given function and consider the (p,q)-difference equation*

$$\frac{\rho\left(\frac{p}{q}x\right)}{\rho(x)} = \theta(x). \tag{9.21}$$

Then,

1. *If $0 < q < p$,*
$$\rho(x) = \prod_{k=0}^{\infty} \theta\left(\frac{q^{k+1}}{p^{k+1}}x\right) \tag{9.22}$$
is a possible solution of (9.21) provided that the infinite series converges.

2. *If $0 < p < q$,*
$$\rho(x) = \left[\prod_{k=0}^{\infty} \theta\left(\frac{p^k}{q^k}x\right)\right]^{-1} \tag{9.23}$$
is a possible solution of (9.21) provided that the infinite series converges.

Proof.

1. If $0 < q < p$, then (9.21) reads
$$\frac{\rho(x)}{\rho\left(\frac{q}{p}x\right)} = \theta\left(\frac{q}{p}x\right),$$
or otherwise stated
$$\rho(x) = \theta\left(\frac{q}{p}x\right)\rho\left(\frac{q}{p}x\right).$$
Making the substitution of x by $\frac{q}{p}x$ on both sides n times yields
$$\rho(x) = \theta\left(\frac{q}{p}x\right)\theta\left(\frac{q^2}{p^2}x\right)\ldots\theta\left(\frac{q^{n+1}}{p^{n+1}}x\right)\rho\left(\frac{q^{n+1}}{p^{n+1}}x\right).$$
Letting n tend to infinity and assuming that the obtained infinite product is convergent gives (9.22).

2. In the case $0 < p < q$, we write (9.21) as
$$\rho(x) = \frac{1}{\theta(x)}\rho\left(\frac{p}{q}x\right)$$
and proceed as previously.

□

9.4 Structure relations for (p, q)-orthogonal polynomials

9.4.1 The three-term recurrence relation

Now since we have the orthogonality relation, it is clear that the polynomial solutions of (9.5) satisfy a three-term recurrence relation of the form

$$xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad n \geq 1 \tag{9.24}$$

In order to obtain the expression of the coefficients a_n, b_n and c_n, following the method used by Koepf and Schmersau in [55] and [54], we first write

$$P_n(x) = k_n x^n + k'_n x^{n-1} + k''_n x^{n-2} + \cdots \qquad (n \in \mathbb{N}_{\geq 0} := \{0, 1, 2 \ldots\},\, k_n \neq 0). \tag{9.25}$$

Since one demands that $P_n(x)$ has exactly degree n, we substitute $P_n(x)$ in the q-differential equation (9.5) and by equating the coefficients of x^n, one gets (9.7).
Equating the coefficients of x^{n-1} and x^{n-2} gives k'_n, and k''_n, respectively, as rational multiples w.r.t. $M = p^m$, and $N = q^n$ of k_n:

$$k'_n = \frac{(M-N)\left(Mepq - Meq^2 + Mbq - Nbp\right) q^2 p^2}{(p-q)\left(M^2dq^2p - M^2dq^3 + M^2aq^2 - N^2ap^2\right)} k_n \tag{9.26}$$

and

$$\begin{aligned} k''_n = {} & k_n (M-N)(Mq-Np)\Big(M^2e^2p^2q^3 - 2\,M^2e^2pq^4 + M^2e^2q^5 + 2\,M^2bepq^3 - 2\,M^2beq^4 \\ & + M^2cdp^2q^2 - 2\,M^2cdpq^3 + M^2cdq^4 - MNbep^3q + MNbepq^3 + M^2acpq^2 - M^2acq^3 + M^2b^2q^3 \\ & - MNb^2p^2q - MNb^2pq^2 - N^2acp^3 + N^2acp^2q + N^2b^2p^3\big) q^4p^4\Big) \Big/ \Big[(p+q)(p-q)^2 \\ & \Big(M^2dpq^3 - M^2dq^4 + M^2aq^3 - N^2ap^3\Big)\left(M^2dpq^2 - M^2dq^3 + M^2aq^2 - N^2ap^2\right) \Big]. \end{aligned} \tag{9.27}$$

An important point is that the coefficients a_n, b_n and c_n appearing in this formula (9.24) can be computed directly in terms of the coefficients of the polynomials $\phi(x)$ and $\psi(x)$, which completely characterize the second order (p, q)-differential equation (9.5).
Next, we substitute $P_n(x)$ in the proposed equation (9.24) and equate the three highest coefficients. This yields a_n, b_n, and c_n in terms of a, b, c, d, e, q, q^n, p^n, k_{n-1}, k_n, k_{n+1}, k'_{n-1}, k'_n, k'_{n+1}, k''_{n-1}, k''_n, k''_{n+1} by linear algebra.
Finally, substituting the values of k'_{n-1}, k'_n, k'_{n+1}, k''_{n-1}, k''_n, and k''_{n+1} given by (9.26) and (9.27) yields the following formulas.

$$a_n = \frac{k_n}{k_{n+1}}. \tag{9.28}$$

$$\begin{aligned} b_n = {} & -\Big(M^2dep^2q^2 - 2\,M^2depq^3 + M^2deq^4 + M^2aepq^2 - M^2aeq^3 + M^2bdp^2q - M^2bdq^3 \\ & - MNaep^3 + MNaepq^2 - MNbdp^2q + MNbdq^3 + N^2aep^2q - N^2aepq^2 + M^2abpq + M^2abq^2 \\ & - MNabp^2 - 2\,MNabpq - MNabq^2 + N^2abp^2 + N^2abpq\big) NMpq \Big/ \left(M^2dpq^2 - M^2dq^3 + M^2aq^2 \right. \\ & \left. - N^2ap^2\right)\left(M^2dp - M^2dq + M^2a - N^2a\right) \end{aligned} \tag{9.29}$$

and

$$c_n = -\frac{k_n}{n_{n-1}} NMp^3q^3 (M-N) \left(pq^2Md - q^3Md + q^2Ma - p^2Na\right) \Big(M^4cd^2p^2q^4 - 2\,M^4cd^2pq^5$$
$$+M^4cd^2q^6 - M^3Nbdep^3q^3 + 2\,M^3Nbdep^2q^4 - M^3Nbdepq^5 + M^2N^2ae^2p^4q^2 - 2\,M^2N^2ae^2p^3q^3$$
$$+M^2N^2ae^2p^2q^4 + 2\,M^4acdpq^4 - 2\,M^4acdq^5 - M^3Nabep^2q^3 + M^3Nabepq^4 - M^3Nb^2dp^2q^3$$
$$+M^3Nb^2dpq^4 + 2\,M^2N^2abep^3q^2 - 2\,M^2N^2abep^2q^3 - 2\,M^2N^2acdp^3q^2 + 2\,M^2N^2acdp^2q^3$$
$$+M^2N^2b^2dp^3q^2 - M^2N^2b^2dp^2q^3 - MN^3abep^4q + MN^3abep^3q^2 + M^4a^2cq^4 - M^3Nab^2pq^3$$
$$-2\,M^2N^2a^2cp^2q^2 + 2\,M^2N^2ab^2p^2q^2 - MN^3ab^2p^3q + N^4a^2cp^4\Big) \Big/ \Big(M^2dpq^3 - M^2dq^4 + M^2aq^3$$
$$-N^2ap^3\big) \left(M^2dpq^2 - M^2dq^3 + M^2aq^2 - N^2ap^2\right)^2 \left(M^2dpq - M^2dq^2 + M^2aq - N^2ap\right) \tag{9.30}$$

9.4.2 Further structure relations

In this section we give several other structure relations for the (p,q)-orthogonal polynomials.

Proposition 108 (Compare with [60]). *If $y(x)$ is a solution of equation* (9.5), *then $y_1 = D_{p,q}y$ is a solution of*

$$\phi_1(x)D^2_{p,q}y_1(x) + \psi_1(x)(D_{p,q}y_1)(px) + \mu_1 y_1(pqx) = 0 \tag{9.31}$$

where

$$\phi_1(x) = \phi(qx), \tag{9.32}$$
$$\psi_1(x) = D_{p,q}\phi(x) + p\psi(px), \tag{9.33}$$
$$\mu_1 = D_{p,q}\psi(x) + pq\lambda. \tag{9.34}$$

Proof.

$$D_{p,q}\left[\phi(x)D^2_{p,q}y(x)\right] + D_{p,q}\left[\psi(x)(D_{p,q}y)(px)\right] + \lambda D_{p,q}\left[y(pqx)\right] = 0 \tag{9.35}$$

Applying the product rule (2.13) it follows that:

$$\phi(qx)D^2_{p,q}\left[D_{p,q}y(x)\right] + D_{p,q}\phi(x)(D^2_{p,q}y)(px) + \psi(px)D_{p,q}\left[(D_{p,q}y)(px)\right]$$
$$+ D_{p,q}\psi(x)(D_{p,q}y)(pqx) + \lambda D_{p,q}\left[y(pqx)\right] = 0.$$

Next, using the product rule (2.13) and using the relation

$$D_{p,q}[f(\alpha x)] = \alpha(D_{p,q}f)(\alpha x),$$

we get

$$\phi(qx)D^2_{p,q}\left[D_{p,q}y(x)\right] + \left[D_{p,q}\phi(x) + p\psi(px)\right](D^2_{p,q}y)(px)$$
$$+ \left[D_{p,q}\psi(x) + pq\lambda\right] D_{p,q}y)(pqx) = 0.$$

This proves the result. □

By induction, it can be seen that if $y(x)$ is solution of (9.5), then $D^n_{p,q}y(x)$ (for $n \geq 1$) is solution of the equation (see [60])

$$\phi_n(x)D^2_{p,q}y(x) + \psi_n(x)(D_{p,q}y)(px) + \mu_n y(pqx) = 0, \tag{9.36}$$

where

$$\phi_n(x) = \phi(q^n x) \tag{9.37}$$
$$\psi_n(x) = p\psi_{n-1}(px) + D_{p,q}\phi_{n-1}(x). \tag{9.38}$$

Setting $\phi(x) = ax^2 + bx + c$ and $\psi(x) = dx + e$, we see that

$$\phi_n(x) = aq^{2n}x^2 + bq^n x + c.$$

Also, setting $\psi(x) = dx + e$ and $\psi_n(x) = d_n x + e_n$, it follows that $\phi_{n+1}(x) = d_{n+1}x + e_{n+1}$. Using (9.38), it follows that

$$\begin{aligned}\psi_{n+1}(x) &= D_{p,q}\phi_n(x) + p\psi(px)\\ &= a(p+q)q^{2n}x + bq^n + p(pd_n x + e_n)\\ &= \left(p^2 d_n + a(p+q)q^{2n}\right)x + bq^n + pe_n\end{aligned}$$

Hence, by identification we obtain the following recurrences for (d_n) and (e_n)

$$d_{n+1} = p^2 d_n + a(p+q)q^{2n}, \quad d_0 = d \tag{9.39}$$
$$e_{n+1} = pe_n + bq^n, \quad e_0 = e. \tag{9.40}$$

If we write $e_n = p^n S_n$, then from (9.40) we get

$$S_{n+1} = S_n + \frac{b}{p}\left(\frac{q}{p}\right)^n.$$

Solving this recurrence gives $S_n = e + \dfrac{b[n]_{p,q}}{p^n}$ and hence

$$e_n = p^n S_n = p^n e + b[n]_{pq}.$$

By a similar approach we obtain

$$d_n = p^{2n}d + a[2n]_{p,q}. \tag{9.41}$$

Now since we have the expressions for ϕ_n and ψ_n, it is interesting to get a closed formula for μ_n. As for the previous cases, by induction it is easy to see that

$$\mu_{n+1} = D_{p,q}\psi_n(x) + pq\mu_n.$$

Using the expression for ψ_n we obtain the following recurrence for the μ_ns

$$\mu_{n+1} = (pq)\mu_n + a[2n]_{p,q} + dp^{2n}, \quad \mu_0 = \lambda.$$

This time putting $\mu_n = (pq)T_n$, it follows that

$$T_{n+1} = T_n + \frac{1}{(pq)^n}\left(a[2n]_{p,q} + dp^{2n}\right).$$

Hence

$$T_n = \frac{a}{(pq)^n}[n]_{p,q}[n-1]_{p,q} + \frac{d}{pq^n} + \lambda.$$

The representation of μ_n follows and we then have the following proposition.

Proposition 109 (Compare with [60]). *If $y(x)$ is a solution of equation* (9.5), *then $D_{p,q}y$ is a solution of*

$$\phi_1(x)D^2_{p,q}y(x) + \psi_1(x)(D_{p,q}y)(px) + \mu_n y(x) = 0 \tag{9.42}$$

where

$$\begin{aligned}
\phi_n(x) &= \phi(q^n x) = aq^{2n}x^2 + bq^n x + c,\\
\psi_n(x) &= (a[2n]_{p,q} + dp^{2n})x + (b[n]_{p,q} + ep^n),\\
\mu_n &= a[n]_{p,q}[n-1]_{p,q} + dp^{n-1}[n]_{p,q} + (pq)^n\lambda.
\end{aligned}$$

Theorem 110 (Compare with [60]). *Let $\{P_n\}$ be a polynomial set, solutions of the (p, q)-differential equation* (9.5), *then the following structure relation holds:*

$$P_n(px) = \hat{a}_n(D_{p,q}P_{n+1})(x) + \hat{b}_n(D_{p,q}P_n)(x) + \hat{c}_n(D_{p,q}P_{n-1})(x). \tag{9.43}$$

Moreover, if we write

$$P_n(x) = k_n x^n + ...,$$

then, the coefficients $\hat{a}_n$, $\hat{b}_n$ and $\hat{c}_n$ can be computed using the formulas:

$$\hat{a}_n = \frac{k_n}{k_{n+1}}\frac{p^n}{[n+1]_{p,q}},$$

$$\hat{b}_n = -\frac{\left(M^2bdpq - M^2bdq^2 - MNaep^2 + MNaeq^2 + M^2abq - MNabp - MNabq + N^2abp\right)}{\left(M^2dpq^2 - M^2dq^3 + M^2aq^2 - N^2ap^2\right)\left(M^2dp - M^2dq + M^2a - N^2a\right)} \times MpqN\,(p-q)$$

$$\begin{aligned}
\hat{c}_n = {}& \frac{k_n}{k_{n-1}} \times \frac{q^3p^4M\,(M-N)\,N^2a\,(p-q)}{\left(M^2dpq^3 - M^2dq^4 + M^2aq^3 - N^2ap^3\right)}\\
&\times \left[\frac{M^4cd^2p^2q^4 - 2\,M^4cd^2pq^5 + M^4cd^2q^6 - M^3Nbdep^3q^3 + 2\,M^3Nbdep^2q^4 - M^3Nbdepq^5}{\left(M^2dq^2p - M^2dq^3 + M^2aq^2 - N^2ap^2\right)^2\left(M^2dpq - M^2dq^2 + M^2aq - N^2ap\right)}\right.\\
&+ \frac{M^2N^2ae^2p^4q^2 - 2\,M^2N^2ae^2p^3q^3 + M^2N^2ae^2p^2q^4 + 2\,M^4acdpq^4 - 2\,M^4acdq^5}{\left(M^2dq^2p - M^2dq^3 + M^2aq^2 - N^2ap^2\right)^2\left(M^2dpq - M^2dq^2 + M^2aq - N^2ap\right)}\\
&+ \frac{-M^3Nabep^2q^3 + M^3Nabepq^4 - M^3Nb^2dp^2q^3 + M^3Nb^2dpq^4 + 2\,M^2N^2abep^3q^2 -}{\left(M^2dq^2p - M^2dq^3 + M^2aq^2 - N^2ap^2\right)^2\left(M^2dpq - M^2dq^2 + M^2aq - N^2ap\right)}\\
&+ \frac{2\,M^2N^2abep^2q^3 - 2\,M^2N^2acdp^3q^2 + 2\,M^2N^2acdp^2q^3 + M^2N^2b^2dp^3q^2 - M^2N^2b^2dp^2q^3}{\left(M^2dq^2p - M^2dq^3 + M^2aq^2 - N^2ap^2\right)^2\left(M^2dpq - M^2dq^2 + M^2aq - N^2ap\right)}\\
&+ \frac{-MN^3abep^4q + MN^3abep^3q^2 + M^4a^2cq^4 - M^3Nab^2pq^3 - 2\,M^2N^2a^2cp^2q^2}{\left(M^2dq^2p - M^2dq^3 + M^2aq^2 - N^2ap^2\right)^2\left(M^2dpq - M^2dq^2 + M^2aq - N^2ap\right)}\\
&\left.+ \frac{2\,M^2N^2ab^2p^2q^2 - MN^3ab^2p^3q + N^4a^2cp^4}{\left(M^2dq^2p - M^2dq^3 + M^2aq^2 - N^2ap^2\right)^2\left(M^2dpq - M^2dq^2 + M^2aq - N^2ap\right)}\right]
\end{aligned}$$

Proof. From Proposition 108, it follows that if $\{P_n\}$ is a family of solutions of (9.5) the family $\{D_{p,q}P_n\}$ forms solutions of (9.42). Hence, they are orthogonal and therefore satisfy a three-term recurrence relation of the form

$$xD_{p,q}P_n(x) = a_{n,1}D_{p,q}P_{n+1}(x) + b_{n,1}D_{p,q}P_n(x) + c_{n,1}D_{p,q}P_{n-1}(x), \quad n \geq 2. \tag{9.44}$$

Now, applying the $D_{p,q}$ operator on both sides of equation (9.24), it follows that

$$D_{p,q}[xP_n(x)] = a_n(D_{p,q}P_{n+1})(x) + b_n(D_{p,q}P_n)(x) + c_n(D_{p,q}P_{n-1})(x), \tag{9.45}$$

Using the product rule (2.13), the left-hand side of the above equation reads

$$D_{p,q}[xP_n(x)] = qx(D_{p,q}P_n)(x) + P_n(px).$$

Using (9.44), it follows that

$$D_{p,q}[xP_n(x)] = P_n(px) + q\left[a_{n,1}(D_{p,q}P_{n+1})(x) + b_{n,1}(D_{p,q}P_n)(x) + c_{n,1}(D_{p,q}P_{n-1})(x)\right]. \tag{9.46}$$

Now combining (9.45) and (9.46) it follows that

$$P_n(px) = (a_n - qa_{n,1})(D_{p,q}P_{n+1})(x) + (b_n - qb_{n,1})(D_{p,q}P_n)(x) + (c_n - qc_{n,1})(D_{p,q}P_{n-1})(x).$$

Therefore, the structure relation is obtained. In order to get the explicit relation for the coefficients, we proceed exactly as in Proposition 9.24. □

9.5 Some special cases of (p,q)-orthogonal polynomials

In this section we discuss two special cases of (p,q)-orthogonal polynomials.

9.5.1 General solutions of the (p,q)-differential equations (9.5) and (9.18)

First, we obtain the recurrence equation for the coefficients of the power series solution of (9.5).

Theorem 111. *Let*

$$y(x) = \sum_{m=0}^{\infty} f_m x^m \tag{9.47}$$

be a solution of (9.5)*, then the coefficients f_m satisfy the second-order recurrence equation*

$$\begin{aligned} c[m+2]_{p,q}[m+1]_{p,q}f_{m+2} + (b[m+1]_{p,q}[m]_{p,q} + ep^m[m+1]_{p,q})f_{m+1} \\ +(a[m]_{p,q}[m-1]_{p,q} + dp^{m-1}[m]_{p,q} + \lambda(pq)^m)f_m, \quad m \geq 0. \end{aligned} \tag{9.48}$$

In particular, if $c = 0$, the recurrence equation

$$(b[m+1]_{p,q}[m]_{p,q} + ep^m[m+1]_{p,q})f_{m+1} + (a[m]_{p,q}[m-1]_{p,q} + dp^{m-1}[m]_{p,q} + \lambda(pq)^m)f_m = 0, \tag{9.49}$$

is valid and

$$f_m = \frac{(-1)^m f_0}{[m]_{p,q}!} \prod_{k=0}^{m-1} \frac{a[k]_{p,q}[k-1]_{p,q} + dp^{k-1}[k]_{p,q} + \lambda(pq)^k}{(b[k]_{p,q} + eq^k)}. \tag{9.50}$$

Proof. We have

$$D_{p,q}y(x) = \sum_{m=1}^{\infty}[m]_{p,q}f_m x^{m-1}, \quad D^2_{p,q}y(x) = \sum_{m=2}^{\infty}[m]_{p,q}[m-1]_{p,q}f_m x^{m-2}.$$

Thus

$$\begin{aligned} \sigma(x)D^2_{p,q}y(x) &= \sum_{m=2}^{\infty} a[m]_{p,q}[m-1]_{p,q}f_m x^m + \sum_{m=1}^{\infty} b[m+1]_{p,q}[m]_{p,q}f_{m+1}x^m \\ &\quad + \sum_{m=0}^{\infty} c[m+2]_{p,q}[m+1]_{p,q}f_{m+2}x^m \\ \tau(x)(D_{p,q}y)(px) &= \sum_{m=1}^{\infty} dp^{m-1}[m]_{p,q}f_m x^m + \sum_{m=0}^{\infty} ep^m[m+1]_{p,q}f_{m+1}x^m \\ \lambda y(pqx) &= \sum_{m=0}^{\infty} \lambda(pq)^m f_m x^m \end{aligned}$$

Hence we get

$$\begin{aligned}
&\sigma(x)D_{p,q}^2 y(x)+\tau(x)(D_{p,q}y)(qx)+\lambda y(x)\\
&=\sum_{m=2}^{\infty}\Big[c[m+2]_{p,q}[m+1]_{p,q}f_{m+2}+(b[m+1]_{p,q}[m]_{p,q}+ep^m[m+1]_{p,q})f_{m+1}\\
&\quad+(a[m]_{p,q}[m-1]_{p,q}+dp^{m-1}[m]_{p,q}+\lambda(pq)^m)f_m\Big]x^m\\
&=\sum_{m=0}^{\infty}\Big[c[m+2]_{p,q}[m+1]_{p,q}f_{m+2}+(b[m+1]_{p,q}[m]_{p,q}+ep^m[m+1]_{p,q})f_{m+1}\\
&\quad+(a[m]_{p,q}[m-1]_{p,q}+dp^{m-1}[m]_{p,q}+\lambda(pq)^m)f_m\Big]x^m.
\end{aligned}$$

The recurrence equation for f_m is then obtained. When $c=0$, the recurrence reads

$$(b[m+1]_{p,q}[m]_{p,q}+ep^m[m+1]_{p,q})f_{m+1}+(a[m]_{p,q}[m-1]_{p,q}+dp^{m-1}[m]_{p,q}+\lambda(pq)^m)f_m=0,$$

so

$$\frac{f_{m+1}}{f_m}=-\frac{a[m]_{p,q}[m-1]_{p,q}+dp^{m-1}[m]_{p,q}+\lambda(pq)^k}{[m+1]_{p,q}(b[m]_{p,q}+ep^m)}.$$

Hence

$$f_m=\frac{(-1)^m f_0}{[m]_{p,q}!}\prod_{k=0}^{m-1}\frac{a[k]_{p,q}[k-1]_{p,q}+dp^{k-1}[k]_{p,q}+\lambda(pq)^k}{(b[k]_{p,q}+ep^k)}.$$

□

Corollary 112. *Let $P_n(x)=\sum_{m=0}^{n}f_m(n)x^m$ be a polynomial solution of the equation*

$$\phi(x)D_{p,q}^2P_n(x)+\psi(x)(D_{p,q}P_n)(px)+\lambda_nP_n(pqx)=0 \tag{9.51}$$

where

$$\phi(x)=ax^2+bx+c,\quad \psi(x)=dx+e\quad \text{and}\quad \lambda_n=-\frac{1}{(pq)^n}[n]_{p,q}\left(a[n-1]_{p,q}+dp^{n-1}\right).$$

Then, the coefficients $f_m(n)$, $m=0,1,\dots,n$, are solutions of the recurrence equation

$$\begin{aligned}
&c[m+2]_{p,q}[m+1]_{p,q}f_{m+2}(n)+(b[m+1]_{p,q}[m]_{p,q}+ep^m[m+1]_{p,q})f_{m+1}(n)\\
&\quad+(a[m]_{p,q}[m-1]_{p,q}+dp^{m-1}[m]_{p,q}+\lambda_n(pq)^m)f_m(n),\quad m\geq 0.
\end{aligned}\tag{9.52}$$

In particular if $c=0$, the recurrence equation

$$(b[m+1]_{p,q}[m]_{p,q}+ep^m[m+1]_{p,q})f_{m+1}(n)+[m-n]\left(a[n+m-1]+dp^{m+n-1}\right)f_m(n)=0, \tag{9.53}$$

is valid and the solutions take the form

$$P_n(x)=K_n\,{}_2\Phi_1\left(\left.\begin{matrix}(p^{-n},q^{-n}),((a+d(p-q))p^{n-1},aq^{n-1})\\ (b+e(p-q),b)\end{matrix}\right|(p,q),-x\right). \tag{9.54}$$

Proof. Equation (9.52) follows directly from (9.48). When $c=0$, then (9.52) becomes

$$(b[m+1]_{p,q}[m]_{p,q}+ep^m[m+1]_{p,q})f_{m+1}(n)+(a[m]_{p,q}[m-1]_{p,q}+dp^{m-1}[m]_{p,q}+\lambda_n(pq)^m)f_m(n)=0$$

and hence
$$\frac{f_{m+1}(n)}{f_m(n)} = -\frac{a[m]_{p,q}[m-1]_{p,q} + dp^{m-1}[m]_{p,q} + \lambda_n (pq)^m}{[m]_{p,q}\left(b[m+1]_{p,q} + ep^m\right)}.$$

But, it is not difficult to see that

$$\begin{aligned}
a[m]_{p,q}[m-1]_{p,q} &+ dp^{m-1}[m]_{p,q} + \lambda_n (pq)^m \\
&= a[m]_{p,q}[m-1]_{p,q} + dp^{m-1}[m]_{p,q} - [n]_{p,q}\left(a[n-1]_{p,q} + dp^{n-1}\right)(pq)^{m-n} \\
&= [m-n]_{p,q}\left(a[m+n-1]_{p,q} + dp^{m+n-1}\right).
\end{aligned}$$

Hence,
$$f_m(n) = \frac{(-1)^m f_0(n)}{[m]_{p,q}!}\prod_{k=0}^{m-1}\frac{[k-n]_{p,q}\left(a[k+n-1]_{p,q} + dp^{k+n-1}\right)}{b[k]_{p,q} + ep^{k-1}}.$$

Next, we remark that
$$\prod_{k=0}^{m-1}[k-n]_{p,q} = \frac{\left(p^{-n}\ominus q^{-n}\right)^m_{p,q}}{(p-q)^m} \tag{9.55}$$

also
$$\prod_{k=0}^{m-1}\left(a[k+n-1]_{p,q} + dp^{k+n-1}\right) = \frac{\left((a+(p-q))p^{n-1}\ominus aq^{n-1}\right)^m_{p,q}}{(p-q)^m} \tag{9.56}$$

and
$$\prod_{k=0}^{m-1}\left(b[k]_{p,q} + ep^k\right) = \frac{(b+(p-q)e\ominus b)^m_{p,q}}{(p-q)^m} \tag{9.57}$$

From (9.55), (9.56) and (9.57), we get

$$f_m(n) = (-1)^m f_0(n)\frac{\left(p^{-n}\ominus q^{-n}\right)^m_{p,q}\left((a+(p-q))p^{n-1}\ominus aq^{n-1}\right)^m_{p,q}}{(p\ominus q)^m_{p,q}\,(b+(p-q)e\ominus b)^m_{p,q}}. \tag{9.58}$$

The representation (9.54) is therefore proved. □

9.5.2 The (p,q)-Jacobi polynomials

(p,q)-Hypergeometric representation

Let us consider now the second order (p,q)-difference equation (9.5) with

$$a = q^2,\quad b = -pq,\quad c = 0,\quad (p-q)d = p^2\left(\left(\frac{p}{q}\right)^{\alpha+\beta} - q^2\right),\quad (p-q)e = pq - p^2\left(\frac{p}{q}\right)^{\alpha}.$$

In this case we get:

Corollary 113 (Compare with [60]). *The polynomial solutions of equation* (9.5) *can be explicitly given by*

$$P_n^{(\alpha,\beta)}(x;p,q) = {}_2\Phi_1\left(\left.\begin{matrix}(p^{-n},q^{-n}),(p^{\alpha+\beta+n+1},q^{\alpha+\beta+n+1})\\ (p^{\alpha+1},q^{\alpha+1})\end{matrix}\right|(p,q);\frac{xq^{-\beta}}{p}\right) \tag{9.59}$$

up to a normalizing constant.

Proof. For these special values, we have

$$\Big((a+(p-q))p^{n-1}\ominus aq^{n-1}\Big)^m_{p,q} = \left(p^{n+1}\left(\frac{p}{q}\right)^{\alpha+\beta}\ominus q^{n+1}\right)^m_{p,q}$$
$$= \left(q^{-\alpha-\beta}\right)^m\left(p^{\alpha+\beta+n+1}\ominus q^{\alpha+\beta+n+1}\right)^m_{p,q}$$

and

$$(b+(p-q)e\ominus b)^m_{p,q} = \left(-p^2\left(\frac{p}{q}\right)^{\alpha}\ominus -pq\right)^m_{p,q}$$
$$= \left(-\frac{p}{q^\alpha}\right)^m\left(p^{\alpha+1}\ominus q^{\alpha+1}\right)^m_{p,q}.$$

Putting together these two relations combining with (9.58) provides the (p,q)-hypergeometric representation (9.59). □

Orthogonality relation

Corollary 114 (Compare with [60]). *Let $\{P_n^{(\alpha,\beta)}(x;p,q)\}_n$ be the sequence of polynomials given by* (9.59). *Then, according to Theorem 106, we have*

$$\int_0^{p/q}\rho^{(\alpha,\beta)}(pqx;p,q)P_n^{(\alpha,\beta)}(pqx;p,q)P_m^{(\alpha,\beta)}(pqx;p,q)d_{p,q}x$$
$$= \left(\int_0^{p/q}\rho^{(\alpha,\beta)}(x;p,q)\left(P_n^{(\alpha,\beta)}(pqx;p,q)\right)^2 d_{p,q}x\right)\delta_{n,m},$$

where $\alpha,\beta>-1$ and $\rho^{(\alpha,\beta)}(x;p,q)$ is a solution of the (p,q)-Pearson equation

$$\frac{\rho^{(\alpha,\beta)}(pq^{-1}x;p,q)}{\rho^{(\alpha,\beta)}(x;p,q)} = \frac{\left(q^{\alpha+\beta+2}+p^{\alpha+\beta+2}-q^{\alpha+\beta+2}p^2\right)x-q^\beta p^{\alpha+2}}{(x-1)\,p^2q^{\alpha+\beta}}.$$

This can be written in the form

$$\frac{\rho^{(\alpha,\beta)}(pq^{-1}x;p,q)}{\rho^{(\alpha,\beta)}(x;p,q)} = \frac{\theta x-1}{x-1}\times\left(\frac{p}{q}\right)^{\alpha}.$$

with

$$\theta = \frac{p^{\alpha+\beta+2}+(1-p^2)q^{\alpha+\beta+2}}{q^\beta p^{\alpha+2}} = \left(\frac{p}{q}\right)^{\beta}+(1-p^2)\left(\frac{q}{p}\right)^{\alpha+2}.$$

We can therefore write $\rho(x)=\rho_1(x)\rho_(x)$ with

$$\frac{\rho_1\left(\frac{p}{q}x\right)}{\rho_1(x)} = \left(\frac{p}{q}\right)^{\alpha} \quad \text{and} \quad \frac{\rho_2\left(\frac{p}{q}x\right)}{\rho_2(x)} = \frac{\theta x-1}{x-1}\times\left(\frac{p}{q}\right)^{\alpha}..$$

The relation $\dfrac{\rho_1\left(\frac{p}{q}x\right)}{\rho_1(x)} = \left(\dfrac{p}{q}\right)^{\alpha}$ has a solution $\rho_1(x)=x^\alpha$ and applying Theorem 9.21 to the second equation it follows that a solution of $\dfrac{\rho_2\left(\frac{p}{q}x\right)}{\rho_2(x)} = \dfrac{\theta x-1}{x-1}$ may be written as

$$\rho_2(x) = \prod_{k=0}^{\infty}\frac{p^{k+1}-xq^{k+1}}{p^{k+1}-q^{k+1}x} = \frac{(p\ominus\theta qx)^\infty_{p,q}}{(p\ominus qx)^\infty_{p,q}}.$$

Therefore, the we get

$$\rho^{\alpha,\beta}(x;p,q)=x^{\alpha}\frac{(p\ominus\theta qx)_{p,q}^{\infty}}{(p\ominus qx)_{p,q}^{\infty}}. \tag{9.60}$$

Three-term recurrence relation

From the (p,q)-hypergeometric representation (9.59), we see that the leading coefficient of $P_n^{(\alpha)}(x;p,q)$ is given by

$$k_n=\frac{(p^{-n}\ominus q^{-n})_{p,q}^{n}\,(p^{n+\alpha+\beta+1}\ominus q^{n+\alpha+\beta+1})_{p,q}^{n}}{(p\ominus q)_{p,q}^{n}\,(p^{\alpha+1}\ominus q^{\alpha+1})_{p,q}^{n}}(pq^{\beta})^{n}.$$

Then, the (p,q)-Jacobi polynomials satisfy the three-term recurrence relation (9.24) with

$$a_n=\frac{k_n}{k_{n+1}}=-\frac{p^{n+2}q^{n+\beta+1}(p^{n+\alpha+1}-q^{n+\alpha+1})(p^{n+\alpha+\beta+1}-q^{n+\alpha+\beta+1})}{(p^{2n+\alpha+\beta+2}-q^{2n+\alpha+\beta+2})(p^{2n+\alpha+\beta+1}-q^{2n+\alpha+\beta+1})}$$

$$\begin{aligned}b_n=&-q^{\beta+1}p^2MN\Big(M^2q^{2\alpha+\beta+2}p^3+M^2q^{\alpha+\beta+2}p^{\alpha+3}-MNq^{2\alpha+\beta+2}p^3-MNq^{2\alpha+\beta+3}p^2\\&-M^2q^{2\alpha+\beta+2}p-M^2q^{\alpha+\beta+2}p^{\alpha+1}-M^2q^{\alpha}p^{\alpha+\beta+3}-M^2p^{2\alpha+\beta+3}+MNpq^{2\alpha+\beta+2}\\&+MNq^{2\alpha+\beta+3}+MNq^{\alpha+\beta}p^{\alpha+3}+MNq^{\alpha+\beta+1}p^{\alpha+2}+MNq^{\alpha}p^{\alpha+\beta+3}+MNq^{\alpha+1}p^{\alpha+\beta+2}\\&-N^2p^2q^{2\alpha+\beta+1}-N^2q^{\alpha+\beta+1}p^{\alpha+2}\Big)\Big/\Big[\Big(M^2q^{\alpha+\beta+2}p^2-M^2q^{\alpha+\beta+2}-M^2p^{\alpha+\beta+2}+N^2q^{\alpha+\beta+2}\Big)\\&\times\Big(M^2q^{\alpha+\beta+2}p^2-M^2q^{\alpha+\beta+2}-M^2p^{\alpha+\beta+2}+N^2p^2q^{\alpha+\beta}\Big)\Big]\end{aligned}$$

$$\begin{aligned}c_n=&\,q^{\alpha+\beta+1}p^5NM(N-M)\Big(Mq^{\alpha+\beta+2}p^2-Mq^{\alpha+\beta+2}-Mp^{\alpha+\beta+2}+Nq^{\beta}p^{\alpha+2}\Big)\Big(Mq^{\alpha+\beta+2}p^2\\&-Mq^{\alpha+\beta+2}-Mp^{\alpha+\beta+2}+Nq^{\alpha+\beta}p^2\Big)\Big(M^2p^{\alpha+\beta}-N^2q^{\alpha+\beta}\Big)\Big(M^2p^{\alpha+\beta}q-pq^{\alpha+\beta}N^2\Big)\Big/\\&\Big(M^2q^{\alpha+\beta+2}p^2-M^2q^{\alpha+\beta+2}-M^2p^{\alpha+\beta+2}+N^2pq^{\alpha+\beta+1}\Big)\Big(M^2q^{\alpha+\beta+2}p^2-M^2q^{\alpha+\beta+2}-M^2p^{\alpha+\beta+2}\\&+N^2p^2q^{\alpha+\beta}\Big)^2\Big(M^2q^{\alpha+\beta+3}p^2-M^2q^{\alpha+\beta+3}-M^2p^{\alpha+\beta+2}q+N^2p^3q^{\alpha+\beta}\Big)\Big(Mp^{\alpha+\beta}-Nq^{\alpha+\beta}\Big)\end{aligned}$$

with $N=q^n$ and $M=p^n$.

9.5.3 (p,q)-Laguerre polynomials

(p,q)-Hypergeometric representation

In this section we discuss the case where

$$a=0,\quad b=-pq,\quad c=0,\quad (p-q)d=p^2\left(\frac{p}{q}\right)^{\alpha},\quad (p-q)e=-p^2\left(\frac{p}{q}\right)^{\alpha}+pq.$$

In this case we have the following corollary.

Corollary 115. *With the special coefficients above, the polynomials solutions of* (9.5) *have the representation*

$$L_n^{(\alpha)}(x;p,q)={}_2\Phi_1\left(\left.\begin{matrix}(p^{-n},q^{-n}),(p^n,0)\\(p^{\alpha+1},q^{\alpha+1})\end{matrix}\right|(p,q);-p^{\alpha}x\right), \tag{9.61}$$

up to a normalization constant.

Proof. For these special values, we have

$$\left((a+(p-q)d)p^{n-1} \ominus aq^{n-1}\right)^m_{p,q} = \left(p^{n+1}\left(\frac{p}{q}\right)^{\alpha} \ominus q^{n+1}\right)^m_{p,q} = \left(\frac{p^{\alpha+1}}{q^{\alpha}}\right)^m (p^n \ominus 0)^m_{p,q}$$

and

$$(b+(p-q)e \ominus b)^m_{p,q} = \left(-p^2\left(\frac{p}{q}\right)^{\alpha} \ominus -pq\right)^m_{p,q} = \left(-\frac{p}{q^{\alpha}}\right)^m \left(p^{\alpha+1} \ominus q^{\alpha+1}\right)^m_{p,q}.$$

Putting together these two relations combining with (9.58) provides the (p,q)-hypergeometric representation (9.61). □

Orthogonality relation

From the (p,q)-Pearson equation (9.18) we see that the weight function of the (p,q)-Laguerre polynomials satisfies the relation

$$\frac{\rho\left(\frac{p}{q}x\right)}{\rho(x)} = -\left(\frac{p}{q}\right)^{\alpha}(x-1). \tag{9.62}$$

We can therefore write $\rho(x) = \rho_1(x)\rho_(x)$ with

$$\frac{\rho_1\left(\frac{p}{q}x\right)}{\rho_1(x)} = \left(\frac{p}{q}\right)^{\alpha} \quad \text{and} \quad \frac{\rho_2\left(\frac{p}{q}x\right)}{\rho_2(x)} = 1-x.$$

The relation $\frac{\rho_1\left(\frac{p}{q}x\right)}{\rho_1(x)} = \left(\frac{p}{q}\right)^{\alpha}$ has a solution $\rho_1(x) = x^{\alpha}$ and applying Theorem 9.21 to the second equation it follows that a solution of $\frac{\rho_2\left(\frac{p}{q}x\right)}{\rho_2(x)} = 1-x$ may be written as

$$\rho_2(x) = \prod_{k=0}^{\infty} \frac{pp^k - (xq)q^k}{pp^k} = \frac{(p \ominus xq)^{\infty}_{p,q}}{(p \ominus 0)^{\infty}_{p,q}}.$$

Next using the (p,q)-Binomial theorem (8.18) with its special case (8.22) it follows that

$$\rho_2(x) = E_{p,q}((q-p)qx).$$

Finally we get

$$\rho(x) = \rho_1(x)\rho_2(x) = x^{\alpha}E_{p,q}((q-p)qx).$$

Corollary 116. *Let* $\{L_n^{(\alpha)}(x;p,q)\}_n$ *be the sequence of polynomials given by* (9.61). *Then, according to Theorem 106, the following orthogonality relation holds true*

$$\int_0^{\infty} x^{\alpha}E_{p,q}((q-p)qx)L_n^{(\alpha)}(pqx;p,q)L_m^{(\alpha)}(pqx;p,q)d_{p,q}x = \left(\int_0^{\infty} x^{\alpha}E_{p,q}((q-p)qx)\left(L_n^{(\alpha)}(x;p,q)\right)^2 d_{p,q}x\right)\delta_{n,m} \tag{9.63}$$

Three-term recurrence relation

From the (p,q)-hypergeometric representation (9.61), we see that the leading coefficient of $L_n^{(\alpha)}(x;p,q)$ is given by

$$k_n = \frac{(p^{-n} \ominus q^{-n})_{p,q}^n (p^n \ominus 0)_{p,q}^n}{(p \ominus q)_{p,q}^n (p^\alpha \ominus q^\alpha)_{p,q}^n}(-p^\alpha)^n.$$

Then, the (p,q)-Laguerre polynomials satisfy the three-term recurrence relation (9.24) with

$$\begin{aligned} a_n &= \frac{k_n}{k_{n+1}} = \frac{q^{n+1}}{p^{\alpha+2n}}\left(p^{n+\alpha+1} - q^{\alpha+n+1}\right) \\ b_n &= \frac{q^{n+1}}{p^{2n+\alpha}}\left(p^{n+\alpha+1} - q^{n+\alpha+1} + p\left(p^{n+\alpha} - q^{n+\alpha}\right)\right) \\ c_n &= \frac{q^{n+\alpha+1}}{p^{2n+\alpha-1}}(p^n - q^n). \end{aligned}$$

9.5.4 (p,q)-Hermite polynomials

In this section we discuss the case $a = b = e = 0$ and $c = 1$.

(p,q)-Hypergeometric representation

The (p,q)-difference equation (9.52) becomes

$$[m+2]_{p,q}[m+1]_{p,q}f_{m+2}(n) + dp^{m+n-1}[m-n]f_m(n) = 0. \tag{9.64}$$

We discuss two cases, $m = 2\ell$ and $m = 2\ell + 1$.

- If $m = 2\ell$, (9.64) becomes

$$f_{2(\ell+1)} = -\frac{[2\ell - n]_{p,q}dp^{2\ell+n-1}}{[2\ell+2]_{p,q}[2\ell+1]_{p,q}}f_{2\ell}(n).$$

  Solving this recurrence yields

$$f_{2\ell} = \left(d(q-p)p^{n-1}\right)^\ell \frac{(p^{-n} \ominus q^{-n})_{p^2,q^2}^\ell (1 \ominus 0)_{p^2,q^2}^\ell}{(p \ominus q)_{p^2,q^2}^\ell (p^2 \ominus q^2)_{p^2,q^2}^\ell}f_0(n)$$

  or otherwise stated

$$f_{2\ell} = ((q-p))^\ell \frac{(p^{-n} \ominus q^{-n})_{p^2,q^2}^\ell (dp^{n-1} \ominus 0)_{p^2,q^2}^\ell}{(p \ominus q)_{p^2,q^2}^\ell (p^2 \ominus q^2)_{p^2,q^2}^\ell}f_0(n) \tag{9.65}$$

- If $m = 2\ell + 1$, (9.64) becomes

$$f_{2\ell+3}(n) = -\frac{[2\ell+1-n]_{p,q}dp^{2\ell+n}}{[2\ell+3]_{p,q}[2\ell+2]_{p,q}}f_{2\ell+1}(n).$$

Solving this recurrence yields

$$f_{2\ell+1} = (d(q-p)p^n)^\ell \frac{\left(p^{1-n}\ominus q^{1-n}\right)_{p^2,q^2}^\ell (1\ominus 0)_{p^2,q^2}^\ell}{\left(p^2\ominus q^2\right)_{p^2,q^2}^\ell \left(p^3\ominus q^3\right)_{p^2,q^2}^\ell} f_1(n).$$

or otherwise stated

$$f_{2\ell+1} = ((q-p))^\ell \frac{\left(p^{1-n}\ominus q^{1-n}\right)_{p^2,q^2}^\ell (dp^n\ominus 0)_{p^2,q^2}^\ell}{\left(p^2\ominus q^2\right)_{p^2,q^2}^\ell \left(p^3\ominus q^3\right)_{p^2,q^2}^\ell} f_1(n). \tag{9.66}$$

From (9.65) and (9.66), we get the following (p,q)-hypergeometric representation for the (p,q)-Hermite polynomials (compare with [60])

$$H_n(x;p,q) = x^{\sigma_n}{}_2\Phi_1\left(\left.\begin{matrix}(p^{\sigma_n-n},q^{\sigma_n-n}),(dp^{2[(n-1)/2]-1},0)\\ (p^{2\sigma_n+1},q^{2\sigma_n+1})\end{matrix}\right|(p^2,q^2);(q-p)x^2\right) \tag{9.67}$$

up to a normalization constant where

$$\sigma_n = \frac{1-(-1)^n}{2}.$$

Orthogonality relation

From the (p,q)-Pearson equation (9.18) we see that the weight function of the (p,q)-Laguerre polynomials satisfies the relation

$$\frac{\rho\left(\frac{p}{q}x\right)}{\rho(x)} = 1 + d(p-q)x^2, \tag{9.68}$$

whose solution is

$$\rho(x;p,q) = E_{p^2,q^2}\left(-((p-q)qx)^2\right).$$

Hence the (p,q)-Hermite polynomials fulfil the orthogonality relation

$$\int_{-\infty}^{\infty} H_n(pqx;p,q)H_m(pqx;p,q)E_{p^2,q^2}\left(-d((p-q)qx)^2\right)d_{p,q}x$$
$$= \left(\int_{-\infty}^{\infty} H_n^2(pqx;p,q)E_{p^2,q^2}\left(-d((p-q)qx)^2\right)d_{p,q}x\right)\delta_{n,m}.$$

Three-term recurrence relation

From the (p,q)-hypergeometric representation (9.67), we see that the leading coefficient of $H_n(x;p,q)$ is given by

$$k_{2n} = \left(d(q-p)p^{n-1}\right)^n \frac{\left(p^{-n}\ominus q^{-n}\right)_{p^2,q^2}^n (p^2)^{\binom{n}{2}}}{(p\ominus q)_{p^2,q^2}^n \left(p^2\ominus q^2\right)_{p^2,q^2}^n}$$

and

$$k_{2n+1} = (d(q-p)p^n)^n \frac{\left(p^{1-n}\ominus q^{1-n}\right)_{p^2,q^2}^n (p^2)^{\binom{n}{2}}}{\left(p^2\ominus q^2\right)_{p^2,q^2}^n \left(p^3\ominus q^3\right)_{p^2,q^2}^n}$$

Therefore, the (p,q)-Hermite polynomials satisfy the three-term recurrence relation (9.24) with

$$\begin{aligned} a_n &= \frac{k_n}{k_{n+1}} \\ b_n &= 0 \\ c_n &= -\frac{[n]_{p,q} q^{n+1}}{d p^{2n-3}} \frac{k_n}{k_{n-1}}. \end{aligned}$$

Chapter 10

(p, q)-Laplace Transform and Applications

The classical Laplace transform of a function f is given by

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)dt, \qquad s = a + ib \in \mathbb{C}, \tag{10.1}$$

and plays a fundamental role in pure and applied analysis, specially in solving differential equations. If a function of a discrete variable $f(t)$, $t \in \mathbb{Z}$ is considered, then the integral transform (10.1) reads

$$F(z) = \sum_{j=0}^{\infty} f(j) z^{-j}, \quad z = e^{-p}. \tag{10.2}$$

Equation (10.2) is referred to as Z-transform and plays a similar role in difference analysis as Laplace transform in continuous analysis, specially in solving difference equations.

In order to deal with q-difference equations, q-versions of the classical Laplace transform have been consecutively introduced in the literature. Studies of q-versions of Laplace transform go back to Hahn [40]. Abdi [1, 2, 3] published also many results in this domain.

The q-deformed algebras [73, 74] and their generalizations ((p, q)-deformed algebras) [23, 37, 48, 5, 6, 43, 7] attracted much attention these last years. The main reason is that these topics stand for a meeting point of today's fast developing areas in mathematics and physics like the theory of quantum orthogonal polynomials and special functions, quantum groups, conformal field theories and statistics. From these works, many generalizations of special functions arise. There is a considerable list of references.

In this chapter, we introduce two (p, q)-versions of the Laplace transform and provide some of their main properties. Next, some applications are done to solve some (p, q)-difference equations, for example the (p, q)-oscillator is introduced and solved using the (p, q)-Laplace transform of first kind.

10.1 (p, q)-Laplace transform of the first kind

Definition 117 (Njionou [68])**.** *For a given function $f(t)$, we define its (p, q)-Laplace transform of the first kind as the function*

$$F(s) = L_{p,q}\{f(t)\}(s) = \int_0^\infty f(t) E_{p,q}(-qts) d_{p,q}t, \;\; s > 0. \tag{10.3}$$

Of course, by definition the (p, q)-Laplace transform of the first kind is linear.

Proposition 118. *For any two complex numbers α and β, we have*

$$L_{p,q}\{\alpha f(t)+\beta g(t)\}=\alpha L_{p,q}\{f(t)\}+\beta L_{p,q}\{g(t)\}.$$

In what follows, we give some examples. From (10.3), we note that:

$$\begin{aligned}
L_{p,q}\{1\}(s) &= \int_0^\infty E_{p,q}(-qst)d_{p,q}t = -\frac{1}{s}\int_0^\infty D_{p,q}E_{p,q}(-st)d_{p,q}t \\
&= -\frac{1}{s}\left[E_{p,q}(-st)\right]_0^\infty = \frac{1}{s}, \quad s>0. \\
L_{p,q}\{t\}(s) &= \int_0^\infty tE_{p,q}(-qst)d_{p,q}t = -\frac{1}{ps}\int_0^\infty (pt)D_{p,q}E_{p,q}(-st)d_{p,q}t \\
&= -\frac{1}{ps}\left\{\left[tE_{p,q}(-st)\right]_0^\infty - \int_0^\infty E_{p,q}(-qst)d_{p,q}t\right\} \\
&= \frac{1}{ps^2}, \quad s>0. \\
L_{p,q}\{1+5t\}(s) &= L_{p,q}\{1\}(s)+5L_{p,q}\{t\}(s) = \frac{1}{s}+\frac{5}{ps^2}, \quad s>0.
\end{aligned}$$

Proposition 119 (Njionou [68])**.** *Let α be a non-zero complex number, then*

$$\int_0^\infty f(\alpha t)d_{p,q}t = \frac{1}{\alpha}\int_0^\infty f(t)d_{p,q}t. \tag{10.4}$$

Theorem 120 (**Scaling**, (Njionou [68]))**.** *Let a be a non-zero complex number, then the following formula applies*

$$L_{p,q}\{f(at)\}(s) = \frac{1}{a}L_{p,q}\{f(t)\}\left(\frac{s}{a}\right). \tag{10.5}$$

Proof. Using the definition and Proposition 119, it follows that

$$\begin{aligned}
L_{p,q}\{f(at)\}(s) &= \int_0^\infty f(at)E_{p,q}(-qst)d_{p,q}t \\
&= \int_0^\infty f(at)E_{p,q}(-aq\frac{s}{a}t)d_{p,q}t \\
&= \frac{1}{a}\int_0^\infty f(t)E_{p,q}(-q\frac{s}{a}t)d_{p,q}t = \frac{1}{a}L_{p,q}\{f(t)\}\left(\frac{s}{a}\right).
\end{aligned}$$

□

Theorem 121 (Njionou [68])**.** *For $\alpha>-1$, the following equation is valid:*

$$L_{p,q}(t^\alpha) = \frac{\Gamma_{p,q}(\alpha+1)}{p^{\frac{\alpha(\alpha+1)}{2}}s^{\alpha+1}}. \tag{10.6}$$

Proof. Of course, it follows from the definition that

$$\begin{aligned}
L_{p,q}\{t^\alpha\}(s) &= \int_0^\infty t^\alpha E_{p,q}(-qst)d_{p,q}t = \frac{1}{s^{\alpha+1}}\int_0^\infty E_{p,q}(-qt)t^\alpha d_{p,q}t \\
&= \frac{1}{p^{\frac{\alpha(\alpha+1)}{2}}s^{\alpha+1}}\int_0^\infty p^{\frac{\alpha(\alpha+1)}{2}}t^{(\alpha+1)-1}E_{p,q}(-qt)d_{p,q}t \\
&= \frac{\Gamma_{p,q}(\alpha+1)}{p^{\frac{\alpha(\alpha+1)}{2}}s^{\alpha+1}}.
\end{aligned}$$

□

The following theorem is a particular case of Theorem (121) when $\alpha = n$ is a nonnegative integer.

Theorem 122 (Njionou [68])**.** *Let $n \in \mathbb{N}$, then for $s > 0$ the following equation holds:*

$$L_{p,q}\{t^n\}(s) = \frac{[n]_{p,q}!}{p^{\binom{n+1}{2}} s^{n+1}}. \tag{10.7}$$

Proof. We provide a proof by induction for this result. The result is obvious for $n = 0$. Assume that it holds true for some nonnegative integer n, then using the (p,q)-integration by parts (6.19) , we have

$$\begin{aligned} L_{p,q}\{t^{n+1}\}(s) &= \int_0^\infty t^{n+1} E_{p,q}(-qst) d_{p,q}t \\ &= -\frac{1}{p^{n+1}s} \int_0^\infty (pt)^{n+1} D_{p,q} E_{p,q}(-ts) d_{p,q}t \\ &= -\frac{1}{p^{n+1}s} \left\{ \left[t^{n+1} E_{p,q}(-st) \right]_0^\infty - [n+1]_{p,q} \int_0^\infty t^n E_{p,q}(-qts) d_{p,q}t \right\} \\ &= \frac{[n+1]_{p,q}}{p^{n+1}s} L_{p,q}\{t^n\}(s) = \frac{[n+1]_{p,q}}{p^{n+1}s} \frac{[n]_{p,q}!}{p^{\binom{n+1}{2}} s^{n+1}} = \frac{[n+1]_{p,q}!}{p^{\binom{n+2}{2}} s^{n+2}}. \end{aligned}$$

This proves the assertion. □

Next, we give formulas for the transform for the (p,q)-exponential and the (p,q)-trigonometric functions.

Theorem 123 (Njionou [68])**.** *Let a be a real number, then*

$$L_{p,q}\{e_{p,q}(at)\}(s) = \frac{p}{ps - a}, \quad s > \frac{a}{p}, \tag{10.8}$$

$$L_{p,q}\{E_{p,q}(at)\}(s) = \frac{1}{s} \sum_{n=0}^\infty (-1)^n \left(\frac{q}{p}\right)^{\binom{n}{2}} \left(\frac{a}{ps}\right)^n. \tag{10.9}$$

Proof. Using (5.1), (5.2) and (10.7), we can write:

$$\begin{aligned} L_{p,q}\{e_{p,q}(at)\}(s) &= \int_0^\infty E_{p,q}(-qst) e_{p,q}(at) d_{p,q}t = \sum_{n=0}^\infty \frac{a^n p^{\binom{n}{2}}}{[n]_{p,q}!} \int_0^\infty E_{p,q}(-qst) t^n d_{p,q}t \\ &= \sum_{n=0}^\infty \frac{a^n p^{\binom{n}{2}}}{[n]_{p,q}!} \frac{[n]_{p,q}!}{p^{\binom{n+1}{2}} s^{n+1}} = \frac{1}{s} \sum_{n=0}^\infty \left(\frac{a}{ps}\right)^n = \frac{p}{ps - a}, \end{aligned}$$

$$\begin{aligned} L_{p,q}\{E_{p,q}(at)\}(s) &= \int_0^\infty E_{p,q}(-qst) E_{p,q}(at) d_{p,q}t \\ &= \sum_{n=0}^\infty (-1)^n \frac{a^n q^{\binom{n}{2}}}{[n]_{p,q}!} \int_0^\infty E_{p,q}(-qst) t^n d_{p,q}t \\ &= \sum_{n=0}^\infty (-1)^n \frac{a^n q^{\binom{n}{2}}}{[n]_{p,q}!} \frac{[n]_{p,q}!}{p^{\binom{n+1}{2}} s^{n+1}} = \frac{1}{s} \sum_{n=0}^\infty (-1)^n \left(\frac{q}{p}\right)^{\binom{n}{2}} \left(\frac{a}{ps}\right)^n. \end{aligned}$$

□

Theorem 124 (Njionou [68]). *The following relations apply:*

$$\begin{aligned}
L_{p,q}\{\cos_{p,q}(at)\}(s) &= \frac{p^2 s}{(ps)^2 + a^2}, \\
L_{p,q}\{\sin_{p,q}(at)\}(s) &= \frac{pa}{(ps)^2 + a^2}.
\end{aligned}$$

Proof. Using equations (5.8), (5.9) and (10.3), it follows that:

$$\begin{aligned}
L_{p,q}\{\cos_{p,q}(at)\}(s) &= \int_0^\infty E_{p,q}(-qst)\cos_{p,q}(at)d_{p,q}t \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} p^{\binom{2n}{2}}}{[2n]_{p,q}!} \int_0^\infty E_{p,q}(-qst)t^{2n} d_{p,q}t \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} p^{\binom{2n}{2}}}{[2n]_{p,q}!} \frac{[2n]_{p,q}!}{p^{\binom{2n+1}{2}} s^{2n+1}} \\
&= \frac{1}{s}\sum_{n=0}^{\infty} (-1)^n \left(\frac{a}{ps}\right)^{2n} = \frac{p^2 s}{(ps)^2 + a^2},
\end{aligned}$$

$$\begin{aligned}
L_{p,q}\{\sin_{p,q}(at)\}(s) &= \int_0^\infty E_{p,q}(-qst)\sin_{p,q}(at)d_{p,q}t \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} \int_0^\infty E_{p,q}(-qst)t^{2n+1} d_{p,q}t \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} \frac{[2n+1]_{p,q}!}{p^{\binom{2n+2}{2}} s^{2n+2}} \\
&= \frac{1}{s}\sum_{n=0}^{\infty} (-1)^n \left(\frac{a}{ps}\right)^{2n+1} = \frac{pa}{(ps)^2 + a^2}.
\end{aligned}$$

□

Remark 125. *Note that one could also use* (10.8), (5.8) *and* (5.9) *to obtain the results.*

Theorem 126 (Njionou [68]). *The following equations apply*

$$\begin{aligned}
L_{p,q}\{\cosh_{p,q}(at)\}(s) &= \frac{p^2 s}{(ps)^2 - a^2}, \quad s > \left|\frac{a}{p}\right| \\
L_{p,q}\{\sinh_{p,q}(at)\}(s) &= \frac{pa}{(ps)^2 - a^2}, \quad s > \left|\frac{a}{p}\right|.
\end{aligned}$$

Proof. Using (10.3), (5.13) and (5.14) we have

$$\begin{aligned}
L_{p,q}\{\cosh_{p,q}(at)\}(s) &= \frac{1}{2}\left\{L_{p,q}\{e_{p,q}(at)\}(s) + L_{p,q}\{e_{p,q}(-at)\}(s)\right\} \\
&= \frac{1}{2}\left(\frac{p}{ps - a} + \frac{p}{ps + a}\right) \\
&= \frac{p^2 s}{(ps)^2 - a^2},
\end{aligned}$$

$$
\begin{aligned}
L_{p,q}\{\sinh_{p,q}(at)\}(s) &= \frac{1}{2}\left\{L_{p,q}\{e_{p,q}(at)\}(s) - L_{p,q}\{e_{p,q}(-at)\}(s)\right\}\\
&= \frac{1}{2}\left(\frac{p}{ps-a} - \frac{p}{ps+a}\right)\\
&= \frac{pa}{(ps)^2 - a^2}.
\end{aligned}
$$

□

Next, f being a function, we provide some properties related to the (p,q)-derivative of the (p,q)-Laplace transform of f and the (p,q)-Laplace transform of the (p,q)-derivative of f. Let us introduce the following notation which makes clear the relative variable to which the (p,q)-derivative is applied:

$$
\frac{\partial_{p,q}}{\partial_{p,q}s} f(x,s) = \frac{f(x,ps) - f(x,qs)}{(p-q)s},
$$

and

$$
\frac{\partial^{n+1}_{p,q}}{\partial_{p,q}s^{n+1}} = \frac{\partial^{n}_{p,q}}{\partial_{p,q}s^{n}} \circ \frac{\partial_{p,q}}{\partial_{p,q}s}, \quad n \geq 1, \quad \text{and} \quad \frac{\partial^{0}_{p,q}}{\partial_{p,q}s^{0}} f = f.
$$

Theorem 127 (Njionou [68])**.** *For $n \in \mathbb{N}$, the following equation holds:*

$$
L_{p,q}\{t^n f(t)\}(s) = (-1)^n q^{\binom{n}{2}} \frac{\partial^{n}_{p,q}}{\partial_{p,q}s^{n}} \left[F\left(q^{-n}s\right)\right]. \tag{10.10}
$$

Proof. The result is obvious for $n = 0$. Let $n \geq 1$, then

$$
\frac{\partial^{n}_{p,q}}{\partial_{p,q}s^{n}}[F(q^{-n}s)] = \int_0^\infty \frac{\partial^{n}_{p,q}}{\partial_{p,q}s^{n}} \left[E_{p,q}(-q^{-n+1}st)\right] f(t) d_{p,q}t
$$

Using equation (5.6), it follows that

$$
\begin{aligned}
\frac{\partial^{n}_{p,q}}{\partial_{p,q}s^{n}} \left[E_{p,q}(-q^{-n+1}st)\right] &= \prod_{j=0}^{n-1} \left(-q^{n-1-j}t\right) E_{p,q}(-qst)\\
&= (-1)^n q^{-\binom{n}{2}} t^n E_{p,q}(-qst).
\end{aligned}
$$

The proof is therefore completed. □

Note that (10.7) can be obtained using Theorem 127. Of course, taking $f(t) = 1$ in (10.10) and using (2.20), we have $F(s) = \dfrac{1}{s}$ and

$$
L_{p,q}\{t^n\}(s) = (-1)^n q^{\binom{n}{2}} \frac{\partial^{n}_{p,q}}{\partial_{p,q}s^{n}} \left[\frac{q^n}{s}\right] = (-1)^n q^{\binom{n+1}{2}} \frac{(-1)^n [n]_{p,q}!}{(pq)^{\binom{n+1}{2}} s^{n+1}} = \frac{[n]_{p,q}!}{p^{\binom{n+1}{2}} s^{n+1}}.
$$

Corollary 128 (Njionou [68])**.** *The following equation applies:*

$$
\begin{aligned}
L_{p,q}\{t^n e_{p,q}(at)\}(s) &= \frac{p^{n+1} q^{\binom{n+1}{2}} [n]_{p,q}!}{(p^{n+1}s - aq^n)(p^n qs - aq^n)\cdots(p^2 q^{n-1}s - aq^n)(pq^n s - aq^n)}\\
&= \frac{p^{n+1} q^{\binom{n+1}{2}} [n]_{p,q}!}{\prod\limits_{k=0}^{n} \left(p^{n+1-k} q^k s - aq^n\right)}.
\end{aligned}
$$

Proof. The proof follows from (2.21) and (10.10). $\square$

Theorem 129 (Transform of the (p,q)-derivative, (Njionou [68])**).** *The following transform rule applies.*

$$L_{p,q}\left\{D^n_{p,q}f(t)\right\}(s)=\frac{s^n}{p^{\binom{n+1}{2}}}L_{p,q}\{f(t)\}\left(\frac{s}{p^n}\right)-\sum_{k=0}^{n-1}\frac{s^{n-1-k}}{p^{\binom{n-k}{2}}}(D^k_{p,q}f)(0). \tag{10.11}$$

Proof. Let f be a function for which the (p,q)-Laplace transform exists. Then, for $n=1$,

$$\begin{aligned}L_{p,q}\left\{D_{p,q}f(t)\right\}(s)&=\int_0^\infty E_{p,q}(-qst)D_{p,q}f(t)d_{p,q}t\\&=\left[f(t)E_{p,q}(-st)\right]_0^\infty-\int_0^\infty f(pt)D_{p,q}E_{p,q}(-st)d_{p,q}t\\&=-f(0)+s\int_0^\infty f(pt)E_{p,q}(-qst)d_{p,q}t\\&=-f(0)+\frac{s}{p}\int_0^\infty f(t)E_{p,q}(-q\frac{s}{p}t)d_{p,q}t\\&=-f(0)+\frac{s}{p}L_{p,q}\{f(t)\}\left(\frac{s}{p}\right).\end{aligned}$$

Let $n\geq 1$, assume (10.11) holds true. Then, applying the result for $n=1$ with $D^n_{p,q}f(t)$, we get

$$\begin{aligned}L_{p,q}\left\{D^{n+1}_{p,q}f(t)\right\}(s)&=-(D^n_{p,q}f)(0)+\frac{s}{p}L_{p,q}\{D^n_{p,q}f(t)\}\left(\frac{s}{p}\right)\\&=-(D^n_{p,q}f)(0)+\frac{s}{p}\left\{\frac{s^n}{p^{\binom{n+1}{2}+n}}L_{p,q}\{f(t)\}\left(\frac{s}{p^{n+1}}\right)\right.\\&\qquad\left.-\sum_{k=0}^{n-1}\frac{s^{n-1-k}}{p^{\binom{n-k}{2}+n-1-k}}(D^k_{p,q}f)(0)\right\}\\&=-(D^n_{p,q}f)(0)+\left\{\frac{s^{n+1}}{p^{\binom{n+1}{2}+n+1}}L_{p,q}\{f(t)\}\left(\frac{s}{p^{n+1}}\right)\right.\\&\qquad\left.-\sum_{k=0}^{n-1}\frac{s^{n-k}}{p^{\binom{n-k}{2}+n-k}}(D^k_{p,q}f)(0)\right\}\\&=-(D^n_{p,q}f)(0)+\left\{\frac{s^{n+1}}{p^{\binom{n+2}{2}}}L_{p,q}\{f(t)\}\left(\frac{s}{p^{n+1}}\right)\right.\\&\qquad\left.-\sum_{k=0}^{n-1}\frac{s^{n-k}}{p^{\binom{n-k+1}{2}}}(D^k_{p,q}f)(0)\right\}\\&=\frac{s^{n+1}}{p^{\binom{n+2}{2}}}L_{p,q}\{f(t)\}\left(\frac{s}{p^{n+1}}\right)-\sum_{k=0}^{n}\frac{s^{n-k}}{p^{\binom{n-k+1}{2}}}(D^k_{p,q}f)(0)\end{aligned}$$

This completes the proof. $\square$

As a direct application, observe that taking $f(t)=t^n$ in (10.11), we obtain

$$L_{p,q}\{D^n_{p,q}t^n\}(s)=\frac{s^n}{p^{\binom{n+1}{2}}}L_{p,q}\{t^n\}\left(\frac{s}{p^n}\right).$$

Taking care that $D^n_{p,q}t^n = [n]_{p,q}!$, and $L_{p,q}\{1\}(s) = \frac{1}{s}$, it follows that

$$L_{p,q}\{t^n\}\left(\frac{s}{p^n}\right) = p^{\binom{n+1}{2}}\frac{[n]_{p,q}!}{s^n}L_{p,q}\{1\}(1) = p^{\binom{n+1}{2}}\frac{[n]_{p,q}!}{s^{n+1}}.$$

Replacing s by sp^n, it follows that

$$L_{p,q}\{t^n\}\,(s) = p^{\binom{n+1}{2}}\frac{[n]_{p,q}!}{s^{n+1}p^{n(n+1)}} = \frac{[n]_{p,q}!}{p^{\binom{n+1}{2}}s^{n+1}}.$$

10.2 (p,q)-Laplace transform of the second kind

Whereas in the previous section we introduced the (p,q)-Laplace transform of the first kind and proved some of its important properties, in this section, we introduce the (p,q)-Laplace transform of the second kind. The main difference is at the level of the (p,q)-exponential function used in the definition. The motivation of the next definition comes from the fact that when we transform the big (p,q)-exponential function, the result remains in terms of a series which we cannot simplify.

Definition 130 (Njionou [68]). *For a given function $f(t)$, we define its (p,q)-Laplace transform of the second kind as the function*

$$F(s) = \mathcal{L}_{p,q}\{f(t)\}(s) = \int_0^\infty f(t)e_{p,q}(-pts)d_{p,q}t, \quad s > 0. \tag{10.12}$$

Proposition 131 ((Linearity)). *By* (10.12)*, the following equation applies:*

$$\mathcal{L}_{p,q}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}_{p,q}\{f(t)\} + \beta\mathcal{L}_{p,q}\{g(t)\}.$$

Proposition 132 (Njionou [68]). *For any real number $\alpha > -1$, we have*

$$\mathcal{L}_{p,q}\{t^\alpha\}(s) = \frac{\gamma_{p,q}(\alpha+1)}{q^{\frac{\alpha(\alpha-1)}{2}}s^{\alpha+1}}. \tag{10.13}$$

Proof. By definition, one has

$$\begin{aligned}\mathcal{L}_{p,q}\{t^\alpha\}(s) &= \int_0^\infty t^\alpha e_{p,q}(-pts)d_{p,q}t\\ &= \frac{1}{s^{\alpha+1}}\int_0^\infty t^\alpha e_{p,q}(-pt)d_{p,q}t\\ &= \frac{\gamma_{p,q}(\alpha+1)}{q^{\frac{\alpha(\alpha-1)}{2}}s^{\alpha+1}}.\end{aligned}$$

□

Proposition 133 (Njionou [68]). *For $n \in \mathbb{N}$, it is valid that*

$$\mathcal{L}_{p,q}\{t^n\}(s) = \frac{[n]_{p,q}!}{q^{\binom{n+1}{2}}s^{n+1}}. \tag{10.14}$$

Proof. Clearly, the results holds for $n = 0$. Indeed

$$\mathcal{L}_{p,q}\{1\}(s) = \int_0^\infty e_{p,q}(-pts)d_{p,q}t = -\frac{1}{s}\left[e_{p,q}(ts)\right]_0^\infty = \frac{1}{s}.$$

Next, for $n > 0$,

$$\begin{aligned}
\mathcal{L}_{p,q}\{t^n\}(s) &= \int_0^\infty t^n e_{p,q}(-pts)d_{p,q}t \\
&= -\frac{1}{q^n s}\int_0^\infty (qt)^n D_{p,q}e_{p,q}(-ts)d_{p,q}t \\
&= -\frac{1}{q^n s}\left\{\left[t^n e_{p,q}(-ts)\right]_0^\infty - [n]_{p,q}\int_0^\infty t^{n-1}e_{p,q}(-pts)d_{p,q}t\right\} \\
&= \frac{[n]_{p,q}}{q^n s}\mathcal{L}_{p,q}\{t^{n-1}\}(s).
\end{aligned}$$

The proof then follows by induction. □

Proposition 134 (Njionou [68]). *The following equation holds*

$$\mathcal{L}_{p,q}\{E_{p,q}(at)\}(s) = \frac{q}{qs-a}, \quad s > \left|\frac{a}{q}\right|. \tag{10.15}$$

Proof. From the definition of $\mathcal{L}_{p,q}$ and $E_{p,q}(x)$, it follows that

$$\begin{aligned}
\mathcal{L}_{p,q}\{E_{p,q}(at)\}(s) &= \sum_{n=0}^\infty \frac{q^{\binom{n}{2}}a^n}{[n]_{p,q}!}\int_0^\infty t^n e_{p,q}(-pts)d_{p,q}t \\
&= \sum_{n=0}^\infty \frac{q^{\binom{n}{2}}a^n}{[n]_{p,q}!} \times \frac{[n]_{p,q}!}{q^{\binom{n+1}{2}}s^{n+1}} \\
&= \frac{1}{s}\sum_{n=0}^\infty \left(\frac{a}{qs}\right)^n = \frac{q}{qs-a}.
\end{aligned}$$

□

Corollary 135 (Njionou [68]). *The following equations hold*

$$\begin{aligned}
\mathcal{L}_{p,q}\{Cos_{p,q}(at)\}(s) &= \frac{q^2 s}{(qs)^2 + a^2}, \quad s > \left|\frac{a}{q}\right|, \\
\mathcal{L}_{p,q}\{Sin_{p,q}(at)\}(s) &= \frac{qa}{(qs)^2 + a^2}, \quad s > \left|\frac{a}{q}\right|.
\end{aligned}$$

Proof. The proof follows from the definitions (5.11), (5.12) and equation (10.15). □

Corollary 136 (Njionou [68]). *The following equations hold*

$$\begin{aligned}
\mathcal{L}_{p,q}\{Cosh_{p,q}(at)\}(s) &= \frac{q^2 s}{(qs)^2 - a^2}, \quad s > \left|\frac{a}{q}\right|, \\
\mathcal{L}_{p,q}\{Sinh_{p,q}(as)\}(s) &= \frac{qa}{(qs)^2 - a^2}, \quad s > \left|\frac{a}{q}\right|.
\end{aligned}$$

Proof. The proof is similar to the proof of Corollary 135. □

Next, f being a function, we provide some properties related to the (p,q)-derivative of the (p,q)-Laplace transform of f and the (p,q)-Laplace transform of the (p,q)-derivative of f.

Theorem 137 (Njionou [68]). *For $n \in \mathbb{N}$, we have*

$$\mathcal{L}_{p,q}\{t^n f(t)\}(s) = (-1)^n p^{\binom{n}{2}} \frac{\partial^n_{p,q}}{\partial_{p,q} s^n}\left[F\left(p^{-n}s\right)\right] \tag{10.16}$$

where $F(s) = \mathcal{L}_{p,q}\{f(t)\}(s)$.

Proof. The result is obvious for $n = 0$. Let $n \geq 1$, we have

$$\frac{\partial^n_{p,q}}{\partial_{p,q} s^n}[F(p^{-n}s)] = \int_0^\infty \frac{\partial^n_{p,q}}{\partial_{p,q} s^n}\left[e_{p,q}(-p^{-n+1}st)\right] f(t) d_{p,q}t$$

Using equation (5.5), it follows that

$$\begin{aligned}\frac{\partial^n_{p,q}}{\partial_{p,q} s^n}\left[e_{p,q}(-p^{-n+1}st)\right] &= \prod_{j=0}^{n-1}\left(-p^{n-1-j}t\right) e_{p,q}(-pst) \\ &= (-1)^n p^{-\binom{n}{2}} t^n e_{p,q}(-pst).\end{aligned}$$

The proof is therefore completed. □

Note that (10.7) can be obtained using Theorem 137. Of course, taking $f(t) = 1$ in (10.16) and using (2.20), we have $F(s) = \dfrac{1}{s}$ and

$$L_{p,q}\{t^n\}(s) = (-1)^n p^{\binom{n}{2}} \frac{\partial^n_{p,q}}{\partial_{p,q} s^n}\left[\frac{p^n}{s}\right] = (-1)^n p^{\binom{n+1}{2}} \frac{(-1)^n [n]_{p,q}!}{(pq)^{\binom{n+1}{2}} s^{n+1}} = \frac{[n]_{p,q}!}{q^{\binom{n+1}{2}} s^{n+1}}.$$

Corollary 138 (Njionou [68]). *The following equation applies:*

$$\begin{aligned}\mathcal{L}_{p,q}\{t^n E_{p,q}(at)\}(s) &= \frac{q^{n+1} p^{\binom{n+1}{2}} [n]_{p,q}!}{(q^{n+1}s - ap^n)(q^n ps - ap^n)\cdots(q^2 p^{n-1}s - ap^n)(pq^n s - ap^n)} \\ &= \frac{q^{n+1} p^{\binom{n+1}{2}} [n]_{p,q}!}{\prod\limits_{k=0}^{n}(q^{n+1-k}p^k s - ap^n)}.\end{aligned}$$

Proof. The proof follows from (2.21) and (10.16). □

Theorem 139 (Transform of the (p,q)-derivative, (Njionou [68])**).** *For any nonnegative integer n, we have*

$$\mathcal{L}_{p,q}\left\{D^n_{p,q} f(t)\right\} = \frac{s^n}{q^{\binom{n+1}{2}}} \mathcal{L}_{p,q}\{f(t)\}\left(\frac{s}{q^n}\right) - \sum_{k=0}^{n-1} \frac{s^{n-1-k}}{q^{\binom{n-k}{2}}}\left(D^k_{p,q} f\right)(0). \tag{10.17}$$

Proof. For $n = 1$, we have

$$\begin{aligned}\mathcal{L}_{p,q}\{f(t)\}(s) &= \int_0^\infty D_{p,q} f(t) e_{p,q}(-pst) d_{p,q}t \\ &= \left[f(t) e_{p,q}(-st)\right] + s\int_0^\infty f(qt) e_{p,q}(-pst) d_{p,q}t \\ &= -f(0) + \frac{s}{q}\int_0^\infty f(t) e_{p,q}\left(-p\frac{s}{q}t\right) d_{p,q}t \\ &= -f(0) + \frac{s}{q}\mathcal{L}_{p,q}\{f(t)\}\left(\frac{s}{q}\right).\end{aligned}$$

Hence the relation is true for $n = 1$. Let $n \geq 1$, assume that (10.17) holds true, then using the case $n = 1$, we can write

$$\begin{aligned}
\mathcal{L}_{p,q}\left\{D_{p,q}^{n+1}f(t)\right\} &= -(D_{p,q}^{n}f)(0) + \frac{s}{q}\mathcal{L}_{p,q}\{D_{p,q}^{n}f(t)\}\left(\frac{s}{q}\right) \\
&= -(D_{p,q}^{n}f)(0) + \frac{s}{q}\left\{\frac{s^n}{q^{\binom{n+1}{2}+n}}\mathcal{L}_{p,q}\{f(t)\}\left(\frac{s}{q^{n+1}}\right)\right. \\
&\quad \left. - \sum_{k=0}^{n-1}\frac{s^{n-1-k}}{q^{\binom{n-k}{2}+n-1-k}}\left(D_{p,q}^{k}f\right)(0)\right\} \\
&= -(D_{p,q}^{n}f)(0) + \frac{s^{n+1}}{q^{\binom{n+1}{2}+n+1}}\mathcal{L}_{p,q}\{f(t)\}\left(\frac{s}{q^{n+1}}\right) \\
&\quad - \sum_{k=0}^{n-1}\frac{s^{n-k}}{q^{\binom{n-k}{2}+n-k}}\left(D_{p,q}^{k}f\right)(0) \\
&= \frac{s^{n+1}}{q^{\binom{n+2}{2}}}\mathcal{L}_{p,q}\{f(t)\}\left(\frac{s}{q^{n+1}}\right) - \sum_{k=0}^{n}\frac{s^{n-k}}{q^{\binom{n-k+1}{2}}}\left(D_{p,q}^{k}f\right)(0).
\end{aligned}$$

The relation holds therefore true for each integer $n \geq 1$. $\square$

We now have another possibility to compute $\mathcal{L}_{p,q}\{t^n\}(s)$ using (10.17). Of course, applying (10.17) to $f(t) = t^n$, we have

$$\mathcal{L}_{p,q}\{D_{p,q}^{n}t^n\}(s) = \frac{s^n}{q^{\binom{n+1}{2}}}\mathcal{L}_{p,q}\{t^n\}\left(\frac{s}{q^n}\right).$$

Taking care that $D_{p,q}^{n}t^n = [n]_{p,q}!$, it follows that

$$\mathcal{L}_{p,q}\{t^n\}\left(\frac{s}{q^n}\right) = q^{\binom{n+1}{2}}\frac{[n]_{p,q}!}{s^n}\mathcal{L}_{p,q}\{1\}(s) = \frac{[n]_{p,q}!q^{\binom{n+1}{2}}}{s^{n+1}}.$$

Replacing s by sq^n, it follows that

$$\mathcal{L}_{p,q}\{t^n\}(s) = \frac{[n]_{p,q}!q^{\binom{n+1}{2}}}{s^{n+1}q^{n(n+1)}} = \frac{[n]_{p,q}!}{q^{\binom{n+1}{2}}s^{n+1}}.$$

10.3 Application to the resolution of some (p,q)-difference equations

As Laplace transform and Z-transform are largely applied in solving differential and difference equations, respectively, and the q-Laplace transforms are applied to solve q-difference equations, the (p,q)-Laplace transforms are expected to play a similar role but now for (p,q)-difference equations. The idea lying behind is always the same. In this section, we show on few examples how the Laplace transforms introduced before can be used to solve some (p,q)-differential equations.

10.3.1 Application 1

Consider the problem of finding $f(t)$, where $f(t)$ satifies (p,q)-Cauchy problem

$$D_{p,q}f(t) + cf(pt) = 0, \quad f(0) = 1, \tag{10.18}$$

where c stands for a complex constant.

Applying the Laplace transform of the first kind to (10.18), we obtain

$$-f(0)+\frac{s}{p}L_{p,q}\{f(t)\}\left(\frac{s}{p}\right)+cL_{p,q}\{f(pt)\}(s)=0.$$

Next, using equation (10.5), and the initial condition $f(0)=0$, we get

$$-1+\frac{s}{p}L_{p,q}\{f(t)\}\left(\frac{s}{p}\right)+\frac{c}{p}L_{p,q}\{f(t)\}\left(\frac{s}{p}\right)=0.$$

Hence,

$$L_{p,q}\{f(t)\}\left(\frac{s}{p}\right)=\frac{p}{s+c},$$

and so

$$L_{p,q}\{f(t)\}(s)=\frac{p}{ps+c},$$

It follows that $f(t)=e_{p,q}(-ct)$.

10.3.2 Application 2

Now, consider the (p,q)-differential equation

$$D_{p,q}h(t)-\lambda h(pt)=e_{p,q}(\lambda qt),\quad h(0)=0. \tag{10.19}$$

Applying the (p,q)-Laplace transform of first kind to (10.19), it follows that

$$-h(0)+\frac{s}{p}L_{p,q}\{h(t)\}\left(\frac{s}{p}\right)-\frac{\lambda}{p}L_{p,q}\{h(t)\}\left(\frac{s}{p}\right)=\frac{p}{ps-\lambda q}.$$

Simplification gives

$$L_{p,q}\{h(t)\}\left(\frac{s}{p}\right)=\frac{p^2}{(s-\lambda)(ps-\lambda q)},$$

and finally, replacing s by ps, we have

$$L_{p,q}\{h(t)\}(s)=\frac{p^2}{(ps-\lambda)(p^2s-\lambda q)}.$$

So, clearly $h(t)=te_{p,q}(\lambda t)$.

10.3.3 Application 3

For the last example, we consider the classical (p,q)-oscillator

$$D^2_{p,q}f(t)+\omega^2 f(p^2t)=0,\quad D_{p,q}f(0)=A,\ \ f(0)=B. \tag{10.20}$$

Applying the (p,q)-Laplace transform of the first kind to (10.20), it follows that

$$-A-\frac{Bs}{p}+\frac{s^2}{p^3}L_{p,q}\{f(t)\}\left(\frac{s}{p^2}\right)+\frac{\omega^2}{p^2}L_{p,q}\{f(t)\}\left(\frac{s}{p^2}\right)=0.$$

By an easy simplification, we get

$$L_{p,q}\{f(t)\}\left(\frac{s}{p^2}\right)=\frac{Bs+Ap}{p}\times\frac{p^3}{s^2+p\omega^2}.$$

It happens that

$$L_{p,q}\{f(t)\}(s) = \frac{Bp^2 s}{(ps)^2 + \left(\frac{\omega}{\sqrt{p}}\right)^2} + A\frac{\sqrt{p}}{\omega}\frac{p\dfrac{\omega}{\sqrt{p}}}{(ps)^2 + \left(\frac{\omega}{\sqrt{p}}\right)^2}.$$

Hence, the solutions of the (p,q)-oscillators are

$$f(t) = B\cos_{p,q}\left(\frac{\omega}{\sqrt{p}}t\right) + A\frac{\sqrt{p}}{\omega}\sin_{p,q}\left(\frac{\omega}{\sqrt{p}}t\right).$$

10.4 Double (p,q)-Laplace transform

The double Laplace transform of a function $f(x,y)$ of two variables was first introduced in 1939 by Berstein in his dissertation [20] (later pubished as an article [21]) as

$$\mathcal{L}_2(f(x,y))(r,s) = \int_0^\infty\int_0^\infty f(x,y)e^{-(rx+sy)}dxdy. \tag{10.21}$$

where x and y are two positive numbers, r and s are complex numbers. Very recently, several interesting properties and applications of the double Laplace transform to functional, integral and partial differential equations have been studied in [34].

In this section, we introduce three kinds of double (p,q)-Laplace transforms and prove their main properties. Next, applications are done to solve some partial (p,q)-differential equations. The double (p,q)-Laplace transform introduced here are clearly generalizations of the double Laplace transform given in [20] and the double q-Laplace transform studied in [72].

10.4.1 (p,q)-addition, (p,q)-subtraction, (p,q)-coaddition, (p,q)-cosubtraction

In the following definition, we generalize the notion of q-addition introduced by Jackson and studied later by Ward and Al-Salam (see [8, 28] for more details). When $p = 1$, our (p,q)-addition reduces to the q-addition defined by Euler and recalled in [80].

Definition 140 (Njionou and Duran [71]). *Let x and y be two complex numbers.*

1. *The (p,q)-addition of x and y which we denote by $x \oplus_{p,q} y$ is defined by*

$$(x \oplus_{p,q} y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-n)} x^k y^{n-k}. \tag{10.22}$$

2. *The (p,q)-subtraction of x and y which we denote by $x \ominus_{p,q} y$ is defined by*

$$(x \ominus_{p,q} y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^{n-k} p^{k(k-n)} x^k y^{n-k}. \tag{10.23}$$

Proposition 141 (Njionou and Duran [71]). *The following relations hold true for any $x, y \in \mathbb{R}$:*

$$\mathrm{e}_{p,q}(x)\mathrm{e}_{p,q}(y) = \mathrm{e}_{p,q}(x \oplus_{p,q} y), \tag{10.24}$$

$$\mathrm{e}_{p,q}(x)\mathrm{e}_{p,q}(-y) = \mathrm{e}_{p,q}(x \ominus_{p,q} y). \tag{10.25}$$

Proof. By the definition of the (p,q)-addition and the Cauchy product we can readily see that

$$
\begin{aligned}
e_{p,q}(x)e_{p,q}(y) &= \sum_{k=0}^{\infty}\frac{p^{\binom{k}{2}}x^k}{[k]_{p,q}!}\sum_{\ell=0}^{\infty}\frac{p^{\binom{\ell}{2}}y^\ell}{[\ell]_{p,q}!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{p^{\binom{k}{2}+\binom{n-k}{2}}x^k y^{n-k}}{[k]_{p,q}![n-k]_{p,q}!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{k(k-n)}x^k y^{n-k}\right)\frac{p^{\binom{n}{2}}}{[n]_{p,q}!} = e_{p,q}(x\oplus_{p,q}y).
\end{aligned}
$$

The second assertion is proved in the same way. □

Definition 142 (Njionou and Duran [71])**.** *Let x and y be two complex numbers.*

1. *The (p,q)-coaddition of x and y which we denote by $x\boxplus_{p,q}y$ is defined by*

$$(x\boxplus_{p,q}y)^n = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}q^{k(k-n)}x^k y^{n-k}. \tag{10.26}$$

2. *The (p,q)-cosubtraction of x and y which we denote by $x\boxminus_{p,q}y$ is defined by*

$$(x\boxminus_{p,q}y)^n = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}q^{k(k-n)}x^k(-y)^{n-k}. \tag{10.27}$$

Proposition 143 (Njionou and Duran [71])**.** *The following relations hold true for any $x,y\in\mathbb{R}$:*

$$E_{p,q}(x)E_{p,q}(y) = E_{p,q}(x\boxplus_{p,q}y), \tag{10.28}$$
$$E_{p,q}(x)E_{p,q}(-y) = E_{p,q}(x\boxminus_{p,q}y). \tag{10.29}$$

Proof. The proof is similar to the proof of Proposition 141. □

10.4.2 The double (p,q)-Laplace transform of the first kind

We define the double (p,q)-Laplace transform of the first kind of a function f as

$$\mathcal{L}^{(1)}_{2,p,q}[f(x,y)](r,s) = \int_0^{\infty}\!\!\int_0^{\infty} f(x,y)E_{p,q}(-qrx)E_{p,q}(-qsy)d_{p,q}x d_{p,q}y, \quad (r,s>0). \tag{10.30}$$

Note that if $f(x,y)=g(x)h(y)$, then

$$\mathcal{L}^{(1)}_{2,p,q}[f(x,y)](r,s) = L_{p,q}\{g(x)\}(r)L_{p,q}\{h(y)\}(s). \tag{10.31}$$

in particular, if $h(y)=1$, or $g(x)=1$, then (10.31) reads

$$\mathcal{L}^{(1)}_{2,p,q}[f(y)](r,s) = L_{p,q}\{1\}(r)L_{p,q}\{f(y)\}(s) = \frac{1}{r}L_{p,q}\{f(y)\}(s). \tag{10.32}$$

and

$$\mathcal{L}^{(1)}_{2,p,q}[f(x)](r,s) = L_{p,q}\{g(x)\}(r)L_{p,q}\{1\}(s) = \frac{1}{s}L_{p,q}\{g(x)\}(r). \tag{10.33}$$

Proposition 144. *For any two complex numbers α and β, we have*

$$\mathcal{L}^{(1)}_{2,p,q}\left\{\alpha f(x,y) + \beta g(x,y)\right\} = \alpha\mathcal{L}^{(1)}_{2,p,q}\{f(x,y)\} + \beta\mathcal{L}^{(1)}_{2,p,q}\{g(x,y)\}.$$

We recall the following important relation [68],

$$\int_0^\infty f(\alpha x) d_{p,q}x = \frac{1}{\alpha}\int_0^\infty f(x) d_{p,q}x, \tag{10.34}$$

where α is a non-zero complex number and f is a one variable function.
Now we state the scaling theorem for $\mathcal{L}^{(1)}_{2,p,q}$.

Theorem 145. *Let a and b two non-zero complex numbers, f a two variable function, then the following formula applies*

$$\mathcal{L}^{(1)}_{2,p,q}\{f(ax, by)\}(r,s) = \frac{1}{ab}\mathcal{L}^{(1)}_{2,p,q}\{f(x,y)\}\left(\frac{r}{a}, \frac{s}{b}\right). \tag{10.35}$$

Proof. Using relation (10.34), we have

$$\begin{aligned}
\mathcal{L}^{(1)}_{2,p,q}\{f(ax, by)\}(r,s) &= \int_0^\infty\int_0^\infty f(ax,by)E_{p,q}(-qrx)E_{p,q}(-qsy)d_{p,q}xd_{p,q}y \\
&= \int_0^\infty \left(\int_0^\infty f(ax,by)E_{p,q}(-qrx)d_{p,q}x\right) E_{p,q}(-qsy)d_{p,q}y \\
&= \frac{1}{a}\int_0^\infty \left(\int_0^\infty f(x,by)E_{p,q}\left(-qx\frac{r}{a}\right) d_{p,q}x\right) E_{p,q}(-qsy)d_{p,q}y \\
&= \frac{1}{a}\int_0^\infty \left(\int_0^\infty f(x,by)E_{p,q}(-qsy)d_{p,q}y\right) E_{p,q}\left(-qx\frac{r}{a}\right) d_{p,q}x \\
&= \frac{1}{ab}\int_0^\infty \left(\int_0^\infty f(x,y)E_{p,q}\left(-qy\frac{s}{b}\right) d_{p,q}y\right) E_{p,q}\left(-qx\frac{r}{a}\right) d_{p,q}x \\
&= \frac{1}{ab}\int_0^\infty\int_0^\infty f(x,y)E_{p,q}\left(-qx\frac{r}{a}\right) E_{p,q}\left(-qy\frac{s}{b}\right) d_{p,q}xd_{p,q}y.
\end{aligned}$$

and the proof of the theorem is completed. □

Theorem 146. *For $\alpha > -1$, $\beta > -1$, we have the following*

$$\mathcal{L}^{(1)}_{2,p,q}\{x^\alpha y^\beta\}(r,s) = \frac{\Gamma_{p,q}(\alpha+1)}{p^{\frac{\alpha(\alpha+1)}{2}} r^{\alpha+1}} \frac{\Gamma_{p,q}(\beta+1)}{p^{\frac{\beta(\beta+1)}{2}} s^{\beta+1}}. \tag{10.36}$$

In particular, for $\alpha = n \in \mathbb{N}$ and $\beta = m \in \mathbb{N}$, we get

$$\mathcal{L}^{(1)}_{2,p,q}\{x^n y^m\}(r,s) = \frac{[n]_{p,q}![m]_{p,q}!}{p^{\binom{n+1}{2}+\binom{m+1}{2}} r^{n+1}s^{m+1}}. \tag{10.37}$$

Proof. The proof follows from (10.6) and the obvious equation

$$\mathcal{L}^{(1)}_{2,p,q}\{x^\alpha y^\beta\}(r,s) = L_{p,q}\{x^\alpha\}(r) \times L_{p,q}\{y^\beta\}(s).$$

□

Let us take for example $\alpha = -\dfrac{1}{2}$ and $\beta = \dfrac{1}{2}$. Then we see that

$$\begin{aligned}\mathcal{L}^{(1)}_{2,p,q}\left(\sqrt{\frac{y}{x}}\right)(r,s) &= L_{p,q}\{x^{-\frac{1}{2}}\}(r) \times L_{p,q}\{y^{\frac{1}{2}}\}(s)\\ &= \Gamma_{p,q}\left(\frac{1}{2}\right)\Gamma_{p,q}\left(\frac{3}{2}\right)\frac{p^{\frac{1}{4}}}{s\sqrt{rs}},\end{aligned}$$

and for $\alpha = -\dfrac{1}{2}$ and $\beta = -\dfrac{1}{2}$ we have

$$\begin{aligned}\mathcal{L}^{(1)}_{2,p,q}\left(\frac{1}{\sqrt{xy}}\right)(r,s) &= L_{p,q}\{x^{-\frac{1}{2}}\}(r) \times L_{p,q}\{y^{-\frac{1}{2}}\}(s)\\ &= \left[\Gamma_{p,q}\left(\frac{1}{2}\right)\right]^2 \frac{p^{-\frac{1}{4}}}{\sqrt{rs}}.\end{aligned}$$

Theorem 147. *Let a and b be two complex numbers, then*

$$\mathcal{L}^{(1)}_{2,p,q}\{e_{p,q}(ax \oplus_{p,q} by)\}(r,s) = \frac{p^2}{(pr-a)(ps-b)}, \quad r > \mathrm{Re}(a/p), \;\; s > \mathrm{Re}(b/p). \tag{10.38}$$

Proof. Combining (10.8), (10.24) and (10.31) gives the result. □

Proposition 148. *The following formulas apply*

$$\mathcal{L}^{(1)}_{2,p,q}\{\cos_{p,q}(ax \oplus_{p,q} by)\}(r,s) = \frac{p^2(p^2rs-ab)}{((pr)^2+a^2)((ps)^2+b^2)} \tag{10.39}$$

$$\mathcal{L}^{(1)}_{2,p,q}\{\sin_{p,q}(ax \oplus_{p,q} by)\}(r,s) = \frac{p^3(as+br)}{((pr)^2+a^2)((ps)^2+b^2)}. \tag{10.40}$$

Proof. We remark first that for any complex number λ, we have

$$e_{p,q}(\lambda(x \oplus_{p,q} y)) = e_{p,q}(\lambda x \oplus_{p,q} \lambda y),$$

to write

$$\begin{aligned}\cos_{p,q}(ax \oplus_{p,q} by) &= \frac{1}{2}\left(e_{p,q}(i(ax \oplus_{p,q} by)) + e_{p,q}(-i(ax \oplus_{p,q} by))\right)\\ &= \frac{1}{2}\left(e_{p,q}((aix \oplus_{p,q} biy)) + e_{p,q}((-aix \oplus_{p,q} -biy))\right)\\ \sin_{p,q}(ax \oplus_{p,q} by) &= \frac{1}{2i}\left(e_{p,q}(i(ax \oplus_{p,q} by)) - e_{p,q}(-i(ax \oplus_{p,q} by))\right)\\ &= \frac{1}{2i}\left(e_{p,q}((aix \oplus_{p,q} biy)) - e_{p,q}((-aix \oplus_{p,q} -biy))\right).\end{aligned}$$

Hence, using the linearity of $\mathcal{L}^{(1)}_{2,p,q}$, and equation (10.38), it follows that

$$\begin{aligned}\mathcal{L}^{(1)}_{2,p,q}\{\cos_{p,q}(ax \oplus_{p,q} by)\}(r,s) &= \frac{1}{2}\left\{\frac{p^2}{(pr-ai)(ps-bi)} + \frac{p^2}{(pr+ai)(ps+ib)}\right\}\\ &= \frac{p^3(rs-ab)}{((pr)^2+a^2)((ps)^2+b^2)}.\end{aligned}$$

This proves (10.39). (10.40) follows in the same way. □

Proposition 149. *The following formulas apply*

$$\mathcal{L}^{(1)}_{2,p,q}\{cosh_{p,q}(ax\oplus_{p,q}by)\}(r,s)=\frac{p^2(p^2rs+ab)}{((pr)^2-a^2)((ps)^2-b^2)}$$
$$\mathcal{L}^{(1)}_{2,p,q}\{sinh_{p,q}(ax\oplus_{p,q}by)\}(r,s)=\frac{p^3(as+br)}{((pr)^2-a^2)((ps)^2-b^2)}.$$

Proof. The proof is similar to the proof of Proposition 148. □

Theorem 150. *Let f be a one variable function that has a q-Laplace transform. Assume that f has the q-Taylor expansion*

$$f(x)=\sum_{n=0}^{\infty}a_n\frac{p^{\binom{n}{2}}x^n}{[n]_{p,q}!},$$

then the following relation holds:

$$\begin{aligned}&\mathcal{L}^{(1)}_{2,p,q}\left[f(\alpha x\oplus_{p,q}\beta y)\right](r,s)\\&=\frac{1}{\alpha s-\beta r}\left(L_{p,q}\left[f(x)\right]\left(\frac{r}{\alpha}\right)-L_{p,q}\left[f(x)\right]\left(\frac{s}{\beta}\right)\right).\end{aligned}\tag{10.41}$$

Proof. We have the following

$$\begin{aligned}f(\alpha x\oplus_{p,q}\beta y)&=\sum_{n=0}^{\infty}a_n\frac{p^{\binom{n}{2}}(\alpha x\oplus_{p,q}\beta y)^n}{[n]_{p,q}!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{k(k-n)}(\alpha x)^k(\beta y)^{n-k}\right)\frac{a_np^{\binom{n}{2}}}{[n]_{p,q}!}.\end{aligned}$$

Hence it follows that

$$\begin{aligned}\mathcal{L}^{(1)}_{2,p,q}\left[f(x\oplus_q y)\right](r,s)&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}\frac{\alpha^k[k]_q!\beta^{n-k}[n-k]_q!}{r^{k+1}s^{n+1-k}}\right)\frac{a_n}{[n]_q!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\alpha^k\beta^{n-k}a_n}{r^{k+1}s^{n+1-k}}\\&=\frac{1}{\alpha s-\beta r}\left(\sum_{n=0}^{\infty}a_n\left(\frac{\alpha}{r}\right)^{n+1}-\sum_{n=0}^{\infty}a_n\left(\frac{\beta}{r}\right)^{n+1}\right)\\&=\frac{1}{\alpha s-\beta r}\left(L_q\left[f(x)\right]\left(\frac{r}{\alpha}\right)-L_q\left[f(x)\right]\left(\frac{s}{\beta}\right)\right).\end{aligned}$$

This ends the proof of the theorem. □

10.4.3 The double (p,q)-Laplace transform of the second kind

We define the double (p,q)-Laplace transform of the second kind of a function f as follows

$$\mathcal{L}^{(2)}_{2,p,q}\{f(x,y)\}(r,s)=\int_0^{\infty}f(t)e_{p,q}(-prx)e_{p,q}(-psy)d_{p,q}xd_{p,q}y,\ (r,s>0).\tag{10.42}$$

Note that if $f(x,y)=g(x)h(y)$, then

$$\mathcal{L}^{(2)}_{2,p,q}[f(x,y)](r,s)=\mathcal{L}_{p,q}\{g(x)\}(r)\mathcal{L}_{p,q}\{h(y)\}(s).\tag{10.43}$$

in particular, if $h(y) = 1$, or $g(x) = 1$, then (10.31) reads

$$\mathcal{L}^{(2)}_{2,p,q}[f(y)](r,s) = \mathcal{L}_{p,q}\{1\}(r)\mathcal{L}_{p,q}\{f(y)\}(s) = \frac{1}{r}\mathcal{L}_{p,q}\{f(y)\}(s). \tag{10.44}$$

and

$$\mathcal{L}^{(2)}_{2,p,q}[f(x)](r,s) = \mathcal{L}_{p,q}\{g(x)\}(r)L_{p,q}\{1\}(s) = \frac{1}{s}\mathcal{L}_{p,q}\{g(x)\}(r). \tag{10.45}$$

Of course, by definition, $\mathcal{L}^{(2)}_{2,p,q}$ is linear.

Proposition 151. *For any two complex numbers α and β, we have*

$$\mathcal{L}^{(2)}_{2,p,q}\left\{\alpha f(x,y) + \beta g(x,y)\right\} = \alpha\mathcal{L}^{(2)}_{2,p,q}\{f(x,y)\} + \beta\mathcal{L}^{(2)}_{2,p,q}\{g(x,y)\}.$$

Now we state the scaling theorem for $\mathcal{L}^{(2)}_{2,p,q}$.

Theorem 152. *Let a and b two non zero-complex numbers, f a two variable function, then the following formula applies*

$$\mathcal{L}^{(2)}_{2,p,q}\{f(ax,by)\}(r,s) = \frac{1}{ab}\mathcal{L}^{(2)}_{2,p,q}\{f(x,y)\}\left(\frac{r}{a},\frac{s}{b}\right). \tag{10.46}$$

Proof. Using relation (10.34), we get

$$\begin{aligned}
\mathcal{L}^{(2)}_{2,p,q}\{f(ax,by)\}(r,s) &= \int_0^\infty\!\!\int_0^\infty f(ax,by)e_{p,q}(-prx)e_{p,q}(-psy)d_{p,q}xd_{p,q}y\\
&= \int_0^\infty\left(\int_0^\infty f(ax,by)e_{p,q}(-prx)d_{p,q}x\right)e_{p,q}(-psy)d_{p,q}y\\
&= \frac{1}{a}\int_0^\infty\left(\int_0^\infty f(x,by)e_{p,q}\left(-px\frac{r}{a}\right)d_{p,q}x\right)e_{p,q}(-psy)d_{p,q}y\\
&= \frac{1}{a}\int_0^\infty\left(\int_0^\infty f(x,by)e_{p,q}(-psy)d_{p,q}y\right)e_{p,q}\left(-px\frac{r}{a}\right)d_{p,q}x\\
&= \frac{1}{ab}\int_0^\infty\left(\int_0^\infty f(x,y)e_{p,q}\left(-py\frac{s}{b}\right)d_{p,q}y\right)e_{p,q}\left(-px\frac{r}{a}\right)d_{p,q}x\\
&= \frac{1}{ab}\int_0^\infty\!\!\int_0^\infty f(x,y)e_{p,q}\left(-px\frac{r}{a}\right)e_{p,q}\left(-py\frac{s}{b}\right)d_{p,q}xd_{p,q}y.
\end{aligned}$$

and the proof of the theorem is completed. □

Theorem 153. *For $\alpha > -1$, $\beta > -1$, we have the following*

$$\mathcal{L}^{(2)}_{2,p,q}\{x^\alpha y^\beta\}(r,s) = \frac{\gamma_{p,q}(\alpha+1)}{q^{\frac{\alpha(\alpha+1)}{2}}r^{\alpha+1}}\frac{\gamma_{p,q}(\beta+1)}{q^{\frac{\beta(\beta+1)}{2}}s^{\beta+1}}. \tag{10.47}$$

In particular, for $\alpha = n \in \mathbb{N}$ and $\beta = m \in \mathbb{N}$, we get

$$\mathcal{L}^{(2)}_{2,p,q}\{x^n y^m\}(r,s) = \frac{[n]_{p,q}![m]_{p,q}!}{q^{\binom{n+1}{2}+\binom{m+1}{2}}r^{n+1}s^{m+1}}. \tag{10.48}$$

Proof. The proof follows from (10.13) and the obvious equation

$$\mathcal{L}^{(1)}_{2,p,q}\{x^\alpha y^\beta\}(r,s) = \mathcal{L}_{p,q}\{x^\alpha\}(r)\times\mathcal{L}_{p,q}\{y^\beta\}(s).$$

□

Theorem 154. *Let a and b two complex numbers, then*

$$\mathcal{L}^{(1)}_{2,p,q}\{E_{p,q}(ax \boxplus_{p,q} by)\}(r,s) = \frac{q^2}{(qr-a)(qs-b)}, \quad r > \mathrm{Re}(a/p), \ \ s > \mathrm{Re}(b/p). \tag{10.49}$$

Proof. This result is obtained using equations (10.28), (10.45) and the fact that:

$$\mathcal{L}_{p,q}(E_{p,q}(ax))(s) = \frac{q}{qs-a}. \tag{10.50}$$

□

Proposition 155. *The following formulas apply*

$$\mathcal{L}^{(2)}_{2,p,q}\{Cos_{p,q}(ax \boxplus_{p,q} by)\}(r,s) = \frac{q^2(q^2rs-ab)}{((qr)^2+a^2)((qs)^2+b^2)} \tag{10.51}$$

$$\mathcal{L}^{(2)}_{2,p,q}\{Sin_{p,q}(ax \boxplus_{p,q} by)\}(r,s) = \frac{q^3(as+br)}{((qr)^2+a^2)((qs)^2+b^2)}. \tag{10.52}$$

Proof. We remark first that for any complex number λ, we have

$$E_{p,q}(\lambda(x \boxplus_{p,q} y)) = E_{p,q}(\lambda x \boxplus_{p,q} \lambda y),$$

to write

$$\begin{aligned} Cos_{p,q}(ax \boxplus_{p,q} by) &= \frac{1}{2}\left(E_{p,q}(i(ax \boxplus_{p,q} by)) + E_{p,q}(-i(ax \boxplus_{p,q} by))\right) \\ &= \frac{1}{2}\left(E_{p,q}((aix \boxplus_{p,q} biy)) + E_{p,q}((-aix \boxplus_{p,q} -biy))\right), \\ Sin_{p,q}(ax \oplus_{p,q} by) &= \frac{1}{2i}\left(E_{p,q}(i(ax \boxplus_{p,q} by)) - E_{p,q}(-i(ax \boxplus_{p,q} by))\right) \\ &= \frac{1}{2i}\left(E_{p,q}((aix \boxplus_{p,q} biy)) - E_{p,q}((-aix \boxplus_{p,q} -biy))\right). \end{aligned}$$

Hence, using the linearity of $\mathcal{L}^{(2)}_{2,p,q}$, it follows that

$$\begin{aligned} \mathcal{L}^{(1)}_{2,p,q}\{cos_{p,q}(ax \oplus_{p,q} by)\}(r,s) &= \frac{1}{2}\left\{\frac{q^2}{(qr-ai)(qs-bi)} + \frac{q^2}{(qr+ai)(qs+ib)}\right\} \\ &= \frac{q^3(rs-ab)}{((qr)^2+a^2)((qs)^2+b^2)}. \end{aligned}$$

This proves (10.51). (10.52) follows in the same way. □

Proposition 156. *The following formulas apply*

$$\mathcal{L}^{(2)}_{2,p,q}\{Cosh_{p,q}(ax \boxplus_{p,q} by)\}(r,s) = \frac{q^2(p^2rs+ab)}{((qr)^2-a^2)((qs)^2-b^2)}$$

$$\mathcal{L}^{(2)}_{2,p,q}\{Sinh_{p,q}(ax \boxplus_{p,q} by)\}(r,s) = \frac{q^3(as+br)}{((qr)^2-a^2)((qs)^2-b^2)}.$$

Proof. The proof is similar to the proof of Proposition 155. □

10.4.4 The double (p,q)-Laplace transform of the third kind

We define the double (p,q)-Laplace transform of the first kind of a function f as follows

$$\mathcal{L}^{(3)}_{2,p,q}\{f(x,y)\}(r,s) = \int_0^\infty f(t)E_{p,q}(-qrx)e_{p,q}(-psy)d_{p,q}xd_{p,q}y,\ (r,s>0). \tag{10.53}$$

Proposition 157. *For any two complex numbers α and β, we have*

$$\mathcal{L}^{(3)}_{2,p,q}\{\alpha f(x,y)+\beta g(x,y)\} = \alpha\mathcal{L}^{(3)}_{2,p,q}\{f(x,y)\} + \beta\mathcal{L}^{(3)}_{2,p,q}\{g(x,y)\}.$$

Theorem 158. *Let a and b two non zero complex numbers, f a two variable function, then the following formula applies*

$$\mathcal{L}^{(3)}_{2,p,q}\{f(ax,by)\}(r,s) = \frac{1}{ab}\mathcal{L}^{(3)}_{2,p,q}\{f(x,y)\}\left(\frac{r}{a},\frac{s}{b}\right). \tag{10.54}$$

Proof. The proof is similar to the proof of (10.46). □

Theorem 159. *For $\alpha>-1$, $\beta>-1$, we have the following*

$$\mathcal{L}^{(3)}_{2,p,q}\{x^\alpha y^\beta\}(r,s) = \frac{\Gamma_{p,q}(\alpha+1)}{p^{\frac{\alpha(\alpha+1)}{2}}r^{\alpha+1}}\frac{\gamma_{p,q}(\beta+1)}{q^{\frac{\beta(\beta+1)}{2}}s^{\beta+1}}. \tag{10.55}$$

In particular, for $\alpha=n\in\mathbb{N}$ and $\beta=m\in\mathbb{N}$, we get

$$\mathcal{L}^{(3)}_{2,p,q}\{x^n y^m\}(r,s) = \frac{[n]_{p,q}![m]_{p,q}!}{p^{\binom{n+1}{2}}q^{\binom{m+1}{2}}r^{n+1}s^{m+1}}. \tag{10.56}$$

10.4.5 Some applications

We consider the following q-Cauchy's functional equation

$$f(x\oplus_{p,q}y) = f(x)+f(y), \tag{10.57}$$

where f is an unknown function.
We apply the double (p,q)-Laplace transform $\mathcal{L}^{(1)}_{2,p,q}$ to (10.57) combined with (10.41), (10.32) and (10.33), to get

$$\frac{1}{s-r}\left[L_{p,q}[f(x)](r) - L_{p,q}[f(y)](s)\right] = \frac{1}{s}L_{p,q}[f(x)](r) + \frac{1}{r}L_{p,q}[f(y)](s)$$

that is

$$L_{p,q}[f(x)](r)\left[\frac{1}{s-r}-\frac{1}{s}\right] = L_{p,q}[f(y)](s)\left[\frac{1}{s-r}+\frac{1}{r}\right].$$

Simplifying this equation, we obtain

$$r^2L_{p,q}[f(x)](r) = s^2L_{p,q}[f(y)](s),$$

where the left-hand side is a function of r alone and the right hand side is a function of s alone. This equation is true provided each side is equal to an arbitrary constant k so that

$$r^2L_{p,q}[f(x)](r) = k,$$

or

$$L_{p,q}[f(x)](r) = \frac{k}{r^2}.$$

This is the transform of $f(x) = kx$, hence the solution of the (p,q)-Cauchy functional equation (10.57) as

$$f(x) = kx, \tag{10.58}$$

where k is an arbitrary constant.

We now consider the following (p,q)-Cauchy-Abel's functional equation

$$f(x \oplus_{p,q} y) = f(x)f(y), \tag{10.59}$$

where f is an unknown function.
We apply the double (p,q)-Laplace transform $\mathcal{L}^{(1)}_{2,p,q}$ to (10.59) combined with (10.41) and (10.31) to get

$$\frac{1}{s-r}\left[L_{p,q}[f(x)](r) - L_{p,q}[f(y)](s)\right] = L_{p,q}[f(x)](r)L_{p,q}[f(y)](s)$$

that is

$$\frac{1 - rL_{p,q}[f(x)](r)}{L_{p,q}[f(x)](r)} = \frac{1 - sL_{p,q}[f(y)](s)}{L_{p,q}[f(y)](s)},$$

where the left hand side is a function of r alone and the right hand side is a function of s alone. This equation is true provided each side is equal to an arbitrary constant k so that

$$\frac{1 - rL_{p,q}[f(x)](r)}{L_{p,q}[f(x)](r)} = k,$$

or

$$L_{p,q}[f(x)](r) = \frac{1}{r+k}.$$

Identifying with the previous computed transforms, it follows that the solution of the q-Cauchy-Abel's functional equation (10.59) as

$$f(x) = \mathrm{e}_{p,q}(-\lambda x), \quad \lambda = \frac{k}{p}. \tag{10.60}$$

where k is an arbitrary constant.

Chapter 11

(p,q)-Appell Polynomials

Let $P_n(x)$, $n = 0, 1, 2, \ldots$ be a polynomial set, i.e. a sequence of polynomials with $P_n(x)$ of exact degre n. Assume further that

$$\frac{dP_n(x)}{dx} = P_n'(x) = nP_{n-1}(x) \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Such polynomial sets are called Appell sets and received considerable attention since P. Appell [13] introduced them in 1880.

A basic (q-)analogue of Appell sequences was first introduced by Sharma and Chak [81] and they called them q-harmonic. Later, Al-Salam [9] studied these families and referred to them it as q-Appell sets in analogy with ordinary Appell sets. Note that both Sharma and Al-Salam defined the so-called q-Appell sets as those sets $\{P_n(x)\}_{n=0}^{\infty}$ which satisfy

$$D_q P_n(x) = [n]_q P_{n-1}(x), \quad n = 0,\ 1,\ 2,\ 3,\ \ldots \tag{11.1}$$

where $[n]_q = (1-q^n)/(1-q)$. Note that when $q \to 1$, (11.1) reduces to

$$\frac{dP_n(x)}{dx} = nP_{n-1}(x),$$

so that we may think of q-Appell sets as a generalization of Appell sets. We call these polynomial sets *q-Appell sets of type I*. Al-Salam also introduced another q-analogue of Appell sets satisfying

$$D_q P_n(x) = [n]_q P_{n-1}(qx), \quad n = 0,\ 1,\ 2,\ 3,\ \ldots \tag{11.2}$$

Again (11.2) reduces to $\frac{d}{dx}P_n(x) = nP_{n-1}(x)$ as $q \to 1$ so that we may also think of these sets as another q-generalization of Appell sets. We call these polynomial sets *q-Appell sets of type II*.

The purpose of this chapter is to study the class of polynomial sequences $\{P_n(x)\}$ which satisfy

$$D_{p,q} P_n(x) = [n]_{p,q} P_{n-1}(px), \quad n = 0,\ 1,\ 2,\ 3,\ \ldots \tag{11.3}$$

with the assumption that $P_{-1}(px) = 0$. We note that when $p = 1$, (11.3) reduces to (11.1) and for $q = 1$, (11.3) reduces to (11.2) so that we may think of (p,q)-Appell sets as a generalization of both types of q-Appell sets.

11.1 Characterization of (p,q)-Appell polynomials

Definition 160 (Njionou [70]). *A polynomial sequence $\{f_n(x)\}_{n=0}^{\infty}$ is called a (p,q)-Appell polynomial sequence if and only if*

$$D_{p,q}f_{n+1}(x) = [n+1]_{p,q}f_n(px), \quad n \geq 0. \tag{11.4}$$

It is not difficult to see that the polynomial sequence $\{f_n(x)\}_{n=0}^{\infty}$ with $f_n(x) = (x \ominus a)_{p,q}^n$ is a (p,q)-Appell polynomial sequence since (see [69])

$$D_{p,q}(x \ominus a)_{p,q}^n = [n]_{p,q}(px \ominus a)_{p,q}^{n-1}, n \geq 1.$$

Next, we give several characterizations of (p,q)-Appell polynomial sequences.

Theorem 161 (Njionou [70]). *Let $\{f_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials. Then the following are all equivalent:*

1. *$\{f_n(x)\}_{n=0}^{\infty}$ is a (p,q)-Appell polynomial sequence.*
2. *There exists a sequence $(a_k)_{k\geq 0}$, independent of n, with $a_0 \neq 0$ and such that*

$$f_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} a_k x^{n-k}.$$

3. *$\{f_n(x)\}_{n=0}^{\infty}$ is generated by*

$$A(t)e_{p,q}(xt) = \sum_{n=0}^{\infty} f_n(x)\frac{t^n}{[n]_{p,q}!},$$

with the determining function

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_{p,q}!}. \tag{11.5}$$

4. *There exists a sequence $(a_k)_{k\geq 0}$, independent of n with $a_0 \neq 0$ and such that*

$$f_n(x) = \left(\sum_{k=0}^{\infty} \frac{a_k p^{\binom{n-k}{2}}}{[k]_{p,q}!} D_{p,q}^k\right) x^n.$$

Proof. First, we prove that $(1) \implies (2) \implies (3) \implies (1)$.

$(1) \implies (2)$. Since $\{f_n(x)\}_{n=0}^{\infty}$ is a polynomial set, it is possible to write

$$f_n(x) = \sum_{k=0}^{n} a_{n,k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} x^{n-k}, \quad n = 1, 2, \ldots, \tag{11.6}$$

where the coefficients $a_{n,k}$ depend on n and k and $a_{n,0} \neq 0$. We need to prove that these coefficients are independent of n. By applying the operator $D_{p,q}$ to each member of (11.6) and taking into account that $\{f_n(x)\}_{n=0}^{\infty}$ is a (p,q)-Appell polynomial set, we obtain

$$f_{n-1}(px) = \sum_{k=0}^{n-1} a_{n,k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\binom{n-1-k}{2}} (px)^{n-1-k}, \quad n = 1, 2, \ldots, \tag{11.7}$$

since $D_{p,q}x^0 = 0$. Shifting the index $n \to n+1$ in (11.7) and making the substitution $x \to xp^{-1}$, we get

$$f_n(x) = \sum_{k=0}^{n} a_{n+1,k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} x^{n-k}, \quad n = 0, 1, \ldots, \tag{11.8}$$

Comparing (11.6) and (11.8), we obtain $a_{n+1,k} = a_{n,k}$ for all k and n, and therefore $a_{n,k} = a_k$ is independent of n.

(2) $\implies$ (3). From (2), and the identity (5.4), it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_{p,q}!} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} a_k x^{n-k} \right) \frac{t^n}{[n]_{p,q}!} \\ &= \left(\sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} (xt)^n \right) \\ &= A(t) e_{p,q}(xt). \end{aligned}$$

(3) $\implies$ (1). Assume that $\{f_n(x)\}_{n=0}^{\infty}$ is generated by

$$A(t) e_{p,q}(xt) = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_{p,q}!}.$$

Then, applying the operator $D_{p,q}$ (with respect to the variable x) to each side of this equation, we get

$$tA(t) e_{p,q}(pxt) = \sum_{n=0}^{\infty} D_{p,q} f_n(x) \frac{t^n}{[n]_{p,q}!}.$$

Moreover,

$$tA(t) e_{p,q}(pxt) = \sum_{n=0}^{\infty} f_n(px) \frac{t^{n+1}}{[n]_{p,q}!} = \sum_{n=0}^{\infty} [n]_{p,q} f_{n-1}(px) \frac{t^n}{[n]_{p,q}!}.$$

By comparing the coefficients of t^n, we obtain (1).

Next, (2) $\iff$ (4) is obvious since $D_{p,q}^k t^n = 0$ for $k > n$. This ends the proof of the theorem. □

11.2 Algebraic structure

We denote a given polynomial set $\{f_n(x)\}_{n=0}^{\infty}$ by a single symbol f and refer to $f_n(x)$ as the n-th component of f. We define (as done in [13, 82]) on the set $\mathcal{P}$ of all polynomials sequences the following three operations $+$, $\cdot$ and $*$. The first one is given by the rule that $f + g$ is the polynomial sequence whose *nth* component is $f_n(x) + g_n(x)$ provided that the degree of $f_n(x) + g_n(x)$ is exactly n. On the other hand, if f and g are two sets whose *nth* components are, respectively,

$$f_n(x) = \sum_{k=0}^{n} \alpha(n,k) x^k, \quad g_n(x) = \sum_{k=0}^{n} \beta(n,k) x^k,$$

then $f * g$ is the polynomial set whose nth component is given by

$$(f * g)_n(x) = \sum_{k=0}^{n} \alpha(n,k) p^{-\binom{k}{2}} g_k(x).$$

If λ is a real or complex number, then $\lambda \cdot f$ is defined as the polynomial sequence whose nth component is $\lambda \cdot f_n(x)$. We obviously have

$$\begin{aligned} f + g = g + f \quad \text{for all} \quad f, g \in \mathcal{P}, \\ \lambda f * g = (f * \lambda g) = \lambda(f * g). \end{aligned}$$

Clearly, the operation $*$ is not commutative (see [82]). One commutative subclass is the set $\mathfrak{A}$ of all Appell polynomials (see [13]).

In what follows, $\mathfrak{A}(p,q)$ denotes the class of all (p,q)-Appell sets.

In $\mathfrak{A}(p,q)$ the identity element (with respect to $*$) is the (p,q)-Appell set $I = \left\{ p^{\binom{n}{2}} x^n \right\}$. Note that I has the determining function $A(t) = 1$. This is due to identity (5.3). The following theorem is easy to prove.

Theorem 162 (Njionou [70])**.** *Let f, g, $h \in \mathfrak{A}(p,q)$ with the determining functions $A(t)$, $B(t)$ and $C(t)$, respectively. Then*

1. $f + g \in \mathfrak{A}(p,q)$ *if* $A(0) + B(0) \neq 0$,
2. $f + g$ *belongs to the determining function* $A(t) + B(t)$,
3. $f + (g + h) = (f + g) + h$.

The next theorem is less obvious.

Theorem 163 (Njionou [70])**.** *If f, g, $h \in \mathfrak{A}(p,q)$ with the determining functions $A(t)$, $B(t)$ and $C(t)$, respectively, then*

1. $f * g \in \mathfrak{A}(p,q)$
2. $f * g = g * f$,
3. $f * g$ *belongs to the determining function* $A(t)B(t)$,
4. $f * (g * h) = (f * g) * h$.

Proof. It is enough to prove the first part of the theorem. The rest follows directly. According to Theorem 161, we may put

$$f_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} a_k x^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} a_{n-k} x^k$$

so that

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_{p,q}!}.$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} (f * g)_n(x) \frac{t^n}{[n]_{p,q}!} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_{n-k} g_k(x) \right) \frac{t^n}{[n]_{p,q}!} \\ &= \left(\sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} g_n(x) \frac{t^n}{[n]_{p,q}!} \right) \\ &= A(t)B(t) \mathrm{e}_{p,q}(xt). \end{aligned}$$

This ends the proof of the theorem. $\square$

Corollary 164 (Njionou [70])**.** *Let $f \in \mathfrak{A}(p,q)$, then f has an inverse with respect to $*$, i.e. there is a set $g \in \mathfrak{A}(p,q)$ such that*
$$f * g = g * f = I.$$

Indeed g belongs to the determining function $(A(t))^{-1}$ where $A(t)$ is the determining function for f.

In view of Corollary 164 we shall denote this element g by f^{-1}. We are further motivated by Theorem 163 and its corollary to define $f^0 = I, f^n = f * (f^{n-1})$ where n is a non-negative integer, and $f^{-n} = f^{-1} * (f^{-n+1})$. We note that we have proved that the system $(\mathfrak{A}(p,q), *)$ is a commutative group. In particular this leads to the fact that if
$$f * g = h$$
and if any two of the elements f, g, h are (p,q)-Appell then the third one is also (p,q)-Appell.

Proposition 165 (Njionou [70])**.** *If f is a (p,q)-Appell sequence with the determining function $A(t)$, and if we set*
$$A^{-1}(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{[n]_{p,q}!}$$
then
$$x^n = p^{-\binom{n}{2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} b_k f_{n-k}(x).$$

Proof. Since f is a (p,q)-Appell sequence, we have
$$\begin{aligned} \sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!} &= (A(t))^{-1} A(t) e_{p,q}(xt) \\ &= \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_{p,q}!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} b_k f_{n-k}(x) \right) \frac{t^n}{[n]_{p,q}!}. \end{aligned}$$
The result follows by comparing the coefficients of t^n. □

11.3 Some (p,q)-Appell polynomial sequences

In this section, we give four examples of (p,q)-Appell polynomial sequences and prove some of their main structure relations. The bivariate (p,q)-Bernoulli, the bivariate (p,q)-Euler and the bivariate (p,q)-Genocchi polynomials are introduced in [35, Duran et *al.*] and some of their relevant properties are given. Without any loss of generality, we will restrict ourselves to the case $y = 0$. Also, we introduce a new generalization of the (p,q)-Hermite polynomials.

11.3.1 The (p,q)-Bernoulli polynomials

The (p,q)-Bernoulli polynomials are (p,q)-Appell polynomials for the determining function $A(t) = \dfrac{t}{e_{p,q}(t) - 1}$. Thus, the (p,q)-Bernoulli polynomials are defined by the generating function
$$\frac{t}{e_{p,q}(t) - 1} e_{p,q}(xt) = \sum_{n=0}^{\infty} \mathcal{B}_n(x; p, q) \frac{t^n}{[n]_{p,q}!}.$$

Let us define the (p,q)-Bernoulli numbers $\mathcal{B}_{n,p,q}$ by the generating function

$$\frac{t}{e_{p,q}(t)-1} = \sum_{n=0}^{\infty} \mathcal{B}_{n,p,q}\frac{t^n}{[n]_{p,q}!}$$

so that

$$\mathcal{B}_n(0;p,q) = \mathcal{B}_{n,p,q}, \quad (n \geq 0).$$

Proposition 166. *The (p,q)-Bernoulli polynomials $\mathcal{B}_n(x;p,q)$ have the representation*

$$\mathcal{B}_n(x;p,q) = \sum_{n=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{B}_{k,p,q} x^{n-k}. \tag{11.9}$$

Proof. The proof follows from Theorem 161. □

11.3.2 The (p,q)-Euler polynomials

The (p,q)-Euler polynomials are (p,q)-Appell polynomials for the determining function $A(t) = \dfrac{2}{e_{p,q}(t)+1}$. Thus, the (p,q)-Euler polynomials are defined by the generating function

$$\frac{2}{e_{p,q}(t)+1}e_{p,q}(xt) = \sum_{n=0}^{\infty} \mathcal{E}_n(x;p,q)\frac{t^n}{[n]_{p,q}!}.$$

Let us define the (p,q)-Euler numbers $\mathcal{E}_{n,p,q}$ by the generating function

$$\frac{2}{e_{p,q}(t)+1} = \sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}\frac{t^n}{[n]_{p,q}!}$$

so that

$$\mathcal{E}_n(0;p,q) = \mathcal{E}_{n,p,q}, \quad (n \geq 0).$$

Proposition 167 (Duran et *al.* [35]). *The (p,q)-Euler polynomials $\mathcal{E}_n(x;p,q)$ have the representation*

$$\mathcal{E}_n(x;p,q) = \sum_{n=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{E}_{k,p,q} x^{n-k}. \tag{11.10}$$

Proof. The proof follows from Theorem 161. □

11.3.3 The (p,q)-Genocchi polynomials

The (p,q)-Genocchi polynomials are (p,q)-Appell polynomials for the determining function $A(t) = \dfrac{2t}{e_{p,q}(t)+1}$. Thus, the (p,q)-Genocchi polynomials are defined by the generating function

$$\frac{2t}{e_{p,q}(t)+1}e_{p,q}(xt) = \sum_{n=0}^{\infty} \mathcal{G}_n(x;p,q)\frac{t^n}{[n]_{p,q}!}.$$

Let us define the (p,q)-Genocchi numbers $\mathcal{G}_{n,p,q}$ by the generating function

$$\frac{2t}{e_{p,q}(t)+1} = \sum_{n=0}^{\infty} \mathcal{G}_{n,p,q}\frac{t^n}{[n]_{p,q}!}$$

so that

$$\mathcal{G}_n(0;p,q) = \mathcal{G}_{n,p,q}, \quad (n \geq 0).$$

Proposition 168 (Duran et *al.* [35], Njionou [70]). *The (p,q)-Genocchi polynomials $\mathcal{G}_n(x;p,q)$ have the representation*

$$\mathcal{G}_n(x;p,q) = \sum_{n=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{G}_{k,p,q} x^{n-k}. \tag{11.11}$$

Proof. The proof follows from Theorem 161. □

11.3.4 A second kind of (p,q)-Hermite polynomials

In this section we construct (p,q)-Hermite polynomials and give some of their properties. Also, we derive the three-term recurrence relation as well as the second-order (p,q)-differential equation satisfied by these polynomials.

We define (p,q)-Hermite polynomials by means of the generating function

$$\mathbf{F}_{p,q}(x,t) := F_{p,q}(t) e_{p,q}(xt) = \sum_{n=0}^{\infty} H_n(x;p,q) \frac{t^n}{[n]_{p,q}!}. \tag{11.12}$$

where

$$F_{p,q}(t) = \sum_{n=0}^{\infty} (-1)^n p^{n(n-1)} \frac{t^{2n}}{[2n]_{p,q}!!}, \quad \text{with} \quad [2n]_{p,q}!! = \prod_{k=1}^{n} [2k]_q, \quad [0]_{p,q}!! = 1. \tag{11.13}$$

It is clear that

$$\begin{aligned} \lim_{p,q\to 1} \mathbf{F}_{p,q}(x,t) &= e^{xt} \lim_{p,q\to 1} \sum_{n=0}^{\infty} (-1)^n p^{n(n-1)} \frac{t^{2n}}{[2n]_{p,q}!!} = e^{xt} \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)(2n-2)\cdots 2} \\ &= e^{xt} \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{2^n n!} = \exp\left(tx - \frac{t^2}{2}\right). \end{aligned}$$

Moreover,

$$D_{p,q}^{\{t\}} F_{p,q}(t) = \sum_{n=1}^{\infty} (-1)^n p^{n(n-1)} \frac{t^{2n-1}}{[2n-2]_{p,q}!!} = \sum_{n=0}^{\infty} (-1)^{n+1} p^{n(n-1)+2n} \frac{t^{2n+1}}{[2n]_{p,q}!!} = -tF_{p,q}(pt),$$

Hence

$$\frac{D_{p,q}^{\{t\}} F_{p,q}(t)}{F_{p,q}(pt)} = -t.$$

Theorem 169 (Njionou [70]). *The (p,q)-Hermite polynomials $H_n(x;p,q)$ have the following representation*

$$H_n(x,p,q) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k p^{\binom{n-2k}{2}+k(k-1)} [n]_{p,q}!}{[2k]_{p,q}!! [n-2k]_{p,q}!} x^{n-2k}.$$

Proof. Indeed, expanding the generating function $\mathbf{H}_{p,q}(x,t)$, we have

$$\begin{aligned} \mathbf{H}_{p,q}(x,t) &= \left(\sum_{k=0}^{\infty} (-1)^k p^{k(k-1)} \frac{t^{2k}}{[2k]_{p,q}!!}\right) \left(\sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!}\right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!} (-1)^k p^{k(k-1)} \frac{t^{2k}}{[2k]_{p,q}!!}. \end{aligned}$$

The result follows by using the series manipulation formula (7) of Lemma 11 in [75]. □

Theorem 170 (Njionou [70]). *The following linear homogeneous recurrence relation for the (p,q)-Hermite polynomials holds true*

$$H_{n+1}(px,p,q) = p^{n+1}xH_n(qx,p,q) - p^{n-1}[n]_{p,q}H_{n-1}(qx,p,q), \quad (n \geq 1).$$

Proof. Note that $D_{p,q}^{\{t\}}F_{p,q}(t) = -tF_{p,q}(pt)$. Hence

$$\begin{aligned}\sum_{n=0}^{\infty} H_{n+1}(x;p,q)\frac{t^n}{[n]_{p,q}!} &= D_{p,q}^{\{t\}}\mathbf{F}_{p,q}(x,t)\\ &= F_{p,q}(qt)D_{p,q}^{\{t\}}e_{p,q}(xt) + e_{p,q}(pxt)D_{p,q}^{\{t\}}F_{p,q}(t)\\ &= xF_{p,q}(qt)e_{p,q}(pxt) - tF_{p,q}(pt)e_{p,q}(pxt)\\ &= \sum_{n=0}^{\infty} xH_n(px/q;p,q)\frac{q^nt^n}{[n]_{p,q}!} - \sum_{n=1}^{\infty}[n]_{p,q}H_{n-1}(x;p,q)\frac{p^{n-1}t^n}{[n]_{p,q}!}\end{aligned}$$

The result follows by equating the coefficients of t^n on both sides and replacing x by qx. □

Theorem 171 (Njionou [70]). *The (p,q)-Hermite polynomials $H_n(x;p,q)$ satisfy the (p,q)-difference equation*

$$\mathcal{L}_p^{-2}D_{p,q}^2H_n(x;p,q) - p^2q^{-1}x\mathcal{L}_p^{-1}D_{p,q}H_n(x;p,q) + p^{2-n}[n]_{p,q}H_n(px/q) = 0. \tag{11.14}$$

Proof. The proof follows from Theorem 170. □

Note that as p and q tend to 1, Equation (11.14) reduces to the second order differential equation satisfied by the Hermite polynomials.

11.3.5 Two bivariate kinds of (p,q)-Bernoulli polynomials

Let $x, y \in \mathbb{R}$. It is well-known that the Taylor expansion of the two functions $e^{xt}\cos yt$ and $e^{xt}\sin yt$ are as follows [61]

$$e^{xt}\cos yt = \sum_{n=0}^{\infty} C_n(x,y)\frac{t^n}{n!}, \tag{11.15}$$

and

$$e^{xt}\sin yt = \sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!}, \tag{11.16}$$

where

$$C_n(x,y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{n}{2k}x^{n-2k}y^{2k}, \tag{11.17}$$

and

$$S_n(x,y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor}(-1)^k\binom{n}{2k+1}x^{n-2k-1}y^{2k+1}. \tag{11.18}$$

Here we introduce a (p,q)-extension of the two above polynomials $C_n(x,y)$ and $S_n(x,y)$ by the following generating functions:

$$e_{p,q}(xt)\cos_{p,q}(yt) = \sum_{k=0}^{\infty} C_{k,p,q}(x,y)\frac{t^k}{[k]_{p,q}!}, \tag{11.19}$$

and

$$e_{p,q}(xt)\sin_{p,q}(yt) = \sum_{k=0}^{\infty} S_{k,p,q}(x,y)\frac{t^k}{[k]_{p,q}!}, \tag{11.20}$$

Some particular cases are

$$C_{2n,p,q}(0,y) = (-1)^n p^{\binom{2n}{2}} y^{2n}, \quad C_{2n+1,p,q}(0,y) = 0$$

and

$$S_{2n,p,q}(0,y) = 0, \quad S_{2n+1,p,q}(0,y) = (-1)^n p^{\binom{2n+1}{2}} y^{2n+1}.$$

The following lemma will be useful in the derivation of several results.

Lemma 172 (Rainville [75]). *The following series manipulations hold*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} A(k,n-2k), \tag{11.21}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor (n-1)/2\rfloor} B(k,n-1-2k). \tag{11.22}$$

Theorem 173 (Njionou and Duran [71]). *The following representations hold*

$$C_{n,p,q}(x,y) = p^{\binom{n}{2}} \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} (-1)^k \begin{bmatrix} n \\ 2k \end{bmatrix}_{p,q} p^{2k(k-n)} x^{n-2k} y^{2k} \tag{11.23}$$

and

$$S_{n,p,q}(x,y) = p^{\binom{n-1}{2}} \sum_{k=0}^{\lfloor \frac{n-1}{2}\rfloor} (-1)^k \begin{bmatrix} n \\ 2k+1 \end{bmatrix}_{p,q} p^{4k^2-2kn} x^{n-2k-1} y^{2k+1} \tag{11.24}$$

Proof. By series manipulation procedure (11.21), we have

$$e_{p,q}(xt)\cos_{p,q}(yt) = \left(\sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!}(xt)^n\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!}(yt)^{2n}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} \frac{p^{\binom{n-2k}{2}}}{[n-2k]_{p,q}!}(xt)^{n-2k}\frac{(-1)^k p^{\binom{2k}{2}}}{[2k]_{p,q}!}(yt)^{2k}$$

$$= \sum_{n=0}^{\infty}\left(p^{\binom{n}{2}} \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} (-1)^k \begin{bmatrix} n \\ 2k \end{bmatrix}_{p,q} p^{2k(k-n)} x^{n-2k} y^{2k}\right)\frac{t^n}{[n]_{p,q}!},$$

which proves (11.19). The proof of (11.20) is similar by means of the series manipulation method (11.22).

□

Theorem 174 (Njionou and Duran [71]). *The following derivative rules are valid*

$$D_{p,q,x}C_{k,p,q}(x,y) = [k]_{p,q}C_{k-1,p,q}(px,y), \tag{11.25}$$

$$D_{p,q,y}C_{k,p,q}(x,y) = -[k]_{p,q}S_{k-1,p,q}(x,py), \tag{11.26}$$

$$D_{p,q,x}S_{k,p,q}(x,y) = [k]_{p,q}S_{k-1,p,q}(px,y), \tag{11.27}$$

$$D_{p,q,y}S_{k,p,q}(x,y) = [k]_{p,q}C_{k-1,p,q}(x,py). \tag{11.28}$$

Proof. Relation (11.19) yields

$$\sum_{n=1}^{\infty} D_{p,q,x}C_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} = te_{p,q}(pxt)\cos_{p,q}(yt) = \sum_{n=0}^{\infty} C_{n,p,q}(px,y)\frac{t^{n+1}}{[n]_{p,q}!}$$
$$= \sum_{n=1}^{\infty} C_{n-1,p,q}(px,y)\frac{t^n}{[n-1]_{p,q}!}$$
$$= \sum_{n=0}^{\infty} [n]_{p,q}C_{n-1,p,q}(px,y)\frac{t^n}{[n]_{p,q}!},$$

proving (11.25). The other equations (11.26), (11.27) and (11.28) can be similarly proved. □

Theorem 175 (Njionou and Duran [71]). *The following relations are valid*

$$C_{n,p,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} C_{k,p,q}(0,y)x^{n-k}, \tag{11.29}$$

$$S_{n,p,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} S_{k,p,q}(0,y)x^{n-k}. \tag{11.30}$$

Theorem 176 (Njionou and Duran [71]). *The following power representations hold*

$$(-1)^n p^{\binom{2n}{2}} y^{2n} = \sum_{k=0}^{2n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_{p,q} C_{2n-k,p,q}(x,y)x^k, \tag{11.31}$$

and

$$(-1)^n p^{\binom{2n+1}{2}} y^{2n+1} = \sum_{k=0}^{2n+1} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{p,q} S_{2n+1-k,p,q}(x,y)x^k. \tag{11.32}$$

Proof. Multiplying both sides of (11.19) by $E_{p,q}(-xt)$ and using (5.3), it follows that

$$\sum_{n=0}^{\infty} (-1)^n p^{\binom{2n}{2}} y^{2n} \frac{t^{2n}}{[n]_{p,q}!} = \left(\sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(-x)^n t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty} C_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} C_{n-k,p,q}(x,y)x^k\right)\frac{t^n}{[n]_{p,q}!},$$

which proves (11.31). The proof of (11.32) is similar. □

Theorem 177 (Njionou and Duran [71]). *The following connection formulas hold*

$$C_{2n+1,p,q}(x,y) = \sum_{k=0}^{2n} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n+1 \\ k+1 \end{bmatrix}_{p,q} C_{2n-k,p,q}(x,y)x^{k+1}, \tag{11.33}$$

and

$$S_{2n,p,q}(x,y) = \sum_{k=0}^{2n-1} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n \\ k+1 \end{bmatrix}_{p,q} S_{2n-k-1,p,q}(x,y)x^{k+1}. \tag{11.34}$$

Proof. From the relation

$$\sum_{n=0}^{\infty}(-1)^n p^{\binom{2n}{2}} y^{2n}\frac{t^{2n}}{[n]_{p,q}!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}C_{n-k,p,q}(x,y)x^k\right)\frac{t^n}{[n]_{p,q}!},$$

it follows that

$$\sum_{k=0}^{2n+1}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_{p,q}C_{2n+1-k,p,q}(x,y)x^k=0.$$

Hence (11.33) is proved. We prove (11.34) in the same way. □

We can now introduce two kinds of bivariate q-Bernoulli polynomials as

$$\frac{te_{p,q}(xt)}{e_{p,q}(t)-1}\cos_{p,q}(yt)=\sum_{n=0}^{\infty}B^{(c)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}, \tag{11.35}$$

and

$$\frac{te_{p,q}(xt)}{e_{p,q}(t)-1}\sin_{p,q}(yt)=\sum_{n=0}^{\infty}B^{(s)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}. \tag{11.36}$$

Upon setting $x=y=0$ for both polynomials in (11.35) and (11.36), we have $B^{(c)}_{n,p,q}(0,0)=B^{(s)}_{n,p,q}(0,0):=B_{n,p,q}$ which are called (p,q)-Bernoulli polynomials defined in [35].
When $y=0$ in (11.35) and (11.36), we get the usual (p,q)-Bernoulli polynomials, denoted by $B_{n,p,q}(x)$, see [35, 70].

Next, we give some basic properties of these polynomials.

Theorem 178 (Njionou and Duran [71]). *$B^{(c)}_{n,p,q}(x,y)$ and $B^{(s)}_{n,p,q}(x,y)$ can be represented in terms of (p,q)-Bernoulli numbers as follows*

$$B^{(c)}_{n,p,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}B_{k,p,q}C_{n-k,p,q}(x,y), \tag{11.37}$$

and

$$B^{(s)}_{n,p,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}B_{k,p,q}S_{n-k,p,q}(x,y), \tag{11.38}$$

Proof. Using the Cauchy product rule, we have

$$\begin{aligned}\sum_{n=0}^{\infty}B^{(c)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}&=\frac{t}{e_{p,q}(t)-1}e_{p,q}(xt)\cos_{p,q}(yt)\\&=\left(\sum_{n=0}^{\infty}B_n\frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}C_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}B_{k,p,q}C_{n-k,p,q}(x,y)\right)\frac{t^n}{[n]_{p,q}!},\end{aligned}$$

which proves (11.37). The proof of (11.38) is similar. □

We now state the following theorem.

Theorem 179 (Njionou and Duran [71]). *The following connection formulas are valid*

$$B^{(c)}_{n,p,q}(x,y) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \begin{bmatrix} n \\ 2k \end{bmatrix}_{p,q} B_{n-2k,p,q}(x) p^{\binom{2k}{2}} y^{2k}, \tag{11.39}$$

and

$$B^{(s)}_{n,p,q}(x,y) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \begin{bmatrix} n \\ 2k+1 \end{bmatrix}_{p,q} B_{n-1-2k,p,q}(x) p^{\binom{2k+1}{2}} y^{2k+1}. \tag{11.40}$$

Proof. The formula (11.39) follows from (11.21) since

$$\begin{aligned}
\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y) \frac{t^n}{[n]_q} &= \frac{t e_{p,q}(xt)}{e_{p,q}(t)-1} \cos_{p,q}(yt) \\
&= \left(\sum_{n=0}^{\infty} B_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} (yt)^{2n} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \begin{bmatrix} n \\ 2k \end{bmatrix}_{p,q} B_{n-2k,p,q}(x) p^{\binom{2k}{2}} y^{2k} \right) \frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

The proof of (11.40) is similar via (11.22). □

Theorem 180 (Njionou and Duran [71]). *The following connection formulas are valid*

$$C_{n,p,q}(x,y) = \sum_{k=0}^{n} \frac{p^{\binom{k+1}{2}}}{[k+1]_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B^{(c)}_{n,p,q}(x,y), \tag{11.41}$$

$$S_{n,p,q}(x,y) = \sum_{k=0}^{n} \frac{p^{\binom{k+1}{2}}}{[k+1]_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B^{(s)}_{n,p,q}(x,y). \tag{11.42}$$

Proof. From (11.35), we have

$$\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!} = \frac{t e_{p,q}(xt)}{e_{p,q}(t)-1} \cos_{p,q}(yt) = \frac{t}{e_{p,q}(t)-1} \sum_{n=0}^{\infty} C_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!}.$$

Hence

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!} &= \frac{e_{p,q}(t)-1}{t} \sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!} \\
&= \left(\sum_{n=0}^{\infty} \frac{p^{\binom{n+1}{2}}}{[n+1]_{p,q}} \frac{t^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{p^{\binom{k+1}{2}}}{[k+1]_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B^{(c)}_{n,p,q}(x,y) \right) \frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

Thus (11.41) follows. (11.42) is proved in a similar way. □

Proposition 181 (Njionou and Duran [71]). *For every $n \in \mathbb{N}$, the following identities hold*

$$B^{(c)}_{n,p,q}((1 \oplus_{p,q} x), y) - B^{(c)}_{n,p,q}(x,y) = [n]_{p,q} C_{n-1,p,q}(x,y), \tag{11.43}$$

$$B^{(s)}_{n,p,q}((1 \oplus_{p,q} x), y) - B^{(s)}_{n,p,q}(x,y) = [n]_{p,q} S_{n-1,p,q}(x,y). \tag{11.44}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}((1 \oplus_{p,q} x), y) \frac{t^n}{[n]_{p,q}!} &= \frac{te_{p,q}[(1 \oplus_{p,q} x)t]}{e_{p,q}(t)-1} \cos_{p,q}(yt) \\
&= \frac{te_{p,q}(xt)[e_{p,q}(t)-1+1]}{e_{p,q}(t)-1} \cos_{p,q}(yt) \\
&= te_{p,q}(xt)\cos_{p,q}(yt) + \frac{te_{p,q}(xt)}{e_{p,q}(t)-1}\cos_{p,q}(yt) \\
&= \sum_{n=0}^{\infty} C_{n,p,q}(x,y)\frac{t^{n+1}}{[n]_{p,q}!} + \sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!},
\end{aligned}$$

which proves (11.43). The identity (11.44) is proved similarly. □

Corollary 182 (Njionou and Duran [71]). *The following relations hold*

$$B^{(c)}_{2n+1,p,q}(1,y) - B^{(c)}_{2n+1,p,q}(0,y) = [2n+1]_{p,q}(-1)^n p^{\binom{2n}{2}} y^{2n},$$

$$B^{(s)}_{2n,p,q}(1,y) - B^{(s)}_{2n,p,q}(0,y) = [2n]_{p,q}(-1)^{n+1} p^{\binom{2n-1}{2}} y^{2n-1}.$$

Proof. If we replace n by $2n+1$ in (11.43), and x by 0, we obtain

$$B^{(c)}_{2n+1,p,q}(1,y) - B^{(c)}_{2n+1,p,q}(0,y) = [2n+1]_{p,q} C_{2n,p,q}(0,y).$$

The first relation is proved since from (11.23) we have $C_{2n,q}(0,y) = (-1)^n p^{\binom{2n}{2}} y^{2n}$. The second relation is proved similarly. □

Proposition 183 (Njionou and Duran [71]). *For every $n \in \mathbb{N}$, the following identities hold*

$$B^{(c)}_{n,p,q}((x \oplus_{p,q} z), y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{k,p,q}(x) C_{n-k,p,q}(y,z), \tag{11.45}$$

and

$$B^{(s)}_{n,p,q}((x \oplus_{p,q} z), y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{k,p,q}(x) S_{n-k,p,q}(y,z), \tag{11.46}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}((x \oplus_{p,q} z), y) \frac{t^n}{[n]_{p,q}!} &= \frac{te_{p,q}((x \oplus_{p,q} z)t)}{e_{p,q}(t)-1} \cos_{p,q}(yt) \\
&= \frac{te_{p,q}(xt)}{e_{p,q}(t)-1} \times e_{p,q}(zt)\cos_{p,q}(yt) \\
&= \left(\sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty} C_{n,p,q}(y,z)\frac{t^n}{[n]_{p,q}!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{k,p,q}(x) C_{n-k}(y,z)\right)\frac{t^n}{[n]_q!},
\end{aligned}$$

which proves (11.45). The proof of (11.46) is similar. □

Proposition 184 (Njionou and Duran [71]). *For every $n \in \mathbb{N}$, the following identities hold*

$$B^{(c)}_{n,p,q}((x \oplus_{p,q} z), y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} B^{(c)}_{k,p,q}(x,y) z^{n-k}, \tag{11.47}$$

and

$$B^{(s)}_{n,p,q}((x \oplus_{p,q} z), y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} B^{(s)}_{k,p,q}(x,y) z^{n-k}. \tag{11.48}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}((x \oplus_{p,q} z), y) \frac{t^n}{[n]_{p,q}!} &= \frac{t e_{p,q}((x \oplus_{p,q} z)t)}{e_{p,q}(t) - 1} \cos_{p,q}(yt) \\
&= \frac{t e_{p,q}(xt)}{e_{p,q}(t) - 1} \cos_{p,q}(yt) \times e_{p,q}(zt) \\
&= \left(\sum_{n=0}^{\infty} B^{(c)}_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} p^{\binom{n}{2}} z^n \frac{t^n}{[n]_{p,q}!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} B^{(c)}_{k,p,q}(x,y) z^{n-k} \right) \frac{t^n}{[n]_q!},
\end{aligned}$$

which proves (11.47). The proof of (11.48) is similar. □

Proposition 185 (Njionou and Duran [71]). *The following equations can be concluded*

$$\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} B^{(c)}_{k,p,q}(x,y) = [n+1]_{p,q} C_{n,p,q}(x,y), \tag{11.49}$$

$$\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} B^{(s)}_{k,p,q}(x,y) = [n+1]_{p,q} S_{n,p,q}(x,y). \tag{11.50}$$

Proof. From (11.47), we have

$$B^{(c)}_{n+1,p,q}((x \oplus_{p,q} 1), y) - B^{(c)}_{n+1,p,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} B^{(c)}_{k,p,q}(x,y).$$

Hence, by using (11.43), relation (11.49) is derived. The proof of (11.50) is concluded in a similar way. □

Corollary 186 (Njionou and Duran [71]). *Relations* (11.49) *and* (11.50) *imply that*

$$\begin{aligned}
&\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} B^{(c)}_{k,p,q}(0,y) \\
&\qquad = \begin{cases} (-1)^m [2m+1]_{p,q} p^{\binom{2m}{2}} y^{2m} & \text{if} \quad n = 2m \quad \text{is odd}, \\ 0 & \text{if} \quad n = 2m+1 \quad \text{is even}, \end{cases}
\end{aligned}$$

and

$$\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} B_{k,p,q}^{(s)}(0,y)$$
$$= \begin{cases} 0 & \text{if} \quad n = 2m \quad \text{is odd}, \\ (-1)^m [2m+2]_{p,q} p^{\binom{2m+1}{2}} y^{2m+1} & \text{if} \quad n = 2m+1 \quad \text{is even}. \end{cases}$$

Corollary 187 (Njionou and Duran [71]). *For every $n \in \mathbb{N}$, the following partial (p,q)-differential equations hold*

$$D_{p,q,x} B_{n,p,q}^{(c)}(x,y) = [n]_{p,q} B_{n-1,p,q}^{(c)}(px,y),$$
$$D_{p,q,y} B_{n,p,q}^{(c)}(x,y) = -[n]_{p,q} B_{n-1,p,q}^{(c)}(x,py),$$
$$D_{p,q,x} B_{n,p,q}^{(s)}(x,y) = [n]_{p,q} B_{n-1,p,q}^{(s)}(px,y),$$

and

$$D_{p,q,y} B_{n,q}^{(c)}(x,y) = [n]_{p,q} B_{n-1,p,q}^{(s)}(x,py). \tag{11.51}$$

Corollary 188 (Njionou and Duran [71]). *The following equations are valid*

$$\int_0^1 B_{2n,p,q}^{(c)}(px,y) d_{p,q}x = (-1)^n p^{\binom{2n}{2}} y^{2n},$$
$$\int_0^1 B_{2n+1,p,q}^{(s)}(px,y) d_{p,q}x = (-1)^{n+1} p^{\binom{2n+1}{2}} y^{2n+1},$$

which are proved by combining Proposition 187 and Corollary 182 using the definition of the (p,q)-integral.

Conclusion and Further Perspectives

In this work we have provided several tools for the two parameter quantum calculus. Going from the (p,q)-derivative in Chapter 2, we have introduced the so-called (p,q)-power basis and the related (p,q)-Taylor expansions in Chapter 3. These Taylor expansions have enabled us to prove connections between several (p,q)-power bases and also used in Chapter 4 to prove the (p,q)-Vandermonde identity. Chapters 5 and 6 introduce (p,q)-Exponential, (p,q)-Trigonometric functions and the (p,q)-integral with the associated fundamental theorem of (p,q)-calculus. These tools are used in Chapter 7 to define (p,q)-analogues of the Gamma and the Beta functions with the proof of their main properties. (p,q)-analogues of hypergeometric series are introduced in Chapter 8. In this chapter we provided several (p,q)-analogues of the very known identities and transformations, namely the (p,q)-Kummer sum, a (p,q)-analogue of Bailey's ${}_2F_2(-1)$, a (p,q)-analogue of Gauss's ${}_2F_1(-1)$ sum, the (p,q)-Saalschütz sum, the (p,q)-Jackson's transformation of the ${}_2\Phi_1$,... Chapter 9 deals with (p,q)-analogues of Sturm-Liouville problems. We provide the regularity conditions to obtain orthogonal polynomial solutions. Some structure relations for these polynomials are proved and some special examples are explained. In Chapter 10 we introduce (p,q)-analogues of the Laplace transform and provide several properties of these transforms with several applications in solving some functional equations, (p,q)-differential equations and (p,q)-partial differential equations. In the last chapter, (p,q)-analogues are introduced for a wide class of polynomials known as Appell polynomials. Of course, we provide important relations these polynomial fulfil and give some connections with some previous known special polynomials.

There are still many things to do. Since Appell polynomials are special cases of Sheffer polynomials, one may think of defining (p,q)-Sheffer polynomials and start studying their fundamental properties. We think of defining the (p,q)-Sheffer polynomials $s_n(x)$ by the generating functions

$$A(t)e_{p,q}(xB(t)) = \sum_{n=0}^{\infty} \frac{s_n(x)}{[n]_{p,q}!}t^n$$

where A and B are (formal) power series in t.
Also, several summation and transformation formulas are still to be stated. Concerning the (p,q)-Sturm Liouville problems, there are still many unsolved problems and we think that due to the very important applications of their solutions for the classical case, it should be a good idea to look forward into this direction. Our future work will then consist in completing missing informations and then provide new tools for application in numerical analysis, partial differential equations, quantum mechanics,...

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