

Numerical Solution of Viscoelastic Fluid-Structure-Diffusion Systems with Applications in Ophthalmology

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The research of fluid-structure interaction problems is a continuously growing field, especially regarding applications in medicine and biology. We present the coupling of a potentially viscoelastic fluid with multiple hyperelastic structures incorporating chemical processes in the arbitrary Lagrangian Eulerian framework. This monolithic formulation allows a robust numerical solution with Newton's method. The discretization is based on the backward Euler scheme for temporal discretization and the Galerkin finite element method for spatial discretization. This fluid-structure interaction problem is applied to ophthalmology in order to improve the medical treatment of retinal diseases. The physiological processes include the elastic response of various structures like the sclera, lens and iris coupled to the fluid-like vitreous which is modeled by a viscoelastic Burgers type model for the healthy case and by the Newtonian Navier-Stokes equations for the pathological case. Since most medical treatments are based on the injection of medicine we furthermore study the drug distribution, which is modeled by convection-diffusion-reaction equations, in the whole eye for healthy and non-healthy pathologies.

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1 Introduction

The goal of this work is the modeling and simulation of essential physiological processes in the human eye in order to improve the medical treatment of diseases like glaucoma and retinal diseases. We study the interaction of two elastic structures, namely the sclera and the lens with the fluid-like vitreous. This fluid-structure interaction problem is solved numerically using the arbitrary Lagrangian Eulerian (ALE) framework (see e.g. [1]). We perform numerical simulations similar to the recent experiment in [2] and additionally study the VEGF distribution in the vitreous.

2 Mathematical Models

The domain Ω is split into a fluid domain Ω_f and two solid domains Ω_{s_1} and Ω_{s_2} via $\Omega := \Omega_f \cup \Omega_{s_1} \cup \Omega_{s_2}$ (see Fig. 1). Conservation of momentum and mass in the fluid domain yields:

$$\rho_f \partial_t v_f + \rho_f (v_f \cdot \nabla) v_f - \operatorname{div} \sigma_f = \rho_f f \text{ in } \Omega_f, t \in I, \quad \operatorname{div} v_f = 0 \text{ in } \Omega_f, t \in I.$$

The liquefied vitreous is modeled by the incompressible Newtonian Navier-Stokes equations with

$$\sigma_f = -p_f I + 2\rho_f \nu_f D(v_f), \quad D(v_f) := \frac{1}{2}(\nabla v_f + \nabla v_f^T).$$

Here v_f is the velocity, p_f the pressure, ν_f the viscosity and ρ_f the density. Due to the viscoelastic behavior of the healthy vitreous we introduce additional tensor valued unknowns B_1 and B_2 and use the following Burgers type model [3]:

$$\sigma_f = -pI + 2\rho_f \nu_f D(v_f) + \mu_1(B_1 - I) + \mu_2(B_2 - I) \text{ with } \overset{\nabla}{B}_1 + \frac{\mu_1}{\nu_1}(B_1 - I) = 0, \quad \overset{\nabla}{B}_2 + \frac{\mu_2}{\nu_2}(B_2 - I) = 0.$$

The upper convected Oldroyd derivative is defined as $\overset{\nabla}{B}_1 := \partial_t B_1 + (v_f \cdot \nabla) B_1 - (\nabla v_f) B_1 - B_1 (\nabla v_f)^T$.

For the structure domains Ω_{s_1} and Ω_{s_2} we use a hyperelastic material modeled in Lagrangian framework by

$$\hat{\rho}_s \partial_t^2 \hat{u}_s - \widehat{\operatorname{div}}(\hat{\Pi}) = \hat{\rho}_s \hat{f} \text{ in } \hat{\Omega}_{s_i}, t \in I, i = 1, 2$$

with the first Piola-Kirchhoff stress tensor $\hat{\Pi} = \frac{\partial \hat{W}}{\partial \hat{F}}$, $\hat{J} = \det \hat{F}$, $\hat{F} := \hat{I} + \hat{\nabla} \hat{u}_s$ and $\hat{C} = \hat{F}^T \hat{F}$ where \hat{u}_s is the displacement and $\hat{\rho}_s$ the density. To model the elastic response of the sclera and lens we use (see [4]): $\hat{W}(\hat{F}) := \frac{1}{2} \mu (\hat{J}^{-1} \operatorname{tr} \hat{C} - 2) + \frac{1}{2} \kappa (\ln \hat{J})^2$.

Finally the equations for the drug therapy in the vitreous are (see [5]):

$$\begin{aligned} \partial_t c_A + v \cdot \nabla c_A - D_A \Delta c_A &= -k_v c_A c_V - k_A c_A & \text{in } \Omega_f, t \in I \\ \partial_t c_V + v \cdot \nabla c_V - D_V \Delta c_V &= -k_v c_A c_V + k_p & \text{in } \Omega_f, t \in I \end{aligned} \quad \text{with } c_A(0) = \begin{cases} c_0(x, y), & (x, y) \in U_r(m) \\ 0 & \text{else} \end{cases}$$

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and $c_V(0) = 1.312 \cdot 10^{-7}$ as initial conditions. Here c_V is the concentration of the disease, c_A is the concentration of the drug which is injected and v is the velocity obtained either from the Navier-Stokes equations or the Burgers model. The goal of the drug therapy is to reduce the concentration of the disease (VEGF).

For all models we impose appropriate boundary conditions and use realistic parameters from the literature (see [3], [6]).

3 Numerical Results

For the numerical treatment we transform the equations into ALE coordinates and derive a monolithic variational formulation. As mesh motion PDE we choose a standard harmonic extension. Interface conditions are continuity of velocity, displacement and stress. We use Q^2 finite elements for velocity, displacement, concentration and viscoelastic tensors and Q^1 elements for the pressure. Linearization is done using Newton's method. The numerical simulations are realized in deal.ii [7] and based on the implementation in [8].

The first numerical simulation is similar to a recent experiment in [2]: The eye is fixed on the left and is slowly pulled to the right on the right boundary. The remaining boundaries are left free to move. Figure 2 shows the deformed domain after 1s and the corresponding displacement magnitude. Figure 3 shows the Cauchy stress tensor magnitude after 0.5s for a healthy vitreous (Burgers model) and a liquefied vitreous (Navier-Stokes). The simulations show that the stress is the same in the whole vitreous for a pathological case while this is not the case for a healthy vitreous. This result is interesting since some medical diseases are assumed to be caused by different stresses in the vitreous.

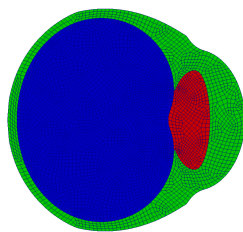


Fig. 1: Domain $\hat{\Omega}$, blue: fluid domain, red/green: solid domain

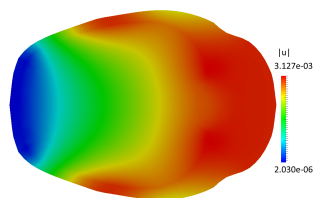


Fig. 2: Displacement magnitude at $t = 1.0s$

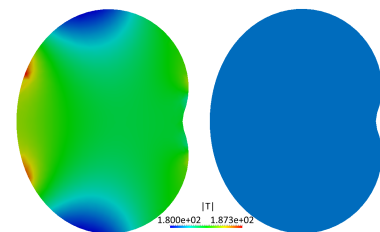


Fig. 3: Stress tensor magnitude, left: healthy, right: liquefied vitreous

The second test case investigates the drug therapy in the vitreous. Most medical treatments are based on the injection of medicine into the vitreous. Figure 4 shows the drug concentration after one day and Figure 5 after 28 days. Figure 6 shows the amount of concentration over time. Due to its metabolism and the coupling the amount of drug decreases over time while the concentration of the disease decreases much faster. Since there is a continuous production of VEGF multiple drug injections are necessary in order to reduce the concentration of the disease for a long time period.

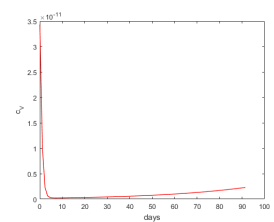
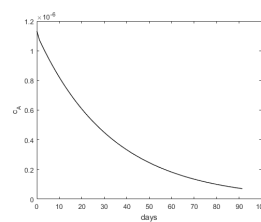
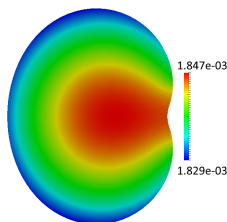
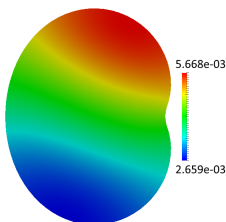


Fig. 4: Drug concentration at $t = 1d$ **Fig. 5:** Drug concentration at $t = 28d$ **Fig. 6:** Amount of drug (left) and VEGF (right) over time

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References

- [1] J. Donea, S. Giuliani, and J. P. Halleux, Computer methods in applied mechanics and engineering **33**(1-3), 689–723 (1982).
- [2] N. S. Shah, D. C. Beebe, S. P. Lake, and B. A. Filas, Annals of biomedical engineering **44**(10), 3084–3095 (2016).
- [3] K. Tůma, J. Stein, V. Průša, and E. Friedmann, Applied Mathematics and Computation **335**, 50–64 (2018).
- [4] R. Grytz and J. C. Downs, Computer methods in biomechanics and biomedical engineering **16**(7), 768–780 (2013).
- [5] K. Li, M. Limjoco, and J. Zarate, Preprint Cornell University, <https://ecommons.cornell.edu/handle/1813/51486>, (2017).
- [6] P. Sharif-Kashani, J. P. Hubschman, D. Sassoon, and H. P. Kavehpour, Journal of biomechanics **44**(3), 419–423 (2011).
- [7] W. Bangerth, R. Hartmann, and G. Kanschat, ACM Trans. Math. Softw. **33**(4), 24/1–24/27 (2007).
- [8] T. Wick, Archive of Numerical Software **1**(1), 1–19 (2013).