Artificial Boundary Conditions for the Stokes and Navier–Stokes Equations in Domains that are Layer-like at Infinity

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Abstract

Artificial boundary conditions are presented to approximate solutions to Stokes- and Navier-Stokes problems in domains that are layer-like at infinity. Based on results about existence and asymptotics of the solutions v^∞, p^∞ to the problems in the unbounded domain Ω the error $v^\infty - v^R$, $p^\infty - p^R$ is estimated in $H^1(\Omega_R)$ and $L^2(\Omega_R)$, respectively. Here v^R, p^R are the approximating solutions on the truncated domain Ω_R, the parameter R controls the exhausting of Ω. The artificial boundary conditions involve the Steklov-Poincaré operator on a circle together with its inverse and thus turn out to be a combination of local and nonlocal boundary operators. Depending on the asymptotic decay of the data of the problems, in the linear case the error vanishes of order $O(R^{-N})$, where N can be arbitrarily large.

Key words. Stokes Problem in layers, Navier-Stokes system, artificial boundary conditions, exact boundary conditions, Steklov-Poincaré operator

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1 Introduction

Layer-like domains appear in many topics of mathematical physics, related to film flows, lubrication patterns, plates etc. In the present paper a layer like domain is a domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$, and Ω coincides with the layer

$$\Lambda = \{x = (y, z) : y = (y_1, y_2) \in \mathbb{R}^2, |z| < 1/2\} \tag{1.1}$$

outside the ball $\mathbb{B}_{R_0} = \{x \in \mathbb{R}^3 : |x| < R_0\}$ of radius $R_0 > 1$. We consider the Stokes equations – and further Navier-Stokes equations – with Dirichlet boundary conditions

$$\begin{aligned} -\nu\Delta v^\infty + \nabla p^\infty &= f \quad \text{in} \quad \Omega, \\ \nabla \cdot v^\infty &= 0 \quad \text{in} \quad \Omega, \\ v^\infty &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.2}$$

The vector $v^\infty = (v_1^\infty, v_2^\infty, v_3^\infty)$ stands for the velocity and the scalar p^∞ for the pressure in a fluid with constant viscosity $\nu > 0$. In domains of type (1.1) besides the question of uniqueness and existence of solutions also the asymptotic behavior of v, p at infinity is important in dependance of the decay properties of f for various reasons. One context is the following:

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Computational schemes for boundary value problems in unbounded domains require the reduction to a problem in a bounded region. A very common practice is to cut the unbounded domain by taking the intersection with a bounded one and prescribe an artificial boundary condition (ABC) on the truncation surface. The choice of the truncation surfaces is usually governed by the geometry of the domains, the choice of the ABCs by the structure of differential operators. An opportune ABC should lead to a well posed problem which is accessible for numerics and leaves a minimal truncation error. The latter feature leads to non reflecting (absorbing, exact) ABC, they produce the restriction of the original solution to the truncated domain. However, with the exception of trivial examples they are nonlocal and require information like the structure of a Fourier expansion for the solution, e.g., information which usually exists only for homogeneous linear systems and simple geometries (see [3, 6, 8, 31, 35], e.g.)

Local ABC normally leave a truncation error but can mostly be handled with finite element methods and are available for inhomogeneous systems as well as for nonlinear problems, e.g., the Navier-Stokes system. Their choice is based on the asymptotic behavior of solutions at infinity. In particular, for elliptic boundary value problems in exterior domains and domains with cylindrical or conical outlets to infinity, ABCs in differential form were systematically developed during the last decades (see e.g., [1, 2, 4, 5, 7, 9, 10, 14, 23, 24, 32, 34]) and the papers quoted there.) The common feature of local ABCs are estimates for the truncation error of the form $\|u^\infty - u^R\| = O(R^{-\gamma})$ as R tends to infinity, with some $\gamma > 0$. Here R is a parameter which controls the size of the truncated domain (usually the radius of a ball), u^∞ is the solution to the original problem, and u^R the approximating solution. The order γ of the error is limited by the asymptotic decay of the problem's data *and* the choice of the boundary operator. This means even if the right hand sides of the boundary value problem have compact support, the choice of a an ABC in differential form fixes a γ_{max}, and of course the aim is then to obtain γ_{max} as large as possible. Usually the estimates of the truncation error require a careful analysis for various boundary value problems in weighted Sobolev spaces.

These questions were barely investigated up to now in layer like domains, although they represent a class of domains with noncompact boundaries that are important for applications. However, to the best of our knowledge, there exists only one paper [25] where ABC were constructed in a layer-like domain for the Neumann problem for the Poisson equation without assuming axial symmetry which turns the three-dimensional problem into a two-dimensional one.

Our results are based on asymptotic expansions at infinity of solutions to the Stokes problem (1.2) and to the Navier-Stokes problem

$$\begin{aligned} -\nu\Delta v^\infty + \big(v^\infty\cdot\nabla\big)v^\infty + \nabla p^\infty &= f \quad \text{in} \quad \Omega, \\ \nabla\cdot v^\infty &= 0 \quad \text{in} \quad \Omega, \\ v^\infty &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.3}$$

These asymptotic expansions (see formulae (2.2)-(2.5)) were obtained in [20] with the help of a method developed in [15–18], they contain the plane harmonics P_N.

The approximation problem in the bounded domain Ω_R is composed from the Stokes (or Navier-Stokes) equations, the Dirichlet conditions restricted to $\Sigma_R = \{x \in \partial\Omega : r < R\}$, and the ABC on the truncation boundary $\Gamma_R = \{x : r = R, |z| < 1/2\}$, in the linear case this means

$$\begin{aligned} -\nu\Delta v^R + \nabla p^R &= f \quad \text{in} \quad \Omega_R, \\ \nabla\cdot v^R &= 0 \quad \text{in} \quad \Omega_R, \\ v^R &= 0 \quad \text{on} \quad \Sigma_R, \\ M_R(v^R, p^R) &= 0 \quad \text{on} \quad \Gamma_R, \end{aligned} \tag{1.4}$$

where the operator M_R has to be chosen properly. "Properly" means here that the problem (1.4) is well posed and the operator M_R vanishes on the main asymptotic terms of (v^∞, p^∞) – the latter feature arises from the experiences with ABC in other situations.

We describe the boundary operator M_R briefly: Let v_r, v_φ and v_z denote the components of a vector field v related to cylindrical coordinates (r, φ, z). Any smooth function $F(y,z)$ on Γ_R can be written as

$$F(y,z) =: (z^2 - 1/4)\overline{F}(y) + F^{\#}(y,z) \quad \text{with} \quad \overline{F}(y) = 30 \int_{-1/2}^{1/2} (z^2 - 1/4)F(y,z)dz.$$

Further let Π_R denote the external Steklov-Poincaré operator (or Dirichlet-to-Neumann operator, see formulae (3.8) - (3.12) for more details) on the circle $\mathbb{S}_R = \{y : r = |y| = R\}$, and finally $\overline{F}(y)_\bullet = \overline{F}(y) - (2\pi R)^{-1} \int_{\mathbb{S}_R} \overline{F}(y)ds$ the projection of $\overline{F}$ onto the mean value free functions. Then the operator M_R is defined by

$$M_R(v,p) = \begin{pmatrix} v_r^{\#} \\ v_\varphi^{\#} \\ v_z \\ \nu \frac{\partial}{\partial r}\overline{v_r} - \overline{p} + \nu \left\{ \Pi_R \overline{v_r} + \frac{1}{R}\overline{v_r} + 10\, \Pi_R^{-1}\left(\overline{v_r}\right)_\bullet \right\} \\ \nu \frac{\partial}{\partial r}\overline{v_\varphi} + \nu \left\{ \Pi_R \overline{v_\varphi} + \frac{1}{R}\overline{v_\varphi} \right\} \end{pmatrix} \quad \text{on } \Gamma_R. \tag{1.5}$$

Why it should have this particular form, is explained in Section 3.

The boundary operator here is a combination of of local and nonlocal operators. In Section 4 we prove existence of a unique solution to problem (1.4) with M_R as in (1.5) (Theorem 4.5) and an error estimate of the form (see formula (4.33) in Theorem 4.7)

$$\|v^\infty|_{\Omega_R} - v^R; H^1(\Omega_R)^3\| + R^{-1}\|p^\infty|_{\Omega_R} - p^R; L_2(\Omega_R)\| \leq C_N\, R^{-N}\, \|f\|_{(N)} \tag{1.6}$$

where the constant C_N does not depend on the radius $R \geq R_0$ and an appropriate weighted norm $\|f\|_{(N)}$ of the right-hand side in the original problem. We emphasize that, for the linear problem, the exponent N can be made arbitrary large by assuming a fast decay of the right-hand side f, i.e., by making the weighted norm harder. This is due to the fact that here the features of asymptotic ABC and non-reflecting ABC are combined, moreover, this result cannot be achieved without knowing the asymptotic form of the solution.

Let us also give a short guide through the other sections of the paper. The results on existence, uniqueness and the asymptotics of the solutions to (1.2) are outlined in Sections 2. As already mentioned, the ABC for the linear problem are derived in Section 3. The well-posedness of the approximation problem and error estimates are proved in Section 4. The most tricky point is here to find a solution to the continuity equation together with an estimate that controls the behavior of $H^1(\Omega_R)$-norm with respect to R (Lemma 4.3).

The last two sections are devoted to the Navier-Stokes problem (1.3). Under suitable restrictions for the data it is possible to obtain solutions to the nonlinear problem with the same ABC as for the linear problem together with error estimates of type (1.6) (see Theorem 6.2). However, by using existence results of [21] and the results on the asymptotic behavior of the solutions to (1.2) (see [20, 27]) and (1.3) it becomes clear how the nonlinearity influences the asymptotics at infinity of suitable strong solutions to (1.3) – these results are explained in Section 5. Thus for the nonlinear problem the order of convergence is limited by $N \leq 3$ in (1.6), even if the right hand side f is infinitely smooth with compact support.

2 Basic function spaces and asymptotics of solutions to the Stokes equations

As shown in [16, 18–20], the following anisotropic weighted Sobolev norms (2.1) are especially adapted to a wide class of elliptic boundary value problems in layer-like domains. We recall that $x = (y, z)$ and $r = |y|$, similarly derivatives $\partial^\beta = \partial_x^\beta$ can be split into $\partial_x^\beta = \partial_y^\alpha \partial_z^j$, with

$|\alpha| + j = |\beta|$, using the common multi-index terminology. By $L^2_\beta(\Omega)$, we understand the space of all locally square summable functions with finite norm

$$\|w; L^2_\beta(\Omega)\| = \|(1+r)^\beta w; L^2(\Omega)\|.$$

We also introduce the space $\mathcal{W}^l_\beta(\Omega)$ as the completion of $C^\infty_0(\overline{\Omega})$ (infinitely smooth functions with compact supports) with respect to the anisotropic weighted norm

$$\|w; \mathcal{W}^l_\beta(\Omega)\| = \Big\{ \sum_{|\alpha|+j\le l} \|\partial^\alpha_y \partial^j_z w; L^2_{\beta-l+|\alpha|}(\Omega)\|^2 \Big\}^{1/2}. \tag{2.1}$$

We emphasize that each differentiation in y_1 and y_2 enlarges the weight exponent in (2.1) by 1, while differentiation in z does not. That is why the weighted norm (2.1) is called "anisotropic" [18] in contrast to the usual "isotropic" Kondratiev norm (see, *e.g.*, [11, 22]) where derivatives in any direction are provided with the same exponent in the weight function.

We recall some standard notations: For an arbitrary domain $G \subset \mathbb{R}^n$ (here only $n = 2, 3$), the notation $C^\infty_0(G)$ indicates the set of all smooth functions with compact support in G, the symbol $H^m(G)$, $m \in \mathbb{N}$, stands for the Sobolev space containing all all functions $w \in L^2(G)$, such that all derivatives $\partial^\alpha w \in L^2(G)$ up to $|\alpha| = m$, by $\mathring{H}^m(G)$ we indicate the closure of $C^\infty_0(G)$ in $H^m(\Omega)$.

The following lemma on the weak solution of problem (1.2) can be found, e.g., in [21].

Lemma 2.1 *Let $f \in L^2(\Omega)^3$ and $\beta < -1$. There exist $v^\infty \in \mathring{H}^1(\Omega)^3$ and $p^\infty \in L^2_\beta(\Omega)$ which satisfy relations $(1.2)_{2,3}$ and the integral identity*

$$\nu\big(\nabla v^\infty, \nabla w\big)_\Omega = \big(p^\infty, \nabla\cdot w\big)_\Omega + \big(f, w\big)_\Omega \qquad \forall w \in C^\infty_0(\Omega)^3.$$

The solution (v^∞, p^∞) is determined up to an additive constant in its pressure component. However, the solution becomes unique with a suitable normalization condition for the pressure p^∞ as, e.g., $\int_{\Omega_{R_0}} p^\infty = 0$. In this case the estimate

$$\|v^\infty; H^1(\Omega)\| + \|p^\infty; L^2_\beta(\Omega)\| \le c_\beta \|f; L^2(\Omega)\|,$$

is valid where the constant c_β depends on ν, β, and Ω, but is independent of f.

Note that the assumption on f used in Lemma 2.1 can be weakened (*cf.* [21]). An additive constant in pressure appears because a constant function p belongs to the space $L^2_\beta(\Omega)$ if $\beta < -1$.

If the right-hand side f of problem (1.2) decays sufficiently fast, the condition $\beta < -1$ in Lemma 2.1 can be replaced by $-1 < \beta < 0$, and then also p^∞ is uniquely determined. The solution (v^∞, p^∞) gets a special asymptotic form, as it was shown in [20]. We distinguish between the longitudinal components, $v^{\infty\prime}$, and the transversal component v^∞_z of the vector v^∞, then

$$p^\infty(y,z) = P^\infty(y,z) + \widetilde{p}^\infty(y,z), \tag{2.2}$$
$$v^\infty(y,z) = V^\infty(y,z) + \widetilde{v}^\infty(y,z) \tag{2.3}$$

with

$$\begin{aligned} P^\infty(y,z) &= \big(1-\chi^{R_0}\big)\, P_N(y), \\ V^{\infty\prime}(y,z) &= \big(1-\chi^{R_0}\big)\, \frac{1}{2\nu}\left(z^2 - \frac{1}{4}\right) \nabla_y P_N(y)), \\ V^\infty_z(y,z) &= 0. \end{aligned} \tag{2.4}$$

Here $\chi \in C_0^\infty(\mathbb{R})$ is a cut-off function such that $\chi(t) = 0$ as $t \geq 2$ and $\chi(t) = 1$ as $t \leq 1$, the function P_N is harmonic, namely

$$P_N(y) = \sum_{j=1}^{N} r^{-j} \Big(a_j \cos(j\varphi) + b_j \sin(j\varphi) \Big), \tag{2.5}$$

To justify formula (2.4), we present simplified results of [20] which are sufficient for the further use in this paper: In the following theorem, the estimates of the remainders in (2.4) are not optimal with respect to the smoothness properties of the data and the solutions, in particular, the assumptions on the right-hand side f are too restrictive. Nevertheless, in the case $f \in C_0^\infty(\overline{\Omega})^3$ the indices l and N can be taken arbitrary and then Theorem 2.2 provides an explicit information on the power-law asymptotic behavior of the solution.

Theorem 2.2 *Let $l \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, $N \in \mathbb{N} := \{1, 2, 3, \dots\}$ and*

$$f \in \mathcal{W}_\gamma^{l+2}(\Omega)^3, \quad \gamma > N + l + 3. \tag{2.6}$$

Then the asymptotic representation (2.4) is valid for the solution (v^∞, p^∞) while the remainders satisfy the inclusions

$$\begin{aligned} &\widetilde{v}^{\infty\prime} \in \mathcal{W}_{\gamma-1}^{l+2}(\Omega)^2, \quad \widetilde{v}_z^\infty \in \mathcal{W}_\gamma^{l+2}(\Omega), \\ &\widetilde{p}^\infty \in \mathcal{W}_{\gamma-1}^{l+3}(\Omega), \quad \partial_z \widetilde{p}^\infty \in \mathcal{W}_\gamma^{l+2}(\Omega). \end{aligned} \tag{2.7}$$

Moreover, these remainders and the coefficients a_j and b_j fulfil the estimate

$$\begin{aligned} &\|\widetilde{v}^{\infty\prime}; \mathcal{W}_{\gamma-1}^{l+2}(\Omega)\| + \|\widetilde{v}_z^\infty; \mathcal{W}_\gamma^{l+2}(\Omega)\| + \|\widetilde{p}^\infty; \mathcal{W}_{\gamma-1}^{l+3}(\Omega)\| \\ &\qquad + \|\partial_z \widetilde{p}^\infty; \mathcal{W}_\gamma^{l+2}(\Omega)\| + \sum_{j=1}^{N} \Big(|a_j| + |b_j| \Big) \leq c_{l,\gamma} \, \|f; \mathcal{W}_\gamma^{l+2}(\Omega)\| \end{aligned} \tag{2.8}$$

with a constant $c_{l,\gamma}$, independent of the right-hand side f.

We emphasize that different weight indices in (2.7) and (2.8) reflect the different asymptotic behavior at infinity of $v^{\infty\prime}$, p^∞ and v_z^∞. If the right hand side f has a compact support, then *formally* there appear series for V^∞, P^∞, we emphasize, that the series do not converge in general. It can be easily verified that the detached terms V^∞, P^∞ in (2.4) do not belong to the spaces indicated in (2.7). However, in the case $\gamma \in (N + l + 3, N + l + 4)$, the next asymptotic terms, which appear if we replace the term P_N in (2.4) by

$$P_{N+1}(y) - P_N(y) = r^{-N-1} \Big(a_{N+1} \cos((N+1)\varphi) + b_{N+1} \sin((N+1)\varphi) \Big),$$

belong to those spaces.

3 Formal construction of ABC

Our objective is to find the ABC operator M_R together with a weak formulation of problem (1.4) in an appropriate Hilbert space contained in $H^1(\Omega_R)$. We introduce the notations $\mathbb{S}$, for the unit circle in $\mathbb{R}^2$, and $\mathbb{S}_R = \{y \in \mathbb{R}^2 : |y| = R\}$. Watching the geometry of our problem it is convenient to use cylindrical coordinates in $\mathbb{R}^3$, i.e. $x = (y, z) \in \mathbb{R}^3$ matches (r, φ, z). If e_r, e_φ, e_z denote the corresponding unit vectors, then a vector field v can be decomposed as follows $v = v_r e_r + v_\varphi e_\varphi + v_z e_z$. We indicate the $L^2(\Xi)$-scalar product by $\big(\cdot\,,\cdot\big)_\Xi$ – without distinguishing between scalar functions and vector fields.

Suppose (v^R, p^R) is a sufficiently smooth solution to problem $(1.4)_{1,2,3}$, and w is a sufficiently smooth divergence free vector field with $w = 0$ on Σ_R (the "top" and the "bottom" faces of Ω_R). By partial integration we have the following Green's formula

$$\big(f, w\big)_{\Omega_R} = \nu\big(\nabla v^R, \nabla w\big)_{\Omega_R} - \big(\nu\partial_r v^R - e_r p^R, w\big)_{\Gamma_R} \tag{3.1}$$

Keeping in mind the two requirements for a good ABC, condition $(1.4)_4$ together with the stable* part of these conditions for the test function w should reduce the remaining surface integral in (3.1), written as

$$I(v^R, p^R; w) = -\int\limits_{\mathbb{S}_R}\int\limits_{-1/2}^{1/2} \Big\{ \big(\nu\partial_r v_r^R - p^R\big)w_r + \nu(\partial_r v_\varphi^R)w_\varphi + \nu(\partial_r v_z^R)w_z \Big\}\, dz\, dl, \tag{3.2}$$

to a nonnegative quadratic form. Moreover, the operator must "swallow" the asymptotic terms detached from the solution (v^∞, p^∞) in Section 2, which means that finally the quadratic term $I(v^\infty - v^R, p^\infty - p^R; v^\infty - v^R)$ decays fast as R tends to infinity. If we plug the asymptotic representation $(v^\infty, p^\infty) = (V^\infty, P^\infty) + (\widetilde{v}^\infty, \widetilde{p}^\infty)$ into the representation (3.2) for I, we find

$$I(v^\infty - v^R, p^\infty - p^R; v^\infty - v^R) = I(V^\infty - v^R, P^\infty - p^R; V^\infty - v^R) + \dots, \tag{3.3}$$

and all other summands hidden in the dots contain at least one term of the rapidly decaying remainders, hence our ABC should remove as much as possible of the term on the right-hand side of (3.3). Therefore with $V_z^\infty = 0$ in mind, our first artificial boundary condition is but

$$v_z^R = 0 \quad \text{on} \quad \Gamma_R. \tag{3.4}$$

This is a stable and local ABC, which kills the term related to $V_z^\infty - v_z^R$ in (3.3). The other components of the velocity field and the pressure contain harmonic functions only.

Furthermore, the number of detached terms in $(2.4)_{2,1}$ increases if the decay properties of f are strengthened. This observation motivates us to search for nonlocal ABCs for the remaining parts v_r^R, v_φ^R and p^R. With $\Upsilon = [-1/2, 1/2]$, and watching the special form of $v^{\infty\prime}$ in (2.4), we introduce a convenient orthogonal decomposition of $L^2(\Upsilon)$: Namely we have

$$L^2(\Upsilon) = \{\alpha\psi : \alpha \in \mathbb{R}\} \oplus \{\psi\}^\perp, \text{ with } \psi(z) = z^2 - 1/4,$$

the index $\perp$ indicates the space of all functions $\varphi \in L^2(\Upsilon)$ that are orthogonal to ψ. If F is a sufficiently smooth function on the layer Λ, we obtain $F(y, z) = \overline{F}(y)\psi(z) + F^\#(y, z)$. An easy calculation shows $\|\psi, L^2(\Upsilon)\|^2 = 1/30$, hence we have

$$\overline{F}(y) = 30\,(\psi, F(y, \cdot\,))_\Upsilon, \quad (\psi, F^\#(y, \cdot\,))_\Upsilon = 0 \quad \text{for } y \in \mathbb{R}^2. \tag{3.5}$$

Recalling the asymptotic representation $(2.4)_2$, it is obvious that

$$(V^{\infty\prime})^\# = \big((2\nu)^{-1}\psi\nabla_y P_N(y)\big)^\# = 0$$

for any N, so that we can choose the next conditions as follows

$$(v_r^R)^\# = (v_\varphi^R)^\# = 0 \quad \text{on} \quad \Gamma_R. \tag{3.6}$$

Now in $I(V^\infty - v^R, P^\infty - p^R; V^\infty - v^R)$ only the projections along the function ψ are left. Before discussing this in more detail we mention two simple, but useful identities, namely

$$(\overline{F}\psi, \overline{G}\psi)_{\Gamma_R} = (\overline{F}, \overline{G})_{\mathbb{S}_R}\|\psi; L^2(\Upsilon)\|^2, \qquad \overline{\partial_r F} = \partial_r\overline{F}, \quad \overline{\partial_\varphi F} = \partial_\varphi\overline{F}. \tag{3.7}$$

*According to [13], all kind of homogeneous Dirichlet boundary conditions are called *stable*, since they are realized for the weak solution u^R by the choice of the Hilbert space which is in turn determined by a dense subset of test functions.

Next we allude to some basic facts about the (exterior) Steklov-Poincaré operator on the circles $\mathbb{S}$ and $\mathbb{S}_R$ and its inverse, which is also called Poincaré-Steklov operator (see [30], e.g.). Any function $g \in L^2(\mathbb{S})$ can be represented as the Fourier series

$$g(y) = a_0 + \sum_{j=1}^{\infty} \{a_j \cos(j\varphi) + b_j \sin(j\varphi)\}. \tag{3.8}$$

On $L^2(\mathbb{S})$, we define the operator Π with the domain $H^1(\mathbb{S})$ by

$$\Pi g(y) = \sum_{j=1}^{\infty} j \{a_j \cos(j\varphi) + b_j \sin(j\varphi)\}. \tag{3.9}$$

If $g \in H^1(\mathbb{S})$ and $h \in L^2(\mathbb{S})$ with Fourier coefficients a_{j1}, b_{j1} and a_{j2}, b_{j2}, respectively, then

$$(\Pi g, h)_{\mathbb{S}} = \pi \sum_{j=1}^{\infty} j(a_{j1}a_{j2} + b_{j1}b_{j2}), \tag{3.10}$$

Now let $s \in \mathbb{R}$ be arbitrary. We recall that via Fourier series the Sobolev-Slobodetskii space $H^s(\mathbb{S})$ can be identified with the sequences of Fourier coefficients (a_j, b_j) such that $|a_0|^2 + \sum j^{2s}(|a_j|^2 + |b_j|^2) < \infty$. Thus, (3.10) defines a symmetric nonnegative quadratic form on $H^{1/2}(\mathbb{S})$, and we can extend the operator defined in (3.9) to the mapping $\Pi : H^{1/2}(\mathbb{S}) \to H^{-1/2}(\mathbb{S})$. Thereby $\ker \Pi = \{\text{const}\}$ and the range of Π consists of all $h \in H^{-1/2}(\mathbb{S})$ with $\langle h, 1 \rangle = 0$ (the brackets $\langle \cdot, \cdot \rangle$ denote the duality between $H^{-1/2}$ and $H^{1/2}$). The inverse operator Π^{-1}, restricted to the subspace

$$H^{1/2}_{\bullet}(\mathbb{S}) := \left\{ h \in H^{1/2}(\mathbb{S}) : \int_{\mathbb{S}} h dl = 0 \right\},$$

induces an isomorphism

$$\Pi^{-1} : H^{1/2}_{\bullet}(\mathbb{S}) \to H^{3/2}_{\bullet}(\mathbb{S}). \tag{3.11}$$

Furthermore, if H is the unique bounded harmonic extension of $h \in H^{3/2}(\mathbb{S})$ to the domain $\{y \in \mathbb{R}^2 : |y| > 1\}$, then $\Pi h = -\partial_r H|_{\mathbb{S}}$. Therefore, Π realizes the Steklov-Poincaré operator on unit circle (see [30], e.g.)

If Π_R denotes the Steklov-Poincaré operator on the circle $\mathbb{S}_R$, by a scaling argument it inherits from Π all the properties mentioned above. Namely, if $h \in L^2(\mathbb{S}_R)$, then $h(R\,\cdot\,) \in L^2(\mathbb{S})$ and

$$\Pi_R h = R^{-1}\, \Pi h(R\,\cdot\,), \quad \Pi_R^{-1} h = R\, \Pi^{-1} h(R\,\cdot\,). \tag{3.12}$$

Indeed, the Fourier expansion for $h(R\,\cdot\,)$ implies

$$h(R\cos\varphi, R\sin\varphi) = a_0 + \sum_{j=1}^{\infty} R^{-j} \{a_j \cos(j\varphi) + b_j \sin(j\varphi)\},$$

hence

$$\Pi_R h(R\cos\varphi, R\sin\varphi) = \sum_{j=1}^{\infty} j R^{-1-j} \{a_j \cos(j\varphi) + b_j \sin(j\varphi)\}, \tag{3.13}$$

the second formula in (3.12) becomes obvious.

Now we return to the terms $\nu\partial_r\overline{V^{\infty\prime}}$ and $\overline{P^\infty}$. Let us for a moment assume that $N=\infty$ in (2.5) (recall that the series do not converge in general). Based on $(2.4)_{2,1}$ and $(3.7)_2$, we *formally* write for $y\in\mathbb{S}_R$

$$\begin{aligned}
\overline{V_r^\infty}(y) &= -\frac{1}{2\nu}R^{-1}\sum_{j=1}^{\infty} jR^{-j}\Big(a_j\cos(j\varphi)) + b_j\sin(j\varphi)\Big),\\
\nu\frac{\partial}{\partial r}\overline{V_r^\infty}(y)-\overline{P^\infty}(y) &= \sum_{j=1}^{\infty}\Big\{\frac{1}{2}j(j+1)R^{-2-j}+5R^{-j}\Big\}\Big(a_j\cos(j\varphi)+b_j\sin(j\varphi)\Big) =\\
&= -\nu\left\{\Pi_R\overline{V_r^\infty}(y)+\frac{1}{R}\overline{V_r^\infty}(y)+10\,\Pi_R^{-1}\big(\overline{V_r^\infty}\big)_\bullet\right\} \qquad (3.14)
\end{aligned}$$

where "$\bullet$" stands for the projection on the space of mean-value free functions on $\mathbb{S}_R$,

$$w_\bullet(y) = w(y) - \frac{1}{2\pi R}\int_{\mathbb{S}_R} w(y)\,ds_y. \qquad (3.15)$$

We point out that $\big(\overline{V_r^\infty}\big)_\bullet$ belongs to the domain of the inverse operator Π_R^{-1} (see (3.11)).

In accordance with (3.14), we now impose the following ABC:

$$\nu\frac{\partial}{\partial r}\overline{v_r^R}-\overline{p^R} = -\nu\left\{\Pi_R\overline{v_r^R}+\frac{1}{R}\overline{v_r^R}+10\,\Pi_R^{-1}\big(\overline{v_r^R}\big)_\bullet\right\} \quad \text{on} \quad \Gamma_R. \qquad (3.16)$$

In a similar but simpler way, we formulate the remaining ABC for the the component v_φ^R:

$$\nu\frac{\partial}{\partial r}\overline{v_\varphi^R} = -\nu\left\{\Pi_R\overline{v_\varphi^R}+\frac{1}{R}\overline{v_\varphi^R}\right\} \quad \text{on} \quad \Gamma_R. \qquad (3.17)$$

Hence, if $w=(w',w_z)\in H^1(\Omega_R)^3$ fulfils the (stable) conditions

$$w=0 \text{ on } \Sigma_R, \qquad w_z=0, \quad w'^{\#}=0 \text{ on } \Gamma_R \qquad (3.18)$$

then with the ABCs (3.4), (3.6), (3.16) and (3.17) and identities (3.7), the integral (3.2) reduces to

$$\begin{aligned}
I(v^R,p^R;w) &= -\frac{1}{30}\Big\{\big(\nu\partial_r\overline{v_r^R}-\overline{p^R},\overline{w_r}\big)_{\mathbb{S}_R}+\big(\nu\partial_r\overline{v_\varphi},\overline{w_\varphi}\big)_{\mathbb{S}_R}\Big\}\\
&= \frac{\nu}{30}\Big\{\big(\Pi_R\overline{v_r^R},\overline{w_r}\,\big)_{\mathbb{S}_R}+\big(\Pi_R\overline{v_\varphi^R},\overline{w_\varphi}\,\big)_{\mathbb{S}_R}\\
&\quad+\frac{1}{R}\big(\overline{v_r^R},\overline{w_r}\,\big)_{\mathbb{S}_R}+\frac{1}{R}\big(\overline{v_\varphi^R},\overline{w_\varphi}\,\big)_{\mathbb{S}_R}+10\,\big(\Pi_R^{-1}\big(\overline{v_r^R}\big)_\bullet,\big(\overline{w_r}\big)_\bullet\,\big)_{\mathbb{S}_R}\Big\}\\
&:= \mathbf{q}_R(v^R,w). \qquad (3.19)
\end{aligned}$$

Clearly, the right-hand-side of (3.19) defines a positive quadratic form on $H^{1/2}(\mathbb{S}_R)^2\times\{0\}$.

4 Solution of the linear approximation problem

After the formal derivation of the ABC we establish the weak approximation problem and show the existence of weak solutions. Summarizing the result of the previous section, we obtain that for a sufficiently smooth vector field (v,p), the boundary operator M_R on Γ_R in (1.4) is defined by (1.5). We introduce the domains

$$\begin{aligned}
&\Lambda_R=\{(y,z)\in\Lambda : r=|y|<R\},\\
&A_R=\{y\in\mathbb{R}^2 : R/2<r<R\}, \qquad (4.1)\\
&\Xi_R=\{x=(y,z)\in\Lambda : R/2<r<R\},
\end{aligned}$$

obviously we have $\Xi_R \subset \Omega_R \cap \Lambda_R$, if R is large enough. If $v \in H^1(\Omega_R)^R$, and $R \geq 2R_0$, we can extend the decomposition $v(x) = \overline{v}'(y)\psi(z) + v^{\#}(x)$ (see formula (3.5)) on Ξ_R, and clearly

$$\|\overline{v}'; H^1(A_R)\| + \|v^{\#}; H^1(\Xi_R)\| \leq C\|v; H^1(\Omega_R)\| \tag{4.2}$$

with a constant independent on R. Thus, the space

$$\mathcal{H}(\Omega_R) = \{w \in H^1(\Omega_R)^3 : w = 0 \text{ on } \Sigma_R, \text{ and } w \text{ fulfils (3.18)}\},$$

is a closed subspace of $H^1(\Omega_R)^3$.

Definition 4.1 (Weak solution of the approximation problem) *We put*

$$\mathcal{H}_\sigma(\Omega_R) = \{w \in \mathcal{H}(\Omega_R) : \nabla \cdot w = 0\},$$
$\mathcal{H}'(\Omega_R)$ *the dual space of* $\mathcal{H}(\Omega_R)$.

Let $F \in L^2(\Omega_R)$ *be fixed and* $\mathbf{q}_R$ *defined by* (3.19). *We call* $V \in \mathcal{H}_\sigma(\Omega_R), P \in L^2(\Omega_R)$ *a weak solution to the problem*

$$\begin{aligned} &-\nu\Delta V + \nabla P = F \qquad \nabla \cdot V = 0 \qquad in \quad \Omega_R, \\ &V = 0 \quad on \quad \Sigma_R,, \qquad M_R(V, P) = 0 \quad on \quad \Gamma_R, \end{aligned} \tag{4.3}$$

if the integral identity

$$(F, w)_{\Omega_R} = \nu(\nabla V, \nabla w)_{\Omega_R} - (P, \nabla \cdot w)_{\Omega_R} + \mathbf{q}_R(V, w) \tag{4.4}$$

is valid for all $w \in \mathcal{H}(\Omega_R)$.
If Φ *is a linear functional on* $\mathcal{H}(\Omega_R)$, *continuous with respect to the* $H^1(\Omega_R)$*-norm, we call a pair* V, P *as above a weak solution to the general approximation problem, provided*

$$\Phi(w) = \nu(\nabla V, \nabla w)_{\Omega_R} - (P, \nabla \cdot w)_{\Omega_R} + \mathbf{q}_R(V, w), \tag{4.5}$$

where $\Phi(w)$ *indicates the value of the functional* Φ *at the test function* w.

A weak solution to the approximation problem (1.4) is a pair (v^R, p^R) which satisfies the definition above with $\Phi(w) = (f, w)_{\Omega_R}$. Furthermore we observe, that due to the construction in Section 3, the restriction of (v^∞, p^∞) to Ω_R fulfils

$$\begin{aligned} &\nu(\nabla v^\infty, \nabla w)_{\Omega_R} - (p^\infty, \nabla \cdot w)_{\Omega_R} + \mathbf{q}_R(v^\infty, w) = \\ &\qquad (f, w)_{\Omega_R} + \mathbf{q}_R(\widetilde{v}^\infty, w) + (\nu\partial_r\widetilde{v}^\infty - \widetilde{p}^\infty \mathrm{e}_r, w)_{\Gamma_R} \end{aligned} \tag{4.6}$$

thus Definition 4.1 with $\Phi(w) = (f, w)_{\Omega_R} + \mathbf{q}_R(\widetilde{v}^\infty, w) + (\nu\partial_r\widetilde{v}^\infty - \widetilde{p}^\infty \mathrm{e}_r, w)_{\Gamma_R}$.

As usual, the existence of a weak solution to Problem (4.3) is reduced to prove the existence of $v^R \in \mathcal{H}_\sigma(\Omega_R)$ by means of the Lax-Milgram lemma, and then recover the pressure while treating the problem $\nabla \cdot w = g$. We start with the auxiliary result on the solution of the divergence equation. To this end we recall a well known result on this problem.

Proposition 4.2 ([12], see also [33, Lemma 2.1.1]) *Let* $\omega \subset \mathbb{R}^n$ *be a bounded domain with Lipschitz boundary, and* $G \in L^2_\bullet(\omega)$ *(i.e.* $\int_\omega G dx = 0$*). Then there exists* $W \in \mathring{H}^1(\omega)^3$ *with* $\nabla \cdot W = G$ *and*

$$\|W; H^1(\omega)\| \leq C(\omega)\|G; L^2(\omega)\|. \tag{4.7}$$

Lemma 4.3 (Solution of the continuity equation) *For any* $g \in L^2(\Omega_R)$ *there exists a* $w \in \mathcal{H}(\Omega_R)^3$ *with* $\nabla \cdot w = g$ *and*

$$\|w; H^1(\Omega_R)\| + R^{3/2}\|w'; L^2(\Gamma_R)\| \leq CR\|g; L^2(\Omega_R)\|, \tag{4.8}$$

where C *is independent of* g *and* R.

Proof. The basic idea is the following: We split the problem on Ω_R into a problem on the fixed domain Ω_{3R_0}, which contains the perturbed part of the boundary, and a problem on the domain $\Omega_R \setminus \Omega_{R_0}$, which can be considered as part of the cylinder Λ_R. On Λ_R the dependence on R of the norms is controlled by a scaling argument. In both parts we use Proposition 4.2, thus we have to juggle a bit with mean values.

To fill in the details, we define a flux driver by the vector field $\mathscr{W}^T(x)$ by

$$\mathscr{W}^{T\prime}(x) = \left(z^2 - \frac{1}{4}\right) \nabla_y \left((1 - \chi(T^{-1}r)) \ln r\right), \quad \mathscr{W}^T_z(x) = 0,$$

where χ is the same cut-off function as in (2.4). For $R \geq 2T \geq 2R_0$, it is obvious that $\mathscr{W}^T|_{\Omega_R} \in \mathcal{H}(\Omega_R)$, and since $\ln r$ is harmonic, we have

$$\nabla \cdot \mathscr{W}^T(x) = 0 \qquad \text{for } r < T \text{ and } r > 2T. \tag{4.9}$$

Integration by parts gives for any $R \geq T$

$$\int_{\Omega_R} \nabla \cdot \mathscr{W}^T(x) dx = \int_{\Gamma_R} \mathscr{W}^T_r(x) ds = \int_{\Gamma_R} \left(z^2 - \frac{1}{4}\right) \partial_r \ln r dx = -\frac{\pi}{3}. \tag{4.10}$$

We put $G_{3R_0} = \int_{\Omega_{3R_0}} g(x)\, dx$, then clearly

$$|G_{3R_0}| \leq C\|g; L^2(\Omega_R)\|. \tag{4.11}$$

Now we look for the solution w to the continuity equation as

$$w(x) = w^*(x) - \frac{3}{\pi} G_{3R_0} \mathscr{W}^{R_0}(x).$$

Then w^* has to solve

$$\begin{aligned} &\nabla \cdot w^* = g + G_{3R_0} \nabla \cdot \mathscr{W}^{R_0} =: g^1 + g^2, \\ &g^1 = \mathcal{X}_{\Omega_{3R_0}} \left(g + G_{3R_0} \nabla \cdot \mathscr{W}^{R_0}\right), \quad g^2 = \mathcal{X}_{\Omega_R \setminus \Omega_{3R_0}} \left(g + G_{3R_0} \nabla \cdot \mathscr{W}^{R_0}\right) = \mathcal{X}_{\Omega_R \setminus \Omega_{3R_0}} g \end{aligned}$$

where $\mathcal{X}_\omega$ is the indicator function of the set ω, for the representation of g^2 we used (4.9). By construction, we have $\int_{\Omega_{3R_0}} g^1 = 0$, hence by Proposition 4.2, we find $w^1 \in \mathring{H}^1(\Omega_{3R_0})$ which, after extension with zero, fulfils $\nabla \cdot w^1 = g^1$ on Ω_R, and by (4.7) and (4.11),

$$\|w^1; H^1(\Omega_R)\| = \|w^1; H^1(\Omega_{3R_0})\| \leq C\|g^1; L^2(\Omega_{3R_0})\| \leq C\|g; L^2(\Omega_R)\|. \tag{4.12}$$

It remains to find $w^2 \in H^1(\Omega_R \setminus \Omega_{R_0})$ with $\nabla \cdot w^2 = g^2$ on $\Omega_R \setminus \Omega_{R_0}$, $w^2 = 0$ on $\Sigma_R \cup \Gamma_{R_0}$, and w^2 fulfils (3.18). Then the extension with zero on Ω_{R_0} leads to an element in $\mathcal{H}(\Omega_R)$ which solves $\nabla \cdot w^2 = g^2$.
To construct w^2 together with the desired estimates we first extend g^2 with zero to the whole cylinder Λ_R and use a scaling argument. With

$$\mathbf{y} = \frac{y}{R}, \ \mathbf{x} = (\mathbf{y}, z), \quad \mathbf{g}^2(\mathbf{x}) = g^2(R\mathbf{y}, z) \text{ and } \mathbf{v}(\mathbf{x}) = \left(\frac{1}{R} v'(R\mathbf{y}, z), v_z(R\mathbf{y}, z)\right) \tag{4.13}$$

we get: The problem $\nabla \cdot v = g^2$ in Λ_R is equivalent to $\nabla_{\mathbf{x}} \cdot \mathbf{v} = \mathbf{g}^2$ in Λ_1. Moreover, we have

$$\|\mathbf{g}^2; L^2(\Lambda_1)\| = CR^{-1}\|g^2; L^2(\Lambda_R)\|. \tag{4.14}$$

Now we use a similar trick as above. We put $\mathbf{G} = \int_{\Lambda_1} \mathbf{g}^2(\mathbf{x}) d\mathbf{x}$, then

$$|\mathbf{G}| \leq C\|\mathbf{g}^2; L^2(\Lambda_1)\|, \tag{4.15}$$

and we look for $\mathbf{v}$ as

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}^*(\mathbf{x}) - \frac{3}{\pi}\mathbf{G}\mathscr{W}^{1/2}(\mathbf{x}) \quad \text{ with } \nabla_{\mathbf{x}} \cdot \mathbf{v}^*(\mathbf{x}) = \mathbf{g}^2(\mathbf{x}) + \frac{3}{\pi}\mathbf{G}\nabla_{\mathbf{x}} \cdot \mathscr{W}^{1/2}(\mathbf{x}).$$

Since the right-hand side of the divergence equation is now mean value free, Proposition 4.2 gives $\mathbf{v}^* \in \overset{\circ}{H}{}^1(\Lambda_1)$ and the estimate

$$\|\nabla_{\mathbf{x}}\mathbf{v}^*; L^2(\Lambda_1)\| \leq C(\Lambda_1)\|\mathbf{g}^2 + \frac{3}{\pi}\mathbf{G}\nabla_{\mathbf{x}} \cdot \mathscr{W}^{1/2}; L^2(\Lambda_1)\| \leq C(\Lambda_1)\|\mathbf{g}^2; L^2(\Lambda_1)\|. \tag{4.16}$$

With (4.15), we also have

$$\|\nabla \mathbf{v}; H^1(\Lambda_1)\| \leq C\|\mathbf{g}^2; L^2(\Lambda_1)\|. \tag{4.17}$$

If we apply the relations (4.13) to obtain v with $\nabla \cdot v = g^2$ on Λ_R, we see

$$\begin{aligned} \|\nabla_y v'; L^2(\Lambda_R)\| &= R\|\nabla_{\mathbf{y}}\mathbf{v}'; L^2(\Lambda_1)\|, \\ \|\nabla_z v'; L^2(\Lambda_R)\| &= R^2\|\nabla_z \mathbf{v}'; L^2(\Lambda_1)\|, \\ \|\nabla_y v_z; L^2(\Lambda_R)\| &= \|\nabla_{\mathbf{y}}\mathbf{v}_z; L^2(\Lambda_1)\|, \\ \|\nabla_z v_z; L^2(\Lambda_R)\| &= R\|\nabla_z \mathbf{v}_z; L^2(\Lambda_1)\|. \end{aligned} \tag{4.18}$$

Together with (4.14) and Poincaré's inequality this leads to

$$\|v; H^1(\Lambda_R)\| \leq CR\|g^2; L^2(\Lambda_R)\| = CR\|g^2; L^2(\Omega_R)\|. \tag{4.19}$$

Although the support of g^2 is contained in $\Omega_R \setminus \Omega_{3R_0}$, the support of v may be larger. Thus we cut v again, using the same function χ as in formulae (2.2). Put $\chi^{R_0}(x) = \chi(R_0^{-1}(r))$, then clearly the vector field $(1-\chi^{R_0})v$, extended by zero, belongs to $\mathcal{H}(\Omega_R)$, moreover,

$$\|(1-\chi^{R_0})v; H^1(\Omega_R)\| \leq C(R_0, \chi)\|v; H^1(\Lambda_R)\| \tag{4.20}$$

and

$$\nabla \cdot \big((1-\chi^{R_0})v\big) = \big(1-\chi^{R_0}\big)\, g^2 - \big(\nabla(1-\chi^{R_0})\big) \cdot v = g^2 - \big(\nabla\chi^{R_0}\big) \cdot v.$$

The support of $(\nabla\chi^{R_0}) \cdot v$ is contained in the annular domain $\Xi_{2R_0} = \Omega_{2R_0} \setminus \overline{\Omega}_{R_0}$, we calculate the mean-value over Ξ_{2R_0}:

$$\int_{\Xi_{R_0}} (\nabla\chi^{R_0}) \cdot v \, dx = \int_{\partial\Xi_{2R_0}} \chi^{R_0} v \cdot n \, do - \int_{\Xi_{2R_0}} \chi^{R_0}(\nabla \cdot v)\, dx. \tag{4.21}$$

The last integral vanishes, since $\nabla \cdot v = 0$ on Ξ_{2R_0}. The boundary integral splits into integrals over $\partial\Xi_{R_0} \cap \Sigma_R$, where $v = 0$, and integrals over the lateral surfaces $\Gamma_{R_0} \cup \Gamma_{2R_0}$. On Γ_{2R_0} we have $\chi^{R_0} = 0$, while $\chi^{R_0} = 1$ on Γ_{R_0}. Here we use $\nabla \cdot v = 0$ in Λ_{R_0} and $v(y,z) = 0$ for $|z| = 1/2$ to see that $\int_{\Gamma_{R_0}} v \cdot n do = 0$. Hence with Proposition 4.2 again, we find $\widetilde{v} \in \overset{\circ}{H}{}^1(\Xi_{R_0})^3$ solving $\nabla \cdot \widetilde{v} = \big(\nabla\chi^{R_0}\big) \cdot v$ and

$$\|\widetilde{v}; H^1(\Xi_{R_0})\| \leq C\| \big(\nabla\chi^{R_0}\big) \cdot v; L^2(\Xi_{R_0})\| \leq \|v; L^2(\Xi_{R_0})\| \leq \|v; H^1(\Lambda_R)\|. \tag{4.22}$$

We extend $\widetilde{v}$ by zero and put $w^2 = (1-\chi^{R_0})v + \widetilde{v}$, then $w^2 \in \mathcal{H}(\Omega_R)$, $\nabla \cdot w^2 = g^2$, and estimates (4.19), (4.20) and (4.22) lead to

$$\|w^2; H^1(\Omega_R)\| \leq C\, R\|g^2; L^2(\Omega_R)\|. \tag{4.23}$$

The final representation of w reads

$$w(x) = w^1(x) + w^2(x) - \frac{3}{\pi}G_{3R_0}\mathscr{W}^{R_0}(x),$$

and due to the construction we have

$$w_i'(x)|_{\Gamma_R} = (G_{3R_0} + \mathbf{G}) \left(z^2 - \frac{1}{4}\right) \frac{1}{R^2} x_i, \quad i = 1, 2,$$

which implies

$$\|w'; L^2(\Gamma_R)\| \le C(G_{3R_0} + \mathbf{G}) R^{-1/2}.$$

From here the estimate (4.8) follows with (4.11), (4.12), (4.15) and (4.23) if we observe that $\|\mathscr{W}^{R_0}; H^1(\Omega_R)\|$ remains bounded independent of $R \ge R_0$. □

In the next step we derive estimates for the bilinear form $\mathbf{q}_R$.

Proposition 4.4 *The bilinear form* $\mathbf{q}_R$ *is symmetric and nonnegative, and for* $v, w \in \mathcal{H}(\Omega_R)$*, the following inequality is valid with a constant* C *independent of* $R \ge R_0$*:*

$$|\mathbf{q}_R(v, w)| \le C \left(\|v'; H^1(\Xi_R)\| + R^{1/2} \|\overline{v}\,'; L^2(\mathbb{S}_R)\|\right) \left(\|w'; H^1(\Xi_R)\| + R^{1/2} \|\overline{w}\,'; L^2(\mathbb{S}_R)\|\right),$$

and the operator $\overline{\,\cdot\,}$ *is defined as in* (3.5)*.*

Proof: The symmetry follows immediately from the symmetry properties of Π_R and Π_R^{-1}. Furthermore, only v', w' are involved the definition of $\mathbf{q}_R$. Using the notation (4.1) and formula (3.5) again, we define $\overline{v}$, $\overline{w}$ on the two-dimensional annulus A_R, and clearly

$$\|\overline{v}\,'; H^1(A_R)\| \le C \|v'; H^1(\Xi_R)\|,$$

with a constant independent of R. Similar as in (4.13), we put

$$\mathbf{y} = \frac{y}{R}, \qquad \overline{\mathbf{v}}\,'(\mathbf{y}) = \overline{v}\,'(R\mathbf{y}), \qquad \overline{\mathbf{w}}\,'(\mathbf{y}) = \overline{w}\,'(R\mathbf{y}).$$

Formula $(3.12)_1$ leads to

$$\begin{aligned} |(\Pi_R \overline{v}_r, \overline{w}_r)_{\mathbb{S}_R}| &= |(\Pi \overline{\mathbf{v}}_r, \overline{\mathbf{w}}_r)_{\mathbb{S}}| \le C \|\overline{\mathbf{v}}_r; H^1(A_1)\| \, \|\overline{\mathbf{w}}_r; H^1(A_1)\| \\ &\le C \left(R^{-1} \|\overline{v}\,'; L^2(A_R)\| + \|\nabla_y \overline{v}\,'; L^2(A_R)\|\right) \left(R^{-1} \|\overline{w}\,'; L^2(A_R)\| + \|\nabla_y \overline{w}\,'; L^2(A_R)\|\right), \end{aligned}$$

To obtain the last inequality we used similar reasonings as in (4.14) and (4.18). By $(3.12)_2$, we get

$$\begin{aligned} |(\Pi_R^{-1} (\overline{v}_r)_\bullet, (\overline{w}_r)_\bullet)_{\mathbb{S}_R}| &= R^2 |(\Pi^{-1} (\overline{\mathbf{v}}_r)_\bullet, (\overline{\mathbf{w}}_r)_\bullet)_{\mathbb{S}}| \le C\, R^2 \, \|\overline{\mathbf{v}}_r; L^2(\mathbb{S})\| \, \|\overline{\mathbf{w}}_r; L^2(\mathbb{S})\| \\ &= C\, R \, \|\overline{\mathbf{v}}_r; L^2(\mathbb{S}_R)\| \, \|\overline{\mathbf{w}}_r; L^2(\mathbb{S}_R)\|, \end{aligned}$$

while the estimate of the term $R^{-1} |(\overline{v}_\varphi, \overline{w}_\varphi)_{\mathbb{S}_R}|$ is obvious. Collecting all the inequalities gives the estimate. Since

$$(\Pi_R \overline{v}_r, \overline{v}_r)_{\mathbb{S}_R} = \|\overline{\mathbf{v}}_r; H^{1/2}(\mathbb{S})\|^2, \quad (\Pi_R^{-1} (\overline{v}_r)_\bullet, (\overline{v}_r)_\bullet)_{\mathbb{S}_R} = R^2 \|(\overline{\mathbf{v}}_r)_\bullet; H^{-1/2}(\mathbb{S})\|^2,$$

we obtain also $\mathbf{q}_R(v, v) \ge 0$ for all $v \in \mathcal{H}(\Omega_R)$. □

With the previous estimate in mind, we define the following R-dependent norms on $\mathcal{H}(\Omega_R)$ and its dual space $\mathcal{H}'(\Omega_R)$:

$$\begin{aligned} \|v; \mathcal{H}(\Omega_R)\|^2 &= \|v; H^1(\Omega_R)\|^2 + \mathbf{q}_R(v, v) && \text{for } v \in \mathcal{H}(\Omega_R), \\ \|\Phi; \mathcal{H}'(\Omega_R)\| &= \sup\{|\Phi(v)| : \|v; \mathcal{H}(\Omega_R)\| \le 1\} && \text{for } \Phi \in \mathcal{H}'(\Omega_R). \end{aligned} \tag{4.24}$$

Theorem 4.5 *For any $\Phi \in \mathcal{H}'(\Omega_R)$, there exists a unique weak solution $U = (V,P) \in \mathcal{H}_\sigma(\Omega_R)\times L^2(\Omega_R)$ to problem (4.5), and the following estimate is valid with a constant independent of $R > R_0$ and Φ*

$$\|V;\mathcal{H}(\Omega_R)\| + R^{-1}\|P;L^2(\Omega_R)\| \le C\,\|\Phi;\mathcal{H}'(\Omega_R)\|. \tag{4.25}$$

In particular, we obtain for the solution (v^R,p^R) to the approximation problem (1.4) that

$$\|v^R;H^1(\Omega_R)\| + R^{-1}\|p^R;L^2(\Omega_R)\| \le C\,\|f;L^2(\Omega_R)\|. \tag{4.26}$$

Proof: On $\mathcal{H}_\sigma(\Omega_R)$, we consider the bilinear form

$$\langle\langle v,w\rangle\rangle = \nu(\nabla v,\nabla w)_{\Omega_R} + \mathbf{q}_R(v,w).$$

Since Poincaré's inequality, $\|v;L^2(\Omega_R)\| \le c\,\|\nabla v;L^2(\Omega_R)\|$, is valid for $v \in \mathcal{H}(\Omega_R)$ with a constant c independent of R on Ω_R, it is clear that $\langle\langle\,\cdot\,\cdot\rangle\rangle$ is coercive and continuous. By means of the Lax Milgram lemma, we find a unique $V \in \mathcal{H}_\sigma(\Omega_R)$ such that

$$\Phi(w) = \langle\langle V,w\rangle\rangle \text{ for any } w \in \mathcal{H}_\sigma(\Omega_R). \tag{4.27}$$

The identity (4.27) applied to $w = V$, together with Poincaré's inequality, leads to

$$\|V;H^1(\Omega_R)\| + \mathbf{q}_R(V,V)^{1/2} \le C\|\Phi;\mathcal{H}'(\Omega_R)\|,$$

where C is independent of R and Φ.

The pressure P is obtained by the following well known argument: From Lemma 4.3 we conclude that for any $g \in L^2(\Omega_R)$ there exists a solution $Dg \in \mathcal{H}(\Omega_R)$ to the problem $\nabla\cdot Dg = g$, while inequality (4.8) together with Proposition 4.4 applied to $v = w = Dg$ lead to the estimate

$$\|Dg;\mathcal{H}(\Omega_R)\| \le C\,R\|g;L^2(\Omega_R)\|$$

Thus we obtain a continuous linear functional F on $L^2(\Omega_R)$ by

$$F(g) = \Phi(Dg) - (\nabla V,\nabla Dg)_{\Omega_R} - \mathbf{q}_R(V,Dg), \quad g \in L^2(\Omega_R). \tag{4.28}$$

Moreover, we have

$$\begin{aligned} |F(g)| \;&\le\; \Big(\,\|\Phi;\mathcal{H}'(\Omega_R)\|\,\|Dg;\mathcal{H}(\Omega_R)\| + \|\nabla V;L^2(\Omega_R)\|\,\|\nabla Dg;L^2(\Omega_R)\| \\ &\qquad + \mathbf{q}_R(V,V)^{1/2}\mathbf{q}_R(Dg,Dg)^{1/2}\,\Big) \\ &\le\; C\,R\,\|\Phi,\mathcal{H}'(\Omega_R)\|\,\|g;L^2(\Omega_R)\| \end{aligned}$$

with a constant C independent on R. By the Riesz representation theorem there exists a unique $P \in L^2(\Omega_R)$ with $F(g) = (P,g)_{\Omega_R}$ and

$$\|P;L^2(\Omega_R)\| \le C\,R\,\|\Phi;\mathcal{H}'(\Omega_R)\|$$

with the same constant C as above. Now, if $w \in \mathcal{H}(\Omega_R)$ is arbitrary, then $w = D\nabla\cdot w + w_0$, where $w_0 \in \mathcal{H}_\sigma(\Omega_R)$, and from (4.27) and (4.28) we obtain

$$F(\nabla\cdot w) = (f,w)_{\Omega_R} - (\nabla V,\nabla w)_{\Omega_R} - \mathbf{q}_R(V,w),$$

which means that (V,P) is a weak solution to (4.3). □

Theorem 4.5 also gives the clue to the error estimate. Keeping (4.6) in mind we see that the pair of errors

$$v^{er} = v^\infty|_{\Omega_R} - v^R, \quad p^{er} = p^\infty|_{\Omega_R} - p^R, \tag{4.29}$$

is a weak solution to the problem (4.3) with $\Phi = \Phi^{er}$ and

$$\Phi^{er}(w) = \mathbf{q}_R(\widetilde{v}^\infty,w) + (\nu\partial_r\widetilde{v}^\infty - \widetilde{p}^\infty e_r, w)_{\Gamma_R} \tag{4.30}$$

for any $w \in \mathcal{H}(\Omega_R)$.

Proposition 4.6 *For $v \in \mathcal{H}_\sigma(\Omega_R) \cap H^2(\Omega_R)$, $p \in H^1(\Omega_R)$ and $w \in \mathcal{H}(\Omega_R)$, the following estimates hold true with constants independent of R:*

$$|\mathbf{q}_R(v,w)| \leq C\ \big(\ \|v; L^2(\Xi_R)\| + R\|\nabla_y v; L^2(\Xi_R)\|\ \big)\ \|w; \mathcal{H}(\Omega_R)\| \tag{4.31}$$

$$\begin{aligned} |(\nu\partial_r v - pe_r, w)_{\Gamma_R}| \ &\leq\ C\ \Big(\ \|\nabla_y v; L^2(\Xi_R)\| + R\|\nabla_y^2 v; L^2(\Xi_R)\| \\ &\qquad + R^{-1}\|p; L^2(\Xi_R)\| + \|\nabla_y p; L^2(\Xi_R)\|\ \Big)\ \|w, \mathcal{H}(\Omega_R)\| \end{aligned} \tag{4.32}$$

Proof: Clearly,

$$|\mathbf{q}_R(v,w)| \leq \mathbf{q}_R(v,v)^{1/2}\mathbf{q}_R(w,w)^{1/2} \leq C\mathbf{q}_R(v,v)^{1/2}\|w; \mathcal{H}(\Omega_R)\|.$$

Now Formula (3.19) gives

$$\begin{aligned} \mathbf{q}_R(v,v) \ &=\ \frac{\nu}{30}\Big\{\big(\Pi_R\overline{v_r}, \overline{v_r}\ \big)_{\mathbb{S}_R} + \big(\Pi_R\overline{v_\varphi}, \overline{v_\varphi}\ \big)_{\mathbb{S}_R} \\ &\quad + \frac{1}{R}\big(\overline{v_r}, \overline{v_r}\ \big)_{\mathbb{S}_R} + \frac{1}{R}\big(\overline{v_\varphi}, \overline{v_\varphi}\ \big)_{\mathbb{S}_R} + 10\ \big(\Pi_R^{-1}\big(\overline{v_r}\,\big)_\bullet, \big(\overline{v_r}\,\big)_\bullet\ \big)_{\mathbb{S}_R}\Big\}. \end{aligned}$$

The last term is the critical one with respect to R. Using the notations of Proposition 4.4, we find

$$\begin{aligned} &(\Pi_R^{-1}(\bar{v}_r)_\bullet, (\bar{v}_r)_\bullet)_{\mathbb{S}_R} = R^2(\Pi^{-1}(\bar{\mathbf{v}}_r)_\bullet, (\bar{\mathbf{v}}_r)_\bullet)_{\mathbb{S}} \\ &\leq\ R^2\|(\bar{\mathbf{v}}_r)_\bullet; L^2(\mathbb{S})\|^2 \leq CR^2\left(\|\bar{\mathbf{v}}_r; L^2(A_1)\|^2 + \|\nabla_{\mathbf{y}}\bar{\mathbf{v}}_r; L^2(A_1)\|^2\right) \\ &=\ C\left(\|\bar{v}_r; L^2(A_R)\|^2 + R^2\|\nabla_y\bar{v}_r; L^2(A_R)\|^2\right). \end{aligned}$$

Together with the arguments used in Proposition 4.4 to estimate the other terms, we arrive at the estimate (4.31).
The same scaling argument leads to

$$\begin{aligned} &|(\partial_r v, w)_{\Gamma_R}| = |(\partial_r\mathbf{v}, \mathbf{w})_{\Gamma_1}| \leq \|\partial_r\mathbf{v}; L^2(\Gamma_1)\|\,\|\mathbf{w}; L^2(\Gamma_1)\| \\ &\leq\ C\ \big(\ \|\nabla_{\mathbf{y}}\mathbf{v}; L^2(\Xi_1)\| + \|\nabla_{\mathbf{y}}^2\mathbf{v}; L^2(\Xi_1)\|\ \big)\big(\ \|\mathbf{w}; L^2(\Xi_1)\| + \|\nabla_{\mathbf{y}}\mathbf{w}; L^2(\Xi_1)\|\ \big) \\ &=\ C\ \big(\ \|\nabla_y v; L^2(\Xi_R)\| + R\|\nabla_y^2 v; L^2(\Xi_R)\|\ \big)\big(\ R^{-1}\|w; L^2(\Xi_R)\| + \|\nabla_y w; L^2(\Xi_R)\|\ \big)\ . \end{aligned}$$

The last terms are majorized by the right hand side of (4.32), observe that the derivatives in z are not needed to estimate the L^2-norm of the traces on Γ_1. The scaling argument applied to the term containing p gives

$$\begin{aligned} &|(p\,e_r, w)_{\Gamma_R}| = |R(\mathbf{p}e_r, \mathbf{w})_{\Gamma_1}| \leq \|\mathbf{p}; L^2(\Gamma_1)\|\,\|\mathbf{w}; L^2(\Gamma_1)\| \\ &\leq\ C\ \big(\ \|\mathbf{p}; L^2(\Xi_1)\| + \|\nabla_{\mathbf{y}}\mathbf{p}; L^2(\Xi_1)\|\ \big)\big(\ \|\mathbf{w}; L^2(\Xi_1)\| + \|\nabla_{\mathbf{y}}\mathbf{w}; L^2(\Xi_1)\|\ \big) \\ &=\ C\ \big(\ R^{-1}\|p; L^2(\Xi_R)\| + \|\nabla_y p; L^2(\Xi_R)\|\ \big)\big(\ R^{-1}\|w; L^2(\Xi_R)\| + \|\nabla_y w; L^2(\Xi_R)\|\ \big)\ . \end{aligned}$$

Here $\mathbf{p}(\mathbf{y}) = p(R\mathrm{y})$, e_r is the unit vector in r-direction. Again this can be estimated by the right hand side of (4.32). □

Now we can formulate the main result for the linear problem.

Theorem 4.7 *Under hypothesis* (2.6) *of Theorem* 2.2*, the approximation problem problem* (1.4) *with the ABC constructed in Section* 3 *admits a unique solution $(v^R, p^R) \in H^1(\Omega_R)^3 \times L_2(\Omega_R)$. This solution and the solution (v^∞, p^∞) of the original problem* (1.2)*, given by Lemma* 2.1*, satisfy the following error estimate*

$$\begin{aligned} &\|v^\infty|_{\Omega_R} - v^R; H^1(\Omega_R)\| + R^{-1}\,\|p^\infty|_{\Omega_R} - p^R; L_2(\Omega_R)\| \\ &\leq\ C\,R^{3+l-\gamma}\,\|f; \mathcal{W}_\gamma^{l+2}(\Omega)\| \leq C\,R^{-N}\,\|f; \mathcal{W}_\gamma^{l+2}(\Omega)\| \end{aligned} \tag{4.33}$$

where the constant C is independent of $R \geq R_0$ and $f \in \mathcal{W}_\gamma^{l+2}(\Omega)^3$ (see (2.1) *for the definition of the norm).*

Proof Recalling (4.29) and (4.30), Proposition 4.6 leads to

$$\begin{aligned}\|\Phi^{er};\mathcal{H}'(\Omega_R)\| &\le C\ \Big(\ \|\widetilde{v}^\infty;L^2(\Xi_R)\|+R\|\nabla_y\widetilde{v}^\infty;L^2(\Xi_R)\|+R\|\nabla_y^2\widetilde{v}^\infty;L^2(\Xi_R)\|\\ &\quad +R^{-1}\|\widetilde{p}^\infty;L^2(\Xi_R)\|+\|\nabla_y\widetilde{p}^\infty;L^2(\Xi_R)\|\ \Big)\\ &\le\ CR^{3+l-\gamma}\ \Big(\ R^{\gamma-3-l}\|\widetilde{v}^\infty;L^2(\Xi_R)\|+R^{\gamma-2-l}\|\nabla_y\widetilde{v}^\infty;L^2(\Xi_R)\|\\ &\quad +R^{\gamma-1-l}\|\nabla_y^2\widetilde{v}^\infty;L^2(\Xi_R)\|+R^{\gamma-4-l}\|\widetilde{p}^\infty;L^2(\Xi_R)\|+R^{\gamma-3-l}\|\nabla_y\widetilde{p}^\infty;L^2(\Xi_R)\|\ \Big)\end{aligned}\tag{4.34}$$

On Ξ_R we have $R/2\le r\le R$, thus with definition (2.1), the right hand side of (4.34) can be estimated by

$$C\ \Big(\ \|\widetilde{v}^{\infty\prime};\mathcal{W}^{l+2}_{\gamma-1}(\Omega)\|+\|\widetilde{v}^\infty_z;\mathcal{W}^{l+2}_{\gamma}(\Omega)\|+\|\widetilde{p}^\infty;\mathcal{W}^{l+3}_{\gamma-1}(\Omega)\|\ \Big),$$

from which estimate (4.33) follows by means of (2.8) and (4.25). □

5 Strong solutions to the Navier–Stokes problem and their asymptotic properties

Let us consider the Navier-Stokes problem (1.3). The proof for the existence of weak solutions to the Navier-Stokes problem (1.3) is standard using solutions on a sequence of expanding domains (see [33, p.169 ff], e.g.). Strong solutions with special decay properties are usually obtained by means of the Banach fixed point theorem, in this case one has to require smallness conditions for the data. The following assertion is a consequence of results proved in [20, 27].

Theorem 5.1 *Let $l\in\mathbb{N}_0$, $N\le 3$, and $f\in\mathcal{W}^{l+2}_{\gamma}(\Omega)^3$ with $\gamma\in(l+3+N,l+4+N)$. There exists $\varepsilon_0>0$ such that in the case $\|f;L_2(\Omega)\|\le\varepsilon_0$ the Navier-Stokes problem (1.3) admits a weak solution $(v^\infty,p^\infty)\in H^1(\Omega)^3\times L^2_\beta(\Omega)$, where $-1<\beta<0$. This solution is unique and takes the asymptotic form (2.4). The remainders $(\widetilde{v}^\infty,\widetilde{p}^\infty)$ accomplish again the inclusions (2.7), while $(\widetilde{v}^\infty,\widetilde{p}^\infty)$ and the coefficients a_j,b_j of representation (2.4) comply with estimate (2.8) as long as the indices l,γ and N satisfy the restriction indicated above.*

Proof. The existence of a weak solution for $f\in L^2(\Omega)$ is well known. The proof of the asymptotic properties uses a bootstrap argument similar to the proof of regularity results for solutions to the Navier-Stokes system. With suitable estimates for the the nonlinear term at hand one can shift it to the right-hand side of (1.3) and use results for the linear system, thus successively improve the properties of the solutions to the nonlinear problem. Since $\gamma-l-N-2\in(1,2)$, we have

$$\|f;L^2(\Omega)\|\le\|f;L^2_{\gamma-l-N-2}(\Omega)\|\le\|f;L^2_{\gamma-l-2}(\Omega)\|\le\|f;\mathcal{W}^{l+2}_{\gamma}(\Omega)\|.$$

If $\|f;L^2(\Omega)\|$ is small enough, the weak solution to (1.3) is uniquely determined (see [21, Section 4.2]). From Theorem 4.2 in [27] we obtain improved decay properties for v^∞ at infinity, namely, if $N\ge 1$, for any $\mu<0$, we have

$$\begin{aligned}&v^\infty\in\mathcal{W}^{l+3}_{\mu+l+4}(\Omega),\quad v_z\in\mathcal{W}^{l+2}_{\mu+l+4}(\Omega),\\ &p^\infty\in\mathcal{W}^{l+3}_{\mu+l+3}(\Omega),\quad \partial_z p\in\mathcal{W}^{l+2}_{\mu+l+4}(\Omega),\end{aligned}\tag{5.1}$$

in particular $v^{\infty,\prime}\in L^2_{\mu+1}(\Omega)$, $v^\infty_z\in L^2_{\mu+2}(\Omega)$, analogous conclusions follow for the derivatives and for p^∞. Lemma 3.4 in [27] supplies us also with estimates for the nonlinear term $(v^\infty\cdot\nabla)v^\infty$ from which we gain

$$(v^\infty\cdot\nabla)v^\infty\in\mathcal{W}^{l+2}_{2\mu+4+l+2}(\Omega).\tag{5.2}$$

Since μ can be arbitrarily close to zero, this implies $f - (v^\infty \cdot \nabla)v^\infty \in \mathcal{W}^{l+2}_{\widetilde{\gamma}}(\Omega)$ where $\widetilde{\gamma} = \gamma$, if $N = 1, 2$; for $N = 3$, the expression $\widetilde{\gamma} - l$ can be any number in the interval $(5, 6)$ (in particular it can be arbitrarily close to 6). Now Theorem 5.3 in [20], also in its simplified formulation as in Theorem 2.2, implies the asymptotic representation $v^\infty(y, z) = V^\infty(y, z) + \widetilde{v}^\infty(y, z)$, where V^∞ has the form (2.4) with $N = 2$. Now we repeat the argument. Since

$$|\partial_z^j \partial_y^\alpha V^\infty(y, z)| = O(r^{-2-|\alpha|}), \text{ as } r \to \infty,$$

elementary, but lengthy calculations show that indeed $(v^\infty \cdot \nabla)v^\infty \in \mathcal{W}^{l+2}_{\gamma}(\Omega)$ is valid, even if $\gamma - l \in (N+3, N+4)$ for $N = 3$ – here one has to apply [27, Lemma 3-4] again. Now Theorem 2.2 leads to the asymptotic representation (2.2)-(2.5) up to $N = 3$, a suitable decay of f provided. The estimate (2.8) for the solutions to the nonlinear problem follows from the quoted results of [20, 21, 27] and Theorem 2.2. □

Comparing this result with those of Theorem 2.2 on the Stokes problem we see that in addition to the smallness condition for the data, the decay rate of the remainder $\widetilde{v}^\infty, \widetilde{p}^\infty$ is limited in representation (2.4). To understand this fact let us have a closer look on how the nonlinearity influences the asymptotic. According to the Theorem 5.1 we have for $r > 2R_0$

$$\begin{aligned} &v^\infty = \sum_{j=1}^{3} v^j + \widetilde{v}^{\infty\prime}, \quad p^\infty = \sum_{j=1}^{3} p^j + \widetilde{p}^{\infty\prime}, \qquad \text{with} \\ &v^{j\prime}(y, z) = \frac{1}{2\nu}\left(z^2 - \frac{1}{4}\right)\nabla_y p^j(y), \quad v_z^j = 0, \quad p^j(y) = r^{-j}(a_j \cos(j\varphi) + b_j \sin(j\varphi)). \end{aligned} \tag{5.3}$$

In view of the asymptotic procedure developed in [15, 18, 20] this solution of the Navier-Stokes problem (1.3) can be decomposed further into a formal series in powers of r. Let us show that, in contrast to (2.4), in this series there appear functions in y which are not harmonic. By a proper choice of the angular variable φ, the main asymptotic term in (5.3) can be always reduced to the expression

$$v^{1,\prime}(r, \phi) = c_1 \frac{1}{2\nu}\left(z^2 - \frac{1}{4}\right)\nabla_y\left(r^{-1}\sin\varphi\right) = c_1 \frac{1}{2\nu}\left(z^2 - \frac{1}{4}\right) r^{-2}\big(-\sin 2\varphi, \cos 2\varphi\big) \tag{5.4}$$

where $c_1 = (a_1^2 + b_1^2)^{1/2}$, the index y indicates the derivatives with respect to the cartesian coordinates (y_1, y_2). The convective term $\big(v^{1\prime} \cdot \nabla_y\big)v^{1\prime}$ calculated for the vector function (5.4) takes the form

$$\big(v^{1\prime} \cdot \nabla_y\big)v^{1\prime} = c_1^2 \frac{1}{2\nu^2}\left(z^2 - \frac{1}{4}\right)^2 r^{-5}\left(-\cos\varphi, -\sin\varphi\right). \tag{5.5}$$

In finding the next summands in (5.3) this expression (5.5) has to be compensated by the particular power-law solution

$$\begin{aligned} V'(y, z) &= \mathcal{Z}(z)\, r^{-5}\left(\cos\varphi, \sin\varphi\right) + \frac{1}{2\nu}\left(z^2 - \frac{1}{4}\right)\nabla_y\, r^{-4}\,\mathcal{P}(\varphi), \\ V_z(y, z) &= r^{-6}\, W(\varphi, z), \\ P(y, z) &= r^{-4}\,\mathcal{P}(\varphi) + r^{-6}\, Q(\varphi, z). \end{aligned} \tag{5.6}$$

Inserting (5.6) into the Stokes problem with the right-hand side (5.5) in the layer (1.1) and collecting coefficients at same powers r^5, we first arrive to the Dirichlet problem on the interval $\Upsilon = (-1/2, 1/2)$:

$$-\nu \frac{d^2}{dz^2}\mathcal{Z}(z) = c_1^2 \frac{1}{2\nu^2}\left(z^2 - \frac{1}{4}\right)^2, \; z \in \Upsilon, \qquad \mathcal{Z}\Big(\pm\frac{1}{2}\Big) = 0.$$

Thus, we obtain the coefficient in $(5.6)_1$

$$\mathcal{Z}(z) = -c_1^2 \frac{1}{2\nu^3} \left\{ \frac{z^6}{30} - \frac{z^4}{24} + \frac{z^2}{32} - 2^{-7}\frac{11}{15} \right\}. \tag{5.7}$$

The mean value of solution (5.7) can be easily computed as follows:

$$\int\limits_{-1/2}^{1/2} \mathcal{Z}(z)dz = \frac{1}{2} \int\limits_{-1/2}^{1/2} \left(z^2 - \frac{1}{4}\right) \frac{d^2}{d\,z^2} \mathcal{Z}(z)\,dz = -c_1^2 \frac{1}{4\nu^3} \int\limits_{-1/2}^{1/2} \left(z^2 - \frac{1}{4}\right)^3 dz = c_1^2 \frac{1}{\nu^3}\frac{1}{2^4}\frac{1}{35}. \tag{5.8}$$

Collecting the coefficients at the powers r^6, there appears the one-dimensional Stokes problem with the parameter $\varphi \in [0, 2\pi)$

$$\begin{array}{rcll} -\nu \dfrac{d^2}{dz^2} W(\varphi, z) + \dfrac{d}{dz} Q(\varphi, z) & = & 0, & z \in \Upsilon, \\ -\dfrac{d}{dz} W(\varphi, z) & = & r^6\, \nabla_y \cdot V'(y, z), & z \in \Upsilon, \\ W\big(\varphi, \pm 1/2\big) & = & 0. & \end{array} \tag{5.9}$$

By virtue of $(5.6)_1$ and (5.8), the compatibility condition for problem (5.9), $\int_{-1/2}^{1/2} \nabla_y \cdot V'(y, z)\, dz = 0$, turns into the Poisson equation

$$\begin{aligned} -\frac{1}{12\nu}\, \Delta_y \left(r^{-4}\, \mathcal{P}(\varphi)\right) &= -\int\limits_{-1/2}^{1/2} \mathcal{Z}(z)\, dz\ \nabla_y \cdot \left(r^{-5}(\cos\varphi, \sin\varphi)\right) = \\ &= \frac{1}{140} \frac{c^2}{\nu^3 r^6}, \qquad r \neq 0. \end{aligned} \tag{5.10}$$

Note that the factor at the Laplacian is but the integral of $(2\nu)^{-1}(z^2 - 1/4)$ over Υ. Since we are looking for P decaying at infinity, we obtain:

$$\mathcal{P}(\varphi) = -\frac{3\, c_1^2}{560\, \nu^2}.$$

Since the compatibility condition is fulfilled, problem (5.9) admits the solution

$$\begin{aligned} W(\varphi, z) &= -\frac{c_1^2}{6720\, \nu^3} (64\, z^7 - 112\, z^5 + 44\, z^3 - 5\, z), \\ Q(\varphi, z) &= -\frac{c_1^2}{6720\, \nu^2} (448\, z^6 - 560\, z^4 + 132\, z^2 - 5). \end{aligned}$$

We see that already the next power law term for p^∞ cannot be harmonic unless $c_1 = 0$, and therefore in the case of the nonlinear problem, the structure of the representation (2.2)-(2.5) is valid only only up to $N = 3$ in general.

6 Error estimates for the Navier–Stokes problem with ABC

Although Theorem 5.1 does not provide the whole asymptotic series in harmonics for the solution (v^∞, p^∞) of the Navier-Stokes problem (1.3), we use the same operator M_R constructed in Section 3 as for the linear problem and formulate the nonlinear problem in the truncated domain Ω_R as follows:

$$\begin{array}{rll} -\nu\Delta v^R + (v^R \cdot \nabla) v^R + \nabla p^R & = f & \text{in} \quad \Omega_R, \\ \nabla \cdot v^R & = 0 & \text{in} \quad \Omega_R, \\ v^R & = 0 & \text{on} \quad \Sigma_R, \\ M_R(v^R, p^R) & = 0 & \text{on} \quad \Gamma_R. \end{array} \tag{6.1}$$

Let us briefly recall how the Banach fixed point principle can be applied to solve Navier-Stokes problems. Problem (6.1) can we written in an abstract way

$$\mathbf{S}\mathbf{u} + \mathbf{N}(\mathbf{u}, \mathbf{u}) = \mathbf{f}, \tag{6.2}$$

where $\mathbf{S} : X \to Y$ is a linear operator between two Banach-spaces $X = \mathcal{H}_\sigma(\Omega_R) \times L^2(\Omega_R)$, and $Y = \mathcal{H}'(\Omega_R)$, while $\mathbf{N} : X \times X \to Y$ is bilinear. Further $\mathbf{u} = (v, p)$, $\mathbf{S}\mathbf{u} = -\Delta v + \nabla p$, $\mathbf{N}(u, u) = ((v \cdot \nabla)v, 0)$. Since $\mathbf{S}$ is invertible, equation (6.2) is equivalent to the fix point equation

$$\mathbf{u} = \mathbf{S}^{-1}(\mathbf{f} - \mathbf{N}(\mathbf{u}, \mathbf{u})), \tag{6.3}$$

and we end up with the following assertion: Suppose $\mathbf{S}^{-1}$ and $\mathbf{N}$ are continuous with $\|\mathbf{S}^{-1}\| \le C_\mathbf{S}$ and $\|\mathbf{N}(\mathbf{u}, \mathbf{v}); Y\| \le C_\mathbf{N}\|\mathbf{u}; X\| \, \|\mathbf{v}; X\|$, then for any $\mathbf{f}$ with $\|\mathbf{f}; Y\| \le (4C_\mathbf{S}^2 C_\mathbf{N})^{-1}$ there exists a unique solution $\mathbf{u}$ to (6.3) in the ball $\|\mathbf{u}; X\| < (2C_\mathbf{S} C_\mathbf{N})^{-1}$, and this solution fulfils $\|\mathbf{u}; X\| \le 2C_\mathbf{S}\|\mathbf{f}; Y\|$ (see, e.g. [26, Lemma 5.1] for more details). Theorem 4.5 ensures that the inverse $\mathbf{S}^{-1}$ of the linear part has a norm bounded independent of R. To control the nonlinearity we have to watch carefully the embedding constants of some Sobolev embeddings.

Lemma 6.1 *For any $v, V, w \in \mathcal{H}$ the following inequalities hold with constants independent of $R \ge R_0$:*

$$\|v; L_4(\Omega_R)\|^2 \le \|v; L_2(\Omega_R)\| \, \|v; L_6(\Omega_R)\| \le c \, \|\nabla v; L_2(\Omega_R)\|^2 \tag{6.4}$$

$$\begin{aligned} \left|\big((v \cdot \nabla)V, w\big)_{\Omega_R}\right| &\le c \, \|v; L_4(\Omega_R)\| \, \|\nabla V; L_2(\Omega_R)\| \, \|w; L_4(\Omega_R)\| \\ &\le C_N \, \|\nabla v; L^2(\Omega_R)\| \, \|\nabla V; L^2(\Omega_R)\| \, \|\nabla w; L^2(\Omega_R)\|. \end{aligned} \tag{6.5}$$

Proof. The first relation in (6.4) follows from the Hölder inequality while the second one needs the Poincaré's inequality and the inequality

$$\|w; L_6(\Omega_R)\| \le c \, \|\nabla w; L_2(\Omega_R)\| \qquad \forall \quad w \in \mathcal{H} \tag{6.6}$$

with a constant independent of w and $R > R_0$. Formula (6.6) can be verified by extending v by zero on the cylinder $\mathsf{C}_R = \{x = (y, z) \, : \, |y| < R, |z| < R\}$ and then again by a scaling argument. We change the variables $x \mapsto \mathsf{x} = R^{-1}x$. Sobolev embedding theorems give (6.6) on the cylinder C_1 with diameter and height 2, and the factors $R^{-3/6}$ and $R^1 R^{-3/2}$, appearing due to the inverse change of variables in the left- and right-hand sides, can be readily cancelled. The second estimate, (6.5), immediately follows from (6.4). □

The inequality (6.5) shows that the operator $(v, V) \mapsto (v \cdot \nabla)V$ is continuous and bilinear from $\mathcal{H}_\sigma(\Omega_R) \times \mathcal{H}_\sigma(\Omega_R)$ into $\mathcal{H}'(\Omega_R)$, and the constant $C_\mathbf{N}$ is independent of R. Since for $f \in L^2(\Omega)$ we always have $\|f; \mathcal{H}'(\Omega_R)\| \le \|f; L^2(\Omega_R) \le \|f; L^2(\Omega)\|$, the scheme described above leads to the existence of a unique small solution to Problem (6.1) (independent of R) provided f is small enough. To obtain an error estimate for the differences $v^{er} = v^R - v^\infty$, $p^{er} = p^R - p^\infty$ the existence of unique solutions in a ball around $(v^\infty, p^\infty)|_{\Omega_R}$ (for small f)) can be proved using exactly the same modification of the abstract scheme above as in [26], thus we repeat only the main ideas. The error fulfils the system

$$\left.\begin{aligned} -\nu\Delta v^{er} + \nabla p^{er} + (v^\infty \cdot \nabla)v^{er} + (v^{er} \cdot \nabla)v^\infty &= -(v^{er} \cdot \nabla)v^{er} & & \\ \nabla \cdot v^{er} &= 0 & & \text{in } \Omega_R, \\ v^{er} &= 0 & & \text{on } \Sigma_R, \\ M_R(v^{er}, p^{er}) &= M_R(\widetilde{v}^\infty, \widetilde{p}^\infty) & & \text{on } \Gamma_R, \end{aligned}\right\} \tag{6.7}$$

Again this system has the structure (6.2), where now the linear part is of the form $\mathbf{S}(v^\infty) = \mathbf{S}_0 + \mathbf{K}(v^\infty)$. Here $\mathbf{S}_0$ assigns to $v \in \mathcal{H}_\sigma(\Omega_R)$ and $p \in L^2(\Omega_R)$ the continuous linear functional

$$(\mathbf{S}_0(v, p))(w) = (\nabla v, \nabla w)_{\Omega_R} - (p, \nabla \cdot w)_{\Omega_R} + \mathbf{q}_R(v, w).$$

According to Theorem 4.5, $\mathbf{S}_0$ defines an isomorphism from $X = \mathcal{H}_\sigma(\Omega_R) \times L^2(\Omega_R)$ onto $Y = \mathcal{H}'(\Omega_R)$. The operator $\mathbf{K}(v^\infty)$ defined by

$$\langle \mathbf{K}(v^\infty)(v), w\rangle = ((v^\infty \cdot \nabla)v + (v \cdot \nabla)v^\infty, w)_{\Omega_R}, \quad v, w \in \mathcal{H}(\Omega_R),$$

is a compact perturbation (since Ω_R is bounded for each fixed R), here we used the notation $\langle\ ,\ \rangle$ to indicate the value of the functional $\mathbf{K}(v^\infty)(v)$ at w. Thus $\mathbf{S}_0 + \mathbf{K}(v^\infty)$ remains an isomorphism, as long as the kernel is trivial. To this end, let $V \in \mathcal{H}_\sigma(\Omega_R)$, $P \in L^2(\Omega_R)$ be fixed such that

$$(\nabla V, \nabla w)_{\Omega_R} - (P, \nabla \cdot w)_{\Omega_R} + \mathbf{q}_R(V, w) + ((v^\infty \cdot \nabla)V + (V \cdot \nabla)v^\infty, w)_{\Omega_R} = 0 \tag{6.8}$$

for any $w \in \mathcal{H}(\Omega_R)$. We apply this identity to $w = V$, then the term with P cancels, and inequality (6.5) gives

$$\|\nabla V; L^2(\Omega_R)\|^2 + \mathbf{q}_R(V, V) \le C_{\mathbf{N}} \|\nabla v^\infty; L^2(\Omega_R)\|\, \|\nabla V; L^2(\Omega_R)\|^2.$$

Thus $V = 0$ as long as

$$\|\nabla v^\infty; L^2(\Omega)\| < C_{\mathbf{N}}^{-1}. \tag{6.9}$$

Then (6.8) together with Lemma 4.3 implies $(P, g)_{\Omega_R} = 0$ for any $g \in L^2(\Omega_R)$, hence it follows also $P = 0$. Now we fix v^∞ with a norm small enough, furthermore, $\Phi \in \mathcal{H}'(\Omega_R)$ and a weak solution to the linearized error system, i.e.

$$(\nabla V, \nabla w)_{\Omega_R} - (P, \nabla \cdot w)_{\Omega_R} + \mathbf{q}_R(V, w) + ((v^\infty \cdot \nabla)V + (V \cdot \nabla)v^\infty, w)_{\Omega_R} = \langle \Phi, w\rangle. \tag{6.10}$$

Then similar arguments as above together with the arguments in the proof of Theorem 4.5 to estimate P lead to estimate (4.25) again, where now the constant depends on v^∞, but neither on R nor on Φ, this means the operator norm of $\mathbf{S}(v^\infty)^{-1}$ is bounded independent of R. Finally everything is prepared to apply the scheme described above to generalize Theorem 4.7 to the nonlinear problem.

Theorem 6.2 *Let $l \in \mathbb{N}_0$, $N \le 3$, and $f \in \mathcal{W}^{l+2}_\gamma(\Omega)^3$ with $\gamma \in (l + 3 + N, l + 4 + N)$ as in Theorem* 5.1, *moreover, let (v^∞, p^∞) be the solution of the original problem* (1.3). *There exist $\varepsilon_1 \in (0, \varepsilon_0]$, $\mathbf{c} > 0$ and $R_1 \ge R_0$ such that, under the restrictions $\|f; L_2(\Omega)\| \le \varepsilon_1$ and $R \ge R_1$, problem* (6.1) *with the ABC constructed in Section* 3 *admits a unique solution in the ball*

$$\|v^R - v^\infty; \mathcal{H}(\Omega_R)\| + \|p^R - p^\infty; L_2(\Omega_R)\| \le \mathbf{c}. \tag{6.11}$$

The differences $v^R - v^\infty|_{\Omega_R}$ and $p^R - p^\infty|_{\Omega_R}$ fulfill again the error estimate (4.33), *if the indices l and γ satisfy the restriction above.*

Proof. Since $\|\nabla v^\infty; L^2(\Omega_R) \le C\|f; L^2(\Omega_R)\|$, there exists ε_1 where $\|f; L^2(\Omega_R)\| \le \varepsilon_1$ implies the smallness condition (6.9). With (6.7) in mind we see that the differences v^{er}, p^{er} must solve

$$\mathbf{S}(v^\infty)(v^{er}, p^{er}) + \mathbf{N}((v^{er}, 0), v^{er}, 0)) = \Phi^{er}$$

with

$$\langle \mathbf{N}((v^{er}, 0), v^{er}, 0)), w\rangle = ((v^{er} \cdot \nabla)v^{er}, w)_{\Omega_R},$$

and Φ^{er} has the same form as in (4.30). The same arguments as in (4.34) and the results of Theorem 5.1 lead to

$$\|\Phi^{er}; \mathcal{H}'(\Omega_R)\| \le CR^{3+l-\gamma}\|f; \mathcal{W}^{l+2}_\gamma(\Omega)\|,$$

where of course l and γ are restricted. Thus there exists a R_1 such that $\|\Phi^{er};\mathcal{H}'(\Omega_R)\|$ is small enough to fulfill the smallness condition mentioned after (6.3) for all $R \geq R_1$, and as explained there, the result follows now from the Banach fixed point theorem. □

Although only three ($N = 3$) terms in asymptotics (2.4) of the solution (v^∞, p^∞) to the Navier-Stokes problem (1.3) are generated by harmonics (2.5), we used the whole Steklov-Poincaré operator (3.13) in the ABC (3.16) and (3.17) though. One, of course, can replace the integral (pseudodifferential) operators Π_R and Π_R^{-1} by their finite-dimensional approximations. Moreover, one can search for local (differential) ABC for the components $\overline{v_r^R}, \overline{p^R}$ and $\overline{v_\varphi^R}$. The simplest ABC of this type are of the form

$$\nu\frac{\partial}{\partial r}\overline{v_r^R} - \overline{p^R} = -2\nu(R^{-1}+5R)\,\overline{v_r^R}\,, \qquad \nu\frac{\partial}{\partial r}\overline{v_\varphi^R} = -2\nu R^{-1}\,\overline{v_\varphi^R} \qquad \text{on} \quad \Gamma_R. \tag{6.12}$$

The coefficients at $\overline{v_r^R}$ and $\overline{v_\varphi^R}$ are chosen such that the main ($N = 1$ in (2.5)) asymptotic terms of the asymptotic representation (2.4) satisfy (6.12). Together with the stable ABC (3.4), (3.6), the ABC (6.12) turn the integral $I(v^R, p^R; w)$ from (3.2) into non-negative quadratic form (*cf.* (3.19)). Thus, Theorems 4.7 and 6.2 remains valid with N=1.

At the same time, employing an approach [14] to improve the ABC (6.12) and defining the coefficients A_i and B_i so that two ($N = 2$ in (2.5)) asymptotic terms of the asymptotic representation (2.4) satisfy the boundary conditions

$$\nu\frac{\partial}{\partial r}\overline{v_r^R} - \overline{p^R} = -A_1\overline{v_r^R} + B_1\partial_\varphi^2\overline{v_r^R}\,, \qquad \nu\frac{\partial}{\partial r}\overline{v_\varphi^R} = -A_2\overline{v_\varphi^R} + B_1\partial_\varphi^2\overline{v_\varphi^R} \quad \text{on} \quad \Gamma_R$$

does not convert $I(v^R, p^R; w)$ into a non-negative quadratic form because $B_1 = (3\nu)^{-1}\big(R^{-1}-5R\big)$ is negative for a large R. We emphasize that, in principle, it happens only by chance that quadratic forms due to the ABC (3.16), (3.17) and (6.12) stay nonnegative: from one side there is no a priori reason for keeping this property and from the other side there is no free constant to fulfill it artificially!

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