A crack on the interface of piezo-electric bodies

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Singularities of elastic and electric fields are investigated at the tip of a crack on the interface of two anisotropic piezo-electric media under various boundary conditions on the crack surfaces. The Griffith formulae are obtained for increments of energy functionals due to growth of the crack and the notion of the energy release matrix is introduced. Normalization conditions for bases of singular solution are proposed to adapt them to the energy, stress, and deformation fracture criteria. Connections between these bases are determined and additional properties of the deformation basis related to the notion of electric surface enthalpy are established.

1 Formulation of the problem.

Let us consider a two-dimensional composite piezo-electric body $\Omega = \Omega^+ \cup \Omega^-$ with the selvage crack Λ on the straight interface $\Upsilon = \partial\Omega^+ \cap \partial\Omega^-$. The tensors of elastic strains ε^M and stresses σ^M and the vectors of electric strain $\varepsilon^E = E$ and induction $\sigma^E = D$ are related by the law

$$\sigma^M = A^{MM}\varepsilon^M - A^{ME}\varepsilon^E\,, \quad \sigma^E = A^{EM}\varepsilon^M + A^{EE}\varepsilon^E \tag{1.1}$$

where A^{MM} and A^{EE} are the 4-rank tensors of elastic moduli and dielectric permabilities, respectively, they are subjected to the usual symmetry and positivity conditions [13, 24, 10]. It is not necessary to impose the condition $A^{ME}_{jk,p} = A^{EM}_{p,kj}$ on the piezo-electric moduli. If $u^M = (u_1, u_2, u_3)$ is the displacement vector and $u^E = \varphi$ the electric potential depending on the two variables $x = (x_1, x_2) \in \Omega$, then $u = (u^M, u^E)$ denotes the vector of generalized displacements. We have the relations

$$\begin{gathered}\varepsilon^M_{jk} = \frac{1}{2}\Big(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\Big), \qquad \varepsilon^M_{3j} = \varepsilon^M_{3j} = \frac{1}{2}\frac{\partial u_3}{\partial x_j}, \quad \varepsilon^M_{33} = 0 \quad (j,k = 1,2)\\ \varepsilon^E_p = -\frac{\partial \varphi}{\partial x_p}, \quad \varepsilon^E_3 = 0 \quad (p = 1,2)\end{gathered} \tag{1.2}$$

The moduli $A^{MM}_{jk,pq}$, $A^{EE}_{j,g}$ and $A^{ME}_{jk,p}$ are piecewise constant and the restrictions of the fields on the homogeneous bodies $\Omega^\pm$ are denoted $\sigma^{M\pm}$, $\sigma^{E\pm}$ and so on.

The equilibrium equations and the electrostatic equation take the form

$$-\frac{\partial}{\partial x_j} A^{MM\,\pm}_{jk,pq} \frac{\partial u^\pm_p}{\partial x_q} - \frac{\partial}{\partial x_j} A^{ME\,\pm}_{jk,q} \frac{\partial \varphi^\pm}{\partial x_q} = 0 \quad (k = 1,2,3) \tag{1.3}$$

$$-\frac{\partial}{\partial x_j} A^{EM\,\pm}_{j,pq} \frac{\partial u^\pm_p}{\partial x_q} + \frac{\partial}{\partial x_j} A^{EE\,\pm}_{j,q} \frac{\partial \varphi^\pm}{\partial x_q} = 0 \tag{1.4}$$

Here summation over repeated indices $j, q = 1,2$ and $p = 1,2,3$ is assumed. We consider the following variants of boundary and contact conditions

$$n_j \sigma^M_{jk} = g^M_k \quad \text{on} \quad \Sigma^M, \qquad u_k = 0 \quad \text{on} \quad \Gamma^M \qquad (k = 1,2,3) \tag{1.5}$$

$$n_j \sigma^E_j = 0 \quad \text{on} \quad \Sigma^E, \qquad \varphi = G^E \quad \text{on} \quad \Gamma^E \tag{1.6}$$

$$\sigma_{2k}^{M+} = \sigma_{2k}^{M-}, \qquad u^+ = u_k^- \qquad \text{on} \quad \Upsilon^M \quad (k = 1, 2, 3) \tag{1.7}$$

$$\sigma_2^{E+} = \sigma_2^{E-}, \qquad \varphi^+ = \varphi^- \qquad \text{on} \quad \Upsilon^E. \tag{1.8}$$

Here $n = (n_1, n_2)$ is the outward normal unit vector to the boundary $\partial\Omega$ while $n = (0, 1)$ on Υ. In all the cases the mechanical contact of the crack surfaces $\Lambda^{\pm}$ is excluded, and the surfaces are free of traction, i.e. $\Lambda^{\pm} \subset \Sigma^M$ and $g^M = 0$ on $\Lambda^{\pm}$. The ideal contact conditions (1.7), (1.8) are imposed on the set $\Upsilon \setminus \Lambda \subset \Upsilon^M \cap \Upsilon^E$. The electric field satisfies one of the following conditions on the crack:

i) There is no electric contact of the surfaces and they are free of electric charge, i.e., $\Lambda^{\pm} \subset \Sigma^E$;

ii) The crack surfaces are grounded, i.e. $\Lambda^{\pm} \subset \Gamma^E$ and $G^E = 0$ on $\Lambda^{\pm}$;

iii) The crack surfaces are in the electric contact, i.e., $\Lambda^{\pm} \subset \Upsilon^M$;

iv) One surface is grounded and the other one is free of electric load, for example, $\Lambda^+ \subset \Gamma^E$ and $\Lambda^- \subset \Sigma^E$.

We mainly pay attention to cases i and ii, while cases iii and iv are commented shortly.

For many purposes, it is mainly important to study the homogeneous model problem (i.e. $g^M = 0$, $g^E = 0$ and $G = 0$) on the composite plane with semi-infinite crack :

$$\Omega^{\pm} = \mathbb{R}^2_{\pm} = \{x : \pm x_2 > 0\}, \quad \Upsilon = \{x : x_2 = 0\}, \quad \Lambda = \{x : x_1 < 0, x_2 = 0\} \tag{1.9}$$

On the surfaces of the semi-infinite crack we formulate the boundary conditions corresponding to cases i–iv and on the extension of the crack $\Upsilon \setminus \Lambda$ the ideal contact condition (1.7), (1.8). In the sequel this problem is recognized as (1.3)–(1.9).

The fracture mechanics requires detailed information on non-trivial solutions of the model problem of the form

$$U(x) = r^{\lambda}\Xi(\theta, \ln r) \tag{1.10}$$

we call such solutions *power logarithmic with exponent* λ. Here U is the vector of generalized displacements including the electric potential as the forth component, (r, θ) is the polar coordinate system, $r = |x|$ and $\theta \in (-\pi, \pi)$, the exponent $\lambda \in \mathbb{C}$ is a complex number and the vector field Ξ is a polynomial in the variable $\ln r$ with piecewise smooth coefficients in $\theta \in [-\pi, 0] \cup [0, \pi]$. If this polynomial is of degree zero, i.e. $\Xi = \xi(\theta)$, then we call the solution (1.10) a *power solution.* Starting from a nontrivial power logarithmic solution, the vector field $r^{\lambda}\Xi'(\theta, \ln r)$, where the prime stands for differentiation in the second argument, remains a power-logarithmic solution and, hence, there always exists a nontrivial power solution with the exponent λ, too. In the theory of elliptic problems in domains with piecewise smooth boundaries λ and $\Xi(\theta)$ are interpreted as eigenvalue and eigenvector of a certain polynomial pencil $\mathbf{A}(\lambda)$ associated to an operator of problem (1.3)–(1.9) (see, e.g., the introductory chapters in book [20]). We denote by S the spectrum of the pencil, i.e S contains all λ for which non-trivial power solutions (1.10) exist. Since problem (1.1)–(1.9) is formally self-adjoint and coefficients of differential operators are real, the set $S \subset \mathbb{C}$ possesses the central symmetry

$$\lambda \in S \quad \Rightarrow \quad \overline{\lambda} \in S, \qquad -\lambda \in S \tag{1.11}$$

Based on the formula (1.11), other general properties of pencil spectra and certain algebraic facts we discover in sections 2 and 3 the structure of the set S for the boundary conditions corresponding to cases i–vi. Moreover, we define the hermitian matrix M which figures in the Griffith formulae (section 4) and relations between bases of singular solutions adapted to fracture criteria of various physical nature (section 5). For functionals composed of the internal energy or the electrical enthalpy and the work of external loadings, we can calculate the asymptotic rate of changeing under small prolongations of the crack as quadratic forms of the intensity vectors. It turns out that always only one of these forms is generated by the matrix M and coincides with the invariant J-integral. For other functionals the rate is expressed in terms of intensity factors which correspond separately to mechanical and electrical loadings and therefore are not local characteristics of the physical fields in the crack mouth. In section 6 we introduce the notion of the surface electric enthalpy and state additional properties of the deformation basis determined by normalization conditions for jumps of the generalized displacements on the crack surfaces.

2 Singularities of the elastic and electric fields.

Since the system (1.3), (1.4) of differential equations possesses the polynomial property (cf. Example 1.13 [18]), Proposition 6.12 in [20] establishes that for an integer m the intersection of the set S with the line $l(m) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda = m\}$ consists of the only point $\{m\}$. To the exponent $\lambda = m \neq 0$ there correspond four linearly independent power solutions and, as $m > 0$, the Cartesian components of these solutions are polynomials in the variables x_1 and x_2. In case i there exist four power solutions with the exponent $\lambda = 0$, which are constants, and four power-logarithmic solutions, which linearly depend on $\ln r$. The constant electric potential does not meet the condition $\varphi = 0$ on $\Lambda^{\pm}$, thus in case ii only three constant solution occur and also three solutions (1.10) where $\lambda = 0$ and $\deg \Xi = 1$.

In case i the power solutions $U(x)$ with the exponent $\lambda = 1$ take the form

$$U^{\pm}(x) = (b_1 x_1 + 2c_1^{\pm} x_2, 2b_2 x_1 + c_2^{\pm} x_2, 2b_3 x_1 + 2c_3^{\pm} x_2, b^4 x_1 + c_4^{\pm} x_2) \tag{2.1}$$

The coefficients $b_1, \ldots, b_4$ are arbitrary but $c_1^{\pm}, \ldots, c_4^{\pm}$ can be found from the following system of eight algebraic equations arising from the boundary conditions $\sigma^{M\pm} = 0$, $\sigma^{E\pm} = 0$ on $\Lambda^{\pm}$ and the transmission conditions on $\Upsilon \setminus \Lambda$.

$$A^{MM\,\pm}_{2k,2p} c_p^{\pm} + A^{ME\,\pm}_{2k,2} c_4^{\pm} = -A^{MM\,\pm}_{2k,1p} b_p - A^{AE\,\pm}_{2k,1} b_4 \quad (k = 1, 2, 3) \tag{2.2}$$

$$A^{EM\,\pm}_{2,2p} c_p^{\pm} - A^{EE\,\pm}_{2,2} c_4^{\pm} = -A^{EM\,\pm}_{2,1p} b_p + A^{EE\,\pm}_{2,1} b_4 \tag{2.3}$$

Due to the posivity properties of the tensors $A^{::}_{::}$ this system is uniquely solvable. In case ii we put $b_4 = 0$ in (2.1), take b_1, b_2, b_3 and c_4^{+} arbitrary, and determine $c_1^{\pm}, c_2^{\pm}, c_3^{\pm}$ and c_4^{-} from the system of seven algebraic equations (2.2) and

$$A^{EM+}_{2,1p} b_p + A^{EM+}_{2,2p} c_p^{+} - A^{EE+}_{2,2} c_4^{+} = A^{EM-}_{2,1p} b_p + A^{EM-}_{2,2p} c_p^{-} - A^{EE-}_{2,2} c_4^{-}$$

The derivative along the crack

$$\frac{\partial U}{\partial x_1}(x) = r^{\lambda - 1} \Theta(\theta, \ln r) \tag{2.4}$$

of a non-trivial solution (1.10) to the model problem remains a power-logarithmic solution and becomes trivial if and only if U is a function of the only variable x_2. In case ii solution (2.1) with $b_p = b_4 = 0$ and $c_4^{+} = 1$ depends only on x_2 and is eliminated by the differentiation $\partial/\partial x_1$; that is why the exponent $\lambda = 0$ has three, not four, linearly independent power solutions.

The above-mention property of the derivative (2.4) and relation (1.11) show that the set $S \ni 0$ is invariant with respect to the shifts in ± 1 along the real axis

$$\lambda \in S \quad \Rightarrow \quad \lambda \pm 1 \in S \tag{2.5}$$

Thus, S is a periodic set and it is sufficient to investigate the structure of S inside the strip Π_+; here $\Pi_\pm = \{\lambda \in \mathbb{C} : \pm\mathrm{Re}\lambda \in (0, 1)\}$. With continuous change of the tensors $A^{MM\pm}$, $A^{EE\pm}$ and $A^{EM\pm}$, the model problem on the composite anisotropic plane can be reduced to the problem on the homogeneous isotropic plane, for which the mechanical and electrical fields do not interact and system (1.3), (1.4) of differential equations decouples into the two-dimensional Lamé system for the vector (u_1, u_2) and two Laplace equations for the scalars u_3 and φ. In both the cases i and ii the isotropic model problem has four linear independent power solutions (1.10) with the exponent $\lambda = 1/2$. Other power solutions with exponents $\lambda \in \Pi_+$ do not exist. The eigenvalues $\lambda \in \Pi_+$ of the polynomial pencil $\mathbf{A}(\lambda)$, generated by the model problem (1.3)–(1.9), depend continuously on the tensors of physical moduli (see [9]). They cannot stay on the boundary of the strip Π_+ since according to the previous result, in particular, formulae (2.1)–(2.3), the geometric and algebraic multiplicities of the eigenvalues $\lambda = 0$ and $\lambda = 1$ on the lines $l(0)$ and $l(1)$ are the same for any $A^{MM\pm}$, $A^{EE\pm}$ and $A^{EM\pm}$.

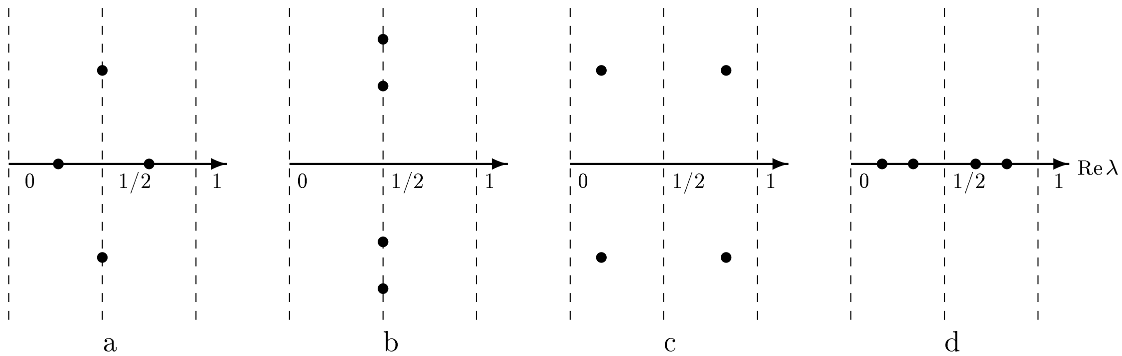


Figure 1: possible distribution of exponents λ in Π_+

Thus, using the theorem on preserving the total multiplicity [9], the model problem on the composite anisotropic plane also has just four linear independent power-logarithmic solutions (1.10) with exponents $\lambda \in \Pi_+$. Taking into account the symmetry property (1.11) and invariance (2.5) of the set S, four configurations of eigenvalues in the strip Π_+ are possible (see Figure 1 (a-d)).

The couples λ, $\overline{\lambda}$ and λ, $1 - \overline{\lambda}$ can glue to form a multiple eigenvalue. If the piezo-electric plane is homogeneous, then the geometric multiplicity of the eigenvalue $\lambda = 1/2$ is equal to 4 (see [6]), i.e., all power solutions with exponents $\lambda \in \Pi_+$ give rise to square-root singularities of the stress tensor and the electricity induction vector. Calculations in [3, 4] show that cases i and ii lead to the configurations a and b respectively while the angular parts of solutions (1.10) with non-integer exponents do not depend on logarithms, since four linear independent power solutions with exponents $\lambda \in \Pi_+$ have been constructed in [3, 4].

3 Quadratic forms and the invariant integral.

For $G^E = 0$, the solution $u = (u^M, u^E)$ of problem (1.3)–(1.8) on the bounded piezo-electric body Ω is a stationary point of the functional $\mathcal{F} = \mathcal{W} - \mathcal{R}^M - \mathcal{R}^E$, where $\mathcal{W}$ is the electric enthalpy and $\mathcal{R}^M$ and $\mathcal{R}^E$ are the works of external mechanical and electrical loadings respectively,

$$\mathcal{W}(u;\Omega) = \frac{1}{2}\int\limits_{\Omega} \left(\sigma^M_{jk}(u)\varepsilon^M_{jk}(u^M) - \sigma^E_j(u)\varepsilon^E_j(u^E)\right)\, dx \tag{3.1}$$

$$\mathcal{R}^M = \int\limits_{\Sigma^M} g^M_j u_j \, ds_x, \quad \mathcal{R}^E = \int\limits_{\Gamma^E} n_j \sigma^E_j(u) G^E \, ds_x \tag{3.2}$$

The functional (3.1) is not positive, however problem (1.3)–(1.8) possesses a unique weak solution in the case $\Gamma^M, \Gamma^E \neq \emptyset$ (see Example 1.13 in [18]). Owing to (1.2) and (1.3) the following symmetric Green's formula is valid

$$\begin{aligned} &\int\limits_{\Omega} \left(\overline{v^M_k}\frac{\partial}{\partial x_j}\sigma^M_{kj}(u) + \overline{v^E}\frac{\partial}{\partial x_j}\sigma^E_j(v) - u^M_k\frac{\partial}{\partial x_j}\overline{\sigma^M_{kj}(u)} - u^E\frac{\partial}{\partial x_j}\overline{\sigma^E_j(v)}\right) dx = \\ &= q(u,v;\partial\Omega) := \int\limits_{\partial\Omega} n_j\left(\sigma^M_{kj}(u)\overline{v^M_k} + \sigma^E_j(u)\overline{v^E} - \overline{\sigma^M_{kj}(v)}u^M_k - \overline{\sigma^E_j(v)}u^E\right) ds_x \end{aligned} \tag{3.3}$$

Here bar means the complex conjugation (2.4) because power solutions can be complex.

Let the path T begin on one surface, end on the other surface and encircle the crack tip. With solutions U and V of the homogeneous model problem (1.3)–(1.9), the integral $q(U,V;T)$ is invariant of T, moreover

$$q(U,V;T) = -\overline{q(V,U;T)} \tag{3.4}$$

The following formula which looks like integration by parts has been proved in [16, 17]

$$q\Big(\frac{\partial U}{\partial x_1}, V; T\Big) = -q\Big(U, \frac{\partial V}{\partial x_1}; T\Big) \tag{3.5}$$

Due to general results for any non-trivial power-logarithmic solution (2.3), one can find a power-logarithmic solution V with exponent $-\overline{\lambda}$ such that the equality $q(U,V;T)=1$ is valid (see [15] and also [20, Chapter 3]). Let $X^1,\dots,X^4$ be linear independent power solutions with exponents $\lambda_j \in \Pi_+$ and $Y^1,\dots,Y^4$ the so called *dual power solutions* with exponents $-\overline{\lambda_j} \in \Pi_+$ subject to the bi-orthogonality conditions

$$q(X^j, Y^k; T) = \delta_{j,k} \quad (j,k = 1,\dots,4) \tag{3.6}$$

Linear combinations of power (logarithmic) solutions are again solutions to the homogeneous problem (1.1)–(1.9), we call such combinations *power law solutions*. Since the derivatives (2.4) of the solutions X^j are power solutions with exponents $\lambda_j - 1 \in \Pi_-$ and hence, they can be decomposed in terms of the linear independent power solutions $\{Y^1,\dots,Y^k\}$ related to the exponents from the strip Π_-:

$$\frac{\partial X^j}{\partial x_1}(x) = -\sum_{k=1}^{4} M_{jk} Y^k(x) \tag{3.7}$$

Since a power solution with a non-integer exponent λ cannot depend only on the variable x_2, the derivatives $\partial X^1/\partial x_1, \dots, \partial X^4/\partial x_1$ preserve linear independence. Thus, the 4×4-matrix $M = (M_{jk})$ is non-degenerate. With the help of formulae (3.4)–(3.6) we conclude that M is an hermitian matrix, indeed

$$\begin{aligned} M_{jp} &= q\Big(X^p, \textstyle\sum_{k=1}^4 M_{jk}Y^k; T\Big) = -q\Big(X^p, \frac{\partial X^j}{\partial x_1}; T\Big) \\ &= q\Big(\frac{\partial X^p}{\partial x_1}, X^j; T\Big) = -\overline{q\Big(X^j, \frac{\partial X^p}{\partial x_1}; T\Big)} = \overline{M_{pj}} \end{aligned} \tag{3.8}$$

Let us consider the sesquilinear form $\boldsymbol{\mathcal{M}}(C,C) = \overline{C}^\top MC$ defined on columns $C = (C_1, \dots, C_4)^\top \in \mathbb{C}^4$; here $\top$ stands for transposition. As it is well known the form can be diagonalized, namely, there exists a non-degenerate matrix 4×4-matrix D [1] such that

$$K = (K_1, \dots K_4)^\top = D^{-1}C, \quad \overline{D}^\top MD = \mathrm{diag}\{m_1 \dots, m_4\}$$
$$\boldsymbol{\mathcal{M}}(C,C) = \sum_{j,k=1}^{4} \overline{C}_j M_{jk} C_k = \sum_{j=1}^{4} m_j |K_j|^2 \tag{3.9}$$

The matrix D and the real numbers $m_1, \dots, m_4$ are not uniquely determined, however, due to Sylvester's theorem the *index of positivity*, i.e., the number of positive values in the set $\{m_1, \dots, m_4\}$, is an invariant of the form $\boldsymbol{\mathcal{M}}(C,C)$. Since M is a non-degenerate matrix, values $m_1, \dots, m_4$ do not vanish. The index of positivity is preserved under a continuous variation of the tensors $A^{MM\pm}$, $A^{EE\pm}$ and $A^{EM\pm}$, in particular, under transition from a composite anisotropic plane to the homogeneous isotropic one.

For the isotropic model problem the power solutions X^j and Y^k, subject to conditions (3.6) are known (see, e.g., [22]). In case i the non-trivial components of these vectors take the form (we use either cylindrical components or cartesian components)

$$\begin{aligned}
X_r^1(r,\theta) &= \frac{1}{t_1} r^{1/2} \Big(-\cos\frac{3}{2}\theta + (5-8\nu)\cos\frac{\theta}{2}\Big), \\
X_\theta^1(r,\theta) &= \frac{1}{t_1} r^{1/2} \Big(\sin\frac{3\theta}{2} - (7-8\nu)\sin\frac{\theta}{2}\Big), \\
X_r^2(r\theta) &= \frac{1}{t_2} r^{1/2} \Big(3\sin\frac{3}{2}\theta - (5-8\nu)\sin\frac{\theta}{2}\Big), \\
X_\theta^2(r,\theta) &= \frac{1}{t_2} r^{1/2} \Big(3\cos\frac{3}{2}\theta - (7-8\nu)\cos\frac{\theta}{2}\Big), \\
Y_r^1(r,\theta) &= \frac{1}{T_1} r^{-1/2} \Big(3\cos\frac{\theta}{2} - (7-8\nu)\cos\frac{3}{2}\theta\Big), \\
Y_\theta^1(r,\theta) &= \frac{1}{T_1} r^{-1/2} \Big(-3\sin\frac{\theta}{2} + (5-8\nu)\sin\frac{3}{2}\theta\Big), \\
Y_r^2(r,\theta) &= \frac{1}{T_2} r^{-1/2} \Big(-\sin\frac{\theta}{2} + (7-8\nu)\sin\frac{3}{2}\theta\Big), \\
Y_\theta^2(r,\theta) &= \frac{1}{T_2} r^{-1/2} \Big(-\cos\frac{\theta}{2} + (5-8\nu)\cos\frac{3}{2}\theta\Big), \\
X_3^3(r,\theta) &= \frac{1}{t_3} r^{1/2} \sin\frac{\theta}{2}, \qquad Y_3^3(r,\theta) = \frac{1}{T_3} r^{-1/2} \sin\frac{\theta}{2}, && (3.10) \\
X_4^4(r,\theta) &= -\frac{1}{t_4} r^{1/2} \sin\frac{\theta}{2}, \qquad Y_4^4(r,\theta) = \frac{1}{T_4} r^{-1/2} \sin\frac{\theta}{2} && (3.11) \\
t_1 &= t_2 = 4(2\pi)^{1/2}\mu, \quad T_1 = T_2 = 8(2\pi)^{1/2}(1-\nu), \\
t_3 &= \frac{1}{2}(2\pi)^{1/2}\mu, \quad t_4 = \frac{1}{2}(2\pi)^{1/2}\beta, \quad T_3 = T_4 = (2\pi)^{1/2}. && (3.12)
\end{aligned}$$

Here μ is the shear modulus, ν the Poisson ratio, and β the dielectric permeability. The minus in the first formula (3.11) is caused by the relation $\sigma_2^E(\varphi) = -\beta\partial\varphi/\partial x_2$. Inserting the expressions for X^j and Y^k into formula (3.7), we find that M is a diagonal matrix with the elements

$$M_{11} = M_{22} = \mu^{-1}(1-\nu), \quad M_{33} = \mu^{-1}, \quad M_{44} = -\beta^{-1} \tag{3.13}$$

[1] D can be even chosen as unitary matrix, then the m_j are the eigenvalues of M

Thus in case i the index of positivity of the sesquilinear form $\boldsymbol{\mathcal{M}}(C, C)$ equals 3.

In case ii the function sin replaces the function cos in formulae (3.11) and thus the last equality in (3.13) turns into $M_{44} = \beta^{-1}$. Hence, the index of positivity of the form $\boldsymbol{\mathcal{M}}(C, C)$ becomes 4 and the matrix M is positive definite.

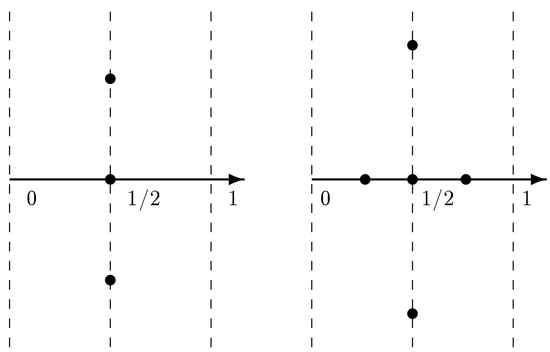
Let the exponents $\lambda_1, \ldots, \lambda_4$ of the power solutions $X^1, \ldots, X^4$ be mutually different. Since the integral $q(X^j, \partial X^k/\partial x_1; T) = -M_{jk}$ is path-independent and its integrand is equal to $O(r^{\lambda_j + \overline{\lambda_k} - 2})$, the elements of the matrix M, corresponding to inconsistent exponents $\lambda_j + \overline{\lambda_k} \neq 1$, vanish. Figure 1 shows four variants for the distribution of the exponents $\lambda_p \in \Pi_+$, the corresponding matrices M are outlined in figure 2, where bullets stand for non-trivial elements. The latter can stay on the main diagonal only in the case $\mathrm{Re}\lambda = 1/2$. In figure 2 also the index of positivity, $\varkappa$, of the form $\boldsymbol{\mathcal{M}}(C, C)$ is indicated. Thus, in case i only the configuration a is possible and in case ii the configuration b only.

$$\begin{pmatrix} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \\ \circ & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \circ \end{pmatrix} \qquad \begin{pmatrix} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ \\ \circ & \circ & \bullet & \circ \\ \circ & \circ & \circ & \bullet \end{pmatrix} \qquad \begin{pmatrix} \circ & \bullet & \circ & \circ \\ \bullet & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \circ \end{pmatrix}$$

a) $\varkappa = 3$ or $\varkappa = 1$ b) $\varkappa = 4$ or $\varkappa = 0$ c),d) $\varkappa = 2$

Figure 2

The number n of linear independent power solutions with exponents $\lambda \in \Pi_+$ does not need to coincide with the number of equations in system (1.3), (1.4), it rather depends on the type of boundary conditions. For example, $n = 3$ in case iii and $n = 5$ in case iv; figure 3 shows possible configurations of the exponents λ and the index of positivity.



BC iii), $\varkappa = 3$ BC iv), $\varkappa = 4$

Figure 3: possible distribution of the exponents λ in Π_+

For the corresponding isotropic model problems in the first case M is a diagonal matrix composed of the three first numbers in (3.13), and in the second case 5×5-matrix M has the following non-trivial elements

$$M_{11} = M_{22} = \mu^{-1}(1 - \nu), \quad M_{33} = \mu^{-1}, \quad M_{45} = M_{54} = 2\beta^{-1}$$

Apart from four positive eigenvalues this matrix has the negative eigenvalue $-2\beta^{-1}$. Moreover, in case iv the pair of power solutions (3.11) is replaced by the pair

$$X_4^4(r, \theta) = -\frac{1}{t_4} r^{1/4} \sin\frac{1}{4}(\theta - \pi), \quad Y_4^4(r, \theta) = \frac{1}{T_4} r^{-1/4} \sin\frac{1}{4}(\theta - \pi)$$

$$X_4^5(r,\theta) = -\frac{1}{t_5}\, r^{3/4} \sin\frac{3}{4}(\theta-\pi), \quad Y_4^5(r,\theta) = \frac{1}{T_5}\, r^{-3/4} \sin\frac{3}{4}(\theta-\pi)$$

$$t_4 = \frac{1}{4}\,\pi^{1/2}\beta, \quad t_5 = \frac{3}{4}\,\pi^{1/2}\beta, \quad T_4 = T_5 = 2\pi^{1/2}$$

4 The energy release matrix.

By Legendre transformation we change the functionals $\mathcal{W}$ and $\mathcal{F}$ into the functionals $\mathcal{E}$ and $\mathcal{V}$:

$$\mathcal{E} = \frac{1}{2} \int\limits_{\Omega} \left(\sigma_{jk}^M(u)\,\varepsilon_{jk}^M(u^M) + \sigma_j^E(u)\,\varepsilon_j^E(u^E)\right)\, dx \tag{4.1}$$

$$\mathcal{V} = \mathcal{E} - \mathcal{R}^M + \mathcal{R}^E \tag{4.2}$$

Here $\mathcal{E}$ is the interior energy of the piezo-electric body Ω. After integration by parts and using equations (1.3)–(1.6) and (3.2), we obtain

$$\mathcal{V} = \frac{1}{2} \int\limits_{\partial\Omega} n_j \left(\sigma_{jk}^M(u)u_k - \sigma_j^E(u)u^E\right)\, ds - \mathcal{R}^M + \mathcal{R}^E = -\frac{1}{2}\mathcal{R}^M + \frac{1}{2}\mathcal{R}^E \tag{4.3}$$

By u^0 and u^h we denote the solutions of problem (1.3)–(1.8) for the body Ω with the initial crack Λ_0 and with the elongated crack Λ_h, respectively, under the same external loading. By rescaling, we reduce a characteristic size of Ω to 1 and the length $h > 0$ of the crack shoot $\Lambda_h \setminus \Lambda_0$ becomes a dimensionless parameter. In view of formulae (4.2) and (4.3) the increment of the functional $\mathcal{V}$ due to the growth of the crack takes the form

$$\Delta\mathcal{V} = \mathcal{V}^h - \mathcal{V}^0 = -\frac{1}{2} \int\limits_{\Sigma^M} g_j \left(U_j^h - u_j^0\right)\, ds + \frac{1}{2} \int\limits_{\Gamma^E} n_j \sigma_j^E (u^h - u^0) G^E ds \tag{4.4}$$

To construct the asymptotics of the generalized displacement vector $u^h = (u^{Mh}, u^{Eh})$, we apply the method of matched asymptotic expansions [7, 11]. General asymptotic structures were adapted to problems of crack theory in papers [22, 17] and the necessary estimates of asymptotic remainders follow from general results in chapters 4,6 of [14].

Near the tip O of the crack Λ_0 we have the expansion

$$u^0(x) = u^0(0) + \sum_{j=1}^{4} C_j X^j(x) + O(r), \quad r \to 0 \tag{4.5}$$

where the coefficients C_j are calculated by the formula

$$C_j = \int\limits_{\Sigma^M} g_p^M(x)\, \overline{\zeta_p^j(x)}\, dx - \int\limits_{\Gamma^E} n_p \overline{\sigma_p^E(\zeta^j)}\, G^E\, ds_x \tag{4.6}$$

Here $\zeta^j = (\zeta^{Mj}, \zeta^{Ej})$ denote the weight functions, i.e., non-energetic solutions of the homogeneous problem (1.3)–(1.8) on the body $\Omega \setminus L$ that satisfy the condition (see [5, 15])

$$\zeta^j(x) = Y^j(x) + O(1), \quad r \to 0. \tag{4.7}$$

Note that the integral representations (4.6) are a consequence of the normalization conditions (3.6). A detailed exposition of the method of weight functions developed in [15] for applications to problems in crack theory is given in paper [17].

According to the asymptotic formula (4.5) the inner (near the crack tip) expansion of the solution u^h takes the form

$$u^h(x) = u^0(0) + \sum_{j=1}^{4} C_j h^{\lambda_j} w^j(\xi) + \dots \tag{4.8}$$

Here $\xi = h^{-1}x$ are stretched coordinates, λ_j is the exponent of the power solution X^j. The field w^j is the solution of the model problem on the composite plane with the crack $\Lambda_1 = \{\xi : \xi_1 \leqslant 1, \xi_2 = 0\}$ which admits the following decomposition at infinity

$$w^j(\xi) = X^j(\xi) + O(1), \quad |\xi| \to \infty. \tag{4.9}$$

We emphasize that relation (4.9) provides matching expansions (4.5) and (4.8) and the change of variables $x \mapsto \xi$ transfers the tip O^h of the crack $\Lambda_h$into the tip O^1 of the crack Λ_1 (see [11]). We rewrite the vector function w^j in the system of the Cartesian coordinates $(\xi_1 - 1, \xi_2)$ centered at O^1

$$w^j(\xi) = X^j(\xi_1 - 1, \xi_2) = |\xi|^{\lambda_j} X^j(|\xi|^{-1}\xi_1 - |\xi|^{-1}, |\xi|^{-1}\xi_2). \tag{4.10}$$

By equalities (3.7) and the Taylor formula with respect to the variable $|\xi| = (\xi_1^2 + \xi_2^2)^{1/2}$, we have

$$\begin{aligned} w^j(\xi) = |\xi|^{\lambda_j} \left(X^j\left(\frac{\xi}{|\xi|}\right) - \frac{1}{|\xi|}\frac{\partial X^j}{\partial \xi_1}\left(\frac{\xi}{|\xi|}\right) + O\left(\frac{1}{|\xi|^2}\right)\right) = \\ = X^j(\xi) - \frac{\partial X^j}{\partial \xi_1}(\xi) + O(|\xi|^{\lambda_j - 2}) = X^j(\xi) + \sum_{k=1}^{4} M_{jk} Y^k(\xi) + O(|\xi|^{\lambda_j-2}) \end{aligned} \tag{4.11}$$

Formula (4.11), firstly, confirms representation (4.9) and, secondly, shows that

$$\begin{aligned} u^h(x) = u^0(0) + \sum_{j=1}^{4} C_j h^{\lambda_j} \left(X^j(\xi) + \sum_{k=1}^{4} M_{jk} Y^k(\xi)\right) + \dots = \\ = u^0(0) + \sum_{j=1}^{4} C_j X^j(\xi) + h \sum_{j,k=1}^{4} C_j M_{jk} Y^k(\xi) + \dots \end{aligned} \tag{4.12}$$

Comparing relations (4.5), (4.7) and (4.12), we can specify the outer expansion (at a distance from the crack tip)

$$u^h(x) = u^0(x) + hu'(x) + \dots \tag{4.13}$$

$$u'(x) = \sum_{j=1}^{4} C_j M_{jk} Y^k = \left.\frac{\partial u^h}{\partial h}\right|_{h=0} \tag{4.14}$$

Note that equality (4.14) extends the Rice formula [25] for weight functions in piezo-electric bodies.

We insert decomposition (4.13) into formula (4.4) and we process it with the help of (4.14) and (4.6)

$$\begin{aligned} \Delta\mathcal{V} &= -\frac{h}{2} \sum_{j,k=1}^{4} C_j M_{jk} \left(\int_{\Sigma^M} g_p \zeta_p^k ds_x - \int_{\Gamma^E} n_p \sigma^E(\zeta^k) G^E \, ds_x \right) + O(h^2) = \\ &= -\frac{h}{2} \sum_{j,k=1}^{4} C_j M_{jk} \overline{C}_k + O(h^2) = -\frac{h}{2} \boldsymbol{\mathcal{M}}(C, C) + O(h^2) \end{aligned} \tag{4.15}$$

Here $\boldsymbol{\mathcal{M}}(C, C)$ is the sesquilinear form (3.9) again, and the precision $O(h^2)$ of the asymptotic formula (4.15) is provided by general results in [14] and the information on the spectrum S obtained in section 2.

Consider the invariant integral

$$J(u;T) = \int\limits_T \left\{ \frac{1}{2} \left(\sigma^M_{jk}(u) \varepsilon^M_{jk}(u^M) - \sigma^E_j(u) \varepsilon^E_j(u^E) \right) n_1 - n_j \left(\sigma^M_{jk}(u) \frac{\partial u_k}{\partial x_1} + \sigma^E_j(u) \frac{\partial u^E}{\partial x_1} \right) \right\} ds_x \tag{4.16}$$

Let us verify the equalities

$$J(u;T) = -\frac{1}{2} q\left(u, \frac{\partial u}{\partial x_1}; T\right) = \frac{1}{2} \boldsymbol{\mathcal{M}}(C, C) \tag{4.17}$$

The second one follows from relations (3.6), (3.7) and (4.5). Since the crack surfaces are free of external loading, the derivative $\partial_1 u = \partial u / \partial x_1$ satisfies equations (1.3), (1.4) and conditions (1.5)–(1.8) inside the domain ω surrounded by the contour T. Denote by $\mathcal{L}$ the matrix differential operator of system (1.3), (1.4) and by χ a smooth cut-off function, which equals zero near the crack tip and equals one near a neighborhood of the arc T. Thus, the product $v = \chi u$ is free of singularities. The following calculation proves the first equality in (4.17):

$$\begin{aligned}
q(u, \partial_1 u; T) &= q(v, \partial_1 v; T) = -\int\limits_\omega (\partial_1 v^\top \mathcal{L} v - v^\top \mathcal{L} \partial_1 v)\, dx - \sum_\pm q(v, \partial_1 v; \Lambda_0^\pm \cap \omega) \\
&= -2 \int\limits_\omega \partial_1 v^\top \mathcal{L} v\, dx - 2 \sum_\pm \int\limits_{\Lambda^\pm \cap \omega} n_j (\sigma^M_{jk}(v) \partial_1 v_k + \sigma^E_j(v) \partial_1 v^E)\, dx_1 = \\
&= -\int\limits_\omega \frac{\partial}{\partial x_1} \left(\sigma^M_{jk}(v) \varepsilon^M_{jk}(v^M) - \sigma^E_j(v) \varepsilon^E(v^E) \right) dx \\
&\quad +2 \int\limits_\top n_j \left(\sigma^M_{jk}(u) \partial_1 u_k + \sigma^E_j(u) \partial_1 u^E \right) ds_x = -2J(u;T).
\end{aligned}$$

Thus, by virtue of (4.15) and (4.17) there holds the formula

$$-\frac{d\mathcal{V}^h}{dh}\bigg|_{h=0} = \frac{1}{2} \boldsymbol{\mathcal{M}}(C, C) = J(u^0; T) \tag{4.18}$$

Usually (cf. [26, 3] and others) the invariant integral $J(u^0;\Gamma)$ is related to the energy release rate due to moving of a crack tip. Consequently and analogous to the pure elastic problem (see [2]) we call the hermitian matrix M, which is composed of coefficients of decompositions (3.7) and gives rise to the sesquilinear form (3.10), *the matrix of the piezo-electric energy release.* Formulae (4.15) and (4.18) remain the same for any type of boundary conditions under consideration, but the size and the index of positivity of the matrix M can vary (cf. figures 1–3).

The integrand in (4.16) contains the enthalpy density (3.1) but the increments of the functionals $\mathcal{W}$ and $\mathcal{F}$ are not related with the invariant integral $J(u^0;T)$, indeed

$$
\begin{aligned}
-\Delta\mathcal{W} = \Delta\mathcal{F} &= -\frac{1}{2}\int\limits_{\Sigma^M} g_p^M\left(u_p^h - u_p^0\right)ds_x - \frac{1}{2}\int\limits_{\Gamma^E} n_p\sigma_p^E(u^h - u^0)G^E\,ds_x \\
&= -\frac{h}{2}\sum_{j,k=1}^{4} C_j M_{jk}\Big\{\int\limits_{\Sigma^M} g_p^M\zeta_p^k\,ds_x + \int\limits_{\Gamma^E} n_p\sigma_p^E(\zeta^k)\,G^E\,ds_x\Big\} + O(h^2) \\
&= -\frac{h}{2}\sum_{j,k=1}^{4}(c_j^M + C_j^E)M_{jk}\overline{(C_k^M - C_k^E)} + O(h^2) \\
&= -\frac{h}{2}\Big(\boldsymbol{\mathcal{M}}(C^M,C^M) - \boldsymbol{\mathcal{M}}(C^E,C^E)\Big) + O(h^2) \qquad (4.19)
\end{aligned}
$$

Here $C_j = C_j^M + C_j^E$, where C_j^M and C_j^E are the coefficients of decomposition (4.5) generated by the mechanical and electrical external loads, respectively. According to the integral representations (4.6) we have the representation

$$
C_j^M = \int\limits_{\Sigma^M} g_p^M(x)\overline{\zeta_p^j(x)}\,ds_x, \quad C_j^E = -\int\limits_{\Gamma^E} n_p\,\overline{\sigma_p^E(\zeta^j)}\,G^E\,ds_x \qquad (4.20)
$$

Formulae (4.15) and (4.19) differ crucially: in the first one the factor on h is a local characteristics of the physical fields but in the second one it is a global characteristics, since in contrast to the coefficients C_j the coefficients C_j^M and C_j^E cannot be recognized by an analysis of the fields in the crack mouth. Moreover, the change of the polarization of the electric load the difference $\boldsymbol{\mathcal{M}}(C^M,C^M) - \boldsymbol{\mathcal{M}}(C^E,C^E)$ does not change; however, the values $\boldsymbol{\mathcal{M}}(C^M+C^E,C^M+C^E)$ and $\boldsymbol{\mathcal{M}}(C^M-C^E,C^M-C^E)$ coincide only in the case $\boldsymbol{\mathcal{M}}(C^M,C^E)=0$ (cf. [23, 8] and others).

Functional (4.2) is the only one, for which the release rate due to growth of the crack is a local characteristics. For example, owing to formulae (4.3), (3.2), (4.14) and (4.20) we obtain

$$
\begin{aligned}
\Delta\mathcal{U} &= \Delta\mathcal{E} - \Delta\mathcal{R}^M - \Delta\mathcal{R}^E = -\frac{1}{2}\Delta\mathcal{R}^M - \frac{3}{2}\Delta\mathcal{R}^E \\
&= -\frac{h}{2}\Big(\boldsymbol{\mathcal{M}}(C^M,C^M) - 3\boldsymbol{\mathcal{M}}(C^E,C^E)\Big) + O(h^2)
\end{aligned}
$$

The properties of the energy functionals under discussion depend on the character of the electric loading. For instance, let the normal component of the electric induction be given on the surface $\Sigma^E\setminus\Lambda$ instead of the electric potential G^E on the surface Γ^E. Then formulae (1.6) and (3.2) are replaced by the following:

$$
n_j\sigma_j^E = g^E \quad \text{on} \quad \Sigma^E, \quad \varphi = 0 \quad \text{on} \quad \Gamma^E \qquad (4.21)
$$

$$
\mathcal{R}^M = \int\limits_{\Sigma^M} g^M u_j^M\,ds_x, \quad \mathcal{R}^E = -\int\limits_{\Sigma^E} g^E u^E\,ds_x \qquad (4.22)
$$

Note that the minus at the second integral in (4.22) is provided by relation (1.2) of the electric stress vector and the potential. In paper [12] a mistake on pp. 315-316 [24] in the calculation of the increment of the potential energy was corrected and furthermore it was shown that for problem (1.3)–(1.5), (4.21), (1.7), (1.8) there holds

$$
\Delta\mathcal{W} - \Delta\mathcal{R}^M + \Delta\mathcal{R}^E = -\frac{h}{2}\boldsymbol{\mathcal{M}}(C,C) + O(h^2), \qquad (4.23)
$$

$$\Delta\mathcal{E} - \Delta\mathcal{R}^M - \Delta\mathcal{R}^E = -\frac{h}{2}\Big(\boldsymbol{\mathcal{M}}(C^M, C^M) - \boldsymbol{\mathcal{M}}(C^E, C^E)\Big) + O(h^2). \tag{4.24}$$

Note that the changes $\mathcal{W} \mapsto \mathcal{E}$ and $\mathcal{E} \mapsto \mathcal{W}$ turn equations (4.23) and (4.24) into the formulae (4.15) and (4.19), respectively.

5 Bases of power solutions.

For the pure elastic problem, in [17, 19] basesfor power law solutions related to exponents in Π_+ were constructed that are adapted for fracture criteria of various physical nature: the energy, stress, and deformation ones. An analogous classification is available for the piezoelectric problem as well.

The energy basis $\{X^{1e}, \dots, X^{e4}\}$ exists for any configuration of the exponents $\lambda_j \in \Pi_+$ and is to be chosen in such a way that the matrix M is diagonal and the fields X^{je} are real. We emphasize that the latter property implies that the elements of the matrix M are real as well. Let us consider the most interesting case i (figure 1 (a)) where

$$\lambda_1 = \frac{1}{2} + i\tau_1, \quad \lambda_2 = \frac{1}{2} - i\tau_1, \quad \lambda_3 = \frac{1}{2} - \tau_2, \quad \lambda_4 = \frac{1}{2} + \tau_2 \qquad (\tau_1, \tau_2 > 0)$$

There exists a basis for the power solutions where X^3 and X^4 have real angular parts Φ^q but the solutions X^1, X^2 remain complex Φ^q; however, and $\Phi^1 = \overline{\Phi}^2$ is possible. The elements $M_{11} = M_{22} > 0$, $M_{34} = M_{43} \neq 0$ of the matrix M (figure 2 (a)) are real. Let L be a characteristic size of the body Ω. We set

$$X^{1e}(x) = \frac{1}{\sqrt{2}}\, r^{1/2}\, \Big(\mathrm{Re}\,\Phi^1(\theta)\cos\Big(\tau_1 \ln\frac{r}{L}\Big) - \mathrm{Im}\,\Phi^1(\theta)\sin\Big(\tau_1\ln\frac{r}{L}\Big)\Big)$$

$$X^{2e}(x) = \frac{1}{\sqrt{2}}\, r^{1/2}\, \Big(\mathrm{Re}\,\Phi^1(\theta)\sin\Big(\tau_1 \ln\frac{r}{L}\Big) + \mathrm{Im}\,\Phi^1(\theta)\cos\Big(\tau_1\ln\frac{r}{L}\Big)\Big)$$

$$X^{3e}(x) = \frac{1}{\sqrt{2}}\, r^{1/2}\, \Big(\Big(\frac{r}{L}\Big)^{-\tau_2}\, \Phi^3(\theta) + \Big(\frac{r}{L}\Big)^{\tau_2}\, \Phi^4(\theta)\Big)$$

$$X^{4e}(x) = \frac{1}{\sqrt{2}}\, r^{1/2}\, \Big(\Big(\frac{r}{L}\Big)^{\tau_2}\, \Phi^3(\theta) - \Big(\frac{r}{L}\Big)^{\tau_2}\, \Phi^4(\theta)\Big)$$

As a result, we get $M^e = \mathrm{diag}\{M_{11}, M_{22}, M_{34}, -M_{34}\}$.

The stress and deformation bases can be introduced in case i under the condition that all power solutions $X^1, \dots, X^4$ have the same exponent $\lambda = 1/2$. The stress basis $\{X^{1\sigma}, \dots, X^{4\sigma}\}$ is subject to the conditions:

$$\sigma^M_{2p}(X^{j\sigma}; r, 0) = (2\pi r)^{-1/2}\delta_{p,j}, \quad p = 1, 2, 3, \tag{5.1}$$

$$\sigma^E_2(X^{j\sigma}; r, 0) = (2\pi r)^{-1/2}\delta_{4,j}, \quad j = 1, \dots, 4. \tag{5.2}$$

The deformation basis $\{X^{1\varepsilon}, \dots, X^{4\varepsilon}\}$ is defined by conditions on the jumps of the displacements and the electric potential over the crack

$$r^{-1}[X^{j\varepsilon}_p](-r) = 2(2\pi r)^{-1/2}\delta_{p,j}, \quad p = 1, 2, 3, \tag{5.3}$$

$$r^{-1}[X^{j\varepsilon E}](-r) = 2(2\pi r)^{-1/2}\delta_{4,j}, \quad j = 1, \dots, 4. \tag{5.4}$$

The existence of the two bases verifying the requirements (5.1), (5.2) and (5.3), (5.4), can be proved following a scheme in [17]. For example, if conditions (5.1), (5.2) cannot be satisfied, one finds a non-trivial power solution $X(x) = r^{1/2}\Phi(\theta)$ such that $\sigma_{21}^{M\pm}(X) = \sigma_{22}^{M\pm}(X) = \sigma_{23}^{M\pm}(X) = 0$, $\sigma_2^{E\pm}(X) = 0$ on the extension of the crack. Hence, X is a solution of the Neumann problem for the elliptic system (1.3), (1.4) in the upper half-plane. It is known [1] that locally bounded solutions of this boundary-value problem are smooth and therefore X cannot take the form (1.10) with the exponent $\lambda = 1/2$, which gives a contradiction.

To each of the two bases there correspond the base of the singular power solutions (1.10) with the exponents $\lambda = -1/2$ which are subject to the normalization conditions (3.6). Relations (3.7) give rise to the hermitian matrices M^σ and M^ε, while the asymptotic expansion (4.5) takes the form

$$u^0(x) = u^0(0) + \sum_{j=1}^{4} K_j^\sigma X^{j\sigma}(x) + O(r) = \sum_{k=1}^{4} K_k^\varepsilon X^{k\varepsilon}(x) + O(r), \quad r \to 0 \tag{5.5}$$

The generalized *stress intensity factors* K_j^σ (GSIFs) and *deformation intensity factors* K_j^ε (GDIFs) are related by

$$K_j^\sigma = \sum_{k=1}^{4} N_{kj} K_k^\varepsilon, \qquad X^{k\varepsilon} = \sum_{j=1}^{4} N_{kj} X^{j\sigma} \tag{5.6}$$

Here N_{kj} are elements of the non-degenerate 4×4-matrix N. By virtue of formulae (4.15), (4.4) and (3.3), we have

$$\begin{aligned} \Delta\mathcal{V} &= -\frac{h}{2}\sum_{j,k=1}^{4} K_j^\varepsilon M_{jk}^\varepsilon K_k^\varepsilon + O(h^2) = \frac{1}{2}\sum_{\pm} \mp \int\limits_{\Lambda_h^\pm\setminus\Lambda^\pm} \Big(\sigma_{2p}^M(u^0)u_p^h + \sigma_2^E(u^0)u^{hE}\Big)\, ds \\ &= -\frac{1}{2}\int\limits_0^h \Big(\sigma_{2p}^M(u^0;x_1,0)\,[u_p^h](x_1) + \sigma_2^E(u^0;x_1,0)\,[u^{hE}](x_1)\Big)\, dx_1 \end{aligned} \tag{5.7}$$

Recalling the normalization conditions (5.1)–(5.4) and the inner asymptotic expansion (4.8), (4.9) of the field u^h, we obtain

$$\begin{aligned} \sigma_{2p}^M(u^0;x_1,0) &= (2\pi r)^{-1/2} K_p^\sigma + O(1), \quad p=1,2,3, \\ \sigma_2^E(u^0;x_1,0) &= (2\pi r)^{-1/2} K_4^\sigma + O(1) \\ [u_p^h](x_1) &= 2\,(2\pi)^{-1/2}\,(h-r)^{1/2}\, K_p^\varepsilon + O(h+r), \quad p=1,2,3, \\ [u^{hE}](x_1) &= 2\,(2\pi)^{-1/2}\,(h-r)^{1/2}\, K_4^\varepsilon + O(h+r) \end{aligned} \tag{5.8}$$

Inserting these expressions into the last integral in (5.7), we derive the relation

$$\begin{aligned} -\frac{h}{2}\sum_{j,k=1}^{4} K_j^\sigma M_{jk}^\sigma K_k^\sigma + O(h^2) &= -\frac{h}{2}\sum_{j,k=1}^{4} K\varepsilon_j M_{jk}^\varepsilon K_k^\varepsilon + O(h^2) \\ = \quad -\frac{h}{2\pi}\sum_{j=1}^{4} K_j^\varepsilon K_j^\sigma \int\limits_0^1 \rho^{-1/2}(1-\rho)^{1/2}\, d\rho + o(h) &= -\frac{h}{4}\sum_{j,k=1}^{4} K_j^\varepsilon N_{kj} K_k^\varepsilon + o(h). \end{aligned}$$

Thus, we obtain the relation

$$N = 2M^\varepsilon$$

for the matrix N which realizes transition (5.6) from the stress basis to the deformation basis. Moreover, the formula $M^\sigma = (4M^\varepsilon)^{-1}$ is valid.

In case iii there exist the stress $\{X^{1\sigma}, X^{2\sigma}, X^{3\sigma}\}$ and deformation $\{X^{1\varepsilon}, X^{2\varepsilon}, X^{3\varepsilon}\}$ bases, subject to the normalization conditions (5.1) and (5.3), respectively; here $j = 1, 2, 3$. The relations

$$K_p^\varepsilon = \lim_{r\to+0} \left(\frac{\pi}{2r}\right)^{1/2} [u_p^0](-r) \quad (p = 1, 2, 3) \tag{5.9}$$

define three SIFs. Since the electric induction vector also has a singularity on the extension of the crack, one may determine [27] *four* GSIFs according to formulae (5.8). However, it is correct to operate only with the three SIFs K_1^σ, K_2^σ, K_3^σ because $K_4^\sigma = a_1K_1^\sigma + a_2K_2^\sigma + a_3K_3^\sigma$ and the coefficients a_q depend on the tensors A^{MM}, A^{ME} and A^{EE} but are independent of the external loads g^M and G^E. Finally, we mention that a repetition of the previous calculation gives relations of the same kind as in case i.

In case ii there also exist the two bases $\{X^{1\sigma}, \ldots, X^{4\sigma}\}$ and $\{X^{1\varepsilon}, \ldots, X^{4\varepsilon}\}$. The first one is defined with the help of conditions on the extension of the crack, but the second one on the crack itself. Equations (5.1) and (5.3) keep their validity, but formulae (5.2) and (5.4), respectively, have to be replaced by

$$X^{j\sigma E}(r, 0) = -(2\pi)^{-1/2} r^{1/2} \delta_{4,j} \quad (j = 1, \ldots, 4) \tag{5.10}$$

$$\left[\sigma_2^E(X^{j\varepsilon})\right](-r) = -2\,(2\pi r)^{-1/2} \delta_{4,j} \quad (j = 1, \ldots, 4) \tag{5.11}$$

The existence of the bases, verifying requirements (5.1), (5.10) or (5.3), (5.11), can be checked up following the scheme [17]. The bases are mixed: the first one give rise to three SIFs and one GDIF, and the second one three DIFs (5.9) and one GSIF. However, the above-stated relations between the basis remain valid.

The authors do not know an appropriate normalization conditions for bases in case iv.

7 The surface electric enthalpy.

We introduce the columns of generalized stresses and strains

$$\begin{aligned} \varepsilon &= (\varepsilon_{11}^M,\ \sqrt{2}\,\varepsilon_{13}^M,\ -\varepsilon_1^M,\ \sqrt{2}\,\varepsilon_{12}^M,\ \sqrt{2}\,\varepsilon_{32}^M,\ -\varepsilon_2^E)^\top, \\ \sigma &= (\sigma_{11}^M,\ \sqrt{2}\,\sigma_{13}^M,\ \sigma_1^M,\ \sqrt{2}\,\sigma_{12}^M,\ \sigma_{22}^M,\ \sqrt{2}\,\sigma_{32}^M,\ \sigma_2^E)^\top. \end{aligned} \tag{7.1}$$

The factors $\sqrt{2}$ equalize the intrinsic norms of the columns and the corresponding tensors. The values ε_{33}^M, ε_3^E and σ_{33}^M, σ_3^E are not included into columns (7.1) because the first couple vanishes due to (1.2) and the second couple does not enter problem (1.3)–(1.8).

By partial Legendre transformation, we obtain two other columns

$$\begin{aligned} \eta &= (\varepsilon_{11}^M,\ \sqrt{2}\,\varepsilon_{13}^M,\ -\varepsilon_1^E,\ -\sqrt{2}\,\sigma_{12}^M,\ -\sigma_{22}^M,\ -\sqrt{2}\,\sigma_{32}^M,\ -\sigma_2^E)^\top, \\ \xi &= (\sigma_{11}^M,\ \sqrt{2}\,\sigma_{13}^M,\ \sigma_1^E,\ \sqrt{2}\,\varepsilon_{12}^M,\ \varepsilon_{22}^M,\ \sqrt{2}\,\varepsilon_{32}^M,\ -\varepsilon_2^E)^\top. \end{aligned} \tag{7.2}$$

According to relations (1.1) the couples (7.1) and (7.2) of columns meet the equalities

$$\sigma = A\varepsilon, \quad \xi = B\eta \tag{7.3}$$

Here A and B are symmetric invertible matrices of size 7×7. The quadratic form $2^{-1}\varepsilon^\top\sigma = 2^{-1}\varepsilon^\top A\varepsilon$ defines the density of the electric enthalpy (3.1). Since in case i the requirements on the jumps of the generalized displacements restrict all components of the column η, the form $P(\xi) = 2^{-1}\xi^\top\eta$ can be called *the density of the surface enthalpy.*

Let T_ρ be a circle with radius $\rho > 0$ and center at the crack tip. According to the Green's formula (3.3), any two power solutions U and V with exponents $\lambda = 1/2$ fulfill the relation

$$\begin{aligned} 0 &= q\left(U, \frac{\partial V}{\partial x_2}; T_{\rho_2}\right) - q\left(U, \frac{\partial V}{\partial x_2}; T_{\rho_1}\right) = \\ &= \sum_{\pm} \pm \int_{\rho_1}^{\rho_2} \left(U_p \sigma^M_{2p}\left(\frac{\partial V}{\partial x_2}\right) + U^E \sigma^E_2\left(\frac{\partial V}{\partial x_2}\right)\right)\Big|_{x_2=\pm 0} dx_1 \end{aligned} \tag{7.4}$$

Note that we differentiate V across the crack, i.e., in general, $q(U, \partial V/\partial x_2; T_\rho)$ is not an invariant integral; however, this integral does not depend on the radius ρ since the integrand equals $O(\rho^{-1})$. Taking into account equations (1.3), (1.4) and the order of the integrand, we process the integral $I_\pm$ from the left-hand side of (7.4)

$$\begin{aligned} I_\pm &= \int_{\rho_1}^{\rho_2} \left(U_p \frac{\partial}{\partial x_1}\sigma^M_{1p}(V) + U^E \frac{\partial}{\partial x_1}\sigma^E_1(V)\right)\Big|_{x_2=\pm 0} dx_1 = \\ &= \int_{\rho_1}^{\rho_2} \left(\frac{\partial U_p}{\partial x_1}\sigma^M_{1p}(V) + \frac{\partial U^E}{\partial x_1}\sigma^E_1(V)\right)\Big|_{x_2=\pm 0} dx_1 = \int_{\rho_1}^{\rho_2} \xi^\pm(V)^\top \eta^\pm(U)\, dx_1. \end{aligned}$$

Since the numbers ρ_1 and ρ_2 are arbitrary, from the latter it follows that the jump of the surface enthalpy on the crack vanishes. This conclusion remains valid for a composite plane in the case when the basis consists of power solutions with the exponent $\lambda = 1/2$.

In view of the normalization conditions (5.3), (5.4) we have

$$\begin{aligned} \left[\varepsilon^M_{11}(X^{j\varepsilon})\right](-r) &= -(2\pi r)^{-1/2}\delta_{1,j}, & \left[\varepsilon^E_1(X^{j\varepsilon})\right](-r) &= -(2\pi r)^{-1/2}\delta_{4,j}, \\ 2\left[\varepsilon^M_{31}(X^{j\varepsilon})\right](-r) &= -(2\pi r)^{-1/2}\delta_{3,j}, & j &= 1, \dots, 4. \end{aligned} \tag{7.5}$$

Thus, the jump $\eta^+(X^{2\varepsilon}) - \eta^-(X^{2\varepsilon})$ vanishes and the following formula is valid for $j = 1, 3, 4$

$$\begin{aligned} &\xi^+(X^{j\varepsilon})^\top \eta^+(X^{2\varepsilon}) - \xi^-(X^{j\varepsilon})^\top \eta^-(X^{2\varepsilon}) \\ &= \left(\eta^+(X^{j\varepsilon}) - \eta^-(X^{j\varepsilon})\right)^\top B\left(\eta^+(X^{2\varepsilon}) + \eta^-(X^{2\varepsilon})\right) = 0. \end{aligned} \tag{7.6}$$

Equalities (7.5) and (7.6) show that the column $\eta(X^{2\varepsilon})$ and, by (7.3), also the column $\xi(X^{2\varepsilon})$ vanish on the crack surfaces. In particular, $\sigma^M_{11}(X^{2\varepsilon}) = \sigma^M_{13}(X^{2\varepsilon}) = 0$, $\sigma^E_2(X^{2\varepsilon}) = 0$ and, owing to equations (1.3), (1.4), the following relations hold on the crack surfaces:

$$\sigma^M_{2p}\left(\frac{\partial X^{2\varepsilon}}{\partial x_2}\right) = -\frac{\partial}{\partial x_1}\sigma^M_{1p}(X^{2\varepsilon}) = 0, \qquad \sigma^E_2\left(\frac{\partial X^{2\varepsilon}}{\partial x_2}\right) = -\frac{\partial}{\partial x_1}\sigma^E_1(X^{2\varepsilon}) = 0$$

Thus, while differentiating *across the crack*, the derivative $\partial X^{2\varepsilon}/\partial x_2$ is still a power solution of the model problem (1.3)–(1.9). Finally, according to the normalization conditions (5.3), (5.4) we obtain

$$\begin{aligned}
\left[\frac{\partial X^{2\varepsilon}}{\partial x_2}\right](-r) &= 2[\varepsilon_{12}(X^{2\varepsilon})](-r) - \frac{\partial}{\partial x_1}[X_2^{2\varepsilon}](-r) = \frac{\partial}{\partial r}\left(2\frac{r}{\pi}\right)^{1/2} \\
\left[\frac{\partial X^{2\varepsilon}}{\partial x_2}\right](-r) &= [\varepsilon_{22}(X^{2\varepsilon})](-r) = 0 \\
\left[\frac{\partial X_3^{2\varepsilon}}{\partial x_2}\right](-r) &= 2[\varepsilon_{23}(X^{2\varepsilon})](-r) = 0, \\
\left[\frac{\partial X^{2\varepsilon E}}{\partial x_2}\right](-r) &= [\varepsilon^{E}(X^{2\varepsilon})](-r) = 0
\end{aligned}$$

The jumps on the crack of the fields $\partial X^{2\varepsilon}/\partial x_2$ and $-\partial X^{1\varepsilon}/\partial x_1$ coincide each with other and, hence,

$$\frac{\partial X^{2\varepsilon}}{\partial x_2} = -\frac{\partial X^{1\varepsilon}}{\partial x_1} = \sum_{k=1}^{4} M_{1k}^{\varepsilon} Y^{k\varepsilon}$$

For the isotropic model problem, the aforementioned properties of the field $X^{2\varepsilon}$, corresponding to the stress-strain state of the first mode, follow from the explicit formulae (3.10), (3.11). In [21, 19], these properties have allowed to investigate the problem on the deviation of a crack in a purely elastic medium.

The above considerations can be adapted to cases ii and iii. For instance, in the first one the boundary conditions on the crack surfaces annul the third, fourth, fifth and sixth components of the column η and the normalization conditions (5.3), (5.11) fix the jumps of the first, second and seventh components. Other calculations also do not need a serious modification.

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