

A generic formula for the values at the boundary points of monic classical orthogonal polynomials

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Abstract: In a previous paper we have determined a generic formula for the polynomial solution families of the well-known differential equation of hypergeometric type

$$\sigma(x)y_n''(x) + \tau(x)y_n'(x) - \lambda_n y_n(x) = 0 .$$

In this paper, we give another such formula which enables us to present a generic formula for the values of monic classical orthogonal polynomials at their boundary points of definition.

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1. Introduction

In previous work [KM], we found a generic polynomial solution for the differential equation

$$\sigma(x)y_n''(x) + \tau(x)y_n'(x) - \lambda_n y_n(x) = 0 \tag{1}$$

where $\sigma(x) = ax^2 + bx + c$ is a polynomial of degree at most 2, $\tau(x) = dx + e$ is a polynomial of degree at most 1 and $\lambda_n = n(n-1)a + nd$ is the eigenvalue parameter depending on $n = 0, 1, 2, \dots$

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Since we will need this formula in this article, we state it here again. In the following theorem from [KM] $\bar{P}_n \left(\begin{smallmatrix} d & e \\ a & b & c \end{smallmatrix} \middle| x \right)$ denotes the monic polynomial solution of equation (1).

2. Theorem: The main differential equation

$$(ax^2 + bx + c)y_n''(x) + (dx + e)y_n'(x) - n((n-1)a + d)y_n(x) = 0 \quad ; \quad n \in \mathbb{Z}^+ \quad (2)$$

has a monic polynomial solution which is represented as

$$\bar{P}_n \left(\begin{smallmatrix} d & e \\ a & b & c \end{smallmatrix} \middle| x \right) = \sum_{k=0}^n \binom{n}{k} G_k^{(n)}(a, b, c, d, e) x^k \quad (3)$$

where

$$G_k^{(n)} = \left(\frac{2a}{b + \sqrt{b^2 - 4ac}} \right)^{k-n} {}_2F_1 \left(k-n, \frac{2ae - bd}{2a\sqrt{b^2 - 4ac}} + 1 - \frac{d}{2a} - n \middle| \frac{2\sqrt{b^2 - 4ac}}{b + \sqrt{b^2 - 4ac}} \right). \quad (4)$$

Note that ${}_2F_1 \left(\begin{smallmatrix} \alpha & \beta \\ \gamma \end{smallmatrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!}$ is the Gauss hypergeometric function [Koe] and

$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)}$ denotes the Pochhammer symbol.

For $a = 0$ these identities can be adapted by limit considerations and give (3) with

$$G_k^{(n)}(0, b, c, d, e) = \lim_{a \rightarrow 0} G_k^{(n)}(a, b, c, d, e) = \left(\frac{b}{c} \right)^{k-n} {}_2F_0 \left(k-n, \frac{cd - be}{b^2} + 1 - n \middle| \frac{b^2}{cd} \right) \quad (5)$$

which is valid for $c, d \neq 0$, leading to

$$\bar{P}_n \left(\begin{smallmatrix} d & e \\ 0 & b & c \end{smallmatrix} \middle| x \right) = \left(\frac{b}{d} \right)^n \left(\frac{eb - cd}{b^2} \right)_n {}_1F_1 \left(\frac{-n}{b^2} \middle| -\frac{d}{b}x - \frac{cd}{b^2} \right). \quad (6)$$

For $a = b = 0$ and $d \neq 0$ we finally get

$$\bar{P}_n \left(\begin{smallmatrix} d & e \\ 0 & 0 & c \end{smallmatrix} \middle| x \right) = \lim_{\substack{a \rightarrow 0 \\ b \rightarrow 0}} \bar{P}_n \left(\begin{smallmatrix} d & e \\ a & b & c \end{smallmatrix} \middle| x \right) = \left(x + \frac{e}{d} \right)^n {}_2F_0 \left(-\frac{n}{2}, -\frac{n-1}{2} \middle| \frac{2cd}{(dx + e)^2} \right). \quad (7)$$

□

In this note, we intend to obtain another representation for the polynomial solution of the main equation (2). To reach this goal, we use the general form of the Rodrigues representation of the polynomials $\bar{P}_n \left(\begin{matrix} d & e \\ a & b & c \end{matrix} \middle| x \right)$.

First, if the equation (2) is written in self-adjoint form, then a weight function $W(x)$ will be derived that satisfies Pearson's differential equation $\frac{d}{dx}(\sigma(x)W(x)) = \tau(x)W(x)$. In other words, we have

$$W \left(\begin{matrix} d & e \\ a & b & c \end{matrix} \middle| x \right) = \exp \left(\int \frac{(d-2a)x + (e-b)}{ax^2 + bx + c} dx \right). \quad (8)$$

Now, without loss of generality, let us assume that $ax^2 + bx + c = a(x + \theta_1)(x + \theta_2)$ where

$$\theta_1 = \frac{b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \theta_2 = \frac{b + \sqrt{b^2 - 4ac}}{2a}. \quad (9)$$

Note that in the generic case, $-\theta_1$ and $-\theta_2$ are the boundary points of the underlying interval for the corresponding orthogonal polynomials. However, if θ_1 and θ_2 are finite and equal, then the polynomials are of the Bessel type and if both θ_1 and θ_2 are finite but different from each other, then the polynomials are of the Jacobi type, whereas if one of these values tends to $\pm\infty$, then the polynomials are of the Laguerre type, and finally if both values are $\pm\infty$, then the polynomials are of the Hermite type.

But relation (9) implies that (8) is simplified as

$$W \left(\begin{matrix} d & e \\ a & b & c \end{matrix} \middle| x \right) = R(x + \theta_1)^A (x + \theta_2)^B \quad (10)$$

where R is a constant and

$$A = \frac{d}{2a} - 1 + \frac{2ae - bd}{2a\sqrt{b^2 - 4ac}} \quad \text{and} \quad B = \frac{d}{2a} - 1 - \frac{2ae - bd}{2a\sqrt{b^2 - 4ac}}. \quad (11)$$

The relation (10) follows because the logarithmic derivative of the function $W^*(x) = (x + \theta_1)^A (x + \theta_2)^B$ equals the logarithmic derivative of the function (8), and since

$$\frac{u'(x)}{u(x)} = \frac{v'(x)}{v(x)} \Leftrightarrow u(x) = R v(x). \quad (12)$$

Hence (10) is valid.

On the other hand, it is known that the Rodrigues representation of $\bar{P}_n \left(\begin{matrix} d & e \\ a & b & c \end{matrix} \middle| x \right)$ can generally be written as [NU, Chapter 1, Section 2]:

$$\bar{P}_n \left(\begin{array}{c} d \quad e \\ a \quad b \quad c \end{array} \middle| x \right) = \frac{1}{\left(\prod_{k=1}^{k=n} d + (n+k-2)a \right)} W \left(\begin{array}{c} d \quad e \\ a \quad b \quad c \end{array} \middle| x \right) \frac{d^n}{dx^n} \left((ax^2 + bx + c)^n W \left(\begin{array}{c} d \quad e \\ a \quad b \quad c \end{array} \middle| x \right) \right). \quad (13)$$

Therefore if (10) is replaced in (13), then

$$\begin{aligned} \bar{P}_n \left(\begin{array}{c} d \quad e \\ a \quad b \quad c \end{array} \middle| x \right) &= \frac{\frac{d^n}{dx^n} (R \ a^n (x+\theta_1)^n (x+\theta_2)^n (x+\theta_1)^A (x+\theta_2)^B)}{R \left(\prod_{k=1}^{k=n} d + (n+k-2)a \right) (x+\theta_1)^A (x+\theta_2)^B} \\ &= \frac{1}{(n-1+d/a)_n (x+\theta_1)^A (x+\theta_2)^B} \frac{d^n}{dx^n} ((x+\theta_1)^{n+A} (x+\theta_2)^{n+B}). \end{aligned} \quad (14)$$

But according to the Leibniz rule

$$\frac{d^n (f(x)g(x))}{dx^n} = \sum_{k=0}^{k=n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \quad (15)$$

we have

$$\frac{d^n ((x+\theta_1)^{n+A} (x+\theta_2)^{n+B})}{dx^n} = (-1)^n \sum_{k=0}^{k=n} \binom{n}{k} (-n-A)_k (-n-B)_{n-k} (x+\theta_1)^{n+A-k} (x+\theta_2)^{n+B-k}. \quad (16)$$

Hence, (14) is simplified as

$$\begin{aligned} \bar{P}_n \left(\begin{array}{c} d \quad e \\ a \quad b \quad c \end{array} \middle| x \right) &= \frac{1}{(2-2n-d/a)_n} \times \\ &\sum_{k=0}^{k=n} \binom{n}{k} \left(-n - \frac{d}{2a} + 1 - \frac{2ae-bd}{2a\sqrt{b^2-4ac}} \right)_k \left(-n - \frac{d}{2a} + 1 + \frac{2ae-bd}{2a\sqrt{b^2-4ac}} \right)_{n-k} \\ &\times \left(x + \frac{b-\sqrt{b^2-4ac}}{2a} \right)^{n-k} \left(x + \frac{b+\sqrt{b^2-4ac}}{2a} \right)^k. \end{aligned} \quad (17)$$

The above relation is in fact another general representation for the polynomial solution of equation (2). Note that (17) is a universal formula. For instance, after simplification of this formula for $n = 0, 1, 2, 3$ we get

$$\left\{ \begin{array}{l}
\bar{P}_0 \left(\begin{array}{ccc|c} d & e & & x \end{array} \right) = 1 \\
\bar{P}_1 \left(\begin{array}{ccc|c} d & e & & x \end{array} \right) = x + \frac{e}{d} \\
\bar{P}_2 \left(\begin{array}{ccc|c} d & e & & x \end{array} \right) = x^2 + 2 \frac{e+b}{d+2a} x + \frac{c(d+2a)+e(e+b)}{(d+2a)(d+a)} \\
\bar{P}_3 \left(\begin{array}{ccc|c} d & e & & x \end{array} \right) = x^3 + 3 \frac{e+2b}{d+4a} x^2 + 3 \frac{c(d+4a)+(e+b)(e+2b)}{(d+4a)(d+3a)} x \\
\qquad \qquad \qquad + \frac{2c(d+3a)(e+2b)+ce(d+4a)+e(e+b)(e+2b)}{(d+4a)(d+3a)(d+2a)}
\end{array} \right. \quad (18)$$

Combining relations (3) and (17), we get straightforwardly

$$\begin{aligned}
& \sum_{k=0}^{k=n} \binom{n}{k} \left(-n - \frac{d}{2a} + 1 - \frac{2ae-bd}{2a\Delta} \right)_k \left(-n - \frac{d}{2a} + 1 + \frac{2ae-bd}{2a\Delta} \right)_{n-k} \left(x + \frac{b-\Delta}{2a} \right)^{n-k} \left(x + \frac{b+\Delta}{2a} \right)^k \\
& = (2-2n-d/a)_n \sum_{k=0}^{k=n} \binom{n}{k} \left(\frac{2a}{b+\Delta} \right)^{k-n} {}_2F_1 \left(\begin{array}{c} k-n \quad \frac{2ae-bd}{2a\Delta} + 1 - \frac{d}{2a} - n \\ 2-d/a-2n \end{array} \middle| \frac{2\Delta}{b+\Delta} \right) x^k .
\end{aligned} \quad (19)$$

where $\Delta = \sqrt{b^2 - 4ac}$.

Relation (17) can also be represented in hypergeometric form as

$$\bar{P}_n \left(\begin{array}{ccc|c} d & e & & x \end{array} \right) = \frac{\left(-n - \frac{d}{2a} + 1 + \frac{2ae-bd}{2a\Delta} \right)_n (x+\theta_1)^n}{(2-2n-d/a)_n} {}_2F_1 \left(\begin{array}{c} -n \quad -n - \frac{d}{2a} + 1 - \frac{2ae-bd}{2a\Delta} \\ \frac{d}{2a} - \frac{2ae-bd}{2a\Delta} \end{array} \middle| \frac{x+\theta_2}{x+\theta_1} \right) \quad (20)$$

where θ_1 and θ_2 are defined by (9).

This hypergeometric representation can still be simplified. For this purpose, we use the hypergeometric identity

$${}_2F_1 \left(\begin{array}{c} -n \quad p \\ q \end{array} \middle| \frac{r}{t} + s \right) = \frac{(-1)^n (p)_n}{(q)_n t^n} \sum_{k=0}^n \binom{n}{k} r^{n-k} {}_2F_1 \left(\begin{array}{c} -k \quad 1-q-n \\ 1-p-n \end{array} \middle| \frac{1}{s} \right) (st)^k \quad (21)$$

which was used already in [KM, formula (1.5)]. If we choose in particular

$$p = -n - \frac{d}{2a} + 1 - \frac{2ae - bd}{2a\sqrt{b^2 - 4ac}}, \quad q = \frac{d}{2a} - \frac{2ae - bd}{2a\sqrt{b^2 - 4ac}}, \quad r = \theta_2 - \theta_1 = \frac{\sqrt{b^2 - 4ac}}{a}, \quad (22)$$

$$s = 1 \quad \text{and} \quad t = x + \frac{b - \sqrt{b^2 - 4ac}}{2a},$$

then by (21) relation (20) reads as

$$\bar{P}_n \left(\begin{matrix} d & e \\ a & b & c \end{matrix} \middle| x \right) = \frac{(-1)^n (-n-B)_n (-n-A)_n \sum_{k=0}^{k=n} \binom{n}{k}}{(2-2n-d/a)_n (1+B)_n} \left(\frac{\Delta}{a} \right)^{n-k} {}_2F_1 \left(\begin{matrix} -k & -n-B \\ 1+A \end{matrix} \middle| 1 \right) (x + \theta_1)^k. \quad (23)$$

Using Gauss's identity

$${}_2F_1 \left(\begin{matrix} a & b \\ c \end{matrix} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (24)$$

(23) can be further simplified as

$$\bar{P}_n \left(\begin{matrix} d & e \\ a & b & c \end{matrix} \middle| x \right) = \frac{(\sqrt{b^2 - 4ac})^n \left(\frac{d}{2a} + \frac{2ae - bd}{2a\sqrt{b^2 - 4ac}} \right)_n}{a^n (2-2n-d/a)_n} \times$$

$${}_2F_1 \left(\begin{matrix} -n & n-1+d/a \\ \frac{d}{2a} + \frac{2ae - bd}{2a\sqrt{b^2 - 4ac}} \end{matrix} \middle| \frac{-ax}{\sqrt{b^2 - 4ac}} + \frac{-b + \sqrt{b^2 - 4ac}}{2\sqrt{b^2 - 4ac}} \right). \quad (25)$$

On the other hand since

$$\bar{P}_n \left(\begin{matrix} \lambda d & \lambda e \\ \lambda a & \lambda b & \lambda c \end{matrix} \middle| x \right) = \bar{P}_n \left(\begin{matrix} d & e \\ a & b & c \end{matrix} \middle| x \right) \quad \forall \lambda \neq 0 \quad (26)$$

is valid, for $\lambda = -1$, the relation (25) can be also brought in the following form

$$\bar{P}_n \left(\begin{matrix} d & e \\ a & b & c \end{matrix} \middle| x \right) = \frac{(\sqrt{b^2 - 4ac})^n \left(\frac{d}{2a} - \frac{2ae - bd}{2a\sqrt{b^2 - 4ac}} \right)_n}{(-a)^n (n-1+d/a)_n} \times$$

$${}_2F_1 \left(\begin{matrix} -n & n-1+d/a \\ \frac{d}{2a} - \frac{2ae - bd}{2a\sqrt{b^2 - 4ac}} \end{matrix} \middle| \frac{ax}{\sqrt{b^2 - 4ac}} + \frac{b + \sqrt{b^2 - 4ac}}{2\sqrt{b^2 - 4ac}} \right). \quad (27)$$

3. Values of the classical orthogonal polynomials at the boundary points

Using our explicit representations for the monic classical orthogonal polynomials of the last section, we can now compute the generic value of these polynomials at their boundary points of definition, $-\theta_1$ and $-\theta_2$, respectively.

If we set in (27)

$$\frac{ax}{\sqrt{b^2-4ac}} + \frac{b+\sqrt{b^2-4ac}}{2\sqrt{b^2-4ac}} = \begin{cases} 0 \\ 1 \end{cases}, \quad (28)$$

then

$$x = \frac{-b-\sqrt{b^2-4ac}}{2a} = -\theta_2 \quad \text{and} \quad x = \frac{-b+\sqrt{b^2-4ac}}{2a} = -\theta_1, \quad (29)$$

respectively. Therefore we get

$$\bar{P}_n \left(\begin{matrix} d & e \\ a & b & c \end{matrix} \middle| -\theta_2 \right) = \frac{(\sqrt{b^2-4ac})^n ((d/2a) - (2ae-bd)/(2a\sqrt{b^2-4ac}))_n}{(-a)^n (n-1+d/a)_n}, \quad (30)$$

and

$$\bar{P}_n \left(\begin{matrix} d & e \\ a & b & c \end{matrix} \middle| -\theta_1 \right) = \frac{(\sqrt{b^2-4ac})^n ((d/2a) + (2ae-bd)/(2a\sqrt{b^2-4ac}))_n}{a^n (n-1+d/a)_n}. \quad (31)$$

For example, for the monic Jacobi orthogonal polynomials $\bar{P}_n^{(\alpha,\beta)}(x)$ [Sze] we have $(a,b,c,d,e) = (-1,0,1,-\alpha-\beta-2,-\alpha+\beta)$. In this case, (30) and (31) therefore yield

$$\bar{P}_n^{(\alpha,\beta)}(+1) = 2^n \frac{(\alpha+1)_n}{(n+1+\alpha+\beta)_n} = 2^n \frac{\Gamma(n+1+\alpha)\Gamma(n+1+\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(2n+1+\alpha+\beta)}, \quad (32)$$

$$\bar{P}_n^{(\alpha,\beta)}(-1) = (-2)^n \frac{(\beta+1)_n}{(n+1+\alpha+\beta)_n} = (-2)^n \frac{\Gamma(n+1+\beta)\Gamma(n+1+\alpha+\beta)}{\Gamma(\beta+1)\Gamma(2n+1+\alpha+\beta)}. \quad (33)$$

Moreover, for the monic Laguerre polynomials $\bar{L}_n^{(\alpha)}(x)$ with $(a,b,c,d,e) = (0,1,0,-1,\alpha+1)$ we have $ax^2+bx+c=x$. Therefore just one root i.e. $\theta_1 = \theta_2 = 0$ is derived. Hence by computing the corresponding limit one gets

$$\bar{L}_n^{(\alpha)}(0) = (-1)^n (1+\alpha)_n. \quad (34)$$

On the other hand, since the Hermite polynomials can be written in terms of the Laguerre polynomials (see e.g. [AS], 22.5.40), we can also conclude that

$$\bar{H}_n(0) = \frac{n!}{2^{n+1}(n/2)!} (1 + (-1)^n) \quad (35)$$

Finally, for the Bessel polynomials $\bar{B}_n^{(\alpha)}(x)$ with $(a, b, c, d, e) = (1, 0, 0, \alpha + 2, 2)$ we have $ax^2 + bx + c = x^2$. Consequently after computing the corresponding limit we get:

$$\bar{B}_n^{(\alpha)}(0) = \frac{2^n}{(n+1+\alpha)_n} . \quad (36)$$

Note that the last result follows from our formula although the Bessel polynomials are not orthogonal in a real interval.

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