

On the Computation of Fourier Coefficients

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Abstract

In this paper we derive an identity for the Fourier coefficients of a differentiable function $f(t)$ in terms of the Fourier coefficients of its derivative $f'(t)$. This yields an algorithm to compute the Fourier coefficients of $f(t)$ whenever the Fourier coefficients of $f'(t)$ are known, and vice versa. Furthermore this generates an iterative scheme for N times differentiable functions complementing the direct computation of Fourier coefficients via the defining integrals which can be also treated automatically in certain cases, see [5] and [2].

1 Introduction

As general references for elementary properties of Fourier series see e.g. [1] and [4].

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function in the interval $I = [a, b]$ which is continuously differentiable in (a, b) . The function f can be periodically continued to \mathbb{R} with period $T = b - a$. Let's denote the resulting periodic function by $F : \mathbb{R} \rightarrow \mathbb{R}$. By definition, the function F is continuous in \mathbb{R} besides the points $a + kT$ ($k \in \mathbb{Z}$) which are (possible) discontinuities of step size $\Delta := f(a) - f(b)$.

The real and complex Fourier series of $f(t)$ are given as

$$\mathcal{F}(f)(t) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t},$$

where $\omega = \frac{2\pi}{T}$ is the circular frequency and the corresponding real Fourier coefficients are given by

$$a_n = \frac{2}{T} \int_a^b f(t) \cos(n\omega t) dt \in \mathbb{R},$$
$$b_n = \frac{2}{T} \int_a^b f(t) \sin(n\omega t) dt \in \mathbb{R},$$

whereas the complex Fourier coefficients are defined as

$$c_n = \frac{1}{T} \int_a^b f(t) e^{-in\omega t} dt \in \mathbb{C}.$$

Note that for real $f(t)$ one has $c_{-n} = \bar{c}_n$, \bar{z} denoting the complex conjugate of z .

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Of course, by definition we have the relations $c_0 = \frac{a_0}{2}$ as well as

$$c_n = \frac{1}{2}(a_n - i b_n) \quad \text{and therefore} \quad \bar{c}_n = \frac{1}{2}(a_n + i b_n) \quad (n \in \mathbb{Z}_{\geq 1})$$

which—solving for a_n and b_n —gives

$$a_n = c_n + \bar{c}_n \quad \text{and} \quad b_n = i(c_n - \bar{c}_n) \quad (n \in \mathbb{Z}_{\geq 1}).$$

Under our assumption that f is continuous in (a, b) , it turns out that pointwise $\mathcal{F}(f)(t) = f(t)$ for all $t \in (a, b)$.¹

Note that if the defining interval is symmetric, i.e. $(a, b) = (-\frac{T}{2}, \frac{T}{2})$, then the Fourier coefficients of an even function f satisfy $b_n = 0$ ($n \in \mathbb{Z}_{\geq 1}$), and for an odd function f one gets $a_n = 0$ ($n \in \mathbb{Z}_{\geq 1}$).

Using the defining equations, the Fourier coefficients can be computed automatically in certain instances using a computer algebra system. This method was considered e.g. in [5] using *Maple* [3] and in [2] using *Mathematica* [6]. In the current article we will develop a different method.

In the sequel we will use the following notations. Since f is continuously differentiable, f' is continuous and has a Fourier series itself, for which we write

$$\mathcal{F}(f')(t) = \frac{a'_0}{2} + \sum_{n=1}^{\infty} a'_n \cos(n\omega t) + \sum_{n=1}^{\infty} b'_n \sin(n\omega t) = \sum_{n=-\infty}^{\infty} c'_n e^{in\omega t},$$

i.e., the Fourier coefficients of the derivative function are denoted by dashes. If $f \in C^N[a, b]$, then we can continue taking derivatives, and for the k th derivative ($k \leq N$) we use the notation

$$\mathcal{F}(f^{(k)})(t) = \frac{a_0^{(k)}}{2} + \sum_{n=1}^{\infty} a_n^{(k)} \cos(n\omega t) + \sum_{n=1}^{\infty} b_n^{(k)} \sin(n\omega t) = \sum_{n=-\infty}^{\infty} c_n^{(k)} e^{in\omega t}.$$

2 An Identity for Fourier Coefficients

The complex Fourier coefficients c_n are given by the relation

$$c_n = \frac{1}{T} \int_a^{a+T} f(t) e^{-in\omega t} dt.$$

Integrating by parts, using $u(t) = f(t)$, $v'(t) = e^{-in\omega t}$, hence $u'(t) = f'(t)$ and $v(t) = \frac{1}{-in\omega} e^{-in\omega t}$, we get

$$\begin{aligned} c_n &= \left[\frac{f(t) e^{-in\omega t}}{-in\omega T} \right]_a^{a+T} + \frac{1}{T} \int_a^{a+T} \frac{f'(t) e^{-in\omega t}}{in\omega} dt \\ &= \frac{f(a+T) e^{-in\omega(a+T)} - f(a) e^{-in\omega a}}{-in\omega T} + \frac{1}{in\omega} \left(\frac{1}{T} \int_a^{a+T} f'(t) e^{-in\omega t} dt \right) \\ &= \frac{f(a+T) - f(a)}{-2\pi in} e^{-in\omega a} - \frac{i}{n\omega} c'_n, \end{aligned}$$

¹One can furthermore prove that at the points of discontinuity one has $\mathcal{F}(f)(a+kT) = \frac{1}{2}(f(a) + f(b))$ ($k \in \mathbb{Z}$).

where we have used that $\omega T = 2\pi$. Hence we have derived the identity

$$c_n + \frac{i}{n\omega} c'_n = \frac{i}{2\pi n} (f(b) - f(a)) e^{-in\omega a}. \quad (1)$$

As we shall discuss later, this easy-to-derive relation has interesting applications, and can be used to compute the Fourier coefficients recursively under certain conditions.

Next, we would like to rewrite the above equation in terms of the real Fourier coefficients a_n and b_n . Using the relation $c_n = \frac{1}{2}(a_n - ib_n)$, we get from (1)

$$a_n - ib_n + \frac{i}{n\omega} (a'_n - ib'_n) = \frac{i}{\pi n} (f(b) - f(a)) e^{-in\omega a},$$

and separating the real and imaginary parts, we conclude that

$$\begin{aligned} a_n + \frac{1}{n\omega} b'_n &= 0 \quad \text{and} \\ -b_n + \frac{1}{n\omega} a'_n &= \frac{1}{\pi n} (f(b) - f(a)) e^{-in\omega a}. \end{aligned} \quad (2)$$

Finally, we summarize the above identities in the following

Theorem 1 (Fourier coefficients and derivatives) *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous in $[a, b]$ and continuously differentiable in (a, b) . Then the real and complex Fourier coefficients of $f(t)$ and of $f'(t)$ satisfy the identities*

$$c_n + \frac{i}{n\omega} c'_n = \frac{i}{2\pi n} (f(b) - f(a)) e^{-in\omega a}$$

and

$$\begin{aligned} a_n + \frac{1}{n\omega} b'_n &= 0 \quad \text{and} \\ -b_n + \frac{1}{n\omega} a'_n &= \frac{1}{\pi n} (f(b) - f(a)) e^{-in\omega a}. \end{aligned}$$

respectively.

In particular: If $a = 0$, i.e. if the interval is $I = [0, T]$, then

$$c_n + i \frac{T}{2\pi n} c'_n = \frac{i}{2\pi n} (f(T) - f(0))$$

and

$$\begin{aligned} a_n + \frac{T}{2\pi n} b'_n &= 0 \quad \text{and} \\ -b_n + \frac{T}{2\pi n} a'_n &= \frac{1}{\pi n} (f(T) - f(0)). \end{aligned} \quad (3)$$

Furthermore, if the interval is symmetric w.r.t the origin, i.e. if $I = [-\frac{T}{2}, \frac{T}{2}]$, then

$$c_n + i \frac{T}{2\pi n} c'_n = \frac{i}{2\pi n} (f(\frac{T}{2}) - f(-\frac{T}{2})) (-1)^n$$

and

$$\begin{aligned} a_n + \frac{T}{2\pi n} b'_n &= 0 \quad \text{and} \\ -b_n + \frac{T}{2\pi n} a'_n &= \frac{(-1)^n}{\pi n} (f(\frac{T}{2}) - f(-\frac{T}{2})). \end{aligned} \quad (4)$$

Using these identities, one can easily compute the Fourier coefficients of $f'(t)$, if those of $f(t)$ are known, and vice versa.

3 Iterative Computation of Fourier Coefficients

Theorem 1 can be used to compute the Fourier coefficients iteratively. We give some examples for this approach.

Example 2 (Fourier coefficients of powers) Let $f(t) = t^m$ for some $m \in \mathbb{Z}_{\geq 1}$. First, we consider the case $I = [0, T]$. For $m = 1$ and $m = 2$, the corresponding periodic functions F are drawn in Figure 1. Note that the periodic linear function is called a sawtooth function.



Figure 1: The linear and square functions for $T = 1$

Whereas in these cases, it is easy to compute the Fourier coefficients directly from the defining formulas, namely for $f(t) = t$:

$$a_n = \frac{2}{T} \int_0^T t \cos\left(\frac{2\pi nt}{T}\right) dt = 0,$$

$$b_n = \frac{2}{T} \int_0^T t \sin\left(\frac{2\pi nt}{T}\right) dt = -\frac{T}{\pi n},$$

and for $f(t) = t^2$:

$$a_n = \frac{2}{T} \int_0^T t^2 \cos\left(\frac{2\pi nt}{T}\right) dt = \frac{T^2}{\pi^2 n^2},$$

$$b_n = \frac{2}{T} \int_0^T t^2 \sin\left(\frac{2\pi nt}{T}\right) dt = -\frac{T^2}{\pi n},$$

we would like to use Theorem 1 instead. Let $f(t) = t$. Then the derivative $f'(t) = 1$ obviously has Fourier coefficients $a'_0 = 2$ and $a'_n = 0, b'_n = 0$ ($n \in \mathbb{Z}_{\geq 1}$). Therefore we get from (3)

$$a_n + \frac{T}{2\pi n} b'_n = 0,$$

hence $a_n = 0$ and

$$-b_n + \frac{T}{2\pi n} a'_n = \frac{1}{\pi n} (f(T) - f(0)) = \frac{T}{\pi n},$$

hence $b_n = -\frac{T}{\pi n}$.

In the next step, we set $f(t) = t^2$. Hence $f'(t) = 2t$ with $a'_n = 0$ and $b'_n = -\frac{2T}{\pi n}$ by our previous computation. Therefore, using (3), we get

$$a_n + \frac{T}{2\pi n} b'_n = a_n - \frac{T}{2\pi n} \frac{2T}{\pi n} = a_n - \frac{T^2}{\pi^2 n^2} = 0,$$

hence $a_n = \frac{T^2}{\pi^2 n^2}$ and

$$-b_n + \frac{T}{2\pi n} a'_n = -b_n = \frac{1}{\pi n} (f(T) - f(0)) = \frac{T^2}{\pi n},$$

hence $b_n = -\frac{T^2}{\pi n}$.

Obviously, this strategy can be used iteratively (or recursively) to compute the Fourier coefficients of every power $f(t) = t^m$, and by linearity, of every polynomial. This algorithm is considered in more generality in the next section.

Next, we consider the symmetric case $I = [-\frac{T}{2}, \frac{T}{2}]$. The corresponding functions for $m = 1$ and $m = 2$ are drawn in Figure 2.

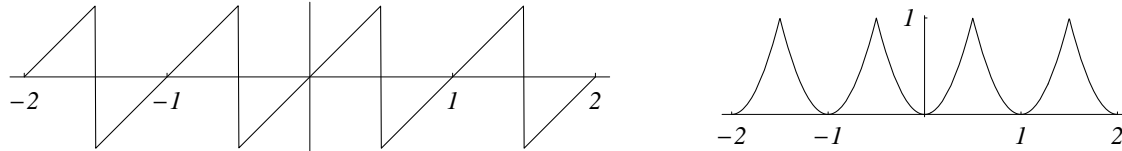


Figure 2: The symmetric linear and square functions ($f(t) = t$ and $f(t) = 4t^2$) for $T = 1$

Again one can compute the Fourier coefficients easily from their definition. For $f(t) = t$:

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} t \cos\left(\frac{2\pi nt}{T}\right) dt = 0,$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} t \sin\left(\frac{2\pi nt}{T}\right) dt = -\frac{(-1)^n T}{\pi n},$$

and for $f(t) = t^2$:

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} t^2 \cos\left(\frac{2\pi nt}{T}\right) dt = \frac{(-1)^n T^2}{\pi^2 n^2},$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} t^2 \sin\left(\frac{2\pi nt}{T}\right) dt = 0.$$

However, these results are contained in Formulas (4) and can be also computed iteratively.

Example 3 (Fourier coefficients including special functions) Let $f(t) = \arctan t$. Since $f(t)$ is odd, it is clear that $a_n = 0$. However the coefficients b_n cannot be easily computed by the defining formula:

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} \arctan t \sin\left(\frac{2\pi nt}{T}\right) dt.$$

Note that Maple fails to compute b_n even if T is given explicitly.

However, we can compute b_n in terms of special functions using (4). Notice that for $f(t) = \arctan t$, we have $f'(t) = \frac{1}{1+t^2}$, a rational function. For the even function $f'(t)$ we get $b'_n = 0$, and for a'_n we derive using Maple

```
> anprime:=2/T*int(1/(1+t^2)*cos(2*Pi*n*t/T),t=-T/2..T/2);
```

$$\begin{aligned}
anprime := & 2(-2 \operatorname{Si}(\frac{n \pi (T + 2 I)}{T}) \sinh(\frac{n \pi}{T}) \cosh(\frac{n \pi}{T}) + \operatorname{Ci}(-\frac{n \pi (T + 2 I)}{T}) \cosh(\frac{n \pi}{T})^2 I \\
& - \frac{1}{2} I \operatorname{Ci}(-\frac{n \pi (T + 2 I)}{T}) - 2 \operatorname{Si}(\frac{n \pi (T - 2 I)}{T}) \sinh(\frac{n \pi}{T}) \cosh(\frac{n \pi}{T}) \\
& - \operatorname{Ci}(-\frac{n \pi (T - 2 I)}{T}) \cosh(\frac{n \pi}{T})^2 I + \frac{1}{2} I \operatorname{Ci}(-\frac{n \pi (T - 2 I)}{T}) \\
& - \operatorname{Ci}(\frac{n \pi (T - 2 I)}{T}) \cosh(\frac{n \pi}{T})^2 I + \frac{1}{2} I \operatorname{Ci}(\frac{n \pi (T - 2 I)}{T}) \\
& + \operatorname{Ci}(\frac{n \pi (T + 2 I)}{T}) \cosh(\frac{n \pi}{T})^2 I - \frac{1}{2} I \operatorname{Ci}(\frac{n \pi (T + 2 I)}{T})) / T
\end{aligned}$$

where

$$\operatorname{Ci}(t) = - \int_t^\infty \frac{\cos x}{x} dx = \gamma + \ln t + \int_0^t \frac{\cos x - 1}{x} dx$$

(γ denoting the Euler-Mascheroni constant) and

$$\operatorname{Si}(t) = \int_0^t \frac{\sin x}{x} dx$$

denote the integral cosine and sine functions, respectively.

Since for $f(t) = \arctan t$ we have $f(-T/2) = \arctan(-T/2) = -\arctan(T/2) = -f(T/2)$ it is now clear that we get for b_n by (4)

$$> f := \arctan(t);$$

$$f := \arctan(t)$$

$$> bn := T / (2 * Pi * n) * anprime - (-1)^n / (Pi * n) * 2 * subs(t=T/2, f);$$

$$\begin{aligned}
bn := & (-2 \operatorname{Si}(\frac{n \pi (T + 2 I)}{T}) \sinh(\frac{n \pi}{T}) \cosh(\frac{n \pi}{T}) + \operatorname{Ci}(-\frac{n \pi (T + 2 I)}{T}) \cosh(\frac{n \pi}{T})^2 I \\
& - \frac{1}{2} I \operatorname{Ci}(-\frac{n \pi (T + 2 I)}{T}) - 2 \operatorname{Si}(\frac{n \pi (T - 2 I)}{T}) \sinh(\frac{n \pi}{T}) \cosh(\frac{n \pi}{T}) \\
& - \operatorname{Ci}(-\frac{n \pi (T - 2 I)}{T}) \cosh(\frac{n \pi}{T})^2 I + \frac{1}{2} I \operatorname{Ci}(-\frac{n \pi (T - 2 I)}{T}) \\
& - \operatorname{Ci}(\frac{n \pi (T - 2 I)}{T}) \cosh(\frac{n \pi}{T})^2 I + \frac{1}{2} I \operatorname{Ci}(\frac{n \pi (T - 2 I)}{T}) \\
& + \operatorname{Ci}(\frac{n \pi (T + 2 I)}{T}) \cosh(\frac{n \pi}{T})^2 I - \frac{1}{2} I \operatorname{Ci}(\frac{n \pi (T + 2 I)}{T})) / (n \pi) - \frac{2(-1)^n \arctan(\frac{T}{2})}{n \pi}
\end{aligned}$$

It turns out that for this particular example Maple fails, but Mathematica is successful and gives the above result directly from the underlying integral. However, our theorem gives an algorithmic approach which will lead to further examples that are out of reach for Maple or Mathematica.

Let us finish this section with the remark that for rational functions the algorithm described can be used to lower the degree of powers of the denominator polynomial recursively. Details will be given in a forthcoming paper.

4 Iterated Derivatives

In this section, we assume that $f \in C^N[0, T]$ for some $N \in \mathbb{Z}_{\geq 1}$. Then by Theorem 1 we can write down successive identities for successive derivatives of f . Using (1), we get for the complex Fourier coefficients

$$\begin{aligned} c_n + i\frac{T}{2\pi n}c'_n &= \frac{i}{2\pi n}(f(T) - f(0)) \\ c'_n + i\frac{T}{2\pi n}c''_n &= \frac{i}{2\pi n}(f'(T) - f'(0)) \\ c''_n + i\frac{T}{2\pi n}c_n^{(3)} &= \frac{i}{2\pi n}(f''(T) - f''(0)) \\ &\vdots \\ c_n^{(N-1)} + i\frac{T}{2\pi n}c_n^{(N)} &= \frac{i}{2\pi n}(f^{(N-1)}(T) - f^{(N-1)}(0)) \end{aligned}$$

In order to manipulate the previous relations easier, let us rewrite them in the following way:

$$\begin{aligned} c_n + a c'_n &= \alpha_0 \\ c'_n + a c''_n &= \alpha_1 \\ c''_n + a c_n^{(3)} &= \alpha_2 \\ &\vdots \\ c_n^{(N-1)} + a c_n^{(N)} &= \alpha_{N-1} \end{aligned}$$

with the abbreviations $a = i\frac{T}{2\pi n}$ and $\alpha_k = \frac{i}{2\pi n}(f^{(k)}(T) - f^{(k)}(0))$.

Then multiplying the k th equation by $(-1)^k a^k$ and summing up obviously yields a telescoping sum with the result

$$c_n = \alpha_0 - a\alpha_1 + a^2\alpha_2 - a^3\alpha_3 + \cdots + (-1)^{N-1}a^{N-1}\alpha_{N-1} + (-1)^N a^N c_n^{(N)}$$

which finally leads to

Theorem 4 (Fourier coefficients and iterated derivatives) *For $f \in C^N[0, T]$ the following identity for the complex Fourier coefficients is valid:*

$$c_n - (-1)^N \left(\frac{Ti}{2\pi n}\right)^N c_n^{(N)} = \sum_{k=0}^{N-1} (-1)^k \left(\frac{Ti}{2\pi n}\right)^k \frac{i}{2\pi n} (f^{(k)}(T) - f^{(k)}(0)).$$

As a consequence, since a polynomial $f(t)$ of degree N satisfies $f^{(N)}(t) = \text{constant}$, and therefore $c_n^{(N)} = 0$ for $n \in \mathbb{Z}_{\geq 1}$, Theorem 4 implies

Corollary 5 (Fourier coefficients of polynomials) *Let $f : [0, T] \rightarrow \mathbb{R}$ be a polynomial of degree N . Then the complex Fourier coefficients of f can be written in the form ($n \in \mathbb{Z}_{\geq 1}$)*

$$c_n = \sum_{k=0}^{N-1} (-1)^k \left(\frac{Ti}{2\pi n}\right)^k \frac{i}{2\pi n} (f^{(k)}(T) - f^{(k)}(0)).$$

Similarly, we can treat the interval $I = [-\frac{T}{2}, \frac{T}{2}]$. In this case, $\alpha_k = i \frac{(-1)^n}{2\pi n} (f^{(k)}(T) - f^{(k)}(0))$, hence we have

Theorem 6 (Fourier coefficients and iterated derivatives in symmetric intervals) For $f \in C^N[-\frac{T}{2}, \frac{T}{2}]$ the following identity for the complex Fourier coefficients is valid:

$$c_n - (-1)^N \left(\frac{Ti}{2\pi n}\right)^N c_n^{(N)} = \sum_{k=0}^{N-1} (-1)^{k+n} \left(\frac{Ti}{2\pi n}\right)^k \frac{i}{2\pi n} \left(f^{(k)}\left(\frac{T}{2}\right) - f^{(k)}\left(-\frac{T}{2}\right)\right).$$

In particular: Let $f : [-\frac{T}{2}, \frac{T}{2}] \rightarrow \mathbb{R}$ be a polynomial of degree N . Then the complex Fourier coefficients of f can be written in the form ($n \in \mathbb{Z}_{\geq 1}$)

$$c_n = \sum_{k=0}^{N-1} (-1)^{k+n} \left(\frac{Ti}{2\pi n}\right)^k \frac{i}{2\pi n} \left(f^{(k)}\left(\frac{T}{2}\right) - f^{(k)}\left(-\frac{T}{2}\right)\right). \quad (5)$$

Note that the computation of the Fourier coefficients of t^m , e.g., using (5) is much more efficient than the computation using the definition. But for polynomials, we can do even better.

5 Fourier Coefficients of Polynomials

Although the algorithm of the previous section can be easily used to compute the Fourier coefficients of every polynomial efficiently, in the current section we would like to mention that the Fourier coefficients of polynomials can be even written down explicitly.

By the linearity of the Fourier coefficients, it is enough to know them for powers $f(t) = t^m$ ($m \in \mathbb{Z}_{\geq 1}$). In this case, we have

Theorem 7 (Fourier coefficients of powers) Let $f(t) = t^m$ ($m \in \mathbb{Z}_{\geq 1}$). The Fourier coefficients in the interval $I = [-\frac{T}{2}, \frac{T}{2}]$ are given as

$$a_n = \begin{cases} \left(\frac{T}{2\pi n}\right)^m (-1)^n \sum_{k=0}^{\frac{m}{2}-1} \frac{2m!}{(2k+1)!} (n\pi)^{2k} (-1)^{\frac{m}{2}-1+k} & \text{if } m \text{ is even} \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

and

$$b_n = \begin{cases} \left(\frac{T}{2\pi n}\right)^m (-1)^{n+1} \sum_{k=0}^{\frac{m-1}{2}} \frac{2m!}{(2k+1)!} (n\pi)^{2k} (-1)^{\frac{m-1}{2}+k} & \text{if } m \text{ is odd} \\ 0 & \text{otherwise} \end{cases}. \quad (7)$$

Proof: The formulas are obvious consequences of Theorem 6.

On the other hand, these formulas can also be obtained more directly from the definition. We would like to find

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} t^m \cos\left(n \frac{2\pi}{T} t\right) dt.$$

To do so it is enough to check the antiderivative

$$\int t^m \cos(n\omega t) dt = \frac{1}{(n\omega)^{m+1}} \times$$

$$\left[\frac{1 + (-1)^m}{2} \left(\sum_{k=0}^{\frac{m}{2}} (n\omega t)^{2k} \frac{m!}{(2k)!} (-1)^{\frac{m}{2}+k} \sin(n\omega t) + \sum_{k=0}^{\frac{m}{2}-1} (-1)^{\frac{m+2}{2}+k} (n\omega t)^{2k+1} \frac{m!}{(2k+1)!} \cos(n\omega t) \right) + \frac{1 - (-1)^m}{2} \sum_{k=0}^{\frac{m-1}{2}} (-1)^{\frac{m-1}{2}+k} \left((n\omega t)^{2k} \frac{m!}{(2k)!} \cos(n\omega t) + (n\omega t)^{2k+1} \frac{m!}{(2k+1)!} \sin(n\omega t) \right) \right]$$

of the integrand $t^m \cos \omega t$ by differentiation. Formula (6) then follows by the fundamental theorem of calculus. In a similar fashion the Fourier coefficient b_n can be treated. \square

We would like to mention that similar formulas can be obtained if the interval is given as $I = [0, T]$.

6 Efficiency Considerations

Whereas it seems obvious that the closed formulas (6)–(7) should yield the fastest computation for the Fourier coefficients of $f(t) = t^m$ whereas (5) should be weaker, and the direct computation using the definition

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} t^m \exp\left(-int \frac{2\pi}{T}\right) dt \quad (8)$$

should lead to the longest computation times, reality is a little more complicated. Undoubtedly, the computation via the definition is the weakest method. Whether (5) or (6)–(7) are faster, depends on implementation details, however, and the above assumption is true only for small values of m . If (5) is programmed directly as sum, then (6)–(7) are faster since (5) needs the repeated computation of high order derivatives. On the other hand, if (5) is programmed by computing the high order derivatives iteratively, then this is faster than (6)–(7) for large m . The reason is that the calculation of large factorials $m!$ and $(2k+1)!$ and their gcd computation in (6)–(7) is avoided. Detailed timings can be seen in Tables 1–3.²

The fastest codes for large m in *Maple* and *Mathematica*, respectively, based on (5), are given in Table 6.

$m = 100$	(8)	(5) with Σ	(5) iteratively	(6)–(7)
<i>Maple</i>	3.47	0.47	0.00	0.00
<i>Mathematica</i>	3.61	0.02	0.02	0.00

Table 1: Timings for the computation of c_n in $[-\frac{T}{2}, \frac{T}{2}]$ for $f(t) = t^{100}$ with different algorithms

$m = 1.000$	(8)	(5) with Σ	(5) iteratively	(6)–(7)
<i>Maple</i>	\diamond	0.86	0.17	0.11
<i>Mathematica</i>	27.45	0.59	0.22	0.23

Table 2: Timings for the computation of c_n for $f(t) = t^{1.000}$

²All timings are in seconds and were done with *Maple* 10 / *Mathematica* 5.2 and a laptop with Intel Core Duo T 2600, 2.16 GHz CPU and 2 GB RAM. The iteration is most efficient with *Maple* using a `for` loop, and with *Mathematica* generating a list `tab` and using `Apply[Plus, tab]`. \diamond indicates that the computation was not successful within one hour.

$m = 10.000$	(8)	(5) with Σ	(5) iteratively	(6)–(7)
<i>Maple</i>	\diamond	105.25	23.64	131.5
<i>Mathematica</i>	112.9	88.77	13.16	77.34

Table 3: Timings for the computation of c_n for $f(t) = t^{10.000}$

```

Method2b:=proc(m,T,n)
  local j,x,sum,dif;
  sum:=0;
  dif:=x^m;
  for j from 0 to m-1 do
    sum:=sum+(-1)^(j+n)*T^j*(I/(2*Pi*n))^(j+1)*
      (subs(x=T/2,dif)-subs(x=-T/2,dif));
    dif:=diff(dif,x);
  od;
  sum;
end;

Method2b[m_, T_, n_] := Module[{j, x, sum, diff, tab},
  sum = 0;
  diff = x^(m + 1)/(m + 1);
  tab = Table[diff = D[diff, x];
    (-1)^(j + n)*T^j*(I/(2*Pi*n))^(j + 1)*
    ((diff /. x -> T/2) - (diff /. x -> -(T/2))),
    {j, 0, m - 1}];
  Apply[Plus,tab]
]

```

Table 4: Fastest *Maple* and *Mathematica* codes for the Fourier coefficients of $f(t) = t^m$

References

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