Self-energy corrections in heavy muonic atoms

K. T. Cheng*
Department of Physics, Notre Dame University, Notre Dame, Indiana 46556

W.-D. Sepp
Gesellschaft für Schwerionenforschung Darmstadt und Gesamthochschule, Kassel, Germany

W. R. Johnson
Department of Physics, Notre Dame University, Notre Dame, Indiana 46556

B. Fricke
Fachbereich Physik, Gesamthochschule, Kassel, Germany

(Received 4 October 1977)

Self-energy corrections for $1s_{1/2}$ levels of heavy muonic atoms are calculated to all orders in the external field using numerical techniques to evaluate the bound-muon propagator. The resulting values of the self-energy are about 10% larger than previous estimates.

I. INTRODUCTION

Progress has been made in recent years in calculating radiative corrections of order α to the binding of K-shell electrons in heavy atoms.[1-3] The most difficult aspect of such calculations is the accurate determination of the electron self-energy (represented by the Feynman diagram of Fig. 1) in the strong field of a nucleus of high charge Z. In recent calculations, expansion in powers of the external field[4] (which shows no signs of convergence for a Coulomb field with $Z \geq 10$) is avoided by employing an expansion based on the known Coulomb Green's function[3]; or alternatively, by a direct numerical evaluation of the electron propagator.[1,2] The results of these recent electron self-energy calculations, combined with an evaluation of the vacuum polarization and the Breit interaction, have been used as corrections to Dirac-Hartree-Fock many-electron calculations to bring theoretical inner-shell binding energies into agreement with experiment to a level of ±10 eV.[5]

Factors influencing the binding in heavy muonic atoms are of course quite different from those occurring in the electronic case. Because of the relatively large muonic mass, nuclear finite size plays the dominant role in determining muonic energy levels. Vacuum polarization and polarization of the nucleus by the muon, together with electronic screening, are other factors important in determining muonic binding energy. The major uncertainty in theoretical calculations of muonic binding is the nuclear polarization correction.[6] Muonic self-energy is only a small correction even for $1s_{1/2}$ states; however, in view of the high precision of x-ray energy measurements, it is necessary to have precise values of the muonic self-energy. (See also Ref. 7.)

In Sec. II of the present paper we discuss previous work on the muonic Lamb shift, then in Sec. III, we discuss our present calculations, and in Sec. IV we describe our results.

II. PREVIOUS MUONIC LAMB-SHIFT CALCULATIONS

Only the lowest-order terms in a field-strength expansion[4,8] have been retained in older calculations of the muonic Lamb shift.[9-11] The resulting energy shift is given by

$$\Delta E_{\mathrm{LS}} = \frac{\alpha}{3\pi m^2}\langle \nabla^2 V\rangle \left(\ln\frac{m}{2\Delta\epsilon} + \frac{11}{24} + \frac{3}{8}\delta_{l0} - \frac{1}{5}\right) + \frac{\alpha}{8\pi m^2}\left\langle \frac{2}{r}\,\frac{dV}{dr}\,\vec{\sigma}\cdot\vec{\mathrm{L}}\right\rangle, \tag{1}$$

where V is the muonic potential energy, m is the muonic mass, and α is the fine structure constant. The quantity $\Delta\epsilon$ occurring in Eq. (1) is the log-average excitation energy defined by the Bethe sum.[9] The term $-\frac{1}{5}$ arises from the muon vacuum polarization, whereas the remaining terms come from the self-energy. There are two sources of uncertainty associated with Eq. (1):

(i) The log-average excitation energy $\Delta\epsilon$ is not always given with high accuracy. Barrett *et al.*[9] determine $\ln(m/2\Delta\epsilon)$ to about 25%. Bethe and Negele[10] determine bounds on $\Delta\epsilon$ which reduce the uncertainty in $\ln(m/2\Delta\epsilon)$ to about 10%. Such bounds

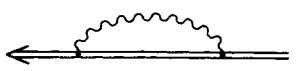

FIG. 1. Feynman diagram representing the muonic self-energy. The double line indicates that the muon is propagating in the static field of a nucleus of charge Z.

are also utilized by Barrett[11] in his later tabulations of ΔE_{LS}. The most recent determination of $\Delta\epsilon$ is that of Klarsfeld[12] who evaluates the Bethe sum numerically. For the $1s_{1/2}$ state of muonic ^{208}Pb Klarsfeld's value of ΔE_{LS} is about 2% higher than Barrett's value.

(ii) Contributions from terms of second and higher order in the external field have been neglected in Eq. (1). As has been pointed out by Barrett *et al.*,[9] the external-field expansion is expected to converge for high Z in muonic atoms in contrast to the situation for the electronic Lamb shift. The largest term of second order has been estimated and found to contribute about 15% to the Lamb shift in heavy muonic atoms. This value can be taken as a generous allowance for all of the neglected higher-order terms.

Considering both sources of uncertainty (i) and (ii) leads to an estimated error of about 20% in the existing Lamb-shift calculations for heavy muonic atoms. As we shall see in Sec. IV, the existing calculations do, in fact, lie below those of the present calculations by about 10%.

III. DESCRIPTION OF THE NUMERICAL SELF-ENERGY CALCULATION

The present calculation is a numerical evaluation of the renormalized self-energy following the method devised by Brown, Langer, and Schaefer.[13] This particular numerical procedure has been applied to heavy atoms ($Z=70$–90) by Desiderio and Johnson[2] and to superheavy atoms ($Z=90$–160) by Cheng and Johnson.[1] For an electron in a nuclear Coulomb field the present method[1] gives self-energy values in close agreement with those determined by Mohr,[3] who bases his work on the known Coulomb Green's function.

The advantage of the present techniques in the muonic case is that we are not restricted to a Coulomb field but may consider other potentials as well; thus, the Coulomb singularity in the self-energy,[3] which occurs at $Z=137$ is avoided by including nuclear finite size in the electron potential energy.[1] In the muonic calculation, where nuclear radii and muonic radii are comparable, it is necessary to allow for nuclear finite size in the interaction potential; the Brown-Langer-Schaefer method provides the appropriate tool.

A detailed description of the method together with a discussion of the numerical problems encountered in its application is given in Ref. 14. We just mention here that after renormalization the Feynman diagram of Fig. 1 reduces to three terms; $\Delta E_{SE}=\Delta E^{(1)}+\Delta E^{(2)}+i\pi R_0$. The "main term," $\Delta E^{(1)}$, involves a sum over photon partial waves l and an integration over photon frequency ω. Since both the infinite l sum and the infinite ω integration are slowly convergent, care must be taken to estimate remainders after truncation accurately. The term $\Delta E^{(2)}$, which arises after renormalization, and the residue $i\pi R_0$, which occurs because the ω integration is rotated to the imaginary axis, are both simple quadratures which are calculated with high accuracy. As in the electronic case, there is a cancellation between the three terms; in the present case this amounts to a reduction in size of the sum to about 10% of the individual terms. This cancellation becomes so severe for light muonic atoms that the present numerical procedure is impractical. Similar numerical cancellations prevent us from giving accurate values for the Lamb shift of states with higher prinicpal quantum numbers.

The primary source of numerical error is our estimate of remainders after truncation of $\Delta E^{(1)}$. With the present techniques these estimates lead to an error of about 5% in the determination of ΔE_{SE} for $1s_{1/2}$ states of muonic atoms.

IV. RESULTS AND DISCUSSION

As pointed out in Sec. III the Brown-Langer-Schaefer method works with sufficient accuracy only for the $1s_{1/2}$ state in very heavy muonic atoms ($Z\geq 70$). We therefore chose five nuclei equally spaced in Z from $Z=74$ to $Z=92$, namely, ^{184}W, ^{194}Pt, ^{208}Pb, ^{222}Rn, and ^{238}U. In order to see a possible isotope effect on the self-energy level shift, we have calculated the level shift for ^{206}Pb also. From an experimental point of view, the muon binding energies of these two Pb isotopes are also the best known ones in the Z region considered. The nuclear charge distribution employed is always that of a Fermi distribution

$$\rho(r)=\rho_0\{1+\exp[(4\ln 3)(r-c)/t]\}^{-1}$$

with nuclear radius c and skin thickness t. The values of c and t were taken from Table III of the compilation of Engfer *et al.*[15] This choice was motivated by the consideration that the muon bound-state wave functions and propagators should be reproduced best by nuclear charge distributions obtained from experimental muonic transition energies. Since all nuclei considered by us except the Pb isotopes are deformed, we have had to construct the corresponding spherical charge distributions. This was done by fitting the radii c of all measured spherical isotopes from $Z=79$ to $Z=83$ by

$$c=r_0A^{1/3},\quad r_0=1.1243\ \text{fm}$$

and then extrapolating this function to the deformed nuclei. In Table I we list the nuclear parameters

TABLE I. Values of the nuclear parameters (fm) and the corresponding uncorrected Dirac $1s_{1/2}$ binding energies (keV).

Nucleus	c	t	$E(1s_{1/2})$	Remarks
^{182}W	6.4038	2.197	9186.239	Barrett's parameters [a]
^{184}W	6.3947	2.3	9168.838	Extrapolation of Engfer's parameters [b]
^{194}Pt	6.5085	2:3	9857.212	Extrapolation of Engfer's parameters [b]
^{206}Pb	6.6302	2.3	10538.801	Engfer's parameters [b]
^{208}Pb	6.6477	2.3	10526.017	Engfer's parameters [b]
^{222}Rn	6.8076	2.3	11173.278	Extrapolation of Engfer's parameters [b]
^{238}U	6.9674	2.3	12197.340	Extrapolation of Engfer's parameters [b]
^{238}U	7.0028	2.637	12064.217	Barrett's parameters [a]

[a] Reference 11.
[b] Reference 15.

used together with the corresponding binding energies of the muon in the $1s_{1/2}$ state; we also give the nuclear parameters used in Barrett's calculation of the self-energy level shift[11] for later comparisons. The results of our calculations on the $1s_{1/2}$ self-energy level shifts for heavy muonic atoms are presented in Table II. For ease of comparison with Ref. 1 we list values of the contributing terms, $\Delta E^{(1)}$, $\Delta E^{(2)}$, and $i\pi R_0$ in muonic Rydbergs ($1\,\mathrm{Ry}_\mu = m_\mu/m_e\,\mathrm{Ry} = 2.813$ keV). The terms $i\pi R_0$ and $\Delta E^{(1)}$ are reduced to about half of their values in the electronic case, whereas $\Delta E^{(2)}$ is more than twice as large as in the electronic case. Since all three terms nearly cancel, the total shift is about 10% of the contributing terms and much smaller than the corresponding electronic values. Of course, this reduction was expected because of the finite field strength argument of Ref. 9.

The values of ΔE_{SE} in keV are given in column 6 of Table II. In column 7 we give the small contribution of vacuum polarization by muonic pairs deduced from Eq. (1). Finally, in column 8 of Table II we list the resulting value of the muonic Lamb-shift in keV.

For comparison we include values calculated using the same nuclear parameters as used in Barrett's Table I (Ref. 11) for two nuclei $^{182}W^{74}$ and $^{238}U^{92}$. For both nuclei we find an increase in the level shift of about 10% as compared with Barrett's first-order calculation. Adding in Barrett's estimated second-order shift reduces the discrepancy further. Comparing our results for two nuclei (^{206}Pb, ^{208}Pb) with different charge distributions but the same total charge Z, we find a completely negligible isotope correction to the Lamb shift. The insensitivity of ΔE_{LS} to nuclear parameters is further illustrated by comparing the first and second rows of Table II in which ^{182}W computed using Barrett's parameters is compared with ^{184}W computed using Engfer's parameters, or by comparing the last two rows of Table II in which corresponding values are given for ^{238}U.

On the basis of the close agreement between our present values and Barrett's values of $\Delta E_{\mathrm{LS}}(1) + \Delta E_{\mathrm{LS}}(2)$ we see that the finite-field-strength ar-

TABLE II. Self-energy level shifts for $1s_{1/2}$ states in heavy muonic atoms.

Nucleus	$\Delta E^{(1)}$ (Ry_μ)	$\Delta E^{(2)}$ (Ry_μ)	$i\pi R_0$ (Ry_μ)	ΔE_{SE} (Ry_μ)	ΔE_{SE} (keV)	ΔE_{VP} (keV)	ΔE_{LS} (keV)	Ref. 11
^{182}W	−14.59	−10.89	26.60	1.12	3.15 [a]	−0.219	2.93 [a]	2.66 ± 0.50
^{184}W	−14.56	−10.89	26.56	1.11	3.12	−0.218	2.90	
^{194}Pt	−14.84	−11.05	27.08	1.19	3.35	−0.233	3.12	
^{206}Pb	−15.10	−11.19	27.55	1.26	3.54	−0.247	3.29	
^{208}Pb	−15.07	−11.19	27.51	1.25	3.52	−0.246	3.27	
^{222}Rn	−15.25	−11.30	27.86	1.31	3.69	−0.256	3.43	
^{238}U	−15.61	−11.43	28.45	1.41	3.97	−0.275	3.70	
^{238}U	−15.39	−11.43	28.19	1.37	3.85 [a]	−0.264	3.59 [a]	3.17 ± 0.52

[a] Calculated with Barrett's nuclear parameters, Ref. 11.

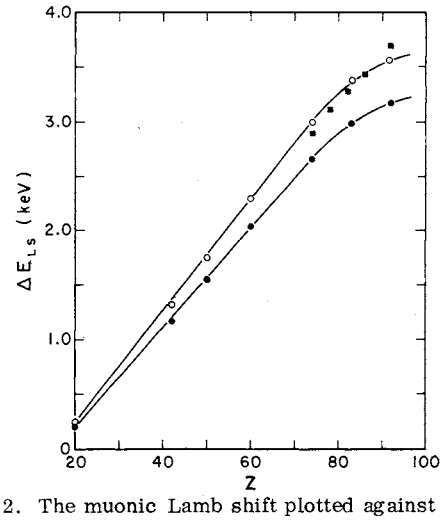

FIG. 2. The muonic Lamb shift plotted against nuclear charge for $1s_{1/2}$ states. The symbol ■ designates results of the present calculation, ● designates Barrett's first-order calculation $\Delta E_{LS}(1)$, ○ designates Barrett's second-order calculation $\Delta E_{LS}(1)+\Delta E_{LS}(2)$.

guments of Ref. 9 are adequate for muons in $1s_{1/2}$ states; moreover, we see that in the range considered here $\Delta E_{LS}(2)$ accounts very well for the higher-order αZ corrections to Eq. (1). We expect Barrett's values of $\Delta E_{LS}(1)+\Delta E_{LS}(2)$ to be increasingly accurate for lower values of Z, so that the present calculations for $Z>74$ supplemented by Barrett's values for lower Z provide ΔE_{LS} for $1s_{1/2}$ states throughout the entire range of atoms accurate to about 5%. Results of our calculation are also shown in Fig. 2 where the self-energy shifts in keV are plotted against nuclear charge Z.

Experiments on muonic atoms are commonly analyzed in a model-independent manner.[16] In these analyses, some model-independent generalized moments of the nuclear-charge distribution (e.g., the Barrett moment $\langle r^k e^{-\alpha r}\rangle$) are deduced from the measured muonic x-ray transition energies. In the standard compilation of Engfer *et al.*,[15] only the first-order Lamb-shift corrections of Barrett[11] are taken into account. Since the higher-order corrections to the self-energy level shift are somewhat larger than present day experimental errors in the muonic x-ray transition energies, these higher-order corrections should be properly taken into account. Using Eq. (17) of Ref. 15, we find a decrease of the Barrett moment $\langle r^k e^{-\alpha r}\rangle$ for the $1s_{1/2}$ state in very heavy muonic atoms of the order of 0.0005 fm due to the higher-order corrections of the self-energy level shift.

As mentioned in the introduction, the main source of uncertainties in the analysis of measured muonic x-ray transition energies is the theoretical uncertainty of the nuclear-polarization correction. In a recent paper, Enteneuer *et al.*[7] have tried to determine experimentally the nuclear-polarization correction by a consistency analysis of measured muonic transition energies and high-energy electron-scattering cross sections. Since all other corrections to the muonic transition energies except the nuclear-polarization corrections have to be taken into account as given by theory, the higher-order corrections to the muonic self-energy level shift would increase the measured nuclear-polarization correction by the same amount.

ACKNOWLEDGMENT

This research was supported in part by NSF Grant No. Phy. 77-15141. One of us (W.D.S.) would like to thank the Physics Department of the University of Notre Dame for its hospitality during his stay.

*Present address: Argonne National Laboratory, Argonne, Ill. 60439.

[1]K. T. Cheng and W. R. Johnson, Phys. Rev. A 14, 1943 (1976).

[2]A. M. Desiderio and W. R. Johnson, Phys. Rev. A 3, 1267 (1971).

[3]P. J. Mohr, Phys. Rev. Lett. 34, 1050 (1975); Ann. Phys. (N.Y.) 88, 26, 52 (1974).

[4]G. W. Erickson and D. R. Yennie, Ann. Phys. (N.Y.) 35, 271 (1965); 35, 447 (1965).

[5]M. S. Freedman, F. T. Porter, and J. B. Mann, Phys. Rev. Lett. 28, 711 (1972); B. Fricke, J. P. Desclaux, and J. T. Waber, *ibid.* 28, 714 (1972); K.-N. Huang, M. Aoyagi, M. H. Chen, B. Crasemann, and H. Mark, At. Data Nucl. Data Tables 18, 243 (1976).

[6]R. Engfer (private communication); L. Schellenberg (private communication); J. L. Friar and J. W. Negele, Adv. Nucl. Phys. 8, 219 (1975).

[7]H. Enteneuer, J. Friedrich, and N. Voegler, Z. Physik A 280, 165 (1977).

[8]R. Karplus, A. Klein, and J. Schwinger, Phys. Rev. 86, 288 (1952).

[9]R. C. Barrett, S. J. Brodsky, G. W. Erickson, and M. H. Goldhaber, Phys. Rev. 166, 1589 (1968).

[10]H. A. Bethe and J. W. Negele, Nucl. Phys. A 117, 575 (1968).

[11]R. C. Barrett, Phys. Lett. B 28, 93 (1968).

[12]S. Klarsfeld, Nucl. Phys. A 285, 493 (1977).

[13]G. E. Brown, J. S. Langer, and G. W. Schaefer, Proc. R. Soc. (Lond.) A 251, 92 (1959).

[14]A. M. Desiderio, Ph.D. thesis (University of Notre Dame, 1970).

[15]R. Engfer, H. Schneuwly, J. L. Vuillenmier, H. K. Walter, and A. Zehnder, At. Data Nucl. Data Tables 14, 509 (1974).

[16]K. W. Ford and G. G. Wills, Phys. Rev. 185, 1429 (1969); C. S. Wu and L. Wilets, Annu. Rev. Nucl. Sci. 19, 527 (1969); R. C. Barrett, Phys. Lett. B 33, 388 (1970); K. W. Ford and G. A. Rinker Jr., Phys. Rev. C 7, 1206 (1973) and Ref. 15.