A Phenomenological Calculation of Vacuum Fluctuation in Electronic and Muonic Atoms (*)

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1. – Introduction.

The experimental discovery of the Lamb shift in 1947 by LAMB and RETHERFORD (1) gave rise to a quick development of quantum electrodynamics. Vacuum polarization and fluctuation were calculated by different methods up to very high order in the usual expansion parameter $Z\alpha$ for single-electron atoms where this effect was measured with very high accuracy (2). The nearly complete agreement between experiment and theory in the region of very small Z was a large confirmation of the methods used in quantum electrodynamics. For high Z the experimental proof of these effects is much more difficult because no single-electron atoms are available and in normal atoms there are several other effects of the same order of magnitude arising from the many-body problem of Z electrons. The effect of vacuum polarization alone can be measured with very high accuracy in muonic atoms up to uranium. It was shown recently that the calculations agree very well with experiment (3), provided that the vacuum fluctuation is very small in comparison to the vacuum polarization. This situation is opposite for electronic atoms; for example, the effect of vacuum polarization for the 1*s* level in Hg is of the order of -3 Ryd, whereas the effect of vacuum fluctuation is of the order of $+15$ Ryd as given by DESIDERIO *et al.* (4). BROWN *et al.* (5) in an older calculation gave $+41$ Ryd. The cause of this discrepancy is not known.

These computations are very complicated and are given for the 1*s* levels only. From this point of view it is worth-while to present a simple phenomenological calculation of the effect of vacuum fluctuation, especially for uses in self-consistent calculations as well as for the study of this effect in superheavy elements.

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(1) W. LAMB and R. RETHERFORD: *Phys. Rev.*, **72**, 241 (1947); **79**, 549 (1950); **86**, 1014 (1952).

(2) For all theoretical and experimental references see: T. APPELQUIST and S. J. BRODSKY: *Phys. Rev. Lett.*, **24**, 562 (1970); and as an older review A. PETERMANN: *Fortschr. Phys.*, **6**, 505 (1958).

(3) G. BACKENSTOSS, S. CHARALAMBUS, H. DANIEL, CH. VON DER MALSBURG, G. POELZ, H. P. POVEL, H. SCHMITT and L. TAUSCHER: *Phys. Lett.*, **31** B, 233 (1970) and references herein.

(4) A. M. DESIDERIO and W. R. JOHNSON: to be published in *Phys. Rev.* (1971).

(5) G. E. BROWN and D. F. MAYERS: *Proc. Roy. Soc.*, A **251**, 105 (1959).

2. – The calculation of vacuum fluctuation.

The effect of vacuum fluctuation was early described as a « zitterbewegung » of the electrons with an amplitude δr, which comes from the interaction of the electrons with the zero-point electromagnetic field. BJORKEN and DRELL (6) give this amplitude δr for small Z as follows:

$$\langle(\delta r)^2\rangle = \frac{2\alpha}{\pi}\left(\frac{1}{m}\right)^2 \ln\frac{1}{Z\alpha}.$$

δr as a function of Z is given in Fig. 1 as a solid line. This explanation of the effect of vacuum fluctuation was used as a simplified illustration only. But if one takes this explanation somewhat more seriously, one gets quite surprising results. A necessary con-

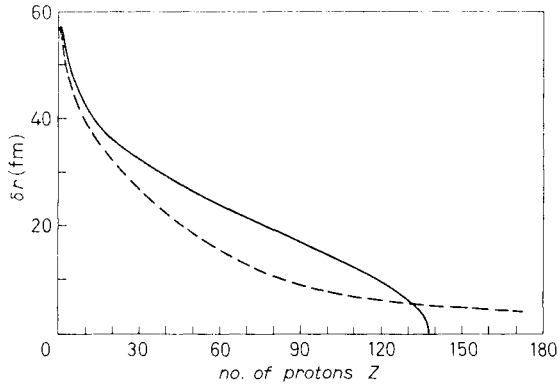

Fig. 1. – The « zitterbewegung » amplitude δr as function of Z. The solid line represents δr according to BJORKEN and DRELL (6). The values given by the dashed line lead to the same results for the vacuum fluctuation as the exact calculations of DESIDERIO *et al.* (4). For $Z > 90$ the dashed line is continued for large Z.

dition to this is a computer program which integrates the Dirac equation numerically (7) with any given potential, so that the effect of the extended nucleus and, if needed, the potential from the other electrons in the atom can be included. To introduce now the « zitterbewegung » we folded the potential with a Gaussian $f(|\boldsymbol{r}'|) = \exp[-\boldsymbol{r}'^2/\langle(\delta r)^2\rangle]$ having a half-width δr, so that one gets a new potential by calculating

$$V_{\text{new}}(\boldsymbol{r}) = \frac{\int \mathrm{d}\boldsymbol{r}'\, V_{\text{old}}(\boldsymbol{r}+\boldsymbol{r}')f(|\boldsymbol{r}'|)}{\int \mathrm{d}\boldsymbol{r}' f(|\boldsymbol{r}'|)}.$$

In doing this, each point in the electron wave function is distributed over a range δr, so that the potential really acting is modified. In an atom with a single electron this leads only to changes in the vicinity of the nucleus where the potential deviates from the $1/r$ behavior. This is an expression of the fact that the vacuum fluctuation can be written as an effective potential acting only at the point of the nucleus.

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(6) J. D. BJORKEN and S. D. DRELL: *Relativistic Quantum Mechanics*.

(7) We would like to thank Prof. J. T. WABER for communicating this computer program.

In Table I the « exact » values for fluctuation [2] are compared with the values of our simple folding method. The agreement is astonishingly good in the region of small Z. For high Z our values lie about a factor two too high above the calculation of DESIDERIO *et al.* [4] but very near to the value of BROWN *et al.* [5].

This result shows that the old quite simple explanation of fluctuation leads to reasonable results and is a practical method for calculation of the shifts of the s levels, where the effect of fluctuation is most strongly (for example, in H the $2s$ shift is 1070 MHz, whereas the $2p_{\frac{1}{2}}$ shift is only — 12 MHz). Because of l-dependent influences, the simple folding method is not useful for levels with $l \neq 0$. The folded new potential dif-

TABLE I. — *The vacuum fluctuation according to the simple folding method in comparison to the observed, respectively, calculated values for various s levels in electronic atoms.*

Element	Level	Vacuum fluctuation	
		exact calculation (*)	present calculation
H	$1s$	8.55 GHz	8.59 GHz
	$2s$	1.07 GHz	1.08 GHz
	$3s$	0.31 GHz	0.31 GHz
He^{+}	$2s$	14.27 GHz	14.45 GHz
	$3s$	4.25 GHz	4.34 GHz
	$4s$	1.79 GHz	1.81 GHz
Li^{2+}	$2s$	64.0 GHz	65.0 GHz
C^{5+}	$2s$	770.0 GHz	820.0 GHz
$Z = 70$	$1s$	9.1 Ryd	22.2 Ryd
	$2s$		3.2 Ryd
$Z = 80$	$1s$	15.0 Ryd	39.0 Ryd
	$2s$		6.4 Ryd
$Z = 90$	$1s$	23.5 Ryd	66.8 Ryd
	$2s$		11.2 Ryd

(*) For H, He^{+} and Li^{2+} see ref. [2] and for C^{5+} see ref. [8]. For $Z = 70$ and 90 see ref. [1].

TABLE II. — *The vacuum fluctuation contribution in muonic atoms from calculations of* BARRETT [9] *in comparison to the present folding method.*

Element	Level	Vacuum fluctuation	
		calculation of BARRETT [8]	present calculation
Ca	$1s$	0.21 keV	0.20 keV
	$2s$	0.05 keV	0.03 keV
U	$1s$	3.17 keV	0.78 keV
	$2s$	0.74 keV	0.42 keV

[8] M. LEVENTHAL and D. E. MURNICK: *Phys. Rev. Lett.*, **25**, 1237 (1970).
[9] R. C. BARRETT: *Phys. Lett.*, **28** B, 93 (1968).

fers very much from the old potential especially at the origin, so that the effect belonging to this change cannot be calculated by ordinary perturbation theory. A totally new integration with the new potential has to be done.

In Table II the calculation of vacuum fluctuation with this method in muonic atoms is compared with the calculations of BARRETT [9]. We get results of the same order of magnitude that they get.

3. – Extrapolation to superheavy elements.

The values for the fluctuation of the $1s$ levels in the elements $Z = 70 \div 90$ calculated by DESIDERIO *et al.* [4] gives us the possibility to determine a curve with new δr values for high-Z elements, which lead to the same results. This curve is given in Fig. 1 as a dashed line. To get an idea of how large the effect of fluctuation will be for very high Z, we continued this dashed line and calculated first-approximation values for vacuum fluctuation for elements up to $Z = 180$, where this effect becomes large. The results are shown in Fig. 2.

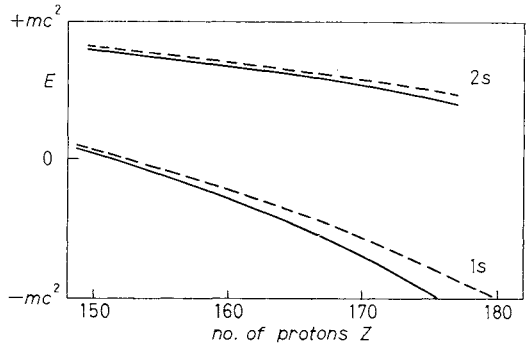

Fig. 2. – The solid lines give the energy eigenvalues for the $1s$ and $2s$ levels from self-consistent calculations for elements between $Z = 150$ and $Z = 180$. The dashed lines are the energy eigenvalues including the vacuum fluctuation contribution.

Nobody knows up to now how large the effects of quantum electrodynamics will be for elements with $Z > 137$, but one expects that the energy eigenvalues for the inner electrons do not reach the continuum of electrons with negative energy [10]. Therefore, this simple folding method might be quite a good description of the behavior of inner electrons in superheavy elements. The decision how large the quantum electrodynamical effects really are has to be given by an exact calculation. This theory of quantum electrodynamics of strong fields is in progress by REINHARDT *et al.* [11]. The exact calculation will be most complicated, so that in practical cases the calculation given in this letter will lead to the right order of magnitude in this simple way which will be sufficient in many cases.

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I would like to thank Profs. J. T. WABER and W. GREINER for their great interest and several discussions.

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[10] W. GREINER: Panel Discussion (Montreal, 1969).

[11] P. G. REINHARDT, H. ARENHÖVEL and W. GREINER: *Nucl. Phys.*, **166** A, 173 (1971).