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CHAPTER 2

Applications and Modelling in Mathematics Teaching – A Review of Arguments and Instructional Aspects

W Blum
University of Kassel, FR Germany

SUMMARY

The aim of this paper is a comprehensive presentation of some important *basic* and *general aspects* of the topic applications and modelling, with emphasis on the secondary school level. Owing to the review character of this paper, some overlap with the survey paper Blum and Niss (1989) for ICME-6 in Budapest is inevitable. The paper will consist of three parts. In *part 1*, I shall try to clarify some basic concepts and remind the reader of a few application and modelling examples suitable for teaching. In *part 2*, I shall formulate some general aims for mathematics instruction and, on that basis, summarise the most important arguments for and against applications and modelling in mathematics teaching. Finally, in *part 3*, I shall discuss some relevant instructional aspects resulting from the considerations in part 2.

1. CONCEPTS AND EXAMPLES

1.1 Basic concepts

I would like to commence by giving pragmatic working definitions for some basic concepts and notions such as *modelling* or *application* which sometimes – even in this volume – are used in different senses. Here I follow Blum and Niss (1989, ch 1).

By an *applied (mathematical) problem* or a *real problem situation* I mean an open situation which belongs to the real world and for the solution of which some mathematics might be helpful. By *real world* I mean the rest of the world outside mathematics, that is school or university subjects or disciplines different from mathematics, or everyday life and the world around us. The *applied problem solving process* consists of the entire process of dealing with an applied problem in attempting to solve it. In the following, I would like briefly to recollect some essential steps of this process (see, among many others, Steiner 1976, Pollak 1979, Blum 1985 or Niss 1987). As an illustration, I shall use the case of *income taxes.*

The *starting point* is a real problem situation. In our example, taxes are to be imposed on income, as 'justly' as possible. There are many possibilities for doing this, dependent on different interests and value judgements. In a first step, the situation has to be *simplified, idealised, structured* and *made more precise* by the problem solver, especially by formulating appropriate conditions and assumptions. In our example, such conditions could be "If person A earns more than person B then A ought to pay at least the same percentage of his income for taxes as B" or – perhaps more basically – "The tax should always be less than the income". This leads to a *real model* (here an economic model) of the original situation. The real model is not – as is sometimes misunderstood – merely a simplified but true image of some part of an objective, pre-existing reality. Rather, the first step of the process also structures and *creates* a piece of reality, dependent on intentions and interests of the problem solver.

In a second step, the real model has to be *mathematised*, that is its data, concepts, relations, conditions and assumptions are to be translated into mathematics. Thus, a *mathematical model* of the original situation results. In our example, the model consists of a real function $s: x \mapsto s(x)$, the income tax function (where $x \geq 0$ is the income and $s(x)$ is the corresponding tax), together with certain concepts like the marginal tax rate

$$s'(x) = \lim_{h \to 0} \frac{s(x+h) - s(x)}{h}$$

and certain conditions for the function s such as

$$\frac{s(x_1)}{x_1} \leqslant \frac{s(x_2)}{x_2} \text{ for } x_1 < x_2.$$

According to Blechman, Myškis and Panovko (1984, p148), "the ability to choose an appropriate mathematical model lies on the borderline between science and art". Again, the resulting model depends on the intentions of the modeller, whether it's a question of description, of prediction or of prescription (to follow a distinction made by Davis and Hersh (1986), ch 3; see also Davis in this volume).

A remark concerning terminology. Sometimes the notions of mathematising, modelling and problem solving are used synonymously. In this paper *mathematisation* is the process of going from the real model into mathematics, that is the second step above, whereas the process of *modelling* or *model building* comprises the first as well as the second step, and problem solving means the entire process which is being described here, of which modelling is only a part.

This process continues by a third step, that is *choice* of suitable mathematical *methods* and *work within mathematics*, through which certain *mathematical results* are obtained. In our example, these results are concrete income tax functions. In Germany, the actual (1990/91) income tax function is a stepwise polynomial function with degrees 0, 1 and 2. (The function which was effective until 1989 was even nicer for mathematics teaching because it contained, in addition, a polynomial of degree 4).

In a fourth step, these results have to be retranslated into the real world, that is to be *interpreted* in relation to the original situation or to the real model. In doing so the problem solver also *validates* the mathematical model. Here *discrepancies* of various kinds may occur which may lead to a modification of the model or to its replacement by a new one. In Germany, this happens nearly every two years with our income tax, so that we get a new tax function every two years. Thus, the problem solving process may require "going round the loop" *several* times (not necessarily, of course, in the order "step 1, step 2, ..., step 1, step 2" as described above).

Besides such complex problem-solving processes there are restricted links between mathematics and reality: a *direct application* of already developed standard mathematical models to real situations with a mathematical content, for instance the investigation of the present German income tax function by means of differential calculus, or a *dressing up* of purely mathematical problems in the words of another discipline or of everyday life, sometimes in a *whimsical* way (Pollak 1979), such as the investigation of the income tax function of Transsylvania.

I use the term *application* of mathematics to denote all the above-mentioned ways of connecting the real world with mathematics. In this sense real problem situations can also be called *applications*. Eventually, all mathematical models can be seen as belonging to a field which may be called *applied mathematics*. As regards the question what applied mathematics can mean and what its main goals are, I refer to Blechman, Myškis and Panovko (1984) and to Davis in this volume.

1.2 Some examples

Up to now I have only referred to the case of income tax, which corresponds to the mathematical topics of arithmetic, functions and calculus. However, for nearly every topic in the mathematics curriculum, there is a wealth of application and modelling examples suitable for teaching. Especially in the last 15 years, a great many examples have been developed all over the world which can be used for many different instructional purposes, also for demonstrating the complex interrelationship between the real world and mathematics (see, for instance, numerous examples in the ICTMA-proceedings, Berry et al 1984, 1986, 1987 and Blum et al 1989; many references are given in Pollak 1979 and Bell 1983; see also the extensive bibliography of Kaiser et al 1982/1990). In the following table I shall recollect a few of these well-known examples, all suitable for the *school level*. With this, I intend to help the reader to relate the general considerations in the second and third part of this paper to concrete examples, and thus to give more meaning to these general statements.

Real-world examples	**Related mathematical topics**
various growth and decay processes: population growth, chemical reactions, spread of epidemics, warming/cooling, absorption, decay of beer froth and so on	functions, calculus, linear algebra
rainbow	geometry, calculus
genetics	arithmetic, probability, (linear) algebra
rates of interest	arithmetic, functions, calculus
price index	arithmetic, algebra, functions
*income tax	arithmetic, functions, calculus
*elections	arithmetic, geometry, stochastic
social classes	finite maths, probability, linear algebra
*traffic flow	finite maths, calculus, stochastics
biking	geometry, functions
football	geometry, arithmetic, stochastics
*shot putting	geometry, calculus

chairoplane	linear algebra, functions, calculus
gambling	stochastics, finite maths
painting	geometry, algebra
musical scales	arithmetic, algebra, functions

I look upon the examples marked with a * as particularly appropriate to showing essential features of the applied problem solving process, such as going round the loop several times or building different models for the same situation. I would like, however, to emphasise that *all* the examples just mentioned are suitable for going through this process with learners. For this is essentially a matter of looking at the examples and of preparing them, and not a question of the nature of the examples themselves.

2. ARGUMENTS

2.1 Aims for mathematics teaching

Now, after dealing with basic concepts and citing some examples in part 1, I shall consider in part 2 the *teaching* of mathematics (as a separate subject) and in particular the *role of applications and modelling* in mathematics instruction. It seems to be quite obvious, but it might be worth emphasising once more, that the role of applications and modelling can only be reflected by making explicit reference to some sort of a theoretical *basis*, to a *philosophy* of mathematics instruction. The most important constituent of such a basis is, in my view, the aspired general *aims* for mathematics teaching. Other relevant constituents are basic didactical principles or empirical findings about the learning and teaching of mathematics. Here I can only deal with the question of aims. There is plenty of literature discussing aims and goals for mathematics instruction, and there are countless catalogues of aims. Nevertheless, I will add another catalogue because I regard mine to be a useful basis which I will refer to several times in the following sections.

The discussion of aims for mathematics instruction must be embedded into a more general context with social, political or pedagogical issues, as done for example in D'Ambrosio (1979), Christiansen, Howson and Otte (1986, ch 1/2) and particularly Niss (1981). Roughly speaking, general aims for mathematics teaching are related to higher order *educational* goals, dependent on a certain '*picture of man*'. Such goals are in particular the following. Students are to be educated to be responsible and intelligent citizens, are to be trained towards professionalism at work, they are to master everyday life and to acquire appropriate intellectual attitudes. For all these educational goals *mathematics* may play an important role. In this respect let me formulate the following general aims for mathematics teaching.

1. *Pragmatic aims* : Mathematics teaching is intended to help students to describe relevant aspects of extra-mathematical areas and situations, to understand these better and to cope with them better. Such situations may originate
   (a) from present or future daily life and the world around us, or from other subjects in general education at school,
   (b) from present or future fields of study or profession in vocational education at school or at university.

2. *Formative aims* : By being concerned with mathematics, students should
   (a) acquire important general qualifications, for instance the ability to argue, to solve problems or to translate between reality and mathematics, or attitudes such as openness towards problem situations and willingness for intellectual efforts,
   (b) obtain pleasure and enjoy their activities.

3. *Cultural aims* : Students should be taught mathematical topics
   (a) as a source for philosophical and epistemological reflection, including individual and human self-reflection,
   (b) to generate a comprehensive and balanced picture of mathematics as a science and as a part of human history and culture, including a critical appreciation of actual uses and misuses of mathematics in society,
   (c) to produce knowledge, skills and abilities concerning special mathematical themes.

   By all this students should also acquire meta-knowledge of mathematics.

These aims correspond to different *aspects of human nature*: man as
   (1) an observing, producing, economic being,
   (2) a reasoning, communicating, social being,
   (3) a reflecting, creating, cultural being.

The various aims must be assessed differently according to the *educational history* and context:

A. Mathematics in *schools* offering *general* education, at the lower or at the upper secondary level.
B. Mathematics in *vocational* education at *schools* and colleges, or as a *service subject* in *university* courses for future scientists, engineers, economists and so on.
C. Mathematics in *university* courses for future *mathematicians* or mathematics teachers.

The result of this assessment is shown in the following matrix. The number of * symbols corresponds to the importance I have assigned to the aims.

educational histories		pragmatic aims a b	formative aims a b	cultural aims a b c
maths in general education	lower secondary	***	**	*
	upper secondary	** *	*** **	***
maths in vocational education or as a service subject		* ***	**	**
maths for mathematicians in university		*	**	*** ***

Altogether it may be said that for all levels *all aims* are relevant, though with different emphases in each case.

On that basis, I shall now try to identify and to discuss the most important arguments for and against incorporating applications and modelling in mathematics instruction. I shall commence with counter-arguments.

2.2 Arguments against applications and modelling

Several investigations and inquiries in many different countries have shown that in everyday mathematics teaching applications and modelling very often do not play as important a role as theoretical analyses have assigned to them (see for example the review by Burkhardt 1983). This is not due to ill-will or incompetence of teachers but to certain severe *obstacles*. Let me briefly summarise some of these obstacles in the form of *counter-arguments* (see Blum and Niss 1989, ch 3).

C1 *Counter-arguments from the point of view of instruction.* There is *not enough time* to deal with applications and modelling in mathematics instruction. More fundamentally, applications and modelling *do not belong* to mathematics instruction at all. So, extra-mathematical problems should be treated in the teaching of other subjects, such as the example of elections in social studies, or shot putting in sports.

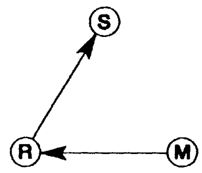
C2 *Counter-arguments from the learner's point of view.* Applications and in particular modelling make the mathematics lessons and examinations more complex, *more demanding* and less predictable for learners. Students have to learn, for example, not only mathematical concepts such as the derivative but also its real world interpretations as a growth rate, a tax rate and so on.

C3 *Counter-arguments from the teacher's point of view.* Applications and modelling make teaching *more demanding* because additional non-mathematical knowledge and qualifications are required. Applications and modelling work makes instruction *more open* and necessitates types of classroom interaction unusual for mathematics teachers - for instance, in the case of traffic flow, discussions about environmental problems with unpredictable outcome. All these might *undermine the teacher's expert authority.*

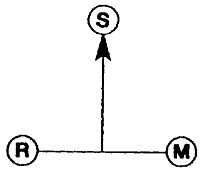
2.3 Arguments for applications and modelling

Referring to the aims identified in section 2.1, I shall now identify potential *roles* applications and modelling may play in the promotion of these aims. This will result in various *pro-arguments.* I shall illustrate the arguments by schematic diagrams in which *S*tudents, *R*eal World and *M*athematics are related to each other in a characteristic manner; an arrow → has the rough meaning of 'helps'. Of course, I do not want to quote these pro-arguments in order to convince the reader how important applications are. Fortunately, this is today, in the beginning of the nineties, no longer necessary, in contrast with the situation 15 or 20 years ago. Rather, I would like to systematise the various arguments for applications and modelling which have been invoked time and again during the history of mathematics education, and thus to lay a solid foundation for curricular and methodical conclusions.

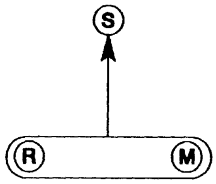
P1 *Pragmatic arguments* (Mathematics as an aid for specific real situations) : The ability of students to understand and to master extra-mathematical situations does not result automatically from learning pure mathematics but can only be achieved by dealing, in mathematics instruction, with such real situations. Students can, for instance, be helped to understand better public discussions about the limits to growth by treating ecological growth processes in the context of percentages, geometric sequences or exponential functions. In my opinion, such 'enlightenment' belongs genuinely to mathematics teaching, and not only to the teaching of other subjects.



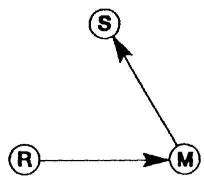
P2 *Formative arguments* (Real-world applications of mathematics as an aid for general qualifications). General competences and attitudes can also be developed by suitable applicational examples. In particular, the ability to translate between the real world and mathematics can only be advanced by means of examples where the entire problem solving process is covered, as was shown in the example of income taxes.



P3 *Cultural arguments* (Real-world applications as a source for reflection and as a component of an appropriate overall picture of mathematics). Reflections on a meta-level can also originate from linking mathematics with the real world. Dealing with the mathematics of perspective drawing, for example, can provoke discussion of the way that man perceives the world. More generally, students can see that - in a certain sense - "modelling is the essential feature of human intellectual behaviour" (D'Ambrosio 1989, p23). Furthermore, applications as well as modelling constitute an essential component of an adequate picture of mathematics. When treating, for instance, the topic of elections, we can illustrate where and how mathematics and mathematicians are needed, and also where they are not needed, by which means blind faith and trust in science can be reduced, and meta-knowledge of mathematics and its relations to applications generated.



P4 *Psychological arguments* (Real-world applications as an aid for the learning of mathematics). Mathematical contents can also be motivated or consolidated by suitable applied examples, and larger mathematical subject ranges may be structured by placing them in applicational contexts. Real world interpretations of mathematical topics can contribute towards better and deeper understanding and longer retention of these topics, and relations to reality may improve the students' attitude towards mathematics.



In addition, the inclusion of applications and modelling contributes towards giving more *meaning* to the learning and teaching of mathematics. This is perhaps the most important kind of argument, for (using a statement made by Thom in a slightly different context) "the real problem which confronts mathematics teaching is ... the problem of the development of meaning" (Thom 1973, p202).

I cannot consider the pros and cons in depth here. In any case, in the light of the arguments just mentioned, counter-argument C1 is simply *wrong*. In particular, time spent on applications and modelling is not only justified and necessary but also mathematically profitable according to the arguments just given. So there *has* to be time for applications and modelling, as a 'didactical axiom', if need be by reducing the mathematics programme.

In contrast, counter-arguments C2 and C3 remain largely *correct* – unfortunately! In particular, mathematics instruction does not become easier for students by the inclusion of applications and modelling – on the contrary. This is why applications sometimes fail totally to motivate students to do mathematics.

The pro-arguments have to be weighed according to the importance of the accompanying aims. The result is shown in the following matrix. Again, the number of * refers to the relevance of the arguments.

educational histories	pragmatic arguments	formative arguments	cultural arguments	psychological arguments
lower secondary school	***	**	*	**
upper secondary school	**	***	***	**
vocational education or service subject	***	**	*	**
mathematicians	*	**	***	**

Altogether it may be said that for all levels *all arguments* are of significance, though with different weights.

It is possible now to identify different *schools of thought* within the mathematics education community, corresponding to the aims placed in the foreground in each case and to the arguments given for the inclusion of applications and modelling in mathematics teaching; for details see Kaiser-Messmer (1986b vol 1 and in this volume).

All things considered, the diverse arguments demand *applications and modelling to become and remain essential parts of mathematics teaching* at all levels (whether students or teachers like that or not). Students should (see Niss 1989)

- *know* concrete standard models and applications of mathematics as well as essential steps of corresponding model building, interpreting and applying processes,
- be able to *perform* modelling themselves and critically *assess* models and modelling processes.

Or, in the words of Skovsmose (1989), students should, besides mathematical knowledge, acquire 'technological' knowledge (about models) as well as 'reflective' knowledge (meta-knowledge).

In order to achieve that, every effort should be made to *overcome the obstacles* to applications and modelling mentioned in the counter-arguments C2 and C3. To deal with that question seriously would necessitate several separate papers. Let me make one general remark only. Most important is an adequate pre-service and in-service *teacher education* in order to supply teachers with knowledge, abilities, experiences and, in particular, with attitudes to cope with the demands of teaching modelling and applications. Teachers must have achieved for *themselves* all that they (ought to) demand from their students.

So there is a consensus of opinion in the mathematics education community that applications and modelling should be an integral part of mathematics teaching. Now, what are consequences of such a decision? This question will be addressed in part 3 of this paper.

3. INSTRUCTIONAL ASPECTS

3.1 Approaches to organisation

The following demand is an obvious consequence of part 2.

Demand 1: Applications and modelling should be incorporated in mathematics instruction at all levels.

What is *actually* going on in mathematics education and in the classroom? In Blum and Niss (1989, ch 2) we stated the following.

Trend 1 (Increasing inclusion of applications and modelling). In the eighties, all over the world a *growing* number of mathematics programmes at different levels are *including* applications and modelling. There are, however, some severe *obstacles* to the inclusion of applications and modelling in everyday teaching practice. So, there is a large *distance* between the forefront of research, development and practice in applications and modelling, on the one hand, and applications and modelling in the *mainstream* of mathematics instruction on the other hand. Yet, this distance is in the process of being considerably *reduced.*

Now, in part 3, I shall discuss some curricular and methodical *consequences* of demand 1. That is, suppose we decide to include applications and modelling in mathematics teaching, how should this be done, what kinds of examples should be chosen, and how should they be taught? In the whole of part 3, I shall concentrate on mathematics at *schools*. Again, I have to restrict myself to making only rather *general* remarks. In this first section, I shall address briefly the question of how mathematics teaching should be *organised* if applications and modelling are to be incorporated; for a far more detailed discussion of important aspects and viewpoints for building application-oriented mathematics curricula see Usiskin (1989 and in this volume).

Niss has distinguished between different types of basic *approaches* to the organisation of mathematical and applicational curriculum components (for details see Blum and Niss 1989, p16): the *separation* approach, the *two-compartment* approach, the *islands* approach, the *mixing* approach, and *integrated* approaches. Which of these different approaches is to be chosen depends on the background philosophy, particularly on the aims for mathematics teaching (see section 2.1). The islands approach, for example, may impede the formation of modelling abilities or of a balanced picture of mathematics, and an integrated approach may render the promotion of cultural aims more difficult. For mathematics in *school*, I advocate a certain kind of a *mixing* approach which allows all the aims to advance and takes into account all the arguments for applications and modelling. Here, the mathematics programme should be organised in a consistent way, and many application and modelling examples should be incorporated. What kinds of examples are to be included I shall discuss next.

3.2 Kinds of examples

In my opinion, many local and some global application and modelling examples should be incorporated. By *local* examples I mean comparatively small-scale examples which correspond primarily to the pragmatic and psychological arguments, so that they are more suited to a description as well as a better understanding and mastering of special real situations, and to a motivation or an illustration of certain mathematical topics. Such examples are mostly given as *problems* or *exercises* as they can be found in almost every textbook, for instance parabolic aerials, orbits of comets or planets, or whispering galleries, as applications of conic sections. By *global* examples I mean larger examples which correspond to all the above arguments, in particular to the formative and cultural arguments, so that they are more suited to advancing general abilities (especially modelling) and adequate attitudes, and to developing a balanced picture of mathematics. Such examples are mostly presented as *teaching units* over several hours. As an example, I refer to a unit on income taxes developed in Kassel and taught several times in general and in vocational schools (see Blum 1988).

- Presentation of problem situation by teacher

- Discussion, and formulation of questions by students
- Structuring, simplifying, stating of fiscal policy demands
- Joint mathematising of demands and concepts
- Individual construction of possible income tax functions
- Presentation of actual income tax law by teacher (or providing it by students)
- Individual calculation of income taxes, drawing of income tax function (also with the aid of computers)
- Analytical investigation of income tax function
- Interpretations, comparison with demands
- Discussion, variation of parameters, interpretation of changes
- Methodological reflections

Not only *really real* but also *artificial*, whimsical problems as well as traditional word problems may be used. If their character is made explicitly clear, such unrealistic examples are sometimes better suited to educational purposes than are genuine real world applications, especially for furthering formative aims.

So, the following demand is sensible.

Demand 2: In mathematics teaching, different kinds of real world examples should be incorporated, both local and global ones, also and particularly modelling examples.

What is the actual situation in research and practice? In Blum and Niss (1989, ch 2) we identified the following.

Trend 2 (Broadening of views on applications and modelling). In the eighties, the *relations* between applications on the one hand and modelling on the other hand have become *stronger*, both in research and in practice. That means that different persons and groups doing application-oriented *research* and development in mathematics education have become increasingly aware of and interested in *contacts* among one another. In the *practice* of mathematics teaching or of constructing concrete mathematics curricula, different kinds of examples – not only classical physical or standard everyday applications – are becoming increasingly included, especially *modelling* examples, and the range of application areas has been significantly *broadened*.

There are several reasons for this *opening* of mathematics instruction to new application areas different from the classical one of physics. For instance, some aspects important for application practice and for instruction (such as going round the loop in the applied problem solving process several times or building different models for the same situation) can often be demonstrated much better with non-physical examples. However, I consider it is still important that a *close contact between* *mathematics and physics* be maintained in our educational system, because relations between mathematics and physics have always been especially intimate and therefore played an outstanding role in cultural history. In some instances, the links in content between a physical situation and its mathematical description are inseparable, for example in the case of natural laws, and the viewpoint of models and modelling tends to create unnatural distances. So students also have to know of physical applications, and to reflect upon this area's peculiar epistemological state (see, for example, Bkouche 1989).

3.3 Some consequences for methods

Another obvious consequence of part 2 is the following.

Demand 3: When incorporating applications and modelling in mathematics teaching, the whole range of arguments for this inclusion – pragmatic, formative, cultural and psychological arguments – should be taken into consideration.

During the history of mathematics instruction and mathematics education this was not always so. Mostly, only *single* arguments were predominant, such as parts of the psychological and of the formative arguments for the traditional, academically-oriented general education in upper secondary schools. Cultural arguments – emphasising metalevels connected with scientific theory and epistemology – were not stressed until the last decade. In Blum and Niss (1989, ch 2) we identified the following actual trend, which is closely connected with the two trends mentioned earlier.

Trend 3 (Widening of the spectrum of arguments). In the eighties, the *complete range* of arguments for including applications and modelling in mathematics teaching is put forward in mathematics education, with different emphases for different educational histories.

This widening of the spectrum of arguments results, on the one hand, in a broadening of examples as described in 3.2. On the other hand, this widening has immediate consequences for the *methods* in mathematics teaching, three of which will be mentioned now.

1. In order to create meta-knowledge (the importance of which is stressed in present-day learning psychology) and to enable students to translate between the real world and mathematics, not only do examples have to be treated in which such translations take place, but also the problem-solving process has to be *analysed*, and the students have to *become* explicitly *aware* of it. This can be done best in the context of global examples, such as the above-mentioned case of income taxes; here I refer to the textbook Griesel and Postel (1988, p208/209).

2. To achieve a comprehensive *understanding of mathematical concepts* – which includes relations to the real world as well – these concepts have to be taught to students in an adequate manner. We in Kassel, especially Gabriele Kaiser–Messmer, have carried out extensive empirical investigations into applications in mathematics teaching in the last few years. Among other things, we have tested the comprehension of the concepts of derivative (see Kaiser–Messmer 1986a) and of definite integral with upper secondary and tertiary students. The following problems are taken from a recently developed *questionnaire* for the concept of *definite integral.*

11. Let x denote the annual income (in DM). Let r(x) denote the corresponding marginal tax rate (in %), that is the local rate of change of the annual income taxes s(x) (in DM) with respect to the income. Herr Y now may earn B DM instead of A DM per year. How can you calculate, using r, how much more income tax Herr Y has to pay per year? Give reasons for your answer.

13. $\int_a^b f(x)dx$ means the volume of blood which has flowed through an artery between time a and time b. Explain the real world meaning of x and of f(x).

16. A factory is diffusing pollutants into the air. In time interval $[t_1,t_2]$ (time t in min) the instantaneous pollution rate p(t) (in g per min) is measured. Interpret the real world meaning of

$$\text{(a)} \quad \int_{t_1}^{t_2} p(t)\, dt \qquad \text{(b)} \quad \frac{1}{t_2-t_1} \cdot \int_{t_1}^{t_2} p(t)\, dt$$

One of the *results* of our investigations is as follows. The ability to interpret and to apply these concepts in unknown real situations can be achieved extensively with many students, along with an adequate introduction and treatment of these concepts, and this distinctly better than in the case of a pure mathematical procedure. Certain ways (both applied and pure ones) of introducing and treating these concepts can even hinder those interpretation and application abilities. *Adequate* introduction and treatment means, for instance, for the concept of derivative: handling of different kinds of average and local rates of change; non–formal introduction of the concepts of rate of change, of difference quotient and derivative (with an exactification later on); parallel geometric interpretations; re–interpretation of these concepts; applications to new situations. This may sound very natural and totally self–evident, but in most of the cases we have observed, at school and at university, the method of teaching was quite different. This is why our own model classes did significantly better in these tests.

3. Relations to reality may also serve for *proving mathematical results* on a *preformal* level (but rigorously!). A well-known example is the theorem "If f' = 0 in an interval I, then f is constant in I", which is equivalent to the completeness property of the real numbers and therefore not easy to prove formally. This can be made evident immediately by the kinematic argument "If the speedometer is showing 0 all the time, then the car is always standing still". Another interesting example, which I observed recently in grade 12 (18 year olds) and which I have since used several times to test students at school and university, refers to the question whether non-trivial solutions of the differential equation f' = f may have zeros. Most of the students used, or accepted unhesitatingly, an argument which looks very similar to the one just cited, but which now is wrong. See Blum and Kirsch (1991) for a detailed discussion of that problem and for some didactical conclusions.

3.4 The role of computers

For several years now, *computers* have proliferated into many areas in society, including the educational system, and they are also influencing mathematics instruction at all levels (compare, for example, various contributions in Blum et al 1989, section E). Here I shall only make some remarks on computers as a *tool* for mathematics teaching, as a means for performing numerical and algebraic calculations, or for drawing graphs, and as an aid for creating new teaching methods. By computers I mean in particular *pocket computers*; I think it will only be these that will really change mathematics teaching. Generally speaking, one can notice the following (see Blum and Niss 1989, ch 2).

Trend 4 (Extended use of computers). The use of computers in mathematics teaching is being *extended, quantitatively* as well as *qualitatively*, also with respect to applications and modelling.

By the use of computers, new possibilities have become available for making mathematical contents accessible to learners, for promoting the intended aims, or for relieving mathematics teaching of some tedious activities. This also has many implications for the teaching of mathematical modelling and applications. I have to refrain here from quoting examples: see Blum and Niss (1989, ch 3), and see also Huntley in this volume. Instead, let me emphasise that – quite apart from the fact that, in a great deal of numerical contexts, the simple pocket calculator is perfectly sufficient – the computer may also entail many kinds of *problems* and *risks*, especially in relation to applications and modelling in mathematics instruction, some of which can already be observed in the classroom, for instance the following.

- The devaluation of routine computational and graphical skills and the shift in emphasis towards applications and modelling enabled by the computer, may make mathematics instruction *more demanding* for both students and teachers.
- Necessary *intellectual efforts* of students may be *replaced* by mere button pressing, and new kinds of senseless routines may occur.
- Mathematics teachers may become interested in *computers instead of* in *modelling and applications*, and students may be prevented by the computer from reflecting deeply upon mathematical problems by being engaged in outward activities like constructing technically elegant programmes.

One of the most important general recommendations for dealing with these problems is probably that teachers *and* students should become fully *aware* of the problems. This will also contribute towards one of the aims of mathematics teaching mentioned in part 2, namely the acquisition of meta-knowledge of mathematics, including its relation to the real world and the advantages and risks of its tools.

In this sense I would like to finish part 3 by formulating a general proposition.

Demand 4: Computers should be used appropriately in mathematics instruction to improve the learning and teaching of mathematical modelling and applications. When using computers, teachers and students should be aware of various inherent problems and risks.

4. FINAL REMARKS

There are, of course, many more aspects of the learning and teaching of mathematical modelling and applications which I cannot touch upon in this paper, such as the important problem of assessment. I would like to close by making some optimistic remarks. There is a general consensus of opinion that applications and modelling should be an essential part of mathematics instruction at all levels. The literature shows that examples and materials are available to a sufficient extent. Empirical investigations have produced encouraging results with regard to various positive effects of applications and modelling (see, among others, Kaiser-Messmer 1986b, vol 2). On the one hand, they show that mathematics instruction does not become easier for students and teachers by the inclusion of applications and modelling but that, on the other hand, the world around us may be comprehended better, mathematical concepts may be understood more deeply and more extensively, formative abilities may be advanced, and attitudes towards mathematics may be improved by giving more meaning to it. My final remark resumes the end of part 2 and is equally self-evident and highly relevant: the most important factor for achieving such positive effects is the *teacher*. All conceptions and proposals for mathematics teaching stand and fall with the teachers, with their professional abilities, and with their didactical and pedagogical qualifications. Nor is the teachers' vital role to be deplored in my view. On the contrary, this role is utterly desirable and favourable for our work in mathematics education, the ultimate aim of which is to improve permanently the learning and teaching of mathematics.

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