In: Educational studies in mathematics, Editor: Dörfler etc., Volume 22, 1991,
Kluwer Academic Publishers, Dordrecht, S. 37-68

WERNER BLUM AND MOGENS NISS

APPLIED MATHEMATICAL PROBLEM SOLVING, MODELLING, APPLICATIONS, AND LINKS TO OTHER SUBJECTS – STATE, TRENDS AND ISSUES IN MATHEMATICS INSTRUCTION[1]

ABSTRACT. The paper will consist of three parts. In *part I* we shall present some background considerations which are necessary as a basis for what follows. We shall try to clarify some basic concepts and notions, and we shall collect the most important arguments (and related goals) in favour of problem solving, modelling and applications to other subjects in mathematics instruction. In the main *part II* we shall review the present state, recent trends, and prospective lines of development, both in empirical or theoretical research and in the practice of mathematics instruction and mathematics education, concerning (applied) problem solving, modelling, applications and relations to other subjects. In particular, we shall identify and discuss four major trends: a widened spectrum of arguments, an increased globality, an increased unification, and an extended use of computers. In the final *part III* we shall comment upon some important issues and problems related to our topic.

I. BACKGROUND CONSIDERATIONS

I.1. Clarification of Basic Concepts and Notions

We shall commence our paper by clarifying some basic concepts and notions such as "problem" and "problem solving", "model" and "modelling", "application", "applying" and "applied mathematics". By no means are we pretending to present an exhaustive epistemological treatment of these concepts. Rather does this section present a pragmatic attempt to give some working definitions which are not claimed to be very original but which are useful for the following parts of our paper.

By a *problem* we mean a situation which carries with it certain open questions that challenge somebody intellectually who is not in immediate possession of direct methods/procedures/algorithms etc. sufficient to answer the questions. This notion of a problem is apparently relative to the persons involved; so, what to one person is a problem may be an exercise to someone else. As to *mathematical* problems, there are two kinds: It is characteristic of an *applied* mathematical problem that the situation and the questions defining it belong to some segment of the real world and allow some mathematical concepts, methods and results to become involved. By *real world* we mean the "rest of the world" outside mathematics, i.e. school or university subjects or disciplines different from mathematics, or everyday life and the world around us. In contrast, with a *purely*

Educational Studies in Mathematics **22**: 37–68, 1991.
© 1991 *Kluwer Academic Publishers. Printed in the Netherlands.*

mathematical problem the defining situation is entirely embedded in some mathematical universe. This does not prevent pure problems from arising from applied ones, but as soon as they are lifted out of the extra-mathematical context which generated them they are no longer applied.

Now, *problem solving* simply refers to the entire process of dealing with a problem in attempting to solve it. Corresponding to the two categories of problems just identified there are two categories of problem solving, applied mathematical problem solving and purely mathematical problem solving. Essential elements of content and structure are common to both categories, but significant differences exist as well, in particular as regards purposes, goals, and roles in mathematics curricula.

In mathematics education problem solving is considered in two ways. (i) As an object of *research* on issues such as: How is problem solving related to other aspects of "thinking mathematically"? What are the essential structural and psychological components in problem solving? How does one adequately classify different problem solving processes? What are the most significant cognitive and affective hurdles and obstacles to students' successful acquisition of a problem solving capability? Is it possible to teach and learn problem solving? (ii) In relation to *mathematics instruction*, where issues concerning the inclusion and implementation of problem solving in mathematics curricula are addressed. In this paper we have to confine ourselves to treating only the *second* aspect, problem solving as an actual or potential part of mathematics instruction, not because the research aspect is unimportant – quite the contrary – , but for brevity and unity in the exposition. Further, we shall concern ourselves mostly with *applied* problem solving. Where appropriate (that is in sections I.2, II.1-3 and III.1), we shall also consider problem solving in a broad sense, but always in connection with applications and modelling.

Next, we are going to look at the *applied problem solving process* in more detail. The following way of describing the interplay between the real world and mathematics is well-known and is by no means our invention (see e.g. Blechman et al., 1984; Steiner, 1976; Pollak, 1979; Blum, 1985). The starting point is an applied problem or, as we also call it, a *real problem situation*. This situation has to be simplified, idealized, structured, subjected to appropriate conditions and assumptions, and to be made more precise by the "problem solver" according to his/her interests. This leads to a *real model* of the original situation which on the one hand still contains essential features of the original situation, but is on the other hand already so schematized that (if at all possible) it allows for an approach with mathematical means.

The real model has to be mathematized, i.e. its data, concepts, relations, conditions and assumptions are to be translated into mathematics. Thus, a *mathematical model* of the original situation results. Such a model consists essentially of certain mathematical objects, corresponding to the "basic elements" of the original situation or the real model, and of certain relations between these objects, again corresponding to relations between those "basic elements". To be a bit more precise, a mathematical model can be viewed as a triple (S, M, R), consisting of some real problem situation S, some collection M of mathematical entities and some relation R by which objects and relations of S are related to objects and relations of M (cf. e.g. Niss, 1989).

While *mathematization* is the process from the real model into mathematics, we use *modelling* or model building to mean the entire process leading from the original real problem situation to a mathematical model. The modelling process does not merely yield a simplified but true image of some part of a pre-existing reality. Rather, mathematical modelling also structures and creates a piece of reality, dependent on knowledge, intentions and interests of the problem solver.

It has proved appropriate to distinguish between different kinds of models. If economic items, for example, such as interests or taxes are considered mathematics particularly serves to establish certain norms involving value judgements. Here it is a matter of *normative* models. If physical phenomena, for example, such as planetary motions or radioactive decay are considered, mathematics serves primarily to describe and explain the respective situation. Here it is a matter of *descriptive* models.

The applied problem solving process continues by work within mathematics, i.e. by drawing conclusions, calculating and checking concrete examples, applying known mathematical methods and results as well as developing new ones etc. In so doing computers may be used as well, also in order to simulate analytically unaccessible cases. Altogether, certain *mathematical results* are obtained.

These results have to be re-translated into the real world, i.e. to be interpreted in relation to the original situation. In doing so, the problem solver also validates the model, i.e. decides whether it is justified to use it for the purposes it was built for. When validating the model, discrepancies of various kinds may occur which lead to a modification of the model or to its replacement by a new one. In other words, the problem solving process may require going round the loop *several* times. If eventually a satisfactory model has been found, the problem solver may use it as a basis for forecasts, decisions or actions. Sometimes, however, even several attempts do not lead to useful and reasonable results, perhaps because the problem simply is not accessible to mathematical treatment in a sensible way.

Besides such complex problem solving processes – which are rare in mathematics instruction – there are abbreviated and restricted links between mathematics and reality which are much more frequently found: On the one hand a *direct application* of already developed "standard" mathematical models to real situations with a mathematical content, on the other hand a "*dressing up*" of purely mathematical problems in the words of an other discipline or of everyday life. Such *word problems* often give a distorted picture of reality. This is sometimes done deliberately in order to serve instructional purposes.

It is common practice to use the term *application* of mathematics (or *applying* mathematics) to denote all the above-mentioned ways of bringing the real world into connection with mathematics, whether it is a matter of proper model building or of a more simple interplay. In a slightly narrower sense, real problem situations can also be called *applications*. Eventually, mathematical models or, more generally, every piece of mathematics which in some way is or may be related to the real world can be seen as belonging to *applied mathematics*. Of course, this definition does not imply a strict separation between "pure" and "applied" mathematics.

In the general sense just mentioned, we can therefore simply speak of "*applications*" when we mean any representational relations whatsoever between the real world and mathematics. Nevertheless, in the following we mostly use the term "*modelling and applications*" instead, since this corresponds to the common usage and since here modelling, as the most important part of the process of relating mathematics to the real world, is explicitly mentioned. We emphasize once more that for us "modelling and applications" also include the process of applied problem solving as described above. If necessary, we make distinctions between the different components of "applications" on the spot.

The use of the *model conception* of the relation between mathematics and reality, especially between mathematics and other subjects, implies an explicit distinction between real situations on the one hand and mathematical models on the other. It indicates that assumptions, specifications and idealizations have been made and that thus the models are open to criticism and to improvement. This is often very helpful for an adequate solution to a given applied problem. The model conception may, however, also have disadvantages. For, a strict separation of mathematics from the real world often means that things which are inseparable and linked together – as mathematics and physics have been for centuries – are examined in a merely formal manner. This tends to create "artificial" distances between certain real situations and their mathematical description, for example in the case of natural laws.

We will conclude this first section by some remarks on which *kinds of mathematics instruction*, in relation to other subjects, we shall consider in this paper. When speaking about "mathematics instruction" we regard it as taking place within a given segment of an existing educational system. Here, firstly, mathematics instruction may essentially serve two different *purposes*:

(a) to provide students with knowledge and abilities concerning mathematics as a subject in itself;
(b) to provide students with knowledge and abilities concerning (one or more) other subjects, to which mathematics is supposed to have actual or potential services to offer.

Secondly, the *organizational framework* of mathematics instruction may take two different shapes:

(1) Mathematics may be taught as a separate subject, i.e. as an independent organizational unit called "mathematics" or something like that;
(2) Mathematics may be taught as a part of and integrated in (one or more) other subjects.

Thirdly, we distinguish between mathematics instruction in different *educational histories*:

① Mathematics in school offering general education, viz. at the primary, junior secondary, and senior secondary level,
② Mathematics in vocational education,
③ Mathematics in university courses for future mathematicians or mathematics teachers,
④ Mathematics as a service subject in university courses for future scientists, engineers, economists etc.

Now we can illustrate the situation by a matrix:

<table>
<tr><th>purpose
organization</th><th>(a) focus on
mathematics</th><th>(b) focus on
other subjects</th></tr>
<tr><td>(1) mathematics as a sepafate subject</td><td>(1a) examples:
①, ③</td><td>(1b) examples:
④, ②, partly ①</td></tr>
<tr><td rowspan="2">(2) mathematics integrated in other subjects</td><td>(2a) examples:</td><td>(2b) examples:
②, partly ④</td></tr>
<tr><td colspan="2">integrated curricula</td></tr>
</table>

In all cells of this matrix, *relations between mathematics and other subjects* may have a part to play. Even if, as in (1a), the purpose of mathematics instruction is to elucidate mathematics as a subject, it may be highly relevant to incorporate mathematical applications and modelling in other subjects; we shall present arguments for that in the next section I.2. When dealing with (1b), (2a) and (2b) it seems quite natural to include applications and modelling in mathematics instruction. However, we can sometimes find a "division of labour", both in (1b) and even – on a much smaller scale – in (2b), such that separate mathematics courses devoid of applications and modelling are organized to collect "once and for all" the mathematical concepts, methods, results etc. needed or desired in the subject being serviced, whereas applications and modelling activities are intended to take place as part of the instruction of extra-mathematical substance matter within the framework of the serviced subject. In section I.2 we shall explain why in general we are not in favour of this approach.

Possible relations shown in the matrix also include truly *integrated* curricula (second row of the matrix), both in school and in university, whereby teaching and learning are taking place in an interdisciplinary and cross-subject way. Internationally speaking there are, however, only few experiences with this approach.

When we talk here about mathematics instruction or mathematics teaching we always have the *whole* matrix in mind, provided that it is possible to distinguish segments of instruction with mathematics as an explicit object of attention. We shall abstain from regarding instances of (2b) only where mathematics is totally "integrated away" in other subjects.

I.2. Review of Arguments

Throughout the history of mathematics education, the inclusion of aspects of applications and – more recently – modelling and problem solving in mathematics instruction has been regularly advocated by various individuals and quarters, and in periods also realized in some curricula. A review of representative literature on mathematics education shows that basically *five arguments* for such an inclusion have been invoked over the years. At the level of detail adopted here, these arguments seem to be fairly relevant, with varying emphasis, of course, to all kinds of mathematics instruction, in school, in technical and vocational education, in the mathematical education of mathematics professionals and others.

1. *The formative argument* emphasizes the application of mathematics and the performing of mathematical modelling and problem solving as

suitable means for developing general competences and attitudes with students, in particular orientated towards fostering overall explorative, creative and problem solving capacities (such as attitudes, strategies, heuristics, techniques etc.), as well as open-mindedness, self-reliance and confidence in their own powers.

2. *The "critical competence" argument* focusses on preparing students to live and act with integrity as private and social citizens, possessing a critical competence in a society the shape and functioning of which are being increasingly influenced by the utilization of mathematics through applications and modelling. The aim of such a critical competence is to enable students to "see and judge" independently, to recognize, understand, analyse and assess representative examples of actual uses of mathematics, including (suggested) solutions to socially significant problems.

3. *The utility argument* emphasizes that mathematics instruction should prepare students to utilize mathematics for solving problems in or describing aspects of specific extra-mathematical areas and situations, whether referring to other subjects or occupational contexts ("mathematics as a service subject") or to the actual or future everyday lives of students. In other words, mathematics instruction should enable students to practise applications and modelling in a variety of contexts where mathematics has instrumental services to offer without occupying in itself the focal point of interest. This argument relies on the assumption/experience that the ability to activate mathematics to extra-mathematical situations does not result automatically from the mastering of pure mathematics but requires some degree of preparation and training.

Sometimes this argument is taken to express society's ultimate reason for providing children and adults with an extensive amount of mathematics education: the applicability of mathematics is implicitly expected.

4. *The "picture of mathematics" argument* insists that it is an important task of mathematics education to establish with students a rich and comprehensive picture of mathematics in all its facets, as a science, as a field of activity in society and culture. Since modelling and applications constitute an essential component in such a picture, this component should be allotted an appropriate position in mathematics curricula. The same holds true for problem solving (together with its close companion: problem posing) which constitutes a fundamental category in all creative mathematical processes, whether they lead to mathematics which is original and new to the mathematics community or just to the problem solver, or new uses of established mathematics.

5. *The "promoting mathematics learning" argument* emphasizes that the

incorporation of problem solving, applications and modelling aspects and activities in mathematics instruction is well suited to assist students in acquiring, learning and keeping mathematical concepts, notions, methods and results, by providing motivation for and relevance of mathematical studies. Such work also contributes to training students to think mathematically, and to select and perform mathematical techniques within and outside of mathematics.

Not only are there arguments in favour of including applications, modelling and problem solving in mathematics instruction; counter-arguments also exist. Some of them will be discussed in section III.1 of the present paper. So, this is not the occasion to offer a detailed discussion of the arguments *for* versus the arguments *against* assigning these aspects significant roles in mathematics instruction. Not surprisingly, to us the collection of arguments pro outweigh the counter-arguments. Let us sum up our reasoning in two points. (I) Considering the fact that substantial mathematics education is no longer reserved for the small minority of people who enter mathematically inclined professions but is now being given, on occupational, social, democratic and cultural grounds, to an ever *increasing proportion* of the population, mathematics instruction at all levels has to deal with the role and use of mathematics in the world outside the realm of mathematics itself. (II) Mastering mathematics can no longer be considered equivalent to knowing a set of mathematical facts. It requires also the mastering of mathematical *processes*, of which problem solving – in the broadest sense – occupies a predominant position.

This, however, should not be taken to imply that the inclusion of applications, modelling and problem solving makes demands for proper mathematical knowledge, proficiency and insight less important, let alone obsolete. On the contrary – the more widely and extensively mathematics is being activated and used, the more necessary genuine mathematical knowledge becomes for the understanding, evaluation and judging of its use, in general as well as in special cases. To this end recipes, thumb rules and rote learning are not sufficient means.

If we agree that mathematical modelling and applications in other disciplines should be granted important positions in mathematics instruction, then their inclusion may pursue one or more of several *overall goals*, among which the following three seem to be particulary significant (cf. Niss, 1989):

Goal 1: Students should be able to *perform* applicational/modelling/applied problem solving processes.

Goal 2: Students should *acquire knowledge* of existing models and applications of mathematics, and of characteristic aspects of related processes.

Goal 3: Students should be able to critically *analyse and assess* given examples of modelling and applications.

The decision as to which of these goals are to be pursued in actual mathematics instruction depends on which of the above-mentioned five arguments are invoked to justify the inclusion of modelling and applications in the curriculum at issue.

II. PRESENT STATE AND CURRENT TRENDS

Problem solving, modelling, relations with and applications to other disciplines are no longer upstarts on the stage of mathematics education, fighting for attention and recognition. Aspects of these topics were on the agendas of ICME-3 (vid. Pollak, 1979), -4 (vid. Bell, 1983), and -5 (cf. Lesh et at., 1986), and latest on that of ICME-6 (cf. Niss, 1988; Blum, 1988), as major themes of increasing significance to mathematics education and instruction, and of increasing interest to the mathematics education community in the world. This interest has materialized in other fora as well, in several conferences – especially in the series of biennial International Conferences on the Teaching of Mathematical Modelling and Applications (ICTMA) held in 1983, 1985, 1987, and 1989 (vid. Berry et al., 1984, 1986, 1987; Blum et al., 1989; Niss et al., 1990) – and in a host of publications: research and debate books, articles, teaching materials of different sorts, e.g. collections of cases, textbooks, computer software, video programmes, etc. (see section III.2).

In view of the relatively established position within the field of attention of mathematics educators and educationalists, in surveying and discussing the state and trends of problem solving, modelling and applications, and the relations between mathematics and other subjects, there seems to be no great need to give an extensive overview of the background and rise of studies and activities in that area. Instead, we shall concentrate on reviewing present states, recent and current trends, with an emphasis on the last half decade or so.

There are two major aspects of studies and activities concerning our topic: an aspect of empirical and theoretical research, and an aspect of practice defined as the actual implementation of mathematics education in the educational system. In the term *research* we comprise not only the creation of new "positive" (often empirical) knowledge, but also systematic reflections – whether mathematical, philosophical, psychological, sociological etc. – on mathematics education, including the development of new types of curricula based on such reflections. Also the term *practice* should be taken in a wide sense referring not only to everyday classroom teaching practice. It comprises in our definition all elements in the actual implementation of mathematics education, including, for instance, the devising and carrying out of specific entire curricula, or curriculum components, the writing of textbooks, creation of teaching materials etc. So, the demarcation line between "research" and "practice" as defined here is somewhat blurred, thus reflecting quite well the reality of mathematics education. What matters in categorizing a piece of work as belonging to either "research" or "practice" is its content of new thoughts, new results and innovative ideas.

Of course, it is not possible in a brief survey like this to even mention, let alone comprehensively review, all significant contributions to research on and practice of our topic. Rather than listing and reviewing a number of examples and cases we shall concentrate on outlining *four major trends*.

II.1. Trend 1: A Widened Spectrum of Arguments

It is probable that all five arguments mentioned in section I.2 for incorporating problem solving, modelling and applications in mathematics instruction have been invoked in some form or another, explicitly or implicitly, and with varying intensity, by mathematics educators and educationalists in the course of the last century and a half. Yet, traditionally the predominant arguments have been only two of the five: the utility argument (3) and the "promoting mathematics learning" argument (5). From the late sixties the formative argument (1) began to be called upon frequently as well. (In Germany and France this happened earlier.) Eventually, during the last decade, the "picture of mathematics" argument (4) and, more recently, the "critical competence" argument (2) have gained momentum too.

So, presently *all five arguments* are seen to be put forward to advocate that mathematics instruction at all levels should deal with applications, modelling and problem solving, with different emphases in different quarters, of course. Still it seems, however, that to many mathematics teachers (from school to university) who advocate these aspects the predominant argument is to provide motivation for students to pursue mathematical activities. This is a special instance of the "promoting mathematics learning" argument.

The widening of the spectrum of arguments is significant in that it places the issue of problem solving, modelling and applications where it belongs: not merely amongst tactical devices to improve the situation for traditional mathematics instruction, but as an *integral part* of the discussion of mathematics education as a whole.

II.2. Trend 2: Increasing Globality

Over the last handful of years, an increasing *globality* of the theoretical and practical interest in and activities of problem solving, modelling and applications to other areas and subjects can be detected. This is true both if globality is taken in a *geographical* sense and if it refers to *internal* aspects of mathematics instruction.

(*i*) In recent years, all over the world a growing number of mathematics programmes at different levels is including problem solving, applications, models and modelling in extra-mathematical areas and subjects. Similarly, an increasing amount of co-operation between mathematics and other subjects can be observed. We may liken the development to a logistic growth process. The process is (still) in the steep segment of the curve but there are indications that it is approaching the segments of slower growth corresponding to gradual saturation. Not surprisingly, the foothold of applications and models is, generally speaking, stronger than that of problem solving and modelling with their greater demands on students and teachers and greater requirements of time etc.

(*ii*) At ICME-5 it was stated (cf. Niss, 1987) that the *distance* between the *forefront* of research, development and practice in applications and modelling on the one hand, and applications and modelling in the *mainstream* of mathematics instruction on the other hand, was very large. Even more so with problem solving in the broader sense, at least as regards post-elementary mathematics instruction (i.e. instruction dealing with mathematics beyond arithmetic and simple computational geometry). Although the distance between the forefront and the mainstream is still very large it is being reduced. An indication of this may be found in the increasing inclusion of application and model cases in textbooks from school to university. Also in some countries (e.g. in Denmark and the Netherlands) applicational aspects of mathematics have entered written examination papers, if only to a modest extent as yet. So it seems that frontline developments and findings are gradually disseminating into mainstream mathematics instruction, thus representing a different kind of increased globality in the manifestation of applications, modelling and problem solving.

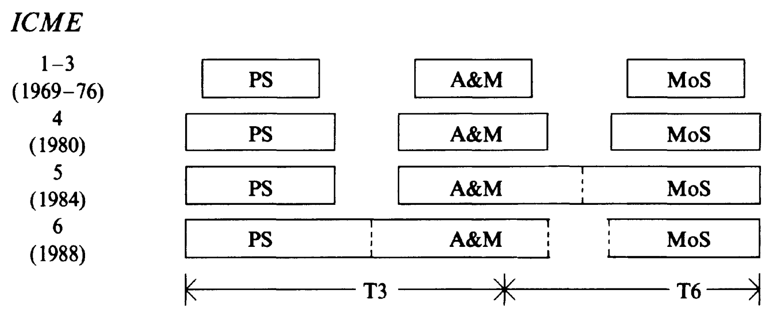
(*iii*) Concurrently with the dissemination of problem solving, applications, modelling and increased co-operation between mathematics and other subjects into the mainstream of mathematics instruction, the *rate of innovation* at the forefront seems to be decreasing. Thus with minor exceptions (such as chaos and fractals) neither the spectrum of applicational areas nor the spectrum of mathematical topics dealt with in mathematics instruction has become fundamentally extended since ICME-5. Similarly, the *paradigms* of research on these aspects have not really changed during the last half decade. (In saying this, it is by no means claimed that research has stagnated in quality and quantity since ICME-5, only that its main directions have remained the same, by and large.) This is hardly surprising. The development over the last one or two decades at the forefront has concentrated on working out and publishing analyses of problem solving, modelling, applications, and interdisciplinary co-operation, – and on giving arguments for their inclusion in mathematics instruction – ; on creating and compiling examples and cases; on preparing and publishing materials; on initiating discussion; on holding conferences; on influencing curriculum planners and decision makers as well as the communities of mathematics and mathematics education at large; on devising curricula and instructional units etc., etc. If it is true, as we claim, that what has been happening in the last half decade or so is that all this is now spreading rather smoothly into mainstream mathematics instruction, without having been confronted as yet with an extensive and massive amount of experiences indicating problems and calling for revision, there is no great demand and incentive for front-line innovation at the moment. Such demands and incentives, on the other hand, will most probably emerge when (if?) problem solving, modelling, applications, and co-operation between mathematics and other subjects become substantially included in mathematics curricula and mathematics instruction up to some point of saturation.

II.3. Trend 3: Increasing Unification

Earlier, the conceptual and other relations between (applied) problem solving, modelling, models and applications, and co-operation between mathematics and other subjects, were made clear only seldom (and this is still the case in some degree). Often the terms were (are) used indiscriminately. Nevertheless, it is possible to distinguish over the years separate tendencies and groupings in mathematics education, concentrating – respectively – on: applications, modelling, problem solving, mathematics as a service subject, and co-operation between mathematics and other subjects.

In recent years these different tendencies and groupings have become more amalgamated. We may speak of an *increasing unification* of the field. This should not be taken to imply that the different components of the field have merged into a homogeneous whole. They are all clearly discernible and will probably remain so to the extent that they represent different perspectives over the field. What has happened is that all quarters have widened their horizons and have become increasingly aware of and interested in areas of contact and interaction with the other quarters. The development of the field could be described as a transition from a discrete towards a continuous spectrum of interest and activity.

To illustrate the process which is taking place we could look at its materialization in the different ICMEs. The following table outlines the development in a simplified form. The abbreviations used are: PS for problem solving, A&M for applications and modelling, MoS for the relationship between mathematics and other subjects. The task of the table is to present certain major points, not a list of confirmed facts and details. It should therefore not be interpreted too rigorously.



(*i*) Although the trend of increasing unification in the field of (applied) problem solving, modelling, applications, and co-operation between mathematics and other subjects is quite international in character, national/regional differences in focus and emphasis do exist. Besides, there are many variations between institutions due to differences in their tasks (see also section III.2 where concrete references will be given).

The *psychological*, in particular the cognitive and metacognitive, aspects of *problem solving* have been widely investigated in empirical and (to a lesser extent) theoretical studies conducted in the USA, where the links to fields such as general cognitive psychology and artificial intelligence have been much cultivated. Also the USSR, Canada and Israel have strong traditions in this area. The *pragmatic* aspects of problem solving, including its implementation in mathematics curricula – mainly primary and lower secondary ones – are receiving considerable attention in the UK and Australia, in particular as regards *applied* problem solving and so-called investigations in less structured situations, and in Finland.

The majority of contributions concerning the *theoretical* and *philosophical* aspects of *applications and modelling* in an educational context come from West Germany, Austria, the Netherlands, Denmark, France and India. Gabriele Kaiser-Messmer's dissertation (1986) is a recent example of a thorough piece of work in this respect. Also these aspects are dealt with in several of the contributions contained in the conference proceedings of ICTMA-3, Kassel 1987, and ICTMA-4, Roskilde 1989 (for references see the beginning of part II). For a long time the leading country as regards the *pragmatic* aspects of applications and modelling has been the UK, but very interesting work is being done in many other countries too, e.g. the USA, the Netherlands, Australia, Italy and the Fed. Rep. of Germany. Several groups based in the British polytechnics are very active in the field of modelling. The different ICTMA proceedings also give valuable accounts of the-state-of-the-art in applications and modelling at secondary and tertiary levels.

When dealing with the relationship between *mathematics and other subjects* it is not easy to identify leading countries or centres, because the concrete structure, organization and traditions of the relevant segments of the educational system play a dominant role for the possibility of interaction between the subjects. Two countries in which a fair amount of valuable pragmatic co-operation between mathematics and other subjects, primarily in upper secondary school, has been carried out, are Denmark and the Netherlands.

(*ii*) The trend at all educational levels of having more contact, brought about through applications and modelling, between mathematics and other subjects is rather a matter of mutual inspiration between independent subjects than of overall thorough co-ordination of curricula and teaching practice. What mostly seems to take place is that mathematics instruction adopts selected elements from other subjects which provide suitable sources for applications and modelling, or that shorter co-ordinated instructional sequences on specific themes are arranged locally between mathematics and one or two other subjects. The latter is prevalent with mathematics taught as a service subject.

(*iii*) Relations between *mathematics and physics* as sciences have always been very intimate and profound, so much that many, although not most, mathematical concepts were established in close connection with attempts at describing and understanding physical systems. This intimate relationship used to be reflected in mathematics instruction at most levels. However, it seems that over the last decade the relations have become weakened in many places. This is due to the *opening* of mathematics instruction to new applicational areas. There are several reasons for this opening:

- Mathematics is being applied to a growing number of areas outside physics and related fields, and a correspondingly growing proportion of students enter professions utilizing mathematics in non-physical contexts.
- As post-elementary school mathematics education is being given to an increasing number of children and youngsters to whom it is their final formal encounter with mathematics, the range of areas to which they see mathematics being applied has to encompass more than just physics etc. in order to be relevant for their interests and their lives in society.
- Aspects important for practice and for instruction such as going round the loop in the applied problem solving process several times or building different models for the same situation, can often be demonstrated much more easily with non-physical examples than with classical physical ones (cf. section I.1).
- Since mathematical applications and modelling in areas outside physics generally rely on less involved and demanding extra-mathematical theory than is the case with physics, such areas often provide opportunities to activate mathematics in ways that are more easily accessible to the majority of students.

In our view, the opening of mathematics instruction to other applicational areas and subjects than physics is both necessary and desirable. It should even be stimulated in order to enable mathematics education in modern society to fulfill its aims, and, more specifically, to achieve the goals of applied problem solving, modelling and applications as put forward earlier in this paper (section I.2).

However, at the school level it is very important that a *close contact* between mathematics and physics be maintained, although unavoidably this has to be on a smaller scale than was usual in earlier days. Furthermore the perspectives on the interaction need some revision. It may be said, somewhat paradoxically, perhaps, that the more mathematics is being applied to areas and subjects outside physics, the more important it is to have access to representative cases from physics to shed light on possibilities, conditions, difficulties and pitfalls of applications and modelling in fields with smaller degrees of well-established mathematatization.

II.4. Trend 4: An Extended Involvement of Computers

For several years it has been evident that computers form a highly powerful tool for the numerical and graphical treatment of mathematical applications and models. Not only do computers constantly allow for greater detail, accuracy, and rapidity in calculations, for the handling of more data, for examining the effects of changing terms or parameters, for providing better illustrations etc., in most cases they are simply indispensable for the mere "cracking" of a given mathematical model to be accessible or realistic. As soon as technically feasible and sufficiently inexpensive microcomputers became available on a large scale, they entered mathematics instruction too and were utilized, as they still are, to treat mathematical models and applications. This development has further given rise to a rapidly growing bulk of programme packages which serve to assist the handling of models and applications on various points, e.g. in solving equations, inverting matrices, drawing graphs, optimizing functions, carrying out statistical tests etc.

Thus there is a trend to *quantitatively extend* the involvement of computers in mathematics instruction to deal with applications and models, a trend which is hardly surprising to anyone in the field. (Conversely, mathematics of course exerts a substantial influence on computer science.)

Recently, however, a somewhat different trend of *qualitiative extension* of the involvement of computers in modelling and applications has manifested itself. Quite a fair amount of software has been developed to offer various sorts of assistance in the processes of modelling or of applying mathematics to various areas. Some kinds of software offer opportunities to explore certain types of applied problem solving situations, other kinds provide interactive tools for building models or for investigating model behaviour (often graphically) within certain standard model universes, such as geometry (LOGO constitutes an early example of this), statistical analysis, stochastic simulation, differential equations, to mention just a few.

It is a significant characteristic of this qualitatively new way of involving computers that they may be utilized without knowing and understanding the mathematics behind the computer programme or – this is far more important – the mathematics involved in the models dealt with.

This development may have several effects which we shall comment on in section III.3.

III. ISSUES AND PROBLEMS

In this final part of our paper we shall identify and comment on some of the most important issues and problems in relation to modelling, applied problem solving, applications, and the interplay between mathematics and other subjects. In view of the scope and limitations of this paper it will neither be possible to present an exhaustive list of the important aspects nor to offer a thorough and detailed discussion of those mentioned. We have to confine ourselves here to outlining briefly a few main points. Emphasis will be laid on *curricular* and *instructional* aspects, whereas less attention can be paid to research aspects in a strict sense. In particular, we do not deal here with the problem of evaluating application-oriented mathematics instruction, that is with the question of how far mathematics education has already succeeded in achieving the diverse positive effects that the protagonists (the authors count themselves among those) are expecting from applied problem solving or from applications to other subjects areas (cf. also Burkhardt, 1989). Such an evaluation of and analytical reflection on our topics is an important task for the nineties (see also our remarks at the end of section II.2).

III.1. Review of Obstacles

In spite of all the good arguments in favour of problem solving, modelling, applications and links to other subjects in mathematics teaching, collected in section I.2, these items often still do not play as important a role in mainstream mathematics instruction at school and university as we would wish (see, for example, Burkhardt, 1983). This is not due to ill-will or incompetence of teachers but to certain "objective" *obstacles* which should be taken very seriously. Such obstacles have been well-known to mathematics educators for a long time (see e.g. Pollak, 1979; Blum, 1985; Niss, 1987), but they still exist. We shall briefly refer to three kinds.

(A) *Obstacles from the point of view of instruction.* Many mathematics teachers from school to university are afraid of not having enough time to deal with problem solving, modelling and applications in addition to the wealth of compulsory mathematics included in the curriculum. This holds also for the teaching of mathematics as a service subject. Furthermore, some teachers even doubt whether applications and connections to other subjects belong to mathematics instruction at all, because such components tend to distort the clarity, the aesthetical purity, the beauty and the context-free universality of mathematics (on which its power, in their view, is essentially based).

(B) *Obstacles from the learner's point of view.* Problem solving, modelling and applications to other disciplines make the mathematics lessons unquestionably more demanding and less predictable for learners than traditional mathematics lessons. Mathematical routine tasks such as calculations are more popular with many students because they are much easier to grasp and can often be solved merely by following certain recipes, which makes it easier for students to obtain good marks in tests and examinations.

(C) *Obstacles from the teacher's point of view.* Problem solving and references to the world outside mathematics make instruction more open and more demanding for teachers because additional "non-mathematical" qualifications are necessary, and make it more difficult to assess students' achievements. Moreover, many teachers do not feel able to deal with applied examples which are not taken from subjects they have studied themselves. And very often teachers simply either do not know enough examples of applications and modelling suitable for instruction, or they do not have enough time to up-date examples, to adapt them to the actual class and to prepare the teaching of them in detail.

Nevertheless: In the light of the arguments put forward in favour of problem solving, modelling, applications and connections to other subjects we should continue to make every effort to *overcome* these obstacles. This could be done both by adequate pre-service and in-service teacher education, to equip teachers (also in universities) with knowledge, abilities, experiences and in particular with attitudes to cope with the demands of teaching mathematics in the desired way, and by stimulating every kind of contact and co-operation between mathematics teachers at school and university and their colleagues in other subjects. And we should urge that *problem solving and relations to the real world* become and remain *essential parts* of mathematics instruction at all levels, even in spite of all the difficulties mentioned.

A few more remarks on the third obstacle: Are there, for each level of mathematics instruction from school to university, really *enough cases* of modelling and applications suitable for teaching? For the elementary school level the answer is obviously "yes". But for the secondary and tertiary level, the question is not quite so easy to answer, because many real situations which are subjectively relevant to students are either too complex or mathematically too difficult (by requiring mathematical concepts and methods far beyond students' capacity), or too easy (by just calling for elementary arithmetic). Nevertheless, altogether our answer is "yes" again (perhaps in contrast to the answer we would have given 20 years ago), and we shall proceed by mentioning a few important materials and resources for modelling and applications activities.

III.2. Materials and Resources

Firstly, let us mention the various references to materials and literature given in Henry Pollak's survey lecture 1976 at ICME-3 (Pollak, 1979) and in Max Bell's lecture 1980 at ICME-4 (Bell, 1983). Further, we would refer to the extensive bibliography of Kaiser et al. (1982/1990) in which several hundred articles and books containing examples, analyses of or general reflections on applications and modelling in mathematics instruction at the secondary school level are summed up and classified.

From among the various current *curriculum projects* we shall list only a few:

(1) From the *USA*:

- The "High School Mathematics and its Applications Project" (HIMAP), with several modules suitable for secondary school mathematics teaching, and the "Undergraduate Mathematics and its Applications Project" (UMAP). Both projects are coordinated and published by the Consortium for Mathematics and its Applications (COMAP, Arlington/Ma.), directed by Solomon Garfunkel and Laurie Aragon. UMAP is publishing modules, monographs and a special journal; the cases are taken from a variety of disciplines and are suitable for upper secondary and early tertiary mathematics teaching.
- The project "Teaching Experiential Applied Mathematics" (TEAM), directed by James Choike, with several booklets for the undergraduate and upper secondary levels, and the project "Applications in Mathematics" (AIM), directed by Jeanne Agnew, with several booklets for the upper secondary level. Both projects were completed at Oklahoma State University and coordinated by the Mathematical Association of America.
- The "University of Chicago School Mathematics Project" (UCSMP), directed by Zalman Usiskin. Many courses for primary and secondary school mathematics have been and are being developed, with real world applications as one of the hallmarks (cf. Usiskin, 1989).
- The "Committee on Enrichment Modules" and its continuation "Contemporary Applied Mathematics", directed by Clifford Sloyer and others

at the University of Delaware. The modules are suitable for upper secondary and early tertiary mathematics teaching (cf. Sloyer, 1989).

(2) From *Great Britain*:
- Several projects at the Shell Centre for Mathematical Education, University of Nottingham, directed by Hugh Burkhardt, Rosemary Fraser et al., especially the "Numeracy through Problem Solving Project" for the lower secondary level (cf., for example, Binns et al., 1989) and the project "Investigations on Teaching with Microcomputers as an Aid" (ITMA) for the secondary and the tertiary level (cf. Fraser, 1989).
- Several projects by the Spode Group and the Centre for Innovation in Mathematics Teaching at the University of Exeter, directed by David Burghes, with many modelling examples, suitable for the junior and senior secondary school level (see, for instance, Hobbs/Burghes, 1989).
- The "Mathematics Applicable Project", directed by Christopher Ormell at the University of East Anglia, with textbooks and materials for the secondary level (cf. Ormell, 1982).

(3) From *Australia*:
- The "Mathematics in Society Project" (MISP) for the primary and the lower secondary level, an international project based in Australia, directed by Alan Rogerson (cf. Rogerson, 1986).
- The project "Careers and Mathematics" (CAM) at the Swinburne Institute of Technology, with materials for the same age group.
- The project "Practical Applications of Mathematics" (PAM), directed by Peter Galbraith and Alistair Carr, which addresses secondary levels mainly (see Galbraith/Carr, 1987).

(4) From the *Netherlands*:
- The HEWET Project at the OW & OC Institute, University of Utrecht, with many materials for secondary school mathematics. The background theory to this project is given by De Lange (1987).

(5) From the *Fed. Rep. of Germany*:
- The "Mathematikunterrichts-Einheiten-Datei" (MUED), an association consisting mainly of school teachers. This group has developed a lot of materials with applications and project works for secondary school mathematics instruction (see Böer/Meyer-Lerch, 1989; Volk, 1989).

Apart from such larger projects there are numerous interesting *individual* contributions of different sorts from all over the world. Again we shall confine ourselves to mentioning some materials published in the eighties, without any claim of giving a complete account.

Firstly, let us name a few examples of *modelling courses*: Burghes/Huntley/McDonald (1982), Meyer (1984), Giordano/Weir (1985), Edwards/Hamson (1989) or Wan (1989). There are very helpful *sourcebooks* and collections of applied examples, e.g. Selkirk (1982) on geography, Howson/McLone (1983) on everyday life, Townend (1984) on sports, and Bushaw et al. (1980) or Klamkin (1987) on various themes. A group of teachers at the Oberstufenkolleg Bielefeld (FRG) has developed several application-oriented courses for the upper secondary and early tertiary levels (see Gerull et al. in Blum et al., 1989). Much material for the secondary and tertiary levels can be found in the proceedings of the three International Conferences on the Teaching of Mathematical Modelling and Applications (ICTMA-1,2,3; for references see part II). As particularly regards mathematical *problem solving*, we would like to mention Schoenfeld's monography (1985), the extensive conference proceedings edited by Silver (1985), and the proceedings of the problem solving theme group at ICME-5 (Burkhardt et al., 1988). Finally, we wish to refer to the two proceedings of the ICMI conference on "Mathematics as a *Service Subject*" (Udine, 1987), i.e. an ICMI Study edited by Howson et al. (1988) and an anthology edited by Clements et al. (1988).

Many of the materials referred to incorporate the use of *computers* to a substantial degree, in accordance with the apparent trend towards a quantitative and qualitative extension of the use of computers in mathematics instruction, as identified in section II.4. Therefore it is useful to consider once again the role and impact of computers in mathematics instruction at school and university, with special respect to modelling and applications.

III.3. The Role and Impact of Computers

For several years now, computers have been proliferating into many areas in society, including the educational system, and also into mathematics instruction at school and university (cf. e.g. the ICMI Study edited by Churchhouse et al., 1986). In this paper, we shall not consider computer assisted instruction, nor shall we deal with computers and computer science as separate objects of study and learning. We shall only reflect here on computers as a *tool* for mathematics instruction, as a means for performing numerical or algebraic calculations, or for drawing graphs and pictures, and as an aid for creating new teaching methods. New possibilities have become available for making mathematical contents accessible to learners, for advancing the acquisition of mathematical concepts, for promoting the intended aims, for relieving mathematics teaching of some tedious activities and thus making it more efficient. Four aspects are of obvious relevance to our topic:

- More complex applied problems with more realistic data become accessible to mathematics instruction at earlier stages and more easily than before.
- The relief from tedious routine makes it possible to concentrate better on the applicational and problem solving processes (see section I.1), and thus to advance important process-oriented qualifications with learners.
- Problems can be analysed and understood better by varying parameters and studying the resulting effects numerically, algebraically or graphically (corresponding to the so-called operative principle: "What happens, if . . . . ?").
- Problems which are inaccessible from a given theoretical basis, for instance by being too complex or mathematically too demanding for a given age group, may be simulated numerically or graphically.

As is well-known, the existence of powerful tools always has implications not only for methods but for *goals* and *contents* as well. Again we shall refer to four apparent aspects in relation to our topic:

- Routine computational skills are becoming increasingly devaluated, whereas problem solving abilities such as building, applying and interpreting models, experimenting, simulating, algorithmic thinking or performing computational modelling are becoming revaluated.
- New types of content which are particularly close to applications can be treated more easily now, e.g. difference and differential equations or data analysis at the upper secondary level, and statistics, optimization, dynamical systems or chaos theory at the tertiary level (cf. section III.5).
- By virtue of various relieving effects produced by computers, there is simply more room for modelling and applications in the curriculum.
- It is possible and – because of their increasing relevance in the application practice – also necessary to deal with such applied problems that necessitate the use of computers to a considerable extent.

Everything said so far may sound very positive and promising. However, it should be recognized that computers may also entail many kinds of *problems* and *risks*, also in relation to our topic, especially the following:

- Arithmetic and geometric skills and abilities of learners may atrophy, though they are still indispensable, also for applied problem solving and real world applications.

- The devaluation of routine skills – skills which hitherto have helped students to pass tests and examinations – will make mathematics instruction more demanding for all students and too demanding for some of them, for proper problem solving and modelling are ambitious activities, with or without computers.
- Paradoxically perhaps, teaching and learning may become even more remote from real life than before, because real life may now only enter the classroom through computers, simulations may replace real experiments, computer graphics may serve as substitutes for real objects.
- The use of ready-made software in applied problem solving may put the emphasis on routine and recipe-like modelling, thus neglecting essential activities such as critically analysing and comparing different models, choosing adequate ones, or reflecting upon the meaning and suitability of concepts and results within a mathematical model. To put it more succinctly and more generally: Intellectual efforts and activities of students may be replaced by mere button pressing.
- The case of "computer mathematics" (especially discrete and numerical mathematics) sometimes seems to be overstated. We cannot review here the long debate on discrete vs. continuous mathematics. Yet, in our opinion, "continuous" concepts and results (e.g. the concept of derivative as a local rate of change or the fundamental theorem of calculus) are still highly relevant, conceptually as well as technically speaking, and not the least so for modelling and applications.

And, last but not least:

- More and more mathematics teachers are becoming interested in computers instead of in modelling and applications, and more and more students are being prevented (or rather: like to be prevented) from reflecting on challenging mathematical problems (pure or applied) by being engaged in outward technological problems (which would not exist without computers) or by spending their time on constructing technically elegant programmes. So, the growing interest in computers and their increasingly easy availability in the classroom may, in some cases, even act to the detriment of modelling and applications in mathematics instruction.

We have no easy patent recipe to offer for solving these problems. Perhaps the most important remedy is a very elementary one: teachers *and* students should become fully *aware* of these problems. At the same time this would contribute towards one of the vital goals of mathematics instruction (cf. section I.2), namely the acquisition of critical competence in and "meta-knowledge" of mathematics, its relations to applications and the advantages and risks of its tools.

In the last three sections of this paper, we shall consider *curricular consequences*. That means, let us assume, for a given mathematics programme, that it is decided to make applied problem solving, modelling, applications, or co-operation with other subjects part of the mathematics instruction. What would/should be the consequences of this decision and its implementation for the organization and methods of mathematics instruction, for the spectrum of topics in the curriculum, and for assessment and tests?

III.4. Consequences for Organization and Methods

In slightly modifying continuation of the categorization suggested in the Applications and Modelling Theme Group report of ICME-5 (see Lesh et al. 1986), the following different *types of basic approaches* to including relations to applicational areas in mathematics programmes seem to prevail. For brevity, we exclude from consideration mathematics instruction given entirely within the framework and under the perspectives of another subject, cf. the remarks given at the end of section I.1.

(A1) *The separation approach.* Instead of including modelling and applications work in the ordinary mathematics courses, such activities are cultivated in separate courses specially devoted to them. In this way, the "pure" mathematics courses may remain unaffected by the introduction of modelling and applications work in the programme as a whole.

(A2) *The two-compartment approach.* The mathematics programme is divided into two parts. The first part consists of a usual course in "pure" mathematics, whereas the second one deals with one or more "applied" items, utilizing mathematics established in the first part or earlier.

(A3) *The islands approach.* The mathematics programme is divided into several segments each organized according to the two-compartment approach. This means that a "pure" mathematics programme is interrupted by "islands" of applicational work, drawing on mathematics developed in the preceding period.

In (A2) and (A3), the closer in time and content the relationship is between "pure" mathematics sections and subsequent "applied" sections, the more the latter sections tend to assume the character of being traditional exercises rather than sessions of genuine problem solving, modelling or application activity.

(A4) *The mixing approach.* Frequently in the teaching of mathematics, elements of applications and modelling are invoked to assist the introduction of mathematical concepts etc. Conversely, newly developed mathematical concepts, methods and results are activated towards applicational and modelling situations whenever possible. In this approach, the mathematics to be involved in applications and modelling activities is more or less given from the outset. This is not the case with approach (A5).

(A5) *The mathematics curriculum integrated approach.* Here problems, whether mathematical or applicational, come first and mathematics to deal with them is sought and developed subsequently. In principle the only restriction is that the problems considered lead to mathematics which is relevant to and tractable in the given mathematics curriculum.

(A6) *The interdisciplinary integrated approach.* This approach is largely similar to (A5) but differs from it in that this one operates with a full integration between mathematical and extra-mathematical activities within an interdisciplinary framework where "mathematics" is not organized as a separate subject.

As particularly regards co-operation between mathematics and other specific subjects, the approaches in mathematics instruction sketched above may be accompanied, in the other subjects involved, by parallel instruction on the substance matter dealt with in the mathematics courses. This may require varying degrees of co-ordination between the subjects, ranging from just an initial agreement on the topics to be covered, over current co-ordination, to integrated instruction (e.g. in (A6)).

The decision of which (combination of) approach(es) should be adopted for a given mathematics programme depends on a multitude of factors: the arguments for and the purposes and goals of problem solving, modelling and applications in mathematics instruction, or the characteristics and peculiarities (legal restrictions and other boundary conditions, specific task traditions, resources etc.) of the educational (sub)system under consideration.

Viewed in an international perspective, the general picture of mathematics programmes that have included modelling and applications seems to be, if painted with a broad brush, the following: In *elementary* mathematics instruction in school the islands and mixing approaches are, for obvious reasons, predominant. This is also the case with *secondary* school mathematics, but in a few instances the mathematics curriculum integrated approach can be seen as well. A small number of experimental curricula have adopted the interdisciplinary integrated approach. At *tertiary* level the diversity is larger. In "mathematics as a service subject" programmes, all approaches can be encountered, but probably the two-compartment, the islands, and the mixing approaches are the ones most widely used. Also in courses orientated towards students of mathematical sciences all approaches may occur, but the separation approach and the islands approach seem to prevail. The two-compartment approach is, however, fairly popular too.

In addition to the question of approach, many other questions of organization and methods arise. We shall confine ourselves to only listing a few of them. In view of the arguments, purposes, goals, approaches etc. which apply to the situation given:

- What *working forms* are appropriate to dealing with applied problem solving, modelling, applications, and co-operation between mathematics and other subjects?
- What *balance* should there be between independent student activities (exercised individually or in groups) and activities directed and controlled more tightly by the teacher?
- What *roles* should the *teacher* play in different kinds of applicational and modelling work?
- What *classroom environments* are needed or desirable to support modelling and applications activities?

III.5. Consequences for the Spectrum of Topics

The leading questions for this section are the following: Are new mathematical topics becoming relevant for mathematics curricula as a result of the inclusion of applied problem solving, modelling and applications to other areas? If yes, which? Are there "old" topics which could be left out? If yes, which? Since in this paper we are concerned with mathematics instruction at any educational level, and in all sorts of educational systems exhibiting an abundance of peculiarities and differences, we have to concentrate on rather general matters.

If we look at *primary* and *lower secondary* school mathematics the mathematical topics represented therein are, almost by definition, basic to all mathematical activity, including modelling and applications. Therefore, the selection of topics at those levels is largely unaffected by the strengthening of these components in mathematics instruction. This does not imply that the emphases put on different elements of the topics remain unaltered, but changes – for instance those due to the advent and proliferation of calculators and microcomputers (see section III.3) – are probably not as such related specifically to applications or modelling components as to the development and mastering of elementary mathematical knowledge and skills. Nor does it imply that no new ingredients have been included in the topics already present in the curricula – this is the case, for example, with elements of combinatorics and graphs, fundamentals of descriptive statistics etc. The point is that *no new concept domains* are introduced and developed as such.

Next, we turn to the *upper secondary* and *tertiary* levels (cf. also Pollak, 1989). It is often stated, and rightly so, that there is no such thing as applied mathematics because experience has shown us that all mathematics is actually or potentially applicable, if not sooner then later. Yet, in addition to "classical" calculus, differential equations, numerical analysis, linear algebra, probability and statistics, some *new mathematical topics* have emerged and/or gained momentum in post-elementary mathematics curricula – subject, of course, to large variations between the curricula. Some of these new topics, which give access to new types of applications and modelling, often in "new" applicational areas, are, perhaps, less theoretically demanding than the classical disciplines. What we have in mind are topics like discrete and finite mathematics, difference (and more generally functional) equations, iterations, dynamical systems, chaos, fractals, graphs and networks. Further, optimization in general, and mathematical programming in particular, dynamic programming, and other kinds of operations research too, but also more advanced topics such as stochastic processes, stochastic differential equations, control theory etc. Singular modelling tools like the method of least squares, computer and Monte Carlo simulation etc. are encountered in many programmes as well.

As to *"old" topics*, there seems to be a trend to revive geometry under the perspective of modelling and applications, instead of only viewing it as a laboratory for deductive reasoning or as a bank of computational facts and formulae. In general, it would be wrong to say that certain old topics have become obsolete or irrelevant because of the growing attention paid to modelling and applications. On the other hand, since the total amount of space and time at disposal for mathematics instruction has not been expanded – on the contrary – the inclusion of new aspects and topics in the mathematics programmes has resulted in a *reduction* of the scope left for traditional topics. Rather than directly being removed from the mathematics curricula, certain topics (for instance number theory) have been deprived of some specialities and circumstancialities. This is especially true for topics which used to require much effort to develop cunning formulae to facilitate technical computations, many of which may now be carried out by computers without difficulty. Examples of this are special functions, and special differential equations.

III.6. Consequences for Assessment and Tests

Before dealing with the generic question of this section, "What forms of assessment and tests would be appropriate for a sensible evaluation of modelling and applications activities?", we should remind ourselves that the issue of assessment and tests has different facets to it.

Basically, any kind of assessment and testing in relation to mathematics instruction serves to evaluate students' outcome of the instruction they have received.

Now, if it has been established, for instance through experience, that the knowledge and skills at issue can be taught and learnt, and that the kind of instruction implicated can be successful, i.e. lead to desired and expected results with not too unusual students, assessment and tests serve to evaluate the *students*, by examining the extent to which they have acquired a satisfactory yield from the instruction. If confidence in the value of the instruction under consideration (content, curriculum, teacher, teaching material etc.) has not been established on beforehand – for instance because it is novel – assessment and tests rather serve to evaluate the *instruction*. In both cases assessment and tests are *practice*-oriented.

However, it may also be that it is not established whether a certain kind of knowledge or skills can be taught and learnt at all. To examine whether or not this is possible, and to establish means (forms of assessment and tests) to this end, is a matter which involves research and development. Thus, in this case assessment and tests are *research-and-development*-oriented.

The facet just described deals with the *orientation* of assessment and tests. The second facet is concerned with the *role* of assessment and tests. Is their role to provide the individual student with *information* about the quality of his or her achievements? Or is it to provide a basis for *decisions* or *measures* to be taken in relation to the individual, for instance the verdict "passed" or "failed" (with or without a ranking), sanctions, awards, privileges etc.? In this facet we also include the question of where to place assessment and tests in the curriculum; should they be continuous, occasional, or only final?

The third and last facet to be mentioned here is the *character* of assessment and tests. What we mean by this is whether they are to refer to defined *standards*, whether they are to be carried out in *qualitative* or in *quantitative* (scores, grades, marks) terms, and whether they are to be *relative* (i.e. based on explicit comparison to a larger population of achievements on similar tasks), or *absolute* (i.e. not involving such direct comparison).

How do the considerations just presented specialize in modelling and applications (and, what we do not detail here, in problem solving as well)? In *primary* and *lower secondary* mathematics instruction, assessment and testing of pupils' abilities to utilize mathematics in solving applied problems which are neither open nor too complicated have a long tradition in the school system in most countries. So, in that respect assessment and tests are mainly practice-oriented. If, however, we consider modelling, open applied or more sophisticated mathematical problems, the situation is different. Assessment and tests of ability in these fields have not been widely implemented in primary and lower secondary curricula, but are rather at present objects of research and development.

Broadly speaking, the same holds true for *upper secondary* and *tertiary* mathematics instruction. The question "Is it possible at all to assess and test modelling and applications knowledge and skills in a valid and reliable way? If yes, how?" has received much research and development attention. There seems to be general agreement, not surprisingly, perhaps, amongst those engaged in the field that such knowledge and skills *can* be taught and learnt (provided that explicit attention in instruction is devoted to them), and that it can be reasonably assessed and tested. However, if assessment and tests are to reflect the spirit, content and complexity inherent in modelling and applications, and are to pay respect to the higher order knowledge and skills they involve, it is necessary to use forms of assessment and testing which cannot be formalized or standardized very easily. Maybe we could compare the situation with that in natural language instruction, where the assessment of composition and essays is normally a very complex affair giving rise to much discussion.

Altogether, for post-elementary mathematics instruction, assessment and tests in relation to modelling and applications are at an experimental stage. Criteria, forms and procedures are being devised 'ad hoc', with particular reference to specific programmes and courses, but the overall picture is that very few curricula round the world make substantial modelling and applications abilities the object of systematic assessment and testing. There is no doubt that this constitutes a bottleneck to a widespread integration of these components in mathematics instruction. In an examination-based educational system, as most educational systems are, instructional components which are not tested on a par with other components tend to occupy marginal positions only. Therefore much effort is being invested these years in constructing and establishing means for assessing and testing modelling and application capabilities, which congenially match the essentials of this area, but which also are realistic and reasonably dimensioned in relation to a normal population of students.

In conclusion we may say that currently the role of assessment and tests is to inform students and teachers rather than to provide bases for decisions or measures, and that the character of assessment and testing is mostly qualitative and absolute with no reference to well-defined standards.

NOTE

[1] Written version of a Survey Lecture given jointly at the Sixth International Congress on Mathematical Education (ICME-6, Budapest 1988). A condensation was published in a volume with contributions from the ICME-6 Theme Groups on Problem Solving, Modelling and Applications and on Mathematics and Other Subjects (Blum/Niss/Huntley 1989, pp. 1–21).

REFERENCES

Bell, M.: 1983, 'Materials available worldwide for teaching applications of mathematics at the school level', in M. Zweng et al. (eds.), *Proceedings of the Fourth International Congress on Mathematical Education*, Birkhäuser, Boston, pp. 252–267.

Berry, J. et al. (eds.): 1984, *Teaching and Applying Mathematical Modelling*, Horwood, Chichester.

Berry, J. et al. (eds.): 1986, *Mathematical Modelling Methodology, Models and Micros*, Horwood, Chichester.

Berry, J. et al. (eds.): 1987, *Mathematical Modelling Courses*, Horwood, Chichester.

Binns, B. et al.: 1989, 'Mathematical modelling in the school classroom: Developing effective support for representative teachers', in Blum, W. et al. (eds.), pp. 136–143.

Blechman, I., Myškis, A. and Panovko, J.: 1984, *Angewandte Mathematik*, Deutscher Verlag der Wissenschaften, Berlin-Ost (Original in Russian: И. И Блехман/А. Д. Мышкис/Я. Г. Пановко: Прикладная математика – Предмет. Логика. Особенности подходов. Наукова Думка, Киев 1976.)

Blum, W.: 1985, 'Anwendungsorientierter Mathematikunterricht in der didaktischen Diskussion', *Mathematische Semesterberichte* **32**, 195–232.

Blum, W.: 1988, 'Report on Theme Group 6: Mathematics and Other Subjects', in A. & K. Hirst (eds.), *Proceedings of the Sixth International Congress on Mathematical Education*, János Bolyai Mathematical Society, Budapest, pp. 277–291.

Blum, W. et al. (eds.): 1989, *Applications and Modelling in Learning and Teaching Mathematics*, Horwood, Chichester.

Blum, W., Niss, M. and Huntley, I. (eds.): 1989, *Modelling, Applications and Applied Problem Solving – Teaching Mathematics in a Real Context*, Horwood, Chichester.

Böer, H. and Meyer-Lerch, J.: 1989, 'MUED: The role of Handlungsorientierung in mathematics teaching', in Blum, W. et al. (eds.), pp. 201–206.

Burghes, D., Huntley, I. and McDonald, J.: 1982, *Applying Mathematics – A Course in Mathematical Modelling*, Horwood, Chichester.

Burkhardt, H. (ed.): 1983, *An International Review of Applications in School Mathematics*, ERIC, Ohio.

Burkhardt, H.: 1989, 'Modelling and curriculum change', *Teaching Mathematics and its Applications* **8**, 158–161.

Burkhardt, H. et al. (eds.): 1988, *Problem Solving – A World View*; Proceedings of Problem Solving Theme Group ICME-5, Shell Centre, Nottingham.

Bushaw, D. et al. (eds.): 1980, *A Sourcebook of Applications of School Mathematics*, NCTM, Reston.

Churchhouse, R. et al. (eds.): 1986, *The Influence of Computers and Informatics on Mathematics and its Teaching*, Cambridge University Press, Cambridge.

Clements, R. et al. (eds.): 1988, *Selected Papers on the Teaching of Mathematics as a Service Subject*, Springer, Berlin/Heidelberg/New York.

Cross, M. and Moscardini, A.: 1985, *Learning the Art of Mathematical Modelling*, Horwood, Chichester.

Edwards, D. and Hamson, M.: 1989, *Guide to Mathematical Modelling*, Mcmillan, London.

Fraser, R.: 1989, 'Role of computers for application and modelling: The ITMA approach', in Blum, W. et al. (eds.), pp. 373–380.

Galbraith, P. and Carr, A.: 1987, *Practical Applications of Mathematics – Proposal for an Australian Curriculum Project*, Churchill, Victoria.

Giordano, F. and Weir, M.: 1985, *A First Course in Mathematical Modelling*, Brooks/Cole, Monterey.

Hobbs, D. and Burghes, D.: 1989, 'Enterprising mathematics: A cross-curricular modular course for 14–16 year-olds', in Blum, W. et al. (eds.), pp. 159–165.

Howson, G. and McLone, R. (eds.): 1983, *Maths at Work*, Heinemann, London.

Howson, G. and Wilson, B. (eds.): 1986, *School Mathematics in the 1990s*, Cambridge University Press, Cambridge.

Howson, G. et al. (eds.): 1988, *Mathematics as a Service Subject*, Cambridge Universtiy Press, Cambridge.

Kaiser, G., Blum, W. and Schober, M.: 1982/1990, *Dokumentation ausgewählter Literatur zum anwendungsorientierten Mathematikunterricht*, Vol.1/2, FIZ, Karlsruhe.

Kaiser-Messmer, G.: 1986, *Anwendungen im Mathematikunterricht*, Vol. 1/2, Franzbecker, Bad Salzdetfurth.

Klamkin, M. (ed.): 1987, *Mathematical Modelling – Classroom Notes in Applied Mathematics*, SIAM, Philadelphia.

De Lange, J.: 1987, *Mathematics – Insight and Meaning*, OW & OC, Utrecht.

Lesh, R., Niss, M. and Lee, D.: 1986, *Report on Theme Group 6: Applications and Modelling*, in Carss, M. (ed.), Proceedings of the Fifth International Congress on Mathematical Education, Birkhäuser, Boston, pp. 197–211.

Meyer, W.: 1984, *Concepts of Mathematical Modeling*, McGraw-Hill, New York.

Niss, M.: 1987, 'Applications and modelling in the mathematics curriculum – state and trends', *International Journal for Mathematical Education in Science and Technology* **18**, 487–505.

Niss, M.: 1988, 'Report on Theme Group 3: Problem Solving, Modelling and Applications', in A. & K. Hirst (eds.), *Proceedings of the Sixth International Congress on Mathematical Education*, Budapest, pp. 237–252.

Niss, M.: 1989, 'Aims and scope of applications and modelling in mathematics curricula', in Blum, W. et al. (eds.), pp. 22–31.

Niss, M., Blum, W. and Huntley, I. (eds.): 1990, *Teaching Mathematical Modelling and Applications*, Horwood, Chichester.
Ormell, C.: 1982, *The Applicability of Mathematics*, Maths Applicable Group, Norwich.
Pollak, H.: 1979, 'The interaction between mathematics and other school objects', *New Trends in Mathematics Teaching IV*, (ed: UNESCO), Paris, pp. 232–248.
Pollak, H.: 1989, 'Recent applications of mathematics and their relevance to teaching', in Blum, W. et al. (eds.), pp. 32–36.
Rogerson, A.: 1986, 'The Mathematics in Society Project: A new conception of mathematics', *International Journal for Mathematical Education in Science and Technology* **17**, 611–616.
Schoenfeld, A.: 1985, *Mathematical Problem Solving*, Academic Press, Orlando.
Selkirk, K.: 1982, *Pattern and Place – An Introduction to the Mathematics of Geography*, Cambridge University Press, Cambridge.
Silver, E. (ed.): 1985, *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*, Erlbaum, Hillsdale.
Sloyer, C.: 1989, 'Contemporary ideas in applied mathematics', in Blum, W. et al. (eds.), pp. 258–261.
Steiner, H.-G.: 1976, 'Zur Methodik des mathematisierenden Unterrichts', in Dörfler, W. and Fischer, R. (eds.), *Anwendungsorientierte Mathematik in der Sekundarstufe II*, Heyn, Klagenfurt, pp. 211–245.
Townend, S.: 1984, *Mathematics in Sport*, Horwood, Chichester.
Usiskin, Z.: 1989, 'The sequencing of applications and modelling in the University of Chicago School Mathematics Project 7–12 curriculum', in Blum, W. et al. (eds.), pp. 176–181.
Volk, D.: 1989, 'Mathematics classes and enlightenment', in Blum, W. et al. (eds.), pp. 187–191.
Wan, F.: 1989, *Mathematical Models and Their Analysis*, Harper & Row, New York.

Kassel University
Dept. of Mathematics
Heinrich-Plett-Str. 40
D-3500 Kassel
Federal Republic of Germany

Roskilde University
IMFVFA
Postbox 260
DK-4000 Roskilde
Denmark