Dedicated to André Ronveaux for his 80th birthday

Characterization theorem for classical orthogonal polynomials on non-uniform lattices: The functional approach.

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Abstract

Using the functional approach, we state and prove a characterization theorem for classical orthogonal polynomials on non-uniform lattices (quadratic lattices of a discrete or a q-discrete variable) including the Askey-Wilson polynomials. This theorem proves the equivalence between seven characterization properties, namely the Pearson equation for the linear functional, the second-order divided-difference equation, the orthogonality of the derivatives, the Rodrigues formula, two types of structure relations, and the Riccati equation for the formal Stieltjes function.

Keywords: Classical orthogonal polynomials, Non-uniform lattices, Linear functionals, Divided-difference equations, Riccati equation, Structure relations, Functional approach
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1 Introduction

Classical orthogonal polynomials (in short COP) on a non-uniform lattice are defined as polynomials $(P_n)_n$ with degree $P_n = n$, satisfying one of the following orthogonality relations [4, 5, 8, 13, 24]

$$\int_C P_n(x(s))P_m(x(s))\rho(s)\nabla x_1(s)\,ds = k_n\,\delta_{n,m},\ k_n \neq 0,\ \forall\, n,\, m \in \mathbb{N}; \tag{1}$$

$$\sum_{i=0}^{N} P_n(x(s_i))P_m(x(s_i))\rho(s_i)\nabla x_1(s_i) = k_n\,\delta_{n,m},\ k_n \neq 0,\ \forall\, n,\, m \in \mathbb{N},\ N \in \mathbb{N}\cup\{\infty\}, \tag{2}$$

where $\mathbb{N}$ is the set of nonnegative integers, $s_0 = a,\ s_N = b$. Here, C is an appropriate contour in the complex s-plane, and the weight ρ is a solution of the Pearson-type equation

$$\frac{\Delta}{\nabla x_1(s)}\left(\sigma(s)\rho(s)\right) = \psi(x(s))\,\rho(s), \tag{3}$$

where ψ is a first-degree polynomial and

$$\phi(x(s)) = \sigma(s) + \frac{1}{2}\psi(x(s))\nabla x_1(s) \tag{4}$$

is a polynomial of degree at most two in $x(s)$, with the border conditions

$$\begin{cases} \int_C \Delta\left[\sigma(s)\,\rho(s)\,x^k(s-\frac{1}{2})\right]ds = 0,\ k = 0,\,1,\,2,\ldots \\ \sigma(s)\,\rho(s)\,x^k(s-\frac{1}{2})\big|_{s=a,\,b} = 0,\ k=0,\,1,\,2,\ldots, \end{cases} \tag{5}$$

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for orthogonality relation (1) and (2) respectively. Δ and ∇ are the forward and the backward operators

$$\Delta f(x(s)) := \Delta f(s) = f(s+1) - f(s),\ \nabla f(x(s)) := \nabla f(s) = f(s) - f(s-1),$$

and

$$x_\mu(s) = x(s + \frac{\mu}{2}),\ \mu \in \mathbb{C},$$

where $x(s)$ is a non-uniform lattice satisfying [4, 25]

$$\begin{aligned} x(s+k) - x(s) &= \gamma_k \nabla x_{k+1}(s),\ k \geq 0, && (6)\\ \frac{x(s+k)+x(s)}{2} &= \alpha_k x_k(s) + \beta_k,\ k \geq 0, && (7) \end{aligned}$$

with the sequences $(\alpha_k),\ (\beta_k),\ (\gamma_k)$ satisfying the following relations

$$\begin{aligned} \alpha_{k+1} - 2\,\alpha\,\alpha_k + \alpha_{k-1} &= 0,\\ \beta_{k+1} - 2\,\beta_k + \beta_{k-1} &= 2\,\beta\,\alpha_k,\\ \gamma_{k+1} - \gamma_{k-1} &= 2\,\alpha_k, \end{aligned} \tag{8}$$

and the initial conditions

$$\alpha_0 = 1,\ \alpha_1 = \alpha,\ \beta_0 = 0,\ \beta_1 = \beta, \gamma_0 = 0,\ \gamma_1 = 1. \tag{9}$$

The lattice $x(s)$ is explicitly given by [25]

$$x(s) = \begin{cases} c_1\, q^{-s} + c_2\, q^s + c_3 & \text{for} \quad \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\\ c_4\, s^2 + c_5\, s + c_6 & \text{for} \quad \alpha = 1. \end{cases} \tag{10}$$

Costas-Santos and Marcellán [5], using the Pearson equation (3) for the weight function, gave a characterization theorem for classical orthogonal polynomials on the lattice (10), proving the equivalence between:

1. the second-order divided-difference equation

$$\left\{\sigma(s) \frac{\Delta}{\nabla x_1(s)} \frac{\nabla}{\nabla x(s)} + \psi(x(s)) \frac{\Delta}{\Delta x(s)} + \lambda_n\right\} P_n(x(s)) = 0,\ n \geq 0; \tag{11}$$

2. the orthogonality of the derivatives $(\frac{\Delta P_{n+1}(x(s))}{\Delta x(s)})_n$;

3. the Rodrigues formula

$$P_n(x(s)) = \frac{B_n}{\rho(s)} \frac{\nabla}{\nabla x_1(s)} \cdots \frac{\nabla}{\nabla x_n(s)} \left(\rho_n(s)\right),\ \text{with } \rho_k(s) = \sigma(s+1)\,\rho_{k-1}(s+1),\ \rho_0(s) := \rho(s); \tag{12}$$

4. and the second structure relation

$$\frac{P_n(x(s+1)) + P_n(x(s))}{2} = C_{n,n+1} \frac{\Delta}{\Delta x(s)} P_{n+1}(x(s)) + C_{n,n} \frac{\Delta}{\Delta x(s)} P_n(x(s)) + C_{n,n-1} \frac{\Delta}{\Delta x(s)} P_{n-1}(x(s)), \tag{13}$$

with $C_{n,n-1} \neq 0$.

Koornwinder [14] in 2007 gave a structure relation for classical orthogonal polynomials of the form

$$\mathbb{L}(p_n)(x) = \gamma_n \, A_n \, p_{n+1}(x) - \gamma_{n-1} \, C_n \, p_{n-1}(x), \tag{14}$$

where A_n and B_n are the coefficients of the three-term recurrence relation

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x),$$

while $\mathbb{L}$ is a linear operator acting on the linear space $\mathbb{R}[x]$ of polynomials of the variable x with real coefficients. In addition $\mathbb{L}$ is skew symmetric

$$\langle \mathbb{L}f, g \rangle = -\langle f, \mathbb{L}g \rangle, \ \forall f, g \in \mathbb{R}[x],$$

and satisfies

$$\mathbb{L}(x^n) = \gamma_n \, x^{n+1} + \text{ terms of lower degree},$$

where $\langle ., . \rangle$ is the inner product with respect to which the corresponding polynomial sequence is orthogonal.

For the specific case of Askey-Wilson polynomials, Koornwinder gave the operator $\mathbb{L}$ as

$$(\mathbb{L}(f))[z] = \frac{(1-az)(1-bz)(1-cz)(1-dz)z^{-2}f[qz] - (1-\frac{a}{z})(1-\frac{b}{z})(1-\frac{c}{z})(1-\frac{d}{z})z^2 f[\frac{z}{q}]}{z - z^{-1}}, \tag{15}$$

with the notation $f[z] := f(\frac{z+z^{-1}}{2}) = f(x)$, where $x = \frac{z+z^{-1}}{2}$. More details are given in Section 4.

The aim of this paper is to:

1. state the Pearson-type equation for the linear functional of the corresponding classical orthogonal polynomials, and prove that the Pearson equation for the weight implies the one of the linear functional;

2. state and prove using the functional approach seven equivalent characterization properties for classical orthogonal polynomials: the four properties given by Costas-Santos and Marcellán [5], plus, the Pearson equation for the linear functional, the Rodrigues formula for the linear functional, the first structure relation and the Riccati equation for the formal Stieltjes function;

3. find the link between the structure relation given above by Koornwinder [14] and our second structure relation;

4. connect this work with the pioneering one by Magnus [15] who gave the Riccati equation for the associate Askey-Wilson polynomials.

Since the operator $\mathbb{D}_x$ reduces to the forward operator Δ and the Hahn operator D_q $\left(D_q f(s) = \frac{f(qs)-f(s)}{(q-1)s} \right)$ for the lattices $x(s) = s$ and $x(s) = q^s$ respectively [8], this work generalizes previous ones characterizing classical orthogonal polynomials by means of the above mentioned seven equivalent properties. Among these, we would like to mention [1, 2, 17] for COP of a continuous variable, [12] for COP of a discrete variable, [21, 20, 3] for COP of a q-discrete variable and [5, 14] for COP on a non-uniform lattice.

In Section 2, we recall known results and link the Pearson equation for the weight with the one of the linear functional. Section 3 deals with the characterization theorem while the last section provides some important connections and perspectives.

2 Known Results and Pearson-type Equations

2.1 Properties of the Companion Operators $\mathbb{D}_x$ and $\mathbb{S}_x$

By means of the companion operators $\mathbb{D}_x$ and $\mathbb{S}_x$ [8]

$$\mathbb{D}_x\, f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}, \;\; \mathbb{S}_x\, f(x(s)) = \frac{f(x(s+\frac{1}{2})) + f(x(s-\frac{1}{2}))}{2}, \tag{16}$$

Equation (11) can be rewritten as [7, 8]

$$\phi(x(s))\, \mathbb{D}_x^2 P_n(x(s)) + \psi(x(s))\, \mathbb{S}_x \mathbb{D}_x P_n(x(s)) + \lambda_n\, P_n(x(s)) = 0, \tag{17}$$

where

$$\lambda_n = -\gamma_n\, (\phi_2\, \gamma_{n-1} + \psi_1\, \alpha_{n-1}). \tag{18}$$

The operators $\mathbb{D}_x$ and $\mathbb{S}_x$, which transform a polynomial of degree n in the variable $x(s)$ into a polynomial of degree $n-1$ and n respectively in $x(s)$, fulfill important relations—which read, taking into account the shift (compared to the definition in [8]) in the definition of the above defined companion operators, as

Theorem 1 *[8]*

1. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ satisfy the product rules I*

$$\begin{aligned} \mathbb{D}_x\,(f(x(s))g(x(s))) &= \mathbb{S}_x f(x(s))\, \mathbb{D}_x g(x(s)) + \mathbb{D}_x f(x(s))\, \mathbb{S}_x g(x(s)), && (19)\\ \mathbb{S}_x\,(f(x(s))g(x(s))) &= U_2(x(s))\, \mathbb{D}_x f(x(s))\, \mathbb{D}_x g(x(s)) + \mathbb{S}_x f(x(s))\, \mathbb{S}_x g(x(s)), && (20) \end{aligned}$$

where U_2 is a polynomial of degree 2

$$U_2(x(s)) = (\alpha^2 - 1)\, x^2(s) + 2\,\beta\,(\alpha+1)\, x(s) + \delta_x, \tag{21}$$

and δ_x is a constant depending on α, β and the initial values $x(0)$ and $x(1)$ of $x(s)$:

$$\delta_x = \frac{x^2(0) + x^2(1)}{4\alpha^2} - \frac{(2\alpha^2-1)}{2\alpha^2} x(0)\, x(1) - \frac{\beta\,(\alpha+1)}{\alpha^2}(x(0)+x(1)) + \frac{\beta^2\,(\alpha+1)^2}{\alpha^2}. \tag{22}$$

2. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ satisfy the quotient rules*

$$\begin{aligned} \mathbb{D}_x\left(\frac{f(x(s))}{g(x(s))}\right) &= \frac{\mathbb{S}_x f(x(s))\, \mathbb{D}_x g(x(s)) - \mathbb{D}_x f(x(s))\, \mathbb{S}_x g(x(s))}{U_2(x(s))\, [\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2}; && (23)\\ \mathbb{S}_x\left(\frac{f(x(s))}{g(x(s))}\right) &= \frac{U_2(x(s))\, \mathbb{D}_x f(x(s))\, \mathbb{D}_x g(x(s)) - \mathbb{S}_x f(x(s))\, \mathbb{S}_x g(x(s))}{U_2(x(s))\, [\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2}, && (24) \end{aligned}$$

provided that $g(x(s)) \neq 0$.

3. *More generally, relations (19)-(20) and (23)-(24) remain valid if we replace x and x_1 by x_μ and $x_{\mu+1}$ respectively, $\mu \in \mathbb{C}$. In particular, the constant δ_x remains unchanged if we replace x in (22) by $x_k,\ k \in \mathbb{Z}$, i.e.,*

$$\delta_{x_k} = \delta_x := \delta,\ k \in \mathbb{Z}. \tag{25}$$

4. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ also satisfy the product rules II*

$$\mathbb{D}_x\, \mathbb{S}_x = \alpha\, \mathbb{S}_x\, \mathbb{D}_x + U_1(s)\, \mathbb{D}_x^2; \tag{26}$$

$$\mathbb{S}_x^2 = U_1(s)\, \mathbb{S}_x\, \mathbb{D}_x + \alpha\, U_2(s)\, \mathbb{D}_x^2 + \mathbb{I}, \tag{27}$$

where $\mathbb{I}$ is the identity operator $\mathbb{I} f(x) = f(x)$, and

$$U_1(s) := U_1(x(s)) = (\alpha^2 - 1)\, x(s) + \beta\,(\alpha+1), \;\; U_2(s) := U_2(x(s)). \tag{28}$$

2.2 Properties of the Basis F_n

Looking for suitable bases for the companion operators, Foupouagnigni, Kenfack, Koepf, and Mboutngam [9] proved the following:

Theorem 2 *[9]*
The polynomial sequence

$$F_n(x(s)) = F_n(x(s), x(z_x)), \text{ with } F_n(x(s), x(z)) = \prod_{j=1}^{n} [x(s) - x_j(z)], \tag{29}$$

where z_x is the unique solution (provided that the lattice $x(s)$ is quadratic or q-quadratic: i.e. the constants c_j in (10) satisfy $c_1\, c_2 \neq 0$ or $c_4 \neq 0$) in the variable t of the equation

$$x_1(t) = x(t),$$

fulfills the following relations

$$\mathbb{D}_x F_n(x(s)) = \gamma_n\, F_{n-1}(x(s)), \tag{30}$$

$$\mathbb{S}_x F_n(x(s)) = \alpha_n\, F_n(x(s)) + \frac{\gamma_n}{2}\, \nabla x_{n+1}(z_x) F_{n-1}(x(s)), \tag{31}$$

$$\mathbb{D}_x \frac{1}{F_n(x(s))} = -\frac{\gamma_n}{F_{n+1}(x(s))}, \tag{32}$$

$$\mathbb{S}_x \frac{1}{F_n(x(s))} = \frac{\alpha_n}{F_n(x(s))} + \frac{\gamma_n}{2} \frac{\nabla x_{n+2}(z_x)}{F_{n+1}(x(s))}, \tag{33}$$

where α_n, β_n and γ_n are defined in (8).

One straightforward corollary of the previous theorems is the following:

Corollary 1 *The coefficients α_n and γ_n fulfill the following relations*

$$\alpha_{n+1} = \alpha\alpha_n + (\alpha^2 - 1)\gamma_n,\ \gamma_{n+1} = \alpha_n + \alpha\gamma_n, \tag{34}$$

from which one deduces after some computations involving basic linear algebra that

$$\alpha_n = 1,\ \gamma_n = n,\ \text{for } \alpha = 1, \tag{35}$$

and

$$\alpha_n = \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{2},\ \gamma_n = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},\ \text{for } \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}. \tag{36}$$

Proof: This can easily be deduced by applying the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ to both sides of the following relation deduced from (29)

$$F_{n+1}(x(s)) = (x(s) - x_{n+1}(z_x))F_n(x(s)),$$

and using the product rules (19) and (20). □

Next, by considering, **instead of the well-known Stieltjes function S_0 of the functional $\mathcal{L}$**

$$S_0\, [\mathcal{L}]\, (z) = \sum_{n=0}^{\infty} \frac{\langle \mathcal{L}, x^n(s) \rangle}{x^{n+1}(z)}, \tag{37}$$

but rather its new representation [9] in terms of the appropriate basis F_n

$$S\, [\mathcal{L}]\, (z) = \sum_{n=0}^{\infty} \frac{\langle \mathcal{L}, F_n(x(s)) \rangle}{F_{n+1}(x(z))}, \tag{38}$$

and using results of the previous theorems, the following result is proved in [9].

Theorem 3 *[9]*

$$S\left[\mathbb{D}_x\mathcal{L}\right](s) = \mathbb{D}_x\left[S(\mathcal{L})\right](s), \tag{39}$$

$$S\left[\mathbb{S}_x\mathcal{L}\right](s) = \alpha\,\mathbb{S}_x S\left[\mathcal{L}\right](s) + U_1\mathbb{D}_x S\left[\mathcal{L}\right](s). \tag{40}$$

Here, for a given $Q \in \mathbb{C}[x(s)]$, $Q\,\mathcal{L}$, $\mathbb{D}_x\mathcal{L}$ and $\mathbb{S}_x\mathcal{L}$ are linear functionals defined as

$$\langle Q\mathcal{L}, P\rangle = \langle \mathcal{L}, QP\rangle,\ \ \langle \mathbb{D}_x\mathcal{L}, P\rangle = -\langle \mathcal{L}, \mathbb{D}_x P\rangle,\ \ \langle \mathbb{S}_x\mathcal{L}, P\rangle = \langle \mathcal{L}, \mathbb{S}_x P\rangle,\ \forall P \in \mathbb{C}[x(s)]. \tag{41}$$

Remark 1

1. *The functions $S_0\left[\mathcal{L}\right]$ and $S\left[\mathcal{L}\right]$ defined respectively by Equations (37) and (38) are equal. In fact, if the corresponding polynomial family is orthogonal with respect to a given positive measure $\mu(x)$, then the function S_0 reads*
$$S_0\left[\mathcal{L}\right](x(z)) = \int_{Supp.\,\mu} \frac{d\mu(x(s))}{x(z)-x(s)}.$$
The latter expression is equal to (38) by means of the relation (see [9], Corollary 7, page 6)
$$\frac{1}{x(z)-x(s)} = \sum_{k=0}^{\infty} \frac{F_k(x(s))}{F_{k+1}(x(z))},\ z \neq s.$$

2. *The expression $S(f\mathcal{L})$ can be evaluated using the well-known relation by Maroni [18]*
$$S\left[f\mathcal{L}\right](x) = f(x)\,S\left[\mathcal{L}\right](x) + (\mathcal{L}\theta_0 f)(x),\ f \in \mathbb{C}[x], \tag{42}$$
with
$$\theta_0 f(x) = \frac{f(x)-f(0)}{x},$$
where the product of the functional $\mathcal{L}$ by a polynomial g, $\mathcal{L}g$, is defined as
$$\mathcal{L}g(x(s)) = \sum_{k=0}^{n} g_k \sum_{j=0}^{k} \langle \mathcal{L}, x^j(s)\rangle x^{k-j}(s),\ \ \text{with } g(x(s)) = \sum_{k=0}^{n} g_k\, x^k(s),\ n \geq 0. \tag{43}$$

2.3 Pearson-type Equations

Let $(P_n)_n$ be a family of COP on a non-uniform lattice. It is well-known that this polynomial sequence satisfies [4, 5, 8] Equation (17). By assuming that $\mathcal{L}$ is the corresponding regular linear functional

$$\langle \mathcal{L}, P_n P_m\rangle = k_n \delta_{n,m},\ k_n \neq 0,\ \forall n, m \geq 0, \tag{44}$$

we obtain using (17) and (41)

$$\begin{aligned} 0 &= \langle \mathcal{L}, 0\rangle \\ &= \langle \mathcal{L}, \phi(x(s))\,\mathbb{D}_x^2 P_{n+1}(x(s)) + \psi(x(s))\mathbb{S}_x\mathbb{D}_x P_{n+1}(x(s)) + \lambda_{n+1}P_{n+1}(x(s))\rangle,\ \forall n \geq 0, \\ &= \langle \mathbb{D}_x(\phi\mathcal{L}) - \mathbb{S}_x(\psi\mathcal{L}), \mathbb{D}_x P_{n+1}(x(s))\rangle,\ \forall n \geq 0. \end{aligned} \tag{45}$$

Since $\deg(\mathbb{D}_x P_{n+1}) = n,\ n \geq 0$, the sequence $(\mathbb{D}_x P_{n+1}(x(s)))_{n\geq 0}$ forms a basis of $\mathbb{C}[x(s)]$, therefore

$$\mathbb{D}_x(\phi\mathcal{L}) = \mathbb{S}_x(\psi\mathcal{L}). \tag{46}$$

Definition 1
We call (46) the **Pearson equation** *for the linear functional $\mathcal{L}$ corresponding to the COP $(P_n)_n$ satisfying (17).*

Proposition 1
Let $(P_n)_n$ be a polynomial family, orthogonal with respect to the weight function ρ satisfying the Pearson equation (3) and the border conditions (5). Then, the linear functional $\mathcal{U}$ defined on $\mathbb{C}[x(s)]$ by

$$\langle \mathcal{U}, P\rangle = \int_C \rho(s) P(x(s))\, \nabla x_1(s)\, ds, \tag{47}$$

for the orthogonality relation (1), where C is an appropriate contour in the complex s-plane, or by

$$\langle \mathcal{U}, P\rangle = \sum_{i=0}^{N} P(x(s_i))\, \rho(s_i) \nabla x_1(s_i),\ N \in \mathbb{N} \cup \{+\infty\}, \tag{48}$$

for the orthogonality relation (2), satisfies the Pearson equation (46).

Proof: The proof uses the following relations obtained by direct computation taking into account the definitions of $\mathbb{D}_x$ and $\mathbb{S}_x$

$$\frac{\Delta}{\nabla x_1(s)} \left[f(s-\frac{1}{2}) \right] = \mathbb{D}_x f(s), \tag{49}$$

$$\mathbb{D}_x(f(s)g(s)) = f(s+\frac{1}{2})\mathbb{D}_x g(s) + g(s-\frac{1}{2})\mathbb{D}_x f(s). \tag{50}$$

In the first step, computations using (4), (41) and (47) for $P \in \mathbb{C}[x(s)]$ give

$$\begin{aligned}
&\langle \mathbb{D}_x(\phi\mathcal{U}) - \mathbb{S}_x(\psi\mathcal{U}), P\rangle = -\langle \mathcal{U}, \phi\, \mathbb{D}_x P + \psi \mathbb{S}_x P\rangle \\
=& -\int_C \rho(s) \left[\left(\sigma(s) + \frac{1}{2}\psi(x(s))\, \nabla x_1(s) \right) \mathbb{D}_x P(x(s)) + \psi(x(s)) \mathbb{S}_x P(x(s)) \right] \nabla x_1(s)\, ds, \\
=& -\int_C \rho(s)\sigma(s)\, \mathbb{D}_x P(x(s))\, \nabla x_1(s)\, ds \\
& -\int_C \psi(x(s))\rho(s) \left[\frac{1}{2}\nabla x_1(s)\, \mathbb{D}_x P(x(s)) + \mathbb{S}_x P(x(s)) \right] \nabla x_1(s)\, ds.
\end{aligned} \tag{51}$$

In the second step, we use (50) for $f(s) = P(x(s))$ and $g(s) = \sigma(s+\frac{1}{2})\rho(s+\frac{1}{2})$ and the relation

$$\frac{1}{2}\nabla x_1(s)\, \mathbb{D}_x P(x(s)) + \mathbb{S}_x P(x(s)) = P(x(s+\frac{1}{2})),$$

which is easily deduced by direct computation, to transform (51) into

$$\begin{aligned}
\langle \mathbb{D}_x(\phi\mathcal{U}) - \mathbb{S}_x(\psi\mathcal{U}), P\rangle =& -\int_C \mathbb{D}_x \left[\sigma(s+\frac{1}{2})\rho(s+\frac{1}{2}) P(x(s)) \right] \nabla x_1(s)\, ds \\
&+ \int_C P(x(s+\frac{1}{2}))\mathbb{D}_x \left[\sigma(s+\frac{1}{2})\rho(s+\frac{1}{2}) \right] \nabla x_1(s)\, ds \\
&- \int_C \psi(x(s))\rho(s) P(x(s+\frac{1}{2}))\, \nabla x_1(s)\, ds.
\end{aligned}$$

In the third step we use the relation

$$\mathbb{D}_x \left(\sigma(s+\frac{1}{2})\rho(s+\frac{1}{2}) \right) = \psi(x(s))\rho(s)$$

which by means of (49) is equivalent to the Pearson equation (3), and the border conditions (5) to get

$$\langle \mathbb{D}_x(\phi\mathcal{U}) - \mathbb{S}_x(\psi\mathcal{U}), P\rangle = 0.$$

The proof is similar if the linear functional $\mathcal{U}$ is represented by (48). □

3 Characterization Theorem

In this section, we first state and prove the following propositions, which are used to give the proof of **the main results of this paper, stated in Theorem 4**.

Proposition 2 *The following relations hold for every linear functional $\mathcal{L}$ and for all polynomials f and g.*

$$\begin{aligned}
\mathbb{D}_x\mathbb{S}_x f &= \frac{1}{\alpha}\mathbb{D}_x\left(U_1(s)\mathbb{D}_x f\right) + \frac{1}{\alpha}\mathbb{S}_x\mathbb{D}_x f; && (52)\\
\mathbb{S}_x^2 f &= \frac{1}{\alpha}\mathbb{S}_x\left(U_1(s)\mathbb{D}_x f\right) + \frac{1}{\alpha}U_2(s)\mathbb{D}_x^2 f + f; && (53)\\
f\mathbb{D}_x g &= \mathbb{D}_x\left[\left(\mathbb{S}_x f - \frac{U_1(x)}{\alpha}\mathbb{D}_x f\right)g\right] - \frac{1}{\alpha}\mathbb{S}_x(g\mathbb{D}_x f); && (54)\\
f\mathbb{S}_x g &= \mathbb{S}_x\left[\left(\mathbb{S}_x f - \frac{U_1(x)}{\alpha}\mathbb{D}_x f\right)g\right] - \frac{U_2(x)}{\alpha}\mathbb{D}_x(g\mathbb{D}_x f); && (55)\\
\left(\mathbb{D}_x^2 f\right)g &= \mathbb{D}_x\left[\mathbb{D}_x f\mathbb{S}_x g - \mathbb{S}_x f\mathbb{D}_x g\right] + \left(\mathbb{D}_x^2 g\right)f && (56)\\
\left(\mathbb{S}_x\mathbb{D}_x f\right)g &= \mathbb{S}_x\left[\mathbb{D}_x f\mathbb{S}_x g - \mathbb{S}_x f\mathbb{D}_x g\right] + \left(\mathbb{S}_x\mathbb{D}_x g\right)f. && (57)
\end{aligned}$$

Proof: The proof of the first four relations is obtained by direct computation, starting from the right-hand side using relations (19), (20), (26), (27) and the following ones linking U_1 and U_2

$$\mathbb{S}_x(U_1(x(s)) = \alpha\, U_1(x(s),\ \mathbb{D}_x(U_2(x(s)) = 2\,\alpha\, U_1(x(s),\ \mathbb{D}_x U_1(x(s)) = \alpha^2 - 1. \qquad (58)$$

Relations (56) and (57) are obtained by direct computation, starting from the right-hand side using relations (19), (20), (26), (27). □

Proposition 3
The following relations hold for every linear functional $\mathcal{L}$ and for all polynomials f, g, ϕ and ψ.

$$\begin{aligned}
\mathbb{D}_x(\phi\mathcal{L}) &= \mathbb{S}_x(\psi\mathcal{L}) \Longrightarrow \langle\mathcal{L}, (\phi\,\mathbb{D}_x^2 f + \psi\mathbb{S}_x\mathbb{D}_x f)g\rangle = \langle\mathcal{L}, (\phi\,\mathbb{D}_x^2 g + \psi\mathbb{S}_x\mathbb{D}_x g)f\rangle; && (59)\\
\mathbb{D}_x\left(f\mathcal{L}\right) &= \left(\mathbb{S}_x f - \frac{U_1(s)}{\alpha}\mathbb{D}_x f\right)\mathbb{D}_x\mathcal{L} + \frac{1}{\alpha}\mathbb{D}_x f\mathbb{S}_x\mathcal{L}; && (60)\\
\mathbb{S}_x\left(f\mathcal{L}\right) &= \left(\mathbb{S}_x f - \frac{U_1(s)}{\alpha}\mathbb{D}_x f\right)\mathbb{S}_x\mathcal{L} + \frac{1}{\alpha}\mathbb{D}_x f\mathbb{D}_x(U_2\mathcal{L}); && (61)\\
f\mathbb{D}_x\mathcal{L} &= \mathbb{D}_x\left(\mathbb{S}_x f\ \mathcal{L}\right) - \mathbb{S}_x\left(\mathbb{D}_x f\ \mathcal{L}\right); && (62)\\
f\mathbb{S}_x\mathcal{L} &= \mathbb{S}_x\left(\mathbb{S}_x f\ \mathcal{L}\right) - \mathbb{D}_x\left(U_2\mathbb{D}_x f\ \mathcal{L}\right). && (63)
\end{aligned}$$

Proof: Relation (59) is obtained by a straightforward application of (41), (56) and (57). Finally, Relations (60) and (61) are easily deduced from (54) and (55) respectively; while (62) and (63) are direct consequences of Relations (19) and (20).

Since the polynomial sequence $(Q_{n,m})_{n\geq 0}$ fulfills $\deg(Q_{n,m}) = n$, $\forall n\in\mathbb{N}$, there exists [19] a sequence of linear functionals $\left(\hat{Q}_{n,m}\right)_{n\geq 0}$ called dual basis of $(Q_{n,m})_{n\geq 0}$ satisfying

$$\langle\hat{Q}_{n,m}, Q_{j,m}\rangle = \delta_{n,j},\ n,\ j\geq 0. \qquad (64)$$

Also, every linear functional $\mathcal{L}$ can be represented as [19]

$$\mathcal{L} = \sum_{n=0}^{\infty}\langle\mathcal{L}, Q_{n,m}\rangle\,\hat{Q}_{n,m}. \qquad (65)$$

In addition, if $(P_n)_n$ is a polynomial sequence orthogonal with respect to the linear functional $\mathcal{L}$, then its dual basis $\hat{P}_n$ is given by [12]

$$\hat{P}_n = \frac{P_n \mathcal{L}}{\langle \mathcal{L}, P_n P_n \rangle}, \; n \geq 0. \tag{66}$$

The derivative of the dual basis $\hat{Q}_{n,m}$ fulfills

Proposition 4

$$\mathbb{D}_x \widehat{Q}_{n,m} = -\gamma_{n+1} \widehat{Q}_{n+1,m-1} \;\; \forall n \geq 0, \; \forall m \geq 1. \tag{67}$$

Proof: Using the following relation easily deduced from (71)-(72),

$$Q_{n,m} = \frac{1}{\gamma_{n+1}} \mathbb{D}_x Q_{n+1,m-1}, \; \forall n \geq 0, \; m \geq 1, \tag{68}$$

we obtain for fixed integers $n \geq 0$ and $m \geq 1$,

$$\begin{aligned} \langle \mathbb{D}_x \widehat{Q}_{n,m}, Q_{j+1,m-1} \rangle &= -\langle \widehat{Q}_{n,m}, \mathbb{D}_x Q_{j+1,m-1} \rangle, \; \forall j \geq 0 \\ &= -\langle \widehat{Q}_{n,m}, \gamma_{j+1} Q_{j,m} \rangle, \; \forall j \geq 0 \\ &= -\gamma_{n+1} \delta_{n,j}, \; \forall j \geq 0 \\ &= -\gamma_{n+1} \langle \widehat{Q}_{n+1,m-1}, Q_{j+1,m-1} \rangle, \; \forall j \geq 0. \end{aligned}$$

In addition,

$$\langle \mathbb{D}_x \widehat{Q}_{n,m}, Q_{0,m-1} \rangle = -\langle \widehat{Q}_{n,m}, \mathbb{D}_x Q_{0,m-1} \rangle = 0 = -\gamma_{n+1} \langle \widehat{Q}_{n+1,m-1}, Q_{0,m-1} \rangle.$$

Therefore,

$$\mathbb{D}_x \widehat{Q}_{n,m} = -\gamma_{n+1} \widehat{Q}_{n+1,m-1}.$$

□

Proposition 5 *Let $\mathcal{L}$ be a regular linear functional satisfying the Pearson equation*

$$\mathbb{D}_x(\phi \mathcal{L}) = \mathbb{S}_x(\psi \mathcal{L}),$$

where ϕ is a polynomial of degree at most 2 and ψ a first degree polynomial. Then, we have

$$\phi_2 \, \gamma_n + \psi_1 \alpha_n \neq 0, \; \forall \, n \geq 0, \tag{69}$$

where ϕ_2 and ψ_1 are the leading coefficients of the polynomials ϕ and ψ with respect to the basis $(x^n(s))_n$.

In addition, for any polynomial P_n of degree n in $x(s)$, we have

$$\deg \left(\phi(x(s)) \mathbb{D}_x^2 P_n(x(s)) + \psi(x(s)) \mathbb{S}_x \mathbb{D}_x P_n(x(s)) \right) = n, \; \forall n \geq 1. \tag{70}$$

Proof: Application of both sides of the Pearson equation to the polynomial F_n yields the following difference equation for the moments $\hat{\mu}_n = \langle \mathcal{L}, F_n \rangle$

$$(\phi_2 \gamma_n + \psi_1 \alpha_n) \hat{\mu}_{n+1} = u_n \, \hat{\mu}_n + v_n \hat{\mu}_{n-1},$$

where u_n and v_n depend on n and the coefficients of the polynomials ϕ and ψ. For all the moments to exist, property (69) is necessary. Relation (70) is easily deduced from (69) since if we write $P_n(x(s)) = a_n F_n(x(s)) + \ldots, \; a_n \neq 0$, then we have

$$\phi(x(s)) \mathbb{D}_x^2 P_n(x(s)) + \psi(x(s)) \mathbb{S}_x \mathbb{D}_x P_n(x(s)) = a_n \gamma_n (\phi_2 \gamma_{n-1} + \psi_1 \alpha_{n-1}) F_n(x(s)) + \ldots,$$

with $a_n \gamma_n (\phi_2 \gamma_{n-1} + \psi_1 \alpha_{n-1}) \neq 0$, $\forall n \geq 1$. □

Theorem 4

Let $\mathcal{L}$ be a regular linear functional, $(P_n)_n$ its corresponding monic orthogonal polynomials and $Q_{n,m}$ the monic polynomial of degree n defined by

$$B_{n,m}Q_{n,m} = \mathbb{D}_x^m P_{n+m},\ m,\, n \geq 0, \tag{71}$$

with

$$B_{n,m} = \prod_{j=0}^{m-1} \gamma_{n+m-j} = \frac{\gamma_{n+m}!}{\gamma_n!},\quad Q_{n,0} \equiv P_n. \tag{72}$$

The following properties are equivalent:

(a) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that

$$\mathbb{D}_x(\phi\mathcal{L}) = \mathbb{S}_x(\psi\mathcal{L}). \tag{73}$$

(b) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer $m \geq 0$,

$$\mathbb{D}_x(\phi^{(m)}\mathcal{L}_m) = \mathbb{S}_x(\psi^{(m)}\mathcal{L}_m), \tag{74}$$

$$\langle \mathcal{L}_m, Q_{n,m}\, Q_{j,m}\rangle = k_n\delta_{j,n},\ k_n \neq 0, \forall n, j \in \mathbb{N}, \tag{75}$$

where the linear functional $\mathcal{L}_m$ and the polynomials $\phi^{(m)}$ and $\psi^{(m)}$ are defined respectively by

$$\phi^{(m+1)} = \mathbb{S}_x\phi^{(m)} + U_1\,\mathbb{S}_x\psi^{(m)} + \alpha U_2\,\mathbb{D}_x\psi^{(m)},\quad \phi^{(0)} \equiv \phi, \tag{76}$$

$$\psi^{(m+1)} = \mathbb{D}_x\phi^{(m)} + \alpha\,\mathbb{S}_x\psi^{(m)} + U_1\,\mathbb{D}_x\psi^{(m)},\quad \psi^{(0)} \equiv \psi, \tag{77}$$

$$\mathcal{L}_{m+1} = \mathbb{D}_x\left[U_2\psi^{(m)}\mathcal{L}_m\right] - \mathbb{S}_x\left[\phi^{(m)}\mathcal{L}_m\right],\quad \mathcal{L}_0 \equiv \mathcal{L}, \tag{78}$$

with the polynomials U_2 and U_1 given respectively by (21) and (28).

(c) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer $m \geq 0$ the following second-order difference equation holds:

$$\phi^{(m)}(x(s))\mathbb{D}_x^2 Q_{n,m}(x(s)) + \psi^{(m)}(x(s))\mathbb{S}_x\mathbb{D}_x Q_{n,m}(x(s)) + \lambda_{n,m}Q_{n,m}(x(s)) = 0,\ \forall n \geq 0, \tag{79}$$

where the polynomials $\phi^{(m)}$ and $\psi^{(m)}$ are given by (76), (77) and the constant

$$\lambda_{n,m} = -\gamma_n\left\{\phi_2^{(m)}\gamma_{n-1} + \psi_1^{(m)}\alpha_{n-1}\right\} \tag{80}$$

with

$$\phi^{(m)}(x(s)) = \phi_2^{(m)}\,x^2(s) + \phi_1^{(m)}\,x(s) + \phi_0^{(m)},\ \psi^{(m)}(x(s)) = \psi_1^{(m)}\,x(s) + \psi_0^{(m)}, \tag{81}$$

where the polynomials $\phi^{(m)}$ and $\psi^{(m)}$ are defined in (76)-(77).

(d) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer $m \geq 0$ the following Rodrigues relation holds:

$$\gamma_n\mathbb{D}_x\left(Q_{n-1,m+1}\mathcal{L}_{m+1}\right) = \alpha\lambda_{n,m}Q_{n,m}\mathcal{L}_m,\ \forall n \geq 1, \tag{82}$$

where $\mathcal{L}_m$ is defined by Equations (76)-(78), and $\lambda_{n,m}$ defined by (80), with the initial condition

$$\langle \mathcal{L}, \psi\rangle = 0. \tag{83}$$

(e) *There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer $m \geq 0$, there exist three sequences $(a^m_{n,n+1})_n$, $(a^m_{n,n})_n$ and $(a^m_{n,n-1})_n$, such that the so-called first structure relation is satisfied:*

$$\psi^{(m)}\mathbb{S}_x^2 Q_{n,m} + \phi^{(m)}\mathbb{D}_x\mathbb{S}_x Q_{n,m} = a^m_{n,n+1}Q_{n+1,m} + a^m_{n,n}Q_{n,m} + a^m_{n,n-1}Q_{n-1,m},\ \forall n \geq 1, \quad (84)$$

with $a^m_{n,n-1} \neq 0$ for $n > 2$, where the polynomials $\phi^{(m)}$ and $\psi^{(m)}$ are defined in (76)-(77).

(f) *For any integer $m \geq 0$, there exist three sequences $(b^m_{n,n+1})_n$, $(b^m_{n,n})_n$ and $(b^m_{n,n-1})_n$, such that the following relation, called second structure relation, is satisfied:*

$$\mathbb{S}_x Q_{n,m} = b^m_{n,n+1}\mathbb{D}_x Q_{n+1,m} + b^m_{n,n}\mathbb{D}_x Q_{n,m} + b^m_{n,n-1}\mathbb{D}_x Q_{n-1,m},\ \forall n \geq 1, \quad (85)$$

with $b^m_{n,n-1} \neq 0$ for $n > 2$.

(g) *There exist three polynomials, A, B and C of degree at most two, one and zero respectively such that the following Riccati equation for the formal Stieltjes function $S(\mathcal{L}) := S$ of the linear functional $\mathcal{L}$ is satisfied*

$$A(x(s))\mathbb{D}_x(S(\mathcal{L})) = B(x(s))\mathbb{S}_x(S(\mathcal{L})) + C. \quad (86)$$

Proof: **Proof of Theorem 1**

We organize the proof in the following scheme:

Step 1: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$ which is equivalent to $(a) \Leftrightarrow (b) \Leftrightarrow (c)$.

Step 2: $(b) + (c) \Rightarrow (d) \Rightarrow (a)$ which taking into account Step 1, is equivalent to $(c) \Leftrightarrow (d)$.

Step 3: $(a) + (b) + (c) \Rightarrow (f) \Rightarrow (a)$ which using Step 1, is equivalent to $(a) \Leftrightarrow (f)$.

Step 4: $(c) + (f) \Rightarrow (e) \Rightarrow (a)$ which thanks to Step 3 is equivalent to $(e) \Leftrightarrow (f)$.

Step 5: $(a) \Leftrightarrow (g)$.

Step 1: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$

Step 1.1 $(a) \Rightarrow (b)$.

We assume that the property (a) is satisfied and we show by induction on m that (74) and (75) are satisfied for $m \geq 0$. ¿From the Pearson equation (73) and the orthogonality relation (44) for the family (P_n), it is obvious that the relations (74) and (75) are satisfied for $m = 0$. Assume that they are satisfied up to a fixed integer $m > 0$. Firstly, we use Relation (68) and the definition of $\mathcal{L}_{m+1}$ given by (78) to get for $0 \leq j \leq n$,

$$\begin{aligned}
&\langle \mathcal{L}_{m+1}, Q_{n,m+1}\, Q_{j,m+1}\rangle = -\langle \mathcal{L}_m, \psi^{(m)}U_2\mathbb{D}_x\left(Q_{n,m+1}Q_{j,m+1}\right) + \phi^{(m)}\mathbb{S}_x\left(Q_{n,m+1}Q_{j,m+1}\right)\rangle \\
= \quad & -\frac{1}{\gamma_{n+1}\gamma_{j+1}}\langle \psi^{(m)}\mathcal{L}_m, U_2\mathbb{D}_x\left(\mathbb{D}_x Q_{n+1,m}\mathbb{D}_x Q_{j+1,m}\right)\rangle + \langle \phi^{(m)}\mathcal{L}_m, \mathbb{S}_x\left(\mathbb{D}_x Q_{n+1,m}\mathbb{D}_x Q_{j+1,m}\right)\rangle.
\end{aligned}$$

Secondly, we use (54) and (55) for $f = Q_{n+1,m}$ and $g = \mathbb{D}_x Q_{j+1,m}$

$$\begin{aligned}
\mathbb{S}_x(\mathbb{D}_x Q_{j+1,m}\mathbb{D}_x Q_{n+1,m}) &= -\alpha Q_{n+1,m}\mathbb{D}_x^2 Q_{j+1,m} \\
&\quad +\alpha\mathbb{D}_x\left[\left(\mathbb{S}_x Q_{n+1,m} - \frac{U_1(x)}{\alpha}\mathbb{D}_x Q_{n+1,m}\right)\mathbb{D}_x Q_{j+1,m}\right], \\
U_2(x)\mathbb{D}_x(\mathbb{D}_x Q_{j+1,m}\mathbb{D}_x Q_{n+1,m}) &= -\alpha Q_{n+1,m}\mathbb{S}_x\mathbb{D}_x Q_{j+1,m} \\
&\quad +\alpha\mathbb{S}_x\left[\left(\mathbb{S}_x Q_{n+1,m} - \frac{U_1(x)}{\alpha}\mathbb{D}_x Q_{n+1,m}\right)\mathbb{D}_x Q_{j+1,m}\right],
\end{aligned}$$

to obtain after making use of the Pearson equation (74)

$$\langle \mathcal{L}_{m+1}, Q_{n,m+1}\, Q_{j,m+1}\rangle = \frac{\alpha}{\gamma_{n+1}\gamma_{j+1}}\langle \mathcal{L}_m, Q_{n+1,m}\left(\phi^{(m)}\mathbb{D}_x^2 Q_{j+1,m} + \psi^{(m)}\mathbb{S}_x\mathbb{D}_x Q_{j+1,m}\right)\rangle.$$

Combination of (74), (75) and Proposition 5 lead to

$$\begin{aligned}\langle \mathcal{L}_{m+1}, Q_{n,m+1}\, Q_{j,m+1}\rangle &= \frac{\alpha(\gamma_j \phi_2^{(m)} + \alpha_j \psi_1^{(m)})}{\gamma_{n+1}} \langle \mathcal{L}_m, Q^2_{n+1,m}\rangle \delta_{j,n},\ 0 \le j \le n \\ &\neq 0,\ \text{for } j = n,\end{aligned}$$

thanks to Proposition 5 with $\mathcal{L}$ replaced by $\mathcal{L}_m$.

Next, we show that (74) is satisfied at order $m+1$, using mainly the fact it is satisfied at order m. Let P be a polynomial of degree at least 1. We have

$$\begin{aligned} I &= \langle -\mathbb{D}_x(\phi^{(m+1)}\mathcal{L}_{m+1}) + \mathbb{S}_x(\psi^{(m+1)}\mathcal{L}_{m+1}), \mathbb{D}_x P\rangle \\ &= \langle \mathcal{L}_{m+1}, \left(\phi^{(m+1)}\mathbb{D}_x^2 P + \psi^{(m+1)}\mathbb{S}_x\mathbb{D}_x P)\right)\rangle \\ &= \langle \mathbb{D}_x(U_2\psi^{(m)}\mathcal{L}_m) - \mathbb{S}_x(\phi^{(m)}\mathcal{L}_m), \left(\phi^{(m+1)}\mathbb{D}_x^2 P + \psi^{(m+1)}\mathbb{S}_x\mathbb{D}_x P)\right)\rangle. \end{aligned}$$

Use of relations (52) and (53) for $f = P$, the previous relation becomes:

$$\begin{aligned} I &= \langle \psi^{(m)}\mathcal{L}_m, -U_2\mathbb{D}_x\left[\phi^{(m+1)}\mathbb{D}_x^2 P + \psi^{(m+1)}\left[\alpha\mathbb{D}_x\mathbb{S}_x P - \mathbb{D}_x(U_1\mathbb{D}_x P)\right]\right] \\ &\quad + \langle \phi^{(m)}\mathcal{L}_m, -\mathbb{S}_x\left[\phi^{(m+1)}\mathbb{D}_x^2 P + \psi^{(m+1)}\left[\alpha\mathbb{D}_x\mathbb{S}_x P - \mathbb{D}_x(U_1\mathbb{D}_x P)\right]\right]\rangle \\ &= \langle \psi^{(m)}\mathcal{L}_m, -U_2\mathbb{D}_x\left[\phi^{(m+1)}\,\mathbb{D}_x^2 P\right]\rangle + \langle \psi^{(m)}\mathcal{L}_m, -U_2\mathbb{D}_x\left[\psi^{(m+1)}\mathbb{D}_x\left[\alpha\mathbb{S}_x P - U_1\mathbb{D}_x P\right]\right] \\ &\quad + \langle \phi^{(m)}\mathcal{L}_m, -\mathbb{S}_x\left[\phi^{(m+1)}\,\mathbb{D}_x^2 P\right]\rangle + \langle \phi^{(m)}\mathcal{L}_m, -\mathbb{S}_x\left[\psi^{(m+1)}\mathbb{D}_x\left[\alpha\mathbb{S}_x P - U_1\mathbb{D}_x P\right]\right]\rangle. \end{aligned}$$

Using (54) and (55), first for $f = \mathbb{D}_x P$, $g = \phi^{(m+1)}$, then again for $f = \alpha\mathbb{S}_x P - U_1\mathbb{D}_x P$, $g = \psi^{(m+1)}$, we obtain after making use of the Pearson equation (74):

$$\begin{aligned} I &= \langle \psi^{(m)}\mathcal{L}_m, \mathbb{D}_x P\,\mathbb{S}_x\phi^{(m+1)} + (\alpha\mathbb{S}_x P - U_1\mathbb{D}_x P)\mathbb{S}_x\psi^{(m+1)}\rangle \\ &\quad + \langle \phi^{(m)}\mathcal{L}_m, \mathbb{D}_x P\,\mathbb{D}_x\phi^{(m+1)} + (\alpha\mathbb{S}_x P - U_1\mathbb{D}_x P)\mathbb{D}_x\psi^{(m+1)}\rangle. \end{aligned} \tag{87}$$

By remarking that

$$\mathbb{S}_x U_1 = \alpha U_1,\ \mathbb{D}_x U_2 = 2\alpha U_1,\ \mathbb{D}_x U_1 = \alpha^2 - 1,\ \mathbb{S}_x U_2 = \alpha^2 U_2 + U_1^2,\ \mathbb{D}_x^2\psi^{(m)} = 0,$$

we get after some computation using (19), (20), (26), (27), (76) and (77)

$$\begin{aligned} \mathbb{S}_x\phi^{(m+1)} &= U_1\mathbb{S}_x\mathbb{D}_x\phi^{(m)} + \alpha U_2\mathbb{D}_x^2\phi^{(m)} + \phi^{(m)} + \alpha U_1\psi^{(m)} \\ &\quad + \left[2\alpha U_1^2 + \alpha(2\alpha^2 - 1)U_2\right]\mathbb{S}_x\mathbb{D}_x\psi^{(m)}, \\ \mathbb{S}_x\psi^{(m+1)} &= \mathbb{S}_x\mathbb{D}_x\phi^{(m)} + 2\alpha U_1\mathbb{S}_x\mathbb{D}_x\psi^{(m)} + \alpha\psi^{(m)}, \\ \mathbb{D}_x\phi^{(m+1)} &= \alpha\mathbb{S}_x\mathbb{D}_x\phi^{(m)} + U_1\mathbb{D}_x^2\phi^{(m)} + (4\alpha^2 - 1)U_1\mathbb{S}_x\mathbb{D}_x\psi^{(m)} + (\alpha^2 - 1)\psi^{(m)}, \\ \mathbb{D}_x\psi^{(m+1)} &= \mathbb{D}_x^2\phi^{(m)} + (2\alpha^2 - 1)\mathbb{S}_x\mathbb{D}_x\psi^{(m)}. \end{aligned} \tag{88}$$

Subtituting (88) into (87) we get after using the product rules (19) and (20)

$$\begin{aligned} I &= \langle \psi^{(m)}\mathcal{L}_m, \mathbb{D}_x P\left(\alpha U_2\mathbb{D}_x^2\phi^{(m)} + \alpha(2\alpha^2 - 1)U_2\mathbb{S}_x\mathbb{D}_x\psi^{(m)} + \phi^{(m)}\right)\rangle \\ &\quad + \langle \psi^{(m)}\mathcal{L}_m, \alpha\mathbb{S}_x P\left(\mathbb{S}_x\mathbb{D}_x\phi^{(m)} + 2\alpha U_1\mathbb{S}_x\mathbb{D}_x\psi^{(m)} + \alpha\psi^{(m)}\right)\rangle \\ &\quad + \langle \phi^{(m)}\mathcal{L}_m, \mathbb{D}_x P\left(\alpha\mathbb{S}_x\mathbb{D}_x\phi^{(m)} + 2\alpha^2 U_1\mathbb{S}_x\mathbb{D}_x\psi^{(m)} + \alpha^2\psi^{(m)} - \psi^{(m)}\right)\rangle \\ &\quad + \langle \phi^{(m)}\mathcal{L}_m, \alpha\mathbb{S}_x P\left(\mathbb{D}_x^2\phi^{(m)} + (2\alpha^2 - 1)\mathbb{S}_x\mathbb{D}_x\psi^{(m)}\right)\rangle \\ &= \langle \psi^{(m)}\mathcal{L}_m, \alpha\left(\mathbb{S}_x P\mathbb{S}_x\mathbb{D}_x\phi^{(m)} + U_2\mathbb{D}_x P\mathbb{D}_x^2\phi^{(m)}\right) + \alpha(2\alpha^2 - 1)U_2\mathbb{D}_x P\mathbb{S}_x\mathbb{D}_x\psi^{(m)}\rangle \end{aligned}$$

$$
\begin{aligned}
&+\langle \psi^{(m)}\mathcal{L}_m, \alpha \mathbb{S}_x P\left(2\alpha U_1 \mathbb{S}_x \mathbb{D}_x \psi^{(m)} + \alpha\psi^{(m)}\right)\rangle \\
&+\langle \phi^{(m)}\mathcal{L}_m, \alpha\left(\mathbb{S}_x P \mathbb{D}_x^2 \phi^{(m)} + \mathbb{D}_x P \mathbb{S}_x \mathbb{D}_x \phi^{(m)}\right) + \alpha \mathbb{D}_x P\left(2\alpha U_1 \mathbb{S}_x \mathbb{D}_x \psi^{(m)} + \alpha\psi^{(m)}\right)\rangle \\
&+\langle \phi^{(m)}\mathcal{L}_m, \alpha(2\alpha^2-1)\mathbb{S}_x P \mathbb{S}_x \mathbb{D}_x \psi^{(m)}\rangle \\
= \; &\langle \psi^{(m)}\mathcal{L}_m, \alpha \mathbb{S}_x(P\mathbb{D}_x\phi^{(m)}) + \alpha(2\alpha^2-1)U_2 \mathbb{D}_x P \mathbb{S}_x \mathbb{D}_x \psi^{(m)} + \alpha \mathbb{S}_x P\left(2\alpha U_1 \mathbb{S}_x \mathbb{D}_x \psi^{(m)} + \alpha\psi^{(m)}\right)\rangle \\
&+\langle \phi^{(m)}\mathcal{L}_m, \alpha \mathbb{D}_x(P\mathbb{D}_x\phi^{(m)}) + \alpha \mathbb{D}_x P\left(2\alpha U_1 \mathbb{S}_x \mathbb{D}_x \psi^{(m)} + \alpha\psi^{(m)}\right) + \alpha(2\alpha^2-1)\mathbb{S}_x P \mathbb{S}_x \mathbb{D}_x \psi^{(m)}\rangle.
\end{aligned}
$$

Use of the Pearson equation (74) allows to get from the previous equation:

$$
\begin{aligned}
I \; = \; &\langle \psi^{(m)}\mathcal{L}_m, \alpha(2\alpha^2-1)U_2 \mathbb{D}_x P \mathbb{S}_x \mathbb{D}_x \psi^{(m)} + \alpha \mathbb{S}_x P\left(2\alpha U_1 \mathbb{S}_x \mathbb{D}_x \psi^{(m)} + \alpha\psi^{(m)}\right)\rangle \\
&+\langle \phi^{(m)}\mathcal{L}_m, \alpha \mathbb{D}_x P\left(2\alpha U_1 \mathbb{S}_x \mathbb{D}_x \psi^{(m)} + \alpha\psi^{(m)}\right) + \alpha(2\alpha^2-1)\mathbb{S}_x P \mathbb{S}_x \mathbb{D}_x \psi^{(m)}\rangle.
\end{aligned} \tag{89}
$$

Taking into account the following relations which can easily be deduced from (19), (20), (26) and (27),

$$
\begin{aligned}
\mathbb{S}_x\left(U_1 \mathbb{D}_x \psi^{(m)} + \alpha \mathbb{S}_x \psi^{(m)}\right) &= 2\alpha U_1 \mathbb{S}_x \mathbb{D}_x \psi^{(m)} + \alpha\psi^{(m)}, \\
\mathbb{D}_x\left(U_1 \mathbb{D}_x \psi^{(m)} + \alpha \mathbb{S}_x \psi^{(m)}\right) &= (2\alpha^2-1)\mathbb{S}_x \mathbb{D}_x \psi^{(m)},
\end{aligned}
$$

we get from the Equation (89, after using again the Pearson equation at the order m:

$$
\begin{aligned}
I \; = \; &\alpha\langle \psi^{(m)}\mathcal{L}_m, U_2 \mathbb{D}_x P \mathbb{D}_x(\alpha \mathbb{S}_x \psi^{(m)} + U_1 \mathbb{D}_x \psi^{(m)}) + \mathbb{S}_x P \mathbb{S}_x(\alpha \mathbb{S}_x \psi^{(m)} + U_1 \mathbb{D}_x \psi^{(m)})\rangle \\
&\alpha\langle \phi^{(m)}\mathcal{L}_m, \mathbb{D}_x P \mathbb{S}_x(\alpha \mathbb{S}_x \psi^{(m)} + U_1 \mathbb{D}_x \psi^{(m)}) + \mathbb{S}_x P \mathbb{D}_x(\alpha \mathbb{S}_x \psi^{(m)} + U_1 \mathbb{D}_x \psi^{(m)})\rangle. \\
= \; &\alpha\langle \psi^{(m)}\mathcal{L}_m, \mathbb{S}_x\left(P(U_1 \mathbb{D}_x \psi^{(m)} + \alpha \mathbb{S}_x \psi^{(m)})\right)\rangle + \alpha\langle \phi^{(m)}\mathcal{L}_m, \mathbb{D}_x\left(P(U_1 \mathbb{D}_x \psi^{(m)} + \alpha \mathbb{S}_x \psi^{(m)})\right)\rangle \\
= \; &0.
\end{aligned}
$$

Therefore,

$$
\mathbb{D}_x(\phi^{(m+1)}\mathcal{L}_{m+1}) = \mathbb{S}_x(\psi^{(m+1)}\mathcal{L}_{m+1}).
$$

Step 1.2 $(b) \Rightarrow (c)$.
We assume (b) and fix two nonnegative integers n and m. Then from the following expansion

$$
\phi^{(m)}\mathbb{D}_x^2 Q_{n,m} + \psi^{(m)}\mathbb{S}_x \mathbb{D}_x Q_{n,m} = \sum_{j=0}^{n} a_{n,j} Q_{j,m} \tag{90}
$$

we deduce for $0 \leq k \leq n$

$$
a_{n,k}\langle \mathcal{L}_m, Q_{k,m}^2\rangle = \langle \mathcal{L}_m, \left(\phi^{(m)}\mathbb{D}_x^2 Q_{n,m} + \psi^{(m)}\mathbb{S}_x \mathbb{D}_x Q_{n,m}\right) Q_{k,m}\rangle.
$$

Next, taking into account (74), we use the property (59) for $\phi = \phi^{(m)}$, $\psi = \psi^{(m)}$, $f = Q_{n,m}$ and $g = Q_{k,m}$ to obtain

$$
a_{n,k}\langle \mathcal{L}_m, Q_{k,m}^2\rangle = \langle \mathcal{L}_m, \left(\phi^{(m)}\mathbb{D}_x^2 Q_{k,m} + \psi^{(m)}\mathbb{S}_x \mathbb{D}_x Q_{k,m}\right) Q_{n,m}\rangle.
$$

Therefore, since $\phi^{(m)}\mathbb{D}_x^2 Q_{k,m} + \psi^{(m)}\mathbb{S}_x \mathbb{D}_x Q_{k,m}$ is a polynomial of degree at most k, we get

$$
a_{n,k} = 0, \text{ for } k < n.
$$

Finally, we write in (90) $Q_{n,m} = F_n(x(s)) + \text{ lower terms }$ and identify the coefficient of F_n on both sides of (90) using relations (30) and (31) to get $a_{n,m} = -\lambda_{n,m}$ given by (80).

Step 1.3 $(c) \Rightarrow (a)$.
We assume Property (c) and obtain for fixed $n \geq 0$ and $m = 1$ after taking into account (79)

$$\begin{aligned}\langle \mathbb{D}_x(\phi\mathcal{L}) - \mathbb{S}_x(\phi\mathcal{L}), Q_{n,1}\rangle &= \frac{1}{\gamma_{n+1}}\langle \mathbb{D}_x(\phi\mathcal{L}) - \mathbb{S}_x(\phi\mathcal{L}), \mathbb{D}_x P_{n+1}\rangle \\ &= \frac{-1}{\gamma_{n+1}}\langle \mathcal{L}, \phi\mathbb{D}_x^2 P_{n+1} + \psi\mathbb{S}_x\mathbb{D}_x P_{n+1}\rangle \\ &= \frac{\lambda_{n+1,0}}{\gamma_{n+1}}\langle \mathcal{L}, P_{n+1}\rangle = 0.\end{aligned}$$

Since $(Q_{n,1})_n$ forms a basis of $\mathbb{C}[x]$, we deduce that $\mathbb{D}_x(\phi\mathcal{L}) = \mathbb{S}_x(\psi\mathcal{L})$.
Step 2: $(c) \Leftrightarrow (d)$.
Step 2.1 $(b) + (c) \Rightarrow (d)$.
We assume property (c). Since we have established the above equivalence between properties (a), (b) and (c), we can then in addition make use of property (b). Let $P \in \mathbb{C}[x]$. Using Relations (78) and (68), we obtain for fixed integers $n \geq 1$ and $m \geq 0$

$$\begin{aligned}\langle \mathbb{D}_x\left(Q_{n-1,m+1}\mathcal{L}_{m+1}\right), P\rangle &= -\langle Q_{n-1,m+1}\mathcal{L}_{m+1}, \mathbb{D}_x P\rangle \\ &= -\langle \mathcal{L}_{m+1}, Q_{n-1,m+1}\mathbb{D}_x P\rangle \\ &= -\langle \mathbb{D}_x\left(U_2\psi^{(m)}\mathcal{L}_m\right) - \mathbb{S}_x\left(\phi^{(m)}\mathcal{L}_m\right), \frac{\mathbb{D}_x Q_{n,m}}{\gamma_n}\mathbb{D}_x P\rangle \\ &= \frac{1}{\gamma_n}\langle \mathcal{L}_m, \psi^{(m)}U_2\mathbb{D}_x\left(\mathbb{D}_x Q_{n,m}\mathbb{D}_x P\right) + \phi^{(m)}\mathbb{S}_x\left(\mathbb{D}_x Q_{n,m}\mathbb{D}_x P\right)\rangle.\end{aligned}$$

Use of the relations (54) and (55) for $f = P$ and $g = \mathbb{D}_x Q_{n,m}$

$$\begin{aligned}U_2\mathbb{D}_x\left(\mathbb{D}_x Q_{n,m}\mathbb{D}_x P\right) &= -\alpha P\,\mathbb{S}_x\mathbb{D}_x Q_{n,m} + \alpha\mathbb{S}_x\left[\left(\mathbb{S}_x P - \frac{U_1}{\alpha}\mathbb{D}_x P\right)\mathbb{D}_x Q_{n,m}\right], \\ \mathbb{S}_x\left(\mathbb{D}_x Q_{n,m}\mathbb{D}_x P\right) &= -\alpha P\,\mathbb{D}_x^2 Q_{n,m} + \alpha\mathbb{D}_x\left[\left(\mathbb{S}_x P - \frac{U_1}{\alpha}\mathbb{D}_x P\right)\mathbb{D}_x Q_{n,m}\right]\end{aligned}$$

together with the Pearson equation for $\mathcal{L}_m$, namely (74), transform the previous equation into

$$\langle \gamma_n\mathbb{D}_x\left(Q_{n-1,m+1}\mathcal{L}_{m+1}\right), P\rangle = -\alpha\langle \mathcal{L}_m, \left(\phi^{(m)}\mathbb{D}_x^2 Q_{n,m} + \psi^{(m)}\mathbb{S}_x\mathbb{D}_x Q_{n,m}\right)P\rangle.$$

By means of (79), the latter equation reads

$$\langle \gamma_n\mathbb{D}_x\left(Q_{n-1,m+1}\mathcal{L}_{m+1}\right), P\rangle = \alpha\lambda_{n,m}\langle Q_{n,m}\mathcal{L}_m, P\rangle.$$

Thus, we have

$$\gamma_n\mathbb{D}_x\left(Q_{n-1,m+1}\mathcal{L}_{m+1}\right) = \alpha\lambda_{n,m}Q_{n,m}\mathcal{L}_m.$$

Equation (79) for $n = 1$ and $m = 0$ gives $\psi + \lambda_1\,P_1 = 0$. Therefore, $\langle \mathcal{L}, \psi\rangle = -\lambda_1\langle \mathcal{L}, P_1\rangle = 0$.
Step 2.2 $(d) \Rightarrow (c)$.
We assume that the property (d) is satisfied. Since $(a) \Longleftrightarrow (b) \Longleftrightarrow (c)$, We will show that (a) is satisfied.

First, we use Equation (82) for $n = 1$ and $m = 0$, and get taking care that $\gamma_1 = 1$, the equation

$$\mathbb{D}_x(\mathcal{L}_1) = \alpha\lambda_1 P_1\mathcal{L},$$

which is equivalent to

$$\langle \psi\mathcal{L}, U_2\mathbb{D}_x^2 F_n\rangle + \langle \phi\mathcal{L}, \mathbb{S}_x\mathbb{D}_x F_n\rangle - \langle \mathcal{L}, \alpha\lambda_1 P_1 F_n\rangle = 0,\ \forall n \geq 0. \tag{91}$$

Next, we use the relations (52) and (53) for $f = F_n$

$$\begin{aligned}
\mathbb{S}_x\mathbb{D}_xF_n &= \alpha\mathbb{D}_x\mathbb{S}_xF_n - \mathbb{D}_x\left(U_1\mathbb{D}_xF_n\right), \\
U_2\mathbb{D}_x^2F_n &= \alpha\mathbb{S}_x^2F_n - \mathbb{S}_x\left(U_1\mathbb{D}_xF_n\right) - \alpha F_n,
\end{aligned}$$

in Equation (91) to get

$$\langle \mathbb{S}_x\left(\psi\mathcal{L}\right) - \mathbb{D}_x\left(\phi\mathcal{L}\right), \alpha\mathbb{S}_xF_n - U_1\mathbb{D}_xF_n\rangle - \alpha\langle\mathcal{L}, \left(\psi + \lambda_1P_1\right)F_n\rangle = 0.$$

Because $\psi + \lambda_1P_1 = 0$, we then have:

$$\langle \mathbb{S}_x\left(\psi\mathcal{L}\right) - \mathbb{D}_x\left(\phi\mathcal{L}\right), \alpha\mathbb{S}_xF_n - U_1\mathbb{D}_xF_n\rangle = 0.$$

Since $\alpha\mathbb{S}_xF_n - U_1\mathbb{D}_xF_n = \big(\alpha\alpha_n - (\alpha^2-1)\gamma_n\big)F_n +$ lower terms and $\alpha\alpha_n - (\alpha^2-1)\gamma_n \neq 0,\ \forall n \geq 0$, $\left(\alpha\mathbb{S}_xF_n - U_1\mathbb{D}_xF_n\right)_n$ forms a basis of $\mathbb{C}[x]$. We therefore deduce that

$$\mathbb{S}_x\left(\psi\mathcal{L}\right) - \mathbb{D}_x\left(\phi\mathcal{L}\right) = 0.$$

Step 3: $(a) + (b) + (c) \Rightarrow (f) \Rightarrow (a)$.
Step 3.1 $(a) + (b) + (c) \Rightarrow (f)$

We assume (a) and therefore, (b) and (c). Expansion of the polynomial $\mathbb{S}_xQ_{n,m}$ in the basis $(Q_{j,m+1})_{j\geq 0}$

$$\mathbb{S}_xQ_{n,m} = \sum_{k=0}^{n} b_{n,k}\,Q_{k,m+1}$$

yields

$$b_{n,j}\langle\mathcal{L}_{m+1}, Q_{j,m+1}^2\rangle = \langle\mathcal{L}_{m+1}, \left[\mathbb{S}_xQ_{n,m}\right]\,Q_{j,m+1}\rangle,\ 0 \leq j \leq n. \tag{92}$$

First we use the second-order divided-difference Equation (79) for $Q_{j,m+1},\ 1 \leq j \leq n$, next the product rules (19)-(20), then the Pearson equation for $\mathcal{L}_{m+1}$ (74) and finally Equation (68), and take into account the fact that thanks to Proposition 5, $\lambda_{n,m} \neq 0,$ for $n \geq 1$, to get

$$\begin{aligned}
b_{n,j}\langle\mathcal{L}_{m+1}, Q_{j,m}^2\rangle &= \frac{-1}{\lambda_{j,m+1}}\langle\mathcal{L}_{m+1}, \mathbb{S}_xQ_{n,m}\left\{\phi^{(m+1)}\mathbb{D}_x^2Q_{j,m+1} + \psi^{(m+1)}\mathbb{S}_x\mathbb{D}_xQ_{j,m+1}\right\}\rangle \\
&= \frac{-1}{\lambda_{j,m+1}}\langle\phi^{(m+1)}\mathcal{L}_{m+1}, \mathbb{D}_x\left(Q_{n,m}\mathbb{D}_xQ_{j,m+1}\right) - \mathbb{D}_xQ_{n,m}\mathbb{S}_x\mathbb{D}_xQ_{j,m+1}\rangle \\
&\quad + \frac{-1}{\lambda_{j,m+1}}\langle\psi^{(m+1)}\mathcal{L}_{m+1}, \mathbb{S}_x\left(Q_{n,m}\mathbb{D}_xQ_{j,m+1}\right) - U_2\mathbb{D}_xQ_{n,m}\mathbb{D}_x^2Q_{j,m+1}\rangle \\
&= \frac{\gamma_n}{\lambda_{j,m+1}}\langle\mathcal{L}_{m+1}, Q_{n-1,m+1}\left(\phi^{(m+1)}\mathbb{S}_x\mathbb{D}_xQ_{j,m+1} + U_2\psi^{(m+1)}\mathbb{D}_x^2Q_{j,m+1}\right)\rangle.
\end{aligned}$$

Since $\phi^{(m+1)}\mathbb{S}_x\mathbb{D}_xQ_{j,m+1} + U_2\psi^{(m+1)}\mathbb{D}_x^2Q_{j,m+1}$ is of degree at most $j+1$, we use the orthogonality of $(Q_{n,m+1})_n$ with respect to $\mathcal{L}_{m+1}$ to deduce that the previous expression vanishes for $j+1 < n-1$. Therefore, $b_{n,j} = 0,\ 0 < j < n-2$.

For $j = n-2$ with $n > 2$, we get using the first line of the previous equation together with Property (59) for $\mathcal{L} = \mathcal{L}_{m+1},\ \phi = \phi_{m+1},\ \psi = \psi_{m+1},\ f = Q_{n-2,m+1}$ and $g = \mathbb{S}_xQ_{n,m}$

$$\begin{aligned}
b_{n,n-2}\langle\mathcal{L}_{m+1}, Q_{n-2,m}^2\rangle &= \frac{-1}{\lambda_{n-2,m+1}}\langle\mathcal{L}_{m+1}, \mathbb{S}_xQ_{n,m}\left\{\phi^{(m+1)}\mathbb{D}_x^2Q_{n-2,m+1} + \psi^{(m+1)}\mathbb{S}_x\mathbb{D}_xQ_{n-2,m+1}\right\}\rangle \\
&= \frac{-1}{\lambda_{n-2,m+1}}\langle\mathcal{L}_{m+1}, Q_{n-2,m+1}\left\{\phi^{(m+1)}\mathbb{D}_x^2\mathbb{S}_xQ_{n,m} + \psi^{(m+1)}\mathbb{S}_x\mathbb{D}_x\mathbb{S}_xQ_{n,m}\right\}\rangle \\
&= \frac{-\alpha_n\gamma_n\left(\gamma_{n-1}\phi_2^{(m)} + \alpha_{n-1}\psi_1^{(m)}\right)}{\lambda_{n-2,m+1}}\langle\mathcal{L}_{m+1}, Q_{n-2,m+1}^2\rangle.
\end{aligned}$$

Therefore, we conclude by means of Proposition 5 with $\mathcal{L}$ replaced by $\mathcal{L}_{m+1}$, that $b_{n,n-2} \neq 0,\ n > 2$.

For $j = 0$, since $\lambda_{0,m} = 0$, we cannot use the previous method. Instead, we use Relation (78) to transform (92) into

$$\begin{aligned} b_{n,0}\langle \mathcal{L}_{m+1}, Q^2_{0,m+1}\rangle &= \langle \mathcal{L}_{m+1}, \mathbb{S}_x Q_{n,m}\rangle \\ &= \langle \mathbb{D}_x(U_2\psi^{(m)}\mathcal{L}_m) - \mathbb{S}_x(\phi^{(m)}\mathcal{L}_m), \mathbb{S}_x Q_{n,m}\rangle \\ &= -\langle U_2\psi^{(m)}\mathcal{L}_m, \mathbb{D}_x\mathbb{S}_x Q_{n,m}\rangle - \langle \phi^{(m)}\mathcal{L}_m, \mathbb{S}_x^2 Q_{n,m}\rangle. \end{aligned}$$

Next, we use Relations (52) and (53) for $f = Q_{n,m}$

$$\begin{aligned} \mathbb{D}_x\mathbb{S}_x Q_{n,m} &= \frac{1}{\alpha}\mathbb{D}_x\left(U_1(s)\mathbb{D}_x Q_{n,m}\right) + \frac{1}{\alpha}\mathbb{S}_x\mathbb{D}_x Q_{n,m}; \\ \mathbb{S}_x^2 Q_{n,m} &= \frac{1}{\alpha}\mathbb{S}_x\left(U_1(s)\mathbb{D}_x Q_{n,m}\right) + \frac{1}{\alpha}U_2(s)\mathbb{D}_x^2 Q_{n,m} + Q_{n,m}, \end{aligned}$$

and Relation (78) again to obtain

$$\begin{aligned} b_{n,0}\langle \mathcal{L}_{m+1}, Q^2_{0,m+1}\rangle &= \frac{1}{\alpha}\langle \mathcal{L}_{m+1}, U_1\mathbb{D}_x Q_{n,m}\rangle \\ &- \frac{1}{\alpha}\langle \mathcal{L}_m, U_2(\phi^{(m)}\mathbb{D}_x^2 Q_{n,m} + \psi^{(m)}\mathbb{S}_x\mathbb{D}_x Q_{n,m}) + \phi^{(m)}Q_{n,m}\rangle. \end{aligned}$$

Next we use Relation (68) and the Property (59) for $\phi = \phi^{(m)}$, $\psi = \psi^{(m)}$, $f = Q_{n,m}$ and $g = U_2$ to transform the previous equation into

$$\begin{aligned} b_{n,0}\langle \mathcal{L}_{m+1}, Q^2_{0,m+1}\rangle &= \frac{\gamma_n}{\alpha}\langle \mathcal{L}_{m+1}, U_1 Q_{n-1,m+1}\rangle \\ &- \frac{1}{\alpha}\langle \mathcal{L}_m, Q_{n,m}(\phi^{(m)}\mathbb{D}_x^2 U_2 + \psi^{(m)}\mathbb{S}_x\mathbb{D}_x U_2)\rangle + \frac{1}{\alpha}\langle \mathcal{L}_m, \phi^{(m)}Q_{n,m}\rangle. \end{aligned}$$

Therefore, $b_{n,0} = 0$ for $n > 2$ thanks to the orthogonality of $(Q_{n,m})_n$ and $(Q_{n,m+1})_n$ with respect to $\mathcal{L}_m$ and $\mathcal{L}_{m+1}$ respectively. Hence,

$$b_{n,j} = 0,\ 0 \leq j \leq n-3,$$

and

$$\begin{aligned} \mathbb{S}_x Q_{n,m} &= b_{n,n}Q_{n,m+1} + b_{n,n-1}Q_{n-1,m+1} + b_{n,n-2}Q_{n-2,m+1},\ \forall n \geq 1, \\ &= \frac{b_{n,n}}{\gamma_{n+1}}\mathbb{D}_x Q_{n+1,m} + \frac{b_{n,n-1}}{\gamma_n}\mathbb{D}_x Q_{n,m} + \frac{b_{n,n-2}}{\gamma_{n-1}}\mathbb{D}_x Q_{n-1,m},\ \forall n \geq 1, \\ &= b^m_{n,n+1}\mathbb{D}_x Q_{n+1,m} + b^m_{n,n}\mathbb{D}_x Q_{n,m} + b^m_{n,n-1}\mathbb{D}_x Q_{n-1,m},\ \forall n \geq 1, \end{aligned}$$

where

$$b^m_{n,n+j} = \frac{b_{n,n+j-1}}{\gamma_{n+j}},\ -1 \leq j \leq 1, \tag{93}$$

with

$$b^m_{n,n-1} = \frac{b_{n,n-2}}{\gamma_{n-1}} \neq 0,\ n > 2.$$

Step 3.2 $(f) \Rightarrow (a)$

We assume Property (f), and denote by $(\hat{Q}_{n,m})_n$ the dual basis associated to $(Q_{n,m})_n$. Then, expansion of the linear functional $\alpha\mathbb{S}_x\hat{Q}_{0,1} - \mathbb{D}_x(U_1\hat{Q}_{0,1})$ in the dual basis $(\hat{Q}_{n,0})_n = (\hat{P}_n)_n$ of $(P_n)_n$

$$\alpha\mathbb{S}_x\hat{Q}_{0,1} - \mathbb{D}_x(U_1\hat{Q}_{0,1}) = \sum_{k\geq 0} c_k\hat{P}_k$$

yields after using (28)

$$c_j = \langle \alpha \mathbb{S}_x \hat{Q}_{0,1} - \mathbb{D}_x(U_1 \hat{Q}_{0,1}), P_j \rangle = \langle \hat{Q}_{0,1}, \alpha \mathbb{S}_x P_j + U_1 \mathbb{D}_x P_j \rangle. \tag{94}$$

Application of $\mathbb{D}_x$ on both sides of the recurrence relation for the orthogonal family $(P_n)_n$

$$x(s)P_j = c_{j,j+1}P_{j+1} + c_{j,j}P_j + c_{j,j-1}P_{j-1},$$

and use of the product rule (19) give

$$\alpha x(s)\mathbb{D}_x P_j + \mathbb{S}_x P_j = c_{j,j+1}\mathbb{D}_x P_{j+1} + c_{j,j}\mathbb{D}_x P_j + c_{j,j-1}\mathbb{D}_x P_{j-1}.$$

Then use of Relation (28), the previous equation as well as the structure relation (85) for $m = 0$ and $n = j$ produces the relation

$$U_1 \mathbb{D}_x P_j = d_{j,j+1}\mathbb{D}_x P_{j+1} + d_{j,j}\mathbb{D}_x P_j + d_{j,j-1}\mathbb{D}_x P_{j-1},$$

which combined with the structure relation (85) for $m = 0$ and $n = j$ gives

$$\alpha \mathbb{S}_x P_j + U_1 \mathbb{D}_x P_j = e_{j,j+1}\mathbb{D}_x P_{j+1} + e_{j,j}\mathbb{D}_x P_j + e_{j,j-1}\mathbb{D}_x P_{j-1}.$$

Finally we deduce from (94), the previous relation and (67) for $m = 1$

$$\begin{aligned} c_j = \langle \hat{Q}_{0,1}, \alpha \mathbb{S}_x P_j + U_1 \mathbb{D}_x P_j \rangle &= \langle \hat{Q}_{0,1}, e_{j,j+1}\mathbb{D}_x P_{j+1} + e_{j,j}\mathbb{D}_x P_j + e_{j,j-1}\mathbb{D}_x P_{j-1} \rangle \\ &= \langle -\mathbb{D}_x \hat{Q}_{0,1}, e_{j,j+1}P_{j+1} + e_{j,j}P_j + e_{j,j-1}P_{j-1} \rangle \\ &= \langle \gamma_1 \hat{Q}_{1,0}, e_{j,j+1}P_{j+1} + e_{j,j}P_j + e_{j,j-1}P_{j-1} \rangle \\ &= \gamma_1 \langle \hat{P}_1, e_{j,j+1}P_{j+1} + e_{j,j}P_j + e_{j,j-1}P_{j-1} \rangle \\ &= 0 \ \text{ for } \ j-1 > 1. \end{aligned}$$

Therefore

$$\begin{aligned} \alpha \mathbb{S}_x \hat{Q}_{0,1} - \mathbb{D}_x(U_1 \hat{Q}_{0,1}) &= c_0 \hat{P}_0 + c_1 \hat{P}_1 + c_2 \hat{P}_2 \\ &= \phi \mathcal{L}, \end{aligned}$$

where

$$\phi = \frac{c_0 P_0}{\langle \mathcal{L}, P_0 P_0 \rangle} + \frac{c_1 P_1}{\langle \mathcal{L}, P_1 P_1 \rangle} + \frac{c_2 P_2}{\langle \mathcal{L}, P_2 P_2 \rangle},$$

thanks to (66). Application of the linear functional $\mathbb{D}_x(\alpha \mathbb{S}_x \hat{Q}_{0,1} - \mathbb{D}_x(U_1 \hat{Q}_{0,1}))$ to the polynomial P_n, and use of Relations (26) and (67) yields

$$\begin{aligned} \mathbb{D}_x(\phi \mathcal{L}) = \langle \mathbb{D}_x(\alpha \mathbb{S}_x \hat{Q}_{0,1} - \mathbb{D}_x(U_1 \hat{Q}_{0,1})), P_n \rangle &= -\langle \hat{Q}_{0,1}, \alpha \mathbb{S}_x \mathbb{D}_x P_n + U_1 \mathbb{D}_x^2 P_n \rangle \\ &= -\langle \hat{Q}_{0,1}, \mathbb{D}_x \mathbb{S}_x P_n \rangle \\ &= \langle \mathbb{D}_x \hat{Q}_{0,1}, \mathbb{S}_x P_n \rangle \\ &= -\gamma_1 \langle \hat{Q}_{1,0}, \mathbb{S}_x P_n \rangle \\ &= -\gamma_1 \langle \mathbb{S}_x \hat{P}_1, P_n \rangle \\ &= \langle \mathbb{S}_x(\psi \mathcal{L}), P_n \rangle, \end{aligned}$$

where $\psi = -\gamma_1 \frac{P_1}{\langle \mathcal{L}, P_1 P_1 \rangle}$. Summing up, we have

$$\mathbb{D}_x(\phi \mathcal{L}) = \mathbb{S}_x(\psi \mathcal{L}),$$

where ϕ is a polynomial of degree at most two and ψ a first-degree polynomial.
Step 4: $(c) + (f) \Rightarrow (e) \Rightarrow (a)$.

Step 4.1 $(c)+(f)\Rightarrow(e)$

We assume Property (f) and make use of Property (c) since we have proved above that $(f) \iff (a) \iff (b) \iff (c)$.

Application of $\mathbb{D}_x$ to both sides of (85) followed by the multiplication by $\phi^{(m)}$ and use of (79) gives for $n \geq 1$

$$\begin{aligned}
\phi^{(m)}\mathbb{D}_x\mathbb{S}_xQ_{n,m} &= b^m_{n,n+1}\phi^{(m)}\mathbb{D}^2_xQ_{n+1,m}+b^m_{n,n}\phi^{(m)}\mathbb{D}^2_xQ_{n,m}+b^m_{n,n-1}\phi^{(m)}\mathbb{D}^2_xQ_{n-1,m}\\
&= -b^m_{n,n+1}\left(\psi^{(m)}\mathbb{S}_x\mathbb{D}_xQ_{n+1,m}+\lambda_{n+1,m}Q_{n+1,m}\right)\\
&\quad -b^m_{n,n}\left(\psi^{(m)}\mathbb{S}_x\mathbb{D}_xQ_{n,m}+\lambda_{n,m}Q_{n,m}\right)\\
&\quad -b^m_{n,n-1}\left(\psi^{(m)}\mathbb{S}_x\mathbb{D}_xQ_{n-1,m}+\lambda_{n-1,m}Q_{n-1,m}\right)\\
&= -\psi^{(m)}\mathbb{S}_x\left[b^m_{n,n+1}\mathbb{D}_xQ_{n+1,m}+b^m_{n,n}\mathbb{D}_xQ_{n,m}+b^m_{n,n-1}\mathbb{D}_xQ_{n-1,m}\right]\\
&\quad -b^m_{n,n+1}\lambda_{n+1,m}Q_{n+1,m}-b^m_{n,n}\lambda_{n,m}Q_{n,m}-b^m_{n,n-1}\lambda_{n-1,m}Q_{n-1,m}.
\end{aligned}$$

A second use of (85) transforms the previous equation into

$$\phi^{(m)}\mathbb{D}_x\mathbb{S}_xQ_{n,m}+\psi^{(m)}\mathbb{S}_xQ_{n,m}=-b^m_{n,n+1}\lambda_{n+1,m}Q_{n+1,m}-b^m_{n,n}\lambda_{n,m}Q_{n,m}-b^m_{n,n-1}\lambda_{n-1,m}Q_{n-1,m}.$$

Therefore,

$$\phi^{(m)}\mathbb{D}_x\mathbb{S}_xQ_{n,m}+\psi^{(m)}\mathbb{S}^2_xQ_{n,m}=a^m_{n,n+1}Q_{n+1,m}+a^m_{n,n}Q_{n,m}+a^m_{n,n-1}Q_{n-1,m},\ n\geq 1,$$

with

$$a^m_{n,n+j}=-b^m_{n,n+j}\,\lambda_{n+j,m},\ -1\leq j\leq 1. \tag{95}$$

In addition, $a^m_{n,n-1}=-b^m_{n,n-1}\,\lambda_{n-1,m}\neq 0$ for $n>2$ since $b^m_{n,n-1}\neq 0$ and $\lambda_{n-1,m}\neq 0$ both for $n>2$.

Step 4.2 $(e)\Rightarrow(a)$

We assume (e) and obtain using (85)

$$\begin{aligned}
\langle\mathbb{D}_x(\phi\mathcal{L})-\mathbb{S}_x(\psi\mathcal{L}),\mathbb{S}_xQ_{n,0}\rangle &= -\langle\mathcal{L},\phi\mathbb{D}_x\mathbb{S}_xQ_{n,0}+\psi\mathbb{S}^2_xQ_{n,0}\rangle\\
&= -\langle\mathcal{L},a^0_{n,n+1}Q_{n+1,0}+a^0_{n,n}Q_{n,0}+a^0_{n,n-1}Q_{n-1,0}\rangle\\
&= -\langle\mathcal{L},a^0_{n,n+1}P_{n+1}+a^0_{n,n}P_n+a^0_{n,n-1}P_{n-1}\rangle\\
&= 0,\ \text{for } n\geq 2.
\end{aligned}$$

For $n=1$ and for $n=0$, we have

$$\begin{aligned}
\phi\mathbb{D}_x\mathbb{S}_xP_1+\psi\mathbb{S}^2_xP_1 &= a^0_{1,2}P_2+a^0_{1,1}P_1+a^0_{1,0}P_0=a^0_{1,2}P_2+a^0_{1,1}P_1,\\
\phi\mathbb{D}_x\mathbb{S}_xP_0+\psi\mathbb{S}^2_xP_0 &= \psi=a^0_{0,1}P_1+a^0_{0,0}P_0=a^0_{0,1}P_1,
\end{aligned}$$

since $\lambda_{0,m}=0$,

$$a^0_{1,0}=-b^0_{1,0}\lambda_{0,0}=0,\ \text{and } a^0_{0,0}=-b^0_{0,0}\lambda_{0,0}=0.$$

Summing up, we have

$$\langle\mathbb{D}_x(\phi\mathcal{L})-\mathbb{S}_x(\psi\mathcal{L}),\mathbb{S}_xQ_{n,0}\rangle=0,\ n\geq 0,\ \text{and } \mathbb{D}_x(\phi\mathcal{L})=\mathbb{S}_x(\psi\mathcal{L}).$$

Step 5 $(a)\Leftrightarrow(g)$

Step 5.1 $(a)\Rightarrow(g)$

Assuming (a), we take the formal Stieltjes function of both sides of the Pearson equation (73) to get

$$S\left[\mathbb{D}_x\left(\phi\mathcal{L}\right)\right](x(s))=S\left[\mathbb{S}_x\left(\psi\mathcal{L}\right)\right](x(s)).$$

Use of (39), (40) and (42) transforms the previous equation into

$$\begin{array}{rcl}\mathbb{D}_x\left[S\left(\phi\mathcal{L}\right)(x(s))\right] & = & \alpha\mathbb{S}_x\left[S\left(\psi\mathcal{L}\right)(x(s))\right]+U_1(x(s))\mathbb{D}_x\left[S\left(\psi\mathcal{L}\right)(x(s))\right]\\ & \Updownarrow & \\ \mathbb{D}_x\left[\phi(x(s))S(\mathcal{L})(x(s))+\left(\mathcal{L}\theta_0\phi\right)(x(s))\right] & = & \alpha\mathbb{S}_x\left[\psi(x(s))S(\mathcal{L})(x(s))+\left(\mathcal{L}\theta_0\psi\right)(x(s))\right]\\ & + & U_1(x(s))\mathbb{D}_x\left[\psi(x(s))S(\mathcal{L})(x(s))+\left(\mathcal{L}\theta_0\psi\right)(x(s))\right].\end{array}$$

Finally, we use the product rules (19), (20) and the definition of $\mathcal{L}\theta_0 f$ given by (43) to obtain the following Riccati equation for $S(\mathcal{L})$

$$A(x(s))\mathbb{D}_x S(\mathcal{L})(x(s))=B(x(s))\mathbb{S}_x S(\mathcal{L})(x(s))+C(x(s))$$

where

$$\begin{array}{rcl}A & = & \mathbb{S}_x\phi-\alpha\psi_1\,U_2-U_1\mathbb{S}_x\psi,\\ B & = & \alpha\mathbb{S}_x\psi+\psi_1\,U_1-\mathbb{D}_x\phi,\\ C & = & (\alpha\psi_1-\phi_2)\langle\mathcal{L},1\rangle,\end{array}$$

where ϕ_2 and ψ_1 are given by (81).

Step 5.2 $(g)\Rightarrow(a)$

Assuming (a), we use Equations (39) and (40) to transform the Riccati Equation (86) into

$$\left(A(x)+\frac{U_1}{\alpha}B(x)\right)S\left(\mathbb{D}_x\mathcal{L}\right)=\frac{1}{\alpha}B(x)S\left(\mathbb{S}_x\mathcal{L}\right)+C(x).$$

By means of (42), the latter equation is equivalent to

$$S\left[\left(A(x)+\frac{U_1}{\alpha}B(x)\right)\mathbb{D}_x\mathcal{L}-\frac{1}{\alpha}B(x)\mathbb{S}_x\mathcal{L}\right] = C(x)-\frac{1}{\alpha}(\mathbb{S}_x\mathcal{L})\theta_0B(x(s))+(\mathbb{D}_x\mathcal{L})\theta_0\left(A(x)+\frac{U_1}{\alpha}B(x)\right).$$

The right-hand side of the previous relation is a polynomial while the left-hand side is, by definition of the Stieltjes function of a given linear functional given by (38), an infinite linear combination of $\{\frac{1}{F_{n+1}},\ n\in\mathbb{N}\}$. Therefore, both sides of the previous equation vanish and we obtain:

$$\left(A(x)+\frac{U_1}{\alpha}B(x)\right)\mathbb{D}_x\mathcal{L}-\frac{1}{\alpha}B(x)\mathbb{S}_x\mathcal{L}=0, \tag{96}$$

and

$$C(x(s))=\frac{1}{\alpha}(\mathbb{S}_x\mathcal{L})\theta_0B(x(s))-(\mathbb{D}_x\mathcal{L})\theta_0\left(A(x)+\frac{U_1}{\alpha}B(x)\right).$$

Using Relations (62) and (63), Relation (96) becomes

$$\mathbb{D}_x\left[(\mathbb{S}_xH(x)+U_2\mathbb{D}_xK(x))\mathcal{L}\right]-\mathbb{S}_x\left[(\mathbb{D}_xH(x)+\mathbb{S}_xK(x))\mathcal{L}\right]=0, \tag{97}$$

where

$$H(x)=A(x)+\frac{U_1}{\alpha}B(x),\quad K(x)=\frac{1}{\alpha}B(x).$$

Since A and B are polynomials of degree at most two and one respectively, the polynomials

$$\phi=\mathbb{S}_x\left(A(x)+\frac{U_1}{\alpha}B(x)\right)-\frac{U_2}{\alpha}\mathbb{D}_x(B(x)),\ \psi=\mathbb{D}_x\left(A(x)+\frac{U_1}{\alpha}B(x)\right)-\frac{1}{\alpha}\mathbb{S}_x(B(x))$$

are of degree at most two and one respectively. Next, we write $\psi=u\,P_1+v$ and obtain $v\langle\mathcal{L},1\rangle=\langle\mathcal{L},\psi\rangle$. Application of both sides of the Pearson equation (97) to the constant polynomial 1 yields $\langle\mathcal{L},\psi\rangle=0$. Therefore, $\psi=uP_1$ is of degree exactly 1. □

4 Important Connections

4.1 Connection with the Structure Relation by Koornwinder

The structure relation (14) given by Koornwinder [14] is related to our results in the following way:

Theorem 5 *The structure relation (14) for classical orthogonal polynomials $(P_n)_n$ on a non-uniform lattice satisfying (17) can be expressed in terms of the operator $\mathbb{D}_x$ and $\mathbb{S}_x$ as*

$$\mathbb{L}(p_n)(x(s)) = \zeta \left(2\psi\, \mathbb{S}_x^2 + 2\phi \mathbb{D}_x \mathbb{S}_x - \psi \mathbb{I}\right) p_n(x(s)) = \gamma_n\, A_n\, p_{n+1}(x) - \gamma_{n-1}\, C_n\, p_{n-1}(x), \tag{98}$$

where ζ is a constant term.

For the specific case of the Askey-Wilson polynomials, the coefficients ϕ and ψ are given by [8]

$$\begin{aligned}
\phi(x(s)) &= 2\,(dcba+1)\, x^2\,(s) - (a+b+c+d+abc+abd+acd+bcd)\, x\,(s) \\
&\qquad + ab+ac+ad+bc+bd+cd-abcd-1, \\
\psi(x(s)) &= \frac{4\,(abcd-1)\, q^{\frac{1}{2}}\, x\,(s)}{q-1} + \frac{2\,(a+b+c+d-abc-abd-acd-bcd)\; q^{\frac{1}{2}}}{q-1}.
\end{aligned}$$

Proof: We assume that $(p_n)_n$ is a family of polynomial orthogonal with respect to the linear functional $\mathcal{L}$ satisfying the Pearson equation

$$\mathbb{D}_x(\phi\mathcal{L}) = \mathbb{S}_x(\psi\mathcal{L}), \tag{99}$$

where ϕ is a polynomial of degree at most two and ψ a first-degree polynomial. Because of the property (59), the operator $\mathbb{O} = \phi\mathbb{D}_x^2 + \psi\mathbb{S}_x\mathbb{D}_x$ is symmetric with respect to the inner product

$$(p,q) = \langle \mathcal{L}, pq\rangle,\ p,\ q \in \mathbb{R}[x(s)], \tag{100}$$

that is,

$$(\mathbb{O}(p), q) = (p, \mathbb{O}(q)),\ \forall p,\ q \in \mathbb{R}[x(s)].$$

Since $\mathbb{O}$ satisfies in addition the property $\mathbb{O}(p_n) = \lambda_n p_n$, with $\lambda_n \neq \lambda_{n-1}$, we deduce thanks to Proposition 2.2 of [14] that the commutator $\tilde{\mathbb{L}}$ defined by

$$\tilde{\mathbb{L}}(p)(x(s)) = [\mathbb{O}, X](x(s)) = \mathbb{O}\,[x(s)p(x(s))] - x(s)\mathbb{O}(p)(x(s))$$

is skew symmetric with respect to the inner product (100) and satisfies the structure relation (14). Computation using the product rules (19), (20), (26) and (27) give

$$\tilde{\mathbb{L}}(p)(x(s)) = \left(2\psi\, \mathbb{S}_x^2 + 2\phi\mathbb{D}_x\mathbb{S}_x - \psi\mathbb{I}\right) p_n(x(s)) = \mathbb{L}(p)(x(s)).$$

For the recurrence relation (15) which is the specific case of the Askey-Wilson polynomials, in the first step, we deduce from the notation

$$[z] = \frac{z+z^{-1}}{2} = \frac{q^s+q^{-s}}{2} = x(s) \tag{101}$$

that

$$[qz] \;=\; x(s+1),\ \left[\frac{z}{q}\right] = x(s-1), \tag{102}$$

$$x(s+\frac{1}{2}) \;=\; [\sqrt{q}z] = \frac{\sqrt{q}z}{2} + \frac{1}{2\sqrt{q}z},\ \left[\frac{z}{\sqrt{q}}\right] = x(s-\frac{1}{2}). \tag{103}$$

In the second step, we solve the linear equations

$$\mathbb{D}_x\mathbb{S}_x f(x(s)) = \frac{f(x(s+1)) - f(x(s-1))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})},\ \mathbb{S}_x^2 f(x(s)) = \frac{f(x(s+1)) + 2f(x(s)) + f(x(s-1))}{4},$$

to get

$$
\begin{aligned}
f(x(s+1)) &= 2\mathbb{S}_x^2 f(x(s)) - f(x(s)) + \left(x(s+\frac{1}{2}) - x(s-\frac{1}{2})\right)\mathbb{D}_x\mathbb{S}_x f(x(s)), \quad (104)\\
f(x(s-1)) &= 2\mathbb{S}_x^2 f(x(s)) - f(x(s)) - \left(x(s+\frac{1}{2}) - x(s-\frac{1}{2})\right)\mathbb{D}_x\mathbb{S}_x f(x(s)). \quad (105)
\end{aligned}
$$

In the third step, we substitute (102) in the right-hand side of (15) to obtain an equation in which we substitute (104) and (105), then (103) to get an equation of the form

$$
\begin{aligned}
\mathbb{L}(p_n) &= \frac{q-1}{2\sqrt{q}}\left(2\psi([z])\mathbb{S}_x^2 p_n(x(s)) + 2\phi([z])\mathbb{D}_x\mathbb{S}_x p_n(x(s)) - \psi([z])p_n(x(s))\right) \quad (106)\\
&= \frac{q-1}{2\sqrt{q}}\left(2\psi(x(s))\,\mathbb{S}_x^2 + 2\phi(x(s))\mathbb{D}_x\mathbb{S}_x - \psi(x(s))\mathbb{I}\right)p_n(x(s)),
\end{aligned}
$$

where ϕ and ψ are those of the Askey-Wilson polynomials given above which appeared already in [8]. □

4.2 Connection with some pioneering work by Magnus

In the papers [15, 16], Magnus defined the Laguerre-Hahn orthogonal polynomials on the non-uniform lattice as the ones for which the formal Stieltjes series of the corresponding functional given by (37) satisfies a Riccati difference equation (see Equation (2.4) of [15]). He also proved that for a non-uniform lattice, the associated Laguerre-Hahn orthogonal polynomials are again Laguerre-Hahn orthogonal polynomials, and he recovered the associated Askey-Wilson polynomials as special case of the Laguerre-Hahn orthogonal polynomials.

The present work provides a bridge between the theory of Magnus based mainly on the Riccati equation satisfied by the formal Stieltjes function (37), and the theory of classical orthogonal polynomials based on the functional approach (which is already extended to the functional approach of the theory of semi-classical and Laguerre-Hahn orthogonal polynomials [10, 11]).

5 Conclusion and Perspectives

In this work, we have:

1. stated the Pearson-type equation for the linear functional of the corresponding classical orthogonal polynomials;

2. proved that the Pearson equation for the weight implies the one of the linear functional;

3. stated and proved using the functional approach seven equivalent characterization properties for classical orthogonal polynomials: the four properties given by Costas-Santos and Marcellán [5] but using the Pearson equation for the corresponding weight function, plus, the Pearson equation for the linear functional, the Rodrigues formula for the linear functional, the first structure relation and the Riccati equation for the formal Stieltjes function;

4. found the link between the structure relation given above by Koornwinder [14] and our second structure relation;

5. connected this work with the pioneering one by Magnus [15, 16], done using mainly the Riccati equation for the corresponding orthogonal family.

Since the operator $\mathbb{D}_x$ reduces to the forward operator Δ and the Hahn operator $D_q \left(D_q f(s) = \frac{f(qs)-f(s)}{(q-1)s} \right)$ for the lattices $x(s) = s$ and $x(s) = q^s$ respectively [8], this work generalizes previous ones characterizing classical orthogonal polynomials by means of the above mentioned seven equivalent properties. Among these, we would like to mention [1, 2, 17] for COP of a continuous variable, [12] for COP of a discrete variable, [21, 20, 3] for COP of a q-discrete variable and [5, 14] for COP on a non-uniform lattice.

We end by mentioning that our work completes and generalizes the one of [5], connecting to the pioneering work done by Magnus. In addition, it has interesting perspective which is the completion and generalization of the work of Magnus [15, 16] by stating and proving—using the functional approach—the characterization theorems for the semi-classical and Laguerre-Hahn orthogonal polynomials on non-uniform lattices [10, 11]). This will allow the study of the properties of new orthogonal polynomials obtained by modifications of the initial ones (see [6] and references therein).

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