Root parametrized differential equations

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Introduction

Differential Galois theory is the analog of classical Galois theory for linear differential equations. Classical Galois theory considers polynomial equations with coefficients in some field F and studies the behavior of their solutions. For this purpose, one examines the field generated by the roots over F, the splitting field E/F. The symmetries of these roots are naturally described by the group $\mathrm{Gal}(E/F)$ of automorphisms of E leaving the base field F fixed. The fundamental theorem of classical Galois theory establishes a correspondence between the intermediate fields of E/F and the subgroups of $\mathrm{Gal}(E/F)$.
Differential Galois theory studies linear differential equations with coefficients in some differential field F, i.e., a field equipped with a derivation, having an algebraically closed field of constants C. The analog for the splitting field is the Picard-Vessiot extension E/F; it is generated by the entries of a fundamental solution matrix for the defining equation. Linear combinations of the solutions over the field of constants C clearly are also solutions for the equation, and they generate the same Picard-Vessiot extension. The group of differential automorphisms of E/F has the structure of a linear algebraic group over the constants. This group, denoted by $\mathrm{Gal}_\partial(E/F)$, is called the differential Galois group. As in classical Galois theory, there is a correspondence between intermediate differential subfields of E/F and the closed subgroups of $\mathrm{Gal}_\partial(E/F)$.

The inverse problem in differential Galois theory is to determine which linear algebraic groups can occur as differential Galois groups. An answer is known for some fields and groups, for example, for differential fields of characteristic zero with algebraically closed field of constants C and connected differential Galois groups. For some specific fields, such as $C(\{z\})$ or $C((z))$, linear algebraic groups, which occur as differential Galois groups have been completely classified. The general problem over the rational function field $C(z)$ was solved by J. Hartmann in [Hart02]. Further, Mitschi and Singer developed in [MS96] a constructive method to realize connected groups over $C(z)$. They applied upper and lower bounds for the differential Galois group to the defining matrix. Therefore the task for the realization of a linear algebraic group reduces to find a sufficiently general element of the Lie algebra such that the upper and lower bound coincide. For the application of the lower bound criterion it is important that the differential ground field is a C_1-field. A more detailed explanation of the bounds is presented below. Over the same differential base field, Magid presented in [Mag94] a technique to realize some classes of connected linear algebraic groups. Inspired by E. Noether's work for algebraic equations, Goldmann introduced the language of generic differential equations in [Gold57]. Here the differential ground field is purely differential transcendental over the constants. More precisely, Goldmann takes the differential field of generic solutions as his extension field, i.e., he starts with a differential field generated by n differential indeterminates over the constants, where

n is equal to the dimension of the representation of the linear algebraic group. Then, by his method, the differential ground field is the fixed field of the group and has therefore the same differential transcendence degree over the constants as the extension field. Moreover, in [Juan08] Juan presented generic equations using an alternative approach to Goldmann's. For the group SO_n Juan's method is well applicable and yields a generic equation, where the differential transcendence degree of the differential ground field is equal to $\frac{1}{2}(n+2)(n+1)$. In the general case the differential transcendence degree equals to the dimension of the linear algebraic group. Here the differential equation is generic in an indirect sense. More precisely, the specialization of the coefficients takes place over a finite extension. For more information on the history of the inverse problem in differential Galois theory, we refer to [PS03, p. 292-293].

The present work concentrates on the realization of the classical groups as differential Galois groups. We introduce a method for a very general realization of these groups, i.e., we present for the classical groups of Lie rank l explicit linear differential equations where the coefficients are differential polynomials in l differential indeterminates over the constants. At the same time we managed to do these realisations in terms of Abhyankar's program, *Nice Equations for Nice Groups*. Here the choice of the defining matrix is important. We found out that an educated choice of l negative roots for the parametrization together with the positiv simple roots leads to a nice differential equation and at the same time defines a sufficiently general element of the Lie algebra. At the end of [Elk99] Elkies proposed that a particular subspace of the Lie algebra, which is conceptual very similar to the choice of our parametrized element of the Lie algebra, yields an differential analogue of a Deligne-Lusztig variety.
In the this thesis we compute explicit parametrized differential equations for the series of types A_l, B_l, C_l and D_l, i.e., we realize the groups $\mathrm{SL}_{l+1}(C)$, $\mathrm{SO}_{2l+1}(C)$, $\mathrm{SP}_{2l}(C)$ and $\mathrm{SO}_{2l}(C)$ as differential Galois groups over the differential ground field $C\langle t_1, ..., t_l\rangle$ in l differential indeterminates $t_1, ..., t_l$. Additionally to the series we consider in detail the exceptional groups of type G_2, F_4 and E_6. For the group of type G_2 we obtain an easy and nice explicit linear differential equation as in the case of the series. In [Kat90], Katz computed a nice differential equation with group G_2. His equation is a specialization of the equation presented in Theorem 1.5 below. Since the corresponding linear differential equations for the groups of type F_4 and E_6 would be of enormous lenght, we present instead matrix differential equations which have also a nice shape. Note that we leave out the realisation of the exceptional groups of type E_7 and E_8 because the size of the root system and the dimension of the representation is too enormous so that the corresponding computations would make this thesis needlessly long.
More generally, let $\mathcal{G}$ be a connected semisimple linear algebraic group, and let Φ denote the root system of $\mathcal{G}$. Our method provides a parametrized differential equation $L(y, t_1, ..., t_l) = \sum_{i=0}^{n} a_i(\boldsymbol{t})y^{(i)}$ over $C\langle t_1, ..., t_l\rangle$ with differential Galois group $\mathcal{G}$, where the number of parameters $\boldsymbol{t} = (t_1, ..., t_l)$ equals the rank of Φ, and the coefficients $a_i(\boldsymbol{t})$ are differential polynomials in $\boldsymbol{t}$.

We sketch the main ideas. A differential module M over a differential field (F, ∂_F) is a finite dimensional vector space, together with a map $\partial : M \to M$, which is additive and satisfies $\partial(f \cdot m) = \partial_F(f) \cdot m + f \cdot \partial(m)$ for $f \in F$ and $m \in M$. Let $e_1, ..., e_n$ be a basis of M. Then the map ∂ is written $\partial(e_i) = \sum_{j=1}^{n} A_{ij} e_j$, where $(A_{ij}) = A \in F^{n\times n}$. The matrix A is the defining matrix of the differential module M, and the resulting equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ is called a matrix differential equation. We can also start the other way around by associating a module M to a matrix $A \in F^{n\times n}$. In other words, we can start with a vector space on which we define an appropriate differential structure by the choice of $A \in F^{n\times n}$. In the literature (e.g., see [PS03]), there is a well-known upper bound criterion for the differential Galois group. It states that if A lies in the Lie algebra $\mathrm{Lie}(\mathcal{G})$ of $\mathcal{G}$, then the differential Galois group $\mathcal{H}$ of $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ is contained in (a conjugate of) $\mathcal{G}$. Thus we choose $A \in \mathrm{Lie}(\mathcal{G})(F)$ so that it does not lie in any proper subalgebra of $\mathrm{Lie}(\mathcal{G})(F)$. More work is needed to apply an appropriate lower bound criterion, and to show that the two bounds coincide. As a lower bound criterion, we can regard a result presented in [PS03]. It says that if $\mathcal{H}(C) \leq \mathcal{G}(C)$ is the differential Galois group of $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ over F, and A satisfies $A \in \mathrm{Lie}(\mathcal{G})(F)$, then there exists $B \in \mathcal{G}(F)$ such that $BAB^{-1} - \partial(B)B^{-1} \in \mathrm{Lie}(\mathcal{H})(F)$. There is another important condition for the application of the lower bound criterion. This condition is automatically satisfied if the differential base field is a C_1-field (e.g., this holds for $C(z)$). Since our differential base field is not a C_1-field, we have here no information whether this condition is satisfied or not. Note that our differential base field is purely differential transcendental over the constants. So we can consider specializations $\sigma : C\langle \boldsymbol{t}\rangle \to C(z)$ to a rational function field $C(z)$ and make use of the lower bound criterion in an indirect way. We introduce the *specialization bound*. It states that the differential Galois group $\mathcal{H}(C)$ of the specialized differential equation $\partial(\boldsymbol{y}) = \sigma(A)\boldsymbol{y}$ over $C(z)$ is contained in the differential Galois group $\mathcal{G}(C)$ of the original equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ over $C\langle \boldsymbol{t}\rangle$. The idea of the proof is to show that there exists a maximal differential ideal I in $C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]$ for the original equation and a maximal differential ideal $\bar{I}$ in $C[z][X_{ij}, \det(X_{ij})^{-1}]$ for the specialized equation satisfying $\sigma(I) \subset \bar{I}$. To find such ideals we use differential embeddings of the corresponding differential rings into fields of power series. Then the specialization of the coefficients of the power series yields the desired ideals. Finally, we can prove that the defining ideal of the group $\mathcal{H}(C)$ contains the defining ideal of $\mathcal{G}(C)$.

Let $\mathcal{B}$ denote a Borel subgroup of $\mathcal{G}$ in upper triangular form, and $\mathcal{B}^-$ the opposite Borel subgroup. Let Δ be a basis for the root system, and let $\{X_\alpha,\ H_{\alpha_i} \mid \alpha \in \Phi,\ 1 \leq i \leq l\}$ be a Chevalley basis for $\mathrm{Lie}(\mathcal{G})$, such that the structure is compatible with $\mathcal{B}$. We give l roots $\beta_1, ..., \beta_l$ of Φ^-, such that the parameterized matrix differential equation $\partial(\boldsymbol{y}) = (\sum_{\alpha\in\Delta} X_\alpha + \sum_{i=1}^{l} t_i \cdot X_{\beta_l})\boldsymbol{y}$ transforms in a natural way into a nice linear differential equation. Furthermore, we observe that every element of the subspace $\sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}^-)$ is differentially equivalent to a specialization of the matrix $\sum_{\alpha\in\Delta} X_\alpha + \sum_{i=1}^{l} t_i \cdot X_{\beta_i}$. This is the content of the *transformation lemma* and the proof uses the adjoint action of the root subgroups on the Chevalley basis. In order to apply the specialization bound, and to show that it coincides with the upper bound, we need a matrix differential equation satisfying the condition of the transformation lemma. In [MS96], Mitschi and Singer developed a method to construct a matrix differential equation $\partial(\boldsymbol{y}) = \bar{A}\boldsymbol{y}$ for every connected semisimple group $\mathcal{G}$ over the rational function field $C(z)$, with differential Galois group $\mathcal{G}$. They use the information gathered in the application of the lower bound criterion to prove that the differential Galois group is $\mathcal{G}$. We use similar ideas to compute a matrix differential equation such that the defining matrix lies in the subspace $\sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}^-)$.

Theorem 1 is a summary of some of our results:

Theorem 1. *Let C be an algebraically closed field of characteristic zero, $t_1, ..., t_l$ differential indeterminates, and $F = C\langle t_1, ..., t_l\rangle$ the corresponding differential field. Then the homogeneous linear differential equation*

1. $L(y, t_1, ..., t_l) = y^{(l+1)} - \sum_{i=1}^{l} t_i\, y^{(i-1)} = 0$ *has* $\mathrm{SL}_{l+1}(C)$ *as differential Galois group over* F,
2. $L(y, t_1, ..., t_l) = y^{(2l)} - \sum_{i=1}^{l}(-1)^{i-1}(t_i\, y^{(l-i)})^{(l-i)} = 0$ *has* $\mathrm{SP}_{2l}(C)$ *as differential Galois group over* F,
3. $L(y, t_1, ..., t_l) = y^{(2l+1)} - \sum_{i=1}^{l}(-1)^{i-1}((t_i\, y^{(l+1-i)})^{(l-i)} + (t_i\, y^{(l-i)})^{(l+1-i)}) = 0$ *has* $\mathrm{SO}_{2l+1}(C)$ *as differential Galois group over* F,
4. $L(y, t_1, ..., t_l) = y^{(2l)} - 2\sum_{i=3}^{l}(-1)^i((t_i y^{(l-i)})^{(l+2-i)} + (t_i y^{(l+1-i)})^{(l+1-i)})$ $-(t_2 y^{(l-2)} + t_1 y)^{(l)} - ((-1)^l t_1 z_1 + z_2) - \sum_{i=0}^{l-2}(t_2^{(l-2-i)} z_1)^{(i)}$ *has* $\mathrm{SO}_{2l}(C)$ *as differential Galois group over* F, *where the coefficients* z_1 *and* z_2 *are*

$$\begin{aligned} z_1 &= y^{(l)} - t_2 y^{(l-2)} - t_1 y \\ z_2 &= \frac{(t_2^{(l-2)} + (-1)^{l-2} t_1)^{(1)}}{t_2^{(l-2)} + (-1)^{l-2} t_1} \cdot \left(y^{(2l-1)} - 2\sum_{i=3}^{l}(-1)^i((t_i y^{(l-i)})^{(l+1-i)} \right. \\ &\quad \left. + (t_i y^{(l+1-i)})^{(l-i)}) - (t_2 y^{(l-2)} + t_1 y)^{(l-1)} - \sum_{i=0}^{l-3}(t_2^{(l-3-i)} z_1)^{(i)} \right), \end{aligned}$$

5. $L(y, t_1, t_2) = y^{(7)} + 2t_1 y' + 2(t_1 y)' + 2(t_2 y^{(4)})' + (t_2 y')^{(4)} - 2(t_2(t_2 y')')' = 0$ *has* $G_2(C)$ *as differential Galois group over* $F = C\langle t_1, t_2\rangle$.

We conclude the introduction with a brief outline of the chapters. In the first chapter we recall the basic notions of differential Galois theory. The second chapter starts with the presentation of the classical bounds for the differential Galois group. We then develop our alternative lower bound criterion which is based on the calculus of specializations, and will be therefore called the specialization bound. For this reason we start with the study of Picard-Vessiot extensions over rings. In the subsequent section, we focus on embeddings of the corresponding differential rings in fields of power series to obtain a well behaving specialization. The chapter ends with the proof of the specialization bound. In Chapter 3, we briefly outline the structure of the classical groups and their Lie algebras, establish the key element for the proof of the transformation lemma, and give a small example. To apply the alternative lower bound criterion, we modify the ideas developed by Mitschi and Singer when they realized semisimple connected linear groups as differential Galois groups over the differential field $C(z)$. In the last section of Chapter 3, we prove the existence of a parametrized differential equation for every semisimple connected linear algebraic group. In Chapter 4, we realize $\mathrm{SL}_{l+1}(C)$ as a differential Galois group over $C\langle t_1, ..., t_l\rangle$. In more details, in the first section we compute a Chevalley basis, and present the root system of type A_l. We continue by collecting enough facts about the root system to prove the transformation lemma for SL_{l+1}. To complete this chapter, we construct an equation which admits SL_{l+1} as its differential Galois group.
In Chapter 5, 6 and 7 we consider the groups SP_{2l+1}, SO_{2l+1} and SO_{2l}. We use the same

method as for SL_{l+1}, and the chapters are organized in a similar way as Chapter 4. In Chapter 8, 9 and 10 we study the groups of type G_2, F_4 and E_6. Again we use the same approach as in the previous chapters, but the root system computations are much easier in these cases since we do not need inductive arguments for the proofs. For the same reasons, the proof of the transformation lemma is also easier as in the cases of the series. Chapters 4-10 can be read independently from each other.

Notation

We use the following notation.

C	algebraically closed field of characteristic 0.
$C(z)$	rational function field in a transcendental z whose field of constants is C.
(R, ∂_R)	differential ring with derivation ∂_R.
(F, ∂_F)	differential field with derivation ∂_F.
$E \geq F$	differential field extension.
$t_1, ..., t_l$	differential indeterminates.
$F\{t_1, ..., t_n\}$	the ring of differential polynomials in $t_1, ..., t_n$ over F (page 20/21).
$F\langle t_1, ..., t_n\rangle$	the field of fractions of $F\{t_1, ..., t_n\}$ (page 20/21).
$R\{X\}$	the differential R-subalgebra generated by the elements of a subset $X \subseteq R_1$ of a differential ring extension $R_1 \geq R$ (page 21).
$F\langle X\rangle$	differential subfield of the differential field extension $E \geq F$ generated by the elements of the set $X \subseteq E$ (page 21).
M	differential module (page 16).
$\partial(\boldsymbol{y}) = A\boldsymbol{y}$	matrix differential equation (page 15).
$\mathcal{G}$	linear algebraic group where $\mathcal{G}(F)$ means the F-rational points (page 48).
$\mathcal{B}_0$	Borel subgroup of $\mathcal{G}$ in upper triangular form (page 48).
$\mathcal{B}_0^-$	Borel subgroup of $\mathcal{G}$ in lower triangular form (page 48).
$\mathcal{T}_0$	maximal diagonal torus of $\mathcal{G}$ (page 48).
$\mathrm{diag}(\lambda_1, ..., \lambda_n)$	diagonal matrix with entries $\lambda_1, ..., \lambda_n$.
$\mathcal{U}$	unipotent subgroup of $\mathcal{G}$ in upper triangular form (page 48).
$\mathcal{U}^-$	unipotent subgroup of $\mathcal{G}$ in lower triangular form (page 48).
$\mathcal{U}_\alpha$	root subgroup which corresponds to a root $\alpha \in \Phi$ (page 49).
$\mathrm{Lie}(\mathcal{G})$, $\mathbf{L}$	the Lie algebra of $\mathcal{G}$ where $\mathrm{Lie}(\mathcal{G})(F)$ means the F-rational points (page 48).
$\mathbf{H}$	Cartan subalgebra of $\mathrm{Lie}(\mathcal{G})$ (page 49).
$\mathrm{Lie}(\mathcal{G})_\alpha$	root space corresponding to the root $\alpha \in \Phi$ (page 49).
X_α	basis element of the root space $\mathrm{Lie}(\mathcal{G})_\alpha$ (page 49).
H_α	the co-root of $\alpha \in \Phi$ (page 49).
H_i	the co-root of $\alpha_i \in \Delta$ (page 49).
Φ	root system of $\mathcal{G}$ resp. $\mathrm{Lie}(\mathcal{G})$ (page 47).
$\Delta = \{\alpha_1, ..., \alpha_l\}$	basis of Φ with simple roots α_i (page 47/48).
$\langle\beta, \alpha\rangle$	the integer defined by $2(\beta, \alpha)/(\alpha, \alpha)$ (page 47).
$\sigma_\alpha(\beta)$	reflection of Φ for α defined by the formula $\sigma_\alpha(\beta) = \beta - \langle\beta, \alpha\rangle\,\alpha$ (page 47).
$\mathrm{ht}(\alpha)$	the height of a root α (page 48).
$\langle v_1, ..., v_n\rangle_F$	vector space spanned by the elements $v_1, ..., v_n$ over a field F.

Zusammenfassung

Differentialgaloistheorie ist eine Verallgemeinerung der klassischen Galoistheorie für lineare Differentialgleichungen. In der klassischen Galoistheorie betrachtet man Polynome mit Koeffizienten aus einem Körper F und untersucht das Verhalten ihrer Nullstellen. Zu diesem Zweck bildet man den von den Nullstellen über F erzeugten Körper, den sogenannten Zerfällungskörper E/F. Die Gruppe der Automorphismen $\mathrm{Gal}(E/F)$ von E, welche den Grundkörper F fest lassen, beschreibt dann auf natürliche Weise die Symmetrien der Nullstellen. Der Hauptsatz der klassischen Galoistheorie liefert eine Korrespondenz zwischen den Zwischenkörpern von E/F und den Untergruppen von $\mathrm{Gal}(E/F)$.
Im Gegensatz zur klassischen Galoistheorie betrachtet man in der Differentialgaloistheorie lineare Differentialgleichungen, deren Koeffizienten aus einem Differentialkörper F mit algebraisch abgeschlossenem Konstantenkörper C stammen. Als Gegenstück zum Zerfällungskörper hat man hier die sogenannte Picard-Vessiot-Erweiterung. Diese wird von den Einträgen einer Fundamentalmatrix der definierenden Gleichung erzeugt. Linearkombinationen von Lösungen über den Konstantenkörper C sind offensichtlich wieder Lösungen der Differentialgleichung und erzeugen die gleiche Picard-Vessiot-Erweiterung. Die Gruppe der Differentialautomorphismen von E/F trägt die Struktur einer linearen algebraischen Gruppe über den Konstantenkörper C. Diese Gruppe heißt Differentialgaloisgruppe und wird mit $\mathrm{Gal}_\partial(E/F)$ bezeichnet. Wie in der klassischen Galoistheorie existiert auch hier eine Korrespondenz zwischen den Differentialzwischenkörpern von E/F und den abgeschlossenen Untergruppen von $\mathrm{Gal}_\partial(E/F)$.
Das inverse Problem in der Differentialgaloistheorie beschäftigt sich mit der Frage, welche linearen algebraischen Gruppen als Differentialgaloisgruppen vorkommen können. Man kennt eine Antwort auf dieses Problem für bestimmte Grundkörper und Gruppen, wie zum Beispiel für Differentialkörper der Charakteristik Null mit algebraisch abgeschlossenen Konstantenkörper C und zusammenhängender Differentialgaloisgruppe. Für bestimmte Differentialgrundkörper, wie $C(\{z\})$ oder $C((z))$, wurden die linearen algebraischen Gruppen, welche als Differnetialgaloisgruppen vorkommen können, vollständig klassifiziert. Das allgemeine Problem für den rationalen Funktionenkörper $C(z)$ wurde von J. Hartmann in [Hart02] gelöst. In [MS96] haben Mitschi und Singer eine für den hier später erläuterten Ansatz wichtige konstruktive Methode entwickelt, um zusammenhängende Gruppen über $C(z)$ zu realisieren. Für den gleichen Differentialgrundkörper veröffentlichte Magid in [Mag94] eine Technik zur Realisierung einer Klasse von zusammenhängenden linearen algebraischen Gruppen. Angeregt von E. Noethers Arbeit für Polynomgleichungen führte Goldmann in [Gold57] die Sprache der generischen Differentialgleichungen ein. Außerdem stellte Goldmann in der gleichen Arbeit die Idee der analytischen Spezialisierungen vor. In [Juan08] verwendete Juan einen zu Goldmann alternativen Ansatz mittels diesem sie

auch generische Gleichungen berechnen konnte.
In der vorliegenden Arbeit wird das inverse Problem in der Differentialgaloistheorie behandelt. Ziel ist es, die klassischen Gruppen vom Lie Typ als Differentialgaloisgruppen zu realisieren. Hierfür entwickeln wir eine Methode, die eine sehr allgemeine Realisierung zulässt, das heißt, wir berechnen explizite lineare Differentialgleichungen für die klassischen Gruppen vom Lie Rang l, deren Koeffizienten Differentialpolynome in l Differentialunbestimmten über dem Konstantenkörper sind. Gleichzeitig gelingt es uns, für diese Gruppen Differentialgleichungen im Sinne von Abhyankars berühmter Reihe „Nice Equations for Nice Groups“ zu konstruieren. Hierbei ist die Wahl der definierenden Matrix aus der Lie Algebra entscheidend. Wir haben herausgefunden, dass eine geschickte Wahl von l negativen Wurzeln für die Parametrisierung zusammen mit den positiven einfachen Wurzeln zu einer schönen und einfachen linearen Differentialgleichung führt und gleichzeitig eine genügend allgemeine Matrix in der Lie Algebra definiert. In [Elk99] vermutet Elkies, dass ein gewisser Untervektorraum der Lie Algebra, der aufgrund seiner ähnlichen Konzeption unsere parametrisierte Matrix aus der Lie Algebra enthält, zu einem Differentialanalogon der Deligne-Lusztig Varietät führt.
Wir verwenden die hier entwickelte Methode zur Konstruktion von expliziten parametrisierten Differentialpolynomen für die Serien A_l, B_l, C_l und D_l, das heißt, wir realisieren die Gruppen $\mathrm{SL}_{l+1}(C)$, $\mathrm{SO}_{2l+1}(C)$, $\mathrm{SP}_{2l}(C)$ und $\mathrm{SO}_{2l}(C)$ über einem Differentialkörper $C\langle t_1, ..., t_l\rangle$ in den l Differentialunbestimmten $t_1, ..., t_l$. Zusätzlich werden die Gruppen vom Ausnahmetyp G_2, F_4 und E_6 im Detail behandelt. Dabei gelingt es uns, für die Gruppe vom Ausnahmetyp G_2 eine explizite und einfache Differentialgleichung, wie im Fall der Serien, zu berechnen. In seiner Arbeit [Kat90] berechnet Katz eine schöne und einfache Differentialgleichung mit Gruppe G_2. Wir erhalten seine Gleichung durch eine Spezialisierung der Parameter unserer Differentialgleichung. Da wir für die Ausnahmegruppen vom Typ F_4 und E_6 riesige linearen Differentialgleichungen erhalten würden, geben wir für diese nur Matrixdifferentialgleichungen an, deren Gestalt aber ebenso einfach ist. Wir möchten darauf hinweisen, dass eine explizite Ausarbeitung für die Gruppen vom Ausnahmetyp E_7 und E_8 nur deshalb weggelassen wurde, da die einzelnen Berechungen wegen der Größe des Wurzelsystems und der Dimension der Darstellung diese Arbeit nur unnötig verlängern würden.
Allgemeiner sei $\mathcal{G}$ eine zusammenhängende halbeinfache lineare algebraische Gruppe mit einer Darstellung in einen n-dimensionalen Vektorraum und es bezeichne Φ das zugehörige Wurzelsystem von $\mathcal{G}$. Dann liefert unsere Methode eine parametrisierte Differentialgleichung $L(y, \boldsymbol{t}) = \sum_{i=0}^{n} a_i(\boldsymbol{t}) y^{(i)} = 0$ über $C\langle t_1, ...t_l\rangle$ mit Differentialgaloisgruppe $\mathcal{G}(C)$, wobei die Anzahl der Parameter $\boldsymbol{t} = (t_1, ..., t_l)$ dem Rang von Φ entspricht und die Koeffizienten $a_i(\boldsymbol{t})$ Differentialpoylnome in $\boldsymbol{t}$ sind.
Wir möchten nun die wichtigsten Ideen unserer Methode skizzieren. Ein Differentialmodul M über einen Differentialkörper (F, ∂_F) ist ein endlich dimensionaler Vektorraum mit einer additiven Abbildung $\partial : M \to M$, welche zusätzlich die Regel $\partial(f \cdot m) = \partial_F(f) \cdot m + f \cdot \partial(m)$ für $f \in F$ und $m \in M$ erfüllt. Sei $e_1, ..., e_n$ eine Basis von M. Dann hat die Abbildung ∂ die Gestalt $\partial(e_i) = \sum_{j=1}^{n} A_{ij} e_j$ mit $(A_{ij}) = A \in F^{n\times n}$. Die Matrix A heißt die definierende Matrix des Differentialmoduls M und die zugehörige Gleichung $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ heißt Matrixdifferentialgleichung. Man kann nun auch andersherum anfangen, indem man einen Modul M zu einer Matrix $A \in F^{n\times n}$ assoziert. Genauer gesagt, man beginnt mit einem Vektorraum und definiert auf diesem durch die Wahl von $A \in F^{n\times n}$ eine geeignete

Differentialstruktur. In der Literatur findet man ein wohlbekanntes oberes Schrankenkriterium für die Differentialgaloisgruppe. Es besagt, dass wenn A in der Lie Algebra $\mathrm{Lie}(\mathcal{G})$ von $\mathcal{G}$ liegt, dann ist die Differentialgaloisgruppe $\mathcal{H}$ von $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ bis auf Konjugation in $\mathcal{G}$ enthalten. Die Aufgabe eine Gruppe zu realisieren reduziert sich somit darauf, eine geeignete Differentialstruktur auf einem Modul zu definieren, indem man eine genügend allgemeine Matrix für die Differentialgleichung aus der Lie Algebra wählt. Folglich wird man $A \in \mathrm{Lie}(\mathcal{G})(F)$ so wählen, dass A in keiner echten Unteralgebra von $\mathrm{Lie}(\mathcal{G})(F)$ liegt. Um ein geeignetes unteres Schrankenkriterium anwenden zu können und um zu zeigen, dass die beiden Schranken übereinstimmen, ist mehr Arbeit nötig. Als ein solches unteres Schrankenkriterium kann man ein in der Literatur bekanntes Resultat ansehen. Unter passender Voraussetung besagt es, dass wenn $\mathcal{H}(C) \leq \mathcal{G}(C)$ die Differentialgaloisgruppe von $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ über F ist und A in $\mathrm{Lie}(\mathcal{G})(F)$ liegt, dann existiert ein $B \in \mathcal{G}(F)$ und $BAB^{-1} - \partial(B)B^{-1} \in \mathrm{Lie}(\mathcal{H})(F)$. Leider ist diese Voraussetzung nur im Fall von C_1-Körpern stets erfüllt wie zum Beispiel für den rationalen Funktionenkörper $C(z)$. In dieser Arbeit kann die klassische untere Schranke nur indirekt angewandt werden, da der hier verwendete Differentialgrundköper im allgemeinen kein C_1-Körper ist und wir somit nicht wissen, ob die für die Anwendung der unteren Schranke notwendige Voraussetzung erfüllt ist. Es wird daher ein neues unteres Schrankenkriterium benötigt und entwickelt. Da man den Differentialgrundkörper rein differentialtranszendent über den Konstanten gewählt hat, basiert die Idee für das neue untere Schrankenkriterium auf Parameterspezialisierung. Hier verwenden wir Spezialisierungen $\sigma : C\langle \boldsymbol{t} \rangle \to C(z)$ in den rationalen Funktionenkörper $C(z)$. Das so gewonnene untere Schrankenkriterium, welches in dieser Arbeit als die Spezialisierungsschranke (The Specialization Bound) bezeichnet wird, besagt, dass die Differentialgaloisgruppe $\mathcal{H}(C)$ der spezialisierten Differentialgleichung $\partial(\boldsymbol{y}) = \sigma(A)\boldsymbol{y}$ über $C(z)$ in der Differentialgaloisgruppe $\mathcal{G}(C)$ der Ausgangsgleichung $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ über $C\langle \boldsymbol{t} \rangle$ enthalten ist. Der Beweis verwendet die Spezialisierung eines maximalen Differentialideals I des universellen Lösungsrings $C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]$ der Ausgangsgleichung. Es ist daher nötig Picard-Vessiot-Ringe über Differentialringen zu konstruieren und genauer zu untersuchen. Die für Differentialkörper bekannten Resultate müssen für Differentialringe neu bewiesen werden. Um die Existenz eines maximalen Differentialideals I für die Ausgangsgleichung in $C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]$ und eines maximalen Differentialideals $\bar{I}$ der spezialisierten Gleichung in $C[z][X_{ij}, \det(X_{ij})^{-1}]$ mit der Eigenschaft, dass die Spezialisierung $\sigma(I)$ in $\bar{I}$ enthalten ist, zu beweisen, werden die entsprechenden Differentialringe via Taylorabbildungen in Potenzreihenringe eingebettet. So erhält man genug Information um zu beweisen, dass die beiden definierenden Ideale der Gruppen ineinander enthalten sind.

Es bezeichne $\mathcal{B}$ eine Borelgruppe von $\mathcal{G}$ in oberer Dreiecksgestallt und $\mathcal{B}^-$ die entgegengesetzte Borelgruppe. Des Weiteren sei Δ eine Basis des Wurzelsystems und es sei $\{X_\alpha,\ H_{\alpha_i} \mid \alpha \in \Phi,\ 1 \leq i \leq l\}$ eine Chavelley Basis für $\mathrm{Lie}(\mathcal{G})$, so dass ihre Struktur mit $\mathcal{B}$ kompatibel ist. Wir können nun l Wurzeln $\beta_1, ..., \beta_l$ aus Φ^- wählen, so dass sich die parametrisierte Matrixdifferentialgleichung $\partial(\boldsymbol{y}) = (\sum_{\alpha\in\Delta} X_\alpha + \sum_{i=1}^{l} t_i \cdot X_{\beta_l})\boldsymbol{y}$ auf natürliche Weise in eine schöne Differentialgleichung transformieren lässt. Außerdem beobachtet man, dass jedes Element aus dem Unterraum $\sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}^-)$ differentialäquivalent zu einer Spezialisierung der Matrix $\sum_{\alpha\in\Delta} X_\alpha + \sum_{i=1}^{l} t_i \cdot X_{\beta_i}$ ist. Dies ist die Aussage des Transformationslemmas (*Transformation Lemma*) und der Beweis verwendet die Operation via der adjungierten Darstellung der Wurzeluntergruppen auf der Chevalley Basis. Um nun

unsere Spezialisierungsschranke anwenden zu können und um zu zeigen, dass sie mit dem oberen Schrankenkriterium übereinstimmt, benötigen wir eine Matrixdifferentialgleichung über $C(z)$, welche die Bedingungen des Transformationslemmas erfüllt und die Gruppe $\mathcal{G}$ als Differentialgaloisgruppe hat. In einer ihrer Arbeiten haben Mitschi und Singer eine Methode entwickelt mit der man für jede zusammenhängende halbeinfache Gruppe $\mathcal{G}$ eine Matrixdifferentialgleichung $\partial(\boldsymbol{y}) = \bar{A}\boldsymbol{y}$ über den rationalen Funktionenkörper $C(z)$ konstruieren kann, so dass sie $\mathcal{G}$ als Differentialgaloisgruppe besitzt. Es ist ihnen gelungen aus der Anwendung des klassischen unteren Schrankenkriteriums genug Information zu ziehen um zu beweisen, dass $\mathcal{G}$ die Differentialgaloisgruppe ist. Anhand der gleichen Ideen werden wir eine Matrixdifferentialgleichung berechnen, so dass die definierende Matrix im Unterraum $\sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}^-)$ enthalten ist.
Die durch die Anwendung unserer Methode erzielten Ergebnisse für die Gruppen vom Typ A_l, C_l, B_l, D_l und G_2 sind in folgendem Theorem zusammengefasst.

Theorem 2. *Sei C ein algebraisch abgeschlossener Körper der Charakteristik Null und $F = C\langle t_1, ..., t_l\rangle$ der von den Differentialunbestimmten $t_1, ..., t_l$ über C erzeugte Differentialkörper. Dann besitzt die homogene lineare Differentialgleichung*

- $L(y, t_1, ..., t_l) = y^{(l+1)} - \sum_{i=1}^{l} t_i\, y^{(i-1)} = 0$ *die Gruppe* $\mathrm{SL}_{l+1}(C)$ *als Differentialgaloisgruppe über F,*
- $L(y, t_1, ..., t_l) = y^{(2l)} - \sum_{i=1}^{l}(-1)^{i-1}(t_i\, y^{(l-i)})^{(l-i)} = 0$ *die Gruppe* $\mathrm{SP}_{2l}(C)$ *als Differentialgaloisgruppe über F,*
- $L(y, t_1, ..., t_l) = y^{(2l+1)} - \sum_{i=1}^{l}(-1)^{i-1}((t_i\, y^{(l+1-i)})^{(l-i)} + (t_i\, y^{(l-i)})^{(l+1-i)}) = 0$ *die Gruppe* $\mathrm{SO}_{2l+1}(C)$ *als Differentialgaloisgruppe über F,*
- $L(y, t_1, ..., t_l) = y^{(2l)} - 2\sum_{i=3}^{l}(-1)^i((t_i y^{(l-i)})^{(l+2-i)} + (t_i y^{(l+1-i)})^{(l+1-i)})$ $-(t_2 y^{(l-2)} + t_1 y)^{(l)} - ((-1)^l t_1 z_1 + z_2) - \sum_{i=0}^{l-2}(t_2^{(l-2-i)} z_1)^{(i)}$ *die Gruppe* $\mathrm{SO}_{2l}(C)$ *als Differentialgaloisgruppe über F. Hierbei sind*

$$\begin{aligned} z_1 &= y^{(l)} - t_2 y^{(l-2)} - t_1 y \\ z_2 &= \frac{(t_2^{(l-2)} + (-1)^{l-2} t_1)^{(1)}}{t_2^{(l-2)} + (-1)^{l-2} t_1} \cdot \left(y^{(2l-1)} - 2\sum_{i=3}^{l}(-1)^i((t_i y^{(l-i)})^{(l+1-i)} \right. \\ &\quad \left. + (t_i y^{(l+1-i)})^{(l-i)}) - (t_2 y^{(l-2)} + t_1 y)^{(l-1)} - \sum_{i=0}^{l-3}(t_2^{(l-3-i)} z_1)^{(i)} \right), \end{aligned}$$

- $L(y, t_1, t_2) = y^{(7)} + 2t_1 y' + 2(t_1 y)' + 2(t_2 y^{(4)})' + (t_2 y')^{(4)} - 2(t_2(t_2 y')')' = 0$ *die Gruppe* $G_2(C)$ *als Differentialgaloisgruppe über $F = C\langle t_1, t_2\rangle$.*

Abschließend geben wir einen Überblick über den Inhalt der einzelnen Kapitel. Zunächst führt die Arbeit in die Grundlagen der Differentialgaloistheorie ein, um auf diesen aufbauend unsere Methode zu entwickeln. Genauer werden hier die grundlegenden Begriffe der Differentialgaloistheorie eingeführt und es werden die wichtigsten Ergebnisse der Picard-Vessiot-Theorie präsentiert. Weiter enthält der einführende Abschnitt eine Formulierung des Hauptsatzes der Differentialgaloistheorie und des Torsor-Satzes. Das erste Kapitel

schließt mit dem „Cyclic Vector Theorem“ zur Konstruktion von linearen Differentialgleichungen aus Matrixdifferentialgleichungen.
Zum Beginn des zweiten Kapitels werden die klassischen Schranken für die Differentialgaloisgruppe eingeführt und mit der Entwicklung des neuen unteren Schrankenkriteriums begonnen. Dazu werden zunächst Picard-Vessiot-Erweiterungen über Differentialringen untersucht und die benötigten Ergebnisse bewiesen. In dem darauffolgenden Abschnitt betten wir die entsprechenden Differentialringe in Potenzreihenkörper ein, damit wir eine wohlverhaltende Spezialisierung eines maximalen Differentialideals erhalten. Abschließend wird die Spezialisierungsschranke bewiesen.
Das nächste Kapitel enthält eine Zusammenfassung der Struktur der klassischen Gruppen und ihrer Lie Algebren und führt somit die Grundlagen für den Beweis der Transformationslemmata ein. Um nun die Spezialisierungsschranke anwenden zu können, modifizieren wir die Ideen von Mitschi und Singer zur Realisierung von zusammenhängenden halbeinfachen Gruppen über $C(z)$. Abschließend beweisen wir allgemein die Existenz einer parametrisierten Differentialgleichung für zusammenhängende halbeinfache lineare algebraische Gruppen.
In den darauffolgenden Kapiteln wird die Methode auf die einzelnen Gruppen vom Typ A_l, C_l, B_l, D_l, G_2, F_4 und E_6 angewandt. Die Ausarbeitungen hierzu enthalten eine Darstellung der Lie Algebra und eine für den Beweis des Transformationslemmas vorbereitende Analyse des Wurzelsystems. Diese wie auch der Beweis des Transformationslemmas gestaltet sich je nach Typ des Wurzelsystems unterschiedlich schwierig. Als nächstes wird mit Hilfe des Transformationslemmas und Mitschis und Singers Diffferentialgleichung bewiesen, dass eine Differentialgleichung gewünschter Gestalt mit vorgegebener Differentialgaloisgruppe über $C(z)$ existiert. Im Fall der Serien und der Ausnahmegruppe vom Typ G_2 wird nun für die jeweilige Matrixdifferentialgleichung ein zyklischer Vektor bestimmt, so dass dieser zu einer einfachen parametrisierten linearen Differentialgleichung führt. Für die Gruppen vom Ausnahmetyp F_4 und E_6 geben wir nur die jeweiligen Matrixdifferentialgleichungen an, um die Präsentation einer riesigen linearen Differentialgleichungen zu vermeiden. Zuletzt wird mit Hilfe der Schranken und der zu Mitschi und Singer differentialäquivalenten spezialisierten Gleichung bewiesen, dass unsere parametrisierte Differentialgleichung die vorgegebene Gruppe als Differentialgaloisgruppe realisiert.

Chapter 1

Basics on differential Galois theory

In this chapter, we present some basic definitions and results of differential Galois theory. We refer to the books [PS03] and [Mag94] for more and detailed information. The reader familiar with the basic properties of differential Galois theory may skip this chapter.

1.1 Matrix differential equations

Definition 1.1. 1. Let R be a commutative ring. We call the map $\partial_R : R \to R$ a **derivation** if it is additive and satisfies

$$\partial_R(r_1 r_2) = \partial_R(r_1) r_2 + r_1 \partial_R(r_2)$$

for all $r_1, r_2 \in R$. A ring (resp. field) together with such a map is called a **differential ring** (resp. **field**). The set of elements satisfying $\partial(r) = 0$ is called the set of **constants** C of R. It is easy to see that they form a subring of R (resp. subfield of F).

2. Let R_1 and R_2 be differential rings, $\phi \in \mathrm{Hom}(R_1, R_2)$ a ring homomorphism, and $I \triangleleft R_1$ an ideal. Then ϕ is called a **differential homomorphism** if it satisfies $\phi \circ \partial_{R_1} = \partial_{R_2} \circ \phi$, and I is called a **differential ideal** if $\partial_{R_1}(r) \in I$ for all $r \in I$.

3. We call a differential ring R a **simple differential ring** if its only differential ideals are (0) and R.

For the rest of this work, we assume C is an algebraically closed field of characteristic zero. Let (F, ∂_F) be a differential field with field of constants C. Let A be an $n \times n$ matrix (i.e., $A \in F^{n \times n}$), and denote by $\boldsymbol{y}^{tr} = (y_1, ..., y_n)^{tr}$ a vector of length n. Then an equation of the form

$$\partial(\boldsymbol{y}) = A\boldsymbol{y}$$

is called a **matrix differential equation**, where $\partial(\boldsymbol{y})$ denotes the component-wise derivation of $\boldsymbol{y}$ (when applied to vectors or matrices, the symbol ∂ always denotes the component-wise derivation).

Lemma 1.2. *Let $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ be a matrix differential equation of dimension n over F, and let $E \geq F$ be a differential field extension (i.e., $E \geq F$ is a field extension and $\partial_E|_F = \partial_F$ holds). The solution space*

$$V := \{\boldsymbol{y} \in E^n \mid \partial(\boldsymbol{y}) = A\boldsymbol{y}\}$$

is a vector space of dimension $\dim_C(V) \leq n$ over the field of constants C.

Suppose we are in the situation of Lemma 1.2, and the solution space $V \subset E^n$ has dimension n. Choose a basis $\boldsymbol{y}_1, ..., \boldsymbol{y}_n$ of V, and write $Y \in \mathrm{GL}_n(E)$ for the matrix with columns $\boldsymbol{y}_1, ..., \boldsymbol{y}_n$. Then $\partial(Y) = AY$ holds. This leads to

Definition 1.3. A matrix $Y \in \mathrm{GL}_n(E)$ satisfying $\partial(Y) = AY$ is called a **fundamental solution matrix** for the differential matrix equation defined by A.

Let $B \in \mathrm{GL}_n(F)$, and let $\boldsymbol{y} \in V$ be a solution of $\partial(\boldsymbol{y}) = A\boldsymbol{y}$. Then the derivative of $B\boldsymbol{y}$ is given by

$$\partial(B\boldsymbol{y}) = \partial(B)\boldsymbol{y} + B\partial(\boldsymbol{y}) = (\partial(B)B^{-1} + BAB^{-1})B\boldsymbol{y}.$$

In other words, $B\boldsymbol{y}$ is a solution of the matrix differential equation

$$\partial(\boldsymbol{x}) = (BAB^{-1} + \partial(B)B^{-1})\boldsymbol{x} =: \tilde{A}\boldsymbol{x}.$$

We see that solutions of the first equation can be transformed into solutions of the second one. This motivates

Definition 1.4. Two matrix differential equations $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ and $\partial(\boldsymbol{x}) = \tilde{A}\boldsymbol{x}$ are called **equivalent** if there exists a matrix $B \in \mathrm{GL}_n(F)$ such that

$$\tilde{A} = BAB^{-1} + \partial(B)B^{-1}.$$

We now formalize the language of matrix differential equations by the introduction of differential modules. A differential module M over the differential field (F, ∂) can be considered as a finite dimensional F-vector space which is also a $F[\partial]$-left module, where $F[\partial]$ denotes the ring of linear differential operators, that is the noncommutative ring of polynomials in the variable ∂ with coefficients in F, such that $\partial a = a\partial + \partial_F(a)$ for all $a \in F$. An equivalent definition is

Definition 1.5. Let (F, ∂_F) be a differential field. A **differential module** M over the differential field (F, ∂) is a finite dimensional F-vector space together with a map $\partial : M \to M$ satisfying for $m_1, m_2 \in M$ and $f \in F$:

1. $\partial(m_1 + m_2) = \partial(m_1) + \partial(m_2)$
2. $\partial(fm_1) = \partial_F(f)m_1 + f\partial(m_1)$.

If M is a differential module over the differential field (F, ∂_F), and $e_1, ..., e_n$ is a basis of M, then

$$\partial(e_i) = \sum_{j=1}^{n} a_{ij}e_j$$

for some $A \in F^{n\times n}$. The matrix A is called a defining matrix for the differential module M. Conversely, let M be an F-vector space with basis $e_1, ..., e_n$, and let $A \in F^{n\times n}$. Then we can make M into a differential module by setting

$$\partial(\sum_{i=1}^{n} f_i e_i) := \sum_{i=1}^{n} \left(\partial_F(f_i) e_i + f_i \sum_{j=1}^{n} a_{ji} e_j \right).$$

In this situation we call the differential module M associated to the matrix differential equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$. Let $\tilde{e}_1, ..., \tilde{e}_n$ be another basis for M with defining matrix $\tilde{A}$, and let $B \in \mathrm{GL}_n(F)$ such that $\tilde{f}_i = \sum_{j=1}^{n} b_{ij} f_j$. Denote by $\boldsymbol{e}$ (resp. $\tilde{\boldsymbol{e}}$) the vector $\boldsymbol{e}^{tr} = (e_1, ..., e_n)^{tr}$ (resp. $\tilde{\boldsymbol{e}}^{tr} = (\tilde{e}_1, ..., \tilde{e}_n)^{tr}$). Then we obtain

$$\partial(\tilde{\boldsymbol{e}}) = \partial(B\boldsymbol{e}) = \partial_F(B)\boldsymbol{e} + BA\boldsymbol{e} = \tilde{A}B\boldsymbol{e},$$

which is equivalent to

$$\left(\partial_F(B)B^{-1} + BAB^{-1}\right) B\boldsymbol{e} = \tilde{A}B\boldsymbol{e}.$$

Thus, we have

$$\partial_F(B)B^{-1} + BAB^{-1} = \tilde{A}.$$

As in Definition 1.4 we say that the two differential equations $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ and $\partial(\boldsymbol{y}) = \tilde{A}\boldsymbol{y}$ are equivalent.
Our approach involves differential conjugation, i.e., the transformation of a matrix equation into an equivalent one as in Definition 1.4. This motivates

Observation 1.6. Let $A \in F^{n\times n}$, and $B_i \in \mathrm{GL}_n(F)$, where $1 \leq i \leq k+1$ for some $k \in \mathbb{N}$. Set

$$\tilde{A} = (\textstyle\prod_{i=1}^{k} B_{k+1-i})A(\prod_{i=1}^{k} B_{k+1-i})^{-1} + \partial(\prod_{i=1}^{k} B_{k+1-i})(\prod_{i=1}^{k} B_{k+1-i})^{-1}.$$

An easy inductive argument shows that

$$B_{k+1}\tilde{A}B_{k+1}^{-1} + \partial(B_{k+1})B_{k+1}^{-1} = (\prod_{i=1}^{k+1} B_{k+2-i})A(\prod_{i=1}^{k+1} B_{k+2-i})^{-1} + \partial(\prod_{i=1}^{k+1} B_{k+2-i})(\prod_{i=1}^{k+1} B_{k+2-i})^{-1}.$$

Throughout the text, we use Observation 1.6 without explicit reference.

1.2 Picard-Vessiot extensions

Assume that the dimension of the solution space V of $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ over F is strictly less than n or that a fundamental matrix $Y \in \mathrm{GL}_n(F)$ does not exist. Then we have to enlarge the differential field F to guarantee enough solutions, i.e., one has to consider differential field extensions $E \geq F$. What follows is the analogue of a splitting field for differential equations.

Definition 1.7. A **Picard-Vessiot ring** over F for the matrix differential equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ is a differential ring R over F satisfying:

1. R is a simple differential ring.

2. There exists a fundamental matrix $Y \in \mathrm{GL}_n(R)$, such that $\partial(Y) = AY$.

3. R is generated as a ring by F, the entries of a fundamental matrix Y, and $\det(Y)^{-1}$.

Using the first condition, it is shown in [PS03, Lemma 1.17] that a Picard-Vessiot ring is always an integral domain. Thus we can define the field of fractions of a PV-ring.

Definition 1.8. The field of fractions E of a Picard-Vessiot ring R for a differential equation over F is called a **Picard-Vessiot field** or a **Picard-Vessiot extension** of F.

Moreover, with the additional help of the third condition it can be shown (e.g., see [PS03, Lemma 1.17]) that the set of constants of R coincides with C. This implies the third statement of the next proposition, proving the existence and uniqueness up to isomorphism of a Picard-Vessiot extension.

Proposition 1.9. *Let $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ be a matrix differential equation over F.*

1. *There exists a Picard-Vessiot ring for the equation.*

2. *Any two Picard-Vessiot rings for $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ are differential isomorphic.*

3. *The quotient field of a Picard-Vessiot extension does not contain new constants.*

The idea of the proof is to construct a particular Picard-Vessiot ring R, which is called the universal solution algebra. One proceeds in following way. Equip the coordinate ring $F[\mathrm{GL}_n] = F[X_{ij}, \det(X_{ij})]$ of the general linear group with a derivation defined by the rule $\partial(X_{ij}) = A(X_{ij})$. Then by construction, the matrix (X_{ij}) is a fundamental solution matrix for the differential equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$. Hence, the second and third conditions are trivially satisfied. For the first condition, we choose a maximal differential ideal I in $F[X_{ij}, \det(X_{ij})]$ and set $R = F[X_{ij}, \det(X_{ij})]/I$.
The following proposition gives an equivalent definition of a Picard-Vessiot extension:

Proposition 1.10. *Let $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ be a matrix differential equation over F, and let $E \geq F$ be a differential field extension. Then $E \geq F$ is a Picard-Vessiot extension if and only if E is generated over F by the entries of a fundamental solution matrix $Y \in \mathrm{GL}_n(E)$ of $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ and the field of constants of E is C.*

For a proof see [PS03, Proposition 1.22].

1.3 The differential Galois group

We can now define the differential Galois group.

Definition 1.11. The **differential Galois group** of a differential equation over F is defined as the group $\mathrm{Aut}^{\partial}(R/F) = \mathrm{Gal}(R/F)$ of differential F-algebra automorphisms of a Picard-Vessiot ring R for the equation.

Let $Y \in \mathrm{GL}_n(R)$ be a fundamental solution matrix with coefficients in the Picard-Vessiot ring R for a differential equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$. Since $A \in F^{n\times n}$, it is left fixed by all $\sigma \in \mathrm{Gal}(R/F)$. Hence, σ sends Y to another fundamental solution matrix. It can be

seen easily that two fundamental solution matrices only differ by a constant matrix. Thus $\sigma(Y) = YC_\sigma$ with $C_\sigma \in \mathrm{GL}_n(C)$. So the above abstract definition becomes more concrete, i.e., there is a faithful representation $\mathrm{Gal}(R/F) \hookrightarrow \mathrm{GL}_n(C)$.
The above definition was made for Picard-Vessiot rings. It turns out that the group of differential automorphisms $\mathrm{Aut}^\partial(E/F)$ for a Picard-Vessiot field E coincides with the group of differential automorphisms $\mathrm{Aut}^\partial(R/F)$ for the corresponding Picard-Vessiot ring. All of this can be read in [PS03, Section 1.4]. The most important facts are summarized in

Proposition 1.12. *Let $E \geq F$ be a Picard-Vessiot field with differential Galois group $\mathrm{Aut}^\partial(E/F)$. Then*

1. *$\mathrm{Aut}^\partial(E/F)$ is the group of C-points $\mathcal{G}(C) \leq \mathrm{GL}_n(C)$ of a linear algebraic group $\mathcal{G}$ over C.*
2. *The field $E^{\mathrm{Aut}(E/F)}$ of $\mathrm{Aut}^\partial(E/F)$-invariant elements of the Picard-Vessiot field E is equal to F.*
3. *The Lie algebra $\mathrm{Lie}(\mathcal{G})(C)$ of $\mathcal{G}(C)$ coincides with the Lie algebra of derivations of E/F that commute with the derivation on E.*

For a proof we refer to [PS03, Theorem 1.27].
We finish this paragraph with the differential Galois correspondence. It is cited here for completeness.

Theorem 1.13. *Let $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ be a matrix differential equation over F and E/F a Picard-Vessiot extension for the equation. Let $\mathcal{G}$ be a linear algebraic group over C such that $\mathrm{Gal}(E/F) \cong \mathcal{G}$.*

1. *There exists an anti-isomorphism between the lattice of closed subgroups $\mathcal{H}(C)$ and the lattice of intermediate differential fields $E \geq L \geq F$ given by*
$$\mathcal{H}(C) \mapsto E^{\mathcal{H}(C)},\ L \mapsto \mathrm{Gal}(E/L).$$
2. *Let $\mathcal{H}(C) \triangleleft \mathcal{G}(C)$ be a normal closed subgroup. Then $E^{\mathcal{H}(C)}/F$ is a Picard-Vessiot extension with $\mathrm{Gal}(E^{\mathcal{H}(C)}/F) \cong (\mathcal{G}/\mathcal{H})(C)$.*
3. *Denote by $\mathcal{G}^0$ the connected component of $\mathcal{G}$. Then $E^{\mathcal{G}^0(C)}/F$ is a finite Galois extension with Galois group isomorphic to $(\mathcal{G}/\mathcal{G}^0)(C)$.*

A proof can be found in [PS03, Proposition].

1.4 Torsors

We start this section with

Definition 1.14. Let $\mathcal{G}$ be a linear algebraic group defined over the field $F \geq C$. A **$\mathcal{G}$-Torsor** is an affine scheme $\mathcal{Z}$ over F with a faithful $\mathcal{G}$-action, i.e., a morphism

$$\mathcal{G} \times_F \mathcal{Z} \to \mathcal{Z},\ (X, Z) \to ZX \tag{1.1}$$

such that the morphism

$$\mathcal{G} \times_F \mathcal{Z} \to \mathcal{Z} \times_F \mathcal{Z},\ (X, Z) \to (ZX, Z) \tag{1.2}$$

is an isomorphism. A $\mathcal{G}$-torsor $\mathcal{Z}$ is called **trivial** if $\mathcal{Z} \cong \mathcal{G}$ holds and the $\mathcal{G}$-action is given by the multiplication on $\mathcal{G}$.

Assume that $\mathcal{Z}$ is a trivial $\mathcal{G}$-Torsor. Then $\mathcal{Z}$ has an F-rational point, i.e., $\mathcal{Z}(F) \neq \emptyset$ since $\mathcal{G}$ has one. Conversely, let $\mathcal{Z}(F)$ be non empty. For $Z \in \mathcal{Z}(F)$, the morphism

$$\mathcal{G} \times \{Z\} \to \mathcal{Z} \times \{Z\},\ (X, Z) \mapsto (ZX, Z)$$

is an isomorphism by Definition 1.14(2). Hence a $\mathcal{G}$-torsor $\mathcal{Z}$ is trivial if and only if it has an F-rational point.
Now we want to explain how this applies in differential Galois theory. Therefore let $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ be a matrix differential equation. Consider the universal solution algebra $R := F[X_{ij}, \det(X_{ij})^{-1}]/I$ with I a maximal differential ideal, i.e., a Picard-Vessiot ring for the equation and denote by $\mathcal{G}(\mathcal{C})$ the differential Galois group for the corresponding Picard-Vessiot extension $\mathrm{Quot}(R)/F$. Define $\mathcal{Z}$ as the affine scheme $\mathrm{Spec}(R)$ over F. In this situation we have

Theorem 1.15. *$\mathcal{Z}$ is a $\mathcal{G}$-Torsor over F.*

For a proof see [PS03, Theorem 1.28].
A consequence of Theorem 1.15 is Kolchins Structure Theorem. It states that a $\mathcal{G}$-Torsor $\mathcal{Z} = \mathrm{Spec}(R)$ for a Picard-Vessiot ring R over (F, ∂) with differential Galois group $\mathcal{G}(\mathcal{C})$ becomes isomorphic to the trivial torsor $\mathcal{Z} \cong \mathcal{G}$ after a finite field extension $\tilde{F}/F$. Equivalently this can be expressed as

$$\tilde{F} \otimes_F R \xrightarrow{\sim} \tilde{F} \otimes_C C[\mathcal{G}].$$

We will use Theorem 1.15 for the proof of the bounds for the differential Galois group in the next chapter. Since the Picard-Vessiot rings R are $\mathcal{G}$-Torsors, the correspondence between the first cohomology set and torsor presented in Proposition 1.16 below is in some situations useful.

Proposition 1.16. *For a linear algebraic group defined over F there is a bijection between the $\mathcal{G}$-torsors and $H^1(\bar{F}/F, \mathcal{G}(\bar{F}))$.*

The proof of this goes back to Serre and can be found in [Ser97, I 5.2, Proposition 33].

1.5 Homogeneous linear differential equations

Let (R, ∂_R) be a differential ring. Then by the **ring of differential polynomials** in the **differential indeterminate** y we mean the polynomial ring

$$R\{y\} := R[\partial^i(y) := y^{(i)} \mid i = 0, 1, 2, ...]$$

in the countable number of indeterminates with the derivation ∂ satisfying $\partial|_R = \partial_R$ and $\partial(y^{(i)}) = y^{(i+1)}$. Now let R be a differential integral domain and F be the field of fractions

of R. Then $R\{y\}$ is also an integral differential domain. We write $F\langle y\rangle$ for the field of fractions of $R\{y\}$. If $S \geq R$ is a differential ring extension (resp. $E \geq F$ a differential field extension) and $X \subset R$ is a subset of R (resp. $X \subset F$ is a subset of F), then we mean by $R\{X\}$ the differential R-subalgebra of S generated by the elements of X (resp. by $F\langle X\rangle$ the differential subfield of E generated by F and the elements of X).
Let (F, ∂_F) be a differential field and $C \leq F$ the field of constants of F. Later, F will be the differential field $C\langle \boldsymbol{t}\rangle$ generated by finitely many differential indeterminates $\boldsymbol{t} = (t_1, ..., t_l)$ over C. Let $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1y_1' + a_0y \in F\{y\}$ be a monic linear homogenoues element of $F\{y\}$. Then a **homogeneous linear differential equation** of degree n over the differential field F is defined as an equation of the form

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1y' + a_0y = 0.$$

To Lemma 1.2 in the case of matrix differential equations one has the following analogue.

Lemma 1.17. *Consider a homogeneous linear differential equation $L(y) = 0$ of degree n over F and a differential field extension $E \geq F$. Then the solution space*

$$V := \{y \in E \mid L(y) = 0\}$$

of $L(y) = 0$ in E is a vector space over C of dimension $\dim_C(V) \leq n$.

A proof can be found in [PS03, Lemma 1.10].
Analogously to the fundamental matrix, one calls a set $y_1, ..., y_n$ of elements satisfying $L(y_i) = 0$ a fundamental set of solutions of $L(y) = 0$, if the y_i are linear independent over C. Again, motivated by the same aspects as in the case of matrix equations, one defines Picard-Vessiot extensions as

Definition 1.18. Let $L(y) = 0$ be a homogeneous linear differential equation of degree n over a differential field F. A differential extension field $E \geq F$ is called a **Picard-Vessiot extension** of F for $L(y)$ if:

1. E is generated over F as a differential field by a fundamental set of solutions $y_1, ..., y_n$ of $L(y) = 0$ in E (i.e., $E = F\langle y_1, ..., y_n\rangle$).
2. Every constant of E lies in F.

The next step is to explain the connection between matrix differential equations and homogeneous linear differential equations. Later we will consider both. Again, let $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1y' + a_0y = 0$ be a homogeneous linear differential equation over (F, ∂_F) and let $y_1, ..., y_n$ be the fundamental set of solutions of $L(y) = 0$ in a differential extension field $E \geq F$. The matrix A_L defined as

$$A_L = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & \cdots & -a_{n-1} \end{pmatrix}$$

is called the **companion matrix** of L. We will also denote the derivation ∂_E of an element $a \in E$ by $a' := \partial_E(a)$. Then the map

$$\iota : y_i \to (y_i, y_i', ..., y_i^{(n-1)})^{tr} =: \boldsymbol{y}_i^{tr}$$

defines an isomorphism of the solution space of $L(y)$ onto a solution space of the matrix differential equation $\partial(\boldsymbol{y}) = A_L\boldsymbol{y}$. The matrix

$$W(y_1, ..., y_n) = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

is called the **Wronskian matrix**. One sees easily that $W(y_1, ..., y_n) := Y$ is a fundamental solution matrix for the matrix equation $\partial(\boldsymbol{y}) = A_L\boldsymbol{y}$. Suppose $E = F\langle y_1, ..., y_n\rangle / F$ is a Picard-Vessiot extension for $L(y)$ in the sense of Definition 1.18. Then the Wronskian matrix satisfies the conditions of Proposition 1.10. Thus E over F is a Picard-Vessiot extension in the sense of Definition 1.8.
Above we have seen that one can convert a homogeneous linear differential equation $L(y)$ into a matrix differential equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$. In many situations the converse is also true, i.e., a matrix differential equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ is equivalent to a matrix equation of type $\partial(\boldsymbol{y}) = A_L\boldsymbol{y}$. There are several proofs of the so called *Cyclic Vector Theorem* (e.g., see [Kat87] or [Kov96]). Let $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ be a matrix differential equation and denote by M_A the associated differential module with basis $e_1, ..., e_n$. Let $m \in M_A$. By the differential span $\langle m\rangle$ of m we mean the smallest vector space closed under ∂ containing m.

Definition 1.19. Let M be a differential module over (F, ∂_F). A vector $m \in M$ is called a **cyclic vector** if $\langle m\rangle = M$ holds.

Suppose m is a cyclic vector of M_A. Then the $n+1$ vectors $m, \partial(m), ..., \partial^n(m)$ are linearly dependent. Let $1 \leq r \leq n$ such that $\partial^r(m)$ can be written as a linear combination of $\partial^i(m)$ with $0 \leq i \leq r$. Without loss of generality we may assume $r = n$ (if not, one differentiates the equation). Hence, there are $a_i \in F$ such that

$$\partial^n(m) = -a_{n-1}\partial^{n-1}(m) - ... - a_1\partial(m) - a_0 m.$$

By an easy computation argument we see that the defining matrix with respect to the basis $m, \partial(m), ..., \partial^{n-1}(m)$ is the companion matrix

$$A_L = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & \cdots & -a_{n-1} \end{pmatrix}.$$

In other words, if $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ is the defining matrix differential equation for a differential vector space M_A with respect to a basis $e_1, ..., e_n$ which has a cyclic vector m, then A is differentially equivalent to a companion matrix A_L.

Theorem 1.20. *Let M be a finite dimensional differential module over a non trivial differential field (F, ∂_F) of characteristic zero with algebraically closed field of constants C. Then M has a cyclic vector.*

Since our approach includes matrix differential equations, we have to compute cyclic vectors. Luckily, the shapes of our matrix equations are such that one can easily detect a cyclic vector, which will lead to nice linear differential equations.
Let E over F be a Picard-Vessiot extension for a matrix differential equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ in the sense of Proposition 1.10 with fundamental solution matrix Y. Further, let F satisfy the conditions of Theorem 1.20. Then there exists $B \in \mathrm{GL}_n(F)$ such that A is differentially equivalent to a companion matrix A_L. The equation $\partial(\boldsymbol{x}) = A_L\boldsymbol{x}$ has fundamental solution matrix $BY = X$. Thus, $E = F\langle X_{11}, ..., X_{1n}\rangle$ is a Picard-Vessiot extension in the sense of Definition 1.18.

Chapter 2

Bounds for the differential Galois group

In this chapter we present upper and lower bounds for the differential Galois group.

2.1 The classical bounds

Everything described in this section is well known and can be found in [PS03, Section 1.4]. In the following, let (F, ∂_F) be a differential field of characteristic zero over an algebraically closed field of constants C. Proposition 2.1 below is known as an upper bound criterion for the differential Galois group.

Proposition 2.1. *Let $\mathcal{H} \leq \mathrm{GL}_n(C)$ be a connected linear algebraic group over C with Lie algebra $\mathrm{Lie}(\mathcal{H}) \leq C^{n\times n}$. Suppose that the matrix equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ over F satisfies $A \in \mathrm{Lie}(\mathcal{H})(F)$. Then the differential Galois group $\mathcal{G}(\mathcal{C})$ of the equation is contained in (a conjugate of) $\mathcal{H}(\mathcal{C})$.*

For a proof we refer to [PS03, Proposition 1.31.1].
The next theorem can be regarded as a lower bound criterion. In [MS96], Mitschi and Singer used it to prove that every connected reductive group can be realized as a differential Galois group over the differential field $C(z)$ with standard derivation $\partial_{C(z)} = \frac{d}{dz}$, where C is an algebraically closed field of characteristic zero. Later, we combine this fact with the results of the next section to develop another lower bound.

Theorem 2.2. *Let R be a Picard-Vessiot ring for the equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ over F with connected Galois group $\mathcal{G}(\mathcal{C})$ and let $\mathcal{Z}$ be the associated $\mathcal{G}$-torsor. Let $\mathrm{Lie}(\mathcal{G})(C)$ be the Lie algebra of $\mathcal{G}(\mathcal{C})$ and let $\mathcal{H}(\mathcal{C}) \geq \mathcal{G}(\mathcal{C})$ be a connected linear algebraic group with Lie algebra $\mathrm{Lie}(\mathcal{H})(C)$. Suppose that $A \in \mathrm{Lie}(\mathcal{H})(F)$. If $\mathcal{Z}$ is the trivial torsor then there exists $B \in \mathcal{H}(\mathcal{F})$ such that the equivalent equation $\partial(\boldsymbol{x}) = \tilde{A}\boldsymbol{x}$ where $\boldsymbol{y} = B\boldsymbol{x}$ and $\tilde{A} = B^{-1}AB - B'B^{-1}$ satisfies $\tilde{A} \in \mathrm{Lie}(\mathcal{G})(F)$.*

A proof can be found in [PS03, Proposition 1.31.2].
In the following sections we will develop another lower bound criterion which is based on the calculus of specializations. The main ingredient to obtain a well behaving specialization is an embedding of the corresponding differential rings in a suitable field of power series. We start with the study of differential rings and their Picard-Vessiot extensions.

2.2 Picard-Vessiot extensions over rings

Let (F, ∂) be a differential field with field of constants C and $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ a matrix differential equation over F. Let $F[X_{ij}, \det(X_{ij})^{-1}]$ be the coordinate ring for $\mathrm{GL}_n(F)$ and choose a maximal differential ideal $\tilde{I}$ of $F[X_{ij}, \det(X_{ij})^{-1}]$. Denote by

$$L := F[X_{ij}, \det(X_{ij})^{-1}]/\tilde{I}$$

the Picard-Vessiot ring and by $E := \mathrm{Quot}(L)$ the corresponding Picard-Vessiot extension. Now let R be an integral differential ring with field of constants C such that $\mathrm{Quot}(R) = F$, and let $R[X_{ij}, \det(X_{ij})^{-1}]$ be the coordinate ring for $\mathrm{GL}_n(R)$. Suppose the matrix A satisfies

$$A \in R^{n\times n}.$$

Then $R[X_{ij}, \det(X_{ij})^{-1}]$ becomes a differential ring by $\partial(\boldsymbol{y}) = A\boldsymbol{y}$. We can choose a differential ideal I in $R[X_{ij}, \det(X_{ij})^{-1}]$ which satisfies $I \cap R = (0)$ and is maximal with this property. Then we can define a differential ring extension S/R by

$$S := R[X_{ij}, \det(X_{ij})^{-1}]/I.$$

By construction S is an R-simple differential ring, i.e., S does not contain proper differential ideals I with $I \cap R = (0)$. Further, S is generated over R by the entries of the fundamental solution matrix

$$(Z_{ij}) := (X_{ij} \bmod I) \in \mathrm{GL}_n(S)$$

and by the inverse of the determinant $\det(Z_{ij})^{-1}$, i.e., (Z_{ij}) satisfies $\partial(Z_{ij}) = A(Z_{ij})$ and $S = R[Z_{ij}, \det(Z_{ij})^{-1}]$. Since S is an R-simple differential ring the field of constants of S is C. The maximality of the differential ideal I implies that the ideal (I) generated by I over $F[X_{ij}, \det(X_{ij})^{-1}]$ is a maximal differential ideal. Say we choose the ideal $\tilde{I}$ from above such that $\tilde{I} = (I)$. Thus we obtain an injection of differential rings

$$S \hookrightarrow L.$$

Since in polynomial rings over rings one has more ideals than over fields, we will consider ideals $I \subset R[X_{ij}, \det(X_{ij})^{-1}]$ (or ideals $I \subset S[X_{ij}, \det(X_{ij})^{-1}]$) satisfying Condition 2.3 presented below.

Condition 2.3. Let $\tilde{R}$ be an integral differential ring with field of constants C and define a differential structure on $\tilde{R}[X_{ij}, \det(X_{ij})^{-1}]$ by a matrix differential equation $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ where $A \in \tilde{R}^{n\times n}$. Then we consider ideals I in $\tilde{R}[X_{ij}, \det(X_{ij})^{-1}]$ satisfying $I \cap \tilde{R} = (0)$, and if for $f \in I$ there exists $r \in \tilde{R}$ such that $(1/r)\cdot f \in \tilde{R}[X_{ij}, \det(X_{ij})^{-1}]$, then $(1/r)\cdot f \in I$.

Lemma 2.4. *Let $\tilde{R}$ be an integral differential ring with field of constants C, and let the field of constants of $\mathrm{Quot}(\tilde{R})$ also be C. We extend the derivation $\partial_{\tilde{R}}$ to a derivation ∂ on $\tilde{R}[Y_{ij}, \det(Y_{ij})^{-1}]$ by setting $\partial(Y_{ij}) = 0$ for $1 \le i, j \le n$. Moreover, one considers $C[Y_{ij}, \det(Y_{ij})^{-1}]$ as a subring of $\tilde{R}[Y_{ij}, \det(Y_{ij})^{-1}]$. Then the map*

$$\delta : I \mapsto (I)$$

from the set of ideals of the ring $C[Y_{ij}, \det(Y_{ij})^{-1}]$ *to the set of differential ideals of* $\tilde{R}[Y_{ij}, \det(Y_{ij})^{-1}]$ *satisfying Condition 2.3 is a bijection. Furthermore, the inverse map is defined by*

$$\delta^{-1} : J \mapsto J \cap C[Y_{ij}].$$

Proof. Let $\{\tilde{r}_j\}_{j\in\mathcal{J}}$ be a basis of $\tilde{R}$ as a vector space over C with $\tilde{r}_{j_0} = 1$. Then $\{\tilde{r}_j\}_{j\in\mathcal{J}}$ is also a basis of the $C[Y_{ij}, \det(Y_{ij})^{-1}]$-module $\tilde{R}[Y_{ij}, \det(Y_{ij})^{-1}]$. The differential ideal (I) consists of elements of the form

$$f = \sum_{j\in\mathcal{J}'} q_j \cdot \tilde{r}_j$$

where $\mathcal{J}' \subset \mathcal{J}$ is a finite subset of $\mathcal{J}$ and $q_j \in I$ for all $j \in \mathcal{J}'$. Therefore we have

$$(I) \cap C[Y_{ij}, \det(Y_{ij})^{-1}] = I.$$

We are going to show that the ideal (I) satisfies Condition 2.3. Therefore we extend the basis $\{\tilde{r}_j\}_{j\in\mathcal{J}}$ of $\tilde{R}$ over C to a basis $\{\tilde{r}_j\}_{j\in\mathcal{M}}$ of $\mathrm{Quot}(\tilde{R})$ over C where $\mathcal{J} \subset \mathcal{M}$. For $f \in (I)$, let $\tilde{r} \in \tilde{R}$ such that $\frac{1}{\tilde{r}} \cdot f \in \tilde{R}[Y_{ij}, \det(Y_{ij})^{-1}]$. We compute

$$\frac{1}{\tilde{r}} \cdot f = \frac{1}{\tilde{r}}(\sum_{j\in\mathcal{J}'} q_j \cdot \tilde{r}_j) = \sum_{j\in\mathcal{J}'} q_j \cdot (\frac{1}{\tilde{r}} \cdot \tilde{r}_j) = \sum_{j\in\mathcal{J}'} q_j \cdot (\sum_{j'\in\mathcal{M}'_j} c_{j'} \cdot \tilde{r}_{j'}) = \sum_{j\in\mathcal{M}'} \tilde{q}_j \cdot \tilde{r}_j,$$

where $\mathcal{M}' = \bigcup_{j\in\mathcal{J}'} \mathcal{M}'_j$ and $\tilde{q}_{j'} = c_{j'} \cdot q_j \in I$ for all $j' \in \mathcal{M}'_j$. Since $\frac{1}{\tilde{r}} \cdot f \in \tilde{R}[Y_{ij}, \det(Y_{ij})^{-1}]$, we obtain $\mathcal{M}' \subset \mathcal{J}$. Thus, $\frac{1}{\tilde{r}} \cdot f \in (I)$.
We have to show that a differential ideal $J \subset \tilde{R}[Y_{ij}, \det(Y_{ij})^{-1}]$ satisfying Condition 2.3 is generated by $I := J \cap C[Y_{ij}, \det(Y_{ij})^{-1}]$. Let $\{\hat{e}_i\}_{i\in\mathcal{I}}$ be a basis of $C[Y_{ij}, \det(Y_{ij})^{-1}]$ over C with $\hat{e}_{i_0} = 1$. Then $f \in J$ can be written for a finite subset $\mathcal{I}' \subset \mathcal{I}$ as

$$f = \sum_{i\in\mathcal{I}'} r_i \cdot \hat{e}_i.$$

We prove by induction on the length $l(f) = \mathrm{card}(\mathcal{I}')$ that $f \in (I)$.
Let $l(f) = 1$. Then f reads as $f = r_i \cdot \hat{e}_i$. The condition on J implies that $(1/r_i)f = \hat{e}_i \in J$. Hence, we obtain $f \in (I)$.
Now let $l(f) > 1$. If all $r_i \in C$, then there is nothing to show. Also if f can be written as $f = \tilde{r} \cdot \sum_{i\in\mathcal{I}'} c_i \cdot \hat{e}_i$, where $c_i \in C$ and $\tilde{r} \in \tilde{R} \setminus C$, since by Condition 2.3 we have

$$\frac{1}{\tilde{r}} \cdot f = \sum_{i\in\mathcal{I}'} c_i \cdot \hat{e}_i \in J$$

and so $f \in (I)$. We claim that for $r_1 \neq 0,\ r_2 \in \tilde{R}$, one has $r_1\partial(r_2) - r_2\partial(r_1) = 0$ if and only if r_1 and r_2 are C-linearly dependent, i.e., it exists $c \in C$ such that $r_1 = cr_2$. Let $r_1\partial(r_2) - r_2\partial(r_1) = 0$. Since $\partial(\frac{r_1}{r_2}) = \frac{\partial(r_1)r_2 - \partial(r_2)r_1}{r_2^2} = 0$, we obtain that $\frac{r_1}{r_2}$ is a constant. Thus, $\frac{r_1}{r_2}$ is an element of C. Hence, r_1 and r_2 are C-linearly dependent. The other direction is trivial. Thus the claim follows.
Without loss of generality let $r_1 \in \tilde{R} \setminus C$, and let r_1 and r_2 be C-linearly disjoint in f.

Hence we obtain

$$\begin{aligned} r_1\partial(f) - \partial(r_1)f &= \sum_{i\in\mathcal{I}'}(r_1\partial(r_i) - \partial(r_1)r_i)\cdot \hat{e}_i \\ &= \sum_{i\in\mathcal{I}'\setminus\{1\}}(r_1\partial(r_i) - \partial(r_1)r_i)\cdot \hat{e}_i := \tilde{f} \neq 0. \end{aligned}$$

Since $l(\tilde{f}) < l(f)$, the induction assumption implies $\tilde{f} = \sum_{i\in\mathcal{I}'\setminus\{1\}} \check{r}_i \cdot \hat{e}_i \in (I)$ where $\hat{r}_i = r_1\partial(r_i) - r_i\partial(r_1)$. We compute

$$\begin{aligned} \check{r}_2 f - r_2\tilde{f} &= \check{r}_2\sum_{i\in\mathcal{I}'} r_i\hat{e}_i - r_2\sum_{i\in\mathcal{I}'\setminus\{1\}}\check{r}_i\hat{e}_i \\ &= \check{r}_2 r_1\cdot\hat{e}_1 - \sum_{i\in\mathcal{I}'\setminus\{1,\,2\}}(\check{r}_2 r_i - r_2\check{r}_i)\hat{e}_i \neq 0. \end{aligned}$$

Further, the length of $\check{r}_2 f - r_2\tilde{f}$ is less than $l(f)$. Hence, $\check{r}_2 f - r_2\tilde{f} \in (I)$ and therefore $\check{r}_2 f \in (I)$. Obviously, we have $\frac{1}{\check{r}_2}\cdot(\check{r}_2 f) \in \tilde{R}[Y_{ij}, \det(Y_{ij})^{-1}]$. Since the ideal (I) satisfies Condition 2.3, we obtain $f \in (I)$. □

Lemma 2.5. *The map*

$$\iota : I \mapsto (I)$$

from the set of ideals in $R[X_{ij}, \det(X_{ij})^{-1}]$ *satisfying Condition 2.3 to the set of* $\mathrm{Gal}(S/R)$-*invariant ideals in* $S[X_{ij}, \det(X_{ij})^{-1}]$ *satisfying Condition 2.3 is a bijection. Furthermore, the inverse map is given by*

$$\iota^{-1} : J \mapsto J \cap R[X_{ij}, \det(X_{ij})^{-1}].$$

Proof. The proof is very similar to the proof of Lemma 2.4.
Choose a basis $\{\hat{s}_j\}_{j\in\mathcal{J}}$ of S as a vector space over R with $\hat{s}_{j_0} = 1$. Then $\{\hat{s}_j\}_{j\in\mathcal{J}}$ is also a basis of the $R[X_{ij}, \det(X_{ij})^{-1}]$-module $S[X_{ij}, \det(X_{ij})^{-1}]$. The $\mathrm{Gal}(S/R)$-invariant ideal (I) consists of elements of the form

$$f = \sum_{j\in\mathcal{J}'} q_j\cdot\hat{s}_j$$

where $\mathcal{J}' \subset \mathcal{J}$ is a finite subset of $\mathcal{J}$ and $q_j \in I$ for all $j \in \mathcal{J}'$. Hence, it holds

$$(I) \cap R[X_{ij}, \det(X_{ij})^{-1}] = I.$$

We are going to show that the ideal (I) satisfies Condition 2.3. For $f \in (I)$, let $\tilde{s} \in S$ such that $\frac{1}{\tilde{s}}\cdot f \in S[X_{ij}, \det(X_{ij})^{-1}]$. Denote by $\{\hat{s}_j\}_{j\in\mathcal{M}}$ a basis of $\mathrm{Quot}(S)$ over $\mathrm{Quot}(R)$ where $\{\hat{s}_j\}_{j\in\mathcal{J}} \subset \{\hat{s}_j\}_{j\in\mathcal{M}}$. Then we get

$$\frac{1}{\tilde{s}}\cdot f = \frac{1}{\tilde{s}}\cdot(\sum_{j\in\mathcal{J}'} q_j\cdot\hat{s}_j) = \sum_{j\in\mathcal{J}'} q_j\cdot(\frac{1}{\tilde{s}}\cdot\hat{s}_j) = \sum_{j\in\mathcal{J}'} q_j\cdot(\sum_{j'\in\mathcal{M}'_j} f_{j'}\cdot\hat{s}_{j'}) = \sum_{j\in\mathcal{M}'} q_j\cdot f_j\cdot\hat{s}_j,$$

where $q_j \in I$, $f_j \in \mathrm{Quot}(R)$ and $\mathcal{M}' := \bigcup_{j\in\mathcal{J}'}\mathcal{M}'_j$ is a finite subset of $\mathcal{M}$. Since $\frac{1}{\tilde{s}}\cdot f \in S[X_{ij}, \det(X_{ij})^{-1}]$, we obtain $\mathcal{M}' \subset \mathcal{J}$. Further, $q_j\cdot f_j \in R[X_{ij}, \det(X_{ij})^{-1}]$ and therefore

we have by Condition 2.3 that $\frac{1}{\tilde{s}} \cdot f \in (I)$.
We have to show that the ideal $J \subset S[X_{ij}, \det(X_{ij})^{-1}]$ is generated by

$$I = J \cap R[X_{ij}, \det(X_{ij})^{-1}].$$

Let $\{e_i\}_{i \in \mathcal{I}}$ be a basis of $R[X_{ij}, \det(X_{ij})^{-1}]$ over R. Then any $f \in J$ can be written for a finite subset $\mathcal{I}' \subset \mathcal{I}$ and $s_i \in S$ as

$$f = \sum_{i \in \mathcal{I}'} s_i \cdot e_i.$$

We prove by induction on the length $l(f) = \text{card}(\mathcal{I}')$ that $f \in (I)$.
Let $l(f) = 1$. Then f reads as $f = s_i e_i$. Since the ideal J satisfies Condition 2.3, we get that $\frac{1}{s_i} \cdot f = e_i \in J$. Thus, it follows that $f \in (I)$.
Now let $l(f) > 1$. If for all s_i it holds $s_i \in R$, then there is nothing to show. Also if f can be written as $f = \tilde{s} \cdot \sum_{i \in \mathcal{I}'} r_i \hat{e}_i$, where $r_i \in R$ and $\tilde{s} \in S$, since by Condition 2.3 we have

$$\frac{1}{\tilde{s}} \cdot f = \sum_{i \in \mathcal{I}'} r_i \hat{e}_i \in J$$

and thus $f \in (I)$. We claim that for all $\sigma \in \text{Gal}(S/R)$ it holds $\sigma(s_1)s_2 - s_1\sigma(s_2) = 0$ if and only if s_1 and s_2 are R-linearly dependent. Let $\sigma(s_1)s_2 - s_1\sigma(s_2) = 0$ for all $\sigma \in \text{Gal}(S/R)$. We obtain $\frac{\sigma(s_1)}{\sigma(s_2)} = \sigma(\frac{s_1}{s_2}) = \frac{s_1}{s_2}$. Since by Theorem 1.13 one has $\text{Quot}(S)^{\text{Gal}(S/R)} = \text{Quot}(R)$, we get $\frac{s_1}{s_2} \in \text{Quot}(R)$. Hence, s_1 and s_2 are R-linearly dependent. The other direction is trivial. Hence the claim follows.
Without loss of generality let $s_1 \in S \setminus R$, and let s_1, s_2 be R-linearly disjoint. We choose $\sigma \in \text{Gal}(S/R)$ such that $\sigma(s_1)s_2 \neq s_1\sigma(s_2)$. Thus we obtain

$$\begin{aligned} s_1\sigma(f) - \sigma(s_1)f &= \sum_{i \in \mathcal{I}'} (s_1\sigma(s_i) - \sigma(s_1)s_i)\hat{e}_i \\ &= \sum_{i \in \mathcal{I}' \setminus \{1\}} (s_1\sigma(s_i) - \sigma(s_1)s_i)\hat{e}_i := \tilde{f}. \end{aligned}$$

We have $l(\tilde{f}) < l(f)$. So by the induction assumption $\tilde{f} = \sum_{i \in \mathcal{I}' \setminus \{1\}} \tilde{s}_i \cdot \hat{e}_i \in (I)$ where $\tilde{s}_i = s_1\sigma(s_i) - \sigma(s_1)s_i$. We calculate

$$\begin{aligned} \tilde{s}_2 f - s_2 \tilde{f} &= \tilde{s}_2 \cdot \sum_{i \in \mathcal{I}'} s_i \cdot \hat{e}_i - s_2 \cdot \sum_{i \in \mathcal{I}' \setminus \{1\}} \tilde{s}_i \cdot \hat{e}_i \\ &= \tilde{s}_2 \cdot s_1 \cdot \hat{e}_1 + \sum_{i \in \mathcal{I}' \setminus \{1,\, 2\}} (\tilde{s}_2 s_i - \tilde{s}_i s_2) \cdot \hat{e}_i \neq 0. \end{aligned}$$

Since the length of $\tilde{s}_2 f - s_2 \tilde{f}$ is less than $l(f)$, we obtain $\tilde{s}_2 f - s_2 \tilde{f} \in (I)$. Further, we have $\tilde{s}_2 \cdot f \in (I)$. Obviously it holds $\frac{1}{\tilde{s}_2} \cdot (\tilde{s}_2 \cdot f) \in S[X_{ij}, \det(X_{ij})^{-1}]$. Since the ideal (I) satisfies Condition 2.3, we get $f \in (I)$.
Let $f \in I = J \cap R[X_{ij}, \det(X_{ij})^{-1}]$, and suppose that it exists $r \in R$ such that $\frac{1}{r} \cdot f \in R[X_{ij}, \det(X_{ij})^{-1}]$. Since the ideal $J \subset S[X_{ij}, \det(X_{ij})^{-1}]$ satisfies Condition 2.3, we obtain $\frac{1}{r} \cdot f \in J$. Hence, we get $\frac{1}{r} \cdot f \in I$, and therefore the ideal I in $R[X_{ij}, \det(X_{ij})^{-1}]$ satisfies Condition 2.3. □

Let $I \subset R[X_{ij}, \det(X_{ij})^{-1}]$ be a differential ideal such that $I \cap R = (0)$. Since I does not necessarily satisfy Condition 2.3, we claim that the expanded ideal

$$I_{ex} := I \cdot F[X_{ij}, \det(X_{ij})^{-1}] \cap R[X_{ij}, \det(X_{ij})^{-1}]$$

is again a differential ideal and satisfies Condition 2.3. This can be seen as follows. Let $f = \sum_i f_i \cdot q_i \in I \cdot F[X_{ij}, \det(X_{ij})^{-1}]$ with $q_i \in I$ and $f_i \in F[X_{ij}, \det(X_{ij})^{-1}]$. Then we compute

$$\partial(f) = \partial(\sum_i f_i \cdot q_i) = \sum_i \partial(f_i) \cdot q_i + f_i \cdot \partial(q_i) \in I \cdot F[X_{ij}, \det(X_{ij})^{-1}]$$

since $\partial(q_i)$ lies again in I. Hence, the ideal $I \cdot F[X_{ij}, \det(X_{ij})^{-1}]$ is a differential ideal. Thus, the intersection

$$I \cdot F[X_{ij}, \det(X_{ij})^{-1}] \cap R[X_{ij}, \det(X_{ij})^{-1}]$$

is again a differential ideal and satisfies Condition 2.3.
Further, if $I \subset R[X_{ij}, \det(X_{ij})^{-1}]$ is a maximal differential ideal satisfying $I \cap R = (0)$, then I automatically satisfies Condition 2.3. This is a consequence of the above and the maximality of I.

2.3 Formal Taylor series

Now our differential fields become more specific. Let $F := C\langle t_1, ..., t_l\rangle$ be the differential field in the l differential indeterminates $\boldsymbol{t} = (t_1, ..., t_l)$ and denote by $R := C\{t_1, ..., t_l\} \subset F$ the corresponding differential subring. Let $\bar{F} := C(z)$ be a rational function field where the derivation is defined by $\partial_{\bar{F}} = \frac{d}{dz}$, and let $\bar{R} := C[z] \subset \bar{F}$ be the corresponding differential subring. Further, let $\partial(\boldsymbol{y}) = A(\boldsymbol{t})\boldsymbol{y}$ be a matrix differential equation over F such that $A \in C\{\boldsymbol{t}\}^{n\times n}$. Moreover, let

$$\sigma : \boldsymbol{t} \mapsto \boldsymbol{f} = (f_1, ..., f_l)$$

be a specialization to $\bar{R}$ such that $C\{\boldsymbol{f}\} = \bar{R}$. We are going to show that there exists a maximal differential ideal $I \subset U := R[X_{ij}, \det(X_{ij})^{-1}]$ with $I \cap R = (0)$ such that its specialization $\sigma(I)$ is contained in a proper differential ideal $\bar{I} \subset \bar{U} := \bar{R}[X_{ij}, \det(X_{ij})^{-1}]$ with $\bar{I} \cap \bar{R} = (0)$ where the differential structure on U (resp. $\bar{U}$) is defined by $\partial(X_{ij}) = A(\boldsymbol{t})(X_{ij})$ (resp. $\partial(X_{ij}) = A(\sigma(\boldsymbol{t}))(X_{ij})$).

Lemma 2.6. *Let $\bar{F} = C(z)$ be the rational function field over the algebraically closed field field of constants C, and let the matrix A of $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ satisfy $A \in C[z]^{n\times n}$. Then there exists a valuation ring $\mathcal{O}_c \supset C[z]$ for $\bar{F}$ with valuation ideal $\mathcal{P}_c$ such that the injective differential homomorphism*

$$\tau : C[z] \ \to\ C[[T]],\ f \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(f)(\mathcal{P}_c)T^k$$

extends to a differential homomorphism

$$\tau : C[z][X_{ij}, \det(X_{ij})^{-1}] \ \to\ C[[T]],\ X_{ij} \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(X_{ij})(\mathcal{P}_c)T^k$$

where the initial values $(X_{ij})(\mathcal{P}_c) = \bar{D} \in \mathrm{GL}_n(C)$ *can be chosen arbitrarily, and the kernel* $\bar{I}$ *of* τ *defines a maximal differential ideal with* $\bar{I} \cap C[z] = (0)$.

Proof. Since $A \in C[z]^{n\times n}$, we can choose a local parameter $(z-c)$ with $c \neq 0 \in C$ such that for all polynomial entries $a_{ij}(z)$ in A it holds $a_{ij}(c) \neq 0$. Further, the valuation ring for $(z-c)$ is of shape

$$\mathcal{O}_c = \{\frac{f}{g} \mid f, g \in C[z],\ g \notin \mathcal{P} := \langle z-c\rangle_{C[z]}\}$$

which obviously contains $C[z]$. Denote by $\mathcal{P}_c := \mathcal{P} \cdot \mathcal{O}_c$ the maximal ideal of $\mathcal{O}_c$ and by $f(\mathcal{P}_c) \in C$ the image of $f \in C[z]$ under the residue map

$$\pi : \mathcal{O}_c \to \mathcal{O}_c/\mathcal{P}_c = C.$$

Then we define the Taylor map

$$\tau : C[z] \to C[[T]],\ f \mapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(f)(\mathcal{P}_c)T^k.$$

We compute for $f, g \in C[z]$

$$\begin{aligned}
\tau(fg) &= \sum_{k\in\mathbb{N}} \partial^{(k)}(fg)(\mathcal{P}_c)T^k \\
&= \sum_{k\in\mathbb{N}} \frac{1}{k!}(\sum_{i=0}^{k} \frac{k!}{i!(i-k)!}\partial^i(f)\partial^{k-i}(g))(\mathcal{P}_c)T^k \\
&= \sum_{k\in\mathbb{N}}(\sum_{i=0}^{k} \partial^{(i)}(f)(\mathcal{P}_c)\partial^{(k-i)}(g)(\mathcal{P}_c))T^k \\
&= (\sum_{k\in\mathbb{N}} \partial^{(k)}(f)(\mathcal{P}_c)T^k)(\sum_{k\in\mathbb{N}} \partial^{(k)}(g)(\mathcal{P}_c)T^k) = \tau(f)\tau(g).
\end{aligned}$$

Obviously, it holds $\tau(f+g) = \tau(f) + \tau(g)$. Hence, τ is a homomorphism. From the calculation

$$\begin{aligned}
\partial_T(\tau(f)) &= \partial_T(\sum_{k=0}^{\infty} \partial^{(k)}(f)(\mathcal{P}_c)T^k) \\
&= \sum_{k=1}^{\infty} k \cdot \partial^{(k)}(f)(\mathcal{P}_c)T^{k-1} \\
&= \sum_{k=1}^{\infty} \frac{k}{k!}\partial^{k-1}(\partial(f))(\mathcal{P}_c)T^{k-1} \\
&= \sum_{k=0}^{\infty} \partial^{(k)}(\partial(f))(\mathcal{P}_c)T^k = \tau(\partial(f))
\end{aligned}$$

we deduce that τ is a differential homomorphism. Since $\bar{R} := C[z]$ is a ∂-simple differential ring, τ is a differential monomorphism. Then the image $\bar{\mathcal{R}} := \tau(C[z]) \subset C[[T]]$ becomes an integral domain with field of constants C. Since $\mathrm{Quot}(\bar{\mathcal{R}}) \subset C((T))$, we obtain that its field of constants is equal to C. Thus, the differential monomorphism extends to an injective differential homomorphism

$$\tau : C(z) \to C((T)),\ \frac{f}{g} \mapsto \frac{\sum_{k\in\mathbb{N}} \partial^{(k)}(f)(\mathcal{P}_c)T^k}{\sum_{k\in\mathbb{N}} \partial^{(k)}(g)(\mathcal{P}_c)T^k}.$$

Now we extend τ to the differential ring $\bar{U} := C[z][X_{ij}, \det(X_{ij})^{-1}]$. One obtains recursively all higher derivations

$$\partial^k(X_{ij}) = A_k(X_{ij}),\ k \in \mathbb{N} \setminus \{0\}$$

of (X_{ij}) where $A_1 = A$. Since $A \in C[z]^{n\times n}$, it follows that $A_k \in C[z]^{n\times n}$ for all $k \in \mathbb{N}\setminus\{0\}$. Hence the differential structure on $\bar{U}$ is well defined. We choose for X_{ij} the initial values $(X_{ij})(\mathcal{P}_c) = \bar{D} \in \mathrm{GL}_n(C)$ and obtain values

$$\partial^{(k)}(X_{ij})(\mathcal{P}_c) \in C^{n\times n}.$$

This leads to an extension

$$\tau : C[z][X_{ij}, \det(X_{ij})^{-1}] \to C[[T]],\ X_{ij} \mapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(X_{ij})(\mathcal{P}_c)T^k.$$

Since the ring $\bar{U} = C[z][X_{ij}, \det(X_{ij})^{-1}]$ is generated by X_{ij} over $C[z]$, we have that the image $\bar{\mathcal{U}}$ of $\bar{U}$ under τ is generated by $\tau(X_{ij})$ over $\bar{\mathcal{R}}$. We are going to show that

$$\bar{\mathcal{E}} := \tau(C(z))[\tau(X_{ij}), \det(\tau(X_{ij}))^{-1}]$$

is a ∂_T-simple differential ring. By construction $\tau(X_{ij})$ is a fundamental solution matrix for the differential module M_τ over $\bar{\mathcal{F}} := \tau(C(z))$ with differential structure defined by $\partial(\boldsymbol{y}) = \tau(A)\boldsymbol{y}$. Since $\bar{\mathcal{E}} \subset C((T))$, the ring $\bar{\mathcal{E}}$ is an integral domain. Further, its field of fractions $\mathrm{Quot}(\bar{\mathcal{E}})$ has C as its field of constants since $\mathrm{Quot}(\bar{\mathcal{E}}) \subset C((T))$. By [Dyc08, Corollary 2.7], $\bar{\mathcal{E}}$ is a simple differential ring. Thus, $\bar{\mathcal{U}}$ is a $\bar{\mathcal{R}}$-simple differential ring. Hence, the kernel $\bar{I}$ of τ defines a maximal differential ideal with $\bar{I} \cap C[z] = (0)$. $\square$

Let $\tilde{E}/\tilde{F}$ be a field extension. We call an element $h \in \tilde{E}$ transcendental over $\tilde{F}$ if there exists no polynomial $f(x) \in \tilde{F}[x]$ such that $f(h) = 0$. Further, suppose $\tilde{E}/\tilde{F}$ is a differential field extension. Then we call an element $h \in \tilde{E}$ differentially transcendental over $\tilde{F}$, if there exists no differential polynomial $f(y) \in \tilde{F}\{y\}$ such that $f(h) = 0$.
We want to repeat the above discussion for the differential ring R, i.e., we look for an injective embedding of R in a ring of power series. Since the extension R/C is differentially transcendental, we need a field of power series containing differentially transcendental elements over its ground field. Therefore, we will define power series with coefficients which are transcendental over C.

Lemma 2.7. *Let $C(\beta_i)$ be a rational function field in the infinitely many transcendentals β_i ($i \in \mathbb{N}$) over C. Then $f := \sum_{i=0}^{\infty} \frac{1}{i!}\beta_i T^i$ is differentially transcendental over $C(\beta_i)[T]$.*

Proof. The proof will be done in three steps.

1. In the first step we are going to compute the shape of a certain coefficient c_i of the power series
$$\prod_{l=0}^{m}(\partial_T^{(l)}(f))^{d_l} = \sum_{i=0}^{\infty} c_i T^i.$$
First we look at an arbitrary power of f. In the second step we handle a power of a derivative $\partial^{(l)}(f)$ and then we consider the whole product.
For $d \in \mathbb{N}$, we denote the coefficients of f^d by c_i, i.e., let $f^d = \sum_{i=0}^{\infty} c_i T^i$. Further, let $\boldsymbol{k} := (k_1, ..., k_d) \in \mathbb{N}^d$ such that $k_i \neq k_j$ for $i \neq j$ and define the index $i_{\boldsymbol{k}} := \sum_{j=1}^{d} k_j$. Then from
$$f^d = (\sum_{i=0}^{\infty} \frac{1}{i!}\beta_i T^i) \cdot ... \cdot (\sum_{i=0}^{\infty} \frac{1}{i!}\beta_i T^i) = \sum_{i=0}^{\infty} c_i T^i$$
we see that the coefficient $c_{i_{\boldsymbol{k}}}$ has shape
$$c_{i_{\boldsymbol{k}}} = d! \cdot \frac{1}{k_1!} \cdot ... \cdot \frac{1}{k_d!} \cdot \beta_{k_1} \cdot ... \cdot \beta_{k_d} + r,$$
where $r \in \mathbb{Q}[\beta_0, ..., \beta_{i_{\boldsymbol{k}}}]$ and no monomial appearing in r is equal to a nonzero multiply of $\beta_{k_1} \cdot ... \cdot \beta_{k_d}$.
For $l \in \mathbb{N}$, denote by
$$f_l := \partial_T^{(l)}(f) = \sum_{i=0}^{\infty} \frac{1}{i!}\beta_{i+l} T^i$$
the l-th derivative of f. As above take $d \in \mathbb{N}$ and $\boldsymbol{k} = (k_1, ..., k_d) \in \mathbb{N}^d$ such that $k_i \neq k_j$ for $i \neq j$ and $k_i \gg l$. For $i_{\boldsymbol{k}} := \sum_{j=1}^{d} k_j - d \cdot l$, we obtain by shifting the indices that the coefficient $c_{i_{\boldsymbol{k}}}$ of the d-th power of f_l reads as
$$c_{i_{\boldsymbol{k}}} = d! \cdot \frac{1}{(k_1 - l)!} \cdot ... \cdot \frac{1}{(k_d - l)!} \cdot \beta_{k_1} \cdot ... \cdot \beta_{k_d} + r,$$
where $r \in \mathbb{Q}[\beta_l, ..., \beta_{i_{\boldsymbol{k}}+d \cdot l}]$ and r does not contain the monomial $q \cdot \beta_{k_1} \cdot ... \cdot \beta_{k_d}$ for any $q \in \mathbb{Q}$.
Now we consider the product $\prod_{l=0}^{m} f_l^{d_l}$, where $d_l \in \mathbb{N}$, and denote its coefficients by c_i. We take $\boldsymbol{k} = (k_1, ..., k_{\bar{d}}) \in \mathbb{N}^{\bar{d}}$ such that $k_i \neq k_j$ for $i \neq j$ and $k_i \gg m$. We define the integers $\bar{d} = \sum_{l=0}^{m} d_l$ and $i_{\boldsymbol{k}} = \sum_{j=1}^{\bar{d}} k_j - \sum_{l=0}^{m} d_l \cdot l$. Further, let
$$\mathcal{K}_0 \cup \mathcal{K}_1 \cup ... \cup \mathcal{K}_m = \{1, ..., \bar{d}\}$$
be a partition of $\{1, ..., \bar{d}\}$, where for $0 \leq l \leq m$ the sets $\mathcal{K}_l$ satisfy $\mathrm{card}(\mathcal{K}_l) = d_l$.
We claim that the coefficient $c_{i_{\boldsymbol{k}}}$ is
$$c_{i_{\boldsymbol{k}}} = \Big(\prod_{l=0}^{m} d_l!\Big) \cdot \Big(\sum_{\substack{\mathcal{K}_0 \cup ... \cup \mathcal{K}_m \\ \mathrm{card}(\mathcal{K}_l) = d_l}} \Big(\prod_{j \in \mathcal{K}_0} \frac{1}{(k_j)!}\Big) \cdot ... \cdot \Big(\prod_{j \in \mathcal{K}_m} \frac{1}{(k_j - m)!}\Big)\Big) \cdot \beta_{k_1} \cdot ... \cdot \beta_{k_{\bar{d}}} + r,$$
with $r \in \mathbb{Q}[\beta_l, ..., \beta_{i_{\boldsymbol{k}}+\sum d_l \cdot l}]$ and no monomial of r is equal to a non-zero multiply of $\beta_{k_1} \cdot ... \cdot \beta_{k_{\bar{d}}}$.
The proof is done by induction on m. Let $m = 1$. The coefficients c_i calculate as

$c_i = \sum_{\mu+\nu=i} a_\mu \cdot b_\nu$, where a_i denotes the coefficients of f^{d_0} and b_i the coefficients of $f_1^{d_1}$. Thus we have to consider the indices of the products $a_\mu \cdot b_\nu$, which contain the monomial $\beta_{k_1} \cdot \ldots \cdot \beta_{k_{\bar{d}}}$.With the above results we deduce that for each partition $\mathcal{K}_0 \cup \mathcal{K}_1$ of $\{1, \ldots, \bar{d}\}$ the product $a_{i_{\mathcal{K}_0}} \cdot b_{i_{\mathcal{K}_1}}$ contains the monomial $\beta_{k_1} \cdot \ldots \cdot \beta_{k_{\bar{d}}}$, where $\mathrm{card}(\mathcal{K}_l) = d_l$ for $0 \le l \le 1$. Note that the index $i_{\mathcal{K}_l}$ is defined as $i_{\mathcal{K}_l} = \sum_{j \in \mathcal{K}_l} k_j - l \cdot d_l$. Thus, we obtain

$$c_{i_{\boldsymbol{k}}} = \sum_{i_{\mathcal{K}_0}+i_{\mathcal{K}_1}=i_{\boldsymbol{k}}} a_{i_{\mathcal{K}_0}} b_{i_{\mathcal{K}_1}} = d_0! \cdot d_1! \Big(\sum_{\substack{\mathcal{K}_0 \cup \mathcal{K}_1 \\ \mathrm{card}(\mathcal{K}_l)=d_l}} \prod_{j \in \mathcal{K}_0} \frac{1}{k_j!} \cdot \prod_{j \in \mathcal{K}_1} \frac{1}{(k_j-1)!} \Big) \cdot \beta_{k_1} \cdot \ldots \cdot \beta_{k_{\bar{d}}} + r.$$

Let $m > 1$. We denote now by a_i the coefficients of $\prod_{l=0}^{m-1} f_l^{d_l}$ and by b_i the coefficients of $f_m^{d_m}$. Motivated by the same ideas as above, we take a partition $\mathcal{K}_0 \cup \ldots \cup \mathcal{K}_m = \{1, \ldots, \bar{d}\}$ such that $\mathrm{card}(\mathcal{K}_l) = d_l$ for $0 \le l \le m$. Thus, if we apply the induction assumption to all $a_{i_{\mathcal{K}'}}$, where $\mathcal{K}' = \mathcal{K}_0 \cup \ldots \cup \mathcal{K}_{m-1}$, we obtain

$$\begin{aligned} c_{i_{\boldsymbol{k}}} = \sum_{i_{\mathcal{K}_m}+i_{\mathcal{K}'}=i_{\boldsymbol{k}}} b_{i_{\mathcal{K}_m}} \cdot a_{i_{\mathcal{K}'}} &= \sum_{i_{\mathcal{K}_m}+i_{\mathcal{K}'}=i_{\boldsymbol{k}}} \Big(d_m! \prod_{j \in \mathcal{K}_m} \frac{1}{(k_j-m)!} \cdot \beta_{k_j} + r \Big) \cdot \\ &\Big(\prod_{l=0}^{m-1} d_l! \Big) \cdot \Big(\sum_{\substack{\mathcal{K}_0 \cup \ldots \cup \mathcal{K}_{m-1} \\ \mathrm{card}(\mathcal{K}_l)=d_l}} \prod_{l=0}^{m-1} \Big(\prod_{j \in \mathcal{K}_l} \frac{1}{(k_j-l)!} \Big) \Big) \cdot \beta_{k_j} + r. \end{aligned}$$

2. Let $g(y) = \sum_{j=1}^{w} g_j \cdot y^{d_{0,j}} \cdot (y')^{d_{1,j}} \cdot \ldots \cdot (y^{(m)})^{d_{m,j}} \in C[\beta_i][T]\{y\}$ be a differential polynomial with $\bar{d} = \sum_{l=0}^{m} d_{l,j}$ for all $1 \le j \le w$, satisfying

$$g(f) = \sum_{j=1}^{w} g_j (\sum \frac{1}{i!} \beta_i T^i)^{d_{0,j}} \cdot (\sum \frac{1}{i!} \beta_{i+1} T^i)^{d_{1,j}} \cdot \ldots \cdot (\sum \frac{1}{i!} \beta_{i+m} T^i)^{d_{m,j}} = \sum c_i T^i = 0.$$

We write the coefficients of $g(y)$ as $g_j = \sum_{h=0}^{p_j} g_{j,h} T^h \in C[\beta_0, \ldots, \beta_{i'}][T]$ with $p_j \in \mathbb{N}$. We are going to show that not all coefficients c_i can vanish. In particular, we show that there exists a term of some coefficient which is a non-zero multiply of $\beta_{k_1} \cdot \ldots \cdot \beta_{k_{\bar{d}}}$ for an appropriate $\boldsymbol{k} = (k_1, \ldots, k_{\bar{d}}) \in \mathbb{N}^{\bar{d}}$.
For $1 \le j \le w$ and $\boldsymbol{k} \in \mathbb{N}^{\bar{d}}$ with $k_i \ne k_j$ for $i \ne j$ and $k_i \gg m$, we define the sum

$$q_j(\boldsymbol{k}) = \Big(\prod_{l=0}^{m} d_l! \Big) \cdot \Big(\sum_{\substack{\mathcal{K}_0 \cup \ldots \cup \mathcal{K}_m \\ \mathrm{card}(\mathcal{K}_l)=d_l}} \prod_{l=0}^{m} \Big(\prod_{j \in \mathcal{K}_l} \frac{1}{(k_i-l)!} \Big) \Big)$$

and the polynomials in $\mathbb{Q}[k_1, \ldots, k_{\bar{d}}]$

$$q_j'(\boldsymbol{k}) = q_j(\boldsymbol{k}) \cdot \prod_{i=1}^{\bar{d}} k_i! = \Big(\prod_{l=0}^{m} d_l! \Big) \cdot \Big(\sum_{\substack{\mathcal{K}_0 \cup \ldots \cup \mathcal{K}_m \\ \mathrm{card}(\mathcal{K}_l)=d_l}} \prod_{l=1}^{m} \Big(\prod_{j \in \mathcal{K}_l} \Big(\prod_{n=1}^{l} (k_j - l + n) \Big) \Big) \Big)$$

where the sum is over all partitions $\mathcal{K}_0 \cup \ldots \cup \mathcal{K}_m = \{1, \ldots, \bar{d}\}$ with $\mathcal{K}_l$ satisfying $\mathrm{card}(\mathcal{K}_l) = d_l$ for $0 \le l \le m$. For each monomial $\prod_{l=0}^{m} (y^{(l)})^{d_{l,j}}$ and $\boldsymbol{k} \in \mathbb{N}^{\bar{d}}$ as above, we consider the coefficient $c^{(j)}_{\hat{i}_j}$ with index

$$\hat{i}_j := \sum_{j=1}^{\bar{d}} k_j - \sum_{l=0}^{m} d_{l,j} \cdot l$$

of the power series $\prod_{l=0}^{m}(y^{(l)})^{d_{l,j}} = \sum_i c_i^{(j)} T^i$. Since these coefficients are shifted by the polynomials $g_j(T)$, we collect all indices $\hat{i}_j$ of the coefficients $c^{(j)}_{\hat{i}_j}$, which appear in $c_{\hat{i}_j+p_1}$, in the set

$$\mathcal{I} := \{\ \hat{i}_j \mid \hat{i}_1 + p_1 = \hat{i}_j + h'_j\ ,\ 0 \leq h'_j \leq p_j,\ g_{j,h'_j} \neq 0\ \}.$$

We define the set of indices of all relevant monomials as $\mathcal{J} := \{\ j \mid \hat{i}_j \in \mathcal{I}\ \}$. Then the set $\mathcal{J}$ stays equal for each choice of $\boldsymbol{k} \in \mathbb{N}^{\bar{d}}$ with $k_i \neq k_j$ for $i \neq j$ and $k_i \gg m$. Since for all such $\boldsymbol{k} \in \mathbb{N}^{\bar{d}}$ the coefficient $c_{\hat{i}_j+p_1}$ has to vanish, we obtain the equations

$$\sum_{j\in\mathcal{J}} g_{j,h'_j} \cdot q_j(\boldsymbol{k}) \cdot \beta_{k_1} \cdot \ldots \cdot \beta_{k_{\bar{d}}} = 0 \Leftrightarrow \sum_{j\in\mathcal{J}} g_{j,h'_j} \cdot q_j(\boldsymbol{k}) = 0. \tag{2.1}$$

After a multiplication with $\prod_{i=1}^{\bar{d}} k_i!$ we can consider the equations in (2.1) as a single polynomial $p(\boldsymbol{k}) := \sum_{j\in\mathcal{J}} g_{j,h'_j} \cdot q'_j(\boldsymbol{k}) \in C[\beta_0, ..., \beta_{i'}][k_1, ..., k_{\bar{d}}]$. Since for $j, j' \in \mathcal{J}$ with $j \neq j'$ the leading monomials in q'_j and $q'_{j'}$ have different degrees in the indeterminates $k_1, ..., k_{\bar{d}}$, the polynomial $p(\boldsymbol{k})$ is not the zero polynomial. We consider

$$p(\boldsymbol{k}) = p(k_2, ..., k_{\bar{d}})(k_1) = (\sum_{j\in\mathcal{J}} g_{j,h'_j} \cdot q'_j(k_2, ..., k_{\bar{d}}))(k_1)$$

as an element of the polynomial ring $C[\beta_0, ..., \beta_{i'}][k_2, ..., k_{\bar{d}}][k_1]$ over the integral domain $C[\beta_0, ..., \beta_{i'}][k_2, ..., k_{\bar{d}}]$. Since the nonzero polynomial $p(k_2, ..., k_{\bar{d}})(k_1)$ has finitely many zeros in $C[\beta_0, ..., \beta_{i'}][k_2, ..., k_{\bar{d}}]$, we can choose $\bar{k}_1 \in \mathbb{N}$ with $\bar{k}_1 \gg m$ such that $p(\bar{k}_1, k_2, ..., k_{\bar{d}}) \neq 0$.
Hence, by induction we obtain that there exists $\bar{k}_1, ..., \bar{k}_{\bar{d}} \in \mathbb{N}$ with $\bar{k}_i \gg m$ and $\bar{k}_i \neq \bar{k}_j$ for $i \neq j$ such that $p(\bar{k}_1, ..., \bar{k}_{\bar{d}}) \neq 0$. Thus, f can not satisfy $g(f) = 0$.

3. Now let $g(y) = \sum_{j=1}^{w} g_j \cdot y^{d_{0,j}} \cdot (y')^{d_{1,j}} \cdot .. \cdot (y^{(m)})^{d_{m,j}} \in C[\beta_i][T]\{y\}$ be an arbitrary differential polynomial satisfying $g(f) = 0$. Denote by $\bar{d} = \max_j\{\ \sum_{l=0}^{m} d_{l,j}\ \}$. Since for all monomials with $\sum_{l=0}^{m} d_{l,j} < \bar{d}$ the coefficients of the resulting power series are polynomials in the β_i of degree lower than $\bar{d}$, it is sufficient to consider coefficients with index high enough of the resulting power series of the monomials with $\sum_{l=0}^{m} d_{l,j} = \bar{d}$. Then the assumption follows from the above.

□

Corollary 2.8. *Let $C(\beta_{ij})$ be a rational function field in the infinitely many transcendentals β_{ij} where $1 \leq i \leq l$ and $j \in \mathbb{N}$. Then $f_i := \sum_{j\in\mathbb{N}} \frac{1}{j!}\beta_{ij}T^j$ is differentially transcendental over $C(\beta_{ij})[f_1, ..., f_{i-1}, f_{i+1}, ..., f_l]$.*

Proof. First we define $\mathcal{I} := \{1, 2, ..., l\}$. Let $i' \in \mathcal{I}$ and suppose that $f_{i'}$ is differentially algebraically dependent over $F_{i'} := C(\beta_{ij})[f_1, ..., f_{i'-1}, f_{i'+1}, ..., f_l]$, i.e., there exists a differential polynomial $h(y) \in F_{i'}\{y\}$ such that $h(f_{i'}) = 0$. The coefficients of $h(y)$ are power series with infinitely many coefficients in $C[\beta_{1j}, ..., \beta_{i'-1,j}, \beta_{i'+1,j}, ..., \beta_{lj}]$ and finitely many in $C(\beta_{ij})$. We define a specialization $\varphi : C(\beta_{ij}) \to C(\beta_{i'j})$ by

$$\beta_{i'j} \to \beta_{i'j} \text{ for } j \in \mathbb{N},$$
$$\beta_{ij} \to c_{ij} \text{ for } i \in \mathcal{I} \setminus \{i'\} \text{ and } j \in \mathbb{N}$$

where we choose finitely many $c_{ij} \in C^\times$ and all other c_{ij} as zero such that $\varphi(h(y)) = \varphi(h)(y)$ has no pole and does not disappear. We obtain

$$\varphi(h(f_{i'})) = \varphi(h)(f_{i'}) = 0.$$

We get a contradiction to Lemma 2.7 since $\varphi(h)(y) \in C(\beta_{i'j})[T]\{y\}$ is a nonzero differential polynomial which vanishes at $f_{i'}$. □

Lemma 2.9. *Let $\hat{F}_\beta := C(\beta_{ij})\langle t_1, ...t_l\rangle$ be a differential field in the differential indeterminates t_i with field of constants $C(\beta_{ij})$ where β_{ij} with $1 \le i \le l,\ j \in \mathbb{N}$ are transcendental over C. Moreover let the matrix A of $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ satisfy $A \in C\{t_1, ..., t_l\}^{n\times n}$. Then there exists a valuation ring $\hat{\mathcal{O}}_\beta \supset C(\beta_{ij})\{t_1, ..., t_l\}$ for $\hat{F}_\beta$ with valuation ideal $\hat{\mathcal{P}}_\beta$ such that the injective differential homomorphism*

$$\hat{\tau} : C(\beta_{ij})\{t_1, ..., t_l\} \ \to\ C(\beta_{ij})[[T]],\ f \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(f)(\hat{\mathcal{P}}_\beta)T^k$$

extends to a differential homomorphism

$$\hat{\tau} : C(\beta_{ij})\{t_1, ..., t_l\}[X_{ij}, \det(X_{ij})^{-1}] \ \to\ C(\beta_{ij})[[T]],\ X_{ij} \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(X_{ij})(\hat{\mathcal{P}}_\beta)T^k,$$

where we can choose arbitrary initial values $(X_{ij})(\hat{\mathcal{P}}_\beta) = D \in \mathrm{GL}_n(C(\beta_{ij}))$, and the kernel $\hat{I}$ of $\hat{\tau}$ defines a maximal differential ideal with $\hat{I} \cap C(\beta_{ij})\{t_1, ..., t_l\} = (0)$.

Proof. Let $\hat{R}_\beta := C(\beta_{ij})\{t_1, ..., t_l\} = C(\beta_{ij})[t_{10}, t_{11}, ..., t_{l0}, t_{l1}...]$ be the polynomial ring in the infinitely many transcendental elements $t_{ij} := t_i^{(j)}$. We define linear polynomials

$$p_{ij} = t_{ij} - \beta_{ij}$$

and the ideal

$$\hat{\mathcal{P}} = \langle\ p_{ij} \mid 1 \le i \le l,\ j \in \mathbb{N}\ \rangle_{\hat{R}_\beta} \subset \hat{R}_\beta.$$

Since $\hat{R}_\beta/\hat{\mathcal{P}}$ is an integral domain, we have that $\hat{\mathcal{P}}$ is a prime ideal. Then by [Eis95, Exercise 11.2], there exists a valuation ring $\hat{\mathcal{O}}_\beta \supset \hat{R}_\beta$ with valuation ideal $\hat{\mathcal{P}}_\beta$ such that $\hat{\mathcal{P}}_\beta \cap \hat{R}_\beta = \hat{\mathcal{P}}$. Denote by π the residue map

$$\pi : \hat{\mathcal{O}}_\beta \to \hat{\mathcal{O}}_\beta/\hat{\mathcal{P}}_\beta = C(\beta_{ij}).$$

From the image of $\pi(p_{ij}) = \overline{p_{ij}} = \overline{t_{ij}} - \overline{\beta_{ij}} = \bar{0}$ we see that $\overline{t_{ij}} = \beta_{ij}$. Thus the image of $f \in \hat{\mathcal{O}}_\beta \setminus \hat{\mathcal{P}}_\beta$ is the evaluation at $\beta = (\beta_{10}, \beta_{11}, ..., \beta_{l0}, \beta_{l1}, ...)$. We denote the image of $f \in \hat{R}_\beta \subset \hat{\mathcal{O}}_\beta$ under the residue map by $f(\hat{\mathcal{P}}_\beta) \in C(\beta_{ij})$. We define the Taylor map

$$\hat{\tau} : \hat{R}_\beta \;\to\; C(\beta_{ij})[[T]],\; f \longmapsto \sum_{k \in \mathbb{N}} \partial_{R_\beta}^{(k)}(f)(\hat{\mathcal{P}}_\beta)T^k.$$

The same calculation as in Lemma 2.6 shows that $\hat{\tau}$ is a differential homomorphism. By Corollary 2.8 the differential ring extension

$$C(\beta_{ij})\{\, \sum_{j=0}^{\infty} \frac{1}{j!}\beta_{1j}T^j,\; ...,\; \sum_{j=0}^{\infty} \frac{1}{j!}\beta_{lj}T^j \;\}/C(\beta_{ij})$$

is a differential transcendental extension. Hence, $\hat{\tau}$ defines a differential monomorphism. The image $\hat{\mathcal{R}}_\beta := \hat{\tau}(\hat{R}_\beta) \subset C(\beta_{ij})[[T]]$ is an integral domain with field of constants equal to $C(\beta_{ij})$. Since the field of fractions $\mathrm{Quot}(\hat{\mathcal{R}}_\beta)$ is contained in $C(\beta_{ij})((T))$, it has also $C(\beta_{ij})$ as its field of constants. We extend $\hat{\tau}$ to the differential ring $\hat{U}_\beta := \hat{R}_\beta[X_{ij}, \det(X_{ij})^{-1}]$. One computes recursively all higher derivations

$$\partial^k(X_{ij}) = A_k(X_{ij}),\; k \in \mathbb{N}$$

of (X_{ij}) where $A_1 = A$. The fact that $A \in \hat{R}_\beta^{n\times n}$ yields that $A_k \in \hat{R}_\beta^{n\times n}$ for all $k \in \mathbb{N}$. Thus, the differential structure on $\hat{U}_\beta$ is well defined. We choose for X_{ij} the initial values $(X_{ij})(\hat{\mathcal{P}}_\beta) = D \in \mathrm{GL}_n(C(\beta_{ij}))$ and obtain therefore values

$$\partial^{(k)}(X_{ij})(\hat{\mathcal{P}}_\beta) \in C(\beta_{ij}).$$

Thus we have an extension

$$\hat{\tau} : \hat{R}_\beta[X_{ij}, \det(X_{ij})^{-1}] \to C(\beta_{ij})[[T]],\; X_{ij} \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(X_{ij})(\hat{\mathcal{P}}_\beta)T^k.$$

Since the ring $\hat{U}_\beta = \hat{R}_\beta[X_{ij}, \det(X_{ij})^{-1}]$ is generated by X_{ij} over $\hat{R}_\beta$, we have that the image $\hat{\mathcal{U}}_\beta$ of $\hat{U}_\beta$ under $\hat{\tau}$ is generated by $\hat{\tau}(X_{ij})$ over $\hat{\mathcal{R}}_\beta$. We are going to show that

$$\hat{\mathcal{E}}_\beta := \mathrm{Quot}(\hat{\mathcal{R}}_\beta)[\hat{\tau}(X_{ij}), \det(\hat{\tau}(X_{ij}))^{-1}]$$

is a ∂_T-simple differential ring. By construction $\hat{\tau}(X_{ij})$ is a fundamental solution matrix for the differential module $M_{\hat{\tau}}$ over $\hat{\mathcal{F}}_\beta := \mathrm{Quot}(\hat{\mathcal{R}}_\beta)$ with differential structure defined by $\partial(\boldsymbol{y}) = \tau(A)\boldsymbol{y}$. Since $\hat{\mathcal{E}}_\beta \subset C(\beta_{ij})((T))$, we obtain that the ring $\hat{\mathcal{E}}_\beta$ is an integral domain. The fact $\mathrm{Quot}(\hat{\mathcal{E}}_\beta) \subset C(\beta_{ij})((T))$ yields that the field of constants of the field of fractions of $\hat{\mathcal{E}}_\beta$ is also $C(\beta_{ij})$. Then by [Dyc08, Corollary 2.7], $\hat{\mathcal{E}}_\beta$ is a simple differential ring. Thus $\hat{\mathcal{U}}_\beta$ is a $\hat{\mathcal{R}}_\beta$-simple differential ring. Hence, the kernel $\hat{I}$ of $\hat{\tau}$ defines a maximal differential ideal with $\hat{I} \cap \hat{R}_\beta = (0)$. □

Corollary 2.10. *Let $F_\beta := C[\beta_{ij}]\langle t_1, ...t_l\rangle$ be a differential ring in the differential indeterminates t_i with ring of constants $C[\beta_{ij}]$ and let the matrix A of $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ satisfy*

$A \in C\{t_1, ..., t_l\}^{n\times n}$. Then there exists a valuation ring $\mathcal{O}_\beta \supset C[\beta_{ij}]\{t_1, ..., t_l\}$ for F_β with valuation ideal $\mathcal{P}_\beta$ such that the injective differential homomorphism

$$\tau : C[\beta_{ij}]\{t_1, ..., t_l\} \;\to\; C[\beta_{ij}][[T]],\; f \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(f)(\mathcal{P}_\beta)T^k$$

extends to a differential homomorphism

$$\tau : C[\beta_{ij}]\{t_1, ..., t_l\}[X_{ij}, \det(X_{ij})^{-1}] \;\to\; C[\beta_{ij}][[T]],\; X_{ij} \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(X_{ij})(\mathcal{P}_\beta)T^k,$$

where we can choose arbitrary initial values $(X_{ij})(\mathcal{P}_\beta) = D \in \mathrm{GL}_n(C[\beta_{ij}])$, and the kernel I of τ defines a maximal differential ideal with $I \cap C[\beta_{ij}]\{t_1, ..., t_l\} = (0)$.

Proof. We repeat the construction in Lemma 2.9 for the differential field $C(\beta_{ij})\langle \boldsymbol{t}\rangle$ and the initial values $(X_{ij})(\mathcal{P}_\beta) = D \in \mathrm{GL}_n(C[\beta_{ij}])$. Thus, we obtain a differential homomorphism

$$\check{\tau} : C(\beta_{ij})\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \to C(\beta_{ij})[[T]],\; f \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(f)(\mathcal{P}_\beta)T^k,$$

where the kernel $\check{I}$ defines a maximal differential ideal with $\check{I} \cap C(\beta_{ij})\{\boldsymbol{t}\} = (0)$. It is easy to see that the restriction of $\check{\tau}$ to $C[\beta_{ij}]\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]$

$$\tau : C[\beta_{ij}]\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \to C[\beta_{ij}][[T]],\; f \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(f)(\mathcal{P}_\beta)T^k$$

is well defined by the choice of $(X_{ij})(\mathcal{P}_\beta) = D$. Then the kernel $I := \mathrm{kern}(\tau)$ of τ satisfies

$$I = \check{I} \cap C[\beta_{ij}]\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]$$

and therefore defines a maximal differential ideal with $I \cap C[\beta_{ij}]\{\boldsymbol{t}\} = (0)$. $\square$

Lemma 2.11. *We keep the situation and notations as in Corollary 2.10. Then there exists a matrix of initial values $(X_{ij})(\mathcal{P}_\beta) = D \in \mathrm{GL}_n(C[\beta_{ij}])$ such that the injective differential homomorphism*

$$\tilde{\tau} : C\{t_1, ..., t_l\} \;\to\; C[\beta_{ij}][[T]],\; f \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(f)(\mathcal{P}_\beta)T^k$$

extends to a differential homomorphism

$$\tilde{\tau} : C\{t_1, ..., t_l\}[X_{ij}, \det(X_{ij})^{-1}] \;\to\; C[\beta_{ij}][[T]],\; X_{ij} \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(X_{ij})(\mathcal{P}_\beta)T^k,$$

and the kernel $\hat{I}$ defines a maximal differential ideal with $\hat{I} \cap C\{\boldsymbol{t}\} = (0)$. Further, $\tilde{\mathcal{U}} :=$ $\mathrm{im}(\tilde{\tau})$ is a Picard-Vessiot ring with field of constants C.

Proof. We apply Corollary 2.10 with initial values $(X_{ij})(\tilde{\mathcal{P}}_\beta) = \mathbf{1}_n$, obtaining the map

$$\tau : C[\beta_{ij}]\{t_1, ..., t_l\}[X_{ij}, \det(X_{ij})^{-1}] \;\to\; C[\beta_{ij}][[T]],\; f \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(f)(\mathcal{P}_\beta)T^k.$$

This implies the first part of the diagram

$$\begin{tikzcd}
C[\beta_{ij}]\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \arrow[d, "\pi_1"] \arrow[dr, "\tau"] & \\
C[\beta_{ij}]\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]/I \arrow[r, "isom", "\iota_1"'] & \mathrm{im}(\tau) \subset C[\beta_{ij}][[T]] \\
(C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \otimes_C C[\beta_{ij}])/(\hat{I} \otimes_C C[\beta_{ij}]) \arrow[u, "isom", "\iota_2"'] & C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \arrow[l, "\pi_2"'] \arrow[u, dashed, "\tilde{\tau}"]
\end{tikzcd}$$

where $I = \mathrm{kern}(\tau)$ with $I \cap C[\beta_{ij}]\{\boldsymbol{t}\} = (0)$ and $C[\beta_{ij}]\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]/I$ is a Picard-Vessiot ring for $\partial(\boldsymbol{y}) = A(\boldsymbol{t})\boldsymbol{y}$ over $C[\beta_{ij}]\{\boldsymbol{t}\}$. Now the kernel I' of the restriction

$$\tau' : \ C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \to \mathrm{im}(\tau) \subset C[\beta_{ij}][[T]]$$

is a differential ideal of $C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]$ with $I' \cap C\{\boldsymbol{t}\} = (0)$. We choose a maximal differential ideal $\hat{I} \supseteq I'$ with $\hat{I} \cap C\{\boldsymbol{t}\} = (0)$. Then the ring $C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]/\hat{I}$ is a Picard-Vessiot ring for the equation $\partial(\boldsymbol{y}) = A(\boldsymbol{t})\boldsymbol{y}$ over $C\{\boldsymbol{t}\}$. Further, by [Mau10, Lemma 10.7], the differential ideal $\hat{I} \otimes_C C[\beta_{ij}] \subset C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \otimes_C C[\beta_{ij}]$ is a maximal differential ideal with $(\hat{I} \otimes_C C[\beta_{ij}]) \cap (C\{\boldsymbol{t}\} \otimes_C C[\beta_{ij}]) = (0)$. Thus, the ring

$$C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \otimes_C C[\beta_{ij}]/(\hat{I} \otimes_C C[\beta_{ij}])$$

is a Picard-Vessiot ring for $\partial(\boldsymbol{y}) = A(\boldsymbol{t})\boldsymbol{y}$ over $C\{\boldsymbol{t}\} \otimes_C C[\beta_{ij}]$. As in [PS03, Proposition 1.20.2], one proves that two Picard-Vessiot rings for the same equation over a differential ring with constants $C[\beta_{ij}]$ are differentially isomorphic. More precisely, in the notations of [PS03, Proposition 1.20.2], let B_1 and B_2 be the fundamental matrices of Picard-Vessiot rings $\bar{R}_1$ and $\bar{R}_2$ with constants $C[\beta_{ij}]$ and denote by $R_i \geq \bar{R}_i$ $(i = 1, 2)$ the Picard-Vessiot rings obtained from extending the constants of $\bar{R}_i$ to the algebraically closed field $\overline{C(\beta_{ij})}$. If we imitate the proof we obtain from

$$\phi_1(B_1) = \phi_2(B_2) \cdot M$$

that the rings R_1 and R_2 are differentially isomorphic, where $M \in \mathrm{GL}_n(\overline{C(\beta_{ij})})$ and $\phi_i : R_i \to (R_1 \otimes R_2)/J$ denotes the differential ring morphism as in [PS03, Proposition 1.20.2]. Since $\phi_i : R_i \to \phi_i(R_i)$ is a differential ring isomorphism and $\phi_i(B_i) \in \mathrm{GL}_n(\phi_i(\bar{R}_i))$ $(i = 1, 2)$, we have

$$M = \phi_2(B_2)^{-1} \cdot \phi_1(B_1) \in \mathrm{GL}_n(C[\beta_{ij}]).$$

Thus, the rings $\bar{R}_1$ and $\bar{R}_2$ become differentially isomorphic by $M \in \mathrm{GL}_n(C[\beta_{ij}])$. We continue with the proof of the lemma.
Hence, we obtain an $(C\{\boldsymbol{t}\} \otimes_C C[\beta_{ij}])$-differential isomorphism

$$\iota_2 : \ C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \otimes_C C[\beta_{ij}]/(\hat{I} \otimes_C C[\beta_{ij}]) \xrightarrow{isom} C[\beta_{ij}]\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]/I.$$

Moreover, the Picard-Vessiot ring $C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]/\hat{I} \cong (C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \otimes_C 1)/(\hat{I} \otimes 1)$ lies inside $C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \otimes_C C[\beta_{ij}]/(\hat{I} \otimes_C C[\beta_{ij}])$. Thus,

$$\iota_1 \circ \iota_2(C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]/\hat{I}) \subset \mathrm{im}(\tau)$$

is isomorphic to the Picard-Vessiot ring $C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]/\hat{I}$ and has C as its field of constants. Denote by Y_1 a fundamental solution matrix of $C[\beta_{ij}]\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]/I$. The isomorphism ι_2 maps a fundamental solution matrix Y_2 of

$$C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \otimes C[\beta_{ij}]/\hat{I} \otimes C[\beta_{ij}]$$

to a fundamental solution matrix $\iota_2(Y_2) = Y_1 \cdot D$ of $C[\beta_{ij}]\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]/I$ with $D \in \mathrm{GL}_n(C[\beta_{ij}])$. The $C[\beta_{ij}]$-isomorphism ι_1 is defined by

$$\iota_1 : \ t_i \longmapsto \sum_j \frac{1}{j!}\beta_{ij}T^j \quad \text{and} \quad \iota_1 : \ X_{ij} + I \longmapsto \sum_k \partial^{(k)}(X_{ij})(\tilde{\mathcal{P}}_\beta)T^k.$$

Thus, we obtain a $C[\beta_{ij}]$-differential isomorphism $\varphi := \iota_1 \circ \iota_2$ defined by

$$\varphi : \ t_i \longmapsto \sum_j \frac{1}{j!}\beta_{ij}T^j \quad \text{and} \quad \varphi : \ X_{ij} + \hat{I} \longmapsto \sum_k \partial^{(k)}(\sum_{m=1}^n D_{mj} \cdot X_{im})(\tilde{\mathcal{P}}_\beta)T^k.$$

This defines the differential homomorphism

$$\tilde{\tau} : C\{t_1, ..., t_l\}[X_{ij}, \det(X_{ij})^{-1}] \ \to \ C[\beta_{ij}][[T]], \ X_{ij} \longmapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(X_{ij})(\mathcal{P}_\beta)T^k$$

with initial values $(X_{ij})(\mathcal{P}_\beta) = D \in \mathrm{GL}_n(C[\beta_{ij}])$ in the Picard-Vessiot ring $\tilde{\mathcal{U}} := \mathrm{im}(\tilde{\tau})$. □

2.4 The specialization bound

If not otherwise stated, the notation is as in the preceding sections.

Proposition 2.12. *Let $\partial(\boldsymbol{y}) = A(t_1, ..., t_l)\boldsymbol{y}$ be a matrix differential equation over the differential field $F = C\langle t_1, ..., t_l\rangle$ with $A \in C\{t_1, ..., t_l\}^{n\times n}$, and let*

$$\sigma : R \to C[z], \ \boldsymbol{t} \mapsto \boldsymbol{f} = (f_1, ..., f_l)$$

be a specialization of $R = C\{t_1, ..., t_l\}$ such that $C\{f_1, ..., f_l\} = C[z]$. One applies Lemma 2.11 and Lemma 2.6 to the matrix equations $\partial(\boldsymbol{y}) = A(\boldsymbol{t})\boldsymbol{y}$ and $\partial(\boldsymbol{y}) = \sigma(A(\boldsymbol{t}))\boldsymbol{y} = A(\boldsymbol{f})\boldsymbol{y}$ respectively, where the initial values for the Taylor extension of $\partial(\boldsymbol{y}) = A(\boldsymbol{t})\boldsymbol{y}$ are $(X_{ij})(\mathcal{P}_\beta) = D \in \mathrm{GL}_n(C[\beta_{ij}])$ and keeps their notations. Then there exists a surjective differential homomorphism $\hat{\sigma}$ and initial values $(X_{ij})(\mathcal{P}_c) = \bar{D} \in \mathrm{GL}_n(C)$ such that the following diagram commutes

$$\begin{tikzcd} C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \arrow[r, two heads, "\sigma"] \arrow[d, "\tilde{\tau}"] & C[z][X_{ij}, \det(X_{ij})^{-1}] \arrow[d, "\bar{\tau}"'] \\ \tilde{\mathcal{U}} \arrow[r, dashed, two heads, "\hat{\sigma}"] & \bar{\mathcal{U}} \end{tikzcd}$$

Proof. The conditions on σ imply that σ is a surjective differential homomorphism. We extend σ to β_{ij} by

$$\sigma : \beta_{ij} \mapsto c_{ij},$$

where we choose $c_{ij} \in C$ such that

$$\sigma(t_{ij} - \beta_{ij}) = \partial^j(f_i) - c_{ij} \in \langle z - c\rangle_{C[z]}.$$

This is possible since for every polynomial $f(z) \in C[z]$, the polynomial $f(z) - f(c)$ obviously has a zero at c. In the case when $f(z) = \tilde{c}$ is a constant, we obtain the zero polynomial by $f(z) - \tilde{c}$. Further, we set the initial values for the Taylor extension of $\partial(\boldsymbol{y}) = \sigma(A(\boldsymbol{t}))\boldsymbol{y}$ to $(X_{ij})(\mathcal{P}_c) := \bar{D} = \sigma(D) \in \mathrm{GL}_n(C)$. Thus we have that $\sigma(\mathcal{P}_\beta) = \mathcal{P}_c$. We define the map

$$\hat{\sigma} : \tilde{\mathcal{U}} \to \bar{\mathcal{U}},\ \sum_{k\in\mathbb{N}} \partial^{(k)}(f)(\mathcal{P}_\beta)T^k \mapsto \sum_{k\in\mathbb{N}} \partial^{(k)}(\sigma(f))(\sigma(\mathcal{P}_\beta))T^k.$$

We are going to show that $\hat{\sigma}$ is well defined. Let $\tilde{g} := \sum_k \partial^{(k)}(g)(\mathcal{P}_\beta)T^k$ and $\tilde{f} := \sum_k \partial^{(k)}(f)(\mathcal{P}_\beta)T^k$ be elements of $\tilde{\mathcal{U}}$. Then we have $\tilde{g} = \tilde{f}$ if and only if $\partial^{(k)}(g)(\mathcal{P}_\beta) = \partial^{(k)}(f)(\mathcal{P}_\beta)$ for all $k \in \mathbb{N}$. Let $g, f \in C\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]$ such that $\tilde{g} = \tilde{f}$. Since $\sigma(\mathcal{P}_\beta) = \mathcal{P}_c$ we obtain from the fundamental theorem of homomorphisms that there exists a unique homomorphism θ such that the diagram

$$\begin{tikzcd}
C[\beta_{ij}]\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}] \arrow[r, two heads, "\sigma"] \arrow[d, "\pi_1"] & C[z][X_{ij}, \det(X_{ij})^{-1}] \arrow[d, "\pi_2"'] \\
C[\beta_{ij}]\{\boldsymbol{t}\}[X_{ij}, \det(X_{ij})^{-1}]/\mathcal{P}_\beta \arrow[r, dashed, two heads, "\theta"] & C[z][X_{ij}, \det(X_{ij})^{-1}]/\mathcal{P}_c
\end{tikzcd}$$

commutes. This yields $\partial^{(k)}(\sigma(g))(\mathcal{P}_c) = \partial^{(k)}(\sigma(f))(\mathcal{P}_c)$ for all $k \in \mathbb{N}$. Thus we get

$$\begin{aligned}
\hat{\sigma}(\sum_k \partial^{(k)}(g)(\mathcal{P}_\beta)T^k) &= \sum_k \partial^{(k)}(\sigma(g))(\mathcal{P}_c)T^k \\
&= \sum_k \partial^{(k)}(\sigma(f))(\mathcal{P}_c)T^k = \hat{\sigma}(\sum_k \partial^{(k)}(f)(\mathcal{P}_\beta)T^k).
\end{aligned}$$

Hence, $\hat{\sigma}$ is well defined.
Since $\hat{\sigma}$ is induced by the differential homomorphism $\bar{\tau}\circ\sigma$ and is well defined, we obtain that $\hat{\sigma}$ is a differential homomorphism. Then by the definition of $\hat{\sigma}$ the diagram commutes. □

Note that the condition $C\{f_1, ..., f_l\} = C[z]$ in Proposition 2.12 was made to exclude the trivial case. As a direct consequence we obtain

Corollary 2.13. *There exist maximal differential ideals $I \subset C\{t_1, ..., t_l\}[X_{ij}, \det(X_{ij})^{-1}]$ and $\bar{I} \subset C[z][X_{ij}, \det(X_{ij})^{-1}]$ such that $I \cap C\{t_1, ..., t_l\} = (0)$, $\bar{I} \cap C[z] = (0)$ and such that the specialized ideal $\sigma(I)$ satisfies*

$$\sigma(I) \subset \bar{I}.$$

In particular, it holds that $\sigma(I) \cap C[z] = (0)$.

Now we are ready to prove the specialization bound.

Theorem 2.14. *Let C be an algebraically closed field of characteristic zero and $F = C\langle t_1, ..., t_l\rangle$ the differential field in the l differential indeterminates $\boldsymbol{t} = (t_1, ..., t_l)$. Let $\partial(\boldsymbol{y}) = A(\boldsymbol{t})\boldsymbol{y}$ be a matrix differential equation with $A \in C\{\boldsymbol{t}\}^{n\times n}$. Moreover, let*

$$\sigma : \boldsymbol{t} \mapsto \boldsymbol{f} = (f_1, ..., f_l)$$

be a specialization in the integral differential ring $\bar{R} = C[z]$, and suppose that $C\{\boldsymbol{f}\} = \bar{R}$. Then the differential Galois group $\mathcal{H}(C)$ of the specialized equation $\partial(\boldsymbol{y}) = A(\sigma(\boldsymbol{t}))\boldsymbol{y}$ over $\bar{F} = C(z)$ is a subgroup of the differential Galois group $\mathcal{G}(C)$ of the original equation $\partial(\boldsymbol{y}) = A(\boldsymbol{t})\boldsymbol{y}$ over F.

Proof. Let $R = C\{t_1, ..., t_l\}$. Since $A(\boldsymbol{t}) \in R^{n\times n}$, we can define a differential structure on $R[X_{ij}, \det(X_{ij})^{-1}]$ by

$$\partial((X_{ij})) = A(\boldsymbol{t})(X_{ij}). \tag{2.2}$$

Furthermore, let I be a maximal differential ideal of $R[X_{ij}, \det(X_{ij})^{-1}]$ with $I \cap R = (0)$ as in Corollary 2.13 and denote by

$$S := R[X_{ij}, \det(X_{ij})^{-1}]/I$$

the differential ring extension S/R. We have an injection

$$S \hookrightarrow F[X_{ij}, \det(X_{ij})^{-1}]/(I)$$

in a Picard-Vessiot ring for $\partial(\boldsymbol{y}) = A(\boldsymbol{t})\boldsymbol{y}$. Let Z_{ij} be the images of X_{ij} in S, i.e., the matrix

$$(Z_{ij}) \in \mathrm{GL}_n(S)$$

is a fundamental solution matrix for $\partial(\boldsymbol{y}) = A(\boldsymbol{t})\boldsymbol{y}$. We define new variables Y_{ij} via the relation

$$(X_{ij}) = (Z_{ij})(Y_{ij}). \tag{2.3}$$

We get the inclusion of rings

$$R[X_{ij}, \det(X_{ij})^{-1}] \subset S[X_{ij}, \det(X_{ij})^{-1}] = S[Y_{ij}, \det(Y_{ij})^{-1}]$$

and

$$C[Y_{ij}, \det(Y_{ij})^{-1}] \subset S[Y_{ij}, \det(Y_{ij})^{-1}].$$

The differentiations on $R[X_{ij}, \det(X_{ij})^{-1}]$ and $S[X_{ij}, \det(X_{ij})^{-1}]$ are given by the differentiation on S and by equation (2.2). Since the matrix $(Z_{ij}) \in \mathrm{GL}_n(S)$ is a fundamental solution matrix, it follows from the computation of

$$A(X_{ij}) = \partial((X_{ij})) = \partial((Z_{ij}))(Y_{ij}) + (Z_{ij})\partial((Y_{ij})) = A(Z_{ij})(Y_{ij}) + (Z_{ij})\partial((Y_{ij}))$$

that $\partial(Y_{ij}) = 0$. Thus, the derivation on $C[Y_{ij}, \det(Y_{ij})^{-1}]$ is trivial and is defined on $S[Y_{ij}, \det(Y_{ij})^{-1}]$ by the derivation on S. The Galois action of $\mathrm{Gal}_\partial(S/R)$ on the above rings is induced by the action on S. Therefore, $\mathrm{Gal}_\partial(S/R)$ acts trivial on $R[X_{ij}, \det(X_{ij})^{-1}]$. Since (Z_{ij}) is a fundamental solution matrix, we get for $\gamma \in \mathrm{Gal}_\partial(S/R)$ the representive

$$M \in \mathcal{G}(C) < \mathrm{GL}_n(C) \text{ via } \gamma((Z_{ij})) = (Z_{ij})M.$$

Hence, the action of $\gamma \in \mathrm{Gal}(S/R)$ on Y_{ij} is represented by $\gamma((Y_{ij})) = M^{-1}(Y_{ij})$. Furthermore, Lemma 2.5 induces a bijection between the differential ideals of $R[X_{ij}, \det(X_{ij})^{-1}]$ satisfying Condition 2.3 and the ideals of $S[X_{ij}, \det(X_{ij})^{-1}]$ which are differential and $\mathrm{Gal}_\partial(S/R)$-invariant ideals and satisfy Condition 2.3. Now Lemma 2.4 yields a bijection between these ideals and the $\mathrm{Gal}_\partial(S/R)$-invariant ideals of $C[Y_{ij}, \det(Y_{ij})^{-1}]$, i.e., we consider the composition of maps $\delta^{-1} \circ \iota$ where the notation is as in Lemma 2.4 and Lemma 2.5.

$$\begin{array}{ccccccc}
R[X_{ij}, \det(X_{ij})^{-1}] & & S[X_{ij}, \det(X_{ij})^{-1}] & = & S[Y_{ij}, \det(Y_{ij})^{-1}] & & C[Y_{ij}, \det(Y_{ij})^{-1}] \\
\cup & & \cup & & \cup & & \cup \\
I & \xmapsto{\iota} & (I) & = & \tilde{I} & \xmapsto{\delta^{-1}} & Q_I
\end{array}$$

Thus, the maximal differential ideal I corresponds to a maximal $\mathrm{Gal}_\partial(S/R)$-invariant ideal

$$Q_I := \tilde{I} \cap C[Y_{ij}, \det(Y_{ij})^{-1}],$$

where $\tilde{I} := (I)$ is the ideal in $S[Y_{ij}, \det(Y_{ij})^{-1}]$, and (I) is the ideal generated by I over $S[X_{ij}, \det(X_{ij})^{-1}]$. Then Q_I is a radical ideal. Its zero set $\mathcal{W}$ is minimal with respect to $\mathrm{Gal}_\partial(S/R)$-invariance. Hence, $\mathcal{W}$ is a left coset in $\mathrm{GL}_n(C)$ for the differential Galois group $\mathcal{G}(C) < \mathrm{GL}_n(C)$. We will show that the matrix $\mathbf{1}_n$ belongs to $\mathcal{W}$. From the fact that the matrix (Z_{ij}) is a zero of I, we conclude that the ideal I lies in the ideal

$$J := \langle\, X_{ij} - Z_{ij} \mid 1 \le i, j \le n \,\rangle \cdot S[X_{ij}, \det(X_{ij})^{-1}].$$

The ideal J is also generated by the set $\{\, Y_{ij} - \delta_{ij} \,\}_{ij}$ over $S[X_{ij}, \det(X_{ij})^{-1}]$. Hence, the zero set of $J \cap C[Y_{ij}, \det(Y_{ij})^{-1}]$ is $\{\mathbf{1}_n\}$. Therefore, $\mathbf{1}_n \in \mathcal{W}$ and so we have $\mathcal{W} = \mathcal{G}(C)$. By Corollary 2.13 the differential ideal I specializes to an ideal $\sigma(I) \subset \bar{I}$ which is contained in the maximal differential ideal $\bar{I}$ of $\bar{R}[X_{ij}, \det(X_{ij})^{-1}]$ with $\bar{I} \cap \bar{R} = (0)$. Since σ is a surjective differential homomorphism, $\sigma(I)$ is a differential ideal of $\bar{R}[X_{ij}, \det(X_{ij})^{-1}]$. Further, $\sigma(I) \subset \sigma(I)_{ex}$ is a differential ideal of $\bar{R}[X_{ij}, \det(X_{ij})^{-1}]$ satisfying Condition 2.3. Now one repeats the above argumentation in the case for the specialized equation, the differential ideal $\bar{I}$ and the corresponding rings and fields.

$$\begin{array}{ccccccc}
R[X_{ij}, \det(X_{ij})^{-1}] & & S[X_{ij}, \det(X_{ij})^{-1}] & = & S[Y_{ij}, \det(Y_{ij})^{-1}] & & C[Y_{ij}, \det(Y_{ij})^{-1}] \\
\cup & & \cup & & \cup & & \cup \\
I & \xmapsto{\iota} & (I) & = & \tilde{I} & \xmapsto{\delta^{-1}} & Q_I \\
\Big\downarrow{\scriptstyle \hat{\sigma}} & & & \Big\downarrow{\scriptstyle \check{\sigma}} & & & \Big\downarrow{\scriptstyle id} \\
\sigma(I)_{ex} & \xmapsto{\iota} & (\sigma(I)_{ex}) & = & \widetilde{\sigma(I)_{ex}} & \xmapsto{\delta^{-1}} & Q_{\sigma(I)_{ex}} \\
\cap & & \cap & & \cap & & \cap \\
\bar{I} & \xmapsto{\iota} & (\bar{I}) & = & \tilde{\bar{I}} & \xmapsto{\delta^{-1}} & Q_{\bar{I}} \\
\cap & & \cap & & \cap & & \cap \\
\bar{R}[X_{ij}, \det(X_{ij})^{-1}] & & \bar{S}[X_{ij}, \det(X_{ij})^{-1}] & = & \bar{S}[Y_{ij}, \det(Y_{ij})^{-1}] & & C[Y_{ij}, \det(Y_{ij})^{-1}]
\end{array}$$

The ideals $Q_{\bar{I}}$ and $Q_{\sigma(I)_{ex}}$ are defined as $Q_{\bar{I}} := \tilde{\bar{I}} \cap C[Y_{ij}, \det(Y_{ij})^{-1}]$ and $Q_{\sigma(I)_{ex}} := \widetilde{\sigma(I)}_{ex} \cap C[Y_{ij}, \det(Y_{ij})^{-1}]$. The zero set of the ideal $Q_{\bar{I}}$ is the differential Galois group $\mathcal{H}(C)$ of $\partial(\boldsymbol{y}) = A(\sigma(\boldsymbol{t}))\boldsymbol{y}$. Moreover, the inclusion $\bar{I} \supset \sigma(I)_{ex}$ implies the inclusion $Q_{\bar{I}} \supset Q_{\sigma(I)_{ex}}$. We are going to show that $Q_{\bar{I}} \supset Q_I$.
From Lemma 2.12 we obtain that we can extend σ to a surjective differential homomorphism

$$\breve{\sigma}: \ S[X_{ij}, \det(X_{ij})^{-1}] = S[Y_{ij}, \det(Y_{ij})^{-1}] \longrightarrow \bar{S}[X_{ij}, \det(X_{ij})^{-1}] = \bar{S}[Y_{ij}, \det(Y_{ij})^{-1}].$$

In particular, the specialization $\breve{\sigma}$ is well defined for the relations

$$(X_{ij}) = (Z_{ij})(Y_{ij}) \text{ and } (Y_{ij}) = (Z_{ij})^{-1}(X_{ij}).$$

Let $f \in Q_I$ be one of the generators of Q_I, say

$$f = \sum_{\boldsymbol{i}} c_{\boldsymbol{i}}\, Y_{11}^{i_{11}} \cdot ... \cdot Y_{nn}^{i_{nn}},$$

where $c_{\boldsymbol{i}} \in C$ and $\boldsymbol{i} = (i_{11}, ..., i_{nn}) \in \mathbb{N}^{n \cdot n}$. Then via the map δ, f is an element of

$$\delta(f) = f \in \tilde{I} = (I) \subset S[Y_{ij}, \det(Y_{ij})^{-1}] = S[X_{ij}, \det(X_{ij})^{-1}].$$

Further, Lemma 2.5 implies that for suitable elements $p_k \in S[X_{ij}, \det(X_{ij})^{-1}]$ and $q_k \in I \subset R[X_{ij}, \det(X_{ij})^{-1}]$ we can write f as

$$f = \sum_k p_k \cdot q_k.$$

Then $\breve{\sigma}$ maps f to

$$\breve{\sigma}(f) = \sum_{\boldsymbol{i}} c_{\boldsymbol{i}} Y_{11}^{i_{11}} \cdot ... \cdot Y_{nn}^{i_{nn}} = \sum_k \breve{\sigma}(p_k) \cdot \breve{\sigma}(q_k) \in (\sigma(I)_{ex}) = \widetilde{\sigma(I)_{ex}} \subset (\bar{I}) = \tilde{\bar{I}}.$$

Hence, we obtain $f \in Q_{\bar{I}} = \tilde{\bar{I}} \cap C[Y_{ij}, \det(Y_{ij})^{-1}]$. We conclude that $Q_{\bar{I}} \supset Q_I$, and therefore we have $\mathcal{H}(C) \leq \mathcal{G}(C)$. □

As a consequence of Theorem 2.14 we get the specialization bound in the language of homogeneous linear differential equations.

Corollary 2.15. *Let C be an algebraically closed field of characteristic zero and $F = C\langle \boldsymbol{t} \rangle$ the differential field in the l differential indeterminates $\boldsymbol{t} = (t_1, ..., t_l)$. Let E be a Picard-Vessiot extension over F for the differential equation*

$$L(t_1, ..., t_l, y) = y^{(n)} + a_1(\boldsymbol{t})y^{(n-1)} + ... + a_n(\boldsymbol{t})y \in C\{\boldsymbol{t}, y\}$$

and denote by $\mathcal{G}(C)$ its differential Galois group.
Let $\bar{F} = C(z)$ be the rational function field and let $\boldsymbol{f} = (f_1, ..., f_l) \in C[z]^l$ such that $C[z] = C\{f_1, ..., f_l\}$. Moreover, let $\bar{E}$ be a Picard-Vessiot extension of $\bar{F} = C(z)$ for the specialized differential equation

$$L(f_1, ..., f_l, y) = y^{(n)} + a_1(\boldsymbol{f})y^{(n-1)} + ... + a_n(\boldsymbol{f})y \in C[z]\{y\}$$

and denote by $\mathcal{H}(C)$ its differential Galois group. Then $\mathcal{H}(C) \leq \mathcal{G}(C)$.

Let $\mathcal{G}(C)$ be one of the classical groups with root system Φ and denote by $l = \mathrm{rank}(\Phi)$ the rank of Φ. Let $t_1, ..., t_l$ be differential indeterminates over C. We are going to realize $\mathcal{G}(C)$ as the differential Galois group for a parametrized differential equation $L(t_1, ..., t_l, y) \in C\{t_1, ..., t_l,\ y\}$ over $C\langle t_1, ..., t_l\rangle$. We will proceed in the following way.
The construction that we provide is based on Corollary 2.15 and Proposition 2.1. But to apply Corollary 2.15 we need a specialization $L(f_1, ..., f_l, y) \in C[z]\{y\}$ over the differential field $\bar{F} = C(z)$ of the parametrized equation above. In [MS96] C. Mitschi and M.F. Singer developed a method to construct matrix differential equations $\partial(\boldsymbol{y}) = A_{\mathcal{G}}^{M\&S}\boldsymbol{y}$ over the rational function field $C(z)$ which has a given connected reductive group as its differential Galois group. Using the structure of the classical groups, we will show that $\partial(\boldsymbol{y}) = A_{\mathcal{G}}^{M\&S}\boldsymbol{y}$ leads to a specialization $L(f_1, ..., f_l, y) \in C[z]\{y\}$ over $C(z)$. We can now apply Corollary 2.15. Since the differential equation $L(t_1, ..., t_l, y)$ comes from a matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathcal{G}}\boldsymbol{y}$ with $A_{\mathcal{G}} \in \mathrm{Lie}(\mathcal{G})(C\langle t_1, ..., t_l\rangle)$, we are able to complete our approach by making use of Proposition 2.1.
In Section 3.4 the construction of a matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathcal{G}}^{M\&S}\boldsymbol{y}$ with differential Galois group $\mathcal{G}(C)$ over $C(z)$ following the ideas of C. Mitschi and M.F. Singer is presented.

Chapter 3

Reductive linear groups

We start by resuming shortly some results on the structure of reductive linear algebraic groups, their Lie algebras and abstract root systems. For an introduction to algebraic groups, we refer to the standard literature, e.g. Humphreys [Hum98], Springer [Spr98] and Borel [Bor91]. The basic facts about Lie algebras and root systems can be found in [Hum72]. In Section 3.3, we present the method for the proofs of the transformation lemma. Finally, we construct a matrix differential equation over $C(z)$ for a connected reductive group following the ideas of Mitschi and Singer, and prove in the subsequent section that for every such a group there exists a parametric equation.

3.1 Abstract root systems

Let V be an euclidean space with positive definite symmetric bilinear form $(\cdot, \cdot)$. For every non zero $\alpha \in V$, one can define a **reflection** σ_α by the formula

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

Denote by $\langle \beta, \alpha \rangle$ the expression $2(\beta, \alpha)/(\alpha, \alpha)$. Obviously, $\langle \beta, \alpha \rangle$ is only linear in the first variable. A finite subset $\Phi \subset V$ of a euclidean space V is called a **root system** in V, if Φ satisfies

1. Φ spans V and does not contain 0.
2. If $\alpha \in \Phi$, then the only scalar multiples of α are $\pm\alpha$.
3. If $\alpha \in \Phi$, then the reflection σ_α leaves Φ invariant.
4. If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

The **rank** of a root system Φ is defined as $\mathrm{rank}(\Phi) = \dim(V)$. Let $\alpha, \beta \in \Phi$ be non-proportional roots (i.e., $\beta \neq \pm\alpha$). One calls all roots $\beta + s\alpha$ with $s \in \mathbb{Z}$ the α-string through β. It can be shown that the α-string through β is unbroken and that there are $r, q \in \mathbb{Z}^+$ such that the **$\boldsymbol{\alpha}$-string** through β goes from $\beta - r\alpha$ to $\beta + q\alpha$. A subset $\Delta \subset \Phi$ is called a **basis**, if Δ is also a basis of V and each root β can be written as

$$\sum_{\alpha \in \Delta} k_\alpha \alpha \tag{3.1}$$

with $k_\alpha \in \mathbb{Z}$ all non-negative or non-positive. The elements of Δ are called the **simple roots**. Since Δ is a basis of V, we have $\mathrm{card}(\Delta) = \mathrm{rank}(\Phi)$ and the expression in (3.1) is unique. For a chosen basis $\Delta = \{\alpha_1, ..., \alpha_l\}$, we define the **height** ht of a root $\beta = \sum_{i=1}^{l} k_i\alpha_i$ as $\mathrm{ht}(\beta) = \sum_{i=1}^{l} k_i$. If all $k_i \geq 0$ (resp. all $k_i \leq 0$), we call the root β positive (resp. negative) and denote the set of all positive roots by Φ^+ (resp. Φ^-).

3.2 The structure of reductive linear algebraic groups and their Lie algebras

Let $\mathcal{G}$ be a reductive linear algebraic group defined over C and fix an embedding $\mathcal{G} \hookrightarrow \mathrm{GL}_n$. Denote by $\mathrm{Lie}(\mathcal{G})$ the Lie algebra of $\mathcal{G}$. An important type of subgroup of a reductive algebraic group $\mathcal{G}$ are the **Borel subgroups**, the maximal closed connected solvable subgroups of $\mathcal{G}$. Denoted by $\mathcal{B} \leq \mathcal{G}$ such a subgroup. Then all Borel subgroups are conjugate, and the maximal tori of $\mathcal{G}$ are contained in the various Borel subgroups. If we choose a Borel subgroup $\mathcal{B}$ containing a **maximal torus** $\mathcal{T}$, then there is a unique Borel subgroup $\mathcal{B}^-$ such that $\mathcal{B} \cap \mathcal{B}^- = \mathcal{T}$. The group $\mathcal{B}^-$ is called the opposite Borel subgroup to $\mathcal{B}$. The subgroup formed by all unipotent elements of $\mathcal{B}$ (resp. $\mathcal{B}^-$) will be denoted by $\mathcal{U}$ (resp. $\mathcal{U}^-$).
After a suitable conjugation of $\mathcal{G}$, one is able to choose these groups in GL_n such that they have a nice shape: Denote by $\mathcal{T}_0 \leq \mathcal{G}$ the maximal diagonal torus, i.e., the group of all diagonal matrices of $\mathcal{G}$ and by $\mathcal{B}_0$ the Borel subgroup with $\mathcal{T}_0 \leq \mathcal{B}_0$ consisting of all upper triangular matrices of $\mathcal{G}$. For $X \in \mathcal{G}$, let

$$\mathrm{int}\ X : \mathcal{G} \to \mathcal{G};\ Y \mapsto XYX^{-1}$$

be the inner automorphism of $\mathcal{G}$. The differential $d(\mathrm{int}\ X)$ will be denoted by $\mathrm{Ad}\ X$, and the induced action is called the **adjoint action**. In the case $\mathcal{G} \leq \mathrm{GL}_n$, the automorphism $\mathrm{Ad}\ X$ of $\mathrm{Lie}(\mathcal{G})$ is just conjugation by X, i.e.,

$$\mathrm{Ad}\ X(A) = XAX^{-1} \tag{3.2}$$

for some $A \in \mathrm{Lie}(\mathcal{G})$. Let $\mathcal{T}_0 \leq \mathcal{G}$ be the maximal diagonal torus of $\mathcal{G}$, and let $\mathcal{X}(\mathcal{T}_0)$ be the character group. Let $\mathcal{T}_0$ act on $\mathrm{Lie}(\mathcal{G})$ via the adjoint action. Then $\mathrm{Lie}(\mathcal{G})$ can be written as the direct sum of weight spaces

$$\mathrm{Lie}(\mathcal{G})_\alpha := L_\alpha := \{A \in \mathrm{Lie}(\mathcal{G}) | Ad\ T(A) = \alpha(T)A \text{ for all } T \in \mathcal{T}_0\}$$

for $\alpha \in \mathcal{X}(\mathcal{T}_0)$. The set of all non zero weights is called the root system $\Phi(\mathcal{T}_0, \mathcal{G})$ of $\mathcal{G}$ relative to $\mathcal{T}_0$, and the elements are called the roots. More precisely, we have a decomposition

$$\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G})^{\mathcal{T}_0} \oplus \bigoplus_{\alpha \in \Phi(\mathcal{T}, \mathcal{G})} L_\alpha, \tag{3.3}$$

whereas we mean by $\mathrm{Lie}(\mathcal{G})^{\mathcal{T}_0}$ the fix point space, i.e., the space corresponding to the zero weight. Since $\mathcal{G}$ is reductive and $\mathcal{T}_0$ is maximal, we have $\mathrm{Lie}(\mathcal{T}_0) = \mathrm{Lie}(\mathcal{G})^{\mathcal{T}_0}$. The spaces $\mathrm{Lie}(\mathcal{G})_\alpha$ are called the **root spaces** and they are of dimension $\dim(\mathrm{Lie}(\mathcal{G})_\alpha)) = 1$. For

$\alpha \in \Phi$, one defines $\mathcal{T}_\alpha = (\ker\,\alpha)^\circ$. The $\mathcal{T}_\alpha$ are the singular tori of codimension 1 in $\mathcal{T}_0$. Their centralizers $\mathcal{Z}_\alpha = C_{\mathcal{G}}(\mathcal{T}_\alpha)$ are reductive groups of semisimple rank 1 with Lie algebra

$$\mathrm{Lie}(\mathcal{Z}_\alpha) = \mathrm{Lie}(\mathcal{T}_0) \oplus \mathrm{Lie}\mathcal{G})_\alpha \oplus \mathrm{Lie}(\mathcal{G})_{-\alpha}.$$

Then $\mathrm{Lie}(\mathcal{G})_\alpha$ is the Lie algebra of the unipotent part $\mathcal{U}_\alpha$ of one of the Borel subgroups of $\mathrm{Lie}(\mathcal{Z}_\alpha)$. The groups $\mathcal{U}_\alpha$ are the unique $\mathcal{T}_0$-stable subgroups of $\mathcal{G}$ with Lie algebra $\mathrm{Lie}(\mathcal{G})_\alpha$ and are called the **root subgroups** of $\mathcal{G}$. Later, we will make use of the root subgroups for the proofs of the transformation lemma. In [Car72], Carter shows how the root subgroups $\mathcal{U}_\alpha$ of $\mathcal{G}$ can be constructed from the Lie algebra $\mathrm{Lie}(\mathcal{G})$.
A toral subalgebra $\mathbf{H}$ of $\mathrm{Lie}(\mathcal{G})$ is a subalgebra generated by semisimple elements of $\mathrm{Lie}(\mathcal{G})$. Now fix a maximal toral subalgebra $\mathbf{H}$. Then it can be shown that $\mathbf{H}$ is a abelian subalgebra. Hence, ad $\mathbf{H}$ consists of commuting semisimple endomorphisms of $\mathrm{Lie}(\mathcal{G})$ and is therefore simultaneously diagonalizable. Thus, $\mathrm{Lie}(\mathcal{G})$ is the direct sum of subalgebras

$$\mathrm{Lie}(\mathcal{G})_\alpha = \{X \in \mathrm{Lie}(\mathcal{G}) |\, [H, X] := \mathrm{ad}(H)(X) = \alpha(H)X \text{ for all } H \in \mathbf{H}\},$$

where $\alpha \in \mathbf{H}^*$ is an element of the dual space $\mathbf{H}^*$. The nonzero $\alpha \in \mathbf{H}^*$ with $\mathrm{Lie}(\mathcal{G})_\alpha \neq 0$ are called roots and form a root system Φ which is isomorphic to the root system presented above. Thus, we have a decomposition

$$\mathrm{Lie}(\mathcal{G}) = C_{\mathrm{Lie}(\mathcal{G})}(\mathbf{H}) \oplus \bigoplus_{\alpha\in\Phi} \mathrm{Lie}(\mathcal{G})_\alpha, \tag{3.4}$$

called the **Cartan Decomposition**. It can be shown that the centralizer $C_{\mathrm{Lie}(\mathcal{G})}(\mathbf{H})$ of $\mathbf{H}$ in $\mathrm{Lie}(\mathcal{G})$ equals $\mathbf{H}$. Moreover, the decomposition (3.4) is isomorphic to the decomposition (3.3), and if we choose $\mathbf{H}$ to consist only of diagonal matrices, then the two decompositions coincide. Let $X_1,\ X_2 \in \mathrm{Lie}(\mathcal{G})$. One defines the Killing form κ by $\kappa(X_1, X_2) = \mathrm{trace}(\mathrm{ad}X_1\mathrm{ad}X_2)$. Then κ is a symmetric bilinear form on $\mathrm{Lie}(\mathcal{G})$ which is non degenerate if and only if $\mathrm{Lie}(\mathcal{G})$ is semisimple. Since the restriction of κ to $\mathbf{H}$ is also non degenerated, we can identify $\mathbf{H}^*$ with $\mathbf{H}$. Let $\alpha \in \mathbf{H}^*$. Then there is a unique element $\tilde{H}_\alpha \in \mathbf{H}$ such that $\alpha(H) = \kappa(\tilde{H}_\alpha, H)$ for all $H \in \mathbf{H}$. One defines the so called **co-root** as

$$H_\alpha = \tilde{H}_\alpha \frac{2}{\kappa(\tilde{H}_\alpha, \tilde{H}_\alpha)}.$$

We fix this notation for the co-roots H_α. For $\alpha \in \Phi$, we denote by X_α a basis element of $\mathrm{Lie}(\mathcal{G})_\alpha$. Then there exists a unique $X_{-\alpha} \in \mathrm{Lie}(\mathcal{G})_{-\alpha}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$. The elements $X_\alpha, X_{-\alpha}$ and H_α span a three dimensional simple subalgebra isomorphic to $\mathrm{Lie}(\mathrm{SL}_2)$. Moreover, for a basis $\Delta = \{\alpha_1, ..., \alpha_l\}$ of Φ, the set

$$\{H_\alpha \mid \alpha \in \Delta\}$$

forms a basis of $\mathrm{Lie}(\mathcal{T}_0)$. If $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$, then

$$[\mathrm{Lie}(\mathcal{G})_\alpha, \mathrm{Lie}(\mathcal{G})_\beta] = \mathrm{Lie}(\mathcal{G})_{\alpha+\beta}.$$

For each root $\alpha \in \Phi$, it is possible to choose the basis elements $X_\alpha \in \mathrm{Lie}(\mathcal{G})_\alpha$ such that

$$[X_\alpha, X_{-\alpha}] = H_\alpha, \tag{3.5}$$
$$[X_\alpha, X_\beta] = \pm(r + 1)X_{\alpha+\beta}, \tag{3.6}$$

where r is the smallest integer for which $\beta - r\alpha$ is a root of Φ (see [Car72, theorem 4.21]). Then the set

$$\{X_\alpha,\ H_{\alpha_i} \mid \alpha \in \Phi,\ 1 \le i \le l\},$$

where X_α satisfy (3.5) and (3.6) from above, is called a **Chevalley basis** of $\mathrm{Lie}(\mathcal{G})$.

Theorem 3.1. *Let* $\{X_\alpha,\ H_i = H_{\alpha_i} \mid \alpha \in \Phi,\ 1 \le i \le l\}$ *be a Chevalley basis of* $\mathrm{Lie}(\mathcal{G})$. *Then the resulting structure constants lie in* $\mathbb{Z}$. *More precisely:*

1. $[H_i, H_j] = 0$, $1 \le i, j \le l$,
2. $[H_i, X_\alpha] = \langle \alpha, \alpha_i \rangle X_\alpha$, $1 \le i \le l$, $\alpha \in \Phi$,
3. $[X_\alpha, X_{-\alpha}] = H_\alpha$ *is a* $\mathbb{Z}$*-linear combination of* $H_1, ..., H_l$,
4. *If* α, β *are independent roots,* $\beta - r\alpha, ..., \beta + q\alpha$ *the* α*-string through* β*, then* $[X_\alpha, X_\beta] = 0$ *if* $q = 0$*, while* $[X_\alpha, X_\beta] = \pm(r+1)X_{\alpha+\beta}$ *if* $\alpha + \beta \in \Phi$.

Let $X_\alpha \in \mathrm{Lie}(\mathcal{G})(C)$ be an element of a Chevalley basis. Then $\delta := \mathrm{ad}(X_\alpha)$ is a nilpotent derivation of $\mathrm{Lie}(\mathcal{G})(C)$. Say $\delta^k = 0$ for $k \in \mathbb{N} \setminus \{0\}$. The image of δ under the exponential map

$$\exp(\delta) = 1 + \delta + \frac{1}{2!}\delta^2 + \frac{1}{3!}\delta^3 + ... + \frac{1}{(k-1)!}\delta^{k-1}$$

defines an automorphism of $\mathrm{Lie}(\mathcal{G})(C)$ by [Car72, Lemma 4.3.1]. Now let $\zeta \in C$. Since the same is also true for $\zeta \cdot \mathrm{ad}(X_\alpha)$, we define the parametrized automorphism of $\mathrm{Lie}(\mathcal{G})(C)$ by

$$U_\alpha(\zeta) = \exp(\zeta\ \mathrm{ad}(X_\alpha)).$$

From the formula

$$U_\alpha(\zeta_1)U_\alpha(\zeta_2) = \exp(\zeta_1\ \mathrm{ad}(X_\alpha))\ \exp(\zeta_2\ \mathrm{ad}(X_\alpha)) = \exp((\zeta_1 + \zeta_2)\mathrm{ad}(X_\alpha)) = U_\alpha(\zeta_1 + \zeta_2)$$

we see that the inverse of $U_\alpha(\zeta)$ is $U_\alpha(\zeta)^{-1} = U_\alpha(-\zeta)$.
We summarize the effect of the automorphism $U_\alpha(\zeta)$ on the elements of a Chevalley basis in Lemma 3.2 presented below.

Lemma 3.2. *Let* $\{X_\alpha,\ H_i = H_{\alpha_i} \mid \alpha \in \Phi,\ 1 \le i \le l\}$ *be a Chevalley basis and* $U_\beta(\zeta)$ *an as above defined parameterized automorphism of* $\mathrm{Lie}(\mathcal{G}(C))$. *Then*

1. $U_\beta(\zeta).X_\beta = X_\beta$.
2. $U_\beta(\zeta).X_{-\beta} = X_{-\beta} + \zeta H_\beta - \zeta^2 X_\beta$.
3. $U_\beta(\zeta).H_\beta = H_\beta - 2\zeta X_\beta$.
4. *Let* β, α *be linearly independent and* $\alpha - r\beta, ..., \alpha + q\beta$ *the* β*-string through* α*. Define* $m_{\beta,\alpha,0} = 0$ *and* $m_{\beta,\alpha,i} = \pm\binom{r+i}{i}$*. Then*

$$U_\beta(\zeta).X_\alpha = \sum_{i=0}^{q} m_{\beta,\alpha,i}\zeta^i X_{i\beta+\alpha}.$$

5. *If β, α are linearly independent, then we have* $U_\beta(\zeta).H_\alpha = H_\alpha - \langle \alpha, \beta \rangle \zeta X_\beta$.

The group generated by the $U_\alpha(\zeta)$ is precisely the root subgroup $\mathcal{U}_\alpha$ of $\mathcal{G}$ in the above discussion. The whole construction of $U_\alpha(\zeta)$ was done over the algebraically closed field C of characteristic 0. But we want to apply these results to a non algebraically closed differential field (F, ∂_F) with C as its field of constants. Carter shows in [Car72, Section 4.4] that the results are also valid over arbitrary fields. Furthermore, in Lemma 3.2 we have written $U_\beta(\zeta).X$ for the action of $U_\beta(\zeta)$ on $X \in \mathrm{Lie}(\mathcal{G})$. Since all the elements of $\mathcal{G}$ and $\mathrm{Lie}(\mathcal{G})$ are represented by matrices, we have

$$U_\beta(\zeta).X = U_\beta(\zeta) X U_\beta(-\zeta) = \mathrm{Ad}(U_\beta(\zeta))X.$$

3.3 Transforming differential modules

Let (F, ∂_F) be a differential field, and $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ a matrix differential equation with associated differential module M, and fundamental solution matrix Y. Suppose that Y generates a Picard-Vessiot extension E over F. Then $\partial(Y)Y^{-1} = A \in F^{n\times n}$. This motivates the following definition:

Definition 3.3. We call the map

$$l\delta : \mathrm{GL}_n(F) \to F^{n\times n}, \quad B \mapsto \partial(B)\cdot B^{-1}$$

the **logarithmic derivative**.

Observation 3.4. Let $e_1, ..., e_n$ be a basis of M. Then we can transform it to another basis $\tilde{e}_1, ..., \tilde{e}_n$ of M. Let this transformation be given by the matrix $B \in \mathrm{GL}_n(F)$. The effect on the defining matrix A is

$$BAB^{-1} + \partial_F(B)B^{-1} =: \tilde{A}.$$

In particular, the matrices A and $\tilde{A}$ are differentially equivalent (see Definition 1.4). Using Definition 3.3 and equation (3.2), we can write $BAB^{-1} + \partial_F(B)B^{-1}$ as

$$\tilde{A} = Ad\, B(A) + l\delta(B). \tag{3.7}$$

Choose B as a parametrized root group element $U_\beta(\xi)$ for some $\beta \in \Phi$ and write A as a linear combination of elements of a Chevalley basis. Then Lemma 3.2 explains in which root spaces the image of A under $\mathrm{Ad}(U_\beta)(\xi)$ lies. Thus, by a good grasp of the root system and an educated choice of A, we can handle the first summand of the right hand side of equation (3.7). The following proposition due to Kovacic in [Kov69] helps us to control the second term.

Proposition 3.5. *Let $\mathcal{H} \leq \mathrm{GL}_n(F)$ be a linear algebraic group. Then the restriction of $l\delta$ to $\mathcal{H}$*

$$l\delta|_{\mathcal{H}} : \mathcal{H} \to \mathrm{Lie}(\mathcal{H})$$

maps $\mathcal{H}$ on its Lie algebra $\mathrm{Lie}(\mathcal{H})$.

Example 3.6. Let $\mathcal{G} = \mathrm{SL}_3$ and $F = C(z)$. The root system Φ is of type A_2 (see [Hum72, Section 1.2]) and as a basis of Φ we take $\Delta = \{\alpha_1, \alpha_2\}$. Then Φ consists of the vectors $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$. The following matrices form a Chevalley basis of $\mathrm{Lie}(\mathrm{SL}_{l+1})$:

$$\begin{array}{cccc} H_{\alpha_1} = E_{11} - E_{22}, & H_{\alpha_2} = E_{22} - E_{33}, & X_{\alpha_1} = E_{12}, & X_{\alpha_2} = E_{23}, \\ X_{\alpha_1+\alpha_2} = E_{13}, & X_{-\alpha_1} = E_{21}, & X_{-\alpha_2} = E_{32}, & X_{\alpha_1+\alpha_2} = E_{31}. \end{array}$$

All of this can be found in the next chapter. We are going to transform

$$A = X_{\alpha_1} + X_{\alpha_2} + z^2 H_{\alpha_2} + zX_{-\alpha_1}.$$

In the first step we differentially conjugate A by the root group element $U_{-\alpha_2}(z^2)$. By linearity and Lemma 3.2, we get for the adjoint action of $U_{-\alpha_2}(z^2)$ on A

$$\begin{aligned} \mathrm{Ad}(U_{-\alpha_2}(z^2))(A) &= \mathrm{Ad}(U_{-\alpha_2}(z^2))(X_{\alpha_1}) + \mathrm{Ad}(U_{-\alpha_2}(z^2))(X_{\alpha_2}) \\ &\quad + z^2\mathrm{Ad}(U_{-\alpha_2}(z^2))(H_{\alpha_2}) + z\mathrm{Ad}(U_{-\alpha_2}(z^2))(X_{-\alpha_1}) \\ &= X_{\alpha_1} + X_{\alpha_2} - z^2 H_{\alpha_2} - z^4 X_{-\alpha_2} + z^2 H_{\alpha_2} + 2z^4 X_{-\alpha_2} + zX_{-\alpha_1} \\ &\quad + z^3 X_{-\alpha_1-\alpha_2} \\ &= X_{\alpha_1} + X_{\alpha_2} + zX_{-\alpha_1} + z^4 X_{-\alpha_2} + + z^3 X_{-\alpha_1-\alpha_2}. \end{aligned}$$

The logarithmic derivate $l\delta(U_{-\alpha_2}(z^2))$ can be computed as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2z & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2z & 0 \end{pmatrix} = 2zX_{-\alpha_2}.$$

Hence, we obtain

$$\tilde{A} := \mathrm{Ad}(U_{-\alpha_2}(z^2)A + l\delta(U_{-\alpha_2}(z^2)) = X_{\alpha_1} + X_{\alpha_2} + zX_{-\alpha_1} + (z^4 + 2z)X_{-\alpha_2} + z^3 X_{-\alpha_1-\alpha_2}.$$

We differentially conjugate $\tilde{A}$ by $U_{-\alpha_1-\alpha_2}(z)$. Again, we begin with the computation of $\mathrm{Ad}(U_{-\alpha_1-\alpha_2}(z))\tilde{A}$. We get

$$\begin{aligned} \mathrm{Ad}(U_{-\alpha_1-\alpha_2}(z))\tilde{A} &= \mathrm{Ad}(U_{-\alpha_1-\alpha_2}(z))X_{\alpha_1} + \mathrm{Ad}(U_{-\alpha_1-\alpha_2}(z))X_{\alpha_2} \\ &\quad + z\mathrm{Ad}(U_{-\alpha_1-\alpha_2}(z))X_{-\alpha_1} + z^4\mathrm{Ad}(U_{-\alpha_1-\alpha_2}(z))X_{-\alpha_2} \\ &\quad + z^3\mathrm{Ad}(U_{-\alpha_1-\alpha_2}(z))X_{-\alpha_1-\alpha_2} \\ &= X_{\alpha_1} + zX_{-\alpha_2} + X_{\alpha_2} - zX_{-\alpha_1} + zX_{-\alpha_1} + z^4 X_{-\alpha_2} + z^3 X_{-\alpha_1-\alpha_2} \\ &= X_{\alpha_1} + X_{\alpha_2} + (z + z^4)X_{-\alpha_2} + z^3 X_{-\alpha_1-\alpha_2}. \end{aligned}$$

It is left to add the image $l\delta(U_{-\alpha_1-\alpha_2}(z))$ of $U_{-\alpha_1-\alpha_2}(z)$ under the logarithmic derivate to the above sum. It is easily calculated as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -z & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = X_{-\alpha_1-\alpha_2}.$$

Summing up by using Observation 1.6 we conclude that

$$\bar{A} := \mathrm{Ad}(U_{-\alpha_1-\alpha_2}(z)U_{-\alpha_2}(z^2))(A) + l\delta(U_{-\alpha_1-\alpha_2}(z)U_{-\alpha_2}(z^2))$$

is equal to

$$\bar{A} = X_{\alpha_1} + X_{\alpha_2} + (z^3+1)X_{-\alpha_1-\alpha_2} + (z^4+z)X_{-\alpha_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ (z^3+1) & (z^4+z) & 0 \end{pmatrix}.$$

3.4 A modification of a result of Mitschi and Singer

In this section, the differential field is $(C(z), \partial_{C(z)})$, i.e., a rational function field in one variable z over an algebraically closed field C of characteristic 0 with standard derivation $\partial = \frac{d}{dz}$.
In [MS96] C. Mitschi and M.F. Singer developed a solution of the inverse problem for connected algebraic groups over $C(z)$. If in addition the group is semisimple, one of their results reads

Theorem 3.7. *(C. Mitschi and M.F. Singer)*
The field C is supposed to be algebraically closed and of characteristic 0. Every connected semisimple linear algebraic group $\mathcal{G}$ is the differential Galois group of an equation $\partial(\boldsymbol{y}) = (A_0 + A_1 z)\boldsymbol{y}$ over $C(z)$ where A_0, A_1 are constant matrices.

In the proof (see, for example, [PS03, Theorem 11.30]), the authors describe how to choose the matrices A_0 and A_1 in $\mathrm{Lie}(\mathcal{G})(C)$. For our purpose, we need an equation of a special shape, which is not given. Therefore, we have to use different matrices A_0 and A_1 than Mitschi and Singer. The proof of Corollary 3.12, which we obtain in this way, will be an imitation of the proof of Theorem 3.7 along with some small modifications. At first we provide the most important tools for the proof of the Corollary 3.12.

Note 3.8. Let K be a field of characteristic zero and $\mathcal{G}$ a connected linear group. Remember that there is a correspondence between torsors and the first cohomology sets $H^1(\bar{K}/K, \mathcal{G}(\bar{K}))$. In some cases $H^1(\bar{K}/K, \mathcal{G}(\bar{K}))$ becomes trivial. In [Ser97, III 2.3 Theorem 1'] it is shown that if the cohomological dimension $\mathrm{cd}(K)$ is at least one, then the first cohomology set is trivial. A field $\tilde{K}$ is called a C_1-field if every homogeneous polynomial $f(x_1, ..., x_n) \in \tilde{K}[x_1, ..., x_n]$ of degree $d \geq 1$ has a nontrivial solution in $\tilde{K}^n$ if $n > d$. If K is a C_1-field, then [Ser97, II 3.2 Corollary] yields $\mathrm{cd}(K) \leq 1$. It can be also found in [Ser97, II 3.3 b] that $C(z)$ is a C_1-field.

We want to apply the lower bound criterion, i.e., Theorem 2.2. Thus, we have to ensure the condition that $\mathcal{Z}$ is the trivial torsor. This is automatically satisfied if $\mathcal{G}$ is connected and the differential ground field is a C_1-field. To ensure the condition of connectness we will apply

Observation 3.9. Let W be a finite dimensional C-vector space and let $A_0, ..., A_m$ be elements of $\mathrm{End}(W)$. Then the differential Galois group $\mathcal{G}$ of the differential equation $\partial(\boldsymbol{y}) = (A_0 + A_1 z + ... + A_m z^m)\boldsymbol{y}$ over $C(z)$ is connected.

Proof. Denote by E the Picard-Vessiot extension of $\partial(\boldsymbol{y}) = (\sum_{i=0}^m A_i z^i)\boldsymbol{y}$ over $C(z)$. Then Theorem 1.13 implies that $F = E^{\mathcal{G}^0}$ is a finite Galois extension of $C(z)$ with Galois group $\mathcal{G}/\mathcal{G}^0$. This extension can be ramified only above the singular points of $\partial(\boldsymbol{y}) = (\sum_{i=0}^{m} A_i z^i)\boldsymbol{y}$. Since the only singular point is ∞ we get $F = C(z)$. Thus $\mathcal{G} = \mathcal{G}^0$ is connected. □

The key of the proof is the following

Definition 3.10. Let $\rho : \mathcal{G} \to \mathrm{GL}(W)$ be a faithful representation. Then the $\mathcal{G}$-module W will be called a **Chevalley module** if:

- $\mathcal{G}$ leaves no line in W invariant.
- Any proper connected closed subgroup of $\mathcal{G}$ has an invariant line.

The existence of a Chevalley module is guaranteed by

Lemma 3.11. *(Mitschi and M.F. Singer)*
Let $\mathcal{G}$ be a connected semisimple linear algebraic group. Then there exists a Chevalley module for $\mathcal{G}$.

For a proof we refer to [PS03, Lemma 11.34].

Corollary 3.12. *Let $\mathcal{G}$ be a connected semisimple algebraic group, Φ the root system, Δ a base of Φ. Denote by*

$$\mathrm{Lie}(\mathcal{G})(C) = \mathbf{H}(C) \oplus \bigoplus_{\alpha\in\Phi} \mathrm{Lie}(\mathcal{G})_\alpha(C)$$

the root space decomposition of $\mathrm{Lie}(\mathcal{G})(C)$*, where* $\mathbf{H}(C)$ *denotes the Cartan subalgebra, and* $\mathrm{Lie}(\mathcal{G})_\alpha(C) = \langle X_\alpha \rangle_C$ *denote the one-dimensional root spaces spanned by a basis element X_α. Set $A_0 = \sum_{\alpha\in\Delta}(X_\alpha + X_{-\alpha})$. Then there exists $A_1 \in \mathbf{H}(C)$ such that the equation $\partial(\boldsymbol{y}) = (A_0 + A_1 z^2)\boldsymbol{y}$ over $C(z)$ has $\mathcal{G}(C)$ as differential Galois group.*

Proof. Lemma 3.11 ensures the existence of a Chevalley module $\rho : \mathcal{G} \to \mathrm{GL}(W)$. We fix such a module. Then there is a induced injective morphism of Lie algebras $d\rho$: $\mathrm{Lie}(\mathcal{G})(C) \to \mathrm{End}(W)$. In the following we will omit the symbols ρ (resp. $d\rho$) when the action of $\mathcal{G}$ (resp. $\mathrm{Lie}(\mathcal{G})(C)$) on W is meant. With respect to the action of $\mathbf{H}$ on W we obtain a decomposition of $W = \bigoplus_{\lambda\in\Lambda} W_\lambda$ into finitely many weight spaces W_λ for a finite number of weights $\lambda \in \Lambda \subset \mathbf{H}^*$. Now we choose $A_1 \in \mathbf{H}$ satisfying:

- The $\alpha(A_1)$ are non-zero and distinct for the simple roots $\alpha \in \Delta$ of $\mathrm{Lie}(\mathcal{G})(C)$.
- The $\lambda(A_1)$ are non-zero and distinct for the non-zero weights λ of the representation $d\rho$.

Note that the roots and the weights are linear combinations of basis elements of $\mathbf{H}^*$. Let $\hat{C} \supset \mathbb{Q}$ be the smallest field containing the coefficients of these linear combinations. Since C algebraically closed and $\hat{C}$ is a finite extension of $\mathbb{Q}$, the extension $C/\hat{C}$ is infinite. Thus we can choose an infinite basis of C over $\hat{C}$. Now let the coefficients of A_1 be such basis elements. Hence, A_1 satisfying the above conditions exists. We fix such an A_1. Observation 3.9 and Proposition 2.1 yield that the differential Galois group of the equation $\partial(\boldsymbol{y}) = (A_0 + A_1 z^2)\boldsymbol{y}$ is a connected algebraic subgroup $\mathcal{H} \leq \mathcal{G}$ of $\mathcal{G}$.

Suppose that $\mathcal{H} \neq \mathcal{G}$. Then $\mathcal{H}$ leaves a line, say $\langle w \rangle_{C(z)}$ with $w \in W$, $w \neq 0$ in W invariant, since W is a Chevalley module. Theorem 2.2 states the existence of $B \in \mathcal{G}(C(z)) \subset \mathrm{GL}(W \otimes C(z))$ such that

$$B^{-1}(A_0 + A_1 z^2)B - B^{-1}\frac{d}{dz}B \in \mathrm{Lie}(\mathcal{H})(C(z)).$$

Thus, the vector $\tilde{w} = Bw \in C(z) \otimes W$ has the property

$$\left[\frac{d}{dz} - (A_0 + A_1 z^2)\right] \tilde{w} \ \in \ \langle \tilde{w} \rangle_{C(z)}, \tag{3.8}$$

where $\frac{d}{dz}$ on $C(z) \otimes W$ is defined as $\frac{d}{dz}(f \otimes v) = (\frac{d}{dz} f \otimes v)$. Suppose $\tilde{w} \in C[z] \otimes W$, where the coordinates of $\tilde{w}$ with respect to a basis of W have 1 as its greatest common divisor. Otherwise one multiplies $\tilde{w}$ with a non-zero element of $C(z)$. Then equation (3.8) reads as

$$\left[\frac{d}{dz} - (A_0 + A_1 z^2)\right] \tilde{w} = c\tilde{w}$$

with $c \in C[z]$. Comparing the degrees yields $c = c_0 + c_1 z + c_2 z^2$. Now write $\tilde{w} = w_m z^m + ... + w_1 z + w_0$ with $w_i \in W$ and $w_m \neq 0$. At first we handle the case when m is supposed to be $m \geq 3$. Then by comparing the coefficients of z^{m+2}, z^{m+1}, z^m we get the equations

$$\begin{aligned}
A_1 w_m &= -c_2 w_m \\
A_1 w_{m-1} &= -c_1 w_m - c_2 w_{m-1} \\
A_0 w_m + A_1 w_{m-2} &= -c_0 w_m - c_1 w_{m-1} - c_2 w_{m-2} \\
-m w_m + A_0 w_{m-1} + A_1 w_{m-3} &= -c_0 w_{m-1} - c_1 w_{m-2} - c_2 w_{m-3}.
\end{aligned}$$

The first equation implies that $w_m \neq 0$ is an eigenvector of A_1 corresponding to a weight space W_λ with eigenvalue $-c_2 = \lambda(A_1)$.
The second equation can be transformed into

$$(A_1 + c_2) w_{m-1} = -c_1 w_m.$$

The left hand side has no component in the weight space W_λ to which w_m belongs. So we deduce that $c_1 = 0$. Moreover, we get $w_{m-1} \in W_\lambda$, since all $\lambda(A_1)$ are non zero.
The third equation reads as

$$A_0 w_m + (A_1 + c_2) w_{m-2} = -c_0 w_m.$$

As above $(A_1 + c_2) w_{m-2}$ has no component in W_λ. The same holds for $A_0 w_m$, since [Hum72, p.107, Lemma] yields that

$$A_0 w_m = \sum_{\alpha \in \Delta} (X_\alpha + X_{-\alpha}) w_m \in \bigoplus_{\alpha \in \Delta} (W_{\lambda+\alpha} \oplus W_{\lambda-\alpha}).$$

This shows that $c_0 = 0$. Now we can write the last equation as

$$A_0 w_{m-1} + (A_1 + c_2) w_{m-3} = m w_m.$$

Due to the same arguments as above, the left hand side terms have no component in W_λ. We conclude that $m = 0$.
Now we discuss the case $m = 2$. This leads as above to the reduced system of equations

$$\begin{aligned} A_1 w_2 &= -c_2 w_2 \\ A_1 w_1 &= -c_1 w_2 - c_2 w_1 \\ A_0 w_2 + A_1 w_0 &= -c_0 w_2 - c_1 w_1 - c_2 w_0 \\ -2w_2 + A_0 w_1 &= -c_0 w_1 - c_1 w_0. \end{aligned}$$

Again, $w_2 \neq 0$ is a eigenvector of A_1 corresponding to the weight space W_λ with eigenvalue $-c_2 = \lambda(A_1)$. Then the left hand side of

$$(A_1 + c_2)w_1 = -c_1 w_2$$

has no component in the weight space W_λ. Hence, it must hold $c_1 = 0$. Furthermore, we get $w_1 \in W_\lambda$, since all $\lambda(A_1)$ are non-zero. Therefore, the third equation can be written as

$$A_0 w_2 + (A_1 + c_2)w_0 = -c_0 w_2.$$

By the same arguments as above, the left hand side has no component in W_λ. Therefore, we have $c_0 = 0$. Then the last equation reads as $A_0 w_1 = 2w_2$. Since w_1 and w_2 are elements of W_λ, [Hum72, p.107, Lemma] yields as above that the left hand side has no component in W_λ. We conclude that $m = 0$.
If we assume $m = 1$, then we get the following system of equations:

$$\begin{aligned} A_1 w_1 &= -c_2 w_1 \\ A_1 w_0 &= -c_1 w_1 - c_2 w_0 \\ A_0 w_1 &= -c_0 w_1 - c_1 w_0 \\ -w_1 + A_0 w_0 &= -c_0 w_0. \end{aligned}$$

As above $w_1 \neq 0$ is a eigenvector of A_1 lying in the eigenspace W_λ with eigenvalue $-c_2$. Again, the left hand side of

$$(A_1 + c_2)w_0 = -c_1 w_1$$

has no component in W_λ. We deduce as above that $c_1 = 0$ and $w_0 \in W_\lambda$. Then the third equation writes as $A_0 w_1 = -c_0 w_1$. Due to the same arguments, the left hand side has no component in W_λ. Thus, we obtain $c_0 = 0$. Then the last equation writes as $A_0 w_0 = w_1$. Again, for the same reasons as above, $A_0 w_0$ has no component in W_λ. Therefore, it must hold $m = 0$.
This leaves us with the equation $-(A_0 + z^2 A_1)w_0 = c_2 z^2 w_0$. Comparing the coefficients yields $w_0 \in W_\lambda$ and $A_0 w_0 = 0$. So $\langle w_0 \rangle_C$ is invariant under A_0 and A_1. Hence, it is also invariant under scalar multiples, sums and bracket products of A_0 and A_1. The next step is to see that A_0 and A_1 generate $\mathrm{Lie}(\mathcal{G})(C)$. Therefore, we construct polynomials $P_\alpha(T), P_{-\alpha}(T) \in C[T]$ for each $\alpha \in \Delta$ such that

$$P_{\pm\alpha}(\mathrm{ad}A_1).A_0 = X_{\pm\alpha}.$$

Let $\operatorname{rank}(\Phi) = l$, and to simplify notation do number $-\alpha_1 = \alpha_{l+1}, ..., -\alpha_l = \alpha_{2l}$. For $i \in \{1, ..., 2l\}$, we set

$$X_{\alpha_i} = \sum_{j=1}^{2l} p_{i,j} \operatorname{ad}^j(A_1)(A_0). \tag{3.9}$$

We have to check that solutions $p_{i,j} \in C$ exist such that equation (3.9) holds. Equation (3.9) is equivalent to

$$X_{\alpha_i} = \sum_{j=1}^{2l} \sum_{k=1}^{2l} p_{i,j} \alpha_k(A_1)^j X_{\alpha_k} = \sum_{k=1}^{2l} \left(\sum_{j=1}^{2l} p_{i,j} \alpha_k(A_1)^j \right) X_{\alpha_k}.$$

Thus, we have to solve

$$\begin{pmatrix} \alpha_1(A_1) & \alpha_1(A_1)^2 & \cdots & \alpha_1(A_1)^{2l} \\ \alpha_2(A_1) & \alpha_2(A_1)^2 & \cdots & \alpha_2(A_1)^{2l} \\ \vdots & & & \vdots \\ \alpha_{2l}(A_1) & \alpha_{2l}(A_1)^2 & \cdots & \alpha_{2l}(A_1)^{2l} \end{pmatrix} \cdot \begin{pmatrix} p_{i,1} \\ p_{i,2} \\ \vdots \\ p_{i,2l} \end{pmatrix} = e_i \tag{3.10}$$

where e_i denotes the i-th unit vector. Let $M(\alpha_1(A_1), ..., \alpha_{2l}(A_1))$ be the matrix in equation (3.10). The determinant of $M(\alpha_1(A_1), ..., \alpha_{2l}(A_1))$ is well known as the *Vandermonde determinant*. We can calculate $\det(M(\alpha_1(A_1), ..., \alpha_{2l}(A_1)))$ as

$$\det(M(\alpha_1(A_1), ..., \alpha_{2l}(A_1))) = \prod_{k=1}^{2l} \alpha_k(A_1) \cdot \prod_{1 \leq i < j \leq 2l} (\alpha_j(A_1) - \alpha_i(A_1)).$$

The assumptions on $\alpha_i(A_1)$ imply $\det(M(\alpha_1(A_1), ..., \alpha_{2l}(A_1))) \neq 0$. Since the root spaces $\{X_{\pm\alpha}\}_{\alpha \in \Delta}$ generate $\operatorname{Lie}(\mathcal{G})$, we conclude that $\operatorname{Lie}(\mathcal{G})$ is generated as an algebra by A_0 and A_1.

But then $\langle w_0 \rangle_C$ is an invariant line under $\operatorname{Lie}(\mathcal{G})$. Hence, $\mathcal{G}$ has also an invariant line, since $\mathcal{G}$ is connected. This is a contradiction to our assumption on the Chavalley module W. This completes the proof.

We want to give an alternative end where the assumption on the $\alpha(A_1)$ is not needed. The calculation

$$A_0 w_0 = \sum_{\alpha \in \Delta} \underbrace{X_\alpha w_0}_{\in W_{\lambda+\alpha}} + \underbrace{X_{-\alpha} w_0}_{\in W_{\lambda-\alpha}} = 0$$

implies that $X_\alpha w_0 = 0$ and $X_{-\alpha} w_0 = 0$ for each $\alpha \in \Delta$, since by [Hum72, Section 20.1, Lemma] W is the direct sum of the weight spaces W_λ.

The Lie algebra $\operatorname{Lie}(\mathcal{G})$ is generated by the root spaces $\{X_{\pm\alpha}\}_{\alpha \in \Delta}$. Thus, $\langle w_0 \rangle_C$ is invariant under $\operatorname{Lie}(\mathcal{G})$ and under $\mathcal{G}$. But this is a contradiction to our assumption on the Chavalley module W. $\square$

3.5 Parametrized equations for connected semisimple linear algebraic groups

Let $\mathcal{G}$ be a connected semisimple linear algebraic group with representation in an n-dimensional vector space and denote by $\operatorname{Lie}(\mathcal{G})$ its Lie algebra. Let Φ be the root system of Lie($\mathcal{G}$) and let $\Delta = \{\alpha_1, ..., \alpha_l\}$ be a basis of Φ. Further denote by

$$\{X_\alpha,\ H_i = H_{\alpha_i} \mid \alpha \in \Phi,\ 1 \leq i \leq l\}$$

a Chevalley basis of Lie($\mathcal{G}$). Let C be algebraically closed field of characteristic zero and define the differential field $F := C\langle t_1, ..., t_l\rangle$ in the l differential indeterminates $\boldsymbol{t} = (t_1, ..., t_l)$.

Theorem 3.13. *There exists a parameterized differential equation*

$$L(y, \boldsymbol{t}) = \sum_{i=0}^{n} a_i(\boldsymbol{t}) Y^{(i)} = 0$$

over F with differential Galois group $\mathcal{G}(C)$.

Proof. We define the matrix $A := \sum_{\alpha\in\Delta} X_\alpha + X_{-\alpha} + \sum_{i=1}^{l} t_i H_i \in \mathrm{Lie}(\mathcal{G})(F)$. Then by Proposition 2.1 we have that the differential Galois group $\mathcal{H}(C)$ of $\partial(\boldsymbol{y}) = A\boldsymbol{y}$ is contained in $\mathcal{G}(C)$. By Corollary 3.12 there exists $A_1 \in \mathbf{H}(C)$ such that the equation $\partial(\boldsymbol{y}) = (\sum_{\alpha\in\Delta} X_\alpha + X_{-\alpha} + z^2 A_1)\boldsymbol{y}$ has $\mathcal{G}(C)$ as differential Galois group. Since the H_i generate $\mathbf{H}(C)$ over C, we obtain a specialization $\sigma : \boldsymbol{t} \to (c_1\ z^2, ..., c_n\ z^2)$ with $c_i \in C$ such that $\sum_{i=1}^{l} \sigma(t_i) H_i = A_1$. Thus, the specialized equation $\partial(\boldsymbol{y}) = \sigma(A)\boldsymbol{y}$ has $\mathcal{G}(C)$ as differential Galois group over $C(z)$. Then Theorem 2.14 yields $\mathcal{G}(C) \leq \mathcal{H}(C)$. Thus it holds $\mathcal{G}(C) = \mathcal{H}(C)$. The theorem follows from the application of Theorem 1.20. $\square$

Chapter 4

A parametrized equation for SL_{l+1}

4.1 The Lie algebra of SL_{l+1} (type A_l)

Let $l \in \mathbb{N} \setminus \{0\}$ and denote by $\epsilon_1, ..., \epsilon_{l+1}$ the standard orthonormal basis of $\mathbb{R}^{l+1}$. Further, let $(\cdot, \cdot)$ denote the standard inner product on $\mathbb{R}^{l+1}$. Then by [Hum72, Section 12.1] the set

$$\Phi := \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq l+1\}$$

forms the root system of type A_l. We can take the set Δ, which consists of the l linear independent vectors

$$\Delta := \{\alpha_i := \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq l\}$$

as a basis of Φ. The Cartan Matrix of type A_l has shape

$$\begin{pmatrix} 2 & -1 & 0 & & & . & . & . & . & . & & & 0 \\ -1 & 2 & -1 & 0 & & . & . & . & . & . & & & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & . & . & . & & & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & . & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & & . & . & . & 0 & 0 & -1 & 2 \end{pmatrix},$$

where the Cartan integer $\langle \alpha_i, \alpha_j \rangle = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ is given by the entry at position (i, j).
Let V be a vector space of dimension $\dim(V) = l+1$ over C and denote by SL_{l+1} the group of all automorphisms $A \in \mathrm{GL}_{l+1}$ satisfying $\det(A) = 1$. Then it is well known (see, for example, [Hum72, p.2]) that the Lie algebra of SL_{l+1} is defined as

$$\mathrm{Lie}(\mathrm{SL}_{l+1})(C) = \left\{ M \in C^{(l+1)\times(l+1)} \mid \mathrm{tr}(M) = 0 \right\},$$

i.e., the set of all endomorphisms of V with trace zero. Evidently, the matrices

$$E_{ij}, \text{ with } 1 \leq i, j \leq l+1, \ i \neq j, \text{ and } H_i = E_{ii} - E_{i+1,i+1}, \text{ with } 1 \leq i \leq l,$$

where $E_{ij} \in C^{l+1 \times l+1}$ is the matrix having 1 as entry at position (i, j) and 0 elsewhere, form a basis of $\mathrm{Lie}(\mathrm{SL}_{l+1})(C)$. Now we determine a Cartan decomposition for $\mathrm{Lie}(\mathrm{SL}_{l+1})$ from this basis. Therefore, let $T = \mathrm{diag}(\lambda_1, ..., \lambda_{l+1})$ be an element of the standard maximal torus $\mathcal{T}_0 \leq \mathrm{SL}_{l+1}$. Then, for $i \neq j$, $1 \leq i, j \leq l+1$, we have

$$T^{-1} E_{ij} T = \frac{\lambda_i}{\lambda_j} E_{ij} \tag{4.1}$$

and, for $1 \leq i \leq l$, we obtain

$$T^{-1} H_i T = H_i. \tag{4.2}$$

Equation (4.2) implies that the elements H_i belong to the weight space of the trivial weight. Since the H_i are linearly independent, they form a basis of the Cartan subalgebra $\mathbf{H} = \langle\, H_i \mid 1 \leq i \leq l \,\rangle_C = \mathrm{Lie}(\mathcal{T}_0)(C)$. To see that $\mathrm{Lie}(\mathrm{SL}_{l+1})$ is of type A_l let us denote by

$$\chi_i : \mathcal{T}_0 \to C\ ,\ \mathrm{diag}(\lambda_1, ..., \lambda_{l+1}) \mapsto \lambda_i$$

the fundamental characters. Then by equation (4.1) the vectors E_{ij} span the one dimensional root spaces $\langle E_{ij} \rangle_C$ which correspond to the weights χ_i/χ_j. Moreover, from the symmetry it follows that $\langle E_{ji} \rangle_C$ is the root space corresponding to the weight χ_j/χ_i which is the inverse of the weight χ_i/χ_j. Then the root system Φ of $\mathrm{Lie}(\mathrm{SL}_{l+1})$ is of type A_l and the Cartan decomposition has shape

$$\mathrm{Lie}(\mathrm{SL}_{l+1}) = \mathbf{H} \bigoplus_{1 \leq i < j \leq l} \langle E_{\epsilon_i - \epsilon_j} \rangle_C \oplus \langle E_{-(\epsilon_i - \epsilon_j)} \rangle_C$$

where we assigned the matrix E_{ij} to the root $\epsilon_i - \epsilon_j$ and defined $E_{\epsilon_i - \epsilon_j}$ as $E_{\epsilon_i - \epsilon_j} := E_{ij}$. We check that $\{H_k,\ E_{ij} \mid 1 \leq k \leq l,\ 1 \leq i, j \leq l\}$ forms a Chavalley basis. The following computation can be found in [Car72, Section 11.2]. First, we determine the co-roots. Therefore, we define the matrices $H_{ij} := [E_{ij}, E_{ji}] = E_{ii} - E_{jj}$. Then the computation of

$$[H_{ij}, E_{ij}] = 2E_{ij} = \langle \epsilon_i - \epsilon_j, \epsilon_i - \epsilon_j \rangle\, E_{ij}$$

implies that the H_{ij} are precisely the co-roots.
Now let us define the map

$$\theta : \mathrm{Lie}(\mathrm{SL}_{l+1}) \to \mathrm{Lie}(\mathrm{SL}_{l+1})\ ,\ X \mapsto -X^T.$$

Evidently, θ is an automorphism of $\mathrm{Lie}(\mathrm{SL}_{l+1})$ and satisfies the identities

$$\theta(E_{ij}) = -E_{ji}, \tag{4.3}$$

$$\theta([X, Y]) = -[X, Y]^T = X^T Y^T - Y^T X^T = \left[-X^T, -Y^T\right] = [\theta(X), \theta(Y)]\,. \tag{4.4}$$

We denote in the following by X_α the matrix E_{ij} where α is the root $\epsilon_i - \epsilon_j$. Thus, equation (4.3) becomes

$$\theta(X_\alpha) = -X_{-\alpha}.$$

For $\alpha, \beta \in \Phi$ the number $n_{\alpha,\beta} \in \mathbb{Z}$ is defined by $[X_\alpha, X_\beta] = n_{\alpha,\beta} X_{\alpha+\beta}$. If we apply θ on both sides of $[X_\alpha, X_\beta] = n_{\alpha,\beta} X_{\alpha+\beta}$, then we obtain with the help of equation (4.4)

$$-n_{\alpha,\beta} X_{-\alpha-\beta} = -[X_\alpha, X_\beta]^T = [-X_\alpha^T, -X_\beta^T] = [X_{-\alpha}, X_{-\beta}] = n_{-\alpha,-\beta} X_{-\alpha-\beta}.$$

Hence, we have $-n_{\alpha,\beta} = n_{-\alpha,-\beta}$. But [Car72, Theorem 4.1.2] yields the identity

$$n_{\alpha,\beta} \cdot n_{-\alpha,-\beta} = -(r+1)^2.$$

This implies $n_{\alpha,\beta} = \pm(r+1)$. We conclude that the above basis is a Chevalley basis.

4.2 The transformation lemma for SL_{l+1}

In this section we present the transformation lemma for SL_{l+1}. The proof is based on differential conjugation, i.e., on the adjoint action and the logarithmic derivative. Since both can be described by the roots, we start this section with the analysis of the root system of type A_l. Denote by (F, ∂_F) a differential field of characteristic 0.

Lemma 4.1. *For $n \in \{1, ..., l\}$, let $\Phi_n = \langle \alpha_l, ..., \alpha_{l+1-n} \rangle_\Phi$ be the set of all $\mathbb{Z}$-linear combinations of the roots $\alpha_l, ..., \alpha_{l+1-n}$ which lie in Φ. Define $\Phi_0 = \emptyset$ as the empty set.*

1. *Then $\Phi_n \subseteq \Phi_l = \Phi$ is an irreducible subsystem of Φ with $\Phi_n \sim A_n$.*
2. *For $k \in \{1, ..., n\}$ there exists a unique root $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\alpha) = k$ and α has shape*
$$\alpha = \sum_{i=l+1-n}^{l-n+k} \alpha_i.$$
3. *Let $\alpha \in \Phi_n^+ \setminus \{\Phi_{n-1}^+ \cup \{\gamma_{l+1-n} = \sum_{i=l+1-n}^{l} \alpha_i\}\}$ and let $\mathrm{ht}(\alpha) = k$. Then there exists a unique $\bar{\alpha} \in \Delta$ such that $\beta = \alpha + \bar{\alpha} \in \Phi_n^+ \setminus \Phi_{n-1}^+$ and $\mathrm{ht}(\beta) = k + 1$. If $\tilde{\alpha} \in \Delta$ is a simple root and $\beta - \tilde{\alpha}$ is a root, then either $\beta - \tilde{\alpha} = \alpha$ or $\beta - \tilde{\alpha} \in \Phi_{n-1}^+$.*
4. *The root system Φ consists of the roots*
$$\Phi = \{\pm\alpha = \pm \sum_{k=i}^{j} \alpha_k = \pm(\epsilon_i - \epsilon_{j+1}) \mid 1 \leq i \leq j \leq l\}.$$

Proof. The first point is a consequence of the Dynkin diagram of type A_l (e.g., see [Hum72, Section 11.4]).
We prove the second point. Since Φ_n is a root system of type A_n, [Hum72, Section 10.4, Lemma A] implies that there is a unique maximal root γ in Φ_n which we will denote by γ_{l+1-n}. From [Hum72, Section 12.2, Table 2] we know that the shape of γ_{l+1-n} is $\sum_{i=l+1-n}^{l} \alpha_i$. Hence, γ_{l+1-n} is an element of $\Phi_n^+ \setminus \Phi_{n-1}^+$.
We prove the assumption by two inductions where the first one is on $n \in \{1, ..., l\}$ and the second one on $k \in \{1, ..., \mathrm{ht}(\gamma_{l+1-n}) = n\}$.
Let $n = 1$. Then $\Phi_1 = \langle \alpha_l \rangle_\Phi$ is the set of all $\mathbb{Z}$-linear combinations of α_l such that they belong to Φ. Since the only scalar multiples of a root α are $\pm\alpha$, we get $\Phi_1 = \{\alpha_l, -\alpha_l\}$. Hence, α_l is the unique root in Φ_1^+ with $\mathrm{ht}(\alpha_l) = 1 = k$.
Let $1 < n \leq l$. Let $k = 1$. Then the unique root $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\alpha) = 1$ is α_{l+1-n}. Let $1 < k \leq \mathrm{ht}(\gamma_{l+1-n})$. Then by the induction assumption on k there exists a unique root α with $\mathrm{ht}(\alpha) = k - 1$ and shape
$$\alpha = \sum_{i=l+1-n}^{l-n+k-1} \alpha_i.$$
For the simple root $\alpha_{l-n+k} \in \Delta$, we compute the integer $\langle \alpha, \alpha_{l-n+k} \rangle$ with the help of the Cartan matrix as
$$\langle \alpha, \alpha_{l-n+k} \rangle = \langle \sum_{i=l+1-n}^{l-n+k-1} \alpha_i, \alpha_{l-n+k} \rangle = \sum_{i=l+n-1}^{l-n+k-1} \langle \alpha_i, \alpha_{l-n+k} \rangle = -1.$$

Then the image of α under the reflection $\sigma_{\alpha_{l-n+k}}$ is

$$\sigma_{\alpha_{l-n+k}}(\alpha) = \alpha - \langle \alpha, \alpha_{l-n+k} \rangle \, \alpha_{l-n+k} = \alpha + \alpha_{l-n+k}.$$

This implies that $\alpha + \alpha_{l-n+k}$ is a root of $\mathrm{ht}(\alpha + \alpha_{l-n+k}) = k$ and lies in $\Phi_n^+ \setminus \Phi_{n-1}^+$. Suppose there is another root $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\beta) = k$ and different from $\alpha + \alpha_{l-n+k}$. Then [Hum72, Section 10.2, Corollary] yields that we can write β as the sum $\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_m$ of simple roots $\bar{\alpha}_i \in \Delta_n = \{\alpha_l, ..., \alpha_{l+1-n}\}$ where the $\bar{\alpha}_i$ are not necessarily distinct, in such a way that for $1 \leq i \leq m$ each partial sum $\bar{\alpha}_1 + ... + \bar{\alpha}_i$ is a root. In particular, $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ is a root and $\mathrm{ht}(\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}) = k - 1$. We assume that $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} \neq \alpha$. Then the uniqueness of α yields $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} \in \Phi_{n-1}^+$. Therefore, we have

$$\beta - (\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}) = \alpha_{l+1-n} \tag{4.5}$$

where $\bar{\alpha}_i \neq \alpha_{l+1-n}$ for all $i \in \{1, ..., m-1\}$. We denote by $\bar{w}$ the minimum of the indices of the simple roots $\alpha_w = \bar{\alpha}_i$ in $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$. Then equation (4.5) implies $\bar{w} > l + 1 - n$. Let $\bar{n} \in \mathbb{N}$ such that $\bar{w} = l + 1 - \bar{n}$. We obtain $l + 1 - \bar{n} > l + 1 - n$ or equivalently $\bar{n} < n$. Hence, the induction assumption applied on $\bar{n}$ yields that the root $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ of height $k - 1$ has shape

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} = \sum_{i=l+1-\bar{n}}^{l-\bar{n}+k+1} \alpha_i.$$

Assume $l + 1 - \bar{n} > l + 2 - n$. We compute

$$\langle \beta, \alpha_{l+1-n} \rangle = \sum_{i=l+1-\bar{n}}^{l-\bar{n}+k-1} \langle \alpha_i, \alpha_{l+1-n} \rangle + \langle \alpha_{l+1-n}, \alpha_{l+1-n} \rangle = 2.$$

Thus, the reflection $\sigma_{\alpha_{l+1-n}}$ maps β to

$$\sigma_{\alpha_{l+1-n}}(\beta) = \beta - 2\alpha_{l+1-n} = \sum_{i=l+1-\bar{n}}^{l-\bar{n}+k-1} \alpha_i - \alpha_{l+1-n}. \tag{4.6}$$

Since the right hand side of equation (4.6) is not a root, it holds $l + 1 - \bar{n} = l + 2 - n$. Then β is the root

$$\beta = \sum_{i=l+2-n}^{l-n+k} \alpha_i + \alpha_{l+1-n} = \alpha + \alpha_{l-n+k}$$

constructed above, which contradicts to the assumption that $\beta \neq \alpha + \alpha_{l+1-n}$.
It is left to check that the sum

$$\alpha + \alpha_j = \sum_{i=l+1-n}^{l-n+k-1} \alpha_i + \alpha_j$$

for $\alpha_j \in \{\alpha_l, ..., \alpha_{l+1-n}\} \setminus \{\alpha_{l-n+k}\}$ is not a root. This is done by comparing the root lenght of $\alpha + \alpha_j$ with α_j.
From [Hum72, Section 12.2, Table 2] we obtain that the irreducible root system Φ_n of type A_n contains only long roots, i.e., all roots of Φ_n are of equal length. Further, [Hum72,

Section 9.4, Table 1] implies that for two roots α, β of Φ_n which are of equal length and nonproportional it holds $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \pm 1$.
Now we check that $\alpha + \alpha_j$ is not a root for some $\alpha_j \in \{\alpha_l, ..., \alpha_{l+1-n}\} \setminus \{\alpha_{l-n+k}\}$. For $j \in \{l+1-n, ..., l-n+k-1\}$, we compute

$$\langle \alpha + \alpha_j, \alpha_j \rangle = (1 - \delta_{l+1-n,j})\langle \alpha_{j-1}, \alpha_j \rangle + 2\langle \alpha_j, \alpha_j \rangle + (1 - \delta_{l-n+k-1,j})\langle \alpha_{j+1}, \alpha_j \rangle \geq 2.$$

Further, for $j \in \{l-n+k+1, ..., l\}$, we get $\langle \alpha + \alpha_j, \alpha_j \rangle = \langle \alpha_j, \alpha_j \rangle = 2$ where we have to assume $n \geq 3$. Thus, the sum $\alpha + \alpha_j$ has a different length than the roots of Φ_n and therefore can not be a root of Φ_n.
Hence, $\alpha + \alpha_{l-n+k}$ is the unique root in $\Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\alpha + \alpha_{l-n+k}) = k$ and has the proposed shape of

$$\alpha + \alpha_{l-n+k} = \sum_{i=l-n+1}^{l-n+k} \alpha_i.$$

Now we show the third assertion of the lemma.
If $\alpha \in \Phi_n^+ \setminus \{\Phi_{n-1}^+ \cup \{\gamma_{l-n+1} = \sum_{i=l-n+1}^{l} \alpha_i\}\}$, then $\mathrm{ht}(\alpha) = k < ht(\gamma_{l-n+1})$ and in particular, by Lemma 4.1. 2, there exists a unique $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ such that $\mathrm{ht}(\beta) = k+1 \leq \mathrm{ht}(\gamma_{l-n+1})$. Hence, the simple root $\beta - \alpha \in \Delta$ has the stated property. Let $\tilde{\alpha} \in \Delta$ be different from $\beta - \alpha$ and let $\beta - \tilde{\alpha}$ be a root. From the uniqueness of α we obtain $\beta - \tilde{\alpha} \notin \Phi_n^+ \setminus \Phi_{n-1}^+$. Therefore, $\beta - \tilde{\alpha} \in \Phi_{n-1}^+$ holds.
Finally, we prove the last point of the lemma.
Evidently, we have $\Phi \supseteq \bigcup_{i=1}^{l} (\Phi_i \setminus \Phi_{i-1})$. Let $\alpha = \sum_{i=1}^{l} k_i \alpha_i \in \Phi$ and let $j \in \{1, ..., l\}$ be minimal with $k_j \neq 0$. Thus, α is an element of $\Phi_j \setminus \Phi_{j-1}$. We obtain the disjoint union $\Phi = \bigcup_{i=1}^{l} (\Phi_i \setminus \Phi_{i-1})$. □

Lemma 4.2. *Let $n \in \{1, ..., l\}$ and denote by γ_i the root of maximal height in Φ_{l+1-i}^-. Let $A_0 = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n} a_{\gamma_i} X_{\gamma_i} + \sum_{\beta \in \Phi_n^-} a_\beta X_\beta$ with $a_{\gamma_i}, a_\beta \in F$. Then there exists $U \in \mathcal{U}^-$ such that*

$$UA_0U^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n+1} \bar{a}_{\gamma_i} X_{\gamma_i} + \sum_{\beta \in \Phi_{n-1}^-} \bar{a}_\beta X_\beta$$

with $\bar{a}_{\gamma_i},\ \bar{a}_\beta \in F$.

Proof. We prove for each $k \in \{1, ..., n-1\}$ the following claim: For the matrix

$$A_{k-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n} a_{\gamma_i} X_{\gamma_i} + \sum_{\beta \in \Phi_{n-1}^-} a_\beta X_\beta + \sum_{\alpha \in \Phi_n^- \setminus \Phi_{n-1}^-, ht(\alpha) \geq k} a_\alpha X_\alpha$$

there exists $U \in \mathcal{U}^-$ such that

$$UA_{k-1}U^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n} \bar{a}_{\gamma_i} X_{\gamma_i} + \sum_{\beta \in \Phi_{n-1}^-} \bar{a}_\beta X_\beta + \sum_{\alpha \in \bar{\Phi}_n^-, ht(\alpha) > k} \bar{a}_\alpha X_\alpha =: A_k$$

where $a_{\gamma_i},\ a_\beta,\ a_\alpha$ and $\bar{a}_{\gamma_i},\ \bar{a}_\beta,\ \bar{a}_\alpha$ are elements of F. Note that in the following we will sometimes write $\bar{\Phi}_n^-$ for $\Phi_n^- \setminus \Phi_{n-1}^-$.

We want to remove the part of A_{k-1} which lies in the root space corresponding to the root $\alpha \in \Phi_n^- \setminus \Phi_{n-1}^-$ with $\mathrm{ht}(\alpha) = k$. Then Lemma 4.1.3 yields a unique simple root $\bar{\alpha} \in \Delta$ such that $-\alpha + \bar{\alpha} = \bar{\beta} \in \Phi_n^+ \setminus \Phi_{n-1}^+$ and $\mathrm{ht}(\bar{\beta}) = k+1$. In other words, for $-\bar{\beta} =: \hat{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$ it holds that $\hat{\beta} + \bar{\alpha} = \alpha$. Motivated by this we differentially conjugate A_{k-1} by the parametrized root group element $U_{\hat{\beta}}(\zeta) \in \mathcal{U}_{\hat{\beta}}$. With the help of Observation 3.4 we obtain

$$\begin{aligned} U_{\hat{\beta}}(\zeta) A_{k-1} U_{\hat{\beta}}(\zeta)^{-1} + \partial(U_{\hat{\beta}}(\zeta)) U_{\hat{\beta}}(\zeta)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\alpha_i}) + \sum_{i=1}^{n-1} a_{\gamma_i} \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\gamma_i}) \\ + \sum_{\beta \in \Phi_{n+1}^-} a_\beta \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\beta) &+ \sum_{\alpha \in \bar{\Phi}_n^-, ht(\alpha) \geq k} a_\alpha \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\alpha) + \partial(U_{\hat{\beta}}(\zeta)) U_{\hat{\beta}}(\zeta)^{-1}. \end{aligned} \tag{4.7}$$

For the first summand of the right hand side of equation (4.7) Lemma 3.2 yields

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\alpha_i}) = \sum_{i=1}^{l} (\sum_{j=0}^{q} m_{\hat{\beta},\alpha_i,j} \zeta^j X_{\alpha_i + j\hat{\beta}}). \tag{4.8}$$

We have to determine for which $1 \leq j \leq \infty$ the sum $\alpha_i + j\hat{\beta}$ is a root of Φ. If $j = 1$, then Lemma 4.1.3 implies that $\alpha_i + \hat{\beta}$ is either α or $\alpha_i + \hat{\beta} \in \Phi_{n-1}^-$. Moreover, for all $j > 1$, the sum $\alpha_i + j\hat{\beta}$ is not a root, since $\mathrm{ht}(\hat{\beta}) = k+1$ and Lemma 4.1 implies that all coefficients of all roots of Φ^+ are equal to 1. Hence, we otain for equation (4.8)

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\alpha_i}) \in \sum_{i=1}^{l} X_{\alpha_i} + \zeta m_{\hat{\beta},\bar{\alpha},1} X_\alpha + \sum_{\beta \in \Phi_{n-1}^-} \mathrm{Lie}(\mathrm{SL}_{l+1})_\beta. \tag{4.9}$$

Again by Lemma 3.2 the second summand of the right hand side of equation (4.7) can be written as

$$\sum_{i=1}^{l-n} a_{\gamma_i} \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\gamma_i}) = \sum_{i=1}^{l-n} (\sum_{j=0}^{q} m_{\hat{\beta},\gamma_i,j} \zeta^j X_{\gamma_i + j\hat{\beta}}). \tag{4.10}$$

Since γ_i is the root of maximal height in Φ_{l+1-i}^- for $i \in \{1, ..., l-n\}$ and $\beta \in \Phi_n^- \setminus \Phi_{n-1}^-$, we conclude that $\gamma_i + j\hat{\beta}$ is not a root for $j > 0$. Hence, equation (4.10) reduces to

$$\sum_{i=1}^{l-n} a_{\gamma_i} \mathrm{Ad}(U_\beta(\zeta))(X_{\gamma_i}) = \sum_{i=1}^{l-n} a_{\gamma_i} X_{\gamma_i}. \tag{4.11}$$

The third summand is

$$\sum_{\beta \in \Phi_{n-1}^-} a_\beta \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\beta) = \sum_{\beta \in \Phi_{n-1}^-} (\sum_{j=0}^{q} m_{\hat{\beta},\beta,j} \zeta^j X_{\beta + j\hat{\beta}}). \tag{4.12}$$

Obviously, if $\beta + j\hat{\beta}$ is a root for $j > 0$, then $\beta + j\hat{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$ and $\mathrm{ht}(\beta + j\hat{\beta}) > k+1$ since $\mathrm{ht}(\hat{\beta}) = k+1$. Thus, equation (4.12) can be reformulated as

$$\sum_{\beta \in \Phi_{n-1}^-} a_\beta \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\beta) \in \sum_{\beta \in \Phi_{n-1}^-} a_\beta X_\beta + \sum_{\beta \in \bar{\Phi}_n^-, ht(\beta) > k+1} \mathrm{Lie}(\mathrm{SL}_{l+1})_\beta. \tag{4.13}$$

We get for the fourth summand of equation (4.7)

$$\sum_{\alpha\in\bar{\Phi}_n^-,ht(\alpha)\geq k} a_\alpha \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\alpha) = \sum_{\alpha\in\bar{\Phi}_n^-,ht(\alpha)\geq k} a_\alpha (\sum_{j=0}^{q} m_{\hat{\beta},\alpha,j}\zeta^j X_{\alpha+j\hat{\beta}}). \tag{4.14}$$

Since $\alpha,\hat{\beta}\in\Phi_n^-\setminus\Phi_{n-1}^-$, the coefficient of α_n in $\alpha+j\hat{\beta}$ is greater equal than 2 for $j\geq 1$. Hence, $\alpha+j\hat{\beta}$ is not a root. Thus we can translate equation (4.14) into

$$\sum_{\alpha\in\bar{\Phi}_n^-,ht(\alpha)\geq k} a_\alpha \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\alpha) = \sum_{\alpha\in\bar{\Phi}_n^-,ht(\alpha)\geq k} a_\alpha X_\alpha. \tag{4.15}$$

The last summand of equation (4.7) must still be checked. From Proposition 3.5 we know that the logarithmic derivative $l\delta$ maps an element $U_{\hat{\beta}}(\zeta)\in\mathcal{U}_{\hat{\beta}}$ to $\mathrm{Lie}(\mathcal{U}_{\hat{\beta}})=\mathrm{Lie}(\mathrm{SL}_{l+1})_{\hat{\beta}}$. Therefore,

$$\partial(U_{\hat{\beta}}(\zeta))U_{\hat{\beta}}(\zeta)^{-1}\in\mathrm{Lie}(\mathrm{SL}_{l+1})_{\hat{\beta}} \tag{4.16}$$

with $\mathrm{ht}(\hat{\beta})=k+1$ and $\hat{\beta}\in\Phi_n^-\setminus\Phi_{n-1}^-$. Putting the equations (4.9), (4.11), (4.13), (4.15) and (4.16) together, we obtain

$$\begin{aligned} A_k \in\; & a_\alpha X_\alpha + m_{\hat{\beta},\bar{\alpha},1}\zeta X_\alpha + \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n} a_{\gamma_i}X_{\gamma_i} + \sum_{\beta\in\Phi_{n-1}^-}\mathrm{Lie}(\mathrm{SL}_{l+1})_\beta \\ & + \sum_{\beta\in\bar{\Phi}_n^-,ht(\beta)>k}\mathrm{Lie}(\mathrm{SL}_{l+1})_\beta. \end{aligned}$$

Hence, with $m_{\hat{\beta},\bar{\alpha},1}\zeta=-a_\alpha$ the proof of the claim is complete.
Using the claim, one then proves by induction that for each $k\in\{1,...,n-1\}$ there exists $U\in\mathcal{U}^-$ such that

$$\begin{aligned} UA_0U^{-1}+\partial(U)U^{-1}\in\; & \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n}\mathrm{Lie}(\mathrm{SL}_{l+1})_{\gamma_i} + \sum_{\beta\in\Phi_{n-1}^-}\mathrm{Lie}(\mathrm{SL}_{l+1})_\beta \\ & + \sum_{\beta\in\bar{\Phi}_n^-,ht(\beta)>k}\mathrm{Lie}(\mathrm{SL}_{l+1})_\beta. \end{aligned}$$

In particular, we get for $k=n-1$ the assertion of the lemma. □

Lemma 4.3. *Let $A\in\sum_{i=1}^{l}X_{\alpha_i}+\mathbf{H}+\sum_{\beta\in\Phi^-}\mathrm{Lie}(\mathrm{SL}_{l+1})_\beta=\sum_{i=1}^{l}X_{\alpha_i}+\mathrm{Lie}(\mathcal{B}_0^-)$ and define $\mathcal{M}=\{\gamma_i\in\Phi^-\mid i=1,...,l\}$ as the set of roots of maximal height of all subsystems Φ_{l+1-i}^-. Then there exists $U\in\mathcal{U}_0^-$ such that*

$$UAU^{-1}+\partial(U)U^{-1}\in\sum_{i=1}^{l}X_{\alpha_i}+\sum_{\alpha\in\mathcal{M}}\mathrm{Lie}(\mathrm{SL}_{l+1})_\alpha.$$

Proof. We start by proving the following claim:
Define for $k = 1, ..., l$ the matrix A_k as $A_k := \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k}^{l} a_i H_i + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SL}_{l+1})_\beta$. Then there exists $U \in \mathcal{U}_0^-$ such that

$$UA_kU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k+1}^{l} a_i H_i + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SL}_{l+1})_\beta.$$

One writes $A_k = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k}^{l} a_i H_i + \sum_{\beta\in\Phi^-} a_\beta X_\beta$ with suitable $a_\beta \in F$. To remove $a_k H_k$ we differentially conjugate A_k by $U_{-\alpha_k}(\zeta) \in \mathcal{U}_{-\alpha_k}$. We use Observation 3.4 to express this as

$$\begin{aligned} U_{-\alpha_k}(\zeta)A_kU_{-\alpha_k}(\zeta)^{-1} + \partial(U_{-\alpha_k}(\zeta))U_{-\alpha_k}(\zeta)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_i}) \\ + \sum_{i=k}^{l} a_i \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) + \sum_{\beta\in\Phi^-} a_\beta \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_\beta) &+ l\delta(U_{-\alpha_k}(\zeta)). \end{aligned} \tag{4.17}$$

Let us look at the first summand on the right hand side of equation (4.17). Then by Lemma 3.2.2 and Lemma 3.2.4 we get for $i \neq k$

$$\sum_{i=1}^{l} (\sum_{j\geq 0} m_{-\alpha_k,\alpha_i,j} \zeta^j X_{\alpha_i + j(-\alpha_k)})$$

and for $i = k$ we have $X_{\alpha_k} + \zeta H_{\alpha_k} - \zeta^2 X_{-\alpha_k}$. Since $\alpha_i - j\alpha_k$ is not a root for $i \neq k$ and $j > 0$, we obtain

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_i}) \in \sum_{i=1}^{l} X_{\alpha_i} + \zeta H_k + \mathrm{Lie}(\mathrm{SL}_{l+1})_{-\alpha_k}. \tag{4.18}$$

If $i = k$, then we get for the second summand with Lemma 3.2.3

$$a_k \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_k) = a_k(H_k - 2\zeta X_{-\alpha_k}).$$

Moreover, with the help of Lemma 3.2.5 and the Cartan matrix we have for $i = k+1$

$$a_{k+1} \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_{k+1}) = a_{k+1}(H_{k+1} - \zeta X_{-\alpha_k})$$

and for $l \geq i \geq k+2$ we obtain $a_i \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) = a_i H_i$. We summarize our results. This yields

$$\sum_{i=k}^{l} a_i \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) \in \sum_{i=k}^{l} a_i H_i + \mathrm{Lie}(\mathrm{SL}_{l+1})_{-\alpha_k}. \tag{4.19}$$

Let $\beta \in \Phi^-$. Obviously, if $\beta + j(-\alpha_k)$ is a root of Φ for $j \geq 0$, then $\beta + j(-\alpha_k) \in \Phi^-$. Hence, the third summand of equation (4.17) lies in

$$\sum_{\beta\in\Phi^-} a_\beta \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_\beta) \in \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SL}_{l+1})_\beta. \tag{4.20}$$

We handle the last summand with Proposition 3.5. It implies

$$l\delta(U_{-\alpha_k}(\zeta)) = \partial(U_{-\alpha_k}(\zeta))U_{-\alpha_k}(\zeta)^{-1} \in \mathrm{Lie}(\mathrm{SL}_{l+1})_{-\alpha_k}. \tag{4.21}$$

We put the equations (4.18), (4.19), (4.20) and (4.21) together and set $\zeta = -a_k$. Hence, the assumption of the claim is shown.
One uses then the claim to prove by induction that for each $k \in \{1, ..., l\}$ there exists $U \in \mathcal{U}^-$ such that

$$UAU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k+1}^{l} a_i H_i + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SL}_{l+1})_\beta.$$

In particular, for $k = l$, it yields that there exists $U \in \mathcal{U}^-$ such that

$$A_0 = U^{-1}AU - U^{-1}U' \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SL}_{l+1})_\beta.$$

Again one proves by an inductive argument together with Lemma 4.2 that for each $n \in \{1, ..., l\}$ and A_0 there exists $U \in \mathcal{U}^-$ such that

$$UA_0U^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n} \mathrm{Lie}(\mathrm{SL}_{l+1})_{\gamma_i} + \sum_{\beta\in\Phi^-_n} \mathrm{Lie}(\mathrm{SL}_{l+1})_\beta$$

where the notations are as in Lemma 4.2. Note that $\Phi^-_1 = \{-\alpha_l = -\gamma_l\}$. Then the lemma follows for $n = 1$. □

4.3 The equation with group SL_{l+1}

The next step is to combine the results of Corollary 3.12 and Lemma 4.3 so that we can apply later the specialization bound. We keep the notations of Lemma 4.3 and recall that $(C(z), \partial = \frac{d}{dz})$ denotes a rational function field with standard derivation as in Section 3.4.

Corollary 4.4. *We apply Corollary 3.12 to the group* SL_{l+1} *and the above Cartan decomposition. We denote by* $A^{M\&S}_{\mathrm{SL}_{l+1}}$ *the matrix satisfying the stated conditions of Corollary 3.12. Then there exists* $U \in \mathcal{U}^-_0 \subset \mathrm{SL}_{l+1}$ *such that*

$$\bar{A}_{\mathrm{SL}_{l+1}} := UA^{M\&S}_{\mathrm{SL}_{l+1}}U^{-1} + \partial(U)U^{-1} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in\mathcal{M}} f_i X_{\gamma_i} \tag{4.22}$$

with at least one $f_i \in C[z] \setminus C$ *and the differential Galois group of the matrix equation* $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SL}_{l+1}}\boldsymbol{y}$ *over* $C(z)$ *is* $\mathrm{SL}_{l+1}(C)$.

Proof. Lemma 4.3 proves the existence of an element $U \in \mathcal{U}^-_0 \subset \mathrm{SL}_{l+1}$ such that equation (4.22) holds. Since differential conjugation defines a differential isomorphism, we deduce with Corollary 3.12 that the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SL}_{l+1}}\boldsymbol{y}$ is again $\mathrm{SL}_{l+1}(C)$ over $C(z)$. We still need to show the existence of $f_i \in C[z] \setminus C$ for some $\gamma_i \in \mathcal{M}$. Suppose $\bar{A}_{\mathrm{SL}_{l+1}} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in T} f_i X_{\gamma_i} \in \mathrm{Lie}(\mathrm{SL}_{l+1})(C)$. Then by Lemma 4.5

the corresponding differential equation $L(y, f_1, ..., f_l) \in C\{y\}$ has coefficients in C. But then by [Mag94, Corollary 3.28] the differential Galois group is abelian. Thus $\bar{A}_{\mathrm{SL}_{l+1}} \in \mathrm{Lie}(\mathrm{SL}_{l+1})(C(z)) \setminus \mathrm{Lie}(\mathrm{SL}_{l+1})(C)$. Since $0 \neq A_1 \in \mathbf{H}(C)$ and $A = (z^2 A_1 + A_0)$ in Corollary 3.12 we start our transformation with at least one coefficient lying in $C[z] \setminus C$. In each step the application of $\mathrm{Ad}(U_\beta(\zeta))$ generates at most new entries which are polynomials in ζ. Moreover, the logarithmic derivative is the product of the two matrices $\partial(U_\beta(\zeta))$ and $U_\beta(\zeta)^{-1} = U_\beta(-\zeta)$. In the proofs of Lemma 4.3 and Lemma 4.2 we choose the parameter ζ to be one of the coefficients. Hence, it holds $f_i \in C[z] \setminus C$. □

Our goal is to produce parametric equations for the series SL_{l+1}. Therefore, let $t_1, ..., t_l$ be differential indeterminates and define the differential field $F = C\langle t_1, ..., t_l\rangle$. Moreover, let us define

$$A_{\mathrm{SL}_{l+1}}(t_1, ..., t_l) = \sum_{\alpha \in \Delta} X_\alpha + \sum_{\gamma_i \in \mathcal{M}} t_i X_{\gamma_i}$$

with $\mathcal{M}$ as in Lemma 4.3. We are going to compute an operator for SL_{l+1} from the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SL}_{l+1}}(t_1, ..., t_l)\boldsymbol{y}$.

Lemma 4.5. *The matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SL}_{l+1}}(t_1, ..., t_l)\boldsymbol{y}$ is differentially equivalent to the homogeneous scalar linear differential equation*

$$L(y, t_1, ..., t_l) = y^{(l+1)} - \sum_{i=1}^{l} t_i\, y^{(i-1)} = 0$$

Proof. Form the description of $\mathrm{Lie}(\mathrm{SL}_{l+1})$ in Section 4.1 we otain the full shape of the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SL}_{l+1}}(t_1, ..., t_l)\boldsymbol{y}$. We have

$$\begin{pmatrix} \partial(y_1) \\ \partial(y_2) \\ \vdots \\ \\ \partial(y_l) \\ \partial(y_{l+1}) \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & & & \\ & & & & 1 \\ t_1 & t_2 & \dots & t_l & 0 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \\ y_l \\ y_{l+1} \end{pmatrix}.$$

To simplify the notation we will write y_i' for $\partial(y_i)$. Then the above equation translates into the system of equations

$$y_1' = y_2 \tag{1}$$
$$y_2' = y_3 \tag{2}$$
$$\vdots$$
$$y_l' = y_{l+1} \tag{l}$$
$$y_{l+1}' = \sum_{i=1}^{l} t_i y_i. \tag{l+1}$$

We will prove that y_1 is a cyclic vector. By an easy inductive argument we deduce for subsystems formed by the first until the n-th equation, where $n \in \{1, ..., l\}$, that

$$y_1^{(n)} = y_{n+1}.$$

In particular, for $n = l$ it holds $y_1^{(l)} = y_{l+1}$. We differentiate this equation and substitute y'_{l+1} by the last equation of the initial system and hence we obtain

$$y^{(l+1)} = \sum_{i=1}^{l} t_i \, y^{(i-1)}.$$

□

Theorem 4.6. *Let C be an algebraically closed field of characteristic zero, $t_1, ..., t_l$ differential indeterminates and $F = C\langle t_1, ..., t_l \rangle$ the corresponding differential field. Then the homogeneous linear differential equation*

$$L(y, t_1, ..., t_l) = y^{(l+1)} - \sum_{i=1}^{l} t_i \, y^{(i-1)} = 0$$

has $\mathrm{SL}_{l+1}(C)$ as differential Galois group over F. Moreover, let $\hat{F}$ be a differential field with field of constants equal to C. Let $\hat{E}$ be a Picard-Vessiot extension over $\hat{F}$ with differential Galois group $\mathrm{SL}_{l+1}(C)$ and suppose the defining matrix differential equation $\partial(\boldsymbol{y}) = \hat{A}\boldsymbol{y}$ satisfies $\hat{A} \in \sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}_0^-)$. Then there is a specialization $L(y, \hat{t}_1, ..., \hat{t}_l)$ with $\hat{t}_i \in \hat{F}$ such that $L(y, \hat{t}_1, ..., \hat{t}_l)$ gives rise to the extension $\hat{E}$ over $\hat{F}$.

Proof. Let E be a Picard-Vessiot extension for the equation $L(y, t_1, ..., t_l) = 0$ over F and denote by $\mathcal{G}$ its differential Galois group. Since the operator comes from the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SL}_{l+1}}(t_1, ..., t_l)\boldsymbol{y}$ with $A_{\mathrm{SL}_{l+1}}(t_1, ..., t_l) \in \mathrm{Lie}(\mathrm{SL}_{l+1})(F)$, Proposition 2.1 yields $\mathcal{G}(C) \leq \mathrm{SL}_{l+1}(C)$. By Corollary 4.4 there exists a specialization $\sigma : (t_1, ..., t_l) \to (f_1, ..., f_l)$ with $f_1, ..., f_l \in C[z]$ such that $\sigma(A_{\mathrm{SL}_{l+1}}(t_1, ..., t_l)) = \bar{A}_{\mathrm{SL}_{l+1}}$ and the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SL}_{l+1}}\boldsymbol{y}$ is $\mathrm{SL}_{l+1}(C)$. Moreover, we have $C\{f_1, ..., f_l\} = C[z]$. Thus we can apply Corollary 2.15. This yields $\mathrm{SL}_{l+1}(C) \leq \mathcal{G}(C)$. Hence, it holds $\mathcal{G}(C) = \mathrm{SL}_{l+1}(C)$.
Since the defining matrix $\hat{A}$ satisfies $\hat{A} \in \sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}_0^-)$, Lemma 4.3 provides that $\hat{A}$ is differentially equivalent to a matrix $\tilde{A} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i \in T} \hat{a}_i X_{\gamma_i}$ with suitable $\hat{a}_i \in \hat{F}$. Obviously, the specialization

$$\hat{\sigma} : (t_1, ..., t_l) \mapsto (\hat{a}_1, ..., \hat{a}_l)$$

has the required property. □

From literature it is known that the general equation with trace zero

$$y^{(n)} = t_{n-2}y^{(n-2)} + t_{n-3}y^{(n-3)} + ... + t_0 y$$

has SL_n as its differential Galois group. This equation yields an alternative proof for Theorem 4.6. Furthermore, note that in case $l = 1$ the equation of Theorem 4.6 is the Airy equation $y^{(2)} = t_1 y$. In the literature there are several proofs of the various types of the Airy equation $y^{(2)} = \tilde{f}y$. The differential Galois group depends on the choice of the differential ground field $(\tilde{F}, \partial_{\tilde{F}})$ where the field of constants of $\tilde{F}$ is equal to C and of the coefficient $\tilde{f} \in \tilde{F}$. In each situation the differential Galois group $\mathcal{G}(C)$ is a subgroup of $\mathrm{SL}_2(C)$. Obviously, our equation specializes to all of these different types.

Chapter 5

A parametrized equation for SP_{2l}

5.1 The Lie algebra of SP_{2l} (type C_l)

As in Section 4.1 we introduce first the root system of type C_l. Let $l \in \mathbb{N} \setminus \{0\}$ with $l \geq 3$. We write $\epsilon_1, ..., \epsilon_l$ for the standard orthonormal basis of $\mathbb{R}^l$ and $(\cdot, \cdot)$ for the standard inner product of $\mathbb{R}^l$. In [Hum72, Section 12.1], it is shown that the root system Φ of type C_l is formed by the set of vectors

$$\Phi = \{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j),\ \pm 2\epsilon_k \mid 1 \leq i < j \leq l,\ 1 \leq k \leq l\}\,.$$

As a basis we take the l linear independent vectors

$$\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1},\ \alpha_l = 2\epsilon_l \mid 1 \leq i \leq l-1\}$$

and we fix this ordering. The Cartan integers $\langle \alpha_i, \alpha_j \rangle = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ are given at position (i, j) in the Cartan matrix which has in the case of C_l the following shape

$$\begin{pmatrix} 2 & -1 & 0 & & . & . & . & & & 0 \\ -1 & 2 & -1 & & . & . & . & & & 0 \\ 0 & -1 & 2 & -1 & . & . & . & & & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & & . & . & . & -1 & 2 & -1 \\ 0 & 0 & & & . & . & . & 0 & -2 & 2 \end{pmatrix}.$$

Let $V = \langle v_1, ..., v_{2l} \rangle_C$ be a vector space over C of dimension $2l$. We define on V a skew-symmetric non degenerate bilinear form f with representing matrix

$$J = \begin{pmatrix} 0 & J_0 \\ -J_0 & 0 \end{pmatrix} \in C^{2l \times 2l} \text{ where the matrix } J_0 \text{ has shape } J_0 = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

The symplectic group SP_{2l} is defined as the group of all invertible linear transformations of V preserving a skew-symmetric non degenerated bilinear form. We are going to choose this bilinear form to be f. Hence, with the representing matrix J of f the group SP_{2l} can be described as

$$\mathrm{SP}_{2l} = \{A \in \mathrm{GL}_{2l} \mid A^T J A = J\}.$$

Then the Lie Algebra $\mathrm{Lie}(\mathrm{SP}_{2l})$ of SP_{2l} consists of all endomorphism $X \in C^{2l\times 2l}$ of V satisfying for $v, w \in V$ the rule

$$f(Xv, w) = -f(v, Xw) \Leftrightarrow (Xw)^T Jw = -v^T J(Xw) \Leftrightarrow JX = -X^T J.$$

Hence, the condition for X to be symplectic reads in matrix terms as $-X^T J = JX$. Let A, B, C, D be arbitrary elements of $C^{l\times l}$. We write the $2l \times 2l$-matrix X as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in C^{2l\times 2l}.$$

Then an explicit calculation of $JX + X^T J = 0$ leads us to

$$\begin{pmatrix} -C^T J_0 & A^T J_0 \\ -D^T J_0 & B^T J_0 \end{pmatrix} + \begin{pmatrix} J_0 C & J_0 D \\ -J_0 A & -J_0 D \end{pmatrix} = 0.$$

Equivalently, we get the system of equations

$$\begin{aligned} C^T J_0 = J_0 C, &\quad A^T J_0 = -J_0 D, \\ -D^T J_0 = J_0 A, &\quad B^T J_0 = J_0 B. \end{aligned}$$

The third equation offers no new information and can be therefore omitted. It can be checked by computation that the conjugation $J_0 M J_0^{-1}$ of an element $M \in C^{l\times l}$ by J_0 is reversing M and then taking the transpose. Here we mean by the reversed matrix, the matrix obtained by reflecting the entries at the second diagonal. Before we start to write down a basis for $\mathrm{Lie}(\mathrm{SP}_{2l})$ we renumber the rows and columns of X into $1, ..., l, -1, ..., -l$. Furthermore, we denote by $E_{ij} \in C^{2l\times 2l}$ the matrix having 1 as entry at position (i, j) and 0 elsewhere. Then it can be checked easily that the l diagonal matrices $E_{ii} - E_{-l-1+i,-l-1+i}$ with $1 \le i \le l$ and the matrices

$$E_{ij} - E_{-l-1+j,-l-1+i},\ E_{ji} - E_{-l-1+i,-l-1+j}$$

with $1 \le i < j \le l$ have non-zero entries in the blocks A and D of X. We see that they satisfy the above equations. Moreover, for $1 \le i, j \le l,\ i + j \le l$, the matrices

$$E_{i,-j} - E_{l+1-j,-l-1+i},\ E_{-j,i} - E_{-l-1+i,l+1-j}$$

and for $1 \le i \le l$, the l matrices

$$E_{i,-l-1+i},\ E_{-l-1+i,i}$$

with non-zero entries in the blocks B and C of X also satisfy the conditions of the equations. Denote by $\mathbf{B}$ the collection of all these matrices. Then the elements of $\mathbf{B}$ are linearly independent, since for an arbitrary chosen position above or on the secondary diagonal there is exact one matrix in $\mathbf{B}$ with a non-zero entry at this position. Furthermore, the number of elements in $\mathbf{B}$ can be easily computed as $\mathrm{card}(\mathbf{B}) = 2l^2 + l$. This number coincides with the dimension of $\mathrm{Lie}(\mathrm{SP}_{2l})$ from literature (see for example [Hum72, p.3]). Hence, the elements of $\mathbf{B}$ form a basis of $\mathrm{Lie}(\mathrm{SP}_{2l})$.

The next step is to determine a Cartan Decomposition for $\mathrm{Lie}(\mathrm{SP}_{2l})$ from this basis. Therefore we compute the maximal diagonal torus $\mathcal{T}$ of SP_{2l}. Let $T = \mathrm{diag}(\lambda_1, ..., \lambda_{2l}) \in \mathrm{GL}_{2l}$ be a diagonal matrix of GL_{2l}. Then the condition $T^T JT = J$ calculates explicitly as

$$\left(\begin{array}{ccc|ccc} & & & & & \lambda_1\lambda_{2l} \\ & & & & & \\ & & & \lambda_l\lambda_{l+1} & & \\ \hline & & -\lambda_l\lambda_{l+1} & & & \\ & & & & & \\ -\lambda_{2l}\lambda_1 & & & & & \end{array}\right) = \left(\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & & \\ & & & 1 & & \\ \hline & & -1 & & & \\ & & & & & \\ -1 & & & & & \end{array}\right).$$

Hence, the condition for T to be an element of SP_{2l} is satisfied if and only if for all $i \in \{1, ..., l\}$ we have

$$\lambda_{2l+1-i} = \lambda_i^{-1}.$$

Thus, the diagonal torus $\mathcal{T}$ of SP_{2l} is the set of matrices

$$\mathcal{T} = \{T = \mathrm{diag}(\lambda_1, ..., \lambda_l, \frac{1}{\lambda_l}, ..., \frac{1}{\lambda_1}) \mid \lambda_1, ..., \lambda_l \in C^\times\}.$$

We calculate the conjugates of the elements of $\mathbf{B}$ by $T = \mathrm{diag}(\lambda_1, ..., \lambda_l, \frac{1}{\lambda_l}, ..., \frac{1}{\lambda_1}) \in \mathcal{T}$:

$$\begin{array}{rcl} T(E_{ii} - E_{-l-1+i,-l-1+i})T^{-1} & = & (E_{ii} - E_{-l-1+i,-l-1+i}), \\ T(E_{ij} - E_{-l-1+j,-l-1+i})T^{-1} & = & (\lambda_i/\lambda_j)\,(E_{ij} - E_{-l-1+j,-l-1+i}), \\ T(E_{ji} - E_{-l-1+i,-l-1+j})T^{-1} & = & (\lambda_j/\lambda_i)\,(E_{ji} - E_{-l-1+i,-l-1+j}), \\ T(E_{i,-j} - E_{l+1-j,-l-1+i})T^{-1} & = & \lambda_i\lambda_{l+1-j}\,(E_{i,-j} - E_{l+1-j,-l-1+i}), \\ T(E_{-j,i} - E_{-l-1+i,l+1-j})T^{-1} & = & (1/\lambda_{l+1-j}\lambda_i)\,(E_{-j,i} - E_{-l-1+i,l+1-j}), \\ T(E_{i,-l-1+i})T^{-1} & = & \lambda_i^2\,E_{i,-l-1+i}, \\ T(E_{-l-1+i,i})T^{-1} & = & (1/\lambda_i^2)\,E_{-l-1+i,i}. \end{array}$$

Hence, the root system Φ of $\mathrm{Lie}(\mathrm{SP}_{2l})$ is of type C_l. The above equations also show to which root space the elements of $\mathbf{B}$ belong. Therefore, we define for $1 \le i < j \le l$, the matrices

$$X_{\epsilon_i-\epsilon_j} := E_{ij} - E_{-l-1+j,-l-1+i},\ X_{-(\epsilon_i-\epsilon_j)} := E_{ji} - E_{-l-1+i,-l-1+j}$$

and for $1 \le i, j \le l,\ i + j \le l$, the matrices

$$X_{\epsilon_i+\epsilon_{l+1-j}} := E_{i,-j} - E_{l+1-j,-l-1+i},\ X_{-(\epsilon_i+\epsilon_{l+1-j})} := E_{-j,i} - E_{-l-1+i,l+1-j}.$$

Moreover, for $1 \le i \le l$, we have

$$X_{2\epsilon_i} := E_{i,-l-1+i} \quad \text{and} \quad X_{-2\epsilon_i} := E_{-l-1+i,i}.$$

Furthermore, we conclude that the Cartan subalgebra $\mathbf{H}$ is generated by the elements

$$\mathbf{H} = \langle E_{ii} - E_{-l-1+i,-l-1+i} \mid 1 \le i \le l\rangle_C\,.$$

Then, in these notations, the shape of the Cartan Decomposition is

$$\begin{array}{rll}\mathrm{Lie}(\mathrm{SP}_{2l})(C) = \mathbf{H}(C) & \bigoplus_{i,j} & \langle X_{\epsilon_i-\epsilon_j}\rangle_C \oplus \langle X_{-(\epsilon_i-\epsilon_j)}\rangle_C \\ & \bigoplus_{i,j} & \langle X_{\epsilon_i+\epsilon_{l+1-j}}\rangle_C \oplus \langle X_{-(\epsilon_i+\epsilon_{l+1-j})}\rangle_C \\ & \bigoplus_{i} & \langle X_{2\epsilon_i}\rangle_C \oplus \langle X_{-2\epsilon_i}\rangle_C.\end{array}$$

The next step is to determine a Chevalley basis of $\mathrm{Lie}(\mathrm{SP}_{2l})$ from the elements of $\mathbf{B}$. We begin with the co-roots. Therefore, we compute

$$\begin{array}{rcl}[X_{\epsilon_i-\epsilon_j}, X_{-(\epsilon_i-\epsilon_j)}] & = & E_{ii} - E_{jj} + E_{l-1+j,-l-1+j} - E_{-l-1+i,-l-1+i} =: H_{\epsilon_i-\epsilon_j}, \\ [X_{\epsilon_i+\epsilon_{l+1-j}}, X_{-(\epsilon_i+\epsilon_{l+1-j})}] & = & E_{ii} + E_{l+1-j,l+1-j} - E_{-j,-j} - E_{-l-1+i,-l-1+i} \\ & =: & H_{\epsilon_i+\epsilon_{l+1-j}}, \\ [X_{2\epsilon_i}, X_{-2\epsilon_i}] & = & E_{ii} - E_{-l-1+i,-l-1+i} =: H_{2\epsilon_i}.\end{array}$$

These are precisely the co-roots, since

$$\begin{array}{rcl}[H_{\epsilon_i-\epsilon_j}, X_{\epsilon_i-\epsilon_j}] & = & E_{ij} - E_{-l-1+j,-l-1+i} - (-E_{i,j} + E_{-l-1+j,-l-1+j}) = 2X_{\epsilon_i-\epsilon_j}, \\ [H_{\epsilon_i+\epsilon_{l+1-j}}, X_{\epsilon_i+\epsilon_{l+1-j}}] & = & E_{i,-j} + E_{l+1-j,-l-1+i} - (-E_{l+1-j,-l-1+i} - E_{i,-j}) \\ & = & 2X_{\epsilon_i+\epsilon_{l+1-j}}, \\ [H_{2\epsilon_i}, X_{2\epsilon_i}] & = & E_{i,-l-1+i} + E_{i,-l-1+i} = 2X_{2\epsilon_i}.\end{array}$$

To simplify the notation we number the l co-roots corresponding to the simple roots by

$$H_1 := H_{\epsilon_1-\epsilon_2},\ \ldots,\ H_{l-1} := H_{\epsilon_{l-1}-\epsilon_-} \text{ and } H_l := H_{2\epsilon_l}.$$

Let $\theta : \mathrm{Lie}(\mathrm{SP}_{2l}) \to \mathrm{Lie}(\mathrm{SP}_{2l})$ be a morphism of $\mathrm{Lie}(\mathrm{SP}_{2l})$ defined by the rule $X \mapsto -X^T$. Hence, θ is an automorphism of $\mathrm{Lie}(\mathrm{SP}_{2l})$. One verifies easily that the following equations for θ hold:

$$\begin{array}{rcl}\theta(X_{\epsilon_i-\epsilon_j}) & = & -X_{-(\epsilon_i-\epsilon_j)}, \\ \theta(X_{\epsilon_i+\epsilon_{l+1-j}}) & = & -X_{-(\epsilon_i+\epsilon_{l+1-j})}, \\ \theta(X_{2\epsilon_i}) & = & -X_{-2\epsilon_i}.\end{array}$$

Additionally, we have the identity

$$\theta([X,Y]) = -[X,Y]^T = [-X^T, -Y^T] = [\theta(X), \theta(Y)]. \tag{5.1}$$

We define the number $n_{\alpha,\beta} \in \mathbb{Z}$ by the rule $[X_\alpha, X_\beta] = n_{\alpha,\beta}X_{\alpha+\beta}$. The next step is to apply θ to $[X_\alpha, X_\beta] = n_{\alpha,\beta}X_{\alpha+\beta}$. This yields with the help of equation (5.1)

$$-n_{\alpha,\beta}X_{-\alpha-\beta} = -[X_\alpha, X_\beta]^T = [X_{-\alpha}, X_{-\beta}] = n_{-\alpha,-\beta}X_{-\alpha-\beta}.$$

Thus, it holds $-n_{\alpha,\beta} = n_{-\alpha,-\beta}$. From [Car72, Theorem 4.1.2] we have the identity

$$n_{\alpha,\beta}n_{-\alpha,-\beta} = -(r+1)^2.$$

Hence, $n_{\alpha,\beta}$ has to be equal to $\pm(r+1)$. We conclude that the elements

$$\{H_i,\ X_\alpha \mid 1 \leq i \leq l,\ \alpha \in \Phi\}$$

from a Chevalley basis of $\mathrm{Lie}(\mathrm{SP}_{2l})$.

5.2 The transformation lemma for SP_{2l}

In this section we present and prove the transformation lemma for SP_{2l}. This is done for a differential field (F, ∂_F) of characteristic zero. But firstly we need a good grasp of the root system of type C_l, since the proof of the transformation lemma is based on the adjoint action and the logarithmic derivate which can be both described by the roots.

Lemma 5.1. *For $n \in \{1, ..., l-1\}$ let $\Phi_n = \langle \alpha_l, ..., \alpha_{l-n} \rangle_\Phi$ denote the set of all $\mathbb{Z}$-linear combinations of the roots $\alpha_l, ..., \alpha_{l-n}$ which lie in Φ and let us define $\Phi_0 := \{\pm\alpha_l\}$.*

1. *The set $\Phi_n \subseteq \Phi_{l-1} = \Phi$ is an irreducible subsystem of Φ with $\Phi_n \sim C_{n+1}$.*
2. *For $k \in \{1, ..., 2n+1\}$ there exists a unique root $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ of $\mathrm{ht}(\alpha) = k$ and α has shape*
$$\begin{aligned} \alpha &= \sum_{i=l-n}^{l-n-1+k} \alpha_i && \text{if} \quad 1 \leq k \leq n+1, \\ \alpha &= \sum_{i=l-n}^{l+n-k} \alpha_i + 2 \sum_{i=l+n-k+1}^{l-1} \alpha_i + \alpha_l && \text{if} \quad n+2 \leq k \leq 2n \quad \text{and} \\ \alpha &= 2 \sum_{i=l-n}^{l-1} \alpha_i + \alpha_l && \text{if} \quad k = 2n+1. \end{aligned}$$
3. *Let $\alpha \in \Phi_n^+ \setminus \{\Phi_{n-1}^+ \cup \{\gamma_{l-n} = \alpha_l + 2\sum_{i=l-n}^{l-1} \alpha_i\}\}$ with $\mathrm{ht}(\alpha) = k$. Then there exists a unique $\bar{\alpha} \in \Delta$ such that $\beta = \alpha + \bar{\alpha} \in \Phi_n^+ \setminus \Phi_{n-1}^+$ and $\mathrm{ht}(\beta) = k+1$. If $\tilde{\alpha} \in \Delta$ is a simple root and $\beta - \tilde{\alpha}$ is a root, then either $\beta - \tilde{\alpha} = \alpha$ or $\beta - \tilde{\alpha} \in \Phi_{n-1}^+$.*
4. *The root system Φ consists of the roots*
$$\begin{aligned} \Phi \quad = \quad & \{\pm(\epsilon_i - \epsilon_j) = \pm \sum_{k=i}^{j-1} \alpha_k \mid 1 \leq i < j \leq l\} \cup \{\pm 2\epsilon_l = \pm\alpha_l\} \\ \cup \quad & \{\pm 2\epsilon_i = \pm(\alpha_l + 2\sum_{k=i}^{l-1} \alpha_k),\ \pm(\epsilon_i + \epsilon_l) = \pm(\alpha_l + \sum_{k=i}^{l-1} \alpha_k) \mid 1 \leq i \leq l-1\} \\ \cup \quad & \{\pm(\epsilon_i + \epsilon_j) = \pm(\alpha_l + 2\sum_{k=j}^{l-1} \alpha_k + \sum_{k=i}^{j-1} \alpha_k \mid 1 \leq i < j \leq l-1)\}. \end{aligned}$$

Proof. The first point is a consequence of the Dynkin diagram of type C_l (e.g., see [Hum72, Section 11.4]).
We prove the second assertion of the lemma. We know that Φ_n is a root system of type C_{n+1}. Thus by [Hum72, Section 10.4, Lemma A] there is a unique root γ_{l-n} of maximal height in Φ_n. Furthermore, from [Hum72, Section 12.2, Table 2] we obtain that γ_{l-n} has shape $\gamma_{l-n} = \alpha_l + 2\sum_{i=l-n}^{l-1} \alpha_i$. We conclude that γ_{l-n} is an element of $\Phi_n^+ \setminus \Phi_{n-1}^+$.
We are going to prove the assumption by three inductions. We will have an outer induction on $n \in \{1, ..., l-1\}$ and we need two inner inductions done on $k_1 \in \{1, ..., n+1\}$ and

$k_2 \in \{n+2, ..., 2n\}$.
Let $n = 1$. We are going to compute the root system $\Phi_1 = \langle \alpha_l, \alpha_{l-1} \rangle_\Phi$ to check the assumption. The Cartan matrix implies that the reflection $\sigma_{\alpha_{l-1}}$ maps the root α_l to

$$\sigma_{\alpha_{l-1}}(\alpha_l) = \alpha_l - \langle \alpha_l, \alpha_{l-1} \rangle \alpha_{l-1} = \alpha_l + 2\alpha_{l-1}.$$

Since root strings are unbroken we have computed the roots $\pm(\alpha_l + \alpha_{l-1})$ and $\pm(\alpha_l + 2\alpha_{l-1})$. Moreover, since $\alpha_l + 2\alpha_{l-1}$ is the unique root of maximal height in Φ_1 and the only multiples of a root α are $\pm\alpha$ we conclude that Φ_1 consists of the vectors

$$\Phi_1 = \{\pm\alpha_l, \pm\alpha_{l-1}, \pm(\alpha_l + \alpha_{l-1}), \pm(\alpha_l + 2\alpha_{l-1})\}.$$

Now it is easily seen that the assumption for $n = 1$ is satisfied.
Let $1 < n \leq l-1$. We show by induction on $k_1 \in \{1, ..., n+1\}$ that there exists a unique root $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\alpha) = k_1$ and α has shape $\alpha = \sum_{i=l-n}^{l-n-1+k_1} \alpha_i$.
Let $k_1 = 1$. Then the unique root α in $\Phi_n^+ \setminus \Phi_{n-1}^+$ of $\mathrm{ht}(\alpha) = 1$ is the root $\alpha = \alpha_{l-n}$.
Let $1 < k_1 \leq n+1$. The induction assumption implies that there exists $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ such that $\mathrm{ht}(\alpha) = k_1 - 1$ and α is of the form

$$\alpha = \sum_{i=l-n}^{l-n-2+k_1} \alpha_i.$$

We are going to construct from α a root of height k_1 which has the required shape. Therefore, we calculate for the simple root $\alpha_{l-n-1+k_1}$ the integer $< \alpha, \alpha_{l-n-1+k_1} >$ with the help of the Cartan matrix and the fact that if $k_1 = n+1$, then $l - n - 2 + k_1$ is equal to $l-1$, as

$$\langle \alpha, \alpha_{l-n-1+k_1} \rangle = \langle \sum_{i=l-n}^{l-n-2+k_1} \alpha_i, \alpha_{l-n-1+k_1} \rangle = -1.$$

Thus, the reflection $\sigma_{\alpha_{n-1+k_1}}$ maps α to the root

$$\sigma_{\alpha_{n-1+k_1}}(\alpha) = \sum_{i=l-n}^{l-n-2+k_1} \alpha_i - \langle \alpha, \alpha_{n-1+k_1} \rangle \alpha_{l-n-1+k_1} = \sum_{i=l-n}^{l-n-1+k_1} \alpha_i.$$

Evidently, this root satisfies the requirements. Suppose there exists another root $\beta \in \Phi_n^+ \setminus \Phi_{n+1}^+$, $\beta \neq \alpha + \alpha_{n-1+k_1}$ and $\mathrm{ht}(\beta) = k_1$. Then [Hum72, Section 10.2 Corollary] implies that we can write β as the sum $\bar{\alpha}_1 + ... + \bar{\alpha}_m$ of simple roots $\bar{\alpha}_i \in \Delta_n = \{\alpha_l, ..., \alpha_{l-n}\}$ (here the $\bar{\alpha}_i$ are not necessarily distinct) such that each partial sum $\bar{\alpha}_1 + ... + \bar{\alpha}_i$ is a root for $1 \leq i \leq m$. Thus, $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ is a root and of $\mathrm{ht}(\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}) = k_1 - 1$. Let us assume that $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} \neq \alpha$. The uniqueness of α yields that $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} \in \Phi_{n-1}^+$. Hence, it holds

$$-(\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}) + \beta = \alpha_{l-n}. \tag{5.2}$$

Denote by $\bar{w}$ the minimum of the indices of the simple roots $\alpha_w = \bar{\alpha}_i$ in $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$. Then equation (5.2) yields $\bar{w} > l - n$. Take $\bar{n} \in \mathbb{N}$ such that $\bar{w} = l - \bar{n}$. We obtain

$\bar{n} < n$. Hence, the outer induction assumption applied on $\bar{n}$ yields the following shapes for $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$:

$$\eta_1 := \sum_{i=l-\bar{n}}^{l-\bar{n}-2+k_1} \alpha_i \qquad \text{if} \quad 1 \leq k_1 - 1 \leq \bar{n}+1,$$

$$\eta_2 := \sum_{i=l-\bar{n}}^{l+\bar{n}-k_1+1} \alpha_i + 2 \sum_{i=l+\bar{n}-k_1+2}^{l-1} \alpha_i + \alpha_l \qquad \text{if} \quad \bar{n}+2 \leq k_1 - 1 \leq 2\bar{n} \quad \text{and}$$

$$\eta_3 := 2 \sum_{i=l-\bar{n}}^{l-1} \alpha_i + \alpha_l \qquad \text{if} \quad k_1 - 1 = 2\bar{n}+1.$$

Assume $l - \bar{n} > l - n + 1$. To simplify notation we denote the three possibilities for β by $\beta_i := \eta_i + \alpha_{l-n}$ with $i = 1, 2, 3$. Then we compute the integers $\langle \beta_i, \alpha_{l-n} \rangle$ as

$$\langle \beta_i, \alpha_{l-n} \rangle = \langle \eta_i, \alpha_{l-n} \rangle + \langle \alpha_{l-n}, \alpha_{l-n} \rangle = 2.$$

Hence, we obtain a contradiction, since the image of β_i under the reflection $\sigma_{\alpha_{l-n}}$ is

$$\sigma_{\alpha_{l-n}}(\beta_i) = \beta_i - \langle \beta_i, \alpha_{l-n} \rangle \, \alpha_{l-n} = \beta_i - 2\alpha_{l-n} = \eta_i - \alpha_{l-n} \tag{5.3}$$

and the right hand side of equation (5.3) is not a root. We conclude that $l - \bar{n} = l - n$. Thus we obtain the inequality $k_1 - 1 < n + 1 = \bar{n} + 2$. Hence, the induction assumption forces $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ to have the shape:

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} = \sum_{i=l-\bar{n}}^{l-\bar{n}-2+k_1} \alpha_i = \sum_{i=l-n+1}^{l-n-1+k_1} \alpha_i.$$

But then β would be the root

$$\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} + \alpha_{l-n} = \sum_{i=l-n+1}^{l-n-1+k_1} \alpha_i + \alpha_{l-n}$$

which we constructed above. This contradicts the assumption $\beta \neq \alpha + \alpha_{l-n+k_1}$.
By [Hum72, Section 10.4, Lemma C] the irreducible root system Φ_n of type C_{n+1} contains at most two root lengths. Further [Hum72, Section 9.4, Table 1] implies that for two roots α, β of Φ_n which are of equal length and nonproportional, it holds $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \pm 1$. With the help of the Cartan matrix we conclude that the simple roots $\alpha_1, ..., \alpha_{l-1}$ of Φ are of equal length. Since $\langle \alpha_{l-1}, \alpha_l \rangle = -1$ and $\langle \alpha_l, \alpha_{l-1} \rangle = -2$ we obtain that the roots $\alpha_1, ..., \alpha_{l-1}$ are short and α_l is long.
Now we check that the sum of $\alpha = \sum_{i=l-n}^{l-n-2+k_1} \alpha_i$ and $\alpha_j \in \{\alpha_l, ..., \alpha_{l-n}\} \setminus \{\alpha_{l-n-1+k_1}\}$ is not a root of Φ_n^+. This will complete the first inner induction. For $j \in \{l-n, ..., l-n-2+k_1\}$, we obtain

$$\langle \alpha + \alpha_j, \alpha_j \rangle = (1 - \delta_{l-n,j}) \langle \alpha_{j-1}, \alpha_j \rangle + 2 \langle \alpha_j, \alpha_j \rangle + (1 - \delta_{l-n-2+k_1,j}) \langle \alpha_{j+1}, \alpha_j \rangle \geq 2.$$

Similarly we have $\langle \alpha + \alpha_j, \alpha_j \rangle = \langle \alpha_j, \alpha_j \rangle = 2$ for $j \in \{l-n+k_1, ..., l-1\}$ where we have to assume $n > k_1$. For $j = l$, we get

$$\langle \alpha + \alpha_l, \alpha_{l-1} \rangle = \langle \alpha_l, \alpha_{l-1} \rangle + \delta_{l-2,l-n-2+k_1} \langle \alpha_{l-2}, \alpha_{l-1} \rangle = -2 - \delta_{l-2,l-n-2+k_1} \leq -2.$$

Thus the root $\alpha+\alpha_j$ has to be long. By [Hum72, Section 10.4, Lemma C] all roots of a given length are conjugate under the Weyl group, i.e., it exists $\sigma_{\tilde{\beta}}$ with $\tilde{\beta}\in\Phi_n$ such that

$$\sigma_{\tilde{\beta}}=\alpha+\alpha_j-\langle\alpha+\alpha_j,\tilde{\beta}\rangle\tilde{\beta}=\alpha_l. \tag{5.4}$$

Let $j\in\{l-n,...,l-1\}$ and $\tilde{\beta}=\sum_{i=l-n}^{l}k_i\alpha_i\in\Phi_n^+$. This forces $k_l=1$ and $\langle\alpha+\alpha_j,\tilde{\beta}\rangle=-1$. Thus, it can not hold equality in equation (5.4). Similarly we deduce for $\tilde{\beta}\in\Phi_n^-$. Let $j=l$ and $k_1\leq n$. Then the integer $\langle\alpha+\alpha_l,\alpha_l\rangle=2$ implies that $\alpha+\alpha_l$ is not long. Since $\alpha+\alpha_j$ is neither short nor long it can not be a root. This completes the first inner induction.
We start the second inner induction: For $k_2\in\{n+2,...,2n\}$, there exists a unique root $\alpha\in\Phi_n^+\setminus\Phi_{n-1}^+$ with $\mathrm{ht}(\alpha)=k_2$ and α is of the form

$$\alpha=\sum_{i=l-n}^{l+n-k_2}\alpha_i+2\sum_{i=l+n-k_2+1}^{l-1}\alpha_i+\alpha_l.$$

Let $k_2=n+2$. The first inner induction hypothesis yields for $k_1=n+1$ that there is a unique root $\alpha\in\Phi_n^+\setminus\Phi_{n-1}^+$ with $\mathrm{ht}(\alpha)=k_1$ and α has shape $\alpha=\sum_{i=l-n}^{l}\alpha_i$. For the construction of a root satisfying the proposed assertion, we compute for the simple root α_{l-1} the integer $\langle\alpha,\alpha_{l-1}\rangle$:

$$\langle\alpha,\alpha_{l-1}\rangle=\sum_{i=l-n}^{l}\langle\alpha_i,\alpha_{l-1}\rangle=\langle\alpha_{l-2},\alpha_{l-1}\rangle+\langle\alpha_{l-1},\alpha_{l-1}\rangle+\langle\alpha_l,\alpha_{l-1}\rangle=-1.$$

Hence, the reflection $\sigma_{\alpha_{l-1}}$ maps α to the root

$$\sigma_{\alpha_{l-1}}(\alpha)=\sum_{i=l-n}^{l}\alpha_i-\langle\alpha,\alpha_{l-1}\rangle\,\alpha_{l-1}=\sum_{i=l-n}^{l-2}\alpha_i+2\alpha_{l-1}+\alpha_l$$

which satisfies the desired properties apart from the uniqueness. Therefore, let $\beta\in\Phi_n^+\setminus\Phi_{n-1}^+$ with $\beta\neq\alpha+\alpha_{l-1}$ and $\mathrm{ht}(\beta)=k_2$. As before, [Hum72, Section 10.2, Corollary] yields the possibility to write β as $\bar{\alpha}_1+...+\bar{\alpha}_m$ with $\bar{\alpha}_i\in\Delta_n$ in such a way that each partial sum $\bar{\alpha}_1+...+\bar{\alpha}_i$ with $1\leq i\leq m$ is a root. It follows at once that $\bar{\alpha}_1+...+\bar{\alpha}_{m-1}$ is a root of $\mathrm{ht}(\bar{\alpha}_1+...+\bar{\alpha}_{m-1})=k_2-1$. Assume that α is different to $\bar{\alpha}_1+...+\bar{\alpha}_{m-1}$. The uniqueness of α implies $\bar{\alpha}_1+...+\bar{\alpha}_{m-1}\in\Phi_{n-1}^+$. This allows us to deduce that $-(\bar{\alpha}_1+...+\bar{\alpha}_{m-1})+\beta=\alpha_{l-n}$. Again, we denote by $\bar{w}$ the minimum of the indices of the simple roots $\alpha_w=\bar{\alpha}_i$ in $\bar{\alpha}_1+...+\bar{\alpha}_{m-1}$. Then it has to hold $\bar{w}>l-n$. Let $\bar{n}\in\mathbb{N}$ such that $l-\bar{n}=\bar{w}$ and suppose $l-\bar{n}>l-n+1$ or equivalent $\bar{n}+1<n$. We have $k_2-1=n+1>\bar{n}+2$. We can apply the outer induction assumption to $\bar{\alpha}_1+...+\bar{\alpha}_{m-1}$. This yields for $\bar{\alpha}_1+...+\bar{\alpha}_{m-1}$ the shapes

$$\eta_1:=\sum_{i=l-\bar{n}}^{l+\bar{n}-k_2+1}\alpha_i+2\sum_{i=l-\bar{n}-k_2+2}^{l-1}\alpha_i+\alpha_l \qquad \text{if}\quad \bar{n}+2\leq k_2-1\leq 2\bar{n}\quad\text{and}$$

$$\eta_2:=\sum_{i=l-\bar{n}}^{l-1}\alpha_i+\alpha_l \qquad \text{if}\quad k_2-1=2\bar{n}+1.$$

Thus, the integer $\langle \eta_i + \alpha_{l-n}, \alpha_{l-n}\rangle$ with $i = 1, 2$ computes as $\langle \eta_i + \alpha_{l-n}, \alpha_{l-n}\rangle = 2$. Hence, the reflection $\sigma_{\alpha_{l-n}}$ sends $\eta_i + \alpha_{l-n}$ to

$$\sigma_{\alpha_{l-n}}(\eta_i + \alpha_{l-n}) = \eta_i + \alpha_{l-n} - \langle \eta_i + \alpha_{l-n}, \alpha_{l-n}\rangle\, \alpha_{l-n} = \eta_i - \alpha_{l-n}.$$

Since the right hand side is not a root, we get a contradiction. Hence, we have $\bar{n} + 1 = n$ and so it holds $k_2 - 1 = n + 1 = \bar{n} + 2$. We make use of this to deduce that $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ has shape

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} = \sum_{i=l-\bar{n}}^{l+\bar{n}-k_2+1} \alpha_i + 2\sum_{l+\bar{w}-k_2+2}^{l-1} \alpha_i + \alpha_l = \sum_{i=l-n+1}^{l-2} \alpha_i + 2\alpha_{l-1} + \alpha_l.$$

Then $\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} + \alpha_{l-n} = \sum_{i=l-n}^{l-2} \alpha_i + 2\alpha_{l-1} + \alpha_l$ is the root constructed above. But this contradicts the assumption $\beta \neq \alpha + \alpha_{l-1}$. It remains to check that the sum $\alpha + \alpha_j = \sum_{i=l-n}^{l} \alpha_i + \alpha_j$ with $\alpha_j \in \{\alpha_{l-n}, ..., \alpha_l\} \setminus \{\alpha_{l-1}\}$ is not a root. We compute for $j \in \{l-n, ..., l-2\}$ the integer

$$\langle \alpha + \alpha_j, \alpha_j\rangle = (1 - \delta_{l-n,j})\langle \alpha_{j-1}, \alpha_j\rangle + 2\langle \alpha_j, \alpha_j\rangle + \langle \alpha_{j+1}, \alpha_j\rangle \geq 2$$

and $\langle \alpha+\alpha_l, \alpha_{l-1}\rangle = \langle \alpha_{l-2}, \alpha_{l-1}\rangle + \langle \alpha_{l-1}, \alpha_{l-1}\rangle + 2\langle \alpha_l, \alpha_{l-1}\rangle = 3$. Hence the root $\alpha+\alpha_j$ has to be long. By [Hum72, Section 10.4, Lemma D] all roots $\gamma = 2\sum_{i=m}^{l-1} \alpha_i + \alpha_l$ of maximal height in Φ_{m-l} with $m \in \{l-n, ..., l-1\}$ are long. Thus there exists a reflection $\sigma_{\tilde{\beta}}$ with $\tilde{\beta} \in \Phi_n$ such that

$$\sigma_{\tilde{\beta}}(\alpha + \alpha_j) = \alpha + \alpha_j - \langle \alpha + \alpha_j, \tilde{\beta}\rangle\tilde{\beta} = 2\sum_{i=j+1}^{l-1} \alpha_i + \alpha_l.$$

Let $j \neq l$ and $\tilde{\beta} = \sum_{i=l-n}^{l} k_i\alpha_i \in \Phi_n^+$. This forces the coefficient k_{l-1} to be 1 and $\langle \alpha + \alpha_j, \tilde{\beta}\rangle = -1$. But then k_j has to be -2 what is impossible. Similarly we deduce for $\tilde{\beta} \in \Phi_n^-$. For $j = l$, we compute $\langle \alpha + \alpha_l, \alpha_l\rangle = \langle \alpha_{l-1}, \alpha_l\rangle + 2\langle \alpha_l, \alpha_l\rangle = 3$. Thus the sum $\alpha + \alpha_j$ is neither short nor long and so can not be a root of Φ_n.
Let $n+2 < k_2 \leq 2n$. As in the steps before we construct a root satisfying the requirements of the induction assertion. The induction hypothesis implies that there exists a root

$$\alpha = \sum_{i=l-n}^{l+n-k_2+1} \alpha_i + 2\sum_{l+n-k_2+2}^{l-1} \alpha_i + \alpha_l$$

in $\Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\alpha) = k_2 - 1$. The integer $\langle \alpha, \alpha_{l+n-k_2+1}\rangle$ calculates as

$$\begin{aligned}\langle \alpha, \alpha_{l+n-k_2+1}\rangle &= \langle \alpha_{l+n-k_2}, \alpha_{l+n-k_2+1}\rangle + \langle \alpha_{l+n-k_2+1}, \alpha_{l+n-k_2+1}\rangle \\ &\quad +2\,\langle \alpha_{l+n-k_2+2}, \alpha_{l+n-k_2+1}\rangle = -1 + 2 - 2 = -1.\end{aligned}$$

Hence, the reflection $\sigma_{\alpha_{l+n-k_2+1}}$ maps α to

$$\sigma_{\alpha_{l+n-k_2+1}}(\alpha) = \alpha + \alpha_{l+n-k_2+1}.$$

Obviously, the root $\alpha+\alpha_{l+n-k_2+1} = \sum_{i=l-n}^{l+n-k_2} \alpha_i + 2\sum_{i=l+n-k_2+1}^{l-1} \alpha_i + \alpha_l$ has the proposed properties apart from the uniqueness. Therefore, assume there is a $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\beta) = k_2$ and $\beta \neq \alpha + \alpha_{l+n-k_2+1}$. Then [Hum72, Section 10.2, Corollary] states that we can write β as the sum $\bar{\alpha}_1 + ... + \bar{\alpha}_m$ of simple roots $\bar{\alpha}_i \in \Delta$ where the $\bar{\alpha}_i$ are not necessarily distinct such that each partial sum $\bar{\alpha}_1 + ... + \bar{\alpha}_j$ with $1 \leq j \leq m$ is a root. In particular, this yields that $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ is a root of $\mathrm{ht}(\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}) = k_2 - 1$. Suppose $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} \neq \alpha$. Since α is the unique root in $\Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\alpha) = k_2 - 1$, we have that $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} \notin \Phi_n^+ \setminus \Phi_{n-1}^+$. Thus, it holds

$$-(\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}) + \beta = \alpha_{l-n}. \tag{5.5}$$

By $\bar{w}$ we mean the minimum of the indices of the simple roots $\alpha_w = \bar{\alpha}_i \in \Delta$ in $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$. Take $\bar{n} \in \mathbb{N}$ such that $l - \bar{n} = \bar{w}$. It follows $l - \bar{n} > l - n$. We make the assumption that $l - \bar{n} > l - n + 1$, i.e., $\bar{n} + 1 < n$. Moreover, we additional observe that $k_2 - 1 \geq n + 2 > \bar{n} + 3$. Hence, the outer induction assumption yields that $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ is the unique root in $\Phi_{\bar{n}}^+ \setminus \Phi_{\bar{n}-1}^+$ of height $k_2 - 1$ and by the above inequality the following shapes for $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ are possible:

$$\eta_1 := \sum_{i=l-\bar{n}}^{l+\bar{n}-k_2+1} \alpha_i + 2 \sum_{i=l+\bar{n}-k_2+2}^{l-1} \alpha_i + \alpha_l \qquad \text{if} \quad \bar{n} + 3 \leq k_2 - 1 \leq 2\bar{n} \quad \text{and}$$

$$\eta_2 := 2 \sum_{i=l-\bar{n}}^{l-1} \alpha_i + \alpha_l \qquad \text{if} \quad k_2 - 1 = 2\bar{n} + 1.$$

The reflection $\sigma_{\alpha_{l-n}}$ maps $\eta_i + \alpha_{l-n}$ for $i = 1, 2$ to

$$\sigma_{\alpha_{l-n}}(\eta_i + \alpha_{l-n}) = \eta_i + \alpha_{l-n} - \langle \eta_i + \alpha_{l-n}, \alpha_{l-n} \rangle \, \alpha_{l-n} = \eta_i - \alpha_{l-n},$$

since the integer $\langle \eta_i + \alpha_{l-n}, \alpha_{l-n} \rangle$ equals $\langle \alpha_{l-n}, \alpha_{l-n} \rangle = 2$. Thus, we have a contradiction to our assumption and so $l - \bar{n} = l - n + 1$. Easily, we check that the inequality $k_2 - 1 \geq \bar{n} + 3$ holds. Hence, the induction assumption yields

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} = \sum_{i=l-\bar{n}}^{l+\bar{n}-k_2+1} \alpha_i + 2 \sum_{l+\bar{n}-k_2+2}^{l-1} \alpha_i + \alpha_l = \sum_{i=l-n+1}^{l+n-k_2} \alpha_i + 2 \sum_{2l-n-k_2+1}^{l-1} \alpha_i + \alpha_l$$

and so $\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} + \alpha_{l-n} = \sum_{i=l-n}^{l+n-k_2} \alpha_i + 2\sum_{l+n-k_2+1}^{l-1} \alpha_i + \alpha_l$ is the root constructed above. It is left to check that the sum

$$\alpha + \alpha_j = (\sum_{i=l-n}^{m} \alpha_i + 2 \sum_{i=m+1}^{l-1} \alpha_i + \alpha_l) + \alpha_j$$

with $m \in \{l + n - k_2 + 1, ..., l - 2\}$ and $\alpha_j \in \{\alpha_{l-n}, ..., \alpha_l\} \setminus \{\alpha_{l+n-k_2+1}\}$ is not a root of Φ_n^-. For $j \in \{l - n, ..., m - 1\}$, we compute

$$\langle \alpha + \alpha_j, \alpha_j \rangle = (1 - \delta_{l-n,j}) \langle \alpha_{j-1}, \alpha_j \rangle + 2\langle \alpha_j, \alpha_j \rangle + \langle \alpha_{j+1}, \alpha_j \rangle = -(1 - \delta_{l-n,j}) + 4 - 1 \geq 2.$$

Thus $\alpha + \alpha_j$ has to be long. Then there exists a reflection $\sigma_{\tilde{\beta}}$ with $\tilde{\beta} \in \Phi_n$ such that

$$\sigma_{\tilde{\beta}}(\alpha + \alpha_j) = \alpha + \alpha_j - \langle \alpha + \alpha_j, \tilde{\beta} \rangle \tilde{\beta} = 2 \sum_{i=j+1}^{l-1} \alpha_i + \alpha_l.$$

Let $\tilde{\beta} = \sum_{i=l-n}^{l} k_i\alpha_i \in \Phi_n^+$. Then it has to hold $k_{j+1} = 1$ and $\langle \alpha + \alpha_j, \tilde{\beta}\rangle = -1$. But we get also $k_j = -2$ which is impossible. Similarly we deduce for $\tilde{\beta} \in \Phi_n^-$.
For $j \in \{m+1, ..., l-1\}$, we obtain the integer

$$\begin{aligned}\langle \alpha + \alpha_j, \alpha_j\rangle &= (2-\delta_{m+1,j})\langle \alpha_{j-1}, \alpha_j\rangle + 3\langle \alpha_j, \alpha_j\rangle + (2-\delta_{l-1,j})\langle \alpha_{j+1}, \alpha_j\rangle \\ &= -(2-\delta_{m+1,j}) + 6 - 2 \geq 2,\end{aligned}$$

since we have $(2-\delta_{l-1,j})\langle \alpha_{j+1}, \alpha_j\rangle = -2$ for all $j \in \{m+1, ..., l-1\}$. Hence, $\alpha + \alpha_j$ has to be long. Then there exists a reflection $\sigma_{\tilde{\beta}}$ with $\tilde{\beta} \in \Phi_n$ such that

$$\sigma_{\tilde{\beta}}(\alpha + \alpha_j) = \alpha + \alpha_j - \langle \alpha + \alpha_j, \tilde{\beta}\rangle\tilde{\beta} = 2\sum_{i=l-n}^{l-1} \alpha_i + \alpha_l.$$

Let $\tilde{\beta} = \sum_{i=l-n}^{l} k_i\alpha_i \in \Phi_n^+$. Then it has to hold $k_j = 1$ and $\langle \alpha + \alpha_j, \tilde{\beta}\rangle = 1$. But this forces $k_{l-n} = -1$, which is impossible. Similarly we deduce for the case $\tilde{\beta} \in \Phi_n^-$. Since the sum $\alpha + \alpha_j$ for $j \neq l$ is neither short nor long, it can not be a root of Φ_n.
Suppose $\alpha + \alpha_l$ is a root. Then the reflection $\sigma_{\alpha_{l-n}} \cdot ... \cdot \sigma_{\alpha_m}$ maps $\alpha + \alpha_l$ to the root $2\sum_{i=l-n}^{l} \alpha_i$ which is higher than γ_{l-n}. Thus $\alpha + \alpha_l$ is not a root of Φ_n^-. This completes the second inner induction.
Now let $k = 2n+1$. By the outer induction assumption there exists a unique root α in $\Phi_{n-1}^+ \setminus \Phi_{n-2}^+$ with $\mathrm{ht}(\alpha) = 2n-1$ and α is of the form $\alpha = 2\sum_{i=l-n+1}^{l-1} \alpha_i + \alpha_l$. Since $l-n < l-1$, integer $\langle \alpha, \alpha_{l-n}\rangle$ computes as $\langle \alpha, \alpha_{l-n}\rangle = 2\langle \alpha_{l-n+1}, \alpha_{l-n}\rangle = -2$. Hence, $\sigma_{\alpha_{l-n}}$ sends α to

$$\sigma_{\alpha_{l-n}}(\alpha) = \alpha - \langle \alpha, \alpha_{l-n}\rangle \alpha_{l-n} = \alpha + 2\alpha_{l-n} = 2\sum_{i=l-n}^{l-1} \alpha_i + \alpha_l.$$

Evidently, the root $2\sum_{i=l-n}^{l-1} \alpha_i + \alpha_l$ has the required shape and is of height $2n+1$. Since $2\sum_{i=l-n}^{l-1} \alpha_i + \alpha_l$ is the root of maximal height in Φ_n (see [Hum72, Section 12.2, Table 2]), [Hum72, Section 10.4, Lemma A] implies the uniqueness of α in $\in \Phi_n^+ \setminus \Phi_{n-1}^+$.
Now we prove the third point of the lemma.
If $\alpha \in \Phi_n^+ \setminus \{\Phi_{n-1}^+ \cup \{\gamma_{l-n+1} = \sum_{i=l-n+1}^{l-1} \alpha_i + \alpha_l\}\}$, then $\mathrm{ht}(\alpha) = k < \mathrm{ht}(\gamma_{l-n+1})$. In particular, Lemma 5.1.2 implies that there exists a unique $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ such that $\mathrm{ht}(\beta) = k+1 \leq \mathrm{ht}(\gamma_{l-n+1})$. Hence, the simple root $\beta - \alpha \in \Delta$ has the stated property. Let $\tilde{\alpha} \in \Delta$ be different from $\beta - \alpha$ and let $\beta - \tilde{\alpha}$ be a root. By the uniqueness of α we obtain $\beta - \tilde{\alpha} \notin \Phi_n^+ \setminus \Phi_{n-1}^+$. Therefore, it has to hold $\beta - \tilde{\alpha} \in \Phi_{n-1}^+$.
Finally, we show the last assertion.
Obviously, we have $\Phi \supseteq (\bigcup_{i=1}^{l}(\Phi_i \setminus \Phi_{i-1})) \cup \Phi_0$. Let $\alpha = \sum_{i=1}^{l} k_i\alpha_i \in \Phi$ and let $j \in \{1, ..., l\}$ be minimal with $k_j \neq 0$. Thus, α is an element of $\Phi_j \setminus \Phi_{j-1}$ or $\alpha \in \Phi_0$ if $j = l$. We obtain the disjoint union $\Phi = (\bigcup_{i=1}^{l}(\Phi_i \setminus \Phi_{i-1})) \cup \Phi_0$. □

Lemma 5.2. *Let $n \in \{1, ..., l-1\}$. We denote by $\gamma_i = \alpha_l + 2\sum_{j=i}^{l-1} \alpha_j$ the root of maximal height in Φ_{l-i}^-. Furthermore, we define $\Phi_0 = \{\pm\alpha_l\}$. Let $A_0 = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-1-n} a_{\gamma_i}X_{\gamma_i} + \sum_{\beta\in\Phi_n^-} a_\beta X_\beta$ with a_{γ_i}, $a_\beta \in F$. Then there exists $U \in \mathcal{U}^-$ such that*

$$UA_0U^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n} \bar{a}_{\gamma_i}X_{\gamma_i} + \sum_{\beta\in\Phi_{n-1}^-} \bar{a}_\beta X_\beta$$

with $\bar{a}_{\gamma_i},\ \bar{a}_\beta \in F$.

Proof. We are going to prove for each $k \in \{1, ..., 2n\}$ the following claim:
For the matrix

$$A_{k-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n-1} a_{\gamma_i} X_{\gamma_i} + \sum_{\beta\in\Phi_{n-1}^-} a_\beta X_\beta + \sum_{\alpha\in\Phi_n^-\setminus\Phi_{n-1}^-, ht(\alpha)\geq k} a_\alpha X_\alpha$$

there exists $U \in \mathcal{U}^-$ such that

$$A_k = UA_{k-1}U^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n-1} \bar{a}_{\gamma_i} X_{\gamma_i} + \sum_{\beta\in\Phi_{n-1}^-} \bar{a}_\beta X_\beta + \sum_{\alpha\in\bar{\Phi}_n^-, ht(\alpha)>k} \bar{a}_\alpha X_\alpha$$

with $a_{\gamma_i},\ a_\beta,\ a_\alpha \in F$ and $\bar{a}_{\gamma_i},\ \bar{a}_\beta,\ \bar{a}_\alpha \in F$. Note that in the following we will sometimes write $\bar{\Phi}_n^-$ for $\Phi_n^- \setminus \Phi_{n-1}^-$.
We want to delete the part of A_{k-1} which lies in the root space corresponding to the root $\alpha \in \Phi_n^- \setminus \Phi_{n-1}^-$ with $\mathrm{ht}(\alpha) = k$. Then by Lemma 5.1.3 there exists a root $\bar{\alpha} \in \Delta$ such that $-\alpha + \bar{\alpha} = \bar{\beta} \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\bar{\beta}) = k + 1$. Thus, for $-\bar{\beta} =: \hat{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$ we get the equation $\hat{\beta} + \bar{\alpha} = \alpha$. Therefore, we are going to differentially conjugate A_{k-1} by the parametrized root group element $U_{\hat{\beta}}(\zeta) \in \mathcal{U}_{\hat{\beta}}$. We use Observation 3.4 to write this as

$$\begin{aligned} U_{\hat{\beta}}(\zeta)A_{k-1}U_{\hat{\beta}}(\zeta)^{-1} + \partial(U_{\hat{\beta}}(\zeta))U_{\hat{\beta}}(\zeta)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\alpha_i}) + \sum_{i=1}^{l-n-1} a_{\gamma_i}\mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\gamma_i}) \\ &+ \sum_{\beta\in\Phi_{n-1}^-} a_\beta \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\beta) + \sum_{\alpha\in\bar{\Phi}_n^-, ht(\alpha)\geq k} a_\alpha \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\alpha) + \partial(U_{\hat{\beta}}(\zeta))U_{\hat{\beta}}(\zeta)^{-1}. \end{aligned} \tag{5.6}$$

For the first summand of the right hand side of equation (5.6), we compute with the help of Lemma 3.2

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\alpha_i}) = \sum_{i=1}^{l}\sum_{j=0}^{q} m_{\hat{\beta},\alpha_i,j}\zeta^j X_{\alpha_i+j\hat{\beta}}. \tag{5.7}$$

First we are interested in the case when $j = 1$. Then Lemma 5.1 yields that there exists a unique $\bar{\alpha} \in \Delta$ such that $\hat{\beta} + \bar{\alpha} = \alpha$. Moreover, if there is another simple root $\tilde{\alpha} \in \Delta$, $\tilde{\alpha} \neq \bar{\alpha}$ such that $\hat{\beta} + \tilde{\alpha}$ is a root, then $\hat{\beta} + \tilde{\alpha} \in \Phi_{n-1}^-$. Now let $j > 1$. Since $\mathrm{ht}(\hat{\beta}) = k + 1$, it holds that if $\alpha_i + j\hat{\beta}$ is a root, then $\alpha_i + j\hat{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$ and $\mathrm{ht}(\alpha_i + j\hat{\beta}) = j(k + 1) - 1 > k$. Therefore, we translate equation (5.7) into

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\alpha_i}) \in \sum_{i=1}^{l} X_{\alpha_i} + m_{\hat{\beta},\bar{\alpha},1}\zeta X_\alpha + \sum_{\beta\in\Phi_{n-1}^-} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta + \sum_{\beta\in\bar{\Phi}_n^-, ht(\beta)>k} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta. \tag{5.8}$$

As before, the second summand of equation (5.6) can be written by Lemma 3.2 as

$$\sum_{i=1}^{l-n-1} a_{\gamma_i}\mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\gamma_i}) = \sum_{i=1}^{l-n-1} a_{\gamma_i} \sum_{j=0}^{q} m_{\hat{\beta},\gamma_i,j}\zeta^j X_{\gamma_i+j\hat{\beta}}. \tag{5.9}$$

Since $\hat{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$ and the γ_i are the roots of maximal height in Φ_{l-i}^- with $i \in \{1, ..., l-n-1\}$, we conclude that for $j \geq i$ the sum $\gamma_i + j\hat{\beta}$ is not a root. Hence, we get for equation (5.9)

$$\sum_{i=1}^{l-n-1} a_{\gamma_i} \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\gamma_i}) = \sum_{i=1}^{l-n-1} a_{\gamma_i} X_{\gamma_i}. \tag{5.10}$$

Again, with the help of Lemma 3.2 we compute the third summand of equation (5.6) to be

$$\sum_{\bar{\beta} \in \Phi_{n-1}^-} a_{\bar{\beta}} \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\bar{\beta}}) = \sum_{\bar{\beta} \in \Phi_{n-1}^-} a_{\bar{\beta}} \sum_{j=1}^{q} m_{\hat{\beta},\bar{\beta},j} \zeta^j X_{\bar{\beta}+j\hat{\beta}}. \tag{5.11}$$

It is easily seen that if for $j \geq 0$ the sum $\bar{\beta} + j\hat{\beta}$ is a root, then $\bar{\beta} + j\hat{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$ and $\mathrm{ht}(\bar{\beta} + j\hat{\beta}) > k+1$. Thus, we reformulate equation (5.11) as

$$\sum_{\bar{\beta} \in \Phi_{n-1}^-} a_{\bar{\beta}} \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_{\bar{\beta}}) \in \sum_{\bar{\beta} \in \Phi_{n-1}^-} a_{\bar{\beta}} X_{\bar{\beta}} + \sum_{\beta \in \bar{\Phi}_n^-, ht(\beta) > k+1} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta. \tag{5.12}$$

The fourth summand of the right hand side of equation (5.6) reads as

$$\sum_{\alpha \in \bar{\Phi}_n^-, ht(\alpha) \geq k} a_\alpha \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\alpha) = \sum_{\alpha \in \bar{\Phi}_n^-, ht(\alpha) \geq k} a_\alpha \sum_{j=1}^{q} m_{\hat{\beta},\alpha,j} \zeta^j X_{\alpha+j\hat{\beta}}. \tag{5.13}$$

If $\alpha + j\hat{\beta}$ is a root for $i \geq 1$, then, obviously $\alpha + j\hat{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$. The fact that $\mathrm{ht}(\hat{\beta}) = k+1$ implies in addition that $\mathrm{ht}(\alpha + j\beta) > k+1$. Hence, equation (5.13) translates into

$$\sum_{\alpha \in \bar{\Phi}_n^-, ht(\alpha) \geq k} a_\alpha \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\alpha) \in \sum_{\alpha \in \bar{\Phi}_n^-, ht(\alpha) \geq k} a_\alpha X_\alpha + \sum_{\alpha \in \bar{\Phi}_n^-, ht(\alpha) > k+1} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta. \tag{5.14}$$

We come to the last summand of equation (5.6). Proposition 3.5 says that the logarithmic derivative $l\delta$ maps the element $U_{\hat{\beta}}(\zeta) \in \mathcal{U}_{\hat{\beta}}$ to $\mathrm{Lie}(\mathcal{U}_{\hat{\beta}}) = \mathrm{Lie}(\mathrm{SP}_{2l})_{\hat{\beta}}$. Therefore,

$$\partial(U_{\hat{\beta}}(\zeta)) U_{\hat{\beta}}(\zeta)^{-1} \in \mathrm{Lie}(\mathrm{SP}_{2l})_{\hat{\beta}} \tag{5.15}$$

with $\mathrm{ht}(\hat{\beta}) = k+1$ and $\hat{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$. Putting the equations (5.8), (5.10), (5.12), (5.14) and (5.15) together yields

$$\begin{aligned} U_{\hat{\beta}}(\zeta) A_{k-1} U_{\hat{\beta}}(\zeta)^{-1} + \partial(U_{\hat{\beta}}(\zeta)) U_{\hat{\beta}}(\zeta)^{-1} \in & \sum_{i=1}^{l} X_{\alpha_i} + \zeta m_{\hat{\beta},\bar{\alpha},1} X_\alpha + a_\alpha X_\alpha + \sum_{i=1}^{l-n-1} a_{\gamma_i} X_{\gamma_i} \\ & + \sum_{\beta \in \Phi_{n-1}^-} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta + \sum_{\beta \in \bar{\Phi}_n^-, ht(\beta) > k} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta. \end{aligned}$$

Hence, if we set $m_{\hat{\beta},\bar{\alpha},1}\zeta = -a_\alpha$ the claim follows.
Using the claim one proves then by induction that for each $k \in \{1, ..., 2n\}$ there exists $U \in \mathcal{U}^-$ such that

$$UA_0U^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n-1} \mathrm{Lie}(\mathrm{SP}_{2l})_{\gamma_i} + \sum_{\beta \in \Phi_{n-1}^-} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta + \sum_{\beta \in \bar{\Phi}_n^-, ht(\beta) > k} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta.$$

In particular, we get for $k = 2n$ the assertion of the lemma. □

Lemma 5.3. *Let $A \in \sum_{i=1}^{l} X_{\alpha_i} + \mathbf{H} + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta = \sum_{i=1}^{l} X_{\alpha_i} + \mathrm{Lie}(\mathcal{B}_0^-)$. We denote by $\{\gamma_i \in \Phi^- \mid 1 \leq i \leq l-1\}$ the set of roots $\gamma_i = -\alpha_l - 2\sum_{j=i}^{l-1}\alpha_j$ of maximal height in the subsystems Φ_{l-i}^-. Moreover, let $\mathcal{M} = \{\gamma_i \in \Phi^- \mid 1 \leq i \leq l-1\} \cup \{-\gamma_l\}$ where $-\gamma_l := -\alpha_l$. Then there exists $U \in \mathcal{U}^-$ such that*

$$UAU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\alpha\in\mathcal{M}} \mathrm{Lie}(\mathrm{SP}_{2l})_\alpha.$$

Proof. First, we are going to prove the following claim: For each $k \in \{1, ..., l\}$ let $A_k = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k}^{l} a_i H_i + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta$ with $a_i \in F$. Then there exists $U \in \mathcal{U}^-$ such that

$$UA_kU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k+1}^{l} \bar{a}_i H_i + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta.$$

We write the matrix A_k as $A_k = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k}^{l} a_i H_i + \sum_{\beta\in\Phi^-} a_\beta X_\beta$ for suitable $a_\beta \in F$. To remove $a_k H_k$ we differentially conjugate A_k by $U_{-\alpha_k}(\zeta) \in \mathcal{U}_{-\alpha_k}$. More precisely, with Observation 3.4 this reads as

$$\begin{aligned} U_{-\alpha_k}(\zeta)A_kU_{-\alpha_k}(\zeta)^{-1} + \partial(U_{-\alpha_k}(\zeta))U_{-\alpha_k}(\zeta)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_i}) \\ + \sum_{i=k}^{l} a_i \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) + \sum_{\beta\in\Phi^-} a_\beta \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_\beta) &+ l\delta(U_{-\alpha_k}(\zeta)). \end{aligned} \tag{5.16}$$

We start with the first summand of the right hand side of equation (5.16). Then Lemma 3.2 yields for $i \neq k$

$$\mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_i}) = \sum_{j\geq 0} m_{-\alpha_k,\alpha_i,j}\zeta^j X_{\alpha_i + j(-\alpha_k)}$$

and if $i = k$ we have $\mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_k}) = X_{\alpha_k} + \zeta H_{\alpha_k} - \zeta^2 X_{-\alpha_k}$. Since $\alpha_i - j\alpha_k$ is not a root for $i \neq k$ and $j \geq 1$, we get

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_i}) \in \sum_{i=1}^{l} X_{\alpha_i} + \zeta H_k + \mathrm{Lie}(\mathrm{SP}_{2l})_{-\alpha_k}. \tag{5.17}$$

We handle the second summand of equation (5.16) with Lemma 3.2. It implies for $l \geq i \geq k+1$ that

$$\sum_{i=k+1}^{l} a_i \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) = \sum_{i=k+1}^{l} a_i(H_i - \langle\alpha_i, \alpha_k\rangle \zeta X_{-\alpha_k})$$

and for $i = k$ it yields $a_k \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_k) = a_k(H_k - 2\zeta X_{-\alpha_k})$. Hence, we can combine these results to

$$\sum_{i=k}^{l} a_i \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) \in \sum_{i=k}^{l} a_i H_i + \mathrm{Lie}(\mathrm{SP}_{2l})_{-\alpha_k}. \tag{5.18}$$

We use the fact that for $\beta \in \Phi^-$ and $j \geq 0$ the sum $\beta + j(-\alpha_k) \in \Phi^-$ to calculate the third summand to lie in

$$\sum_{\beta \in \Phi^-} a_\beta \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_\beta) \in \sum_{\beta \in \Phi^-} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta. \tag{5.19}$$

For the last summand of equation (5.16), Proposition 3.5 implies

$$l\delta(U_{-\alpha_k}(\zeta)) = \partial(U_{-\alpha_k}(\zeta))U_{-\alpha_k}(\zeta)^{-1} \in \mathrm{Lie}(\mathrm{SP}_{2l})_{-\alpha_k}. \tag{5.20}$$

Hence, if we put the equations (5.17), (5.18), (5.19) and (5.20) together and set $\zeta = -a_k$ we get the assumption of the claim.
One uses then the claim to prove by induction that for each $k \in \{1, ..., l\}$ there exists $U \in \mathcal{U}^-$ such that

$$UAU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k+1}^{l} a_i H_i + \sum_{\beta \in \Phi^-} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta.$$

In particular, this yields for $k = l$ that there exists $U \in \mathcal{U}^-$ such that

$$A_0 = UAU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\beta \in \Phi^-} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta.$$

Again, one proves by an inductive argument together with Lemma 5.2 that for each $n \in \{1, ..., l-1\}$ and A_0 there exists $U \in \mathcal{U}^-$ such that

$$U^{-1}A_0U - U^{-1}U' \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n} \mathrm{Lie}(\mathrm{SP}_{2l})_{\gamma_i} + \sum_{\beta \in \Phi^-_{n-1}} \mathrm{Lie}(\mathrm{SP}_{2l})_\beta.$$

Remember that Φ_0 is defined as $\Phi_0^- = \{-\alpha_l = -\gamma_l\}$. Then, the lemma follows for $n = 1$. $\square$

5.3 The equation with group SP_{2l}

The next step is to combine the results of Corollary 3.12 and Lemma 5.3, since we want to apply later the specialization bound. Therefore, let $C(z)$ be a rational function field with standard derivation $\partial = \frac{d}{dz}$ as in Section 3.4 and keep the notations of Lemma 5.3.

Corollary 5.4. *Apply Corollary 3.12 to the group SP_{2l} and the above Cartan Decomposition. We denote by $A^{M\&S}_{\mathrm{SP}_{2l}}$ the matrix satisfying the stated conditions of Corollary 3.12. Then there exists $U \in \mathcal{U}_0^- \subset \mathrm{SP}_{2l}$ such that*

$$\bar{A}_{Sp_{2l}} := UA^{M\&S}_{\mathrm{SP}_{2l}}U^{-1} + \partial(U)U^{-1} = \sum_{\alpha \in \Delta} X_\alpha + \sum_{\gamma_i \in \mathcal{M}} f_i X_{\gamma_i} \tag{5.21}$$

with at least one $f_i \in C[z] \setminus C$ and the differential Galois group of the matrix equation $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SP}_{2l}}\boldsymbol{y}$ is $\mathrm{SP}_{2l}(C)$ over $C(z)$.

Proof. Lemma 5.3 proves the existence of an element $U \in \mathcal{U}_0^- \subset \mathrm{SP}_{2l}$ such that equation (5.21) holds. Since differential conjugation defines a differential isomorphism, we deduce with Corollary 3.12 that the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SP}_{2l}}\boldsymbol{y}$ is again $\mathrm{SP}_{2l}(C)$ over $C(z)$. We still need to show the existence of $f_i \in C\,[z] \setminus C$ for some $\gamma_i \in \mathcal{M}$. Suppose $\bar{A}_{\mathrm{SP}_{2l}} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in T} f_i X_{\gamma_i} \in \mathrm{Lie}(\mathrm{SP}_{2l})(C)$. Then by Lemma 5.5 the corresponding differential equation $L(y, f_1, ..., f_l) \in C\,\{y\}$ has coefficients in C. But then by [Mag94, Corollary 3.28] the differential Galois group is abelian. Thus $\bar{A}_{\mathrm{SP}_{2l}} \in \mathrm{Lie}(\mathrm{SP}_{2l})(C(z)) \setminus \mathrm{Lie}(\mathrm{SP}_{2l})(C)$. Since $0 \neq A_1 \in \mathbf{H}(C)$ and $A = (z^2A_1 + A_0)$ in Corollary 3.12 we start our transformation with at least one coefficient lying in $C\,[z]\setminus C$. In each step the application of $\mathrm{Ad}(U_\beta(\zeta))$ generates at most new entries which are polynomials in ζ. Moreover, the logarithmic derivative is the product of the two matrices $\partial(U_\beta(\zeta))$ and $U_\beta(\zeta)^{-1} = U_\beta(-\zeta)$. In the proofs of Lemma 5.3 and Lemma 5.2 we choose the parameter ζ to be one of the coefficients. Hence, it holds $f_i \in C[z] \setminus C$. □

Our goal is to produce parametric equations for the series SP_{2l}. Therefore, let $t_1, ..., t_l$ be differential indeterminates and define the differential field $F = C\,\langle t_1, ..., t_l\rangle$. Moreover, define the matrix $A_{\mathrm{SP}_{2l}}(t_1, ..., t_l)$ as

$$A_{Sp_{2l}}(t_1, ..., t_l) = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\beta\in\mathcal{M}} t_\beta X_\beta$$

where $\mathcal{M}$ is as in Lemma 5.3. The next step is to compute a linear differential equation for SP_{2l} from the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SP}_{2l}}(t_1, ..., t_l)\boldsymbol{y}$.

Lemma 5.5. *The matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SP}_{2l}}(t_1, ..., t_l)\boldsymbol{y}$ is differentially equivalent to the homogeneous scalar linear differential equation*

$$L(y, t_1, ..., t_l) = y^{(2l)} - \sum_{i=1}^{l} (-1)^{i-1}(t_i\, y^{(l-i)})^{(l-i)} = 0$$

Proof. From the description of the Lie algebra $\mathrm{Lie}(\mathrm{SP}_{2l})$ in Section 5.1 we see that the matrix equation $\partial(\boldsymbol{y}) = A_{\mathrm{SP}_{2l}}(t_1, ..., t_l)\boldsymbol{y}$ has shape

$$\begin{pmatrix} \partial(y_1) \\ y_2 \\ \vdots \\ \\ \\ \vdots \\ \partial(y_{2l-1}) \\ \partial(y_{2l}) \end{pmatrix} = \left(\begin{array}{cccc|cccc} 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & & 1 & & & & \\ & & & 0 & 1 & & & \\ \hline & & 0 & t_1 & 0 & -1 & & \\ & & & 0 & & \ddots & \ddots & \\ 0 & & & & & & & -1 \\ t_l & 0 & & & & & & 0 \end{array}\right) \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \\ \\ \vdots \\ y_{2l-1} \\ y_{2l} \end{pmatrix}.$$

To simplify notation we will write y_i' for $\partial(y_i)$. Equivalently to the above matrix equation, we have the following system of equations

$$\begin{aligned} y_1' &= y_2 \\ &\vdots \\ y_l' &= y_{l+1} \\ y_{l+1}' &= t_1\, y_l - y_{l+2} \\ &\vdots \\ y_{2l-1}' &= t_{l-1}\, y_2 - y_{2l} \\ y_{2l}' &= t_l\, y_1. \end{aligned}$$

We want to show that we can transform the system in a single differential equation in y_1, i.e., y_1 is a cyclic vector for the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SP}_{2l}}(t_1, ..., t_l)\boldsymbol{y}$. Therefore, we claim that for each n, with $2 \leq n \leq l$, the corresponding subsystem

$$\begin{aligned} y_n' &= y_{n+1} \\ &\vdots \\ y_l' &= y_{l+1} \\ y_{l+1}' &= t_1\, y_l - y_{l+2} \\ &\vdots \\ y_{2l-n+1}' &= t_{l-n+1}\, y_n - y_{2l-n+2} \end{aligned}$$

yields the equation

$$y_n^{(2l-2n+2)} = \sum_{i=1}^{l-n+1} (-1)^{i-1} (t_i\, y_n^{(l-n+1-i)})^{(l-n+1-i)} + (-1)^{l-n+1} y_{2l-n+2}.$$

The proof of the claim will be done by backwards induction.
For $n = l$, we have the subsystem

$$\begin{aligned} y_l' &= y_{l+1} \\ y_{l+1}' &= t_1\, y_l - y_{l+2}. \end{aligned}$$

Differentiating the first equation and then substituting y_{l+1}' by the second, we get

$$y_l'' = y_{l+1}' = t_1\, y_l - y_{l+2}.$$

Now let $2 \leq n < l$. We obtain the following system of equations:

$$\begin{aligned} y'_n &= y_{n+1} && \text{(n)} \\ y'_{n+1} &= y_{n+2} && \text{(n+1)} \\ &\vdots \\ y'_l &= y_{l+1} \\ y'_{l+1} &= t_1\, y_l - y_{l+2} \\ &\vdots \\ y'_{2l-n} &= t_{l-n}\, y_{n+1} - y_{2l-n+1} && \text{(2l-n)} \\ y'_{2l-n+1} &= t_{l-n+1}\, y_n - y_{2l-n+2} && \text{(2l-n+1)} \end{aligned}$$

By the induction assumption the subsystem formed by equation (n+1) up to equation (2l-n) leads to

$$y_{n+1}^{2l-2n} = \sum_{i=1}^{l-n} (-1)^{i-1} (t_i\, y_{n+1}^{(l-n-i)})^{(l-n-i)} + (-1)^{(l-n)} y_{2l-n+1}. \tag{I}$$

We substitute y_{n+1} by y'_n in equation (I) and obtain

$$y_n^{(2l-2n+1)} = \sum_{i=1}^{l-n} (-1)^{i-1} (t_i y_n^{(l-n+1-i)})^{(l-n-i)} + (-1)^{(l-n)} y_{2l-n+1}. \tag{II}$$

Differentiating equation (II) and substituting y'_{2l-n+1} by equation (2l-n+1) yields

$$\begin{aligned} y_n^{(2l-2n+2)} &= \sum_{i=1}^{l-n} (-1)^{i-1} (t_i y_n^{(l-n+1-i)})^{(l-n+1-i)}) + (-1)^{(l-n)} (t_{l-n+1} y_n - y_{2l-n+2}) \\ &= \sum_{i=1}^{l-n} (-1)^{i-1} (t_i y_n^{(l-n+1-i)})^{(l-n+1-i)}) + (-1)^{(l-n)} t_{l-n+1} y_n \\ &\quad + (-1)^{l-n+1} y_{2l-n+2} \\ &= \sum_{i=1}^{l-n+1} (-1)^{i-1} (t_i y_n^{(l-n+1-i)})^{(l-n+1-i)}) + (-1)^{l-n+1} y_{2l-n+2}. \end{aligned}$$

Thus, the claim is shown.
Now we return to the proof of the lemma. We apply the claim to the subsystem of the initial system, obtained by leaving out the first and last equation, i.e., we consider the case $n = 2$, and get

$$y_2^{(2l-2)} = \sum_{i=1}^{l-1} (-1)^{i-1} (t_i y_2^{(l-i-1)})^{(l-i-1)}) + (-1)^{l-1} y_{2l}.$$

As in the induction step we substitute y_2 by y'_1. This leads to the equation

$$y_1^{(2l-1)} = \sum_{i=1}^{l-1} (-1)^{i-1} (t_i y_1^{(l-i)})^{(l-i-1)} + (-1)^{l-1} y_{2l}. \tag{III}$$

At last we differentiate equation (III) and write $t_l\, y_1$ for y'_{2l}. Hence, we obtain

$$\begin{aligned} y_1^{(2l)} &= \sum_{i=1}^{l-1}(-1)^{i-1}(t_i y_1^{(l-i)})^{(l-i)}) + (-1)^{l-1} t_l\, y_l) \\ &= \sum_{i=1}^{l}(-1)^{i-1}(t_i y_1^{(l-i)})^{(l-i)}). \end{aligned}$$

This completes the proof of the lemma. □

Theorem 5.6. *Let C be an algebraically closed field of characteristic zero, $t_1, ..., t_l$ differential indeterminates and $F = C\langle t_1, ..., t_l\rangle$ the corresponding differential field. Then the homogeneous linear differential equation*

$$L(y, t_1, ..., t_l) = y^{(2l)} - \sum_{i=1}^{l}(-1)^{i-1}(t_i\, y^{(l-i)})^{(l-i)} = 0$$

has $\mathrm{SP}_{2l}(C)$ as differential Galois group over F. Moreover, let $\hat{F}$ be a differential field with field of constants equal to C. Let $\hat{E}$ be a Picard-Vessiot extension over $\hat{F}$ with differential Galois group $\mathrm{SP}_{2l}(C)$ and suppose the defining matrix differential equation $\partial(\boldsymbol{y}) = \hat{A}\boldsymbol{y}$ satisfies $\hat{A} \in \sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}_0^-)$. Then there is a specialization $L(y, \hat{t}_1, ..., \hat{t}_l)$ with $\hat{t}_i \in \hat{F}$ such that $L(y, \hat{t}_1, ..., \hat{t}_l)$ gives rise to the extension $\hat{E}$ over $\hat{F}$.

Proof. Let E be a Picard-Vessiot extension for the equation $L(y, t_1, ..., t_l) = 0$ over F and denote by $\mathcal{G}$ its differential Galois group. Since the operator comes from the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SP}_{2l}}(t_1, ..., t_l)\boldsymbol{y}$ with $A_{\mathrm{SP}_{2l}}(t_1, ..., t_l) \in \mathrm{Lie}(\mathrm{SP}_{2l})(F)$, Proposition 2.1 yields $\mathcal{G}(C) \leq \mathrm{SP}_{2l}(C)$. By Corollary 5.4 there exists a specialization $\sigma : (t_1, ..., t_l) \to (f_1, ..., f_l)$ with $f_1, ..., f_l \in C[z]$ such that $\sigma(A_{\mathrm{SP}_{2l}}(t_1, ..., t_l)) = \bar{A}_{\mathrm{SP}_{2l}}$ and the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SP}_{2l}}\boldsymbol{y}$ is $\mathrm{SP}_{2l}(C)$. Moreover, we have $C\{f_1, ..., f_l\} = C[z]$. Thus we can apply Corollary 2.15. This yields $\mathrm{SP}_{2l}(C) \leq \mathcal{G}(C)$. Hence, it holds $\mathcal{G}(C) = \mathrm{SP}_{2l}(C)$.
Since the defining matrix $\hat{A}$ satisfies $\hat{A} \in \sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}_0^-)$, Lemma 5.3 provides that $\hat{A}$ is differentially equivalent to a matrix $\tilde{A} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in T} \hat{a}_i X_{\gamma_i}$ with suitable $\hat{a}_i \in \hat{F}$. Obviously, the specialization

$$\hat{\sigma} : (t_1, ..., t_l) \mapsto (\hat{a}_1, ..., \hat{a}_l)$$

has the required property. □

Chapter 6

A parametrized equation for SO_{2l+1}

6.1 The Lie algebra of SO_{2l+1} (type B_l)

We begin this chapter with the introduction of the root system of type B_l. Take $l \in \mathbb{N}$, with $l \geq 2$, and write $\epsilon_1, ..., \epsilon_l$ for the standard orthonormal basis of $\mathbb{R}^l$. We denote by $(\cdot, \cdot)$ the standard inner product of $\mathbb{R}^l$. Following [Hum72, Section 12.1], the root system of type B_l consists of the vectors

$$\Phi = \{\pm\epsilon_k,\ \pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j) \mid 1 \leq k \leq l; 1 \leq i \leq j \leq l\}.$$

A basis of Φ is given by the set of l linear independent vectors

$$\Delta = \{\alpha_i = \epsilon_i - \epsilon_{i+1},\ \alpha_l = \epsilon_l \mid 1 \leq i \leq l-1\}.$$

The Cartan integers $\langle \alpha_i, \alpha_j \rangle = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ can be taken from position (i, j) of the Cartan Matrix. In case of B_l it has the shape

$$\begin{pmatrix} 2 & -1 & 0 & & & . & . & . & & 0 \\ -1 & 2 & -1 & 0 & & . & . & . & & 0 \\ 0 & -1 & 2 & -1 & 0 & . & . & . & & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & & . & . & . & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & & . & . & 0 & -1 & 2 \end{pmatrix}.$$

Let V be a $2l+1$ dimensional C-vector space with basis $v_1, ..., v_{2l+1}$ and let f be a symmetric bilinear form on V given by the representing matrix

$$J = \left(\begin{array}{c|c|c} & & J_0 \\ \hline & 2 & \\ \hline J_0 & & \end{array}\right) \in C^{(2l+1)\times(2l+1)}$$

with respect to our basis. Here, the matrix J_0 has shape $J_0 = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in C^{2l\times 2l}$.

The group SO_{2l+1} is defined as the group of all automorphisms $A \in \mathrm{GL}(V)$ leaving invariant a non degenerated bilinear form. We choose this bilinear form to be defined by the representing matrix J. Hence, SO_{2l+1} is the set of matrices

$$\mathrm{SO}_{2l+1} = \{A \in \mathrm{GL}_{2l+1} \mid A^T J A = J\}.$$

Then the Lie algebra $\mathrm{Lie}(\mathrm{SO}_{2l+1})$ of SO_{2l+1} is defined by all matrices $X \in C^{(2l+1)\times(2l+1)}$ leaving the symmetric bilinear form f invariant, i.e., for $v, w \in V$ the matrix X has to satisfy

$$f(Xv, w) = -f(v, Xw) \Leftrightarrow (Xv)^T J w = -v^T J (Xw) \Leftrightarrow X^T J = -JX.$$

For $i = 1, 2$, let $X_{0i}^T,\ X_{i0} \in C^l$ and for all other indices $1 \leq i, j \leq 2$, take $X_{ij} \in C^{l\times l}$. Moreover, let X_{00} be an element of C. Then, we can write the matrix X as

$$X = \left(\begin{array}{c|c|c} X_{11} & X_{10} & X_{12} \\ \hline X_{01} & X_{00} & X_{02} \\ \hline X_{21} & X_{20} & X_{22} \end{array}\right).$$

Furthermore, we renumber the rows and columns of X into $1, ..., l, 0, -1, ..., -l$. Hence, the above condition for X to be an element of $\mathrm{Lie}(\mathrm{SO}_{2l+1})$ translates into

$$\begin{array}{rclrclrcl} J_0 X_{11} J_0 & = & -X_{22}^T & J_0 X_{12} J_0 & = & -X_{12}^T & J_0 X_{21} J_0 & = & -X_{21}^T \\ 2X_{01} & = & -X_{20}^T J_0 & 2X_{02} & = & -X_{10}^T J_0 & X_{00} & = & -X_{00}. \end{array}$$

The last equation obviously implies $X_{00} = 0$. It is easy to see that the l matrices

$$2E_{i0} - E_{0,-l-1+i} \quad \text{and} \quad E_{0i} - 2E_{-l-1+i,0},$$

with $1 \leq i \leq l$, satisfy the conditions of the fourth and fifth equation. A computation shows that the conjugation $J_0 M J_0^{-1}$ of an element $M \in C^{l\times l}$ by J_0 is reversing M and then taking its transpose. Here we mean by the reversed matrix, the matrix obtained by reflecting the entries at the second diagonal. Then the l diagonal matrices $E_{ii} - E_{-l-1+i,-l-1+i}$, with $1 \leq i \leq l$, and the matrices

$$E_{ij} - E_{-l-1+j,-l-1+i},\ E_{ji} - E_{-l-1+i,-l-1+j},$$

with $1 \leq i < j \leq l$, have only non-zero entries in the blocks X_{11} and X_{22}. We get that they satisfy the condition of the first equation. Moreover, for $1 \leq i, j \leq l$ with $i + j \leq l$, the matrices

$$E_{i,-j} - E_{l+1-j,-l-1+i},\ E_{-j,i} - E_{-l-1+i,l+1-j},$$

with non-zero entries in the blocks X_{12} and X_{21}, satisfy the conditions of the second and third equation. Denote by **B** the collection of all these matrices. Then the elements of **B** are lineary independent, since for each position above the secondary diagonal there is a unique matrix in **B** with a nonzero entry at this position. The number of elements in **B** is equal to $2l^2 - l$. But this number coincides with the dimension of $\mathrm{Lie}(\mathrm{SO}_{2l+1})$ known from literature (for example, see [Hum72, p.3]). Hence, the set **B** is a basis for $\mathrm{Lie}(\mathrm{SO}_{2l+1})$.
The next step is to determine a Cartan Decomposition for $\mathrm{Lie}(\mathrm{SO}_{2l+1})$. Therefore, we

compute the standard maximal torus $\mathcal{T}_0$ of SO_{2l+1}. Let $T = \mathrm{diag}(\lambda_1, ..., \lambda_{2l+1}) \in \mathrm{GL}_{2l+1}$ be a diagonal matrix of GL_{2l+1}. Then an explicit calculation of $T^T JT = J$ leads to

$$T^T JT = \left(\begin{array}{c|c|c} & & \lambda_{2l+1}\lambda_1 \\ \hline & \lambda_{l+1}^2 & \\ \hline \lambda_{2l+1}\lambda_1 & & \end{array}\right) = \left(\begin{array}{c|c|c} & & 1 \\ \hline & 1 & \\ \hline 1 & & \end{array}\right).$$

Thus, $T \in \mathrm{SO}_{2l+1}$ if and only if $\lambda_{2l+1} = \frac{1}{\lambda_1}, ..., \lambda_{l+2} = \frac{1}{\lambda_l}$ and $\lambda_{l+1} = 1$. Hence, the elements of $\mathcal{T}_0$ are

$$\mathcal{T}_0 = \{T = \mathrm{diag}(\lambda_1, ..., \lambda_l, 1, \frac{1}{\lambda_l}, ..., \frac{1}{\lambda_1}) \mid \lambda_1, ..., \lambda_l \in C^\times\}.$$

Then we obtain for the conjugates of the basis elements of $\mathrm{Lie}(\mathrm{SO}_{2l+1})$ by $T \in \mathcal{T}_0$

$$\begin{aligned}
T(E_{ij} - E_{-l-1+j,-l-1+i})T^{-1} &= \frac{\lambda_i}{\lambda_j}(E_{ij} - E_{-l-1+j,-l-1+i}), \\
T(E_{ji} - E_{-l-1+i,-l-1+j})T^{-1} &= \frac{\lambda_j}{\lambda_i}(E_{ji} - E_{-l-1+i,-l-1+j}), \\
T(E_{i,-j} - E_{l+1-j,-l-1+i})T^{-1} &= \lambda_i\lambda_{l+1-j}(E_{i,-j} - E_{l+1-j,-l-1+i}), \\
T(E_{-j,i} - E_{-l-1+i,l+1-j})T^{-1} &= \frac{1}{\lambda_{l+1-j}}\frac{1}{\lambda_i}(E_{-j,i} - E_{-l-1+i,l+1-j}), \\
T(2E_{i0} - E_{0,-l-1+i})T^{-1} &= \lambda_i(2E_{i0} - E_{0,-l-1+i}), \\
T(E_{0i} - 2E_{-l-1+i,0})T^{-1} &= \frac{1}{\lambda_i}(E_{0i} - 2E_{-l-1+i,0}).
\end{aligned}$$

We conclude that the root system Φ of $\mathrm{Lie}(\mathrm{SO}_{2l+1})$ is of type B_l. We can assign the elements of $\mathbf{B}$ to their root spaces. For $1 \le i < j \le l$, we define the matrices

$$X_{\epsilon_i-\epsilon_j} := E_{ij} - E_{-l-1+j,-l-1+i},\ X_{-(\epsilon_i-\epsilon_j)} := E_{ji} - E_{-l-1+i,-l-1+j},$$

and for $1 \le i, j \le l,\ i + j \le l$, the matrices

$$X_{\epsilon_i+\epsilon_{l+1-j}} := E_{i,-j} - E_{l+1-j,-l-1+i},\ X_{-(\epsilon_i+\epsilon_{l+1-j})} := E_{-j,i} - E_{-l-1+i,l+1-j}.$$

Furthermore, for $1 \le i \le l$, we set

$$X_{\epsilon_i} := 2E_{i0} - E_{0,-l-1+i} \quad \text{and} \quad X_{-\epsilon_i} := E_{0i} - 2E_{-l-1+i,0}.$$

Thus, the Cartan decomposition of $\mathrm{Lie}(\mathrm{SO}_{2l+1})$ has shape

$$\begin{aligned}
\mathrm{Lie}(\mathrm{SO}_{2l+1}) = \mathbf{H} \ &\textstyle\bigoplus_{i,j} \ \langle X_{\epsilon_i-\epsilon_j}\rangle_C \oplus \langle X_{-(\epsilon_i-\epsilon_j)}\rangle_C \\
&\textstyle\bigoplus_{i,j} \ \langle X_{\epsilon_i+\epsilon_{l+1-j}}\rangle_C \oplus \langle X_{-(\epsilon_i+\epsilon_{l+1-j})}\rangle_C \\
&\textstyle\bigoplus_{i} \ \langle X_{\epsilon_i}\rangle_C \oplus \langle X_{-\epsilon_i}\rangle_C,
\end{aligned}$$

where the Cartan subalgebra $\mathbf{H}$ is generated by

$$\mathbf{H} = \langle E_{ii} - E_{-l-1+i,-l-1+i} \mid 1 \leq i \leq l \rangle_C.$$

Now we check if the above vectors form a Chevalley basis for SO_{2l+1}. Therefore, we compute

$$\begin{aligned}
[X_{\epsilon_i-\epsilon_j}, X_{-(\epsilon_i-\epsilon_j)}] &= E_{ii} + E_{-l-1+j,-l-1+j} - E_{jj} - E_{-l-1+i,-l-1+i} \\
&=: H_{\epsilon_i-\epsilon_j}, \\
[X_{\epsilon_i+\epsilon_{l+1-j}}, X_{-(\epsilon_i+\epsilon_{l+1-j})}] &= E_{ii} + E_{l+1-j,l+1-j} - E_{-j,-j} - E_{-l-1+i,-l-1+i} \\
&=: H_{\epsilon_i+\epsilon_{l+1-j}}, \\
[X_{\epsilon_i}, X_{-\epsilon_i}] &= 2E_{ii} - 2E_{-l-1+i,-l-1+i} =: H_{\epsilon_i}.
\end{aligned}$$

These are precisely the co-roots, since we have

$$\begin{aligned}
[H_{\epsilon_i-\epsilon_j}, X_{\epsilon_i-\epsilon_j}] &= E_{ij} - E_{-l-1+j,-l-1+i} - (-E_{ij} + E_{-l-1+j,-l-1+i}) = X_{\epsilon_i-\epsilon_j}, \\
[H_{\epsilon_i+\epsilon_{l+1-j}}, X_{\epsilon_i+\epsilon_{l+1-j}}] &= E_{i,-j} - E_{l+1-j,-l-1+i} - (-E_{i,-j} + E_{l+1-j,-l-1+i}) \\
&= 2X_{\epsilon_i+\epsilon_{l+1-j}}, \\
[H_{\epsilon_i}, X_{\epsilon_i}] &= 4E_{i0} - 2E_{0,-l-1+i} = 2X_{\epsilon_i}.
\end{aligned}$$

We denote the l co-roots corresponding to the simple roots by

$$H_1 = H_{\epsilon_1-\epsilon_2}, ..., H_{l-1} = H_{\epsilon_{l-1}-\epsilon_l} \text{ and } H_l := H_{\epsilon_l}.$$

Now we define a morphism $\theta : \mathrm{Lie}(\mathrm{SO}_{2l+1}) \to \mathrm{Lie}(\mathrm{SO}_{2l+1})$ by $X \mapsto -D^{-1}X^T D$, where D denotes diagonal matrix of shape

$$D = \begin{pmatrix} \mathbf{1}_l & & \\ & 2 & \\ & & \mathbf{1}_l \end{pmatrix}.$$

It is easily seen that θ is an automorphism of $\mathrm{Lie}(\mathrm{SO}_{2l+1})$ which satisfies the following equations:

$$\begin{aligned}
\theta(X_{\epsilon_i-\epsilon_j}) &= -X_{-(\epsilon_i-\epsilon_j)} \\
\theta(X_{\epsilon_i+\epsilon_{l+1-j}}) &= -X_{-(\epsilon_i+\epsilon_{l+1-j})} \\
\theta(X_{\epsilon_i}) &= -X_{-\epsilon_i}.
\end{aligned}$$

In addition to these equations we have the identity

$$\theta([X, Y]) = -[X, Y]^T = [-X^T, -Y^T] = [\theta(X), \theta(Y)]. \tag{6.1}$$

We define the number $n_{\alpha,\beta} \in \mathbb{Z}$ for two roots $\alpha, \beta \in \Phi$ by the rule $[X_\alpha, X_\beta] = n_{\alpha,\beta} X_{\alpha+\beta}$. The next step is to apply θ to $[X_\alpha, X_\beta] = n_{\alpha,\beta} X_{\alpha+\beta}$. This can be calculated with the help of equation (6.1) as

$$-n_{\alpha,\beta} X_{-\alpha-\beta} = -[X_\alpha, X_\beta]^T = [X_{-\alpha}, X_{-\beta}] = n_{-\alpha,-\beta} X_{-\alpha-\beta}.$$

Thus, it holds $-n_{\alpha,\beta} = n_{-\alpha,-\beta}$. But [Car72, Theorem 4.1.2] yields the identity

$$n_{\alpha,\beta} n_{-\alpha,-\beta} = -(r+1)^2.$$

We conclude that $n_{\alpha,\beta}$ is equal to $\pm(r+1)$. Hence, the elements in

$$\{H_i,\ X_\alpha \mid 1 \leq i \leq l,\ \alpha \in \Phi\}$$

form a Chevalley basis of $\mathrm{Lie}(\mathrm{SO}_{2l+1})$.

6.2 The transformation lemma for SO_{2l+1}

In this section we are going to prove the transformation lemma for SO_{2l+1} over a differential field (F, ∂) of characteristic zero. The proof is based on differential conjugation, i.e., on the adjoint action and the logarithmic derivative which we can both describe by the root system. Therefore we begin with the study of the root system of type B_l.

Lemma 6.1. *For $n \in \{1, ..., l-1\}$, let $\Phi_n := \langle \alpha_l, ..., \alpha_{l-n} \rangle_\Phi$ be the set of all $\mathbb{Z}$-linear combinations of the roots $\alpha_l, ..., \alpha_{l-n}$ which lie in Φ and define $\Phi_0 := \{\pm\alpha_l\}$.*

1. *The set $\Phi_n \subseteq \Phi = \Phi_{l-1}$ is an irreducible subsystem of Φ with $\Phi_n \sim B_{n+1}$.*

2. *For $k \in \{1, ..., 2n+1\}$, there exists a unique root $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\alpha) = k$, and α has shape*

$$\alpha = \sum_{i=l-n}^{l-n-1+k} \alpha_i \qquad \text{if} \quad 1 \le k \le n+1 \quad \text{and}$$

$$\alpha = \sum_{i=l-n}^{l+n+1-k} \alpha_i + 2 \sum_{i=l+n+2-k}^{l} \alpha_i \qquad \text{if} \quad n+2 \le k \le 2n+1.$$

3. *Let $\alpha \in \Phi_n^+ \setminus \{\Phi_{n-1}^+ \cup \{\gamma = \alpha_{l-n} + 2\sum_{i=l-n+1}^{l} \alpha_i\}\}$ with $\mathrm{ht}(\alpha) = k$. Then there exists a unique $\bar{\alpha} \in \Delta$ such that $\beta = \alpha + \bar{\alpha} \in \Phi_n^+ \setminus \Phi_{n-1}^+$ and $\mathrm{ht}(\beta) = k+1$. If $\tilde{\alpha} \in \Delta$ is a simple root and $\beta - \tilde{\alpha}$ is a root, then either $\beta - \tilde{\alpha} = \alpha$ or $\beta - \tilde{\alpha} \in \Phi_{n-1}^+$.*

4. *The root system Φ consists of the roots*

$$\begin{aligned} \Phi \;&=\; \{\pm(\epsilon_i - \epsilon_j) = \pm \sum_{k=i}^{j-1} \alpha_k \mid 1 \le i < j \le l\} \cup \{\pm\epsilon_i = \pm \sum_{k=i}^{l} \alpha_k \mid 1 \le i \le l\} \\ &\cup\; \{\pm(\epsilon_i + \epsilon_j) = \pm(\sum_{k=i}^{j-1} \alpha_k + 2 \sum_{k=j}^{l} \alpha_k \mid 1 \le i < j \le l)\} \end{aligned}$$

Proof. The first assertion of the lemma is a consequence of the Dynkin diagram of type B_l (e.g., see [Hum72, Section 11.4]).
We prove the second point. The fact that Φ_n is a root system of type B_{n+1} together with [Hum72, Section 10.4, Lemma A] yields that for $n \in \{1, ..., l-1\}$ there exists a unique root γ_{l-n} of maximal height in Φ_n. Moreover, by [Hum72, Section 12.2, Table 2] γ_{l-n} has shape $\gamma_{l-n} = 2\sum_{i=l-n+1}^{l} \alpha_i + \alpha_{l-n}$.
We are going to prove the assumption by induction on $n \in \{1, ..., l-1\}$. The induction step will be done by two additional inductions and a single computation.
Let $n = 1$. We will compute the root system $\Phi_1 = \langle \alpha_l, \ \alpha_{l-1} \rangle_\Phi$. Since the integer $\langle \alpha_{l-1}, \alpha_l \rangle$ can be read from the Cartan matrix as $\langle \alpha_{l-1}, \alpha_l \rangle = -2$, the reflection σ_{α_l} maps α_{l-1} to

$$\sigma_{\alpha_l}(\alpha_{l-1}) = \alpha_{l-1} - \langle \alpha_{l-1}, \alpha_l \rangle \alpha_l = \alpha_{l-1} + 2\alpha_l,$$

the root of maximal height. Since root strings are unbroken, $\alpha_l + \alpha_{l-1}$ again is a root. Remember that the only scalar multiples of a root α are $\pm\alpha$. Hence, Φ_1 consists of the

roots $\Phi_1 = \{\pm\alpha_l, \pm\alpha_{l-1}, \pm(\alpha_{l-1}+\alpha_l), \pm(2\alpha_l+\alpha_{l-1})\}$. It is easy to see that these roots satisfy the assumption.
Let $1 < n \leq l-1$. We prove by induction on $k_1 \in \{1, ..., n\}$ that there exists a unique root $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\alpha) = k_1$ and the shape of α is

$$\alpha = \sum_{i=l-n}^{l-n-1+k} \alpha_i.$$

Let $k_1 = 1$. Then α_{l-n} is the unique root of $\Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\alpha_{l-n}) = 1$.
Let $1 < k_1 \leq n$. Then, by the induction assumption there exists an $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ such that $\mathrm{ht}(\alpha) = k_1 - 1$ and α has shape $\alpha = \sum_{i=l-n}^{l-n-2+k_1} \alpha_i$. We calculate the integer $\langle \alpha, \alpha_{l-n-1+k_1} \rangle$ to be

$$\langle \sum_{i=l-n}^{l-n-2+k_1} \alpha_i, \alpha_{l-n-1+k_1} \rangle = \sum_{i=l-n}^{l-n-2+k_1} \langle \alpha_i, \alpha_{l-n-1+k_1} \rangle = -1.$$

Hence, the reflection $\sigma_{\alpha_{l-n-1+k_1}}$ maps α to

$$\sigma_{\alpha_{l-n-1+k_1}}(\alpha) = \alpha - \langle \alpha, \alpha_{l-n-1+k_1} \rangle \alpha_{l-n-1+k_1} = \sum_{i=l-n}^{l-n-1+k_1} \alpha_i.$$

Thus, we have constructed a root of $\mathrm{ht}(\alpha + \alpha_{l-n-1+k_1}) = k_1$, which lies in $\Phi_n^+ \setminus \Phi_{n-1}^+$. Suppose there is a root $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\beta) = k_1$ and $\beta \neq \alpha + \alpha_{l-n-1+k_1}$. Then [Hum72, Section 10.2, Corollary] implies that we can write β as the sum of simple roots, i.e., $\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_m$ with $\bar{\alpha}_i \in \Delta$ in such a way that each partial sum is a root. Hence, $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ is a root of $\mathrm{ht}(\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}) = k_1 - 1$. We assume that $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} \neq \alpha$. Then the uniqueness of α implies $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} \in \Phi_{n-1}^+$. Hence, we get the equation

$$-(\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}) + \beta = \alpha_{l-n}. \tag{6.2}$$

Denote by $\bar{w}$ the minimum of the indices of the simple roots $\alpha_w = \bar{\alpha}_i$ in $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$. Thus, equation (6.2) implies $\bar{w} > l - n$. Let $\bar{n} \in \mathbb{N}$ such that $\bar{w} = l - \bar{n}$ holds. It follows that $\bar{n} < n$. The induction hypothesis yields that the shape of $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ is

$$\begin{aligned} \eta_1 &:= \sum_{i=l-\bar{n}}^{l-\bar{n}-2+k_1} \alpha_i && \text{if} \quad 1 \leq k_1 - 1 \leq \bar{n}+1 \quad \text{and} \\ \eta_2 &:= \sum_{i=l-\bar{n}}^{l+\bar{n}+2-k_1} \alpha_i + 2 \sum_{i=l+\bar{n}+3-k_1}^{l} \alpha_i && \text{if} \quad \bar{n}+2 \leq k_1 - 1 \leq 2\bar{n}+1. \end{aligned}$$

Let us assume $l - \bar{n} > l - n + 1$. To simplify the notation we set $\beta_i := \eta_i + \alpha_{l-n}$. We compute the integers $\langle \beta_1, \alpha_{l-n} \rangle$ and $\langle \beta_2, \alpha_{l-n} \rangle$. They are

$$\langle \beta_1, \alpha_{l-n} \rangle = \sum_{i=l-\bar{n}}^{l-\bar{n}-2+k_1} \langle \alpha_i, \alpha_{l-n} \rangle + \langle \alpha_{l-n}, \alpha_{l-n} \rangle = 2 \quad \text{and}$$

$$\langle \beta_2, \alpha_{l-n} \rangle = \sum_{i=l-\bar{n}}^{l+\bar{n}+2-k_1} \langle \alpha_i, \alpha_{l-n} \rangle + 2 \sum_{i=l+\bar{n}+3-k_1}^{l} \langle \alpha_i, \alpha_{l-n} \rangle + \langle \alpha_{l-n}, \alpha_{l-n} \rangle = 2.$$

Hence, the reflection $\sigma_{\alpha_{l-n}}$ maps β_i to

$$\sigma_{\alpha_{l-n}}(\beta_i) = \beta_i - 2\alpha_{l-n} = \eta_i - \alpha_{l-n}. \tag{6.3}$$

Since the right hand side of equation (6.3) is not a root, we obtain a contradiction. This forces $l - \bar{n} = l - n + 1$. We observe that $k_1 - 1 < k_1 \leq n = \bar{n} + 1$. Hence, the induction assumption implies

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} = \sum_{i=l-\bar{n}}^{l-\bar{n}-2+k_1} \alpha_i = \sum_{i=l-n+1}^{l-n+k_1-1} \alpha_i.$$

But then

$$\beta = \sum_{i=l-n+1}^{l-n+k_1-1} \alpha_i + \alpha_{l-n} = \sum_{i=l-n}^{l-n+k_1-1} \alpha_i$$

is the root $\alpha + \alpha_{l-n}$ constructed above. We obtain a contradiction.
From the Cartan matrix we obtain for $1 \leq i \leq l-2$ the integers

$$\langle \alpha_i, \alpha_{i+1} \rangle = \langle \alpha_{i+1}, \alpha_i \rangle = -1.$$

Hence, the roots $\alpha_1, ..., \alpha_{l-1}$ are of equal length. The integers $\langle \alpha_l, \alpha_{l-1} \rangle = -1$ and $\langle \alpha_{l-1}, \alpha_l \rangle = -2$ imply together with [Hum72, Section 9.4, Table 1] that the roots $\alpha_1, ..., \alpha_{l-1}$ are long and α_l is short.
It remains to check that the sum

$$\alpha + \alpha_j = \sum_{i=l-n}^{l-n-2+k_1} \alpha_i + \alpha_j$$

with $\alpha_j \in \{\alpha_{l-n}, ..., \alpha_l\} \setminus \{\alpha_{l-n-1+k_1}\}$ is not a root. For $j \in \{l-n, ..., l-n-2+k_1\}$, we obtain

$$\langle \alpha + \alpha_j, \alpha_j \rangle = (1 - \delta_{l-n,j})\langle \alpha_{j-1}, \alpha_j \rangle + 2\langle \alpha_j, \alpha_j \rangle + (1 - \delta_{l-n-2+k_1,j})\langle \alpha_{j+1}, \alpha_j \rangle \geq 2.$$

Furthermore, for $j \in \{l-n+k_1, ..., l-1\}$ and $n > k_1$, we get $\langle \alpha + \alpha_j, \alpha_j \rangle = 2$. Thus $\alpha + \alpha_j$ has a different length than α_j, i.e., $\alpha + \alpha_j$ is a short root. But this forces the integer $\langle \alpha + \alpha_j, \alpha_j \rangle$ to be 0 or ± 1 by [Hum72, Section 9.4, Table 1], contradicting $\langle \alpha + \alpha_j, \alpha_j \rangle \geq 2$.
For $j = l$, we have $\langle \alpha + \alpha_l, \alpha_l \rangle = 2$. Thus, the reflection σ_{α_l} maps $\alpha + \alpha_l$ to the sum

$$\sigma_{\alpha_l}(\alpha + \alpha_l) - \langle \alpha + \alpha_l, \alpha_l \rangle \alpha_l = \alpha - \alpha_l.$$

But $\alpha - \alpha_l$ is not a root. Thus, for every $\alpha_j \in \{\alpha_{l-n}, ..., \alpha_l\} \setminus \{\alpha_{l-n-1+k_1}\}$, the sum $\alpha + \alpha_j$ is not a root. Hence, the assumption follows and the first inner induction is done.
Now let $k = n+1$. We have shown right before that there exists a unique root $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\alpha) = k-1$ and of shape $\alpha = \sum_{i=l-n}^{l-1} \alpha_i$. We compute the integer $\langle \alpha, \alpha_l \rangle$ as $\langle \alpha, \alpha_l \rangle = \sum_{i=l-n}^{l-1} \langle \alpha_i, \alpha_l \rangle = -2$. Thus, the image of α under the reflection σ_{α_l} is

$$\sigma_{\alpha_l}(\alpha) = \alpha - \langle \alpha, \alpha_l \rangle \alpha_l = \alpha + 2\alpha_l = \sum_{l-n}^{l-1} \alpha_i + 2\alpha_l.$$

Since root strings are unbroken, we have computed two roots. Namely, the root $\alpha+\alpha_l$ of $\mathrm{ht}(\alpha+\alpha_l)=n+1$ and the root $\alpha+2\alpha_l$ of $\mathrm{ht}(\alpha+2\alpha_l)=n+2$. We prove the uniqueness assumption for $\alpha+\alpha_l$. Let $\beta\in\Phi_n^+\setminus\Phi_{n-1}^+$ with $\beta\neq\alpha+\alpha_l$ and $\mathrm{ht}(\beta)=n+1$. Write β as $\bar\alpha_1+...+\bar\alpha_m$ with $\bar\alpha_i\in\Delta$ in such a way that each partial sum is a root. Then, in particular $\bar\alpha_1+...+\bar\alpha_{m-1}$ is a root. Let us assume that $\bar\alpha_1+...+\bar\alpha_{m-1}\neq\alpha$. Then the uniqueness of $\alpha\in\Phi_n^+\setminus\Phi_{n-1}^+$ with $\mathrm{ht}(\alpha)=k-1$ yields $\bar\alpha_1+...+\bar\alpha_{m-1}\in\Phi_{n-1}^+$. Hence, we have $-(\bar\alpha_1+...+\bar\alpha_{m-1})+\beta=\alpha_{l-n}$. Denote by $\bar w$ the minimum of the indices of the simple roots $\alpha_w=\bar\alpha_i$ in $\bar\alpha_1+...+\bar\alpha_{m-1}$. We conclude that $\bar w>l-n$. Take $\bar n\in\mathbb{N}$ such that $\bar w=l-\bar n$ holds and assume $l-\bar n>l-n+1$. The induction assumption for $k=n+1>\bar n+2$ yields that $\bar\alpha_1+...+\bar\alpha_{m-1}$ has shape

$$\eta:=\sum_{i=l-\bar n}^{l+\bar n+2-k}\alpha_i+2\sum_{i=l+\bar n+3-k}^{l}\alpha_i.$$

We compute the integer $\langle\eta+\alpha_{l-n},\alpha_{l-n}\rangle$ to be $\langle\eta+\alpha_{l-n},\alpha_{l-n}\rangle=2$. Hence, we obtain

$$\sigma_{\alpha_{l-n}}(\eta+\alpha_{l-n})=\eta-\alpha_{l-n}$$

as the image of $\eta+\alpha$ under $\sigma_{\alpha_{l-n}}$. This forces $l-\bar n=l-n+1$. From $k-1=n=\bar n+1$ we conclude that $\bar\alpha_1+...+\bar\alpha_{m-1}$ is of shape

$$\bar\alpha_1+...+\bar\alpha_{m-1}=\sum_{i=l-n+1}^{l}\alpha_i.$$

We observe that $\beta=\bar\alpha_1+...+\bar\alpha_{m-1}+\alpha_{l-n}=\sum_{i=l-n}^{l}\alpha_i=\alpha+\alpha_l$ is the root constructed above, in contrast to the assumption. It remains to check that $\beta=\alpha+\alpha_j$ is not a root for some $j\in\{l-n,...,l-1\}$. We compute

$$\langle\alpha+\alpha_j,\alpha_j\rangle=(1-\delta_{l-n,j})\langle\alpha_{j-1},\alpha_j\rangle+2\langle\alpha_j,\alpha_j\rangle+(1-\delta_{l-1,j})\langle\alpha_{j+1},\alpha_j\rangle\geq 2.$$

This implies as above that $\alpha+\alpha_j$ is not a root.
We prove by induction on $k_2\in\{n+2,...,2n+1\}$ that there exists a unique root $\alpha\in\Phi_n^+\setminus\Phi_{n-1}^+$ such that $\mathrm{ht}(\alpha)=k_2$ and α has shape

$$\alpha=\sum_{i=l-n}^{l+n+1-k_2}\alpha_i+2\sum_{i=l+n+2-k_2}^{l}\alpha_i.$$

Let $k_2=n+2$. We have constructed above the root $\bar\alpha+2\alpha_l$ with $\mathrm{ht}(\bar\alpha+2\alpha_l)=n+2$ where $\bar\alpha$ was $\bar\alpha=\sum_{i=l-n}^{l-1}\alpha_i$. We have also shown that $\alpha=\sum_{i=l-n}^{l}\alpha_i$ is the unique root in $\Phi_n^+\setminus\Phi_{n-1}^+$ with $\mathrm{ht}(\alpha)=n+1$. Therefore, we write $\bar\alpha+2\alpha_l$ as the sum of α and α_l. Let $\beta\in\Phi_n^+\setminus\Phi_{n-1}^+$ be another root with $\mathrm{ht}(\beta)=n+2$ and $\beta\neq\alpha+\alpha_l$. We write again β as $\bar\alpha_1+...+\bar\alpha_m$ with $\bar\alpha_j\in\Delta$ such that each partial sum is a root. Hence, $\bar\alpha_1+...+\bar\alpha_{m-1}$ is a root. Assume that $\bar\alpha_1+...+\bar\alpha_{m-1}$ is different to α. Then $\bar\alpha_1+...+\bar\alpha_{m-1}\in\Phi_{n-1}^+$ by the uniqueness of α. Furthermore, we obtain

$$-(\bar\alpha_1+...+\bar\alpha_{m-1})-\beta=\alpha_{l-n}.$$

Denote by $\bar{w}$ the minimum of the indices of the simple roots $\alpha_w = \bar{\alpha}_i \in \Delta_n$ in $\bar{\alpha}_1+...+\bar{\alpha}_{m-1}$. It follows $\bar{w} > l-n$. Let $n \in \mathbb{N}$ such that $\bar{w} = l-\bar{n}$ holds and assume $l-\bar{n} > l-n+1$. Then $k_2 - 1 = n+1 = \bar{n}+2$ and therefore

$$\sum_{i=1}^{m-1} \bar{\alpha}_i = \sum_{i=l-\bar{n}}^{l+\bar{n}+2-k_2} \alpha_i + 2 \sum_{i=l+\bar{n}+3-k_2}^{l} \alpha_i.$$

We observe that the integer $\langle \beta, \alpha_{l-n} \rangle$ is equal to $\langle \alpha_{l-n}, \alpha_{l-n} \rangle = 2$. We get that the reflection $\sigma_{\alpha_{l-n}}$ maps β to

$$\sigma_{\alpha_{l-n}}(\beta) = \beta - \langle \beta, \alpha_{l-n} \rangle \alpha_{l-n} = \bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} - \alpha_{l-n}.$$

Hence, $l-\bar{n} = l-n+1$ holds. We compute $k_2 - 1 = n+1 = \bar{n}+2$. Applying the induction hypothesis yields that the shape of $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ is

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} = \sum_{i=l-n+1}^{l-1} \alpha_i + 2\alpha_l.$$

Thus, $\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} + \alpha_{l-n}$ is the root constructed above, which contradicts to the assumption $\beta \neq \alpha + \alpha_l$. It remains to check that the sum $\alpha + \alpha_j = \sum_{i=l-n}^{l} \alpha_i + \alpha_j$ with $j \in \{l-n, ..., l-1\}$ is not a root. We compute

$$\langle \alpha + \alpha_j, \alpha_j \rangle = (1 - \delta_{l-n,j}) \langle \alpha_{j-1}, \alpha_j \rangle + 2\langle \alpha_j, \alpha_j \rangle + \langle \alpha_{j+1}, \alpha_j \rangle \geq 2.$$

Hence, the root $\alpha+\alpha_j$ has to be short. But this forces $\langle \alpha+\alpha_j, \alpha_j \rangle = 0$ or $\langle \alpha+\alpha_j, \alpha_j \rangle = \pm 1$. We conclude that $\alpha + \alpha_j$ is not a root. Therefore $\alpha + \alpha_l$ is the unique root of height $\mathrm{ht}(\alpha + \alpha_l) = n+2$ in $\Phi_n^+ \setminus \Phi_{n-1}^+$.
Now let $n+2 < k_2 \leq 2n+1$. By the induction assumption there exists $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\alpha) = k_2 - 1$ and the shape of α is

$$\alpha = \sum_{i=l-n}^{l+n+2-k_2} \alpha_i + 2 \sum_{i=l+n+3-k_2}^{l} \alpha_i.$$

We compute the integer $\langle \alpha, \alpha_{l+n+2-k_2} \rangle$. This reads as

$$\langle \sum_{i=l-n}^{l+n+2-k_2} \alpha_i + 2 \sum_{i=l+n+3-k_2}^{l} \alpha_i, \quad \alpha_{l+n+2-k_2} \rangle =$$
$$\langle \alpha_{l+n+1-k_2}, \alpha_{l+n+2-k_2} \rangle + \langle \alpha_{l+n+2-k_2}, \alpha_{l+n+2-k_2} \rangle + 2\langle \alpha_{l+n+3-k_2}, \alpha_{l+n+2-k_2} \rangle =$$
$$-1 + 2 - 2 = -1.$$

Hence, the reflection $\sigma_{\alpha_{l+n+2-k_2}}$ sends α to

$$\sigma_{\alpha_{l+n+2-k_2}}(\alpha) = \alpha + \alpha_{l+n+2-k_2} = \sum_{i=l-n}^{l+n+1-k_2} \alpha_i + 2 \sum_{i=l+n+2-k_2}^{l} \alpha_i.$$

Thus, $\alpha + \alpha_{l+n+2-k_2}$ is a root of $\mathrm{ht}(\alpha + \alpha_{l+n+2-k_2}) = k_2$ and lies in $\Phi_n^+ \setminus \Phi_{n-1}^+$. Assume there exists $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\beta) = k_2$ and $\beta \neq \alpha + \alpha_{l+n+2-k_2}$. We write β as the sum of simple roots, i.e., $\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_m$ with $\bar{\alpha}_i \in \Delta_n$ in such a way that each partial sum is a root. Suppose that the root $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} \neq \alpha$. Since α is unique in $\Phi_n^+ \setminus \Phi_{n-1}^+$, it has to hold that $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} \in \Phi_{n-1}^+$ and therefore we obtain

$$-(\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}) + \beta = \alpha_{l-n}. \tag{6.4}$$

Let $\bar{w}$ be the minimum of the indices of the simple roots $\alpha_w = \bar{\alpha}_i$ in $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$. Then, we conclude from equation (6.4) that $\bar{w} > n$. Let $\bar{n} \in \mathbb{N}$ such that $\bar{w} = l - \bar{n}$ holds and suppose $l - \bar{n} > l - n + 1$. We deduce that $k_2 - 1 > n + 1 > \bar{n} + 1$. Thus, the induction assumption yields that $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ is of the form

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} = \sum_{i=l-\bar{n}}^{l+\bar{n}+2-k_2} \alpha_i + 2 \sum_{i=l+\bar{n}+3-k_2}^{l} \alpha_i.$$

The integer $\langle \beta, \alpha_{l-n} \rangle = \langle \alpha_{l-n} + \bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}, \alpha_{l-n} \rangle = 2$ implies

$$\sigma_{\alpha_{l-n}}(\beta) = \beta - 2\alpha_{l-n} = \bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} - \alpha_{l-n}. \tag{6.5}$$

Since the right hand side of equation (6.5) is not a root, we get a contradiction. It follows $l - \bar{n} = l - n + 1$. Hence, we have $k_2 - 1 > n + 1 = \bar{n} + 2$ and so the induction assumption yields for the shape of $\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1}$ the following sum of simple roots:

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{m-1} = \sum_{i=l-\bar{n}}^{l+\bar{n}+2-k_2} \alpha_i + 2 \sum_{i=l+\bar{n}+3-k_2}^{l} \alpha_i = \sum_{i=l-n+1}^{l+n+1-k_2} \alpha_i + 2 \sum_{i=l+n+2-k_2}^{l} \alpha_i.$$

But then $\beta + \alpha_{l-n}$ is the root constructed right before. It remains to check that the sum

$$\alpha + \alpha_j = \sum_{i=l-n}^{l+n+2-k_2} \alpha_i + 2 \sum_{i=l+n+3-k_2}^{l} \alpha_i + \alpha_j$$

with $j \in \{l - n, ..., l\} \setminus \{l + n + 2 - k_2\}$ is not a root. For $j \in \{l - n, ..., l + n + 1 - k_2\}$ we obtain

$$\langle \alpha + \alpha_j, \alpha_j \rangle = (1 - \delta_{l-n,j}) \langle \alpha_{j-1}, \alpha_j \rangle + 2\langle \alpha_j, \alpha_j \rangle + \langle \alpha_{j+1}, \alpha_j \rangle \geq 2.$$

Further, for $j \in \{l + n + 3 - k_2, ..., l - 1\}$, where we have to assume $n > 2$ and $k_2 \geq n + 4$, we get

$$\langle \alpha + \alpha_j, \alpha_j \rangle = (2 - \delta_{l+n+3-k_2,j}) \langle \alpha_{j-1}, \alpha_j \rangle + 3\langle \alpha_j, \alpha_j \rangle + 2\langle \alpha_{j+1}, \alpha_j \rangle \geq 2.$$

We conclude, as above, that the root $\alpha + \alpha_j$ is neither short nor long. For $j = l$, we compute

$$\langle \alpha + \alpha_l, \alpha_l \rangle = (2 - \delta_{k_2,n+3}) \langle \alpha_{l-1}, \alpha_l \rangle + 3\langle \alpha_l, \alpha_l \rangle \geq 2.$$

Thus, the root $\alpha + \alpha_l$ has to be long. By [Hum72, Section 10.4, Lemma D] the root of maximal height is long and by [Hum72, Section 10.4, Lemma C] all roots of a given length are conjugate under the Weyl group. Hence there exists $\sigma_{\tilde{\beta}}$ with $\tilde{\beta} \in \Phi_n$ such that

$$\sigma_{\tilde{\beta}}(\alpha + \alpha_l) = \alpha + \alpha_l - \langle \alpha + \alpha_l, \tilde{\beta} \rangle \tilde{\beta} = \alpha_{l-n} + 2 \sum_{i=l-n+1}^{l} \alpha_i.$$

Let $\tilde{\beta} = \sum_{i=l-n}^{l} k_i\alpha_i \in \Phi_n^+$. Then we obtain $k_l = 1$ and $\langle \alpha + \alpha_l, \tilde{\beta}\rangle = 1$. This forces $k_{l-n+1} = -1$ which is impossible. Similarly we deduce that $\alpha + \alpha_l$ is not long for $\tilde{\beta} \in \Phi_n^-$. Hence $\alpha+\alpha_l$ is not a root. We conclude that there is no possibility for the sum $\alpha+\alpha_j$ with $j \in \{l-n, ..., l\} \setminus \{l+n+2-k_2\}$ to be a root. This completes the second inner induction. Hence, the outer induction is complete and the second point of the lemma follows.
Now we show the third point of the lemma.
If $\alpha \in \Phi_n^+ \setminus \{\Phi_{n-1}^+ \cup \{\gamma_{l-n} = 2\sum_{i=l-n+1}^{l} \alpha_i + \alpha_{l-n}\}\}$, then $\mathrm{ht}(\alpha) = k < \mathrm{ht}(\gamma_{l-n})$. In particular, Lemma 6.1.2 yields that there exists a unique root $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ such that $\mathrm{ht}(\beta) = k+1 \leq \mathrm{ht}(\gamma_{l-n})$. Obviously, the simple root $\beta - \alpha \in \Delta$ satisfies the stated property. Let $\tilde{\alpha} \in \Delta$ be different from $\beta - \alpha$ and let $\beta - \tilde{\alpha}$ be a root. By the uniqueness of α we obtain $\beta - \tilde{\alpha} \notin \Phi_n^+ \setminus \Phi_{n-1}^+$. Therefore, $\beta - \tilde{\alpha} \in \Phi_{n-1}^+$ holds.
Finally, we prove the last assertion of the lemma.
Obviously, we have $\Phi \supseteq (\bigcup_{i=1}^{l}(\Phi_i \setminus \Phi_{i-1})) \cup \Phi_0$. Let $\alpha = \sum_{i=1}^{l} k_i\alpha_i \in \Phi$ and let $j \in \{1, ..., l\}$ be minimal with $k_j \neq 0$. Thus, α is an element of $\Phi_j \setminus \Phi_{j-1}$ or $\alpha \in \Phi_0$ if $j = l$. We obtain the disjoint union $\Phi = (\bigcup_{i=1}^{l}(\Phi_i \setminus \Phi_{i-1})) \cup \Phi_0$. □

Lemma 6.2. *Let $n \in \{1, ..., l-1\}$. We denote by $\gamma_i = 2\sum_{j=i+1}^{l} \alpha_j + \alpha_i$ the root of maximal height in Φ_{l-i}^-. Moreover, as before we define the set $\Phi_0 = \{\pm\alpha_l\}$. Then for $A_0 = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n-1} a_{\gamma_i} X_{\gamma_i} + \sum_{\beta\in\Phi_n^-} a_\beta X_\beta$ with a_γ, $a_\beta \in F$ there exists $U \in \mathcal{U}^-$ such that*

$$UA_0U^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n} \bar{a}_{\gamma_i} X_{\gamma_i} + \sum_{\beta\in\Phi_{n-1}^-} \bar{a}_\beta X_\beta.$$

Proof. Let $k \in \{1, ..., 2n\}$ and set

$$A_{k-1} := \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n-1} a_{\gamma_i} X_{\gamma_i} + \sum_{\beta\in\Phi_{n-1}^-} a_\beta X_\beta + \sum_{\alpha\in\Phi_n^-\setminus\Phi_{n-1}^-, ht(\alpha)\geq k} a_\alpha X_\alpha$$

with suitable a_α, a_β, $a_{\gamma_i} \in F$. To simplify the notation we write sometimes $\bar{\Phi}_n^-$ for $\Phi_n^- \setminus \Phi_{n-1}^-$ and $\mathbf{L}$ for $\mathrm{Lie}(\mathrm{SO}_{2l+1})$. We will prove the following claim: For the matrix A_{k-1} there exists $U \in \mathcal{U}^-$ such that

$$UA_{k-1}U^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n-1} \bar{a}_{\gamma_i} X_{\gamma_i} + \sum_{\beta\in\Phi_{n-1}^-} \bar{a}_\beta X_\beta + \sum_{\alpha\in\bar{\Phi}_n^-, ht(\alpha)>k} \bar{a}_\alpha X_\alpha$$

with $\bar{a}_{\gamma_i}$, $\bar{a}_\beta$, $\bar{a}_\alpha \in F$.
We are going to remove the root $\alpha \in \Phi_n^- \setminus \Phi_{n+1}^-$ with $\mathrm{ht}(\alpha) = k$. Then by Lemma 6.1.3 there exists a root $\bar{\alpha} \in \Delta$ such that $-\alpha + \bar{\alpha} = \bar{\beta} \in \Phi_n^- \setminus \Phi_{n+1}^-$ with $\mathrm{ht}(\bar{\beta}) = k+1$. Thus, for $\hat{\beta} := -\bar{\beta} \in \Phi_n^- \setminus \Phi_{n+1}^-$, we have $\hat{\beta} + \bar{\alpha} = \alpha$. Therefore, we are going to differentially conjugate A_{k-1} by the parametrized root group element $U_{\hat{\beta}}(\zeta) \in \mathcal{U}_{\hat{\beta}}$. With Observation 3.4

this leads to

$$\begin{aligned} U_{\hat\beta}(\zeta)A_{k-1}U_{\hat\beta}(\zeta)^{-1}+\partial(U_{\hat\beta}(\zeta))U_{\hat\beta}(\zeta)^{-1} &= \sum_{i=1}^{l}\mathrm{Ad}(U_{\hat\beta}(\zeta))(X_{\alpha_i})+\sum_{i=1}^{l-n-1}a_{\gamma_i}\mathrm{Ad}(U_{\hat\beta}(\zeta))(X_{\gamma_i}) \\ &+\sum_{\beta\in\Phi_{n-1}^-}a_\beta\mathrm{Ad}(U_{\hat\beta}(\zeta))(X_\beta)+\sum_{\alpha\in\bar\Phi_n^-,ht(\alpha)\geq k}a_\alpha\mathrm{Ad}(U_{\hat\beta}(\zeta))(X_\alpha)+\partial(U_{\hat\beta}(\zeta))U_{\hat\beta}(\zeta)^{-1}. \end{aligned} \tag{6.6}$$

For the first summand of the right hand side of equation (6.6), we obtain with the help of Lemma 3.2

$$\sum_{i=1}^{l}\mathrm{Ad}(U_{\hat\beta}(\zeta))(X_{\alpha_i})=\sum_{i=1}^{l}\sum_{j=0}^{q}m_{\hat\beta,\alpha_i,j}\zeta^jX_{\alpha_i+j\hat\beta}. \tag{6.7}$$

First let $j=1$. Then by the choice of $\hat\beta$, i.e., the above discussion, and Lemma 6.1.3 there exists a unique simple root $\bar\alpha_i\in\Delta$ such that $\hat\beta+\bar\alpha_i=\alpha$, and if there is another simple root $\tilde\alpha_j\in\Delta$ with $\tilde\alpha_j\neq\bar\alpha_i$ such that $\hat\beta+\tilde\alpha_j$ is a root, then $\hat\beta+\tilde\alpha_j\in\Phi_{n-1}^-$. Now let $j>1$. Note that if $\alpha_i+j\hat\beta$ is a root, then $\alpha_i+j\hat\beta\in\Phi_n^-\setminus\Phi_{n-1}^-$. Furthermore, since $\mathrm{ht}(\hat\beta)=k+1$, it holds that $\mathrm{ht}(\alpha_i+j\hat\beta)=j(k+1)-1>k$. Therefore, equation (6.7) translates into

$$\sum_{i=1}^{l}\mathrm{Ad}(U_{\hat\beta}(\zeta))(X_{\alpha_i})\in\sum_{i=1}^{l}X_{\alpha_i}+m_{\hat\beta,\bar\alpha,1}\zeta X_\alpha+\sum_{\beta\in\Phi_{n-1}^-}\mathbf{L}_\beta+\sum_{\beta\in\bar\Phi_n^-,ht(\beta)>k}\mathbf{L}_\beta. \tag{6.8}$$

As above, the second summand can be written as

$$\sum_{i=1}^{l-n-1}a_{\gamma_i}\mathrm{Ad}(U_{\hat\beta}(\zeta))(X_{\gamma_i})=\sum_{i=1}^{l-n-1}a_{\gamma_i}\sum_{j=0}^{q}m_{\hat\beta,\gamma_i,j}\zeta^jX_{\gamma_i+j\hat\beta}. \tag{6.9}$$

Since $\hat\beta\in\Phi_n^-\setminus\Phi_{n-1}^-$ and for $i\in\{1,...,l-n-1\}$ the γ_i are the roots of maximal height in Φ_{l-i}^-, we conclude that the sum $\gamma_i+j\hat\beta$ can not be a root for $j>0$. Thus, equation (6.9) leads to

$$\sum_{i=1}^{l-n-1}a_{\gamma_i}\mathrm{Ad}(U_{\hat\beta}(\zeta))(X_{\gamma_i})=\sum_{i=1}^{l-n-1}a_{\gamma_i}X_{\gamma_i}. \tag{6.10}$$

We compute the third summand of the right hand side of equation (6.6) with the help of Lemma 3.2 and obtain

$$\sum_{\bar\beta\in\Phi_{n-1}^-}a_{\bar\beta}\mathrm{Ad}(U_{\hat\beta}(\zeta))(X_{\bar\beta})=\sum_{\bar\beta\in\Phi_{n-1}^-}a_{\bar\beta}\sum_{j=0}^{q}m_{\hat\beta,\bar\beta,j}\zeta^jX_{\bar\beta+j\hat\beta}. \tag{6.11}$$

It is easily seen that if $\bar\beta+j\hat\beta$ is a root, then $\bar\beta+j\hat\beta\in\Phi_n^-\setminus\Phi_{n-1}^-$ and $\mathrm{ht}(\bar\beta+j\hat\beta)>k+1$. Hence, equation (6.11) can be reformulated as

$$\sum_{\bar\beta\in\Phi_{n-1}^-}a_{\bar\beta}\mathrm{Ad}(U_{\hat\beta}(\zeta))(X_{\bar\beta})\in\sum_{\bar\beta\in\Phi_{n-1}^-}a_{\bar\beta}X_{\bar\beta}+\sum_{\beta\in\bar\Phi_n^-,ht(\beta)>k+1}\mathbf{L}_\beta. \tag{6.12}$$

For the fourth summand, we get by Lemma 3.2

$$\sum_{\alpha\in\bar{\Phi}_n^-, ht(\alpha)\geq k} a_\alpha \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\alpha) = \sum_{\alpha\in\bar{\Phi}_n^-, ht(\alpha)\geq k} a_\alpha \sum_{j=0}^{q} m_{\hat{\beta},\alpha,j}\zeta^j X_{\alpha+j\hat{\beta}}. \tag{6.13}$$

If $\alpha + j\hat{\beta}$ is a root for $j > 0$, then it obviously holds that $\alpha + j\hat{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$. Moreover, $\mathrm{ht}(\hat{\beta}) = k+1$ implies $\mathrm{ht}(\alpha + j\hat{\beta}) > k+1$. This yields for equation (6.13)

$$\sum_{\alpha\in\bar{\Phi}_n^-, ht(\alpha)\geq k} a_\alpha \mathrm{Ad}(U_{\hat{\beta}}(\zeta))(X_\alpha) \in \sum_{\alpha\in\bar{\Phi}_n^-, ht(\alpha)\geq k} a_\alpha X_\alpha + \sum_{\alpha\in\bar{\Phi}_n^-, ht(\alpha)> k+1} \mathbf{L}_\alpha. \tag{6.14}$$

Proposition 3.5 states that the logarithmic derivative $l\delta$ maps an element $U_\beta(\zeta) \in \mathcal{U}_\beta$ to $\mathrm{Lie}(\mathcal{U}_\beta)$ for a root $\beta \in \Phi$. Therefore, we have

$$l\delta(U_{\hat{\beta}}(\zeta)) = \partial(U_{\hat{\beta}}(\zeta))U_{\hat{\beta}}(\zeta)^{-1} \in \mathbf{L}_{\hat{\beta}} \tag{6.15}$$

with $\mathrm{ht}(\hat{\beta}) = k+1$ and $\hat{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$. Putting the equations (6.8), (6.10), (6.12), (6.14) and (6.15) together yields

$$\begin{aligned} U_{\hat{\beta}}(\zeta)A_{k-1}U_{\hat{\beta}}(\zeta)^{-1} + \partial(U_{\hat{\beta}}(\zeta))U_{\hat{\beta}}(\zeta)^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n-1} a_{\gamma_i}X_{\gamma_i} + \\ \zeta X_\alpha + a_\alpha X_\alpha + \sum_{\beta\in\Phi_{n-1}^-} \mathbf{L}_\beta + \sum_{\beta\in\bar{\Phi}_n^-, ht(\beta)>k} \mathbf{L}_\beta. \end{aligned}$$

Hence, if we set $m_{\hat{\beta},\bar{\alpha},1}\zeta = -a_\alpha$ the claim follows. Using the claim one proves by induction that for each $k \in \{1, ..., 2n\}$ there exists $U \in \mathcal{U}^-$ such that

$$UA_0U^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n-1} \mathbf{L}_{\gamma_i} + \sum_{\beta\in\Phi_{n-1}^-} \mathbf{L}_\beta + \sum_{\beta\in\bar{\Phi}_n^-, ht(\beta)>k} \mathbf{L}_\beta.$$

In particular, for $k = 2n$, we get the assertion of the lemma. □

Lemma 6.3. *Let* $A \in \sum_{i=1}^{l} X_{\alpha_i} + \mathbf{H} + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SO}_{2l+1})_\beta = \sum_{i=1}^{l} X_{\alpha_i} + \mathrm{Lie}(\mathcal{B}^-)$ *and define* $\mathcal{M} = \{-\gamma_i = -2\sum_{j=i+1}^{l} \alpha_j - \alpha_i \mid i \in \{1, ..., l-1\}\} \cup \{-\gamma_l := -\alpha_l\}$. *Then there exists* $U \in \mathcal{U}^-$ *such that*

$$UAU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\alpha\in\mathcal{M}} \mathrm{Lie}(\mathrm{SO}_{2l+1})_\alpha.$$

Proof. For $k \in \{1, ..., l\}$, let $A_k = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k}^{l} a_iH_i + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SO}_{2l+1})_\beta$ with suitable $a_i \in F$. First we will prove the following claim: For the matrix A_k there exists $U \in \mathcal{U}^-$ such that

$$UA_kU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k+1}^{l} a_iH_i + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SO}_{2l+1})_\beta.$$

We write $A_k = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k}^{l} a_i H_i + \sum_{\beta\in\Phi^-} a_\beta X_\beta$ with suitable $a_\beta \in F$. To remove $a_k H_k$, we differentially conjugate A_k by $U_{-\alpha_k}(\zeta) \in \mathcal{U}_{-\alpha_k}$. With Observation 3.4 we get

$$\begin{aligned} U_{-\alpha_k}(\zeta)A_k U_{-\alpha_k}(\zeta)^{-1} + \partial(U_{-\alpha_k}(\zeta))U_{-\alpha_k}(\zeta)^{-1} = \sum_{i=1}^{l} \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_i} \\ + \sum_{i=k}^{l} a_i \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) + \sum_{\beta\in\Phi^-} a_\beta \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_\beta). \end{aligned} \tag{6.16}$$

We begin with the first summand of the right hand side of equation (6.16). With the help of Lemma 3.2, for $i \neq k$, we get

$$\mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_i}) = \sum_{j\geq 0} m_{-\alpha_k,\alpha_i,j}\zeta^j X_{\alpha_i + j(-\alpha_k)},$$

and for $i = k$, we have $\mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_k}) \in X_{\alpha_k} + \zeta H_k + \mathrm{Lie}(\mathrm{SO}_{2l+1})_{-\alpha_k}$. Furthermore, the sum $\alpha_i - j\alpha_k$ is not a root for $i \neq k$ and $j \geq 1$. This yields

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_i}) \in \sum_{i=1}^{l} X_{\alpha_i} + \zeta H_k + \mathrm{Lie}(\mathrm{SO}_{2l+1})_{-\alpha_k}. \tag{6.17}$$

Now we investigate the second summand. Then Lemma 3.2 yields for $l \geq i \geq k+1$

$$\sum_{i=k+1}^{l} a_i \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) = \sum_{i=k+1}^{l} a_i (H_i - \langle \alpha_i, \alpha_k\rangle \zeta X_{-\alpha_k}),$$

and it implies for $i = k$ the equation $a_k \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) = a_k(H_k - 2\zeta X_{-\alpha_k})$. These results lead to

$$\sum_{i=k}^{l} a_i \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) \in \sum_{i=k}^{l} a_i H_i + \mathrm{Lie}(\mathrm{SO}_{2l+1})_{-\alpha_k}. \tag{6.18}$$

Since $\beta \in \Phi^-$, we get $\beta - j\alpha_k \in \Phi^-$ for $j \geq 0$. Hence, we conclude for the third summand

$$\sum_{\beta\in\Phi^-} a_\beta \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_\beta) \in \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SO}_{2l+1})_\beta. \tag{6.19}$$

We handle the last summand with Proposition 3.5. It implies

$$l\delta(U_{-\alpha_k}(\zeta)) = \partial(U_{-\alpha_k}(\zeta))U_{-\alpha_k}(\zeta)^{-1} \in \mathrm{Lie}(\mathrm{SO}_{2l+1})_{-\alpha_k}. \tag{6.20}$$

Thus, if we put the equations (6.17), (6.18), (6.19) and (6.20) together and set $\zeta = -a_k$, the assumption of the claim follows. Using the claim one proves by induction that for each $k \in \{1, ..., l\}$ there exists $U \in \mathcal{U}^-$ such that

$$UAU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k+1}^{l} a_i H_i + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SO}_{2l+1})_\beta.$$

In particular, for $k = l$, there exists $U \in \mathcal{U}^-$ such that $A_0 = UAU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SO}_{2l+1})_\beta$. Again, using induction and Lemma 6.2, one shows that for each $n \in \{1, ..., l-1\}$ and A_0 there exists $U \in \mathcal{U}^-$ such that

$$UA_0U^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=1}^{l-n} \mathrm{Lie}(\mathrm{SO}_{2l+1})_{\gamma_i} + \sum_{\beta\in\Phi^-_{n-1}} \mathrm{Lie}(\mathrm{SO}_{2l+1})_\beta.$$

Since Φ^-_0 is defined as $\Phi^-_0 = \{ -\gamma_l \}$ where $-\gamma_l := -\alpha_l$, the case $n = 1$ yields the assertion of the lemma. □

6.3 The equation with group SO_{2l+1}

In the next step we combine the results of Corollary 3.12 and Lemma 6.3. Later, we make use of Corollary 6.4, when we are going to apply the specialization bound. Denote by $(C(z), \partial = \frac{d}{dz})$ a rational function field with standard derivation as in Section 3.4 and keep the notations of Lemma 6.3.

Corollary 6.4. *We apply Corollary 3.12 to the group* SO_{2l+1} *and the above Cartan Decomposition. We denote by* $A^{M\&S}_{\mathrm{SO}_{2l+1}}$ *the matrix satisfying the stated conditions of Corollary 3.12. Then there exists* $U \in \mathcal{U}^-_0 \subset \mathrm{SO}_{2l+1}$ *such that*

$$\bar{A}_{\mathrm{SO}_{2l+1}} := UA^{M\&S}_{\mathrm{SO}_{2l+1}}U^{-1} + \partial(U)U^{-1} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in\mathcal{M}} f_i X_{\gamma_i} \tag{6.21}$$

with at least one $f_i \in C[z] \setminus C$ *and the differential Galois group of the matrix equation* $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SO}_{2l+1}}\boldsymbol{y}$ *is* $\mathrm{SO}_{2l+1}(C)$ *over* $C(z)$.

Proof. Lemma 6.3 proves the existence of an element $U \in \mathcal{U}^-_0 \subset \mathrm{SO}_{2l+1}$ such that equation (6.21) holds. Since differential conjugation defines a differential isomorphism, we deduce from Corollary 3.12 that the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SO}_{2l+1}}\boldsymbol{y}$ again is $\mathrm{SO}_{2l+1}(C)$ over $C(z)$. We still need to show the existence of $f_i \in C[z]\backslash C$ for some $\gamma_i \in \mathcal{M}$. Suppose $\bar{A}_{\mathrm{SO}_{2l+1}} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in T} f_i X_{\gamma_i} \in \mathrm{Lie}(\mathrm{SO}_{2l+1})(C)$. Then by Lemma 6.5 the corresponding differential equation $L(y, f_1, ..., f_l) \in C\{y\}$ has coefficients in C. But then by [Mag94, Corollary 3.28] the differential Galois group has to be abelian. Thus, $\bar{A}_{\mathrm{SO}_{2l+1}} \in \mathrm{Lie}(\mathrm{SO}_{2l+1})(C(z))\backslash\mathrm{Lie}(\mathrm{SO}_{2l+1})(C)$. Since $0 \neq A_1 \in \mathbf{H}(C)$ and $A = (z^2A_1 + A_0)$ in Corollary 3.12, we start our transformation with at least one coefficient lying in $C[z]\backslash C$. In each step the application of $\mathrm{Ad}(U_\beta(\zeta))$ generates at most new entries which are polynomials in ζ. Moreover, the logarithmic derivative is the product of the two matrices $\partial(U_\beta(\zeta))$ and $U_\beta(\zeta)^{-1} = U_\beta(-\zeta)$. In the proofs of Lemma 6.3 and Lemma 6.2 we choose the parameter ζ to be one of the coefficients. Hence, it holds $f_i \in C[z]$. □

To obtain parametric equations for the series SO_{2l+1} we change the differential ground field. Therefore, let $t_1, ..., t_l$ be differential indeterminates and define the differential field $F = C\langle t_1, ..., t_l\rangle$. Furthermore, we define the matrix $A_{\mathrm{SO}_{2l+1}}(t_1, ..., t_l)$ as

$$A_{\mathrm{SO}_{2l+1}}(t_1, ..., t_l) = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\beta\in\mathcal{M}} \frac{1}{2} t_\beta X_\beta$$

where $\mathcal{M}$ is as in Lemma 6.3. The next step is to compute an operator for SO_{2l+1} from the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SO}_{2l+1}}\boldsymbol{y}$.

Lemma 6.5. *The matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SO}_{2l+1}}(t_1, ..., t_l)\boldsymbol{y}$ is differentially equivalent to the homogeneous scalar linear differential equation*

$$L(y, t_1, ..., t_l) = y^{(2l+1)} - \sum_{i=1}^{l}(-1)^{i-1}((t_i\, y^{(l+1-i)})^{(l-i)} + (t_i\, y^{(l-i)})^{(l+1-i)}) = 0.$$

Proof. The description of the Lie algebra of SO_{2l+1} in Section 6.1 yields that the shape of the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SO}_{2l+1}}(t_1, ..., t_l)\boldsymbol{y}$ is

$$\begin{pmatrix} \partial(y_1) \\ \partial(y_2) \\ \vdots \\ \\ \vdots \\ \partial(y_{2l}) \\ \partial(y_{2l+1}) \end{pmatrix} = \left(\begin{array}{cccc|c|cccc} 0 & 1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & & 1 & & & & & \\ & & & 0 & 2 & & & & \\ \hline & & & \frac{1}{2}t_1 & 0 & -1 & & & \\ \hline & & \frac{1}{2}t_2 & 0 & -t_1 & 0 & -1 & & \\ & \iddots & & -\frac{1}{2}t_2 & & & \ddots & \ddots & \\ \frac{1}{2}t_l & & \iddots & & & & & & -1 \\ 0 & -\frac{1}{2}t_l & & & & & & & 0 \end{array}\right) \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \\ \vdots \\ y_{2l} \\ y_{2l+1} \end{pmatrix}. \tag{I}$$

To simplify notation we will write y_i' for $\partial(y_i)$. Then equation (I) is equivalent to the following system of equations

$$y_1' = y_2 \tag{1}$$

$$\vdots$$

$$y_{l-1}' = y_l \tag{l-1}$$

$$y_l' = 2y_{l+1} \tag{l}$$

$$y_{l+1}' = \frac{1}{2}t_1 y_l - y_{l+2} \tag{l+1}$$

$$y_{l+2}' = \frac{1}{2}t_2 y_{l-1} - t_1 y_{l+1} - y_{l+3} \tag{l+2}$$

$$y_{l+3} = \frac{1}{2}t_3 y_{l-2} - \frac{1}{2}t_2 y_l - y_{l+4} \tag{l+3}$$

$$\vdots$$

$$y_{l+k}' = \frac{1}{2}t_k y_{l-k+1} - \frac{1}{2}t_{k-1} y_{l-k+3} - y_{l+k+1} \quad 3 \leq k \leq l \tag{l+k}$$

$$\vdots$$

$$y_{2l}' = \frac{1}{2}t_l y_1 - \frac{1}{2}t_{l-1} y_3 - y_{2l+1} \tag{2l}$$

$$y_{2l+1}' = -\frac{1}{2}t_l y_2. \tag{2l+1}$$

Let $l \geq 3$. We are going to prove the following claim. For $k \in \{3, ..., l\}$, the corresponding subsystem

$$\begin{aligned}
y'_{k-1} &= y_k \\
&\vdots \\
y'_{l-1} &= y_l \\
y'_l &= 2y_{l+1} \\
y'_{l+1} &= \frac{1}{2}t_1 y_l - y_{l+2} \\
y'_{l+2} &= \frac{1}{2}t_2 y_{l-1} - t_1 y_{l+1} - y_{l+3} \\
y'_{l+3} &= \frac{1}{2}t_3 y_{l-2} - \frac{1}{2}t_2 y_l - y_{l+4} \\
&\vdots \\
y'_{2l-k+3} &= \frac{1}{2}t_{l-k+3} y_{k-2} - \frac{1}{2}t_{l-k+2} y_k - y_{2l-k+4}
\end{aligned}$$

leads to the differential equation

$$\begin{aligned}
y_{k-1}^{(2(l-k)+5)} = &\sum_{i=1}^{l+2-k} (-1)^{i-1} (t_i y_{k-1}^{(l+3-k-i)})^{(l+2-k-i)} + (t_i y_{k-1}^{(l+2-k-i)})^{(l+3-k-i)} \\
&+ (-1)^{l+2-k} (t_{l+3-k} y_{k-2} - 2y_{2l+4-k}).
\end{aligned}$$

The proof will be done by backwards induction. Let $k = l$. Then the subsystem consists of the five equations

$$y'_{l-1} = y_l \tag{l-1}$$
$$y'_l = 2y_{l+1} \tag{l}$$
$$y'_{l+1} = \frac{1}{2}t_1 y_l - y_{l+2} \tag{l+1}$$
$$y'_{l+2} = \frac{1}{2}t_2 y_{l-1} - t_1 y_{l+1} - y_{l+3} \tag{l+2}$$
$$y'_{l+3} = \frac{1}{2}t_3 y_{l-2} - \frac{1}{2}t_2 y_l - y_{l+4}. \tag{l+3}$$

The first two equations imply $y''_{l-1} = 2y_{l+1}$. Now we differentiate again and substitute y'_{l+1} by equation (l+1). This yields

$$y'''_{l-1} = t_1 y_l - 2y_{l+2} = t_1 y'_{l-1} - 2y_{l+2}. \tag{II}$$

If we repeat this process for equation (II), this time making use of equation (l+2) and $y''_{l-1} = 2y_{l+1}$, we obtain

$$y_{l-1}^{(4)} = (t_1\, y'_{l-1})' + t_1 y''_{l-1} - t_2 y_{l-1} + 2y_{l+3}. \tag{III}$$

We differentiate equation (III) and substitute y'_{l+3} by equation (l+3). Then we get

$$y_{l-1}^{(5)} = (t_1\, y'_{l-1})'' + (t_1 y''_{l-1})' - (t_2 y_{l-1})' - t_2 y'_{l-1} + t_3 y_{l-2} - 2y_{l+4}.$$

Note that in the case $l = 2$ we have to omit equation (l+3) and to use instead $y_5' = -\frac{1}{2}t_2y_2$. Then the differentiation of equation (III) and the substitution of y_5' by $y_5' = -\frac{1}{2}t_2y_1$ implies

$$y_1^{(5)} = (t_1\, y_1')'' + (t_1y_1'')' - (t_2y_1)' - t_2y_1'.$$

Now let $k < l$. Then the subsystem of equations is:

$$\begin{aligned}
y_{k-1}' &= y_k && \text{(k-1)}\\
y_k' &= y_{k+1} && \text{(k)}\\
&\vdots\\
y_{l-1}' &= y_l && \text{(l-1)}\\
y_l' &= 2y_{l+1} && \text{(l)}\\
y_{l+1}' &= \frac{1}{2}t_1y_l - y_{l+2} && \text{(l+1)}\\
y_{l+2}' &= \frac{1}{2}t_2y_{l-1} - t_1y_{l+1} - y_{l+3} && \text{(l+2)}\\
y_{l+3}' &= \frac{1}{2}t_3y_{l-2} - \frac{1}{2}t_2y_l - y_{l+4} && \text{(l+3)}\\
&\vdots\\
y_{2l-k+2}' &= \frac{1}{2}t_{l-k+2}y_{k-1} - \frac{1}{2}t_{l-k+1}y_{k+1} - y_{2l-k+3} && \text{(2l+2-k)}\\
y_{2l-k+3}' &= \frac{1}{2}t_{l-k+3}y_{k-2} - \frac{1}{2}t_{l-k+2}y_k - y_{2l-k+4}. && \text{(2l+3-k)}
\end{aligned}$$

Then the induction assumption yields for $k+1$, i.e., for the subsystem formed by equation (k) up to equation (2l+2-k), the differential equation

$$\begin{aligned}
y_k^{(2(l-k)+3)} &= \sum_{i=1}^{l+1-k} (t_i\, y_k^{(l+2-k-i)})^{(l+1-k-i)} + (t_i\, y_k^{(l+1-k-i)})^{(l+2-k-i)}\\
&\quad + (-1)^{l+1-k}(t_{l+2-k}\, y_{k-1} - 2y_{2l+3-k}). \qquad \text{(IV)}
\end{aligned}$$

We substitute in equation (IV) y_k by y_{k-1}'. The result is

$$\begin{aligned}
y_{k-1}^{(2(l-k)+4)} &= \sum_{i=1}^{l+1-k} (-1)^{i-1}((t_i\, y_{k-1}^{(l+3-k-i)})^{(l+1--i)} + (t_i\, y_{k-1}^{(l+2-k-i)})^{(l+2-k-i)})\\
&\quad + (-1)^{l+1-k}(t_{l+2-k}\, y_{k-1} - 2y_{2l+3-k}). \qquad \text{(V)}
\end{aligned}$$

Now we differentiate equation (V) and use equation $(2l+3-k)$ and $(k-1)$ to deduce

$$\begin{aligned}
y_{k-1}^{(2(l-k)+5)} &= \sum_{i=1}^{l+1-k} (-1)^{i-1}(t_i\, y_{k-1}^{(l+3-k-i)})^{(l+2-k-i)} + (t_i\, y_{k-1}^{(l+2-k-i)})^{(l+3-k-i)}\\
&\quad + (-1)^{l+1-k}((t_{l+2-k}\, y_{k-1})' - t_{l-k+3}\, y_{k-2} + t_{l-k+2}y_{k-1}' + 2y_{2l-k+4})\\
&= \sum_{i=1}^{l+2-k} (-1)^{i-1}(t_i\, y_{k-1}^{(l+3-k-i)})^{(l+2-k-i)} + (t_i\, y_{k-1}^{(l+2-k-i)})^{(l+3-k-i)}\\
&\quad + (-1)^{l+2-k}(t_{l+3-k}y_{k-2} - 2y_{2l-k+4}).
\end{aligned}$$

Thus, the claim is shown. Now we return to the proof of the lemma. Therefore, we consider the full system of equations. Then the claim yields for the subsystem, obtained by leaving out the first and last equation (i.e., the case $k = 3$) the equation

$$y_{k-1}^{(2(l-k)+5)} = \sum_{i=1}^{l-1}(-1)^{i-1}(t_i\, y_2^{(l-i)})^{(l-1-i)} + (t_i\, y_2^{(l-1-i)})^{(l-i)} + (-1)^{l-1}(t_l y_1 - 2y_{2l+1}). \quad \text{(VI)}$$

With the help of the first equation of the full system, we obtain for equation (VI)

$$\begin{aligned} y_1^{(2l)} &= \sum_{i=1}^{l-1}(-1)^{i-1}(t_i\, y_1^{(l-i+1)})^{(l-1-i)} + (t_i\, y_1^{(l-i)})^{(l-i)} \\ &+ (-1)^{l-1}(t_l y_1 - 2y_{2l+1}). \end{aligned} \quad \text{(VII)}$$

Finally, we differentiate equation (VII) and use $y'_{2l+1} = -\frac{1}{2}t_l y_2 = -\frac{1}{2}t_l y'_1$ to get an expression only in y_1. This leads to

$$\begin{aligned} y_1^{(2l+1)} &= \sum_{i=1}^{l-1}(-1)^{i-1}(t_i\, y_1^{(l-i+1)})^{(l-i)} + (t_i\, y_1^{(l-i)})^{(l-i+1)} \\ &+ (-1)^{l-1}((t_l y_1)' + t_l y'_1) \\ &= \sum_{i=1}^{l}(-1)^{i-1}(t_i\, y_1^{(l-i+1)})^{(l-i)} + (t_i\, y_1^{(l-i)})^{(l-i+1)}. \end{aligned}$$

□

Theorem 6.6. *Let C be an algebraically closed field of characteristic zero, $t_1, ..., t_l$ differential indeterminates and $F = C\langle t_1, ..., t_l\rangle$ the corresponding differential field. Then the homogeneous linear differential equation*

$$L(y, t_1, ..., t_l) = y^{(2l+1)} - \sum_{i=1}^{l}(-1)^{i-1}((t_i\, y^{(l+1-i)})^{(l-i)} + (t_i\, y^{(l-i)})^{(l+1-i)}) = 0$$

has $\mathrm{SO}_{2l+1}(C)$ as differential Galois group over F. Moreover, let $\hat{F}$ be a differential field with field of constants equal to C. Let $\hat{E}$ be a Picard-Vessiot extension over $\hat{F}$ with differential Galois group $\mathrm{SO}_{2l+1}(C)$ and suppose the defining matrix differential equation $\partial(\boldsymbol{y}) = \hat{A}\boldsymbol{y}$ satisfies $\hat{A} \in \sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}_0^-)$. Then there is a specialization $L(y, \hat{t}_1, ..., \hat{t}_l)$ with $\hat{t}_i \in \hat{F}$ such that $L(y, \hat{t}_1, ..., \hat{t}_l)$ gives rise to the extension $\hat{E}$ over $\hat{F}$.

Proof. Let E be a Picard-Vessiot extension for the equation $L(y, t_1, ..., t_l) = 0$ over F and denote by $\mathcal{G}$ its differential Galois group. Since the operator comes from the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SO}_{2l+1}}(t_1, ..., t_l)\boldsymbol{y}$ with $A_{\mathrm{SO}_{2l+1}}(t_1, ..., t_l) \in \mathrm{Lie}(\mathrm{SO}_{2l+1})(F)$, Proposition 2.1 yields $\mathcal{G}(C) \leq \mathrm{SO}_{2l+1}(C)$. By Corollary 6.4 there exists a specialization $\sigma : (t_1, ..., t_l) \to (f_1, ..., f_l)$ with $f_1, ..., f_l \in C[z]$ such that $\sigma(A_{\mathrm{SO}_{2l+1}}(t_1, ..., t_l)) = \bar{A}_{\mathrm{SO}_{2l+1}}$ and the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SO}_{2l+1}}\boldsymbol{y}$ is $\mathrm{SO}_{2l+1}(C)$. Moreover, we have $C\{f_1, ..., f_l\} = C[z]$. Thus we can apply Corollary 2.15. This yields $\mathrm{SO}_{2l+1}(C) \leq \mathcal{G}(C)$. Hence, it holds $\mathcal{G}(C) = \mathrm{SO}_{2l+1}(C)$.

Since the defining matrix $\hat{A}$ satisfies $\hat{A} \in \sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}_0^-)$, Lemma 6.3 provides that $\hat{A}$ is differentially equivalent to a matrix $\tilde{A} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in T} \hat{a}_i X_{\gamma_i}$ with suitable $\hat{a}_i \in \hat{F}$. Obviously the specialization

$$\hat{\sigma} : (t_1, ..., t_l) \mapsto (\hat{a}_1, ..., \hat{a}_l)$$

does the required. $\square$

Chapter 7

A parametrized equation for SO_{2l}

7.1 The Lie algebra of SO_{2l} (type D_l)

We begin this paragraph with the abstract definition of the root system Φ of type D_l. Let $l \in \mathbb{N} \setminus \{0\}$ with $l \geq 4$. We denote by $\epsilon_1, ..., \epsilon_l$ the standard orthonormal basis of $\mathbb{R}^l$ and by $(\cdot, \cdot)$ the standard inner product of $\mathbb{R}^l$. In [Hum72, Section 12.1] it is shown that the root system Φ of type D_l consists of the vectors

$$\Phi = \{\pm(\epsilon_i - \epsilon_j),\ \pm(\epsilon_i + \epsilon_j) \mid 1 \leq i < j \leq l\}\,.$$

The set Δ which is formed by the l independent vectors

$$\Delta := \{\ \alpha_i := \epsilon_i - \epsilon_{i+1},\ \alpha_l := \epsilon_{l-1} + \epsilon_l \mid 1 \leq i \leq l-1\}$$

defines a basis of Φ. The Cartan matrix of type D_l has shape

$$\begin{pmatrix}
2 & -1 & 0 & & . & . & . & & & 0 \\
-1 & 2 & -1 & & . & . & . & & & 0 \\
. & . & . & . & . & . & . & . & . & . \\
0 & 0 & & . & . & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & & . & . & & -1 & 2 & -1 & -1 \\
0 & 0 & & . & . & & 0 & -1 & 2 & 0 \\
0 & 0 & & . & . & & 0 & -1 & 0 & 2
\end{pmatrix}$$

where the entry at position (i, j) gives the Cartan integer $\langle \alpha_i, \alpha_j \rangle = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$.
In the next step we compute the Lie algebra $\mathrm{Lie}(\mathrm{SO}_{2l})(C)$ of $\mathrm{SO}_{2l}(C)$. Therefore denote by $V := \langle v_1, ..., v_{2l} \rangle_C$ a vector space over C of dimension $\dim(V) = 2l$ and let f be a symmetric bilinear form on V defined by the representing matrix

$$J = \begin{pmatrix} 0 & J_0 \\ J_0 & 0 \end{pmatrix} \in C^{2l \times 2l} \text{ where the matrix } J_0 \in C^{l \times l} \text{ has shape } J_0 = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Then the Lie algebra of $\mathrm{SO}_{2l}(C)$ is defined as the set of all endomorphisms $X \in C^{2l \times 2l}$ on V leaving the symmetric bilinear form f invariant, i.e., we have

$$\mathrm{Lie}(\mathrm{SO}_{2l})(C) = \left\{ X \in C^{2l \times 2l} \mid X^T J = -JX \right\}.$$

For the computation of the shapes of the matrices $X \in \mathrm{Lie}(\mathrm{SO}_{2l})(C)$ we set

$$X = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in C^{2l\times 2l} \text{ with } M_{ij} \in C^{l\times l} \text{ for } 1 \leq i,j \leq 2.$$

Thus, the condition $JX = -X^T J$ for X to be an element of $\mathrm{Lie}(\mathrm{SO}_{2l})(C)$ is

$$\begin{pmatrix} M_{21}^T J_0 & M_{11}^T J_0 \\ M_{22}^T J_0 & M_{12}^T J_0 \end{pmatrix} + \begin{pmatrix} J_0 M_{21} & J_0 M_{22} \\ J_0 M_{11} & J_0 M_{12} \end{pmatrix} = 0.$$

Equivalently, we obtain the three equations

$$\begin{array}{rcrcrcl} M_{21}^T J_0 & = & -J_0 M_{21} & \Leftrightarrow & M_{21}^T & = & -J_0 M_{21} J_0^{-1}, \\ M_{11}^T J_0 & = & -J_0 M_{22} & \Leftrightarrow & M_{11}^T & = & -J_0 M_{22} J_0^{-1}, \\ M_{12}^T J_0 & = & -J_0 M_{12} & \Leftrightarrow & M_{12}^T & = & -J_0 M_{12} J_0^{-1}. \end{array}$$

We call the matrix obtained by reflecting the entries at the secondary diagonal the reversed matrix. It can be checked by computation that conjugation $J_0 M J_0^{-1}$ of a matrix $M \in C^{l\times l}$ by J_0 is reversing M and then taking its transposed. Hence, we can formulate the condition for X to be an element of $\mathrm{Lie}(\mathrm{SO}_{2l})(C)$ as:

1. M_{22} is the negative reversed of M_{11}.

2. M_{21} is the negative reversed of M_{21}.

3. M_{12} is the negative reversed of M_{12}.

Before we determine the elements of $\mathrm{Lie}(\mathrm{SO}_{2l})(C)$, we renumber the rows and columns of $X = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ into 1, ..., l, -1, ..., $-l$.
Evidently, the l diagonal matrices $E_{ii} - E_{-l-1+i,-l-1+i}$ with $1 \leq i \leq l$ and the matrices

$$E_{i,j} - E_{-l-1+j,-l-1+i},\ E_{j,i} - E_{-l-1+i,-l-1+j}$$

with $1 \leq i < j \leq l$ have non-zero entries in the blocks $M_{1,1}$ and $M_{2,2}$ and satisfy the condition stated in 1. Furthermore, the matrices

$$E_{i,-j} - E_{l+1-j,-l-1+i},\ E_{-j,i} - E_{-l-1+i,l+1-j}$$

with $1 \leq i,j \leq l$ and $i + j \leq l$ satisfy the conditions 2 and 3.
Denote by $\mathbf{B}$ the set defined by these matrices. We are going to check that the set $\mathbf{B}$ is a basis of $\mathrm{Lie}(\mathrm{SO}_{2l})(C)$. If we choose an arbitrary position above the secondary diagonal, then there is exactly one matrix in $\mathbf{B}$ with a non-zero entry at this position. Hence, the elements of $\mathbf{B}$ are linearly independent. It is easily seen that $\mathrm{card}(\mathbf{B}) = 2l^2 - l$. But the dimension of $\mathrm{Lie}(\mathrm{SO}_{2l})(C)$ which we know from literature (e.g., see [Hum72, Section 1.2]) is also $2l^2 - l$. Thus, $\mathbf{B}$ is a basis of $\mathrm{Lie}(\mathrm{SO}_{2l})(C)$.
The next technical step is to compute a Cartan decomposion from $\mathbf{B}$. To achieve this decomposition we need to determine the shape of the maximal diagonal torus $\mathcal{T}_0$ of $\mathrm{SO}_{2l}(C)$. The group $\mathrm{SO}_{2l}(C)$ is the set of all elements of $\mathrm{SL}_{2l}(C)$ leaving the bilinear form f invariant, i.e., $\mathrm{SO}_{2l}(C)$ is defined as

$$\mathrm{SO}_{2l}(C) = \left\{ A \in \mathrm{SL}_{2l}(C) \mid A^T J A = J \right\}.$$

Let $T = \mathrm{diag}(\lambda_1, ..., \lambda_{2l})$ be a diagonal matrix of $\mathrm{GL}_{2l}(C)$. Then the equation $T^T J T = J$ calculates as

$$\left(\begin{array}{cc|cc} & & & \lambda_1\cdot\lambda_{2l} \\ & & \lambda_l\cdot\lambda_{l+1} & \\ \hline & \lambda_l\cdot\lambda_{l+1} & & \\ \lambda_1\cdot\lambda_{2l} & & & \end{array}\right) = \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right).$$

Thus, the condition for $T = \mathrm{diag}(\lambda_1, ..., \lambda_{2l})$ to be an element of $\mathrm{SO}_{2l}(C)$ translates into the l equations

$$\lambda_{2l+1-i} = \lambda_i^{-1} \text{ for } i = 1, ..., l.$$

We conjugate the elements of $\mathbf{B}$ by $T = \mathrm{diag}(\lambda_1, ..., \lambda_l, \frac{1}{\lambda_l}, ..., \frac{1}{\lambda_1}) \in \mathcal{T}_0$. We obtain

$$\begin{array}{rcl} T(E_{ii} - E_{-l-1+i,-l-1+i})T^{-1} &=& (E_{ii} - E_{-l-1+i,-l-1+i}) \\ T(E_{ij} - E_{-l-1+j,-l-1+i})T^{-1} &=& (\lambda_i/\lambda_j)\ (E_{ij} - E_{-l-1+j,-l-1+i}), \\ T(E_{ji} - E_{-l-1+i,-l-1+j})T^{-1} &=& (\lambda_j/\lambda_i)\ (E_{ji} - E_{-l-1+i,-l-1+j}), \\ T(E_{i,-j} - E_{l+1-j,-l-1+i})T^{-1} &=& \lambda_i\lambda_{l+1-j}\ (E_{i,-j} - E_{l+1-j,-l-1+i}), \\ T(E_{-j,i} - E_{-l-1+i,l+1-j})T^{-1} &=& 1/(\lambda_{l+1-j}\lambda_i)\ (E_{-j,i} - E_{-l-1+i,l+1-j}). \end{array}$$

We conclude that the Lie algebra $\mathrm{Lie}(\mathrm{SO}_{2l})(C)$ is of type D_l. With the help of the above equations we are able to assign the roots to the corresponding root vectors. For $1 \leq i < j \leq l$ we have the assignments

$$X_{\epsilon_i-\epsilon_j} := E_{ij} - E_{-l-1+j,-l-1+i} \quad \text{and} \quad X_{-(\epsilon_i-\epsilon_j)} := E_{ji} - E_{-l-1+i,-l-1+j}.$$

For $1 \leq i, j \leq l,\ i + j \leq l$ we get

$$X_{\epsilon_i+\epsilon_{l+1-j}} := E_{i,-j} - E_{l+1-j,-l-1+i} \quad \text{and} \quad X_{-(\epsilon_i+\epsilon_{l+1-j})} := E_{-j,i} - E_{-l-1+i,l+1-j}.$$

The above equations also yield that the Cartan subalgebra $\mathbf{H}(C)$ is generated by

$$\mathbf{H}(C) = \langle E_{ii} - E_{-l-1+i,-l-1+i} \mid 1 \leq i \leq l\rangle_C.$$

Hence, the Cartan decomposition has in the above notations the shape

$$\begin{array}{rll} \mathrm{Lie}(\mathrm{SO}_{2l})(C) = \mathbf{H}(C) & \bigoplus_{i,j} & \langle X_{\epsilon_i-\epsilon_j}\rangle_C \oplus \langle X_{-(\epsilon_i-\epsilon_j)}\rangle_C \\ & \bigoplus_{i,j} & \langle X_{\epsilon_i+\epsilon_{l+1-j}}\rangle_C \oplus \langle X_{-(\epsilon_i+\epsilon_{l+1-j})}\rangle_C. \end{array}$$

The next step is to compute a Chevalley basis for $\mathrm{Lie}(\mathrm{SO}_{2l})(C)$. We start with the determination of the co-roots. We compute

$$\begin{array}{rcl} [X_{\epsilon_i-\epsilon_j}, X_{-(\epsilon_i-\epsilon_j)}] &=& E_{i,i} - E_{j,j} + E_{l-1+j,-l-1+j} - E_{-l-1+i,-l-1+i} \\ &=:& H_{\epsilon_i-\epsilon_j}, \\ [X_{\epsilon_i+\epsilon_{l+1-j}}, X_{-(\epsilon_i+\epsilon_{l+1-j})}] &=& E_{i,i} + E_{l+1-j,l+1-j} - E_{-j,-j} - E_{-l-1+i,-l-1+i} \\ &=:& H_{\epsilon_i+\epsilon_{l+1-j}}. \end{array}$$

From the bracket products

$$\begin{aligned}[H_{\epsilon_i-\epsilon_j}, X_{\epsilon_i-\epsilon_j}] &= E_{ij} - E_{-l-1+j,-l-1+i} - (-E_{i,j} + E_{-l-1+j,-l-1+j}) \\ &= 2X_{\epsilon_i-\epsilon_j} \quad \text{and} \\ [H_{\epsilon_i+\epsilon_{l+1-j}}, X_{\epsilon_i+\epsilon_{l+1-j}}] &= E_{i,-j} + E_{l+1-j,-l-1+i} - (-E_{l+1-j,-l-1+i} - E_{i,-j}) \\ &= 2X_{\epsilon_i+\epsilon_{l+1-j}}\end{aligned}$$

we deduce that the matrices $H_{\epsilon_i-\epsilon_j}$ and $H_{\epsilon_i+\epsilon_{l+1-j}}$ are already the co-roots. We denote the l co-roots which correspond to the l simple roots $\alpha_i \in \Delta$ with $1 \leq i \leq l$ of Φ by

$$H_1 := H_{\epsilon_1-\epsilon_2},\ \dots,\ H_{l-1} := H_{\epsilon_{l-1}-\epsilon_l} \quad \text{and} \quad H_l := H_{\epsilon_{l-1}+\epsilon_l}.$$

Hence, $\mathbf{H}(C)$ is spanned by $\mathbf{H}(C) = \langle H_i \mid 1 \leq i \leq l\rangle_C$. We define the map

$$\theta : \mathrm{Lie}(\mathrm{SO}_{2l})(C) \to \mathrm{Lie}(\mathrm{SO}_{2l})(C) \quad \text{by} \quad \theta(X) = -X^T.$$

Then θ is an automorphism of $\mathrm{Lie}(\mathrm{SO}_{2l})(C)$ satisfying the identities

$$\begin{aligned}\theta(X_{\epsilon_i-\epsilon_j}) &= -X_{-(\epsilon_i-\epsilon_j)}, \\ \theta(X_{\epsilon_i+\epsilon_{l+1-j}}) &= -X_{-(\epsilon_i+\epsilon_{l+1-j})}.\end{aligned}$$

For the root vectors X_α, X_β with $\alpha, \beta \in \Phi$ we obtain for the automorphism θ the additional identity

$$\theta([X_\alpha, X_\beta]) = -[X_\alpha, X_\beta]^T = [-X_\alpha^T, -X_\beta^T] = [\theta(X_\alpha), \theta(X_\beta)]. \tag{7.1}$$

Let $n_{\alpha,\beta} \in \mathbb{Z}$ be the number defined by

$$n_{\alpha,\beta} X_{\alpha+\beta} = [X_\alpha, X_\beta]. \tag{7.2}$$

We apply θ to both sides of equation (7.2). This computes with the help of equation (7.1) as

$$-n_{\alpha,\beta} X_{-\alpha-\beta} = \theta([X_\alpha, X_\beta]) = [X_{-\alpha}, X_{-\beta}] = n_{-\alpha,-\beta} X_{-\alpha-\beta}.$$

Hence, $n_{-\alpha,-\beta}$ is equal to $-n_{\alpha,\beta}$. From [Car72, Theorem 4.1.2] we know the identity

$$n_{-\alpha,-\beta} \cdot n_{\alpha,\beta} = -(r+1)^2.$$

Thus, $n_{\alpha,\beta} = \pm(r+1)$ holds. We conclude that

$$\{H_i,\ X_\alpha \mid 1 \leq i \leq l,\ \alpha \in \Phi\}$$

is a Chevalley basis of $\mathrm{Lie}(\mathrm{SO}_{2l})(C)$.

7.2 The transformation lemma for SO_{2l}

In this section we present the transformation lemma for SO_{2l} and its proof. Let (F, ∂_F) be a differential field of characteristic 0. Since the proof is based on differential conjugation, i.e., on the adjoint action and the logarithmic derivate which can be both described by the roots, we start with the study of the root system Φ of type D_l. We keep the notations done in the previous section.

Lemma 7.1. *For $l \geq 4$ and $n \in \{1, ..., l-3\}$ let $\Phi_n = \langle \alpha_l, ..., \alpha_{l-2-n} \rangle_\Phi$ be the set of all $\mathbb{Z}$-linear combinations of the roots $\alpha_l, ..., \alpha_{l-2-n}$ which ly in Φ. Furthermore, we define $\Phi_0 = \{\pm\alpha_{l-2},\ \pm\alpha_{l-1},\ \pm\alpha_l,\ \pm(\alpha_{l-2}+\alpha_{l-1}),\ \pm(\alpha_{l-2}+\alpha_l),\ \pm(\alpha_{l-2}+\alpha_{l-1}+\alpha_l)\}$.*

1. *Then $\Phi_n \subseteq \Phi_{l-3} = \Phi$ is an irreducible subsystem of Φ with $\Phi_n \sim D_{3+n}$.*

2. *For $k \in \{1, ..., n+1\}$ there exists a unique root $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ of $\mathrm{ht}(\alpha) = k$ and α has shape*
$$\alpha = \sum_{i=l-2-n}^{l-3-n+k} \alpha_i.$$
If $k \in \{1, ..., n\}$ then there exists a unique simple root $\bar{\alpha} \in \Delta$ such that $\beta = \alpha + \bar{\alpha} \in \Phi_n^+ \setminus \Phi_{n-1}^+$ and $\mathrm{ht}(\beta) = k+1$. If $\tilde{\alpha} \in \Delta$ is a simple root and $\beta - \tilde{\alpha}$ is a root, then either $\beta - \tilde{\alpha} = \alpha$ or $\beta - \tilde{\alpha} \in \Phi_{n-1}^+$.

3. *For $k = n+2$ there exist two roots α_1 and α_2 in $\Phi_n^- \setminus \Phi_{n-1}^-$ of $\mathrm{ht}(\alpha_i) = n+2$. These two roots have shape*
$$\alpha_1 = \sum_{i=l-2-n}^{l-1} \alpha_i \text{ and } \alpha_2 = \sum_{i=l-2-n}^{l-2} \alpha_i + \alpha_l.$$

4. *For $k \in \{n+3, ..., 2n+3\}$ there exists a unique root $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ of $\mathrm{ht}(\alpha) = k$ and α has shape*
$$\alpha = \sum_{i=l-2-n}^{l} \alpha_i \quad if\ k = n+3 \text{ and}$$
$$\alpha = \sum_{i=l-2-n}^{l+n+1-k} \alpha_i + 2\sum_{i=l+n+2-k}^{l-2} + \alpha_{l-1} + \alpha_l \quad if\ k \geq n+4.$$
There exists a unique simple root $\bar{\alpha} \in \Delta$ such that $\beta = \alpha + \bar{\alpha} \in \Phi_n^+ \setminus \Phi_{n-1}^+$ and $\mathrm{ht}(\beta) = k+1$. If $\tilde{\alpha} \in \Delta$ is a simple root and $\beta - \tilde{\alpha}$ is a root, then either $\beta - \tilde{\alpha} = \alpha$ or $\beta - \tilde{\alpha} \in \Phi_{n-1}^+$.

Proof. The first point follows from the Dynkin diagram. We will prove the remaining assertions of the lemma by an outer induction on $n \in \{1, ..., l-3\}$ and two inner inductions which we will specify later.
Let $n = 1$. We are going to compute the root system $\Phi_1 = \langle \alpha_l,\ \alpha_{l-1},\ \alpha_{l-2},\ \alpha_{l-3} \rangle_\Phi$. The images of α_{l-3}, α_{l-1} and α_l under the reflection $\sigma_{\alpha_{l-2}}$ are calculated with the help of the Cartan matrix presented in the previous paragraph as
$$\begin{aligned}
\sigma_{\alpha_{l-2}}(\alpha_l) &= \alpha_l - \langle \alpha_l, \alpha_{l-2} \rangle \alpha_{l-2} = \alpha_l + \alpha_{l-2}, \\
\sigma_{\alpha_{l-2}}(\alpha_{l-1}) &= \alpha_{l-1} - \langle \alpha_{l-1}, \alpha_{l-2} \rangle \alpha_{l-2} = \alpha_{l-1} + \alpha_{l-2}, \\
\sigma_{\alpha_{l-2}}(\alpha_{l-3}) &= \alpha_{l-3} - \langle \alpha_{l-3}, \alpha_{l-2} \rangle \alpha_{l-2} = \alpha_{l-3} + \alpha_{l-2}.
\end{aligned}$$

Hence, we computed the roots $\pm(\alpha_l+\alpha_{l-2})$, $\pm(\alpha_{l-1}+\alpha_{l-2})$ and $\pm(\alpha_{l-3}+\alpha_{l-2})$. The reflections $\sigma_{\alpha_{l-3}}$, $\sigma_{\alpha_{l-1}}$ and σ_{α_l} map the roots $\alpha_l+\alpha_{l-2}$, $\alpha_{l-3}+\alpha_{l-2}$ and $\alpha_{l-1}+\alpha_{l-2}$ to

$$\begin{aligned}
\sigma_{\alpha_{l-3}}(\alpha_l+\alpha_{l-2}) &= \alpha_l+\alpha_{l-2}-(\langle\alpha_l,\alpha_{l-3}\rangle+\langle\alpha_{l-2},\alpha_{l-3}\rangle)\alpha_{l-3}\\
&= \alpha_l+\alpha_{l-2}+\alpha_{l-3},\\
\sigma_{\alpha_{l-1}}(\alpha_{l-3}+\alpha_{l-2}) &= \alpha_{l-3}+\alpha_{l-2}-(\langle\alpha_{l-3},\alpha_{l-1}\rangle+\langle\alpha_{l-2},\alpha_{l-1}\rangle)\alpha_l\\
&= \alpha_{l-3}+\alpha_{l-2}+\alpha_{l-1},\\
\sigma_{\alpha_l}(\alpha_{l-1}+\alpha_{l-2}) &= \alpha_{l-1}+\alpha_{l-2}-(\langle\alpha_{l-1},\alpha_l\rangle+\langle\alpha_{l-2},\alpha_l\rangle)\alpha_l\\
&= \alpha_l+\alpha_{l-1}+\alpha_{l-2}.
\end{aligned}$$

Thus, we obtain the roots $\pm(\alpha_l+\alpha_{l-2}+\alpha_{l-3})$, $\pm(\alpha_{l-3}+\alpha_{l-2}+\alpha_{l-1})$ and $\pm(\alpha_l+\alpha_{l-1}+\alpha_{l-2})$ of Φ_1. The reflection $\sigma_{\alpha_{l-3}}$ maps $\alpha_l+\alpha_{l-1}+\alpha_{l-2}$ to

$$\begin{aligned}
\sigma_{\alpha_{l-3}}(\alpha_l+\alpha_{l-1}+\alpha_{l-2}) &= \alpha_l+\alpha_{l-1}+\alpha_{l-2}-(\langle\alpha_l,\alpha_{l-3}\rangle+\langle\alpha_{l-1},\alpha_{l-3}\rangle\\
&\quad+\langle\alpha_{l-2},\alpha_{l-3}\rangle)\alpha_{l-3}\\
&= \alpha_l+\alpha_{l-1}+\alpha_{l-2}+\alpha_{l-3}.
\end{aligned}$$

At last we are interested in the image of the root $\alpha_l+\alpha_{l-1}+\alpha_{l-2}+\alpha_{l-3}$ under the reflection $\sigma_{\alpha_{l-2}}$. This image computes as

$$\begin{aligned}
\sigma_{\alpha_{l-2}}(\alpha_l+\alpha_{l-1}+\alpha_{l-2}+\alpha_{l-3}) &= \alpha_l+\alpha_{l-1}+\alpha_{l-2}+\alpha_{l-3}-(\langle\alpha_l,\alpha_{l-2}\rangle+\langle\alpha_{l-1},\alpha_{l-2}\rangle\\
&\quad+\langle\alpha_{l-2},\alpha_{l-2}\rangle+\langle\alpha_{l-3},\alpha_{l-2}\rangle)\alpha_{l-2}\\
&= \alpha_l+\alpha_{l-1}+\alpha_{l-2}+\alpha_{l-3}.
\end{aligned}$$

Since the number of positive roots in Φ_1 is twelve (e.g., see [Hum72, Section 12.2, Table 1]), we conclude that

$$\begin{gathered}
\Phi_1=\{\pm\alpha_l,\ \pm\alpha_{l-1},\ \pm\alpha_{l-2},\ \pm\alpha_{l-3},\ \pm(\alpha_l+\alpha_{l-2}),\ \pm(\alpha_{l-3}+\alpha_{l-2}),\\
\pm(\alpha_{l-1}+\alpha_{l-2}),\ \pm(\alpha_{l-1}+\alpha_{l-2}+\alpha_{l-3})\ \pm(\alpha_{l-1}+\alpha_{l-2}+\alpha_l),\\
\pm(\alpha_l+\alpha_{l-2}+\alpha_{l-3}),\ \pm(\alpha_l+\alpha_{l-1}+\alpha_{l-2}+\alpha_{l-3}),\\
\pm(\alpha_l+2\alpha_{l-1}+\alpha_{l-2}+\alpha_{l-3})\}\,.
\end{gathered}$$

It is easily seen that the roots in $\Phi_1\setminus\Phi_0$ satisfy the requirements of the lemma.
Let $1<n\le l-3$. We prove by induction on $k_1\in\{1,...,n+1\}$ that there is a unique root $\alpha\in\Phi_n^+\setminus\Phi_{n-1}^+$ of $\mathrm{ht}(\alpha)=k_1$ and α has shape

$$\alpha=\sum_{i=l-2-n}^{l-3-n+k_1}\alpha_i.$$

Let $k_1=1$. Obviously, α_{l-2-n} is the unique root in $\Phi_n^+\setminus\Phi_{n-1}^+$ of $\mathrm{ht}(\alpha)=1$.
Let $1<k_1\le n+1$. Then the induction hypothesis implies that there is a unique root $\alpha\in\Phi_n\setminus\Phi_{n-1}$ such that $\mathrm{ht}(\alpha)=k_1-1$ and this α has shape $\alpha=\sum_{i=l-2-n}^{l-4-n+k_1}\alpha_i$. We are going to construct a root of height k_1 lying in $\Phi_n^+\setminus\Phi_{n-1}^+$ with the required shape.

Therefore, we compute the image of α under the reflection $\sigma_{\alpha_{l-3-n+k_1}}$. First, we make use of the Cartan matrix to deduce that the integer $\langle \alpha, \alpha_{l-3-n+k_1}\rangle$ is

$$\langle \sum_{i=l-2-n}^{l-4-n+k_1} \alpha_i, \alpha_{l-3-n+k_1}\rangle = -1.$$

This is true since if $k_1 = n+1$, then $l-4-n+k_1 = l-3$. Hence, we have

$$\sigma_{\alpha_{l-3-n+k_1}}(\alpha) = \sum_{i=l-2-n}^{l-4-n+k_1} \alpha_i - \langle \sum_{i=l-2-n}^{l-4-n+k_1} \alpha_i, \alpha_{l-3-n+k_1}\rangle \alpha_{l-3-n+k_1} = \sum_{i=l-2-n}^{l-3-n+k_1} \alpha_i.$$

Evidently, this root satisfies the requirements. Suppose there is another root $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with $\mathrm{ht}(\beta) = k_1$ and $\beta \neq \alpha + \alpha_{l-3-n+k_1}$. By [Hum72, Section 10.2, Corollary] we are able to write β as the sum $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1}$ of simple roots $\bar{\alpha}_i \in \Delta$ such that each partial sum $\bar{\alpha}_1 + ... + \bar{\alpha}_m$ with $1 \leq m \leq k_1$ is a root. Hence, $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1}$ is a root of $\mathrm{ht}(\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1}) = k_1 - 1$. Assume $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1} \neq \alpha$. Thus, the uniqueness of α implies that $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1} \notin \Phi_n^+ \setminus \Phi_{n-1}^+$. Hence, we have the identity

$$-(\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1}) + \beta = \alpha_{l-2-n}.$$

Denote by $\bar{w}$ the minimum of the indices of the simple roots $\alpha_w = \bar{\alpha}_i$ in $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1}$. Then the above identity implies $\bar{w} > l-2-n$. Let $\bar{n} \in \mathbb{N}$ such that $l-2-\bar{n} = \bar{w}$. Equivalently, we have $\bar{n} < n$. Hence, the outer induction assumption implies that the shape of the root $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1}$ is

$$\eta_{1,1} := \sum_{i=l-2-\bar{n}}^{l-4-\bar{n}+k_1} \alpha_i \qquad \text{for } 1 \leq k_1 - 1 \leq \bar{n}+1,$$

$$\eta_{2,1} := \sum_{i=l-2-\bar{n}}^{l-1} \alpha_i \quad \text{or} \quad \eta_{2,2} := \sum_{i=l-2-\bar{n}}^{l-2} \alpha_i + \alpha_l \qquad \text{for } k_1 - 1 = \bar{n}+2 \quad \text{and}$$

$$\eta_{3,1} := \sum_{i=l-2-\bar{n}}^{l} \alpha_i \quad \text{or} \quad \eta_{3,2} := \sum_{i=l-2-\bar{n}}^{l+2+\bar{n}-k_1} \alpha_i + 2\sum_{i=l+3+\bar{n}-k_1+1}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l$$

for $k_1 - 1 \geq \bar{n}+3$. If $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1}$ is a root of Φ_0^+, then we denote the six possibilities for it by $\eta_{4,i}$ with $1 \leq i \leq 6$. Note that the smallest index of the simple roots in $\eta_{4,i}$ is equal or greater then $l-2$.
Assume $l-2-\bar{n} > l-1-n$. To simplify notation we define $\beta_{i,j} = \eta_{i,j} + \alpha_{l-2-n}$. Then it is easily seen that in each case the integer $\langle \beta_{ij}, \alpha_{l-2-n}\rangle$ computes as

$$\langle \beta_{i,j}, \alpha_{l-2-n}\rangle = \langle \eta_{i,j}, \alpha_{l-2-n}\rangle + \langle \alpha_{l-2-n}, \alpha_{l-2-n}\rangle = \langle \alpha_{l-2-n}, \alpha_{l-2-n}\rangle = 2.$$

Hence, we compute the image of $\beta_{i,j}$ under $\sigma_{\alpha_{l-2-n}}$ as

$$\sigma_{\alpha_{l-2-n}}(\beta_{i,j}) = \beta_{i,j} - \langle \beta_{i,j}, \alpha_{l-2-n}\rangle \alpha_{l-2-n} = \eta_{i,j} + \alpha_{l-2-n} - 2\alpha_{l-2-n} = \eta_{i,j} - \alpha_{l-2-n}. \quad (7.3)$$

Since the right hand side of equation (7.3) is not a root, we obtain a contradiction. Thus, we have $l-2-\bar{n} = l-n-1$. Note that in addition it holds $k_1 - 1 < n+1 = \bar{n}+1$. Therefore, the induction assumption forces $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1}$ to have shape

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1} = \sum_{i=\bar{w}}^{l-4-\bar{n}+k_1} \alpha_i = \sum_{i=l-n-1}^{l-3-n+k_1} \alpha_i.$$

Then $\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1} + \alpha_{l-2-n} = \sum_{i=l-2-n}^{l-3-n+k_1} \alpha_i$ is the root constructed above. Again we obtain a contradiction. Thus, we can conclude $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_1-1} = \alpha$. To complete the first inner induction we have to check that β is not the sum of α and a simple root $\alpha_j \in \{\alpha_{l-2-n}, ..., \alpha_l\} \setminus \{\alpha_{l-3-n+k_1}\}$. From [Hum72, Section 12.2, Table 2] we obtain that the irreducible root system of type D_{3+n} contains only long roots, i.e., all roots of Φ_n are of equal length. For two roots α, β of equal length which are not proportional [Hum72, Section 9.4, Table 1] yields that

$$\langle \alpha, \beta \rangle = \pm 1.$$

We check that $\alpha + \alpha_j = \sum_{i=l-2-n}^{l-4-n+k} \alpha_i + \alpha_j$ is not a root of Φ. For $j \in \{l-2-n, ..., l-4-n+k_1\}$ we compute

$$\langle \alpha + \alpha_j, \alpha_j \rangle = (1 - \delta_{l-2-n,j}) \langle \alpha_{j-1}, \alpha_j \rangle + 2\langle \alpha_j, \alpha_j \rangle + (1 - \delta_{l-4-n+k_1,j}) \langle \alpha_{j+1}, \alpha_j \rangle \geq 2$$

and for $j \in \{l-2-n+k_1, ..., l\}$ we have

$$\langle \alpha + \alpha_j, \alpha_j \rangle = \langle \alpha_j, \alpha_j \rangle = 2.$$

Thus, $\alpha + \alpha_j$ for $\alpha_j \in \{\alpha_{l-2-n}, ..., \alpha_l\} \setminus \{\alpha_{l-3-n+k_1}\}$ has a different length than α_j and is therefore not a root of Φ. This completes the first inner induction.
Let $k = n+2$. The first inner induction yields that there exists a root $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ of $\mathrm{ht}(\alpha) = n+1$ and α has shape $\sum_{i=l-2-n}^{l-2} \alpha_i$. We construct the two roots of the desired height. We compute the integers $\langle \alpha, \alpha_{l-1} \rangle$ and $\langle \alpha, \alpha_l \rangle$ as $\langle \alpha, \alpha_{l-1} \rangle = \langle \alpha, \alpha_l \rangle = -1$. Hence, the reflections $\sigma_{\alpha_{l-1}}$ and σ_{α_l} maps α to

$$\sigma_{\alpha_{l-1}}(\alpha) = \alpha - <\alpha, \alpha_{l-1}> \alpha_{l-1} = \sum_{i=n}^{l-1} \alpha_i \quad \text{and}$$
$$\sigma_{\alpha_l}(\alpha) = \alpha - <\alpha, \alpha_{l-1}> \alpha_l = \sum_{i=n}^{l-2} \alpha_i + \alpha_l.$$

We have to show that these two roots are unique with the described properties. Assume there is another root $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ of $\mathrm{ht}(\beta) = n+2$ and $\beta \neq \alpha + \alpha_{l-1}$ and $\beta \neq \alpha + \alpha_l$. With the help of [Hum72, Section 10.2, Corollary] we can write β as the sum $\bar{\alpha}_1 + ... + \bar{\alpha}_k$ of simple roots such that each partial sum is a root. Thus, we obtain the root $\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1}$ of $\mathrm{ht}(\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1}) = k-1$. Assume $\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1} \neq \alpha$. Then the uniqueness of α implies that $\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1} \notin \Phi_n^+ \setminus \Phi_{n-1}^+$. We conclude that β and $\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1}$ differ by α_{l-2-n}, i.e., the equation

$$-(\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1}) + \beta = \alpha_{l-2-n}$$

holds. By $\bar{w}$ we mean the minimum of the indices of the simple roots $\alpha_w = \alpha_i$ in $\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1}$. The above equation yields $\bar{w} > l - 2 - n$. Let $\bar{n} \in \mathbb{N}$ such that $l - 2 - \bar{n} = \bar{w}$ holds. Thus, we have $\bar{n} < n$ and $k - 1 = n + 1 > \bar{n} + 1$. Then by the outer induction assumption the shape of $\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1}$ is for $k - 1 = \bar{n} + 2$

$$\eta_{1,1} := \sum_{i=l-2-\bar{n}}^{l-1} \alpha_i \quad \text{or} \quad \eta_{1,2} := \bar{\alpha}_1 + ... + \bar{\alpha}_{k-1} = \sum_{i=l-2-\bar{n}}^{l-2} \alpha_i + \alpha_l$$

and for $k - 1 \geq \bar{n} + 3$

$$\eta_{2,1} := \sum_{i=l-2-\bar{n}}^{l} \alpha_i \quad \text{or} \quad \eta_{2,2} := \sum_{i=l-2-\bar{n}}^{l+\bar{n}+2-k} \alpha_i + 2 \sum_{i=l+\bar{n}+3-k}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l.$$

Assume $l - 2 - \bar{n} > l - 1 - n$ and define $\beta_{i,j} := \eta_{i,j} + \alpha_{l-2-n}$. Since $\langle \beta_{i,j}, \alpha_{l-2-n} \rangle = \langle \eta_{i,j}, \alpha_{l-2-n} \rangle + \langle \alpha_{l-2-n}, \alpha_{l-2-n} \rangle = 2$, we conclude that the reflection $\sigma_{\alpha_{l-2-n}}$ maps $\beta_{i,j}$ to

$$\sigma_{\alpha_{l-2-n}}(\beta_{i,j}) = \beta_{i,j} - \langle \beta_{i,j}, \alpha_{l-2-n} \rangle \alpha_{l-2-n} = \eta_{i,j} - \alpha_{l-2-n}.$$

This implies a contradiction since $\eta_{i,j} - \alpha_{l-2-n}$ is not a root. We conclude $\bar{w} = l - 1 - n$. But then $k - 1 = n + 1 = \bar{n} + 2$. Hence, the induction assumption offers two possible shapes for $\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1}$. However, we have

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1} = \sum_{i=\bar{w}}^{l-1} \alpha_i = \sum_{i=l-1-n}^{l-1} \alpha_i \quad \text{and}$$
$$\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1} = \sum_{i=\bar{w}}^{l-2} \alpha_i + \alpha_l = \sum_{i=l-1-n}^{l-2} \alpha_i + \alpha_l.$$

But in each case $\bar{\alpha}_1 + ... + \bar{\alpha}_{k-1} + \alpha_{l-2-n}$ is one of the roots constructed before. We get a contradiction. It is left to check that $\beta = \alpha + \alpha_j$ is not a root for $\alpha_j \in \{\alpha_{l-2-n}, ..., \alpha_l\} \setminus \{\alpha_{l-1}, \alpha_l\}$. We compute for $j \in \{l - 2 - n, ..., l - 2\}$

$$\langle \alpha + \alpha_j, \alpha_j \rangle = (1 - \delta_{l-2-n,j}) \langle \alpha_{j-1}, \alpha_j \rangle + 2 \langle \alpha_j, \alpha_j \rangle + (1 - \delta_{l-2,j}) \langle \alpha_{j+1}, \alpha_j \rangle \geq 2$$

and obtain that $\alpha + \alpha_j$ has a different length than α_j. Thus, for all $\alpha_j \in \{\alpha_{l-2-n}, ..., \alpha_l\} \setminus \{\alpha_{l-1}, \alpha_l\}$ the sum $\alpha + \alpha_j$ is not a root of Φ.
Now let $k = n+3$. We know that there are two roots $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \Phi_n^+ \setminus \Phi_{n-1}^+$ of $\mathrm{ht}(\tilde{\alpha}_i) = n+2$ and they have the shapes

$$\tilde{\alpha}_1 = \sum_{i=l-2-n}^{l-1} \alpha_i \text{ and } \tilde{\alpha}_2 = \sum_{i=l-2-n}^{l-2} \alpha_i + \alpha_l.$$

We will construct a root with the desired properties. Therefore, we compute the integer $\langle \sum_{i=l-2-n}^{l-1} \alpha_i, \alpha_l \rangle = -1$. Thus, the reflection σ_{α_l} maps $\tilde{\alpha}_1$ to

$$\sigma_{\alpha_l}(\tilde{\alpha}_1) = \sum_{i=l-2-n}^{l-1} \alpha_i - \langle \sum_{i=l-2-n}^{l-1} \alpha_i, \alpha_l \rangle \alpha_l = \sum_{i=l-2-n}^{l} \alpha_i.$$

It is left to show the uniqueness of $\sum_{i=n}^{l} \alpha_i \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with the claimed characteristics. Let $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ with the properties $\beta \neq \sum_{i=l-2-n}^{l} \alpha_i$ and $\mathrm{ht}(\beta) = n+3$. We write $\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_{n+3}$ as the sum of simple roots such that each partial sum is a root. Suppose $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+2} \neq \tilde{\alpha}_1$ and $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+2} \neq \tilde{\alpha}_2$. Since $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are the only roots in $\Phi_n^+ \setminus \Phi_{n-1}^+$ of $\mathrm{ht}(\tilde{\alpha}_i) = n+2$, we have $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+2} \notin \Phi_n^+ \setminus \Phi_{n-1}^+$. This forces $-(\bar{\alpha}_1 + ... + \bar{\alpha}_{n+2}) + \beta = \alpha_{l-2-n}$. We denote by $\bar{w}$ the smallest index of the simple roots $\alpha_w = \bar{\alpha}_i$ in $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+2}$. We deduce $\bar{w} > l-2-n$. Let $\bar{n} \in \mathbb{N}$ such that $l-2-\bar{n} = \bar{w}$ and assume $l-2-\bar{n} > l-1-n$. Then $k-1 = n+2 > \bar{n}+3$. Thus, the outer induction assumption yields that $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+2}$ has shape

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{n+2} = \sum_{i=l-2-\bar{n}}^{l+2+\bar{n}-k_1} \alpha_i + 2 \sum_{i=l+3+\bar{n}-k_1}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l.$$

We compute $\langle \bar{\alpha}_1 + ... + \bar{\alpha}_{n+2} + \alpha_{l-2-n}, \alpha_{l-2-n} \rangle = 2$. Then the reflection $\sigma_{\alpha_{l-2-n}}$ maps β to

$$\sigma_{\alpha_{l-2-n}}(\beta) = \beta - \langle \beta, \alpha_{l-2-n} \rangle \alpha_{l-2-n} = \bar{\alpha}_1 + ... + \bar{\alpha}_{n+2} - \alpha_{l-2-n}. \tag{7.4}$$

Since the right hand side of equation (7.4) is not a root of Φ, it holds $l-2-\bar{n} = l-1-n$ or equivalently $\bar{n}+1 = n$. This forces $k-1 = n+2 = \bar{n}+3$ and thus the shape of $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+2}$ is

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{n+2} = \sum_{i=l-2-\bar{n}}^{l} \alpha_i = \sum_{i=l-1-n}^{l} \alpha_i.$$

But then $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+2} + \alpha_{l-2-n} = \sum_{i=l-2-n}^{l} \alpha_i$ is the root constructed above. Hence we obtain a contradiction to the assumption that $\beta \neq \sum_{i=l-2-n}^{l} \alpha_i$. It is left to show that $\tilde{\alpha}_1 + \alpha_j$ for $\alpha_j \in \{\alpha_{l-2-n}, ..., \alpha_{l-1}\}$ and $\tilde{\alpha}_2 + \alpha_j$ for $\alpha_j \in \{\alpha_{l-2-n}, ..., \alpha_l\} \setminus \{\alpha_{l-1}\}$ is not a root. We check the lengths of $\tilde{\alpha}_i + \alpha_j$ for $i = 1, 2$. We compute for $j \in \{l-2-n, ..., l-1\}$

$$\langle \tilde{\alpha}_1 + \alpha_j, \alpha_j \rangle = (1 - \delta_{l-2-n,j}) \langle \alpha_{j-1}, \alpha_j \rangle + 2 \langle \alpha_j, \alpha_j \rangle + (1 - \delta_{l-1,j}) \langle \alpha_{j+1}, \alpha_j \rangle \geq 2.$$

For $j \in \{l-2-n, ..., l-2\}$ we obtain

$$\langle \tilde{\alpha}_2 + \alpha_j, \alpha_j \rangle = (1 - \delta_{l-2-n,j}) \langle \alpha_{j-1}, \alpha_j \rangle + 2 \langle \alpha_j, \alpha_j \rangle + (1 - \delta_{l-2,j}) \langle \alpha_{j+1}, \alpha_j \rangle + \delta_{l-2,j} \langle \alpha_j, \alpha_l \rangle \geq 2$$

and for $j = l$ we have

$$\langle \tilde{\alpha}_2 + \alpha_l, \alpha_l \rangle = \langle \alpha_{l-2}, \alpha_l \rangle + 2 \langle \alpha_l, \alpha_l \rangle = 3.$$

Thus in each case the root $\tilde{\alpha}_i + \alpha_j$ has length different to the length of α_j and therefore $\tilde{\alpha}_i + \alpha_j$ is not a root of Φ.
Now we start the second inner induction: For $k_2 \in \{n+4, ..., 2n+3\}$ there exists a unique root $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ of $\mathrm{ht}(\alpha) = k_2$ and α has shape

$$\alpha = \sum_{i=l-2-n}^{l+n+1-k_2} \alpha_i + 2 \sum_{i=l+n+2-k_2}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l.$$

Let $k_2 = n+4$. We construct a root satisfying the assertion using the root $\alpha = \sum_{i=l-2-n}^{l} \alpha_i$ of $\mathrm{ht}(\alpha) = n+3$ of the previous step. The integer $\langle \alpha, \alpha_{l-2} \rangle$ computes as

$$\langle \alpha, \alpha_{l-2} \rangle = \langle \alpha_{l-3}, \alpha_{l-2} \rangle + \langle \alpha_{l-2}, \alpha_{l-2} \rangle + \langle \alpha_{l-1}, \alpha_{l-2} \rangle + \langle \alpha_l, \alpha_{l-2} \rangle = -1.$$

Hence, the reflection $\sigma_{\alpha_{l-2}}$ maps α to

$$\sigma_{\alpha_{l-2}}(\alpha) = \alpha - \langle \alpha, \alpha_{l-2} \rangle \alpha_{l-2} = \sum_{i=l-2-n}^{l-3} \alpha_i + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l. \tag{7.5}$$

Evidently, this root satisfies the proposed properties unless the uniqueness. Suppose there is another root $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ of $\mathrm{ht}(\beta) = n+4$ and

$$\beta \neq \sum_{i=l-2-n}^{l-3} \alpha_i + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l.$$

We write β as the sum $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+4}$ of simple roots in such a way that each partial sum is a root. We assume in addition that the root $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+3} \neq \alpha$. Then, by the uniqueness of $\alpha \in \Phi_n^+ \setminus \Phi_{n-1}^+$ it holds $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+3} \notin \Phi_n^+ \setminus \Phi_{n-1}^+$. Hence, we have $-(\bar{\alpha}_1 + ... + \bar{\alpha}_{n+3}) + \beta = \alpha_{l-2-n}$. Denote by $\bar{w}$ the minimum of the indices of the simple roots $\alpha_w = \bar{\alpha}_i$ in $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+3}$. Form the last equation we get $\bar{w} > l-2-n$. Assume $\bar{w} > l-1-n$. Then the coefficients c_{l-2-n} and c_{l-1-n} of $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+3} = \sum_{i=1}^{l} c_i \alpha_i$ are $c_{l-2-n} = c_{l-1-n} = 0$. This forces $\langle \beta, \alpha_n \rangle = 2$. Hence, the image of β under the reflection $\sigma_{\alpha_{l-2-n}}$ is not a root. However, the reflection $\sigma_{\alpha_{l-2-n}}$ maps β to

$$\sigma_{\alpha_{l-2-n}}(\beta) = \beta - \langle \beta, \alpha_{l-2-n} \rangle \, \alpha_{l-2-n} = \bar{\alpha}_1 + ... + \bar{\alpha}_{n+3} - \alpha_{l-2-n} \notin \Phi.$$

Let $\bar{n} \in \mathbb{N}$ such that $\bar{w} = l-2-\bar{n}$. Then the above yields $l-2-\bar{n} = l-1-n$ or equivalently $\bar{n}+1 = n$. Consequently $k_2 - 1 = n+3 = \bar{n}+4$. Thus, $\bar{\alpha}_1 + ... + \bar{\alpha}_{n+3}$ has shape

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{n+3} = \sum_{i=l-1-n}^{l+2+n-k_2} \alpha_i + 2 \sum_{i=l+2+n-k_2}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l.$$

Hence, $\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_{n+3} + \alpha_{l-2-n}$ is the root constructed in equation (7.5) what contradicts to the assumption that

$$\beta \neq \sum_{i=l-2-n}^{l-3} \alpha_i + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l.$$

It is left to check that $\beta \neq \alpha + \alpha_j$ for some $\alpha_j \in \{\alpha_{l-2-n}, ..., \alpha_l\} \setminus \{\alpha_{l-2}\}$. This will be done by comparing the length of $\alpha + \alpha_j$ with α_j. For $j \in \{l-2-n, ..., l-3\}$ we obtain

$$\langle \alpha + \alpha_j, \alpha_j \rangle = (1 - \delta_{l-2-n,j}) \langle \alpha_{j-1}, \alpha_j \rangle + 2\langle \alpha_j, \alpha_j \rangle + \langle \alpha_{j+1}, \alpha_j \rangle \geq 2.$$

Further, for $j = l-1$ we compute

$$\langle \alpha + \alpha_{l-1}, \alpha_{l-1} \rangle = \langle \alpha_{l-2}, \alpha_{l-1} \rangle + 2\langle \alpha_{l-1}, \alpha_{l-1} \rangle = 3$$

and for $j = l$ we get

$$\langle \alpha + \alpha_l, \alpha_l \rangle = \langle \alpha_{l-2}, \alpha_l \rangle + 2\langle \alpha_l, \alpha_l \rangle = 3.$$

This forces in each case the root length of $\alpha+\alpha_j$ to be different to the length of α_j. Hence $\alpha+\alpha_j$ is not a root of Φ. Thus, the induction assertion is shown for $k_2 = n+4$.
Let $n+4 < k_2 \leq 2n+3$. The induction assumption yields the root

$$\alpha = \sum_{i=l-2-n}^{l+2+n-k_2} \alpha_i + 2 \sum_{i=l+3+n-k_2?+1}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l \in \Phi_n^+ \setminus \Phi_{n-1}^+$$

of $\mathrm{ht}(\alpha) = k_2 - 1$. To construct a root with the proposed characteristics we compute

$$\begin{aligned}\langle \alpha, \alpha_{l+2+n-k_2} \rangle = \langle \alpha_{l+1+n-k_2}, \alpha_{l+2+n-k_2} \rangle + \langle \alpha_{l+2+n-k_2}, \alpha_{l+2+n-k_2} \rangle \\ +2\langle \alpha_{l+3+n-k_2}, \alpha_{l+2+n-k_2} \rangle = -1.\end{aligned}$$

Hence, the reflection $\sigma_{\alpha_{l+2+n-k_2}}$ maps α to

$$\sigma_{\alpha_{l+2+n-k_2}}(\alpha) = \alpha - \langle \alpha, \alpha_{l+2+n-k_2} \rangle \alpha_{l+2+n-k_2} = \sum_{i=l-2-n}^{l+1+n-k_2} \alpha_i + 2 \sum_{i=l+2+n-k_2}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l.$$

Evidently,

$$\sum_{i=l-2-n}^{l+1+n-k_2} \alpha_i + 2 \sum_{i=l+2+n-k_2}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l$$

satisfies the stated properties. It is left to show the uniqueness. We assume that there is $\beta \in \Phi_n^+ \setminus \Phi_{n-1}^+$ of $\mathrm{ht}(\beta) = k_2$ and $\beta \neq \alpha + \alpha_{l+2+n-k_2}$. We write β as in [Hum72, Section 10.2, Corollary], i.e., as the sum of simple roots $\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_{k_2}$ such that each partial sum is a root. We assume that the root $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_2-1}$ of $\mathrm{ht}(\bar{\alpha}_1 + ... + \bar{\alpha}_{k_2-1}) = k_2 - 1$ is different to α. The uniqueness of α implies that $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_2-1} \notin \Phi_n^+ \setminus \Phi_{n-1}^+$. This forces $-(\bar{\alpha}_1 + ... + \bar{\alpha}_{k_2-1}) + \beta = \alpha_{l-2-n}$. It follows $\bar{w} > l-2-n$ where we denote by $\bar{w}$ the smallest index w of the simple roots $\alpha_w = \bar{\alpha}_i$ in $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_2-1}$. Let $\bar{n} \in \mathbb{N}$ such that $l-2-\bar{n} = \bar{w}$. Assume $l-2-\bar{n} > l-1-n$ or equivalently that $\bar{n}+1 < n$ holds. Since the coefficients c_{l-2-n} and c_{l-1-n} of $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_2-1} = \sum_{i=1}^{l} c_i \alpha_i$ are zero, we conclude that

$$\langle \beta, \alpha_{l-2-n} \rangle = \langle \bar{\alpha}_1 + ... + \bar{\alpha}_{k_2-1} + \alpha_{l-2-n}, \alpha_{l-2-n} \rangle = \langle \alpha_{l-2-n}, \alpha_{l-2-n} \rangle = 2.$$

Since the image of β under the reflection $\sigma_{\alpha_{l-2-n}}$

$$\sigma_{\alpha_{l-2-n}}(\beta) = \beta - 2\alpha_{l-2-n} = \bar{\alpha}_1 + ... + \bar{\alpha}_{k_2-1} - \alpha_{l-2-n}$$

is not a root of Φ, we obtain a contradiction. This forces $l-2-\bar{n} = l-1-n$ or equivalently $\bar{n}+1 = n$. We are able to apply the outer induction assumption to $n+4 = \bar{n}+5 \leq k_2 - 1$. We gain that $\bar{\alpha}_1 + ... + \bar{\alpha}_{k_2-1}$ has the shape

$$\bar{\alpha}_1 + ... + \bar{\alpha}_{k_2-1} = \sum_{i=l-1-n}^{l+n+1-k_2} \alpha_i + 2 \sum_{i=l+2+n-k_2}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l.$$

But then $\beta = \bar{\alpha}_1 + ... + \bar{\alpha}_{k_2-1} + \alpha_{l-2-n}$ is the root constructed right before. To finish the proof we need to check that β is not a root of type $\alpha + \alpha_j$ for $\alpha_j \in \{\alpha_{l-2-n}, ..., \alpha_l\} \setminus$

$\{\alpha_{l+2+n-k_2}\}$. We check the root lengths of $\alpha+\alpha_j$ and α_j. For $j \in \{l-2-n, ..., l+1-n-k_2\}$ we compute

$$\langle \alpha + \alpha_j, \alpha_j \rangle = (1 - \delta_{l-2-n,j})\langle \alpha_{j-1}, \alpha_j \rangle + 2\langle \alpha_j, \alpha_j \rangle + \langle \alpha_{j+1}, \alpha_j \rangle \geq 2$$

and for $j \in \{l + 3 - n - k_2, ..., l - 2\}$ we obtain

$$\langle \alpha+\alpha_j, \alpha_j \rangle = (2-\delta_{l+3-n-k_2,j})\langle \alpha_{j-1}, \alpha_j \rangle+3\langle \alpha_j, \alpha_j \rangle+(2-\delta_{l-2,j})\langle \alpha_{j+1}, \alpha_j \rangle+\delta_{l-2,j}\langle \alpha_l, \alpha_j \rangle \geq 2.$$

If $j = l - 1$ or $j = l$, then we have

$$\langle \alpha + \alpha_j, \alpha_j \rangle = 2\langle \alpha_{l-2}, \alpha_j \rangle + 2\langle \alpha_j, \alpha_j \rangle = 2.$$

Since in each case the length of $\alpha + \alpha_j$ is different to the length of α_j, we conclude that $\alpha+\alpha_j$ is not a root. This completes the second inner induction. Hence, the outer induction is also completed and the lemma is shown. □

For the proof of the transformation lemma we decompose the set of the positive roots Φ^+ into subsets $\Omega_n, \Lambda_n, \Theta_n \subset \Phi^+$. For $n \in \{1, ..., l - 3\}$ we define the sets $\Omega_n, \Lambda_n, \Theta_n \subset \Phi^+$ as follows. The set Ω_n is defined as

$$\begin{aligned}\Omega_n := \bigcup_{m=1}^{n} \{\alpha \in \Phi_m \setminus \Phi_{m-1} \mid \alpha \text{ as in Lemma 7.1.2 of } \mathrm{ht}(\alpha) = 1, ..., m\} \\ = \bigcup_{m=1}^{n} \{ \sum_{i=l-2-m}^{l-3-m+k} \alpha_i \mid 1 \leq k \leq m\}.\end{aligned}$$

Note that $\Omega_n \setminus \Omega_{n-1}$ is

$$\Omega_n \setminus \Omega_{n-1} = \{\alpha \in \Phi_n \setminus \Phi_{n-1} \mid \alpha \text{ as in Lemma 7.1.2 of } \mathrm{ht}(\alpha) = 1, ..., n\}.$$

The set

$$\Lambda_n := \{ \sum_{i=l-2-n}^{l-2} \alpha_i, \sum_{i=l-1-n}^{l-1} \alpha_i, \sum_{i=l-1-n}^{l-2} \alpha_i + \alpha_l\}$$

contains the root $\alpha \in \Phi_n \setminus \Phi_{n-1}$ as in Lemma 7.1.2 of $\mathrm{ht} = n + 1$ and the roots $\alpha \in \Phi_{n-1} \setminus \Phi_{n-2}$ as in Lemma 7.1.3 of $\mathrm{ht} = n + 1$. In the case $n = 1$ the set Λ_1 contains the roots $\alpha_{l-2} - \alpha_{l-1}$ and $\alpha_{l-2} - \alpha_l$ of Φ_0. The set Θ_n is

$$\begin{aligned}\Theta_n := \bigcup_{m=1}^{n} \{\alpha \in \Phi_m \setminus \Phi_{m-1} \mid \alpha \text{ as in Lemma 7.1.4 of } \mathrm{ht}(\alpha) = m + 3, ..., 2m + 3\} \\ = \bigcup_{m=1}^{n} \{ \sum_{i=l-2-m}^{l} \alpha_i, \sum_{i=l-2-m}^{l+m+1-k} \alpha_i + 2 \sum_{i=l+m+2-k}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l \mid m + 4 \leq k \leq 2m + 3\}.\end{aligned}$$

Here we have

$$\Theta_n \setminus \Theta_{n-1} = \{\alpha \in \Phi_n \setminus \Phi_{n-1} \mid \alpha \text{ as in Lemma 7.1.4 of } \mathrm{ht}(\alpha) = n + 3, ..., 2n + 3\}.$$

Finally we define

$$\Lambda' := \{\alpha_l,\ \alpha_{l-1},\ \alpha_{l-2},\ \alpha_l + \alpha_{l-1} + \alpha_{l-2},\ \alpha_1 + ... + \alpha_{l-1},\ \alpha_1 + ... + \alpha_{l-2} + \alpha_l\}\,.$$

Since $\alpha_1 + ... + \alpha_{l-1}$ and $\alpha_1 + ... + \alpha_{l-2} + \alpha_l$ are not contained in $\bigcup_{j=1}^{l-3} \Lambda_j$ and the roots $\alpha_{l-2} + \alpha_{l-1}$ and $\alpha_{l-2} + \alpha_l$ of Φ_0 are in Λ' we conclude that

$$\Phi^+ = \Omega_{l-3} \cup (\bigcup_{j=1}^{l-3} \Lambda_j) \cup \Lambda' \cup \Theta_{l-3} = (\bigcup_{j=1}^{l-3} \Omega_j \setminus \Omega_{j-1}) \cup (\bigcup_{j=1}^{l-3} \Lambda_j) \cup \Lambda' \cup (\bigcup_{j=1}^{l-3} \Theta_j \setminus \Theta_{j-1})$$

where the union of the last equation is disjoint.
The transformation lemma for SO_{2l} will be proved in 3 steps. First we will transform the root spaces which correspond to the roots of Ω_{l-3}. In the second step we handle the roots of $\Lambda = (\bigcup_{j=1}^{l-3} \Lambda_j) \cup \Lambda'$. In the last step the roots of Θ_{l-3} are processed.
Before we start we recall some facts about the adoint action.
In the previous section we computed a Cartan subalgebra $\mathbf{H}$ and a Cartan decomposition $\mathbf{L} = \mathbf{H} \oplus \bigoplus_{\alpha\in\Phi} \mathbf{L}_\alpha$ of $\mathrm{Lie}(\mathrm{SO}_{2l})(F)$. Further, we showed that the set $\{X_\alpha,\ H_\alpha \mid \alpha \in \Phi\}$ forms a Chevalley basis where the notation is as in the previous section. Then the Chevalley construction yields a representation of the group SO_{2l}. Let us denote for each $\beta \in \Phi$ the corresponding root subgroups by $\mathcal{U}_\beta$ and a parametrized element of $\mathcal{U}_\beta$ by $U_\beta(\zeta)$ with $\zeta \in F$. Let α and β be two roots of Φ. Then the adjoint action of $U_\beta(\zeta)$ on X_α is determined (see also Section 3.2) by

$$\mathrm{Ad}(U_\beta(\zeta))(X_\alpha) = \sum_{i\geq 0} m_{\alpha+i\beta} \cdot \zeta^i \cdot X_{\alpha+i\beta}. \tag{7.6}$$

For β, α linearly independent let $\alpha - r\beta,\ ...\ , \alpha + q\beta$ be the β-string through α. Then the values for $m_{\beta,\alpha,i}$ are determined by $m_{\beta,\alpha,i} = \pm\binom{r+i}{i}$ and $m_{\beta,\alpha,0} = 0$.

Lemma 7.2. *For $l \geq 4$ let $1 \leq n \leq l-3$ and let $A = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Omega_n} a_\gamma X_\gamma + \sum_{\gamma\in\Phi^-\setminus\Omega_{l-3}} a_\gamma X_\gamma$. Then there exists $U \in \mathcal{U}^-$ such that*

$$UAU^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Omega_{n-1}} a_\gamma X_\gamma + \sum_{\gamma\in\Phi^-\setminus\Omega_{l-3}} a_\gamma X_\gamma.$$

Proof. First remember that the roots in $\Omega_n \setminus \Omega_{n-1}$ are of height k with $1 \leq k \leq n$. We will sometimes write shortly $\bar{\Omega}_n$ for $\Omega_n \setminus \Omega_{n-1}$. We show for a fixed n the following claim: For $1 \leq k \leq n$ and

$$A_k = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Omega_{n-1}} a_\gamma X_\gamma + \sum_{\gamma\in\Phi^-\setminus\Omega_{l-3}} a_\gamma X_\gamma + \sum_{\gamma\in\Omega_n\setminus\Omega_{n-1};\mathrm{ht}(\gamma)\geq k} a_\gamma X_\gamma$$

there exists $U \in \mathcal{U}^-$ such that

$$UA_kU^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Omega_{n-1}} a_\gamma X_\gamma + \sum_{\gamma\in\Phi^-\setminus\Omega_{l-3}} a_\gamma X_\gamma + \sum_{\gamma\in\bar{\Omega}_n;\mathrm{ht}(\gamma)\geq k+1} a_\gamma X_\gamma.$$

In other words we have to delete the unique root $\alpha \in \Omega_n \setminus \Omega_{n-1} \subset \Phi_n^- \setminus \Phi_{n-1}^-$ of $\mathrm{ht}(\alpha) = k$. By Lemma 7.1.2 there exists a unique $\bar{\alpha} \in \Delta$ such that $-\alpha + \bar{\alpha} = \bar{\beta} \in \Phi_n^+ \setminus \Phi_{n-1}^+$ and $\mathrm{ht}(\bar{\beta}) = k+1$. Hence, for $\beta := -\bar{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$ we obtain $\beta + \bar{\alpha} = \alpha$.
We differentially conjugate A_k by $U_\beta(\zeta)$. This yields

$$\begin{aligned} U_\beta(\zeta) A_k U_\beta(\zeta)^{-1} + \partial(U_\beta(\zeta))U_\beta(\zeta)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_\beta(\zeta))(X_{\alpha_i}) \\ + \sum_{\gamma\in\Omega_{n-1}} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) &+ \sum_{\gamma\in\Phi^-\setminus\Omega_{l-3}} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) \\ + \sum_{\gamma\in\bar{\Omega}_n;\mathrm{ht}(\gamma)\geq k} & a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) + \partial(U_\beta(\zeta))U_\beta(\zeta)^{-1}. \end{aligned} \tag{7.7}$$

Note that for $\gamma \in \Phi^-$, $\gamma \neq 0$ we have $\mathrm{ht}(\gamma + i\beta) > \mathrm{ht}(\beta) = k+1$ for $i > 0$. Thus, by equation (7.6) the second summand of the right hand side of equation (7.7) is

$$\sum_{\gamma\in\Omega_{n-1}} a_\gamma Ad(U_\beta(\zeta))(X_\gamma) \in \sum_{\gamma\in\Omega_{n-1}} a_\gamma X_\gamma + \sum_{\gamma\in\Phi_n^-,\mathrm{ht}(\gamma)>k+1} a_\gamma X_\gamma.$$

It is easily seen that for $\gamma \in \Phi^- \setminus \Omega_{l-3}$ and $i \geq 0$ the sum $\gamma + i\beta$ is not an element of Ω_{l-3}. Hence, for the third summand of the right hand side of equation (7.7) we obtain with equation (7.6)

$$\sum_{\gamma\in\Phi^-\setminus\Omega_{l-3}} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) \in \sum_{\gamma\in\Phi^-\setminus\Omega_{l-3}} \mathrm{Lie}(\mathrm{SO}_{2l})(F)_\gamma.$$

The fourth summand of equation (7.7) is by the same arguments an element of the subspace

$$\sum_{\gamma\in\bar{\Omega}_n;\mathrm{ht}(\gamma)\geq k} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) \in \sum_{\gamma\in\bar{\Omega}_n;\mathrm{ht}(\gamma)\geq k} a_\gamma X_\gamma + \sum_{\gamma\in\Phi_n^-\setminus\Phi_{n-1}^-;\mathrm{ht}(\gamma)\geq k+1} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F)$$

and Proposition 3.5 yields for the last summand of equation (7.7)

$$\partial(U_\beta(\zeta))U_\beta(\zeta)^{-1} = l\delta(U_\beta(\zeta)) \in \mathrm{Lie}(\mathrm{SO}_{2l})_\beta.$$

Note that β is an element of $\Phi_n^- \setminus \Phi_{n-1}^-$? of $\mathrm{ht}(\hat{\beta}) = k+1$.
Now we analyse the first summand of equation (7.7). With the help of Lemma 7.1 we deduce

$$\begin{aligned} \sum_{i=1}^{l} \mathrm{Ad}(U_\beta(\zeta))(X_{\alpha_i}) \in \sum_{i=1}^{l} X_{\alpha_i} + m_{\beta,\bar{\alpha},1}\zeta X_\alpha &+ \sum_{\gamma\in\Phi_n^-;\mathrm{ht}(\gamma)>k+1} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F) \\ &+ \sum_{\gamma\in\Phi_{n-1}^-} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F). \end{aligned}$$

If we define $\zeta = \frac{-a_\alpha}{m_{\beta,\bar{\alpha},1}}$, then we obtain from our results for equation (7.7)

$$\begin{aligned} U_\beta(\zeta) A_k U_\beta(\zeta)^{-1} + \partial(U_\beta(\zeta))U_\beta(\zeta)^{-1} &= \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Phi^-\setminus\Omega_{l-3}} \bar{a}_\gamma X_\gamma + \sum_{\gamma\in\Omega_{n-1}} \bar{a}_\gamma X_\gamma \\ &+ \sum_{\gamma\in\Omega_n\setminus\Omega_{n-1};\mathrm{ht}(\gamma)\geq k+1} \bar{a}_\gamma X_\gamma \end{aligned}$$

with suitable $\bar{a}_\gamma \in F$.
Now it can be shown by the claim and induction on height $1 \leq k \leq n$ that for A there exists $U \in \mathcal{U}^-$ such that $UAU^{-1} + \partial(U)U^{-1} = A_{k+1}$. This yields for $k = n$ an element $U \in \mathcal{U}^-$ such that A is differentially equivalent to

$$UAU^{-1} + \partial(U)U^{-1} = A_{\bar{n}+1}.$$

Since the set $\{\gamma \in \Omega_n \setminus \Omega_{n-1} \mid \mathrm{ht}(\gamma) \geq n+1\}$ is empty, we obtain

$$UAU^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma \in \Omega_{n-1}} a_\gamma X_\gamma + \sum_{\gamma \in \Phi^- \setminus \Omega_{l-3}} a_\gamma X_\gamma.$$

Thus, the lemma follows. □

For the transformation of the roots of the set Λ we need some additional information of some specific roots since the root system of type D_l is more improving than the other root systems of the series.

Observation 7.3. Let $\gamma_1 = \alpha_{l-2} + \alpha_{l-1}$ and $\gamma_2 = \alpha_{l-2} + \alpha_l$. Then for γ_i there are two unique simple roots $\bar{\alpha}_j \in \Delta$ such that $\gamma_i - \bar{\alpha}_j$ is a root. We have

$$\begin{array}{rclcrcl} \gamma_1 - \alpha_{l-1} &=& \alpha_{l-2} & \text{and} & \gamma_1 - \alpha_{l-2} &=& \alpha_{l-1}, \\ \gamma_2 - \alpha_{l-2} &=& \alpha_l & \text{and} & \gamma_2 - \alpha_l &=& \alpha_{l-2}. \end{array}$$

For $1 \leq n \leq l-3$ let us define the set

$$T'_n = \{ \sum_{i=l-2-n}^{l-1} \alpha_i,\ \sum_{i=l-2-n}^{l-2} \alpha_i,\ \sum_{i=l-1-n}^{l} \alpha_i \} = \{\beta_1,\ \beta_2,\ \beta_3\}$$

and let us denote the roots of Λ_n by $\Lambda_n = \{\tilde{\alpha}_1,\ \tilde{\alpha}_2,\ \tilde{\alpha}_3\}$.

Observation 7.4. There are two unique simple roots $\bar{\alpha}_1,\ \bar{\alpha}_2 \in \Delta$ such that $\beta_i - \bar{\alpha}_j$ is a root for $i = 1, 2$. We have

$$\begin{array}{rclcrcl} \beta_1 - \alpha_{l-1} &=& \tilde{\alpha}_1 & \text{and} & \beta_1 - \alpha_n &=& \tilde{\alpha}_2, \\ \beta_2 - \alpha_l &=& \tilde{\alpha}_1 & \text{and} & \beta_2 - \alpha_n &=& \tilde{\alpha}_3. \end{array}$$

For $n = 1$ there are two unique simple roots $\bar{\alpha}_j \in \Delta$ such that $\beta_3 - \bar{\alpha}_j$ is a root. We have

$$\beta_3 - \alpha_l = \tilde{\alpha}_2 \quad \text{and} \quad \beta_3 - \alpha_{l-1} = \tilde{\alpha}_3.$$

For $2 \leq n \leq l-3$ there are three unique simple roots $\bar{\alpha}_j \in \Delta$ such that $\beta_3 - \bar{\alpha}_j$ is a root. We have

$$\beta_3 - \alpha_l = \tilde{\alpha}_2, \quad \beta_3 - \alpha_{l-1} = \tilde{\alpha}_3 \quad \text{and} \quad \beta_3 - \alpha_{l-1-n} = \sum_{i=l-n}^{l} \alpha_i$$

where $\sum_{i=l-n}^{l} \alpha_i \in \Lambda'$ for $n = 2$ and $\sum_{i=l-n}^{l} \alpha_i \in \Theta_{n-2}$ for $n \geq 3$.

Observation 7.5. For the root $\gamma = \sum_{i=1}^{l} \alpha_i$ there are three unique simple roots $\bar{\alpha}_j \in \Delta$ such that $\gamma - \bar{\alpha}_j$ is a root. We have

$$\gamma - \alpha_1 \quad \in \quad \Theta_{l-4}, \quad \gamma - \alpha_l = \sum_{i=1}^{l-1} \alpha_i \quad \in \quad \Lambda' \quad \text{and} \quad \gamma - \alpha_{l-1} = \sum_{i=1}^{l-2} \alpha_i + \alpha_l \quad \in \quad \Lambda'.$$

Lemma 7.6. *For $l \geq 4$ let $A = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Lambda} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{l-3}} a_\gamma X_\gamma$ and denote by Γ' the set $\Gamma' = \{-\alpha_l,\ -\alpha_l - \alpha_{l-1} - \alpha_{l-2},\ -\sum_{i=1}^{l-2} \alpha_i - \alpha_l\}$. Then there exists $U \in \mathcal{U}^-$ such that*

$$UAU^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Theta_{l-3}} a_\gamma X_\gamma + \sum_{\gamma\in\Gamma'} a_\gamma X_\gamma.$$

Proof. The definition of the sets Λ_n and Λ' implies that we can decompose A into

$$A = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{1\leq n\leq l-3} \sum_{\gamma\in\Lambda_n} a_\gamma X_\gamma + \sum_{\gamma\in\Lambda'} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{l-3}} a_\gamma X_\gamma.$$

In the first step we delete the root spaces which correspond to the two simple roots $-\alpha_{l-2}$ and $-\alpha_{l-1}$ of Λ'. Therefore, we analyse the roots $-(\alpha_{l-2} + \alpha_{l-1}) =: \beta_1$ and $-(\alpha_{l-2} + \alpha_l) =: \beta_2$. Let α be a root of $\Lambda \cup \Theta_{l-3}$. If $\alpha + i\beta_j$ is a root for $i \geq 1$, then $i = 1$ and $\alpha + \beta_j \in \Theta_{l-3} \cup \{-(\alpha_{l-2} + \alpha_{l-1} + \alpha_l)\}$. This follows from the fact that one of the two coefficients c_{l-1} or c_l in the sum $\alpha + i\beta_j = \sum_{k=1}^{l} c_k\alpha_k$ with $c_k \in \mathbb{Z}$ is equal to 1 and so both coefficients of $\alpha + i\beta_j$ have to be 1. The roots with $c_{l-1} = 1$ and $c_l = 1$ are precisely the roots of $\Theta_{l-3} \cup \{-(\alpha_{l-2} + \alpha_{l-1} + \alpha_l)\}$.
We start with the differential conjugation of A by $U_{\beta_1}(\zeta_1) \in \mathcal{U}_{\beta_1}$. This yields

$$\begin{aligned} U_{\beta_1}(\zeta_1)AU_{\beta_1}(\zeta_1)^{-1} + \partial(U_{\beta_1}(\zeta_1))U_{\beta_1}(\zeta_1)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_{\alpha_i}) \\ + \sum_{1\leq n\leq l-3} \sum_{\gamma\in\Lambda_n} a_\gamma \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_\gamma) &+ \sum_{\gamma\in\Theta_{l-3}} a_\gamma \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_\gamma) \\ + \sum_{\gamma\in\Lambda_{2,l-2}} a_\gamma \mathrm{Ad}(U_{\beta_1'}(\zeta_1))(X_\gamma) &+ \partial(U_{\beta_1}(\zeta_1))U_{\beta_1}(\zeta_1)^{-1}. \end{aligned} \tag{7.8}$$

For the first summand of the right hand side of equation (7.8) we get by Oberservation 7.3

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_{\alpha_i}) = \sum_{i=1}^{l} X_{\alpha_i} + m_{\beta_1,\alpha_{l-1},1}\zeta_1 X_{-\alpha_{l-2}} + m_{\beta_1,\alpha_{l-2},1}\zeta_1 X_{-\alpha_{l-1}}.$$

The above discussion yields that the second, third and fourth summand are elements of

$$\sum_{1\le n\le l-3}\sum_{\gamma\in\Lambda_n} a_\gamma \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_\gamma) \in \sum_{1\le n\le l-3}\sum_{\gamma\in\Lambda_n} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{l-3}\cup\{-\alpha_{l-2}-\alpha_{l-1}-\alpha_l\}} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F),$$

$$\sum_{\gamma\in\Theta_{l-3}} a_\gamma \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_\gamma) \in \sum_{\gamma\in\Theta_{l-3}} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{l-3}\cup\{-\alpha_{l-2}-\alpha_{l-1}-\alpha_l\}} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F) \text{ and}$$

$$\sum_{\gamma\in\Lambda'} a_\gamma \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_\gamma) \in \sum_{\gamma\in\Lambda'} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{l-3}\cup\{-\alpha_{l-2}-\alpha_{l-1}-\alpha_l\}} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F).$$

The last summand is by Proposition 3.5 an element of

$$\partial(U_{\beta_1}(\zeta_1))U_{\beta_1}(\zeta_1)^{-1} \in \sum_{\gamma\in\Lambda'} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F).$$

We define $\zeta_1 := \frac{-a_{-\alpha_{l-1}}}{m_{\beta_1,\alpha_{l-1},1}}$. Thus, for suitable $\bar{a}_\gamma \in F$ we get

$$U_{\beta_1}(\zeta_1)AU_{\beta_1}(\zeta_1)^{-1} + \partial(U_{\beta_1}(\zeta_1))U_{\beta_1}(\zeta_1)^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{1\le n\le l-3}\sum_{\gamma\in\Lambda_n} \bar{a}_\gamma X_\gamma + \sum_{\gamma\in\Theta_{l-3}} \bar{a}_\gamma X_\gamma + \sum_{\gamma\in\Lambda'\setminus\{-\alpha_{l-1}\}} \bar{a}_\gamma X_\gamma =: \bar{A}.$$

Now we differentially conjugate $\bar{A}$ by $U_{\beta_2}(\zeta_2) \in \mathcal{U}_{\beta_2}$. We obtain

$$\begin{aligned} U_{\beta_2}(\zeta_2)\bar{A}U_{\beta_2}(\zeta_2)^{-1} + \partial(U_{\beta_2}(\zeta_2))U_{\beta_2}(\zeta_2)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_{\alpha_i}) \\ &+ \sum_{1\le n\le l-3}\sum_{\gamma\in\Lambda_n} a_\gamma \mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_\gamma) + \sum_{\gamma\in\Theta_{l-3}} a_\gamma \mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_\gamma) \\ &+ \sum_{\gamma\in\Lambda'\setminus\{-\alpha_{l-1}\}} a_\gamma \mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_\gamma) + \partial(U_{\beta_2}(\zeta_2))U_{\beta_2}(\zeta_2)^{-1}. \end{aligned} \tag{7.9}$$

Then Observation 7.3 yields for the first summand of the right hand side of equation (7.9)

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_{\alpha_i}) = \sum_{i=1}^{l} X_{\alpha_i} + m_{\beta_2,\alpha_{l-2},1}\zeta_2 X_{\alpha_l} + m_{\beta_2,\alpha_l,1}\zeta_2 X_{\alpha_{l-2}}.$$

If we define $\zeta_2 = \frac{-a_{-\alpha_{l-2}}}{m_{\beta_2,\alpha_l,1}}$ and use the same arguments as above for the computation of the second, third, fourth and fifth summand of equation (7.9), then we get

$$U_{\beta_2}(\zeta_2)\bar{A}U_{\beta_2}(\zeta_2)^{-1} + \partial(U_{\beta_2}(\zeta_2))U_{\beta_2}(\zeta_2)^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{1\le n\le l-3}\sum_{\gamma\in\Lambda_n} \bar{a}_\gamma X_\gamma + \sum_{\gamma\in\Theta_{l-3}} \bar{a}_\gamma X_\gamma + \sum_{\gamma\in\Lambda'\setminus\{-\alpha_{l-1},\, -(\alpha_{l-2})\}} \bar{a}_\gamma X_\gamma = A_{l-3}.$$

This completes the first step. To simplify notation for the rest of the proof we define $\bar{\Lambda}' =: \Lambda' \setminus \{-\alpha_{l-1},\ -(\alpha_{l-2})\}$. In the second step we delete the parts of A_{l-3} lying in the root spaces which correspond to the roots of $\bigcup_{i=1}^{l-3} \Lambda_i$. We prove the following claim:
For $1 \le n \le l-3$ let

$$A_n = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Theta_{l-3}} a_\gamma X_\gamma + \sum_{\gamma\in\Lambda'} a_\gamma X_\gamma + \sum_{i=n}^{l-3}\sum_{\gamma\in\Lambda_i} a_\gamma X_\gamma.$$

Then there exists $U \in \mathcal{U}^-$ such that

$$UA_nU^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Theta_{l-3}} a_\gamma X_\gamma + \sum_{\gamma\in\bar{\Lambda}'} a_\gamma X_\gamma + \sum_{i=n+1}^{l-3}\sum_{\gamma\in\Lambda_i} a_\gamma X_\gamma.$$

In other words we delete the parts of A_n which are elements of the root spaces corresponding to the roots of $\Lambda_n = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3\}$. This will be done by stepwise differential conjugation with the elements $U_{\beta_i}(\zeta_i)$ where β_i is one of the roots of $T_n = \{\beta_1, \beta_2, \beta_3\}$. Note that

1. if for $\alpha \in \bigcup_{i=1}^{l-2} \Lambda_i$ and $\beta_i \in T_n' = \{\beta_1, \beta_2, \beta_3\}$ (for $1 \le n \le l-3$) $\alpha + \beta_i$ is a root, then $\alpha + \beta \in \Theta_{l-3}$.
2. for $\alpha \in \bigcup_{i=1}^{l-2} \Lambda_i$ and $\beta_i \in T_n' = \{\beta_1, \beta_2, \beta_3\}$ (for $1 \le n \le l-3$) $\alpha + j\cdot\beta_i$ is not a root for $j \ge 2$.
3. for $\alpha \in \Theta_{l-3}$ and $\beta_i \in T_n' = \{\beta_1, \beta_2, \beta_3\}$ (for $1 \le n \le l-3$) $\alpha + j\cdot\beta_i$ is not a root for $j \ge 1$.

We start with the differential conjugation of A_n by the root group element $U_{\beta_1}(\zeta_1) \in \mathcal{U}_{\beta_1}$. This yields

$$\begin{aligned} U_{\beta_1}(\zeta_1)A_nU_{\beta_1}(\zeta_1)^{-1} + \partial(U_{\beta_1}(\zeta_1))U_{\beta_1}(\zeta_1)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_{\alpha_i}) \\ + \sum_{\gamma\in\Theta_{l-3}} a_\gamma \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_\gamma) &+ \sum_{\gamma\in\bar{\Lambda}'} a_\gamma \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_\gamma) \\ + \sum_{i=n}^{l-3}\sum_{\gamma\in\Lambda_i} a_\gamma \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_\gamma) &+ \partial(U_{\beta_1}(\zeta_1))U_{\beta_1}(\zeta_1)^{-1}. \end{aligned} \tag{7.10}$$

For the first summand of the right hand side of equation (7.10) we obtain with the help of Observation 7.4

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_{\alpha_i}) = \sum_{i=1}^{l} X_{\alpha_i} + m_{\beta_1,\alpha_{l-1},1}\zeta_1 X_{\tilde{\alpha}_1} + m_{\beta_1,\alpha_n,1}\zeta_1 X_{\tilde{\alpha}_2}.$$

The above note implies for the third and fourth summand of equation (7.10)

$$\sum_{\gamma\in\bar{\Lambda}'} a_\gamma \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_\gamma) \in \sum_{\gamma\in\bar{\Lambda}'} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{l-3}} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F),$$

$$\sum_{i=n}^{l-3}\sum_{\gamma\in\Lambda_i} a_\gamma \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_\gamma) \in \sum_{i=n}^{l-3}\sum_{\gamma\in\Lambda_i} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{l-3}} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F)$$

and for the second summand

$$\sum_{\gamma\in\Theta_{l-3}} a_\gamma \mathrm{Ad}(U_{\beta_1}(\zeta_1))(X_\gamma) = \sum_{\gamma\in\Theta_{l-3}} a_\gamma X_\gamma.$$

Since $\beta_1 \in \Lambda_{n+1}$, we get for the last summand

$$\partial(U_{\beta_1}(\zeta_1))U_{\beta_1}(\zeta_1)^{-1} \in \sum_{\gamma\in\Lambda_{n+1}} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F).$$

Thus, equation (7.10) is equivalent to

$$A_{n,1} := U_{\beta_1}(\zeta_1)A_nU_{\beta_1}(\zeta_1)^{-1} + U_{\beta_1}(\zeta_1)U_{\beta_1}(\zeta_1)^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Theta_{l-3}} \bar{a}_\gamma X_\gamma + \sum_{\gamma\in\bar{\Lambda}'} a_\gamma X_\gamma$$
$$+ \sum_{i=n+1}^{l-3}\sum_{\gamma\in\Lambda_i} a_\gamma X_\gamma + (a_{\tilde{\alpha}_1} + m_{\beta_1,\alpha_{l-1},1}\zeta_1)X_{\tilde{\alpha}_1} + (a_{\tilde{\alpha}_2} + m_{\beta_1,\alpha_n,1}\zeta_1)X_{\tilde{\alpha}_2} + a_{\tilde{\alpha}_3}X_{\tilde{\alpha}_3}.$$

Now we differentially conjugate $A_{n,1}$ by $U_{\beta_2}(\zeta_2)$. We obtain

$$\begin{aligned} U_{\beta_2}(\zeta_2)A_{n,1}U_{\beta_2}(\zeta_2)^{-1} + \partial(U_{\beta_2}(\zeta_2))U_{\beta_2}(\zeta_2)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_{\alpha_i}) \\ + \sum_{\gamma\in\Theta_{l-3}} a_\gamma \mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_\gamma) &+ \sum_{\gamma\in\bar{\Lambda}'} a_\gamma \mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_\gamma) \\ + \sum_{i=n+1}^{l-3}\sum_{\gamma\in\Lambda_i} a_\gamma \mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_\gamma) &+ (a_{\tilde{\alpha}_1} + m_{\beta_1,\alpha_{l-1},1}\zeta_1)\mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_{\tilde{\alpha}_1}) \\ +(a_{\tilde{\alpha}_2} + m_{\beta_1,\alpha_n,1}\zeta_1)\mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_{\tilde{\alpha}_2}) &+ a_{\tilde{\alpha}_3}\mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_{\tilde{\alpha}_3}) \\ &+\partial(U_{\beta_2}(\zeta_2))U_{\beta_2}(\zeta_2)^{-1}. \end{aligned} \tag{7.11}$$

Observation 7.4 yields for the first summand of the right hand side of equation (7.11)

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_{\alpha_i}) = \sum_{i=1}^{l} X_{\alpha_i} + m_{\beta_2,\alpha_l,1}\zeta_2 X_{\tilde{\alpha}_1} + m_{\beta_2,\alpha_n,1}\zeta_2 X_{\tilde{\alpha}_3}.$$

For the computation of the remaining summands we use the same arguments as in the step before and obtain similar results. However, we have

$$\begin{gathered} (a_{\tilde{\alpha}_1} + m_{\beta_1,\alpha_{l-1},1}\zeta_1)\mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_{\tilde{\alpha}_1}) + (a_{\tilde{\alpha}_2} + m_{\beta_1,\alpha_n,1}\zeta_1)\mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_{\tilde{\alpha}_2}) \\ +a_{\tilde{\alpha}_3}\mathrm{Ad}(U_{\beta_2}(\zeta_2))(X_{\tilde{\alpha}_3}) \in \\ (a_{\tilde{\alpha}_1} + m_{\beta_1,\alpha_{l-1},1}\zeta_1)X_{\tilde{\alpha}_1} + (a_{\tilde{\alpha}_2} + m_{\beta_1,\alpha_n,1}\zeta_1)X_{\tilde{\alpha}_2} + a_{\tilde{\alpha}_3}X_{\tilde{\alpha}_3} + \sum_{\gamma\in\Theta_{l-3}} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F). \end{gathered}$$

Since $\beta_2 \in \Lambda_{n+1}$, the last summand is $\partial(U_{\beta_2}(\xi_2))U_{\beta_2}(\xi_2^{-1}) \in \sum_{\gamma\in\Lambda_{n+1}} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F)$. We conclude

$$A_{n,2} := U_{\beta_2}(\zeta_2)A_{n,1}U_{\beta_2}(\zeta_2)^{-1} + \partial(U_{\beta_2}(\zeta_2))U_{\beta_2}(\zeta_2)^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Theta_{l-3}} \bar{a}_\gamma X_\gamma$$
$$+ \sum_{\gamma\in\bar{\Lambda}'} a_\gamma X_\gamma + \sum_{i=n+1}^{l-3}\sum_{\gamma\in\Lambda_i} \bar{a}_\gamma X_\gamma + (a_{\tilde{\alpha}_1} + m_{\beta_1,\alpha_{l-1},1}\zeta_1 + m_{\beta_2,\alpha_l,1}\zeta_2)X_{\tilde{\alpha}_1} +$$
$$(a_{\tilde{\alpha}_2} + m_{\beta_1,\alpha_n,1}\zeta_1)X_{\tilde{\alpha}_2} + (a_{\tilde{\alpha}_3} + m_{\beta_2,\alpha_n,1}\zeta_2)X_{\tilde{\alpha}_3}.$$

In the next step we differentially conjugate $A_{n,2}$ by $U_{\beta_3}(\zeta_3) \in \mathcal{U}_{\beta_3}$. This computes as

$$\begin{aligned} U_{\beta_3}(\zeta_3)A_{n,2}U_{\beta_3}(\zeta_3)^{-1} + \partial(U_{\beta_3}(\zeta_3))U_{\beta_3}(\zeta_3)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_{\beta_3}(\zeta_3))(X_{\alpha_i}) \\ &+ \sum_{\gamma\in\Theta_{l-3}} a_\gamma \mathrm{Ad}(U_{\beta_3}(\zeta_3))(X_\gamma) + \sum_{\gamma\in\bar{\Lambda}'} a_\gamma \mathrm{Ad}(U_{\beta_3}(\zeta_3))(X_\gamma) \\ &+ \sum_{i=n+1}^{l-3}\sum_{\gamma\in\Lambda_i} a_\gamma \mathrm{Ad}(U_{\beta_3}(\zeta_3))(X_\gamma) + \partial(U_{\beta_3}(\zeta_3))U_{\beta_3}(\zeta_3)^{-1} \\ &+(a_{\tilde{\alpha}_1} + m_{\beta_1,\alpha_{l-1},1}\zeta_1 + m_{\beta_2,\alpha_l,1}\zeta_2)\mathrm{Ad}(U_{\beta_3}(\zeta_3))(X_{\tilde{\alpha}_1}) \\ +(a_{\tilde{\alpha}_2} + m_{\beta_1,\alpha_n,1}\zeta_1)\mathrm{Ad}(U_{\beta_3}(\zeta_3))(X_{\tilde{\alpha}_2}) &+ (a_{\tilde{\alpha}_3} + m_{\beta_2,\alpha_n,1}\zeta_2)\mathrm{Ad}(U_{\beta_3}(\zeta_3))(X_{\tilde{\alpha}_3}). \end{aligned} \tag{7.12}$$

We deduce with Observation 7.4 that for $n = 1$ the first summand of the right hand side of equation (7.12) is

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{\beta_3}(\zeta_3))(X_{\alpha_i}) = \sum_{i=1}^{l} X_{\alpha_i} + m_{\beta_3,\alpha_l,1}\zeta_3 X_{\tilde{\alpha}_2} + m_{\beta_3,\alpha_{l-1},1}\zeta_3 X_{\tilde{\alpha}_3}$$

and that for $2 \le n \le l-3$ we have

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{\beta_3}(\zeta_3))(X_{\alpha_i}) \in \sum_{i=1}^{l} X_{\alpha_i} + m_{\beta_3,\alpha_l,1}\zeta_3 X_{\tilde{\alpha}_2} + m_{\beta_3,\alpha_{l-1},1}\zeta_3 X_{\tilde{\alpha}_3} + \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F)$$

where $\gamma = \sum_{i=n+2}^{l} \alpha_i$. If $l \ge 5$ and $n = 2$, then γ is an element of $\bar{\Lambda}'$ and if $l \ge 6$ and $3 \le n \le l-3$, then we have $\gamma \in \Theta_{l-3}$. The same arguments as in the previous steps yield similar results for the remaining terms. For the last summand we obtain

$$\partial(U_{\beta_3}(\zeta_3))U_{\beta_3}(\zeta_3)^{-1} \in \sum_{\gamma\in\bar{\Lambda}'} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F) \quad \text{if } n = 1 \text{ and}$$
$$\partial(U_{\beta_3}(\zeta_3))U_{\beta_3}(\zeta_3)^{-1} \in \sum_{\gamma\in\Theta_{l-3}} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F) \quad \text{if } 2 \le n \le l-3.$$

Thus, we obtain for equation (7.12)

$$U_{\beta_3}(\zeta_3)A_{n,2}U_{\beta_3}(\zeta_3)^{-1}+\partial(U_{\beta_3}(\zeta_3))U_{\beta_3}(\zeta_3)^{-1}=\sum_{i=1}^{l}X_{\alpha_i}+\sum_{\gamma\in\Theta_{l-3}}\bar{a}_\gamma X_\gamma+\sum_{\gamma\in\bar{\Lambda}'}\bar{a}_\gamma X_\gamma$$
$$+\sum_{i=n+1}^{l-3}\sum_{\gamma\in\Lambda_i}\bar{a}_\gamma X_\gamma+(a_{\tilde{\alpha}_1}+m_{\beta_1,\alpha_{l-1},1}\zeta_1+m_{\beta_2,\alpha_l,1}\zeta_2)X_{\tilde{\alpha}_1}$$
$$+(a_{\tilde{\alpha}_2}+m_{\beta_1,\alpha_n,1}\zeta_1+m_{\beta_3,\alpha_l,1}\zeta_3)X_{\tilde{\alpha}_2}+(a_{\tilde{\alpha}_3}+m_{\beta_2,\alpha_n,1}\zeta_2+m_{\beta_3,\alpha_{l-1},1}\zeta_3)X_{\tilde{\alpha}_3}.$$

It can be checked by computation or with [How01, Theorem 2.2] that the integers $m_{\beta_j,\alpha_i,1}$ have all the same signs. Thus, the determinant

$$\det\begin{pmatrix} m_{\beta_1,\alpha_{l-1},1} & m_{\beta_2,\alpha_l,1} & 0\\ m_{\beta_1,\alpha_n,1} & 0 & m_{\beta_3,\alpha_l,1}\\ 0 & m_{\beta_2,\alpha_n,1} & m_{\beta_3,\alpha_{l-1},1}\end{pmatrix}\neq 0$$

is not zero. Hence, the system of equations

$$\begin{aligned} m_{\beta_1,\alpha_{l-1},1}\zeta_1+m_{\beta_2,\alpha_l,1}\zeta_2 &= -a_{\tilde{\alpha}_1},\\ m_{\beta_1,\alpha_n,1}\zeta_1+m_{\beta_3,\alpha_l,1}\zeta_3 &= -a_{\tilde{\alpha}_2},\\ m_{\beta_2,\alpha_n,1}\zeta_2+m_{\beta_3,\alpha_{l-1},1}\zeta_3 &= -a_{\tilde{\alpha}_3}\end{aligned}$$

has a solution $(\bar{\zeta}_1,\bar{\zeta}_2,\bar{\zeta}_3)\in F^3$. Thus, for $U:=U_{\beta_1}(\bar{\zeta}_1)U_{\beta_2}(\bar{\zeta}_2)U_{\beta_3}(\bar{\zeta}_3)$ we get that

$$UA_nU^{-1}+\partial(U)U^{-1}=\sum_{i=1}^{l}X_{\alpha_i}+\sum_{\gamma\in\Theta_{l-3}}\bar{a}_\gamma X_\gamma+\sum_{\gamma\in\bar{\Lambda}'}\bar{a}_\gamma X_\gamma+\sum_{i=n+1}^{l-3}\sum_{\gamma\in\Lambda_i}\bar{a}_\gamma X_\gamma.$$

This completes the proof of the claim. Now it can be shown by the claim and induction on $1\leq n\leq l-3$ that for A_1 there exists $U\in\mathcal{U}^-$ such that

$$UA_1U^{-1}+\partial(U)U^{-1}=\sum_{i=1}^{l}X_{\alpha_i}+\sum_{\gamma\in\Theta_{l-3}}\bar{a}_\gamma X_\gamma+\sum_{\gamma\in\bar{\Lambda}'}\bar{a}_\gamma X_\gamma+\sum_{i=n+1}^{l-3}\sum_{\gamma\in\Lambda_i}\bar{a}_\gamma X_\gamma.$$

This yields for $n=l-3$ that there exists $U\in\mathcal{U}^-$ such that A_1 is differentially equivalent to

$$UA_1U^{-1}+\partial(U)U^{-1}=\sum_{i=1}^{l}X_{\alpha_i}+\sum_{\gamma\in\Theta_{l-3}}\bar{a}_\gamma X_\gamma+\sum_{\gamma\in\bar{\Lambda}'}\bar{a}_\gamma X_\gamma=:A_{l-3}.$$

In the last step we differentially conjugate A_{l-3} by $U_\beta(\zeta)\in\mathcal{U}_\beta$ where β is the root $\beta=\sum_{i=1}^{l}\alpha_i$. We obtain

$$\begin{aligned} U_\beta(\zeta)A_{l-3}U_\beta(\zeta)^{-1}+\partial(U_\beta(\zeta))U_\beta(\zeta)^{-1}&=\sum_{i=1}^{l}\mathrm{Ad}(U_\beta(\zeta))(X_{\alpha_i})\\ +\sum_{\gamma\in\Theta_{l-3}}\bar{a}_\gamma\mathrm{Ad}(U_\beta(\zeta))(X_\gamma)&+\sum_{\gamma\in\bar{\Lambda}'}\bar{a}_\gamma\mathrm{Ad}(U_\beta(\zeta))(X_\gamma).\end{aligned}\tag{7.13}$$

Observation 7.5 yields for the first summand of the right hand side of equation (7.13)

$$\sum_{i=1}^{l} Ad(U_\beta(\zeta))(X_{\alpha_i}) \in \sum_{i=1}^{l} X_{\alpha_i} + m_{\beta,\alpha_l,1}\zeta X_{\gamma_1} + m_{\beta,\alpha_{l-1},1}\zeta X_{\gamma_2} + \sum_{\gamma\in\Theta_{l-3}} \mathrm{Lie}(\mathrm{SO}_{2l})_\gamma(F)$$

where $\gamma_1 = \sum_{i=1}^{l-1}\alpha_i$ and $\gamma_2 = \sum_{i=1}^{l-2}\alpha_i + \alpha_l$. Since for every root α of Θ_{l-3} or $\bar{\Lambda}'$ one of the coefficients c_{l-1} or c_l of $\alpha + i\beta$ with $i \geq 1$ is greater than 2, the second and third summand computes as

$$\sum_{\gamma\in\Theta_{l-3}} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) = \sum_{\gamma\in\Theta_{l-3}} a_\gamma X_\gamma \quad \text{and}$$
$$\sum_{\gamma\in\bar{\Lambda}'} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) = \sum_{\gamma\in\bar{\Lambda}'} a_\gamma X_\gamma.$$

We define $\zeta = -\frac{a_{\gamma_1}}{m_{\beta,\alpha_l,1}}$. Then the assertion of the lemma follows, i.e., we have

$$U_\beta(\zeta)^{-1}A_1U_\beta(\zeta) + \partial(U_\beta(\zeta))U_\beta(\zeta)^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Theta_{l-3}} \bar{a}_\gamma X_\gamma + \sum_{\gamma\in\Gamma'} \bar{a}_\gamma X_\gamma.$$

□

In the next step we transform the roots of the set Θ_{l-3}. Since it is not possible to delete all roots of Θ_{l-3} we define for $0 \leq n \leq l-4$ the set

$$\Gamma_n = \Gamma' \cup \{\alpha_i + 2\sum_{j=i+1}^{l-2}\alpha_i + \alpha_{l-1} + \alpha_l \mid 1 \leq i \leq l-3-n\}$$

and Γ_{l-3} as $\Gamma_{l-3} := \Gamma'$ where Γ' is as in Lemma 7.6. However, the transformation is done in Lemma 7.7 below.

Lemma 7.7. *For $l \geq 4$ let $1 \leq n \leq l-3$ and*

$$A = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Theta_n} a_\gamma X_\gamma + \sum_{\gamma\in\Gamma_n} a_\gamma X_\gamma.$$

Then there exists $U \in \mathcal{U}^-$ such that

$$UAU^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Theta_{n-1}} a_\gamma X_\gamma + \sum_{\gamma\in\Gamma_{n-1}} a_\gamma X_\gamma.$$

Proof. First remember that the roots in $\Theta_n \setminus \Theta_{n-1}$ are of height k with $n+3 \leq k \leq 2n+2$. We will sometimes write $\bar{\Theta}_n$ for $\Theta_n \setminus \Theta_{n-1}$.
We prove the following claim: For $n+3 \leq k \leq 2n+2$ let

$$A_k = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Gamma_n} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{n-1}} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_n\setminus\Theta_{n-1};\mathrm{ht}(\gamma)\geq k} a_\gamma X_\gamma.$$

Then there exists $U \in \mathcal{U}^-$ such that

$$UA_kU^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Gamma_n} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{n-1}} a_\gamma X_\gamma + \sum_{\gamma\in\bar{\Theta}_n;\mathrm{ht}(\gamma)\geq k+1} a_\gamma X_\gamma.$$

By Lemma 7.1 there is unique root $\alpha \in \Theta_n \setminus \Theta_{n-1} \subset \Phi_n^- \setminus \Phi_{n-1}^-$ of $\mathrm{ht}(\alpha) = k$. To prove the claim we have to delete the term of A_k which corresponds to this root α. Lemma 7.1 yields that there exists a unique root $\bar{\alpha} \in \Delta$ such that $-\alpha + \bar{\alpha} = \bar{\beta} \in \Phi_n^+ \setminus \Phi_{n-1}^+$ of height $\mathrm{ht}(\bar{\beta}) = k+1$. Thus, for $\beta := -\bar{\beta} \in \Phi_n^- \setminus \Phi_{n-1}^-$ it holds $\beta + \bar{\alpha} = \alpha$. From the shape of

$$\beta = -(\sum_{i=l-2-n}^{l+n-k} \alpha_i + 2\sum_{i=l+n+1-k}^{l-2} + \alpha_{l-1} + \alpha_l)$$

we obtain the following note:

1. If $\bar{\alpha} \in \Delta$ is one of the roots of Lemma 7.1.3, then $\bar{\alpha} + i\beta$ is not a root for $i \geq 2$.
2. If $\gamma \in \Gamma_n$, then $\gamma + i\beta$ is not a root for $i \geq 1$.
3. If $\gamma \in \Theta_n$, then $\gamma + i\beta$ is not a root for $i \geq 1$.

We differentially conjugate A_k by $U_\beta(\zeta) \in \mathcal{U}_\beta$. This yields

$$\begin{aligned} U_\beta(\zeta)A_kU_\beta(\zeta)^{-1} + \partial(U_\beta(\zeta))U_\beta(\zeta)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_\beta(\zeta))(X_{\alpha_i}) \\ &+ \sum_{\gamma\in\Gamma_n} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) + \sum_{\gamma\in\Theta_{n-1}} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) \\ &+ \sum_{\gamma\in\Theta_n\setminus\Theta_{n-1};\mathrm{ht}(\gamma)\geq k+1} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) + \partial(U_\beta(\zeta))U_\beta(\zeta)^{-1}. \end{aligned} \tag{7.14}$$

With the help of Lemma 7.1.3 and the above note we conclude that the first summand of the right hand side of equation (7.14) is

$$\sum_1^l \mathrm{Ad}(U_{\hat{\beta}}(\zeta))X_{\alpha_i} = \sum_1^l X_{\alpha_i} + m_{\hat{\beta},\bar{\alpha},1}\zeta X_\alpha + m_{\hat{\beta},\tilde{\alpha},1}\zeta X_\gamma$$

$$\text{where}\quad \gamma = -(\sum_{i=l-1-n}^{l+n-k} \alpha_i + 2\sum_{i=l+n+1-k}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l) \in \Theta_{n-1}$$

The note yields for the second summand of the right hand side of equation (7.14)

$$\sum_{\gamma\in\Gamma_n} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) = \sum_{\gamma\in\Gamma_n} a_\gamma X_\gamma$$

and for the third and fourth summand

$$\sum_{\gamma\in\Theta_{n-1}} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) = \sum_{\gamma\in\Theta_{n-1}} a_\gamma X_\gamma \quad \text{and}$$

$$\sum_{\gamma\in\Theta_n\setminus\Theta_{n-1};\mathrm{ht}(\gamma)\geq k+1} a_\gamma \mathrm{Ad}(U_\beta(\zeta))(X_\gamma) = \sum_{\gamma\in\Theta_n\setminus\Theta_{n-1};\mathrm{ht}(\gamma)\geq k+1} a_\gamma X_\gamma.$$

Proposition 3.5 implies for the last summand

$$\partial(U_\beta(\zeta))U_\beta(\zeta)^{-1} = l\delta(U_\beta(\zeta)) \in \mathrm{Lie}(\mathrm{SO}_{2l})_\beta(F).$$

We define $\zeta := -\frac{a_\alpha}{m_{\beta,\bar{\alpha},1}}$. Then we obtain for equation (7.14)

$$U_\beta(\zeta)A_kU_\beta(\zeta)^{-1} + \partial(U_\beta(\zeta))U_\beta(\zeta)^{-1} =$$
$$\sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Gamma_n} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{n-1}} \bar{a}_\gamma X_\gamma + \sum_{\gamma\in\Theta_n\setminus\Theta_{n-1};\mathrm{ht}(\gamma)\geq k+1} \bar{a}_\gamma X_\gamma.$$

Thus the claim follows.
An inductive argument together with the claim shows that for $n+3 \leq k \leq 2n+2$ and $A = A_{n+3}$ there exists $U \in \mathcal{U}^-$ such that

$$UAU^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Gamma_n} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{n-1}} a_\gamma X_\gamma + \sum_{\gamma\in\bar{\Theta}_n;\mathrm{ht}(\gamma)\geq k} a_\gamma X_\gamma.$$

Thus, for $k = 2n+2$ we have

$$UAU^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Gamma_n} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{n-1}} a_\gamma X_\gamma + \sum_{\gamma\in\bar{\Theta}_n;\mathrm{ht}(\gamma)\geq 2n+3} a_\gamma X_\gamma. \quad (7.15)$$

Note that the root

$$\gamma = \alpha_{l-2-n} + 2\sum_{i=l-1-n}^{l-2} \alpha_i + \alpha_{l-1} + \alpha_l$$

is the only root in $\Theta_n \setminus \Theta_{n-1}$ of $\mathrm{ht}(\gamma) \geq 2n+3$ and the only element of $\Gamma_{n-1}\setminus\Gamma_n$. Hence, we obtain for equation (7.15)

$$UAU^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma\in\Gamma_{n-1}} a_\gamma X_\gamma + \sum_{\gamma\in\Theta_{n-1}} a_\gamma X_\gamma.$$

□

Now we are ready to prove the transformation lemma. Its proof splits into two parts. In the first part we delete the terms of A which correspond to the elements of the Cartan subalgebra $\mathbf{H}(F)$. In the second part we put the results of Lemma 7.2 , 7.6 and 7.7 together and obtain so the transformation of the roots of the sets Ω_{l-3}, Λ and Θ_{l-3}.

Lemma 7.8. *(Transformation Lemma)*
Let

$$A \in \sum_{i=1}^{l} X_{\alpha_i} + \mathbf{H}(F) + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SO}_{2l})_\beta(F) = \sum_{i=1}^{l} X_{\alpha_i} + \mathrm{Lie}(\mathcal{B}^-)(F)$$

and denote by Γ *the set* Γ_0 *of Lemma 7.7. Then there exists* $U \in \mathcal{U}^-$ *such that*

$$UAU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\alpha\in T} \mathrm{Lie}(\mathrm{SO}_{2l})_\alpha.$$

Proof. First, we prove the following claim:
For $1 \leq k \leq l$ let

$$A_k = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k}^{l} a_i H_i + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SO}_{2l})_\beta =$$
$$\sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k}^{l} a_i H_i + \sum_{\beta\in\Phi^-} a_\beta X_\beta.$$

Then, there exists $U \in \mathcal{U}^-$ such that

$$UA_kU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} \sum_{i=k+1}^{l} a_i H_i + \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SO}_{2l})_\beta(F).$$

To delete the term $a_k H_k$ of A_k we differentially conjugate A_k by $U_{-\alpha_k}(\zeta) \in \mathcal{U}_{-\alpha_k}$. This yields

$$\begin{aligned} U_{-\alpha_k}(\zeta)A_kU_{-\alpha_k}(\zeta)^{-1} + \partial(U_{-\alpha_k}(\zeta))U_{-\alpha_k}(\zeta)^{-1} &= \sum_{i=1}^{l} \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_i}) \\ + \sum_{i=k}^{l} a_i \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) + \sum_{\beta\in\Phi^-} a_\beta \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_\beta) &+ \partial(U_{-\alpha_k}(\zeta))U_{-\alpha_k}(\zeta)^{-1}. \end{aligned} \tag{7.16}$$

For the first summand of the right hand side of equation (7.16) we get by Lemma 3.2

$$\sum_{i=1}^{l} \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_{\alpha_i}) \in \sum_{i=1}^{l} X_{\alpha_i} + m_{-\alpha_k,\alpha_k,1}\zeta H_{\alpha_k} - \mathrm{Lie}(\mathrm{SO}_{2l})_{-\alpha_k}(F).$$

The second summand in equation (7.16) computes with the help of Lemma 3.2 as

$$\begin{aligned} \sum_{i=k}^{l} a_i \mathrm{Ad}(U_{-\alpha_k}(\zeta))(H_i) &= \sum_{i=k}^{l} a_i H_i + \zeta\,[X_{-\alpha_k}, H_i] \\ &\in \sum_{i=k}^{l} a_i H_i + \mathrm{Lie}(\mathrm{SO}_{2l})_{\alpha_k}(F) + \mathrm{Lie}(\mathrm{SO}_{2l})_{\alpha_{k+1}}(F). \end{aligned}$$

It is easy to see that for $\beta \in \Phi^-$, $\alpha_k \in \Delta$ and $i \geq 0$ the sum $\beta - i\alpha_k$ is an element of Φ^-. Thus the third summand of equation (7.16) lies in

$$\sum_{\beta\in\Phi^-} a_\beta \mathrm{Ad}(U_{-\alpha_k}(\zeta))(X_\beta) \in \sum_{\beta\in\Phi^-} \mathrm{Lie}(\mathrm{SO}_{2l})_\beta(F).$$

The last summand is an element of the root space

$$\partial(U_{-\alpha_k}(\zeta))U_{-\alpha_k}(\zeta)^{-1} \in \mathrm{Lie}(\mathrm{SO}_{2l})_{-\alpha_k}(F).$$

If we define $\zeta := -a_k$ then the assertion of the claim follows.
Now it can be proved by the claim and induction on $k \in \{1, ..., l\}$ that there exists $U \in \mathcal{U}^-$ such that

$$UAU^{-1} + \partial(U)U^{-1} \in \sum_{i=1}^{l} X_{\alpha_i} + \sum_{i=k+1}^{l} a_i H_i + \sum_{\beta \in \Phi^-} a_\beta X_\beta.$$

In particular, this yields for $k = l$ that there exists $U \in \mathcal{U}^-$ such that

$$\bar{A}_1 := UAU^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\beta \in \Phi^-} a_\beta X_\beta.$$

For $n = l - 3$ we write $\bar{A}_1$ as in Lemma 7.2. We have

$$\bar{A}_1 = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma \in \Omega_{l-3}} a_\gamma X_\gamma + \sum_{\gamma \in \Phi^- \setminus \Omega_{l-3}} a_\gamma X_\gamma.$$

One proves by an inductive argument on $1 \leq m \leq l - 3$ together with Lemma 7.2 that there exists $U \in \mathcal{U}^-$ such that $\bar{A}_1$ is differentially equivalent to

$$U\bar{A}_1U^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma \in \Omega_{l-3-m}} a_\gamma X_\gamma + \sum_{\gamma \in \Phi^- \setminus \Omega_{l-3}} a_\gamma X_\gamma$$

This yields for $m = l - 3$ that there exists $U \in \mathcal{U}^-$ such that

$$\bar{A}_2 := U\bar{A}_1U^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma \in \Phi^- \setminus \Omega_{l-3}} a_\gamma X_\gamma = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma \in \Lambda} a_\gamma X_\gamma + \sum_{\gamma \in \Theta_{l-3}} a_\gamma X_\gamma.$$

Now we can apply Lemma 7.6 to $\bar{A}_2$. This yields

$$U\bar{A}_2U^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma \in \Theta_{l-3}} a_\gamma X_\gamma + \sum_{\gamma \in \Gamma'} a_\gamma X_\gamma = \bar{A}_3$$

where Γ' is as in Lemma 7.6. Again, it can be shown by an inductive argument on $1 \leq m \leq l - 3$ and Lemma 7.7 that for $\bar{A}_3$ there exists $U \in \mathcal{U}^-$ such that

$$U\bar{A}_3U^{-1} + \partial(U)U^{-1} = \sum_{i=1}^{l} X_{\alpha_i} + \sum_{\gamma \in \Theta_{l-3-m}} a_\gamma X_\gamma + \sum_{\gamma \in \Gamma_{l-3-m}} a_\gamma X_\gamma.$$

Then for $m = l - 3$ the assertion of the lemma follows. □

7.3 The equation with group SO_{2l}

The next step is to combine the result of Corollary 3.12 and Lemma 7.8, since we want to apply later the specialization bound. This is done in Corollary 7.9 below.
Before we start, recall that $\bar{F} := (C(z), \partial = \frac{d}{dz})$ denotes a rational function field with standard derivation.

Corollary 7.9. *Apply Corollary 3.12 to the group* SO_{2l} *and the Cartan Decomposition of* $\mathrm{Lie}(\mathrm{SO}_{2l})$. *Denote by* $A^{M\&S}_{\mathrm{SO}_{2l}} \in \mathrm{Lie}(\mathrm{SO}_{2l})(\bar{F})$ *the matrix which satisfies the stated conditions of Corollary 3.12. Then there exists* $U \in \mathcal{U}_0^- \subset \mathrm{SO}_{2l}(\bar{F})$ *such that*

$$\bar{A}_{SO_{2l}} := UA^{M\&S}_{SO_{2l}}U^{-1} + \partial(U)U^{-1} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in\Gamma} f_i X_{\gamma_i} \tag{7.17}$$

with at least one $f_i \in C[z] \setminus C$ *and the differential Galois group of the matrix differential equation* $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SO}_{2l}}\boldsymbol{y}$ *over* $C(z)$ *is* $\mathrm{SO}_{2l}(C)$.

Proof. Lemma 7.8 implies the existence of an $U \in \mathcal{U}_0^- \subset \mathrm{SO}_{2l}$ such that equation (7.17) holds. Since differential conjugation defines a differential isomorphism, we deduce with Corollary 3.12 that the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SO}_{2l}}\boldsymbol{y}$ is also $\mathrm{SO}_{2l}(C)$ over $C(z)$. We still need to show the existence of $f_i \in C[z]\backslash C$ for some $\gamma_i \in \Gamma$. Suppose $\bar{A}_{\mathrm{SO}_{2l}} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in\Gamma} f_i X_{\gamma_i} \in \mathrm{Lie}(\mathrm{SO}_{2l})(C)$. Then by Lemma 7.10 below the corresponding differential equation $L(y, f_1, ..., f_l) \in C\{y\}$ has coefficients in C. But then by [Mag94, Corollary 3.28] the differential Galois group is abelian. Thus $\bar{A}_{\mathrm{SO}_{2l}} \in \mathrm{Lie}(\mathrm{SO}_{2l})(C(z)) \setminus \mathrm{Lie}(\mathrm{SO}_{2l})(C)$. Since $0 \neq A_1 \in \mathbf{H}(C)$ and $A = (z^2A_1 + A_0)$ in Corollary 3.12, we start our transformation with at least one coefficient lying in $C[z] \setminus C$. In each step the application of $\mathrm{Ad}(U_\beta(\zeta))$ generates at most new entries which are polynomials in ζ. Moreover, the logarithmic derivative is the product of the two matrices $\partial(U_\beta(\zeta))$ and $U_\beta(\zeta)^{-1} = U_\beta(-\zeta)$. In the proofs of Lemma 7.2, 7.6, 7.7 and 7.8 we choose the parameter ζ to be one of the coefficients. Hence, we have $f_i \in C[z] \setminus C$. □

Since our goal is to compute a parametrized differential equation for the series SO_{2l}, we denote by $F = C\langle t_1, ..., t_l\rangle$ the differential field generated by the l differential indeterminates $t_1, ..., t_l$ over C and define the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SO}_{2l}}(t_1, ..., t_l)\boldsymbol{y}$ by

$$A_{\mathrm{SO}_{2l}}(t_1, ..., t_l) = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in\Gamma} t_i X_{\gamma_i}$$

where the set Γ is as in Lemma 7.8.
We compute now the linear differential equation for SO_{2l} from the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SO}_{2l}}(t_1, ..., t_l)\boldsymbol{y}$.

Lemma 7.10. *The matrix differential equation* $\partial(\boldsymbol{y}) = A_{\mathrm{SO}_{2l}}(t_1, ..., t_l)\boldsymbol{y}$ *is differentially equivalent to the homogeneous scalar linear differential equation*

$$\begin{aligned} L(y, t_1, ..., t_l) \quad &= \quad y^{(2l)} - 2\sum_{i=3}^{l}(-1)^i((t_i y^{(l-i)})^{(l+2-i)} + (t_i y^{(l+1-i)})^{(l+1-i)}) \\ &\quad -(t_2 y^{(l-2)} + t_1 y)^{(l)} - ((-1)^l t_1 z_1 + z_2) - \sum_{i=0}^{l-2}(t_2^{(l-2-i)} z_1)^{(i)} \end{aligned}$$

where the coefficients z_1 *and* z_2 *are*

$$\begin{aligned} z_1 &= y^{(l)} - t_2 y^{(l-2)} - t_1 y \\ z_2 &= \frac{(t_2^{(l-2)} + (-1)^{l-2} t_1)^{(1)}}{t_2^{(l-2)} + (-1)^{l-2} t_1} \cdot \Bigg(y^{(2l-1)} - 2 \sum_{i=3}^{l} (-1)^i ((t_i y^{(l-i)})^{(l+1-i)} + (t_i y^{(l+1-i)})^{(l-i)}) \\ &\quad - (t_2 y^{(l-2)} + t_1 y)^{(l-1)} - \sum_{i=0}^{l-3} (t_2^{(l-3-i)} z_1)^{(i)} \Bigg). \end{aligned}$$

Proof. The matrix differential equation $\partial(\boldsymbol{y}) = A_{SO_{2l}}(t_1, ..., t_l)\boldsymbol{y}$ has by the representation of the Lie algebra $\mathrm{Lie}(SO_{2l})$ in Section 7.1 the shape

$$\begin{pmatrix} \partial(y_1) \\ \partial(y_2) \\ \vdots \\ \\ \\ \\ \\ \vdots \\ \partial(y_{2l-1}) \\ \partial(y_{2l}) \end{pmatrix} = \left(\begin{array}{cccc|ccc} 0 & 1 & & & & & \\ & & \ddots & & & & \\ & & & & & & \\ & & & 1 & 1 & & \\ & & & & & -1 & \\ \hline t_1 & 0 & t_2 & 0 & 0 & -1 & \\ 0 & & & -t_2 & & & \ddots \\ & & & & & & \\ t_l & & & 0 & & & -1 \\ 0 & -t_l & 0 & -t_1 & & & 0 \end{array}\right) \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \\ \\ \\ \\ \vdots \\ y_{2l-1} \\ y_{2l} \end{pmatrix}.$$

Note that we write sometimes y_i' for $\partial(y_i)$. This matrix differential equation is equivalent to the following system of equations:

$$\begin{aligned} y_1' &= y_2 && && (1) \\ &\vdots \\ y_{l-2}' &= y_{l-1} && && (l\text{-}2) \\ y_{l-1}' &= y_l + y_{l+1} && && (l\text{-}1) \\ y_l' &= -y_{l+2} && && (l) \\ y_{l+1}' &= t_1 y_1 + t_2 y_{l-1} - y_{l+2} && && (l+1) \\ y_{l+2}' &= t_3 y_{l-2} - t_2 y_l - y_{l+3} && && (l+2) \\ y_{l+3}' &= t_4 y_{l-3} - t_3 y_{l-1} - y_{l+4} && && (l+3) \\ &\vdots \\ y_{l+k}' &= t_{k+1} y_{l-k} - t_k y_{l-k+2} - y_{l+k+1} && \text{for } 4 \le k \le l-2 && (l+k) \\ &\vdots \\ y_{2l-1}' &= t_l y_1 - t_{l-1} y_3 - y_{2l} && && (2l\text{-}1) \\ y_{2l}' &= -t_l y_2 - t_1 y_l. && && (2l) \end{aligned}$$

We show that y_1 is a cyclic vector. With the help of an easy inductive argument it follows from the Equations (1) - (l-2) that

$$y_1^{(i-1)} = y_i \quad \text{for } 1 \le i \le l-1.$$

In particular, we have $y_1^{(l-2)} = y_{l-1}$. Differentiating the expression $y_1^{(l-2)} = y_{l-1}$ and substituting y_{l-1}' by the right hand side of equation (l-1) yields $y_1^{(l-1)} = y_l + y_{l+1}$. We differentiate again and obtain from equation (l) and (l-1) the equation

$$y_1^{(l)} = t_1y_1 + t_2y_{l-1} - 2y_{l+2}.$$

Thus we have

$$y_1^{(l)} = t_1y_1 + t_2y_1^{(l-2)} - 2y_{l+2} \quad\Leftrightarrow\quad -2y_{l+2} = y_1^{(l)} - t_1y_1 - t_2y_1^{(l-2)} =: z_1.$$

Now we differentiate $y_1^{(l)} = t_1y_1 + t_2y_1^{(l-2)} - 2y_{l+2}$ and substitute y_{l+2}' by the right hand side of equation (l+2). This yields

$$\begin{aligned} y_1^{(l+1)} &= (t_1y_1 + t_2y_1^{(l-2)})' - 2t_3y_{l-2} + 2t_2y_l + 2y_{l+3} \\ &= (t_1y_1 + t_2y_1^{(l-2)})' - 2t_3y_1^{(l-3)} + 2t_2y_l + 2y_{l+3}. \end{aligned}$$

We prove the following claim: For $1 \le k \le l-3$ the system

$$\begin{aligned} y_{l+3}' &= t_4y_{l-3} - t_3y_{l-1} - y_{l+4} && (1) \\ &\vdots \\ y_{l+2+k}' &= t_{k+3}y_{l-k-2} - t_{k+2}y_{l-k} - y_{l+k+3} && (k) \end{aligned}$$

together with the equations

$$\begin{aligned} y_1^{(l+1)} &= (t_1y_1 + t_2y_1^{(l-2)})' - 2t_3y_1^{(l-3)} + 2t_2y_l + 2y_{l+3} && (A) \\ y_1^{(i-1)} &= y_i \qquad\qquad \text{for} \quad l-k-2 \le i \le l-1 && (B) \\ y_l' &= -y_{l+2} && (C) \\ -2y_{l+2} &= y_1^{(l)} - t_1y_1 - t_2y_1^{(l-2)} =: z_1 && (D) \end{aligned}$$

yields the differential equation

$$\begin{aligned} y_1^{(l+k+1)} \;=\;& (t_1y_1 + t_2y_1^{(l-2)})^{(k+1)} + 2\sum_{i=3}^{k+2}(-1)^i((t_iy_1^{(l-i)})^{(k+3-i)} + (t_iy_1^{(l-i+1)})^{(k+2-i)}) \\ & +2((-1)^{k+1}t_{k+3}y_1^{(l-k-3)} + (-1)^k y_{l+k+3} + t_2^k y_l) + \sum_{i=0}^{k-1}(t_2^{(k-i-1)}z_1)^{(i)}. \end{aligned}$$

The proof is done by induction on $1 \le k \le l-3$.
Let $k=1$. We differentiate $y_1^{(l+1)} = (t_1y_1 + t_2y_1^{(l-2)})' - 2t_3y_1^{(l-3)} + 2t_2y_l + 2y_{l+3}$ and substitute y_{l+3}' by the right hand side of equation (1). We obtain

$$\begin{aligned} y_1^{(l+2)} \;=\;& (t_1y_1 + t_2y_1^{(l-2)})^{(2)} - 2(t_3y_1^{(l-3)})^{(1)} - 2t_3y_{l-1} + 2t_4y_{l-3} \\ & +2t_2'y_l + 2t_2y_l' - 2y_{l+4}. \end{aligned}$$

Now we use Equations (B), (C) and (D) for the substitution of y_{l-1}, y_{l-3} and y_l'. This is

$$\begin{aligned} y_1^{(l+2)} &= (t_1y_1+t_2y_1^{(l-2)})^{(2)} - 2(t_3y_1^{(l-3)})^{(1)} - 2t_3y_1^{(l-2)} + 2t_4y_1^{(l-4)} \\ &\quad +2t_2'y_l - 2t_2y_{l+2} - 2y_{l+4} \\ &= (t_1y_1+t_2y_1^{(l-2)})^{(2)} - 2(t_3y_1^{(l-3)})^{(1)} - 2t_3y_1^{(l-2)} + 2t_4y_1^{(l-4)} \\ &\quad +2t_2'y_l + t_2z_1 - 2y_{l+4}. \end{aligned}$$

Now let $1 < k \leq l-3$. For $k-1$ we obtain a subsystem of the above system formed by

$$y'_{l+3} = t_4y_{l-3} - t_3y_{l-1} - y_{l+4} \tag{1'}$$

$$\vdots$$

$$y'_{l+1+k} = t_{k+2}y_{l-k-1} - t_{k+1}y_{l-k+1} - y_{l+k+2} \tag{k'}$$

and by the equations

$$y_1^{(l+1)} = (t_1y_1+t_2y_1^{(l-2)})' - 2t_3y_1^{(l-3)} + 2t_2y_l + 2y_{l+3} \tag{A'}$$

$$y_1^{(i-1)} = y_i \qquad \text{for} \quad l-k-1 \leq i \leq l-1 \tag{B'}$$

$$y_l' = -y_{l+2} \tag{C'}$$

$$-2y_{l+2} = y_1^{(l)} - t_1y_1 - t_2y_1^{(l-2)} =: z_1. \tag{D'}$$

Then the induction assumption yields for $k-1$ the differential equation

$$\begin{aligned} y_1^{(l+k)} &= (t_1y_1)^{(k)} + (t_2y_1^{(l-2)})^{(k)} + 2\sum_{i=3}^{k+1}(-1)^i((t_iy_1^{(l-i)})^{(k+2-i)} + (t_iy_1^{(l-i+1)})^{(k+1-i)}) \\ &\quad +2((-1)^kt_{k+2}y_1^{(l-k-2)} + (-1)^{k-1}y_{l+k+2} + t_2^{(k-1)}y_l) + \sum_{i=0}^{k-2}(t_2^{(k-2-i)}z_1)^{(i)}. \end{aligned} \tag{I}$$

We differentiate equation (I) and substitute y'_{l+k+2} by equation (k'). We get

$$\begin{aligned} y_1^{(l+k+1)} &= (t_1y_1+t_2y_1^{(l-2)})^{(k+1)} + 2\sum_{i=3}^{k+1}(-1)^i((t_iy_1^{(l-i)})^{(k+3-i)} + (t_iy_1^{(l-i+1)})^{(k+2-i)}) \\ &\quad +2((-1)^k(t_{k+2}y_1^{(l-k-2)})' + (-1)^{k-1}(t_{k+3}y_{l-k-2} - t_{k+2}y_{l-k} - y_{l+k+3}) \\ &\quad +t_2^{(k)}y_l + t_2^{(k-1)}y_l') + \sum_{i=0}^{k-2}(t_2^{(k-2-i)}z_1)^{(i+1)}. \end{aligned}$$

Now we use equation (B') for the substitution of y_{l-k-2} and y_{l-k}. It is easily seen that

Equations (C') and (D') imply $2y_l' = -2y_{l+2} = z_1$. Hence, we have

$$\begin{aligned}
y_1^{(l+k+1)} &= (t_1y_1 + t_2y_1^{(l-2)})^{(k+1)} + 2\sum_{i=3}^{k+1}(-1)^i((t_iy_1^{(l-i)})^{(k+3-i)} + (t_iy_1^{(l-i+1)})^{(k+2-i)}) \\
&\quad +2((-1)^k(t_{k+2}y_1^{(l-k-2)})' + (-1)^k t_{k+2}y_1^{(l-k-1)})) + 2(-1)^{k-1}t_{k+3}y_1^{l-k-3} \\
&\quad +2(-1)^k y_{l+k+3} + 2t_2^{(k)}y_l + t_2^{(k-1)}z_1) + \sum_{i=0}^{k-2}(t_2^{(k-2-i)}z_1)^{(i+1)} \\
&= (t_1y_1 + t_2y_1^{(l-2)})^{(k+1)} + 2\sum_{i=3}^{k+2}(-1)^i((t_iy_1^{(l-i)})^{(k+3-i)} + (t_iy_1^{(l-i+1)})^{(k+2-i)}) \\
&\quad +2((-1)^{k+1}t_{k+3}y_1^{l-k-3} + (-1)^k y_{l+k+3} + t_2^{(k)}y_l) + \sum_{i=0}^{k-1}(t_2^{(k-1-i)}z_1)^{(i)}.
\end{aligned}$$

Thus the induction is completed and the claim follows.
The claim yields for $k = l-3$ the differential equation

$$\begin{aligned}
y_1^{(2l-2)} &= (t_1y_1 + t_2y_1^{(l-2)})^{(l-2)} + 2\sum_{i=3}^{l-1}(-1)^i((t_iy_1^{(l-i)})^{(l-i)} + (t_iy_1^{(l-i+1)})^{(l-1-i)}) \\
&\quad +2((-1)^{l-2}t_ly_1 + (-1)^{l-3}y_{2l} + t_2^{(l-3)}y_l) + \sum_{i=0}^{l-4}(t_2^{(l-4-i)}z_1)^{(i)}.
\end{aligned}$$

We differentiate it and use equation (2l) for the substitution of y_{2l}'. However, we obtain

$$\begin{aligned}
y_1^{(2l-1)} = (t_1y_1 + t_2y_1^{(l-2)})^{(l-1)} &+ 2\sum_{i=3}^{l-1}(-1)^i((t_iy_1^{(l-i)})^{(l-i+1)} + (t_iy_1^{(l-i+1)})^{(l-i)}) \\
&+2((-1)^{l-2}(t_ly_1)' + (-1)^{l-3}(-t_ly_1' - t_1y_l) + t_2^{(l-2)}y_l + t_2^{(l-3)}y_l') \\
&+\sum_{i=0}^{l-4}(t_2^{(l-4-i)}z_1)^{(i+1)}.
\end{aligned} \tag{II}$$

Using the same ideas as above equation (II) simplifies to

$$\begin{aligned}
y_1^{(2l-1)} = (t_1y_1 + t_2y_1^{(l-2)})^{(l-1)} &+ 2\sum_{i=3}^{l}(-1)^i((t_iy_1^{(l-i)})^{(l-i+1)} + (t_iy_1^{(l-i+1)})^{(l-i)}) \\
&+2((-1)^{l-2}t_1 + t_2^{(l-2)})y_l + \sum_{i=0}^{l-3}(t_2^{(l-3-i)}z_1)^{(i)}.
\end{aligned} \tag{III}$$

We solve equation (III) for $2y_l$ and multiply it by $((-1)^{l-2}t_1 + t_2^{(l-2)})'$, i.e. we get

$$\begin{aligned}
2((-1)^{l-2}t_1 + t_2^{(l-2)})'y_l &= \frac{((-1)^{l-2}t_1 + t_2^{(l-2)})'}{(-1)^{l-2}t_1 + t_2^{(l-2)}} \cdot \big(y_1^{(2l-1)} - (t_1y_1 + t_2y_1^{(l-2)})^{(l-1)} \\
&-2\sum_{i=3}^{l}(-1)^i((t_iy_1^{(l-i)})^{(l-i+1)} + (t_iy_1^{(l-i+1)})^{(l-i)}) - \sum_{i=0}^{l-3}(t_2^{(l-3-i)}z_1)^{(i)}\big) =: z_2.
\end{aligned} \tag{IV}$$

Differentiating equation (IV) leads us to

$$\begin{aligned}
y_1^{(2l)} &= (t_1y_1 + t_2y_1^{(l-2)})^{(l)} + 2\sum_{i=3}^{l}(-1)^i((t_iy_1^{(l-i)})^{(l-i+2)} + (t_iy_1^{(l-i+1)})^{(l-i+1)}) \\
&\quad +2((-1)^{l-2}t_1 + t_2^{(l-2)})'y_l + 2(-1)^{l-2}t_1y_l' + 2t_2^{(l-2)}y_l' + \sum_{i=0}^{l-3}(t_2^{(l-3-i)}z_1)^{(i+1)} \\
&= (t_1y_1 + t_2y_1^{(l-2)})^{(l)} + 2\sum_{i=3}^{l}(-1)^i((t_iy_1^{(l-i)})^{(l-i+2)} + (t_iy_1^{(l-i+1)})^{(l-i+1)}) \\
&\quad +2((-1)^{l-2}t_1 + t_2^{(l-2)})'y_l + (-1)^lt_1z_1 + t_2^{(l-2)}z_1 + \sum_{i=0}^{l-3}(t_2^{(l-3-i)}z_1)^{(i+1)} \\
&= (t_1y_1 + t_2y_1^{(l-2)})^{(l)} + 2\sum_{i=3}^{l}(-1)^i((t_iy_1^{(l-i)})^{(l-i+2)} + (t_iy_1^{(l-i+1)})^{(l-i+1)}) \\
&\quad +(-1)^lt_1z_1 + z_2 + \sum_{i=0}^{l-2}(t_2^{(l-2-i)}z_1)^{(i)}.
\end{aligned}$$

□

Theorem 7.11. *The homogeneous linear differential equation*

$$\begin{aligned}
L(y, t_1, ..., t_l) &= y^{(2l)} - 2\sum_{i=3}^{l}(-1)^i((t_iy^{(l-i)})^{(l+2-i)} + (t_iy^{(l+1-i)})^{(l+1-i)}) \\
&\quad -(t_2y^{(l-2)} + t_1y)^{(l)} - ((-1)^lt_1z_1 + z_2) - \sum_{i=0}^{l-2}(t_2^{(l-2-i)}z_1)^{(i)}
\end{aligned}$$

where the coefficients z_1 and z_2 are

$$\begin{aligned}
z_1 &= y^{(l)} - t_2y^{(l-2)} - t_1y \\
z_2 &= \frac{(t_2^{(l-2)} + (-1)^{l-2}t_1)^{(1)}}{t_2^{(l-2)} + (-1)^{l-2}t_1} \cdot \Bigg(y^{(2l-1)} - 2\sum_{i=3}^{l}(-1)^i((t_iy^{(l-i)})^{(l+1-i)} + (t_iy^{(l+1-i)})^{(l-i)}) \\
&\quad - (t_2y^{(l-2)} + t_1y)^{(l-1)} - \sum_{i=0}^{l-3}(t_2^{(l-3-i)}z_1)^{(i)}\Bigg)
\end{aligned}$$

has $\mathrm{SO}_{2l}(C)$ as differential Galois group over $F = C\langle \boldsymbol{t}\rangle$. Moreover, let $\hat{F}$ be a differential field with field of constants equal to C. Let $\hat{E}$ be a Picard-Vessiot extension over $\hat{F}$ with differential Galois group $\mathrm{SO}_{2l}(C)$ and suppose the defining matrix differential equation $\partial(\boldsymbol{y}) = \hat{A}\boldsymbol{y}$ satisfies $\hat{A} \in \sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}_0^-)$. Then there is a specialization $L(y, \hat{t}_1, ..., \hat{t}_l)$ with $\hat{t}_i \in \hat{F}$ such that $L(y, \hat{t}_1, ..., \hat{t}_l)$ gives rise to the extension $\hat{E}$ over $\hat{F}$.

Proof. Let E be a Picard-Vessiot extension for the differential equation $L(y, t_1, ..., t_l) = 0$ over F and denote by $\mathcal{G}$ its differential Galois group. Since the linear differential equation is equivalent to the matrix differential equation $\partial(\boldsymbol{y}) = A_{\mathrm{SO}_{2l}}(t_1, ..., t_l)\boldsymbol{y}$ with

$A_{\mathrm{SO}_{2l}}(t_1, ..., t_l) \in \mathrm{Lie}(\mathrm{SO}_{2l})(F)$, Proposition 2.1 yields $\mathcal{G}(C) \leq \mathrm{SO}_{2l}(C)$. By Corollary 7.9 there exists a specialization $\sigma : (t_1, ..., t_l) \to (f_1, ..., f_l)$ with $f_1, ..., f_l \in C[z]$ such that $\sigma(A_{\mathrm{SO}_{2l}}(t_1, ..., t_l)) = \bar{A}_{\mathrm{SO}_{2l}}$ and the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{\mathrm{SO}_{2l}}\boldsymbol{y}$ is $\mathrm{SO}_{2l}(C)$. Moreover, we have $C\{f_1, ..., f_l\} = C[z]$. Thus we can apply Corollary 2.15. This yields $\mathrm{SO}_{2l}(C) \leq \mathcal{G}(C)$. Hence, it holds $\mathcal{G}(C) = \mathrm{SO}_{2l}(C)$.
Since the defining matrix $\hat{A}$ satisfies $\hat{A} \in \sum_{\alpha\in\Delta} X_\alpha + \mathrm{Lie}(\mathcal{B}_0^-)$, Lemma 7.8 provides that $\hat{A}$ is differentially equivalent to a matrix $\tilde{A} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in\Gamma} \hat{a}_i X_{\gamma_i}$ with suitable $\hat{a}_i \in \hat{F}$. Evidently, the specialization

$$\hat{\sigma} : (t_1, ..., t_l) \mapsto (\hat{a}_1, ..., \hat{a}_l)$$

does the required. $\square$

Chapter 8

A parametrized equation for G_2

8.1 A Lie algebra representation of G_2

The below discussion can be found in [Hum72, Section 12.1]. Denote by $\epsilon_1,\ \epsilon_2,\ \epsilon_3$ the standard orthonormal basis of $\mathbb{R}^3$ and let (α, β) be the usual inner product of $\alpha,\ \beta \in \mathbb{R}^3$. Then, the vectors

$$\Phi = \pm\left\{\epsilon_1 - \epsilon_2,\ \epsilon_2 - \epsilon_3,\ \epsilon_1 - \epsilon_3,\ 2\epsilon_1 - \epsilon_2 - \epsilon_3,\ 2\epsilon_2 - \epsilon_2 - \epsilon_3,\ 2\epsilon_3 - \epsilon_1 - \epsilon_2\right\}$$

form the root system Φ of type G_2. As a basis we take the set

$$\Delta = \{\epsilon_1 - \epsilon_2 =: \alpha_1,\ -2\epsilon_1 + \epsilon_2 + \epsilon_3 =: \alpha_2\}.$$

The Cartan integers $\langle \alpha_i, \alpha_j \rangle = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ are given by the entry at position (i, j) in the Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

With respect to this basis the roots of Φ can be expressed uniquely as

$$\begin{aligned} \pm(\alpha_1 + \alpha_2) &= \pm(-\epsilon_1 + \epsilon_3) \\ \pm(2\alpha_1 + \alpha_2) &= \pm(-\epsilon_2 + \epsilon_3) \\ \pm(3\alpha_1 + \alpha_2) &= \pm(-2\epsilon_2 + \epsilon_1 + \epsilon_3) \\ \pm(3\alpha_1 + 2\alpha_2) &= \pm(2\epsilon_3 - \epsilon_1 - \epsilon_2). \end{aligned}$$

We are going to construct the Lie algebra $\mathbf{L}$ of type G_2 as a subalgebra of Lie(SO_7), the Lie algebra of type B_3, where we take the representation of Lie(SO_7) as presented in [Hum72, Section 1.2]. We will follow the ideas presented in [Hum72, Section 19.3]. From the root system we see directly that $\mathbf{L}$ has dimension 14 and the Cartan subalgebra $\mathbf{H}$ is of dimension 2. Denote by E_{rs} with $1 \le r, s \le 7$ the matrices having 1 as entry at position (r, s) and 0 elsewhere. The Cartan subalgebra $\bar{\mathbf{H}}$ of Lie(SO_7) has the set

$$\bar{\mathbf{H}} = \{D_i = E_{i+1,i+1} - E_{i+4,i+4} \mid 1 \le i \le 3\}$$

as a basis. For the Cartan subalgebra of G_2 we take

$$\mathbf{H} = \left\{ \sum_{i=1}^{3} a_i D_i \mid \sum_{i=1}^{3} a_i = 0 \text{ with } a_i \in C \right\}.$$

Obviously the dimension of $\mathbf{H}$ is $\dim(\mathbf{H}) = 2$. Following Humphreys we choose the root vectors $G_{i,j}$ $(i \neq j)$ of $\mathrm{Lie}(\mathrm{O}_7)$ relative to $\bar{\mathbf{H}}$, which correspond to the six long roots in G_2, as follows:

$$\begin{aligned} G_{1,-2} &= G_{2,-1}^t = E_{23} - E_{65} \\ G_{1,-3} &= G_{3,-1}^t = E_{24} - E_{75} \\ G_{2,-3} &= G_{3,-2}^t = E_{34} - E_{76}. \end{aligned}$$

Furthermore, for the six short roots $G_{\pm i}$ $(i = 1, 2, 3)$ of G_2 relative to $\mathbf{H}$, we take the matrices

$$\begin{aligned} G_1 &= -G_{-1}^t = \sqrt{2}(E_{12} - E_{51}) - (E_{37} - E_{46}) \\ G_2 &= -G_{-2}^t = \sqrt{2}(E_{13} - E_{61}) - (E_{27} - E_{45}) \\ G_3 &= -G_{-3}^t = \sqrt{2}(E_{14} - E_{71}) - (E_{26} - E_{35}). \end{aligned}$$

Then, the span of $\mathbf{H}$ together with these twelve vectors is the irreducible representation $\mathbf{L}$ of G_2 in $\mathrm{Lie}(\mathrm{SO}_7)$. The next step is to determine how these twelve matrices can be assigned to the roots of G_2. The relations of the root vectors under the bracket product are described by the following equations:

$$\begin{aligned} [G_{i,-j}, G_{k,-l}] &= \delta_{jk} G_{i,-l} - \delta_{il} G_{k,-j} \\ [G_i, G_{-i}] &= 3D_i - (D_1 + D_2 + D_3) \\ [G_{i,-j}, G_k] &= -\delta_{ik} G_j \\ [G_{i,-j}, G_{-k}] &= \delta_{jk} G_{-i} \\ [G_i, G_{-j}] &= 3G_{j,-i} \qquad i \neq j \\ [G_i, G_j] &= \pm 2G_{-k} \qquad i, j, k \text{ distinct} \\ [G_{-i}, G_{-j}] &= \pm 2G_k \qquad i, j, k \text{ distinct} \end{aligned}$$

where $\delta_{i,j}$ denotes the Kronecker delta. It is useful to distinguish between the long and the short roots. The set

$$\Phi_L^{\pm} = \{\pm\alpha_2, \pm 3\alpha_1 \pm \alpha_2, \pm 3\alpha_1 \pm 2\alpha_2\}$$

contains all long roots of Φ. Moreover,

$$\Phi_S^{\pm} = \{\pm\alpha_1, \pm\alpha_1 \pm \alpha_2, \pm 2\alpha_1 \pm \alpha_2\}$$

is the set of all short roots of Φ. The long roots of G_2 form a root system of type A_2 (see, for example, [Hum72, Section 12.2, Exercise 4]). Therefore, the computation of

$$\begin{aligned} [G_{1,-2}, G_{2,-3}] &= G_{1,-3} \\ [G_{2,-1}, G_{3,-2}] &= -G_{3,-1} \\ [G_{1,-2}, G_{3,-2}] &= 0 \\ [G_{2,-3}, G_{2,-1}] &= 0 \end{aligned}$$

implies that we can define the root vectors corresponding to the long roots of maximal height as

$$X_{3\alpha+3\alpha_2} := G_{1,-3} \text{ and } X_{-3\alpha_1-2\alpha_2} := G_{3,-1}.$$

Hence, we obtain

$$X_{3\alpha_1+\alpha_2} := G_{1,-2},\ X_{\alpha_2} := G_{2,-3},\ X_{-3\alpha_1-\alpha_2} := G_{2,-1} \text{ and } X_{-\alpha_2} := G_{3,-2}$$

respectively. Note that there are other choices for the assignments of the root vectors to the root system A_2 in Φ_L possible. The bracket products

$$\begin{aligned} [G_{1,-3}, G_i] &= -\delta_{1,i} G_3 \neq 0 \Leftrightarrow i = 1 \\ [G_{1,-3}, G_{-i}] &= \delta_{3,i} G_{-1} \neq 0 \Leftrightarrow i = 3 \\ [G_{3,-1}, G_i] &= -\delta_{3,i} G_1 \neq 0 \Leftrightarrow i = 3 \\ [G_{3,-1}, G_{-i}] &= \delta_{1,i} G_{-3} \neq 0 \Leftrightarrow i = 1 \end{aligned}$$

yield $G_1,\ G_{-3} \in \Phi_S^- \setminus \{-\alpha_1\}$ and $G_3,\ G_{-1} \in \Phi_S^+ \setminus \{\alpha_1\}$. Thus, with the help of the Lie products

$$\begin{aligned} [G_{1,-2}, G_{1},] &= -\delta_{11} G_2 \\ [G_{1,-2}, G_{-3},] &= 0 \\ [G_{2,-1}, G_{-1},] &= -\delta_{11} G_{-2} \\ [G_{2,-1}, G_{3},] &= 0 \end{aligned}$$

we are able to define

$$X_{\alpha_1} := G_2,\ X_{-2\alpha_1-\alpha_2} := G_1,\ X_{-\alpha_1-\alpha_2} := G_{-3}$$
$$\text{and}$$
$$X_{-\alpha_1} := G_{-2},\ X_{2\alpha_1+\alpha_2} := G_{-1},\ X_{\alpha_1+\alpha_2} := G_3.$$

From a short calculation we get $H_2 := [G_{2,-3}, G_{3,-2}] = D_2 - D_3$ and $H_1 = [G_2, G_{-2}] = -D_1 + 2D_2 - D_3$. Obviously the Cartan algebra $\mathbf{H}$ is spanned by H_1 and H_2. Summing up the Lie Algebra $\mathbf{L}$ consists of the 14 elements

$$\begin{array}{ll} H_1 = -D_1 + 2D_2 - D_3 & H_2 = D_2 - D_3 \\ X_{\alpha_1} = G_2 & X_{-\alpha_1} = G_{-2} \\ X_{\alpha_2} = G_{2,-3} & X_{-\alpha_2} = G_{3,-2} \\ X_{\alpha_1+\alpha_2} = G_3 & X_{-\alpha_1-\alpha_2} = G_{-3} \\ X_{2\alpha_1+\alpha_2} = G_{-1} & X_{-2\alpha_1-\alpha_2} = G_1 \\ X_{3\alpha_1+\alpha_2} = G_{1,-2} & X_{-3\alpha_1-\alpha_2} = G_{2,-1} \\ X_{3\alpha+3\alpha_2} = G_{1,-3} & X_{-3\alpha_1-2\alpha_2} = G_{3,-1}. \end{array}$$

8.2 The transformation lemma for G_2

In this section we prove the transformation lemma for G_2. Let (F, ∂_F) be a differential field of characteristic 0.

Lemma 8.1. *Let $A \in X_{\alpha_1} + X_{\alpha_2} + \sum_{\beta \in \Phi^-} \mathbf{L}_\beta(F)$. Then there exists $U \in \mathcal{U}^-(F) \subset \mathrm{GL}_7(F)$ such that*

$$UAU^{-1} + \partial(U)U^{-1} \in X_{\alpha_1} + X_{\alpha_2} + \mathbf{L}_{-\alpha_2} + \mathbf{L}_{-3\alpha_1-2\alpha_2}.$$

Proof. We write A with respect to the basis given in the previous section as

$$A_0 = X_{\alpha_1} + X_{\alpha_2} + a_{0,1}H_1 + a_{0,2}H_2 + \sum_{\beta\in\Phi^-} a_{0,\beta}X_\beta.$$

Let $\alpha \in \Phi$. Then $\zeta\mathrm{ad}(X_\alpha)$ is a nilpotent derivation of $\mathbf{L}$. Let $X \in \mathbf{L}$. Thus, as in Section 3.2, the map $\exp(\zeta\mathrm{ad}(X_\alpha))$ is an automorphism of $\mathbf{L}$. The application of $\exp(\zeta\mathrm{ad}(X_\alpha))$ to X reads as

$$\exp(\zeta\mathrm{ad}(X_\alpha)).X = \sum_{i\geq 0}\frac{1}{i!}\zeta^i\mathrm{ad}(X_\alpha)^i(X) = U_\alpha(\zeta)XU_\alpha(\zeta)^{-1} = \mathrm{Ad}(U_\alpha(\zeta))(X) \qquad (8.1)$$

where $U_\alpha(\zeta)$ equals $\exp(\zeta X_\alpha)$. For $\beta\in\Phi$ we rewrite equation (8.1) with suitable $\tilde{m}_{\alpha,\beta,i} \in \mathbb{Q}^*$ as

$$\mathrm{Ad}(U_\alpha(\zeta))(X_\beta) = X_\beta + \sum_{i\geq 1}\tilde{m}_{\alpha,\beta,i}\zeta^i X_{\beta+i\alpha}. \qquad (8.2)$$

In the first step we want to remove the part $H_0 := a_{0,1}H_1 + a_{0,2}H_2$ of A_0. Therefore, we differentially conjugate A_0 with $U_{\alpha_1}(\zeta_1)$. Observation 3.4 and the linearity of Ad yield

$$\begin{aligned}\mathrm{Ad}(U_{-\alpha_1}(\zeta_1))(A_0) + l\delta(U_{-\alpha_1}(\zeta_1))) &= \mathrm{Ad}(U_{-\alpha_1}(\zeta_1))(X_{\alpha_1}) + \mathrm{Ad}(U_{-\alpha_1}(\zeta_1))(X_{\alpha_2})\\ &+\mathrm{Ad}(U_{-\alpha_1}(\zeta_1))(H_0) + \sum_{\beta\in\Phi^-}\mathrm{Ad}(U_{-\alpha_1}(\zeta_1))(a_{0,\beta}X_\beta) + l\delta(U_{-\alpha_1}(\zeta_1)).\end{aligned} \qquad (8.3)$$

Since the only multiples of a root $\alpha\in\Phi$ are $\pm\alpha$ and the coefficients k_i of a root $\alpha = \sum_{\alpha_i\in\delta}k_i\alpha_i$ are all positive or negative, we obtain with formula (8.2) for the first three summands of equation (8.3)

$$\begin{aligned}\mathrm{Ad}(U_{-\alpha_1}(\zeta_1))(X_{\alpha_1}) &= X_{\alpha_1} + \sum_{i\geq 1}\tilde{m}_{-\alpha_1,\alpha_1,i}\,\zeta_1^i X_{\alpha_1+i(-\alpha_1)}\\ &\in X_{\alpha_1} + \tilde{m}_{-\alpha_1,\alpha_1,1}\zeta_1 H_1 + \mathbf{L}_{-\alpha_1}\\ \mathrm{Ad}(U_{-\alpha_1}(\zeta_1))(X_{\alpha_2}) &= X_{\alpha_2} + \sum_{i\geq 1}\tilde{m}_{-\alpha_1,\alpha_2,i}\,\zeta_1^i X_{\alpha_2+i(-\alpha_1)} = X_{\alpha_2}\\ \mathrm{Ad}(U_{-\alpha_1}(\zeta_1))(H_0) &= H_0 + \zeta_1\,[X_{-\alpha_1}, H_0] \in a_{0,1}H_1 + a_{0,2}H_2 + \mathbf{L}_{-\alpha_1}.\end{aligned}$$

Obviously, the sum of $-\alpha_1+\beta$ with $\beta\in\Phi^-$ again lies in Φ^-. Hence, the fourth summand is

$$\mathrm{Ad}(U_{-\alpha_1}(\zeta_1))(\sum_{\beta\in\Phi^-}a_{0,\beta}X_\beta) = \sum_{\beta\in\Phi^-}a_{0,\beta}(X_\beta + \sum_{i\geq 1}\tilde{m}_{-\alpha_1,\beta,i}\,\zeta_1^i X_{\beta+i(-\alpha_1)}) \in \sum_{\beta\in\Phi^-}\mathbf{L}_\beta.$$

The last summand is calculated by Proposition 3.5, which gives us

$$\partial(U_{-\alpha_1}(\zeta_1))U_{-\alpha_1}(\zeta_1)^{-1} \in \mathbf{L}_{-\alpha_1}.$$

We set $\tilde{m}_{-\alpha_1,\alpha_1,1}\,\zeta_1 = -a_{0,1}$ and summarize the above results. We have

$$A_1 := \mathrm{Ad}(U_{-\alpha_1}(\zeta_1))(A_0) + \partial(U_{-\alpha_1}(\zeta_1)))U_{-\alpha_1}(\zeta_1))^{-1} = X_{\alpha_1} + X_{\alpha_2} + a_{0,2}H_2 + \sum_{\beta\in\Phi^-}a_{1,\beta}X_\beta$$

with suitable $a_{1,\beta} \in F$. Now we differentially conjugate A_1 by $U_{-\alpha_2}(\zeta_2)$. This gives

$$\begin{aligned}\mathrm{Ad}(U_{-\alpha_2}(\zeta_2))(A_1) + l\delta(U_{-\alpha_2}(\zeta_2))) &= \mathrm{Ad}(U_{-\alpha_2}(\zeta_2))(X_{\alpha_1}) + \mathrm{Ad}(U_{-\alpha_2}(\zeta_2))(X_{\alpha_2}) \\ &+ a_{0,2}\mathrm{Ad}(U_{-\alpha_2}(\zeta_2))(H_2) + \sum_{\beta\in\Phi^-} a_{1,\beta}\mathrm{Ad}(U_{-\alpha_2}(\zeta_2))(X_\beta) + l\delta(U_{-\alpha_2}(\zeta_2)).\end{aligned} \tag{8.4}$$

By the same arguments as above we get for the summands of the right hand side of equation (8.4)

$$\begin{aligned}\mathrm{Ad}(U_{-\alpha_2}(\zeta_2))(X_{\alpha_1}) &= X_{\alpha_1} + \sum_{i\geq 1} \tilde{m}_{-\alpha_2,\alpha_1,i}\ \zeta_2^i X_{\alpha_1+i(-\alpha_2)} = X_{\alpha_1} \\ \mathrm{Ad}(U_{-\alpha_2}(\zeta_2))(X_{\alpha_2}) &= X_{\alpha_2} + \sum_{i\geq 1} \tilde{m}_{-\alpha_2,\alpha_2,i}\ \zeta_2^i X_{\alpha_2+i(-\alpha_2)} \\ &\in X_{\alpha_2} + \tilde{m}_{-\alpha_2,\alpha_2,1}\ \zeta_2 H_2 + \mathbf{L}_{-\alpha_2} \\ \mathrm{Ad}(U_{-\alpha_2}(\zeta_2))(a_{0,2}H_2) &= a_{0,2}H_2 + \zeta_2 a_{0,2}\,[X_{-\alpha_2}, H_2] \in a_{0,2}H_2 + \mathbf{L}_{-\alpha_2} \\ \mathrm{Ad}(U_{-\alpha_2}(\zeta_2))(\sum_{\beta\in\Phi^-} a_{1,\beta}X_\beta) &= \sum_{\beta\in\Phi^-} a_{1,\beta}(X_\beta + \sum_{i\geq 1} \tilde{m}_{-\alpha_2,\beta,i}\ \zeta_2^i X_{\beta+i(-\alpha_2)}) \in \sum_{\beta\in\Phi^-} \mathbf{L}_\beta \\ \partial(U_{-\alpha_2}(\zeta_2))U_{-\alpha_2}(\zeta_2)^{-1} &\in \mathbf{L}_{-\alpha_2}.\end{aligned}$$

If we set $\tilde{m}_{-\alpha_2,\alpha_2,1}\ \zeta_2 = -a_{0,2}$, then we obtain for equation (8.4)

$$A_2 := \mathrm{Ad}(U_{-\alpha_2}(\zeta_2))(A_1) + \partial(U_{-\alpha_2}(\zeta_2)))U_{-\alpha_2}(\zeta_2))^{-1} = X_{\alpha_1} + X_{\alpha_2} + \sum_{\beta\in\Phi^-} a_{2,\beta}X_\beta$$

with suitable $a_{2,\beta} \in F$.
The next step is to delete the parts of A_2 which lie in the root spaces $\mathbf{L}_{-\alpha}$, $\mathbf{L}_{-\alpha_1-\alpha_2}$, $\mathbf{L}_{-2\alpha_1-\alpha_2}$ and $\mathbf{L}_{-3\alpha_1-\alpha_2}$. The candidates for these transformations are the root group elements $U_{-\alpha_1-\alpha_2}(\zeta)$, $U_{-2\alpha_1-\alpha_2}(\zeta)$, $U_{-3\alpha_1-\alpha_2}(\zeta)$ and $U_{-3\alpha_1-2\alpha_2}(\zeta)$.
But before we can start with the transformation we need to understand better the adjoint action on several root spaces. The tables (8.1), (8.2), (8.3) and (8.4) give the images $\mathbf{L}_{\beta+k\alpha}$ of the root spaces $\mathbf{L}_\beta$ with $\beta \in \Phi^- \cup \{\alpha_1, \alpha_2\}$ under $\mathrm{Ad}(U_\alpha)$ for $\alpha \in \{-\alpha_1-\alpha_2, -2\alpha_1-\alpha_2, -3\alpha_1-\alpha_2, -3\alpha_1-2\alpha_2\}$.

$-\alpha_1-\alpha_2$	k=1	k=2	k=3
α_1	$-\alpha_2$	–	–
α_2	$-\alpha_1$	$-2\alpha_1-\alpha_2$	$-3\alpha_1-2\alpha_2$
$-\alpha_1$	$-2\alpha_1-\alpha_2$	$-3\alpha_1-\alpha_2$	–
$-\alpha_2$	–	–	–
$-\alpha_1-\alpha_2$	–	–	–
$-2\alpha_1-\alpha_2$	$-3\alpha_1-2\alpha_2$	–	–
$-3\alpha_1-\alpha_2$	–	–	–
$-3\alpha_1-2\alpha_2$	–	–	–

Table (8.1)

$-3\alpha_1-\alpha_2$	k=1
α_1	$-2\alpha_1-\alpha_2$
α_2	–
$-\alpha_1$	–
$-\alpha_2$	$-3\alpha_1-2\alpha_2$
$-\alpha_1-\alpha_2$	–
$-2\alpha_1-\alpha_2$	–
$-3\alpha_1-\alpha_2$	–
$-3\alpha_1-2\alpha_2$	–

Table (8.3)

$-2\alpha_1-\alpha_2$	k=1	k=2
α_1	$-\alpha_1-\alpha_2$	$-3\alpha_1-2\alpha_2$
α_2	$-$	$-$
$-\alpha_1$	$-3\alpha_1-\alpha_2$	$-$
$-\alpha_2$	$-$	$-$
$-\alpha_1-\alpha_2$	$-3\alpha_1-2\alpha_2$	$-$
$-2\alpha_1-\alpha_2$	$-$	$-$
$-3\alpha_1-\alpha_2$	$-$	$-$
$-3\alpha_1-2\alpha_2$	$-$	$-$

Table (8.2)

$-3\alpha_1-2\alpha_2$	k=1
α_1	$-$
α_2	$-3\alpha_1-\alpha_2$
$-\alpha_1$	$-$
$-\alpha_2$	$-$
$-\alpha_1-\alpha_2$	$-$
$-2\alpha_1-\alpha_2$	$-$
$-3\alpha_1-\alpha_2$	$-$
$-3\alpha_1-2\alpha_2$	$-$

Table (8.4)

Table (8.1) yields that X_{α_2} is send by $\mathrm{Ad}(U_{-\alpha_1-\alpha_2})$ to the root space $\mathbf{L}_{-\alpha_1}$. Thus, we can use this to remove the part of A_2 which lies in the root space $\mathbf{L}_{-\alpha_1}$. Hence, with the help of table (8.1) we obtain that the summands of the right hand side of

$$\begin{aligned}\mathrm{Ad}(U_{-\alpha_1-\alpha_2}(\zeta))(A_2)+\partial(U_{-\alpha_1-\alpha_2}(\zeta))U_{-\alpha_1-\alpha_2}(\zeta)^{-1} = \mathrm{Ad}(U_{-\alpha_1-\alpha_2}(\zeta))(X_{\alpha_1})\\ +\mathrm{Ad}(U_{-\alpha_1-\alpha_2}(\zeta))(X_{\alpha_2})+\sum_{\beta\in\Phi^-} a_{2,\beta}\mathrm{Ad}(U_{-\alpha_1-\alpha_2}(\zeta))(X_\beta)+l\delta(U_{-\alpha_1-\alpha_2}(\zeta))\end{aligned} \tag{8.5}$$

are equal to

$$\begin{aligned}
\mathrm{Ad}(U_{-\alpha_1-\alpha_2}(\zeta))(X_{\alpha_1}) &= X_{\alpha_1}+\sum_{i\geq 1}\tilde{m}_{-\alpha_1-\alpha_2,\alpha_1,i}\,\zeta^i X_{\alpha_1+i(-\alpha_1-\alpha_2)}\\
&= X_{\alpha_1}+\tilde{m}_{-\alpha_1-\alpha_2,\alpha_1,1}\,\zeta X_{-\alpha_2}\\
\mathrm{Ad}(U_{-\alpha_1-\alpha_2}(\zeta))(X_{\alpha_2}) &= X_{\alpha_2}+\sum_{i\geq 1}\tilde{m}_{-\alpha_1-\alpha_2,\alpha_2,i}\,\zeta^i X_{\alpha_2+i(-\alpha_1-\alpha_2)}\\
&\in X_{\alpha_2}+\tilde{m}_{-\alpha_1-\alpha_2,\alpha_2,1}\,\zeta X_{-\alpha_1}+\mathbf{L}_{-2\alpha_1-\alpha_2}+\mathbf{L}_{-3\alpha_1-2\alpha_2}\\
\mathrm{Ad}(U_{-\alpha_1-\alpha_2}(\zeta))(\sum_{\beta\in\Phi^-}a_{2,\beta}X_\beta) &= \sum_{\beta\in\Phi^-}a_{2,\beta}(X_\beta\sum_{i\geq 1}\tilde{m}_{-\alpha_1-\alpha_2,\beta,i}\,\zeta^i X_{\beta+i(-\alpha_1-\alpha_2)})\\
&\in \sum_{\beta\in\Phi^-}a_{2,\beta}X_\beta+\mathbf{L}_{-2\alpha_1-\alpha_2}+\mathbf{L}_{-3\alpha_1-\alpha_2}+\mathbf{L}_{-3\alpha_1-2\alpha_2}\\
\partial(U_{-\alpha_1-\alpha_2}(\zeta))U_{-\alpha_1-\alpha_2}(\zeta)^{-1} &\in \mathbf{L}_{-\alpha_1-\alpha_2}.
\end{aligned}$$

We define $\Theta_1 := \{-\alpha_1\}$ and set $\tilde{m}_{-\alpha_1-\alpha_2,\alpha_2,1}\,\zeta = -a_{2,-\alpha_1}$. Then equation (8.5) becomes

$$A_3 := \mathrm{Ad}(U_{-\alpha_1-\alpha_2}(\zeta))(A_2)+\partial(U_{-\alpha_1-\alpha_2}(\zeta))U_{-\alpha_1-\alpha_2}(\zeta)^{-1} = X_{\alpha_1}+X_{\alpha_2}+\sum_{\beta\in\Phi^-\setminus\Theta_1}a_{3,\beta}X_\beta.$$

To delete the part of A_3 which lies in the root space $\mathbf{L}_{-\alpha_1-\alpha_2}$ we differentially conjugate A_3 by $U_{-2\alpha_1-\alpha_2}(\zeta)$. Note that by table (8.2) this conjugation sends no vector of the root spaces, which form the subspace containing A_3, to $\mathbf{L}_{-\alpha_1}$. More precisely, the conjugation yields

$$\begin{aligned}\mathrm{Ad}(U_{-2\alpha_1-\alpha_2}(\zeta))(A_3)+\partial(U_{-2\alpha_1-\alpha_2}(\zeta))U_{-2\alpha_1-\alpha_2}(\zeta)^{-1} = \mathrm{Ad}(U_{-2\alpha_1-\alpha_2}(\zeta))(X_{\alpha_1})+\\ \mathrm{Ad}(U_{-2\alpha_1-\alpha_2}(\zeta))(X_{\alpha_2})+\sum_{\beta\in\Phi^-\setminus\Theta_1}a_{3,\beta}\mathrm{Ad}(U_{-2\alpha_1-\alpha_2}(\zeta))(X_\beta)+l\delta(U_{-2\alpha_1-\alpha_2}(\zeta))).\end{aligned} \tag{8.6}$$

With the help of table (8.2) we get for the summands of the right hand side of equation (8.6)

$$\begin{aligned}
\mathrm{Ad}(U_{-2\alpha_1-\alpha_2}(\zeta))(X_{\alpha_1}) &= X_{\alpha_1} + \sum_{i\geq 1} \tilde{m}_{-2\alpha_1-\alpha_2,\alpha_1,i}\, \zeta^i X_{\alpha_1+i(-2\alpha_1-\alpha_2)} \\
&\in X_{\alpha_1} + \tilde{m}_{-2\alpha_1-\alpha_2,\alpha_1,1}\, \zeta X_{-\alpha_1-\alpha_2} + \mathbf{L}_{-3\alpha_1-2\alpha_2} \\
\mathrm{Ad}(U_{-2\alpha_1-\alpha_2}(\zeta))(X_{\alpha_2}) &= X_{\alpha_2} + \sum_{i\geq 1} \tilde{m}_{-2\alpha_1-\alpha_2,\alpha_2,i}\, \zeta^i X_{\alpha_2+i(-2\alpha_1-\alpha_2)} \\
&= X_{\alpha_2} \\
\sum_{\beta\in\Phi^-\setminus\Theta_1} a_{3,\beta}\mathrm{Ad}(U_{-2\alpha_1-\alpha_2}(\zeta))(X_\beta) &= \sum_{\beta\in\Phi^-\setminus\Theta_1} a_{3,\beta}\Big(X_\beta + \sum_{i\geq 1} \tilde{m}_{-2\alpha_1-\alpha_2,\beta,i} \\
&\quad \zeta^i\, X_{\beta+i(-2\alpha_1-\alpha_2)}\Big) \\
&\in \sum_{\beta\in\Phi^-\setminus\Theta_1} a_{3,\beta}X_\beta + \mathbf{L}_{-3\alpha_1-2\alpha_2} \\
\partial(U_{-2\alpha_1-\alpha_2}(\zeta))U_{-2\alpha_1-\alpha_2}(\zeta)^{-1} &\in \mathbf{L}_{-2\alpha_1-\alpha_2}.
\end{aligned}$$

We set $\tilde{m}_{-2\alpha_1-\alpha_2,\alpha_1,1}\, \zeta = -a_{3,-\alpha_1-\alpha_2}$ and $\Theta_2 := \Theta_1 \cup \{-\alpha_1-\alpha_2\}$. We obtain for equation (8.6)

$$A_4 := \mathrm{Ad}(U_{-2\alpha_1-\alpha_2}(\zeta))(A_3) + l\delta(U_{-2\alpha_1-\alpha_2}(\zeta)) = X_{\alpha_1} + X_{\alpha_2} + \sum_{\beta\in\Phi^-\setminus\Theta_2} a_{4,\beta}X_\beta.$$

Now we want to delete the term $a_{4,-2\alpha_1-\alpha_2}X_{-2\alpha_1-\alpha_2}$. For this we differentially conjugate A_4 with $U_{-3\alpha_1-\alpha_2}(\zeta)$. We get

$$\begin{aligned}
&\mathrm{Ad}(U_{-3\alpha_1-\alpha_2}(\zeta))(A_4) + \partial(U_{-3\alpha_1-\alpha_2}(\zeta))U_{-3\alpha_1-\alpha_2}(\zeta)^{-1} = \mathrm{Ad}(U_{-3\alpha_1-\alpha_2}(\zeta))(X_{\alpha_1}) + \\
&\quad \mathrm{Ad}(U_{-3\alpha_1-\alpha_2}(\zeta))(X_{\alpha_2}) + \sum_{\beta\in\Phi^-\setminus\Theta_2} a_{4,\beta}\mathrm{Ad}(U_{-3\alpha_1-\alpha_2}(\zeta))(X_\beta) + l\delta(U_{-3\alpha_1-\alpha_2}(\zeta)).
\end{aligned} \tag{8.7}$$

For the summands of the right hand side of equation (8.7), table (8.3) yields

$$\begin{aligned}
\mathrm{Ad}(U_{-3\alpha_1-\alpha_2}(\zeta))(x_{\alpha_1}) &= X_{\alpha_1} + \sum_{i\geq 1} \tilde{m}_{-3\alpha_1-\alpha_2,\alpha_1,i}\, \zeta^i X_{\alpha_1+i(-3\alpha_1-\alpha_2)} \\
&= X_{\alpha_1} + \tilde{m}_{-3\alpha_1-\alpha_2,\alpha_1,1}\, \zeta X_{-2\alpha_1-\alpha_2} \\
\mathrm{Ad}(U_{-3\alpha_1-\alpha_2}(\zeta))(X_{\alpha_2}) &= X_{\alpha_2} + \sum_{i\geq 1} \tilde{m}_{-3\alpha_1-\alpha_2,\alpha_2,i}\, \zeta^i X_{\alpha_2+i(-3\alpha_1-\alpha_2)} \\
&= X_{\alpha_2} \\
\mathrm{Ad}(U_{-3\alpha_1-\alpha_2}(\zeta))(\sum_{\beta\in\Phi^-\setminus\Theta_2} a_{4,\beta}X_\beta) &= \sum_{\beta\in\Phi^-\setminus\Theta_2} a_{4,\beta}\Big(X_\beta + \sum_{i\geq 1} \tilde{m}_{-3\alpha_1-\alpha_2,\beta,i} \\
&\quad \zeta^i\, X_{\beta+i(-3\alpha_1-\alpha_2)}\Big) \\
&\in \sum_{\beta\in\Phi^-\setminus\Theta_2} a_{4,\beta}X_\beta + \mathbf{L}_{-3\alpha_1-2\alpha_2} \\
\partial(U_{-3\alpha_1-\alpha_2}(\zeta))U_{-3\alpha_1-\alpha_2}(\zeta)^{-1} &\in \mathbf{L}_{-3\alpha_1-\alpha_2}.
\end{aligned}$$

With $\tilde{m}_{-3\alpha_1-\alpha_2,\alpha_1,1}\,\zeta = -a_{4,-2\alpha_1-\alpha_2}$ and $\Theta_3 := \Theta_2 \cup \{-2\alpha_1-\alpha_2\}$ we obtain

$$A_5 := \mathrm{Ad}(U_{-3\alpha_1-\alpha_2}(\zeta))(A_4) + l\delta(U_{-3\alpha_1-\alpha_2}(\zeta)) = X_{\alpha_1} + X_{\alpha_2} + \sum_{\beta\in\Phi^-\setminus\Theta_3} a_{5,\beta}X_\beta.$$

Finally, we want to eliminate the part of A_5 lying in $\mathbf{L}_{-3\alpha_1-\alpha_2}$. A look at table (8.4) implies that we can get rid of $a_{5,-3\alpha_1-\alpha_2}X_{-3\alpha_1-\alpha_2}$ by a differential conjugation with the root group element $U_{-3\alpha_1-2\alpha_2}(\zeta)$ without creating a new vector lying in the just above deleted root spaces. We obtain

$$\begin{aligned}\mathrm{Ad}(U_{-3\alpha_1-2\alpha_2}(\zeta))(A_5) + \partial(U_{-3\alpha_1-2\alpha_2}(\zeta))U_{-3\alpha_1-2\alpha_2}(\zeta)^{-1} &= \mathrm{Ad}(U_{-3\alpha_1-2\alpha_2}(\zeta))(X_{\alpha_1}) + \\ \mathrm{Ad}(U_{-3\alpha_1-2\alpha_2}(\zeta))(X_{\alpha_2}) &+ \sum_{\beta\in\Phi^-\setminus\Theta_3} a_{5,\beta}\mathrm{Ad}(U_{-3\alpha_1-2\alpha_2}(\zeta))(X_\beta) \\ = X_{\alpha_1} + X_{\alpha_2} + \tilde{m}_{-3\alpha_1-2\alpha_2,\alpha_2,1}\,\zeta X_{-3\alpha_1-\alpha_2} &+ \sum_{\beta\in\Phi^-\setminus\Theta_3} a_{5,\beta}X_\beta.\end{aligned}$$

Hence, with $\tilde{m}_{-3\alpha_1-2\alpha_2,\alpha_2,1}\,\zeta = -a_{5,-3\alpha_1-\alpha_2}$ the lemma follows. □

8.3 The equation with group G_2

We combine now the results of Lemma 8.1 and Corollary 3.12 in Corollary 8.2 below. Denote by $\mathcal{G}_{G_2}$ the group of type G_2 with the Lie algebra $\mathbf{L}$ presented in Section 8.1. Moreover, let

$$\Omega := \{\gamma_1 := -3\alpha_1 - 2\alpha_2,\ \gamma_2 := -\alpha_2,\}$$

and let $C(z)$ be as in Section 3.4. Further, we keep all notations of Lemma 8.1.

Corollary 8.2. *We apply Corollary 3.12 to the group $\mathcal{G}_{G_2}$ and the above Cartan Decomposition. We denote by $A^{M\&S}_{G_2}$ the matrix satisfying the stated conditions of Corollary 3.12. Then there exists $U \in \mathcal{U}^-(C(z)) \subset \mathcal{G}_{G_2}(C(z))$ such that*

$$\bar{A}_{G_2} := UA^{M\&S}_{G_2}U^{-1} + \partial(U)U^{-1} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in\Omega} f_iX_{\gamma_i} \tag{8.8}$$

with at least one $f_i \in C[z]\setminus C$ and the differential Galois group of the matrix equation $\partial(\boldsymbol{y}) = \bar{A}_{G_2}\boldsymbol{y}$ is $\mathcal{G}_{G_2}(C)$ over $C(z)$.

Proof. Lemma 8.1 implies the existence of an element $U \in \mathcal{U}_0^- \subset \mathcal{G}_{G_2}$ such that equation (8.8) holds. Since differential conjugation defines a differential isomorphism, we deduce with Corollary 3.12 that the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{G_2}\boldsymbol{y}$ is again $\mathcal{G}_{G_2}(C)$ over $C(z)$. We still need to show the existence of $f_i \in C[z]\setminus C$ for some $\gamma_i \in \Omega$. Suppose $\bar{A}_{G_2} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in T} f_iX_{\gamma_i} \in \mathrm{Lie}(\mathcal{G}_{G_2})(C)$. Then by Lemma 8.3 below the corresponding differential equation $L(y, f_1, ..., f_l) \in C\{y\}$ has coefficients in C. But then by [Mag94, Corollary 3.28] the differential Galois group is abelian. Thus, we obtain $\bar{A}_{G_2} \in \mathrm{Lie}(\mathcal{G}_{G_2})(C(z)) \setminus \mathrm{Lie}(\mathcal{G}_{G_2})(C)$. Since $0 \neq A_1 \in \mathbf{H}(C)$ and $A = (z^2A_1 + A_0)$ in Corollary 3.12, we start our transformation with at least one coefficient lying in $C[z]\setminus C$. In each step application of $\mathrm{Ad}(U_\beta(\zeta))$ generates at most new entries which are polynomials in ζ. Moreover, the logarithmic derivative is the product of the two matrices $\partial(U_\beta(\zeta))$ and $U_\beta(\zeta)^{-1} = U_\beta(-\zeta)$. In the proof of Lemma 8.1 we choose the parameter ζ to be one of the coefficients. Hence, we get $f_i \in C[z]\setminus C$. □

Lemma 8.3. *Let C be an algebraically closed field of characteristic zero and $F = C\langle t_1, t_2\rangle$ the differential field generated by the differential indeterminates t_1, t_2. Then the matrix*

$$A_{G_2}(t_1,t_2) = X_{\alpha_1} + X_{\alpha_2} + t_1 X_{-3\alpha_1-2\alpha_2} + t_2 X_{-\alpha_2}$$

has the shape

$$A_{G_2}(t_1,t_2) = \left(\begin{array}{c|ccc|ccc} 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & t_1 & t_2 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -t_1 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & -t_2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right)$$

and the matrix differential equation $\partial(\boldsymbol{y}) = A_{G_2}(t_1,t_2)\boldsymbol{y}$ is equivalent to the differential equation

$$y^{(7)} = 2t_1y' + 2(t_1y)' + 2(t_2y^{(4)})' + (t_2y')^{(4)} - 2(t_2(t_2y')')'.$$

Proof. The matrix equation

$$\begin{pmatrix} \partial(y_1) \\ \vdots \\ \partial(y_7) \end{pmatrix} = A_{G_2}(t_1,t_2) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_7 \end{pmatrix}$$

is equivalent to the system of equations defined by

$$\begin{aligned} y_1' &= \sqrt{2}y_3 \\ y_2' &= y_7 \\ y_3' &= y_4 \\ y_4' &= t_1y_2 + t_2y_3 - y_5 \\ y_5' &= -t_1y_7 \\ y_6' &= -\sqrt{2}y_1 - t_2y_7 \\ y_7' &= -y_6 \end{aligned}$$

where we use the notation y_i' for $\partial(y_i)$. We can take y_2 as a cyclic vector. We compute the derivatives of y_2:

$$\begin{aligned} y_2^{(1)} &= y_7 \\ y_2^{(2)} &= -y_6 \\ y_2^{(3)} &= \sqrt{2}y_1 + t_2y_2^{(1)} \\ y_2^{(4)} &= (t_2y_2^{(1)})^{(1)} + 2y_3 \\ y_2^{(5)} &= (t_2y_2^{(1)})^{(2)} + 2y_4 \\ y_2^{(6)} &= (t_2y_2^{(1)})^{(3)} + 2t_1y_2 + 2t_2y_2^{(4)} - 2t_2(t_2y_2^{(1)})^{(1)} - 2y_5 \\ y_2^{(7)} &= (t_2y_2^{(1)})^{(4)} + 2(t_1y_2)^{(1)} + 2(t_2y_2^{(4)})^{(1)} - 2(t_2(t_2y_2^{(1)})^{(1)})^{(1)} + 2t_1y_2^{(1)}. \end{aligned}$$

□

Theorem 8.4. *Let $F = C\langle t_1, t_2\rangle$ be as in Lemma 8.3. The differential equation*

$$L(t_1,t_2,y) = y^{(7)} - 2t_1y' - 2(t_1y)' - 2(t_2y^{(4)})' - (t_2y')^{(4)} + 2(t_2(t_2y')')'.$$

has G_2 as differential Galois group over $C\langle t_1, t_2\rangle$. Moreover, let $\hat{F}$ be a differential field with field of constants equal to C. Let $\hat{E}$ be a Picard-Vessiot extension over $\hat{F}$ with differential Galois group $\mathcal{G}_{G_2}(C)$ and suppose the defining matrix differential equation $\partial(\boldsymbol{y}) = \hat{A}\boldsymbol{y}$ satisfies $\hat{A} \in X_{\alpha_1} + X_{\alpha_2} + \sum_{\alpha\in\Phi^-} \mathbf{L}_\alpha$. Then there is a specialization $L(y, \hat{t}_1, \hat{t}_2)$ with $\hat{t}_i \in \hat{F}$ such that $L(y, \hat{t}_1, \hat{t}_2)$ gives rise to the extension $\hat{E}$ over $\hat{F}$.

Proof. Let E be a Picard-Vessiot extension for the equation $L(y, t_1, t_2) = 0$ over F and denote by $\mathcal{G}$ the differential Galois group. Since the operator comes from the matrix differential equation $\partial(\boldsymbol{y}) = A_{G_2}(t_1,t_2)\boldsymbol{y}$ with $A_{G_2}(t_1,t_2) \in \mathrm{Lie}(\mathcal{G}_{G_2})(F)$, Proposition 2.1 yields $\mathcal{G}(C) \leq \mathcal{G}_{G_2}(C)$. By Corollary 8.2 there exists a specialization $\sigma : (t_1, ..., t_l) \to (f_1, f_2)$ with $f_1, f_2 \in C[z]$ such that $\sigma(A_{G_2}(t_1, ..., t_l)) = \bar{A}_{G_2}$ and the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{G_2}\boldsymbol{y}$ is $\mathcal{G}_{G_2}(C)$. Moreover, we have $C\{f_1, f_2\} = C[z]$. Thus we can apply Corollary 2.15. This yields $\mathcal{G}_{G_2}(C) \leq \mathcal{G}(C)$. Hence, it holds $\mathcal{G}(C) = \mathcal{G}_{G_2}(C)$.
Since the defining matrix $\hat{A}$ satisfies $\hat{A} \in X_{\alpha_1} + X_{\alpha_2} + \sum_{\alpha\in\Phi^-} \mathrm{Lie}(\mathcal{G}_{G_2})_\alpha$, Lemma 8.1 provides that $\hat{A}$ is differentially equivalent to a matrix $\tilde{A} = X_{\alpha_1} + X_{\alpha_2} + \hat{a}_1X_{-3\alpha_1-2\alpha_2} + \hat{a}_2X_{-\alpha_2}$ with suitable $\hat{a}_i \in \hat{F}$. Obviously the specialization

$$\hat{\sigma} : (t_1, t_2) \mapsto (\hat{a}_1, \hat{a}_2)$$

does the required. □

In [Kat90, Theorem 2.10.6] Katz presented an equation for G_2 which has a nice and easy shape. His result is cited in Theorem 8.5 below.

Theorem 8.5. *For any polynomial f in $C[z]$ of degree k prime to 6, the differential Galois group of*

$$\partial^7 - f\partial - \frac{1}{2}f' \tag{8.9}$$

on $\mathbb{A}^1$ is G_2.

Now the question arises if we can specialize $L(t_1, t_2, y)$ to equation (8.9). Obviously the specialization $\sigma : (t_1, t_2) \to (\frac{1}{4}f, 0)$ satisfies

$$L(\sigma(t_1), \sigma(t_2), y) = y^{(7)} - fy' - \frac{1}{2}f'y.$$

Chapter 9

A parametrized equation for F_4

9.1 The root system of type F_4

The following construction of the root system of type F_4 is taken from [Hum72, Section 12.1]. Let $\epsilon_1, ..., \epsilon_4$ denote the standard orthonormal unit vectors of $\mathbb{R}^4$ and let (α, β) denote the usual inner product for $\alpha, \beta \in \mathbb{R}^4$. Denote by I the $\mathbb{Z}$-span of this basis. Then by definition I is a lattice. Moreover, let $I' = I + \mathbb{Z}((\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)/2)$. Then by [Hum72, Section 12.1] the set $\Phi = \{\alpha \in I' \mid (\alpha, \alpha) = 1 \text{ or } 2\}$ defines the root system of type F_4. It consists of all elements $\pm\epsilon_i$, $\pm(\epsilon_i - \epsilon_j)$ (here we need $i \neq j$) and $\pm\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$ where the signs may be chosen independently. We can take the vectors

$$\alpha_1 = \epsilon_2 - \epsilon_3,\ \alpha_2 = \epsilon_3 - \epsilon_4,\ \alpha_3 = \epsilon_4 \text{ and } \alpha_4 = \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$$

as a basis of Φ which we denote by Δ. Then the roots are $\mathbb{Z}$-linear combinations of this basis vectors. In particular, for the 24 positive roots of Φ (for this number see [Hum72, Section 12.2, Table 1]) we have

$$\begin{array}{ll}
\epsilon_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, & \epsilon_1 + \epsilon_4 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4, \\
\epsilon_2 = \alpha_1 + \alpha_2 + \alpha_3, & \epsilon_2 + \epsilon_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3, \\
\epsilon_3 = \alpha_2 + \alpha_3, & \epsilon_2 + \epsilon_4 = \alpha_1 + \alpha_2 + 2\alpha_3, \\
\epsilon_4 = \alpha_3, & \epsilon_3 + \epsilon_4 = \alpha_2 + 2\alpha_3, \\
\epsilon_1 - \epsilon_2 = \alpha_2 + 2\alpha_3 + 2\alpha_4, & \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \\
\epsilon_1 - \epsilon_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, & \frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4) = \alpha_2 + 2\alpha_3 + \alpha_4, \\
\epsilon_1 - \epsilon_4 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, & \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4) = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \\
\epsilon_2 - \epsilon_3 = \alpha_1, & \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4, \\
\epsilon_2 - \epsilon_4 = \alpha_1 + \alpha_2, & \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4) = \alpha_3 + \alpha_4, \\
\epsilon_3 - \epsilon_4 = \alpha_2, & \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\
\epsilon_1 + \epsilon_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, & \frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4) = \alpha_2 + \alpha_3 + \alpha_4, \\
\epsilon_1 + \epsilon_3 = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, & \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) = \alpha_4.
\end{array}$$

Suppose a representation of the Lie algebra of type F_4 to $\mathrm{Lie}(\mathrm{GL}(V))$ is given and denote its image by $\mathbf{L} \leq \mathrm{Lie}(\mathrm{GL}(V))$. Further let $\mathbf{H}$ denote a Cartan subalgebra and $\mathbf{L} = \mathbf{H} \oplus \bigoplus_{\alpha\in\Phi} \mathbf{L}_\alpha$ be a Cartan decomposition of $\mathbf{L}$. Then for each $\alpha \in \Phi$ we are able to choose X_α together with $H_\alpha = [X_\alpha, X_{-\alpha}]$ such that the set $\{X_\alpha,\ H_\alpha \mid \alpha \in \Phi\}$ forms a Chevalley basis. The Chevalley construction yields a representation of the group of type F_4 which we denote by $\mathcal{G}$. Further, we denote for each $\beta \in \Phi$ the corresponding root subgroups by $\mathcal{U}_\beta$ and a parametrized element of $\mathcal{U}_\beta$ by $U_\beta(\zeta)$ with $\zeta \in F$. Let α and β be two roots of Φ. Then the adjoint action of $U_\beta(\zeta)$ on X_α is determined (see also Section 3.2) by

$$\mathrm{Ad}(U_\beta(\zeta))(X_\alpha) = \sum_{i \geq 0} m_{\alpha+i\beta} \cdot \zeta^i \cdot X_{\alpha+i\beta}. \tag{9.1}$$

For β, α linearly independent let $\alpha - r\beta, \ldots, \alpha + q\beta$ be the β-string through α. Then the values for $m_{\beta,\alpha,i}$ are determined by $m_{\beta,\alpha,i} = \pm\binom{r+i}{i}$ and $m_{\beta,\alpha,0} = 0$. Since the proof of the transformation lemma is based on differential conjugation, it is useful to study more detailed the adjoint action for some specific roots. Let α be one of the simple roots $\alpha_j \in \Delta$ and let $\beta \in \Phi$ be the h-th positive root of height $\mathrm{ht}(\beta) = k \geq 2$ which we indicate by $\beta_{h,k}$. Note that the numbering of the roots of a given height is arbitrarily defined by us below. We determine for each $-\beta_{h,k} \in \Phi^-$ and $\alpha_j \in \Delta$ if

$$\hat{\beta}_{j,h} := \alpha_j + (-\beta_{h,k}) \tag{9.2}$$

is a root of Φ or not, i.e., we analyse if the term $m_{\alpha+i\beta} \cdot \zeta^i \cdot X_{\alpha+i\beta}$ of equation (9.1) is for $i = 1$ zero or not.
We start with the negative roots of height 2. Those are the roots $-\beta_{1,2} = -\alpha_1 - \alpha_2$, $-\beta_{2,2} = -\alpha_2 - \alpha_3$, and $-\beta_{3,2} = -\alpha_3 - \alpha_4$. From the list which contains all positive roots on the previous page we obtain the $\hat{\beta}_{j,h}$ for $h = 1, 2, 3$. The root $\hat{\beta}_{j',h'}$ can be found at position j', h' of table (9.1).

	$-\beta_{1,2}$	$-\beta_{2,2}$	$-\beta_{3,2}$
α_1	$-\alpha_2$		
α_2	$-\alpha_1$	$-\alpha_3$	
α_3		$-\alpha_2$	$-\alpha_4$
α_4			$-\alpha_3$

Table (9.1)

Note that if $\hat{\beta}_{j,h}$ is not a root then the position j, h is empty.
The next step is to analyse the negative roots of height 3. There are the three negative roots, namely $-\beta_{1,3} = -\alpha_1 - \alpha_2 - \alpha_3$, $-\beta_{2,3} = -\alpha_2 - 2\alpha_3$ and $-\beta_{3,3} = -\alpha_2 - \alpha_3 - \alpha_4$. For those roots we determine the $\hat{\beta}_{j,h}$. This is presented in table (9.2).

	$-\beta_{1,3}$	$-\beta_{2,3}$	$-\beta_{3,3}$
α_1	$-\alpha_2 - \alpha_3$		
α_2			$-\alpha_3 - \alpha_4$
α_3	$-\alpha_1 - \alpha_2$	$-\alpha_2 - \alpha_3$	
α_4			$-\alpha_2 - \alpha_3$

Table (9.2)

Now we come to the negative roots of height 4. Then the $-\beta_{h,4}$ are the roots $-\beta_{1,4} = -\alpha_1 - \alpha_2 - 2\alpha_3$, $-\beta_{2,4} = -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$ and $-\beta_{3,4} = -\alpha_2 - 2\alpha_3 - \alpha_4$. The result of the analysis of those roots is given in table (9.3).

	$-\beta_{1,4}$	$-\beta_{2,4}$	$-\beta_{3,4}$
α_1	$-\alpha_2 - 2\alpha_3$	$-\alpha_2 - \alpha_3 - \alpha_4$	
α_2			
α_3	$-\alpha_1 - \alpha_2 - \alpha_3$		$-\alpha_2 - \alpha_3 - \alpha_4$
α_4		$-\alpha_1 - \alpha_2 - \alpha_3$	$-\alpha_2 - 2\alpha_3$

Table (9.3)

If $\mathrm{ht}(\beta) = 5$, then we obtain for the root $-\beta_{1,5} = -\alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4$ the roots $\hat{\beta}_{1,1}$, $\hat{\beta}_{3,1}$ and $\hat{\beta}_{4,1}$, i.e., we have

$$\hat{\beta}_{1,1} = -\alpha_2 - 2\alpha_3 - \alpha_4,\ \hat{\beta}_{3,1} = -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 \quad \text{and} \quad \hat{\beta}_{4,1} = -\alpha_1 - \alpha_2 - 2\alpha_3.$$

Moreover, for the remaining roots $\beta_{h,5}$ $(h \neq 1)$ of height 5 we get from the list

$$\begin{aligned} \hat{\beta}_{2,2} &= -\alpha_1 - \alpha_2 - 2\alpha_3 \quad \text{for} \quad \beta_{2,5} = -\alpha_1 - 2\alpha_2 - 2\alpha_3 \quad \text{and} \\ \hat{\beta}_{4,3} &= -\alpha_2 - 2\alpha_3 - \alpha_4 \quad \text{for} \quad \beta_{3,5} = -\alpha_2 - 2\alpha_3 - 2\alpha_4. \end{aligned}$$

This is summarized in table (9.4).

	$-\beta_{1,5}$	$-\beta_{2,5}$	$-\beta_{3,5}$
α_1	$-\alpha_2 - 2\alpha_3 - \alpha_4$		
α_2		$-\alpha_1 - \alpha_2 - 2\alpha_3$	
α_3	$-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$		
α_4	$-\alpha_1 - \alpha_2 - 2\alpha_3$		$-\alpha_2 - 2\alpha_3 - \alpha_4$

Table (9.4)

Now we consider the negative roots of height 6. There are two negative roots of height six, namely $-\beta_{1,6} = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$ and $-\beta_{2,6} = -\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4$. The computation of the $\hat{\beta}_{j,h}$ for $-\beta_{1,6}$ and $-\beta_{2,6}$ shows that they have the root $-\alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4$ in common. However, the results for the negative roots of height 6 can be found in table (9.5).

	$-\beta_{1,6}$	$-\beta_{2,6}$
α_1		$-\alpha_2 - 2\alpha_3 - 2\alpha_4$
α_2	$-\alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4$	
α_3		
α_4	$-\alpha_1 - 2\alpha_2 - 2\alpha_3$	$-\alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4$

Table (9.5)

The negative roots of height seven are $-\beta_{1,7} = -\alpha_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4$ and $-\beta_{2,7} = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4$. From the list which contains all positiv roots we determine the $\hat{\beta}_{j,h}$ for $h = 1, 2$ and $k = 7$. The result is given in table (9.6).

	$-\beta_{1,7}$	$-\beta_{2,7}$
α_1		
α_2		$-\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4$
α_3	$-\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$	
α_4		$-\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$

Table (9.6)

The root $-\beta_{1,8} = -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4$ is the only root of height eight. Only for the two simple roots α_3 and α_4 we obtain roots of the root system in the sense of equation (9.2), i.e., we obtain table (9.7).

	$-\beta_{1,8}$
α_1	
α_2	
α_3	$-\alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4$
α_4	$-\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$

Table (9.7)

The analysis for the negative roots of height 9, 10 and 11 can be found in table (9.8) below. We have $-\beta_{1,9} = -\alpha_1 - 2\alpha_2 - 4\alpha_3 - 2\alpha_4$ of height 9, $-\beta_{1,10} = -\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4$ of height 10 and $-\beta_{1,11} = -2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4$ the root of maximal height.

	$-\beta_{1,9}$	$-\beta_{1,10}$	$-\beta_{1,11}$
α_1			$-\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4$
α_2		$-\alpha_1 - 2\alpha_2 - 4\alpha_3 - 2\alpha_4$	
α_3	$-\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4$		
α_4			

Table (9.8)

9.2 The transformation lemma for F_4

Let (F, ∂) be a differential field of characteristic 0. We are going to prove the transformation lemma for the group of type F_4. In the proof we make use of the study of the root system Φ of type F_4 done in the previous section. Therefore we keep the notations done there.

Lemma 9.1. *Let $A \in X_{\alpha_1} + X_{\alpha_2} + X_{\alpha_3} + X_{\alpha_4} + \sum_{\beta\in\Phi^-} \mathbf{L}_\beta(F) + \mathbf{H}(F)$. Then there exists $U \in \mathcal{U}^-$ such that*

$$\begin{aligned} UAU^{-1} + \partial(U)U^{-1} \quad &\in \quad X_{\alpha_1} + X_{\alpha_2} + X_{\alpha_3} + X_{\alpha_4} + \mathbf{L}_{-\alpha_1}(F) + \mathbf{L}_{-\alpha_1-2\alpha_2-2\alpha_3}(F) \\ &+ \quad \mathbf{L}_{-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4}(F) + \mathbf{L}_{-2\alpha_1-3\alpha_2-4\alpha_3-2\alpha_4}(F). \end{aligned}$$

Proof. With respect to a Chevalley basis $\{X_\alpha,\ H_i\}$ an element of $X_{\alpha_1} + X_{\alpha_2} + X_{\alpha_3} + X_{\alpha_4} + \sum_{\beta\in\Phi^-} \mathrm{L}_\beta(F) + \mathbf{H}(F)$ is given by

$$A_0 = \sum_{i=1}^{4} X_{\alpha_i} + a_{0,i}H_i + \sum_{\beta\in\Phi^-} a_{0,\beta}X_\beta.$$

In the first step we get rid of the part of A lying in the Cartan subalgebra, i.e., we delete the vectors of the subspace $\langle H_i \mid i = 1, ..., l\rangle$. Let $-\alpha_i$ be the negative of a simple root α_i. We differentially conjugate A_0 with $U_{-\alpha_i}(\zeta)$. We have

$$\begin{aligned} \mathrm{Ad}(U_{-\alpha_i}(\zeta))(A_0) + l\delta(U_{-\alpha_i}(\zeta)) &= \sum_{j=1}^{4} \mathrm{Ad}(U_{-\alpha_i}(\zeta))(X_{\alpha_j}) + a_{0,j}\mathrm{Ad}(U_{-\alpha_i}(\zeta))(H_j) \\ &+ \sum_{\beta\in\Phi^-} a_{0,\beta}\mathrm{Ad}(U_{-\alpha_i}(\zeta))(X_\beta) + l\delta(U_{-\alpha_i}(\zeta)). \end{aligned}$$

Proposition 3.5 yields $l\delta(U_{-\alpha_i}(\zeta)) \in \langle X_{\alpha_i}\rangle$. From the signs of the roots we deduce that $\sum_{\beta\in\Phi^-} a_{0,\beta}\mathrm{Ad}(U_{-\alpha_i}(\zeta))(X_\beta)$ is an element of $\sum_{\beta\in\Phi^-} \mathbf{L}_\beta$. The elements $\mathrm{Ad}(U_{-\alpha_i}(\zeta))(X_{\alpha_j})$ for $j \neq i$ are

$$\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(X_{\alpha_j}) = X_{\alpha_j} + \sum_{k\geq 1} m_{-\alpha_i,\,\alpha_j,\,k}\zeta_i^k X_{\alpha_j+k(-\alpha_i)} = X_{\alpha_j}.$$

Further, for $j = i$ we have

$$\begin{aligned}\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(X_{\alpha_i}) &= X_{\alpha_i} + \sum_{k\geq 1} m_{-\alpha_i,\,\alpha_i,\,k}\zeta_i^k X_{\alpha_i+k(-\alpha_i)} \\ &\in X_{\alpha_i} + m_{-\alpha_i,\,\alpha_i,\,1}\zeta_i H_i + \mathbf{L}_{-\alpha_i}.\end{aligned}$$

Let H_0 denote $H_0 = \sum_{i=1}^4 a_{0,i}H_i$. We obtain $\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(H_0) = H_0 + \zeta_i[X_{-\alpha_i}, H_0] \in H_0 + \mathbf{L}_{-\alpha_i}$. We put now our results together. We conclude

$$\begin{aligned}\mathrm{Ad}(U_{-\alpha_4}(\zeta_4)\cdot\ldots\cdot U_{-\alpha_1}(\zeta_1))(A_0) + l\delta(U_{-\alpha_4}(\zeta_4)\cdot\ldots\cdot U_{-\alpha_1}(\zeta_1)) = \\ \sum_{i=1}^4 X_{\alpha_i} + (a_{0,i} + m_{-\alpha_i,\,\alpha_i,\,1}\zeta_i)H_i \\ + \sum_{\beta\in\Phi^-} a_{1,\beta}X_\beta := A_1\end{aligned}$$

where the new coefficients $a_{1,\beta}$ are elements of F. If we define the parameter ζ_i as $\zeta_i = \frac{-a_{0,i}}{m_{-\alpha_i,\alpha_i,1}}$, then it follows $A_1 = \sum_{i=1}^4 X_{\alpha_i} + \sum_{\beta\in\Phi^-} a_{1,\beta}X_\beta$.

In the next step we delete all parts of A_0 lying in the subspaces $\langle X_{\hat\beta}\rangle$ of all negative roots $\hat\beta \in \Phi^-$ except of $-\alpha_1$, $-\alpha_1 - 2\alpha_2 - 2\alpha_3$, $-\alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4$ and $-2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4$. Since we will do this for each height k, there are some repeating arguments and facts. If we want to delete a vector which corresponds to a root $\hat\beta_{j,h}$ of height k, we differentially conjugate with a parametrized root group element $\mathcal{U}_{-\beta_{h,k+1}}(\zeta)$ which corresponds to one of the roots $-\beta_{h,k+1}$ of height $k+1$. By Proposition 3.5 the logarithmic derivate $l\delta(\mathcal{U}_{-\beta_{h,k+1}}(\zeta))$ of $\mathcal{U}_{-\beta_{h,k+1}}(\zeta)$ is an element of $\mathbf{L}_{-\beta_{h,k+1}}$, i.e., it is a vector lying in a root space which corresponds to a negative root of height $k+1$. If γ is any negative root, then $\mathrm{Ad}(U_{-\beta_{h,k+1}}(\zeta))(a_\gamma X_\gamma)$ is an element of the space

$$\mathrm{Ad}(U_{-\beta_{h,k+1}}(\zeta))(a_\gamma X_\gamma) = a_\gamma X_\gamma + \sum_{\bar\gamma\in\Phi^-,\ \mathrm{ht}(\bar\gamma)\geq\mathrm{ht}(\gamma+(-\beta_{h,k+1}))} \mathbf{L}_{\bar\gamma}.$$

We will not refer to this arguments in each step of the argumentation, since it would make the proof needlessly long.

We start with the negative roots of height one. Let $\beta_{1,2}$, $\beta_{2,2}$ and $\beta_{3,2}$ be as in the previous section. Then we deduce with the help of the table (9.1)

$$\mathrm{Ad}(U_{\beta_{3,2}}(\zeta_3)U_{\beta_{2,2}}(\zeta_2)U_{\beta_{1,2}}(\zeta_1))(A_1)+l\delta(U_{\beta_{3,2}}(\zeta_3)U_{\beta_{2,2}}(\zeta_2)U_{\beta_{1,2}}(\zeta_1))=$$
$$\sum_{i=1}^{4}\mathrm{Ad}(U_{\beta_{3,2}}(\zeta_3)U_{\beta_{2,2}}(\zeta_2)U_{\beta_{1,2}}(\zeta_1)(X_{\alpha_i})$$
$$+\sum_{\gamma\in\Phi^-}a_{1,\gamma}\mathrm{Ad}(U_{\beta_{3,2}}(\zeta_3)U_{\beta_{2,2}}(\zeta_2)U_{\beta_{1,2}}(\zeta_1))(X_\gamma)+l\delta(U_{\beta_{3,2}}(\zeta_3)U_{\beta_{2,2}}(\zeta_2)U_{\beta_{1,2}}(\zeta_1))$$
$$=\sum_{i=1}^{4}X_{\alpha_i}+\sum_{\gamma\in\Phi^-,\,\mathrm{ht}(\gamma)\geq 2}a_{2,\gamma}X_\gamma$$
$$+(a_{1,-\alpha_1}+m_{\beta_{1,2},\alpha_2}\zeta_1)X_{-\alpha_1}+(a_{1,-\alpha_2}+m_{\beta_{1,2},\alpha_1}\zeta_1+m_{\beta_{2,2},\alpha_3}\zeta_2)X_{-\alpha_2}$$
$$+(a_{1,-\alpha_3}+m_{\beta_{2,2},\alpha_2}\zeta_2+m_{\beta_{3,2},\alpha_4}\zeta_3)X_{-\alpha_3}+(a_{1,-\alpha_4}+m_{\beta_{3,2},\alpha_3}\zeta_3)X_{-\alpha_4}=:A_2$$

with new elements $a_{2,\gamma}\in F$. If we define

$$\zeta_1=-\frac{1}{m_{\beta_{1,2},\alpha_1}}(a_{1,-\alpha_2}+m_{\beta_{2,2},\alpha_3}\zeta_2),\quad \zeta_2=-\frac{1}{m_{\beta_{2,2},\alpha_2}}(a_{1,-\alpha_3}+m_{\beta_{3,2},\alpha_4}\zeta_1)$$
$$\text{and}\quad \zeta_3=-\frac{1}{m_{\beta_{3,2},\alpha_3}}a_{1,-\alpha_4},$$

then A_1 contains no vector lying in the root subspaces of the negative simple roots except of $\gamma_1:=-\alpha_1$, i.e., we have

$$A_2=\sum_{i=1}^{4}X_{\alpha_i}+a_{2,-\alpha_1}X_{-\alpha_1}+\sum_{\gamma\in\Phi^-,\,\mathrm{ht}(\gamma)\geq 2}a_{2,\gamma}X_\gamma.$$

Now we delete the roots of height 2. Since there are three negative roots of height 3, we have three parameters available for the transformation of the three roots of height 2. However, table (9.2) yields

$$\mathrm{Ad}(U_{\beta_{3,3}}(\zeta_3)U_{\beta_{2,3}}(\zeta_2)U_{\beta_{1,3}}(\zeta_1))(A_2)+l\delta(U_{\beta_{3,3}}(\zeta_3)U_{\beta_{2,3}}(\zeta_2)U_{\beta_{1,3}}(\zeta_1))=$$
$$\sum_{i=1}^{4}\mathrm{Ad}(U_{\beta_{3,3}}(\zeta_3)U_{\beta_{2,3}}(\zeta_2)U_{\beta_{1,3}}(\zeta_1)(X_{\alpha_i})a_{2,-\alpha_1}\mathrm{Ad}(U_{\beta_{3,3}}(\zeta_3)U_{\beta_{2,3}}(\zeta_2)U_{\beta_{1,3}}(\zeta_1)(X_{-\alpha_1})$$
$$+\sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 2}a_{2,\gamma}\mathrm{Ad}(U_{\beta_{3,3}}(\zeta_3)U_{\beta_{2,3}}(\zeta_2)U_{\beta_{1,3}}(\zeta_1)(X_\gamma)+l\delta(U_{\beta_{3,3}}(\zeta_3)U_{\beta_{2,3}}(\zeta_2)U_{\beta_{1,3}}(\zeta_1))$$
$$=\sum_{i=1}^{4}X_{\alpha_i}+a_{3,-\alpha_1}X_{-\alpha_1}+\sum_{\gamma\in\Phi^-,\,\mathrm{ht}(\gamma)\geq 3}a_{3,\gamma}X_\gamma$$
$$+X_{-\alpha_1-\alpha_2}(a_{2,-\alpha_1-\alpha_2}+m_{\beta_{1,3},\alpha_3}\zeta_1)+X_{-\alpha_3-\alpha_4}(a_{2,-\alpha_3-\alpha_4}+m_{\beta_{3,3},\alpha_2}\zeta_3)$$
$$+X_{-\alpha_2-\alpha_3}(a_{2,-\alpha_2-\alpha_3}+m_{\beta_{1,3},\alpha_1}\zeta_1+m_{\beta_{2,3},\alpha_3}\zeta_2+m_{\beta_{3,3},\alpha_4}\zeta_3)=:A_3$$

with new coefficients $a_{3,\gamma}\in F$. We set $\zeta_1=-\frac{1}{m_{\beta_{1,3},\alpha_3}}a_{2,-\alpha_1-\alpha_2}$, $\zeta_3=-\frac{1}{m_{\beta_{3,3},\alpha_2}}a_{2,-\alpha_3-\alpha_4}$ and $\zeta_2=-\frac{1}{m_{\beta_{2,3},\alpha_3}}(a_{2,-\alpha_2-\alpha_3}+m_{\beta_{1,3},\alpha_1}\zeta_1+m_{\beta_{3,3},\alpha_4}\zeta_3)$. Thus we obtain

$$A_3=\sum_{i=1}^{4}X_{\alpha_i}+a_{3,-\alpha_1}X_{-\alpha_1}+\sum_{\gamma\in\Phi^-,\,\mathrm{ht}(\gamma)\geq 3}a_{3,\gamma}X_\gamma.$$

The transformation of the negative roots of height 3 is more complicated. However, we conclude with table (9.3)

$$\begin{aligned}
&\mathrm{Ad}(U_{\beta_{3,4}}(\zeta_3)U_{\beta_{2,4}}(\zeta_2)U_{\beta_{1,4}}(\zeta_1))(A_3) + l\delta(U_{\beta_{3,4}}(\zeta_3)U_{\beta_{2,4}}(\zeta_2)U_{\beta_{1,4}}(\zeta_1)) = \\
&\sum_{i=1}^{4}\mathrm{Ad}(U_{\beta_{3,4}}(\zeta_3)U_{\beta_{2,4}}(\zeta_2)U_{\beta_{1,4}}(\zeta_1)(X_{\alpha_i}) + a_{3,-\alpha_1}\mathrm{Ad}(U_{\beta_{3,4}}(\zeta_3)U_{\beta_{2,4}}(\zeta_2)U_{\beta_{1,4}}(\zeta_1)(X_{-\alpha_1}) \\
&+ \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 3} a_{3,\gamma}\mathrm{Ad}(U_{\beta_{3,4}}(\zeta_3)U_{\beta_{2,4}}(\zeta_2)U_{\beta_{1,4}}(\zeta_1)(X_\gamma) + l\delta(U_{\beta_{3,4}}(\zeta_3)U_{\beta_{2,4}}(\zeta_2)U_{\beta_{1,4}}(\zeta_1)) \\
&= \sum_{i=1}^{4} X_{\alpha_i} + a_{4,-\alpha_1}X_{-\alpha_1} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 4} a_{4,\gamma}X_\gamma \\
&+X_{-\alpha_1-\alpha_2-\alpha_3}(a_{3,-\alpha_1-\alpha_2-\alpha_3} + m_{\beta_{1,4},\alpha_3}\zeta_1 + m_{\beta_{2,4},\alpha_4}\zeta_2) \\
&+X_{-\alpha_2-2\alpha_3}(a_{3,-\alpha_2-2\alpha_3} + m_{\beta_{1,4},\alpha_1}\zeta_1 + m_{\beta_{3,4},\alpha_4}\zeta_3) \\
&+X_{-\alpha_2-\alpha_3-\alpha_4}(a_{3,-\alpha_2-\alpha_3-\alpha_4} + m_{\beta_{2,4},\alpha_1}\zeta_2 + m_{\beta_{3,4},\alpha_3}\zeta_3) := A_4
\end{aligned}$$

with new coefficients $a_{4,\gamma}$ and $a_{4,-\alpha_1} \in F$. We have to determine values $(\zeta_1,\zeta_2,\zeta_3) \in F^3$ such that the coefficients of $X_{-\alpha_1-\alpha_2-\alpha_3}$, $X_{-\alpha_2-2\alpha_3}$ and $X_{-\alpha_2-\alpha_3-\alpha_4}$ become zero. This problem is equivalent to the system of equations

$$\begin{pmatrix} m_{\beta_{1,4},\alpha_3} & m_{\beta_{2,4},\alpha_4} & 0 \\ m_{\beta_{1,4},\alpha_1} & 0 & m_{\beta_{3,4},\alpha_4} \\ 0 & m_{\beta_{2,4},\alpha_1} & m_{\beta_{3,4},\alpha_3} \end{pmatrix} \cdot \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} a_{3,-\alpha_1-\alpha_2-\alpha_3} \\ a_{3,-\alpha_2-2\alpha_3} \\ a_{3,-\alpha_2-\alpha_3-\alpha_4} \end{pmatrix}. \tag{9.3}$$

Denote the matrix of equation (9.3) by B. Then equation (9.3) has a solution if and only if

$$\det(B) = m_{\beta_{1,4},\alpha_3}(-m_{\beta_{3,4},\alpha_4}m_{\beta_{2,4},\alpha_1}) - m_{\beta_{2,4},\alpha_4}(m_{\beta_{1,4},\alpha_1}m_{\beta_{3,4},\alpha_3}) \neq 0.$$

From equation (9.1) we obtain the values of the m_{β_i,α_j} up to their signs, i.e., we have

$$\det(B) = (\pm 1)\cdot(\mp 2)\cdot(\pm 1) - (\pm 1)\cdot(\pm 1)\cdot(\pm 1) = \pm 2 - (\pm 1) \neq 0.$$

Thus there exists a triple $(\zeta_1,\zeta_2,\zeta_3) \in F^3$ such that

$$A_4 = \sum_{i=1}^{4} X_{\alpha_i} + a_{4,-\alpha_1}X_{-\alpha_1} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 4} a_{4,\gamma}X_\gamma.$$

In the next step we delete all roots of height 4. Table (9.4) yields

$$\begin{aligned}
&\mathrm{Ad}(U_{\beta_{3,5}}(\zeta_3)U_{\beta_{2,5}}(\zeta_2)U_{\beta_{1,5}}(\zeta_1))(A_4) + l\delta(U_{\beta_{3,5}}(\zeta_3)U_{\beta_{2,5}}(\zeta_2)U_{\beta_{1,5}}(\zeta_1)) = \\
&\sum_{i=1}^{4}\mathrm{Ad}(U_{\beta_{3,5}}(\zeta_3)U_{\beta_{2,5}}(\zeta_2)U_{\beta_{1,5}}(\zeta_1)(X_{\alpha_i}) \\
&+a_{4,-\alpha_1}\mathrm{Ad}(U_{\beta_{3,5}}(\zeta_3)U_{\beta_{2,5}}(\zeta_2)U_{\beta_{1,5}}(\zeta_1)(X_{-\alpha_1}) \\
&+\sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 4} a_{4,\gamma}\mathrm{Ad}(U_{\beta_{3,5}}(\zeta_3)U_{\beta_{2,5}}(\zeta_2)U_{\beta_{1,5}}(\zeta_1)(X_\gamma) + l\delta(U_{\beta_{3,5}}(\zeta_3)U_{\beta_{2,5}}(\zeta_2)U_{\beta_{1,5}}(\zeta_1)) \\
&= \sum_{i=1}^{4} X_{\alpha_i} + a_{5,-\alpha_1}X_{-\alpha_1} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 5} a_{5,\gamma}X_\gamma \\
&+X_{-\alpha_1-\alpha_2-2\alpha_3}(a_{4,-\alpha_1-\alpha_2-2\alpha_3} + m_{\beta_{1,5},\alpha_4}\zeta_1 + m_{\beta_{2,5},\alpha_2}\zeta_2) \\
&+X_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}(a_{4,-\alpha_1-\alpha_2-\alpha_3-\alpha_4} + m_{\beta_{1,5},\alpha_3}\zeta_1) \\
&+X_{-\alpha_2-2\alpha_3-\alpha_4}(a_{4,-\alpha_2-2\alpha_3-\alpha_4} + m_{\beta_{1,5},\alpha_1}\zeta_1 + m_{\beta_{3,5},\alpha_4}\zeta_3) := A_5
\end{aligned}$$

If we define the parameters ζ_1, ζ_2 and ζ_3 as

$$\zeta_1 = -\frac{1}{m_{\beta_{1,5},\alpha_3}}a_{4,-\alpha_1-\alpha_2-\alpha_3-\alpha_4}, \quad \zeta_2 = -\frac{1}{m_{\beta_{2,5},\alpha_2}}(a_{4,-\alpha_1-\alpha_2-2\alpha_3} + m_{\beta_{1,5},\alpha_4}\zeta_1)$$
$$\text{and} \quad \zeta_3 = -\frac{1}{m_{\beta_{3,5},\alpha_4}}(a_{4,-\alpha_2-2\alpha_3-\alpha_4} + m_{\beta_{1,5},\alpha_1}\zeta_1),$$

then we obtain

$$A_5 = \sum_{i=1}^{4} X_{\alpha_i} + a_{5,-\alpha_1}X_{-\alpha_1} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 5} a_{5,\gamma}X_\gamma.$$

From table (9.5) we see that we are only able to delete two of the three negative roots of height five, i.e., we have

$$\begin{aligned}
&\mathrm{Ad}(U_{\beta_{2,6}}(\zeta_2)U_{\beta_{1,6}}(\zeta_1))(A_4) + l\delta(U_{\beta_{2,6}}(\zeta_2)U_{\beta_{1,6}}(\zeta_1)) = \\
&\sum_{i=1}^{4}\mathrm{Ad}(U_{\beta_{2,6}}(\zeta_2)U_{\beta_{1,6}}(\zeta_1))(X_{\alpha_i}) + a_{5,-\alpha_1}\mathrm{Ad}(U_{\beta_{2,6}}(\zeta_2)U_{\beta_{1,6}}(\zeta_1)(X_{-\alpha_1}) \\
&+\sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 5} a_{5,\gamma}\mathrm{Ad}(U_{\beta_{2,6}}(\zeta_2)U_{\beta_{1,6}}(\zeta_1)(X_\gamma) + l\delta(U_{\beta_{2,6}}(\zeta_2)U_{\beta_{1,6}}(\zeta_1)) \\
&= \sum_{i=1}^{4} X_{\alpha_i} + a_{6,-\alpha_1}X_{-\alpha_1} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 6} a_{6,\gamma}X_\gamma \\
&+X_{-\alpha_1-2\alpha_2-2\alpha_3}(a_{5,-\alpha_1-2\alpha_2-2\alpha_3} + m_{\beta_{1,6},\alpha_4}\zeta_1) \\
&+X_{-\alpha_1-\alpha_2-2\alpha_3-\alpha_4}(a_{5,-\alpha_1-\alpha_2-2\alpha_3-\alpha_4} + m_{\beta_{1,6},\alpha_2}\zeta_1 + m_{\beta_{2,6},\alpha_4}\zeta_2) \\
&+X_{-\alpha_2-2\alpha_3-2\alpha_4}(a_{5,-\alpha_2-2\alpha_3-2\alpha_4} + m_{\beta_{2,6},\alpha_1}\zeta_2) := A_6
\end{aligned}$$

If we define ζ_1 and ζ_2 as

$$\zeta_1 = -\frac{1}{m_{\beta_{1,6},\alpha_2}}(a_{5,-\alpha_1-\alpha_2-2\alpha_3-\alpha_4} + m_{\beta_{2,6},\alpha_4}\zeta_2) \quad \text{and} \quad \zeta_2 = -\frac{1}{m_{\beta_{2,6},\alpha_1}}a_{5,-\alpha_2-2\alpha_3-2\alpha_4},$$

then the coefficient of X_{γ_2} where $\gamma_2 := -\alpha_1 - 2\alpha_2 - 2\alpha_3$ will not necessarily be zero, i.e., we have

$$A_6 = \sum_{i=1}^{4} X_{\alpha_i} + \sum_{i=1}^{2} a_{6,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 6} a_{6,\gamma} X_\gamma.$$

In the next step we are again able to delete all roots of height 6. However, we differentially conjugate A_6 by $U_{\beta_{2,7}}(\zeta_2)U_{\beta_{1,7}}(\zeta_1)$. With the help of table (9.6) this differential conjugation computes as

$$\begin{gathered}
\mathrm{Ad}(U_{\beta_{2,7}}(\zeta_2)U_{\beta_{1,7}}(\zeta_1))(A_6) + l\delta(U_{\beta_{2,7}}(\zeta_2)U_{\beta_{1,7}}(\zeta_1)) = \\
\sum_{i=1}^{4} \mathrm{Ad}(U_{\beta_{2,7}}(\zeta_2)U_{\beta_{1,7}}(\zeta_1)(X_{\alpha_i}) + \sum_{i=1}^{2} a_{6,\gamma_i}\mathrm{Ad}(U_{\beta_{2,7}}(\zeta_2)U_{\beta_{1,7}}(\zeta_1)(X_{\gamma_i}) \\
+ \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 6} a_{6,\gamma}\mathrm{Ad}(U_{\beta_{2,7}}(\zeta_2)U_{\beta_{1,7}}(\zeta_1)(X_\gamma) + l\delta(U_{\beta_{2,7}}(\zeta_2)U_{\beta_{1,7}}(\zeta_1)) = \\
\sum_{i=1}^{4} X_{\alpha_i} + \sum_{i=1}^{2} a_{7,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 7} a_{7,\gamma} X_\gamma \\
+X_{-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4}(a_{6,-\alpha_1-2\alpha_2-2\alpha_3-\alpha_4} + m_{\beta_{1,7},\alpha_3}\zeta_1 + m_{\beta_{2,7},\alpha_4}\zeta_2) \\
+X_{-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4}(a_{6,-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4} + m_{\beta_{2,7},\alpha_2}\zeta_2) =: A_7.
\end{gathered}$$

Obviously, we can choose $(\zeta_1, \zeta_2) \in F^2$ such that A_7 becomes

$$A_7 = \sum_{i=1}^{4} X_{\alpha_i} + \sum_{i=1}^{2} a_{7,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 7} a_{7,\gamma} X_\gamma.$$

Since $\beta_{1,8} = -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4$ is the only negative root of height 8, we can only delete one of the two negative roots of height 7. We compute with the help of table (9.7)

$$\begin{gathered}
\mathrm{Ad}(U_{\beta_{1,8}}(\zeta_1))(A_7) + l\delta(U_{\beta_{1,8}}(\zeta_1)) = \sum_{i=1}^{4} \mathrm{Ad}(U_{\beta_{1,8}}(\zeta_1))(X_{\alpha_i}) \\
+ \sum_{i=1}^{2} a_{7,\gamma_i}\mathrm{Ad}(U_{\beta_{1,8}}(\zeta_1))(X_{\gamma_i}) + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 7} a_{7,\gamma}\mathrm{Ad}(U_{\beta_{1,8}}(\zeta_1))(X_\gamma) + l\delta(U_{\beta_{1,8}}(\zeta_1)) = \\
\sum_{i=1}^{4} X_{\alpha_i} + \sum_{i=1}^{2} a_{8,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 8} a_{8,\gamma} X_\gamma \\
+X_{-\alpha_1-3\alpha_2-2\alpha_3-\alpha_4}(a_{7,-\alpha_1-3\alpha_2-2\alpha_3-\alpha_4} + m_{\beta_{1,8},\alpha_4}\zeta_1) \\
+X_{-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4}(a_{7,-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4} + m_{\beta_{1,8},\alpha_2}\zeta_1) \\
= \sum_{i=1}^{4} X_{\alpha_i} + \sum_{i=1}^{3} a_{8,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 8} a_{8,\gamma} X_\gamma =: A_8
\end{gathered}$$

where $\gamma_3 := -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4$. Here we obtained the last equation by defining ζ_1 as $\zeta_1 = -\frac{1}{m_{\beta_{1,8},\alpha_4}} a_{7,-\alpha_1-3\alpha_2-2\alpha_3-\alpha_4}$.

We use table (9.8) to delete the roots of height 8, 9 and 10. The first column of table (9.8) together with $\zeta := -\frac{1}{m_{\beta_{1,9},\alpha_3}} a_{8,\hat{\beta}_3}$ implies

$$\mathrm{Ad}(U_{\beta_{1,9}}(\zeta))(A_8) + l\delta(U_{\beta_{1,9}}(\zeta)) = \sum_{i=1}^{4} \mathrm{Ad}(U_{\beta 1,9}(\zeta))(X_{\alpha_i})$$

$$+ \sum_{i=1}^{3} a_{8,\gamma_i} \mathrm{Ad}(U_{\beta_{1,9}}(\zeta))(X_{\gamma_i}) + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 8} a_{8,\gamma} \mathrm{Ad}(U_{\beta_{1,9}}(\zeta))(X_\gamma) + l\delta(U_{\beta_{1,9}}(\zeta)) =$$

$$\sum_{i=1}^{4} X_{\alpha_i} + \sum_{i=1}^{3} a_{9,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 9} a_{9,\gamma} X_\gamma + X_{\hat{\beta}_3}(a_{8,\hat{\beta}_3} + m_{\beta_{1,9},\alpha_3}\zeta)$$

$$= \sum_{i=1}^{4} X_{\alpha_i} + \sum_{i=1}^{3} a_{9,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 9} a_{9,\gamma} X_\gamma =: A_9.$$

With the help of second column of table (9.8) we delete the part of A_9 which lies in the root space corresponding to the root $\beta_{1,9} = -\alpha_1 - 2\alpha_2 - 4\alpha_3 - 2\alpha_4$ of height 9. We obtain

$$\mathrm{Ad}(U_{\beta_{1,10}}(\zeta))(A_9) + l\delta(U_{\beta_{1,10}}(\zeta)) = \sum_{i=1}^{4} \mathrm{Ad}(U_{\beta_{1,10}}(\zeta))(X_{\alpha_i}) + \sum_{i=1}^{3} a_{9,\gamma_i} \mathrm{Ad}(U_{\beta_{1,10}}(\zeta))(X_{\gamma_i})$$

$$+ \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 9} a_{9,\gamma} \mathrm{Ad}(U_{\beta_{1,10}}(\zeta))(X_\gamma) + l\delta(U_{\beta_{1,10}}(\zeta)) =$$

$$\sum_{i=1}^{4} X_{\alpha_i} + \sum_{i=1}^{3} a_{10,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 10} a_{10,\gamma} X_\gamma + X_{\hat{\beta}_2}(a_{9,\hat{\beta}_2} + m_{\beta_{1,10},\alpha_2}\zeta)$$

$$= \sum_{i=1}^{4} X_{\alpha_i} + \sum_{i=1}^{3} a_{10,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 10} a_{10,\gamma} X_\gamma =: A_{10}$$

where the definition of $\zeta := -\frac{1}{m_{\beta_{1,10},\alpha_2}} a_{9,\hat{\beta}_2}$ implies the last equation. With the last column of table (9.8) the last transformation, i.e., the transformation of the root of height 10, is

$$\mathrm{Ad}(U_{\beta_{1,11}}(\zeta))(A_{10}) + l\delta(U_{\beta_{1,11}}(\zeta)) = \sum_{i=1}^{4} \mathrm{Ad}(U_{\beta_{1,11}}(\zeta))(X_{\alpha_i}) +$$

$$\sum_{i=1}^{3} a_{10,\gamma_i} \mathrm{Ad}(U_{\beta_{1,11}}(\zeta))(X_{\gamma_i}) + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 10} a_{10,\gamma} \mathrm{Ad}(U_{\beta_{1,11}}(\zeta))(X_\gamma) + l\delta(U_{\beta_{1,11}}(\zeta)) =$$

$$\sum_{i=1}^{4} X_{\alpha_i} + \sum_{i=1}^{3} a_{11,\gamma_i} X_{\gamma_i}$$

$$+ X_{\hat{\beta}_1}(a_{10,\hat{\beta}_1} + m_{\beta_{1,11},\alpha_1}\zeta) + a_{11,-2\alpha_1-3\alpha_2-4\alpha_3-2\alpha_4} X_{-2\alpha_1-3\alpha_2-4\alpha_3-2\alpha_4} =: A_{11}.$$

We define $\zeta := -\frac{1}{m_{\beta_{1,11},\alpha_1}} a_{10,\hat{\beta}_1}$. This yields

$$A_{11} = \sum_{i=1}^{4} X_{\alpha_i} + a_{11,\gamma_1} X_{\gamma_1} + a_{11,\gamma_2} X_{\gamma_2} + a_{11,\gamma_3} X_{\gamma_3} + a_{11,\gamma_4} X_{\gamma_4}$$

where we denote by $\gamma_4 = -2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4$ the negative root of maximal height. This completes the proof. □

9.3 The equation with group F_4

In [How01] the authors R. B. Howlett, L. J. Rylands and D. E. Taylor computed a 26-dimensional representation of the Lie algebra of type F_4. They present explicit matrices for the positive and negative simple roots, i.e., generators for the Lie algebra of type F_4. More precisely, their results are

$$\begin{aligned}
X_{\alpha_1} &= E_{4,5} + E_{6,7} + E_{8,10} + E_{18,20} + E_{19,21} + E_{22,23},\\
X_{\alpha_2} &= E_{3,4} + E_{7,9} + E_{10,12} + E_{16,18} + E_{17,19} + E_{23,24},\\
X_{\alpha_3} &= E_{2,3} + E_{4,6} + E_{5,7} + E_{9,11} + E_{12,13} + 2E_{12,14} + E_{14,16}\\
&\quad + E_{15,17} + E_{19,22} + E_{21,23} + E_{24,25},\\
X_{\alpha_4} &= E_{1,2} + E_{6,8} + E_{7,10} + E_{9,12} + 2E_{11,13} + E_{11,14} + E_{13,15}\\
&\quad + E_{16,17} + E_{18,19} + E_{20,21} + E_{25,26},\\
X_{-\alpha_1} &= X_{\alpha_1}^T,\\
X_{-\alpha_2} &= X_{\alpha_2}^T,\\
X_{-\alpha_3} &= E_{3,2} + E_{6,4} + E_{7,5} + E_{11,9} + E_{14,12} + E_{16,13} + 2E_{16,14}\\
&\quad + E_{17,15} + E_{22,19} + E_{23,21} + E_{25,24},\\
X_{-\alpha_4} &= E_{2,1} + E_{8,6} + E_{10,7} + E_{12,9} + E_{13,11} + 2E_{15,13} + E_{15,14}\\
&\quad + E_{17,16} + E_{19,18} + E_{21,20} + E_{26,25}.
\end{aligned}$$

Then the elements $\{ X_{\pm\alpha_i} \mid \alpha_i \in \Delta\}$ generate the Lie algebra of type F_4. We denote this representation of the Lie algebra of type F_4 by $\mathbf{L}_{F_4}$. With the help of a computer algebra system we compute the shape of the additional elements

$$\begin{aligned}
X_{-\alpha_1-2\alpha_2-2\alpha_3} &= -E_{9,2} - E_{11,3} - E_{18,8} - E_{20,10} - E_{24,15} - E_{25,17},\\
X_{-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4} &= E_{12,1} - E_{15,3} + E_{19,6} + E_{21,7} - E_{24,11} + E_{26,16} \quad \text{and}\\
X_{-2\alpha_1-3\alpha_2-4\alpha_3-2\alpha_4} &= E_{20,1} + E_{21,2} + E_{23,3} + E_{24,4} + E_{25,6} + E_{26,8}.
\end{aligned}$$

Now let $F = C\langle t_1, ..., t_4\rangle$ be the differential field generated by the 4 differential indeterminates $\boldsymbol{t} = (t_1, t_2, t_3, t_4)$ over C. Denote by $\boldsymbol{y}$ the vector $\boldsymbol{y} = (y_1, y_2, y_3, ..., y_{25}, y_{26})^T$. We define the matrix differential equation

$$\partial(\boldsymbol{y}) = A_{F_4}(\boldsymbol{t})\boldsymbol{y}$$

for the group of type F_4 over F by

$$A_{F_4}(\boldsymbol{t}) := \sum_{i=1}^{4} X_{\alpha_i} + t_1 X_{-\alpha_1} + t_2 X_{-\alpha_1-2\alpha_2-2\alpha_3} + t_3 X_{-\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4} + t_4 X_{-2\alpha_1-3\alpha_2-4\alpha_3-2\alpha_4}.$$

The shape of the matrix $A_{F_4}(\boldsymbol{t})$, which we obtain from the above representation, can be found on the next page.

$$A_{F_4}(\boldsymbol{t}) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t_1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
t_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & t_4 & 0 & 0 & 0 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 & -t_3 & 0 & 0 & 0 & -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

For the computation of a linear differential equation from the matrix equation $\partial(\boldsymbol{y}) = A_{F_4}(\boldsymbol{t})\boldsymbol{y}$ we can choose, as in the cases of the other groups, y_1 as a cyclic vector. Unfortunately, y_1 does not lead to a nice and short differential equation. We tried also other cyclic vectors. Simular to the case of y_1, we obtained non printable equations. However, we guess that y_1 is the most easiest cyclic vector. The matrix $A_{F_4}(\boldsymbol{t})$ has already a nice and easy shape. Thus we do not compute an enormous linear differential equation and continue with the matrix differential equation.
Denote by $\mathcal{G}_{F_4}$ the group of type F_4 with Lie algebra $\mathbf{L}_{F_4}$. Before we prove that the differential equation $\partial(\boldsymbol{y}) = A_{F_4}(\boldsymbol{t})\boldsymbol{y}$ over F has $\mathcal{G}_{F_4}$ as its differential Galois group we are going to combine the results of Lemma 9.1 and Corollary 3.12 in Corollary 9.2. Therefore we define Ω as the set of the 4 negative roots

$$\begin{aligned}\Omega := \{&\gamma_1 = -\alpha_1,\ \gamma_2 = -\alpha_1 - 2\alpha_2 - 2\alpha_3,\ \gamma_3 = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4,\\ &\gamma_4 = -2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4\ \}\end{aligned}$$

and we denote by $\bar{F} := (C(z), \partial = \frac{d}{dz})$ the rational function field with standard derivation.

Corollary 9.2. *Let $A^{M\&S}_{F_4} \in \mathbf{L}_{F_4}(\bar{F})$ be the matrix satisfying the conditions of Corollary 3.12 which we applied to the group $\mathcal{G}_{F_4}$ and the above Cartan decomposition. Then there exists $U \in \mathcal{U}^-(C(z)) \subset \mathcal{G}_{F_4}(C(z))$ such that*

$$\bar{A}_{F_4} := U A^{M\&S}_{F_4} U^{-1} + \partial(U)U^{-1} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in\Omega} f_i X_{\gamma_i} \tag{9.4}$$

with at least one $f_i \in C\,[z] \setminus C$ and the differential Galois group of the matrix equation $\partial(\boldsymbol{y}) = \bar{A}_{F_4}\boldsymbol{y}$ over $\bar{F}$ is $\mathcal{G}_{F_4}(C)$.

Proof. Lemma 9.1 implies the existence of an element $U \in \mathcal{U}_0^- \subset \mathcal{G}_{F_4}$ such that equation (9.4) holds. Since differential conjugation defines a differential isomorphism, we deduce with Corollary 3.12 that the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{F_4}\boldsymbol{y}$ is again $\mathcal{G}_{F_4}(C)$ over $\bar{F}$. We still need to show the existence of $f_i \in C\,[z] \setminus C$ for some $\gamma_i \in \Omega$. Suppose $\bar{A}_{F_4} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in\Omega} f_i X_{\gamma_i} \in \mathrm{Lie}(\mathcal{G}_{F_4})(C)$. Then the corresponding differential equation $L(y) \in C\,\{y\}$ has coefficients in C. But then by [Mag94, Corollary 3.28] the differential Galois group is abelian. Thus, we obtain $\bar{A}_{F_4} \in \mathrm{Lie}(\mathcal{G}_{F_4})(\bar{F}) \setminus \mathrm{Lie}(\mathcal{G}_{F_4})(C)$. Since $0 \neq A_1 \in \mathbf{H}(C)$ and $A = (z^2 A_1 + A_0)$ in Corollary 3.12, we start our transformation with at least one coefficient lying in $C\,[z] \setminus C$. In each step the application of $\mathrm{Ad}(U_\beta(\zeta))$ generates at most new entries which are polynomials in ζ. Moreover, the logarithmic derivative is the product of the two matrices $\partial(U_\beta(\zeta))$ and $U_\beta(\zeta)^{-1} = U_\beta(-\zeta)$. In the proof of Lemma 9.1 we choose the parameter ζ to be one of the coefficients. Hence, we get $f_i \in C[z] \setminus C$. □

Theorem 9.3. *The matrix differential equation*

$$\partial(\boldsymbol{y}) = A_{F_4}(\boldsymbol{t})\boldsymbol{y}$$

has F_4 as differential Galois group over $C\,\langle t_1, ..., t_4\rangle$. Moreover, let $\hat{F}$ be a differential field with field of constants equal to C. Let $\hat{E}$ be a Picard-Vessiot extension over $\hat{F}$ with differential Galois group $\mathcal{G}_{F_4}(C)$ and suppose the defining matrix differential equation $\partial(\boldsymbol{y}) = \hat{A}\boldsymbol{y}$ satisfies $\hat{A} \in \sum_{\alpha_i\in\Delta} X_{\alpha_i} + \sum_{\alpha\in\Phi^-} \mathbf{L}_\alpha$. Then there is a specialization

$\partial(\boldsymbol{y}) = A_{F_4}(\hat{t}_1, ..., \hat{t}_4)\boldsymbol{y}$ with $\hat{t}_i \in \hat{F}$ such that $\partial(\boldsymbol{y}) = A_{F_4}(\hat{t}_1, ..., \hat{t}_4)\boldsymbol{y}$ gives rise to the extension $\hat{E}$ over $\hat{F}$.

Proof. Let E be a Picard-Vessiot extension for the equation $\partial(\boldsymbol{y}) = A_{F_4}(\boldsymbol{t})\boldsymbol{y}$ over F and denote by $\mathcal{G}$ the differential Galois group. Since for our matrix differential equation $\partial(\boldsymbol{y}) = A_{F_4}(\boldsymbol{t})\boldsymbol{y}$ holds $A_{F_4}(\boldsymbol{t}) \in \mathrm{Lie}(\mathcal{G}_{F_4})(F)$, Proposition 2.1 yields $\mathcal{G}(C) \leq \mathcal{G}_{F_4}(C)$. By Corollary 9.2 there exists a specialization $\sigma : (t_1, ..., t_4) \to (f_1, ..., f_4)$ with $f_i \in C[z]$ such that $\sigma(A_{F_4}(t_1, ..., t_4)) = \bar{A}_{F_4}$ and the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{F_4}\boldsymbol{y}$ is $\mathcal{G}_{F_4}(C)$. Moreover, we have $C\{f_1, ..., f_4\} = C[z]$. Thus we can apply Corollary 2.15. This yields $\mathcal{G}_{F_4}(C) \leq \mathcal{G}(C)$. Hence, it holds $\mathcal{G}(C) = \mathcal{G}_{F_4}(C)$.
Since the defining matrix $\hat{A}$ satisfies $\hat{A} \in \sum_{\alpha_i \in \Delta} X_{\alpha_i} + \sum_{\alpha \in \Phi^-} \mathrm{Lie}(\mathcal{G}_{F_4})_\alpha$, Lemma 9.1 provides that $\hat{A}$ is differentially equivalent to a matrix $\tilde{A} = \sum_{\alpha_i \in \Delta} X_{\alpha_i} + \sum_{\gamma_i \in \Omega} \hat{a}_i X_{\gamma_i}$ with suitable $\hat{a}_i \in \hat{F}$. Obviously the specialization

$$\hat{\sigma} : (t_1, ..., t_4) \mapsto (\hat{a}_1, ..., \hat{a}_4)$$

does the required. $\square$

Chapter 10

A parametrized equation for E_6

10.1 The root system of type E_6

The below discussion for the construction of the root system of type E_6 is taken from [Hum72, Section 12.1]. Since the root system of type E_6 can be identified canonically with a subsystem of E_8 we construct first the root system of type E_8. Therefore let $\epsilon_1, ..., \epsilon_8$ be the standard orthonormal basis of $\mathbb{R}^8$ and let (α, β) denote the usual inner product of $\alpha, \beta \in \mathbb{R}^8$. The $\mathbb{Z}$-span of $\epsilon_1, ..., \epsilon_8$ is a lattice which we denote by I. Further let $I' = I + \mathbb{Z}(\epsilon_1 + ... + \epsilon_8)/2$ and I'' be the subgroup of I' consisting of all elements $\sum_{i=1}^8 c_i\epsilon_i + \frac{c}{2}(\epsilon_1 + ... + \epsilon_8)$ for which $c + \sum_{i=1}^8 c_i$ is an even integer. Then following [Hum72, Section 12.1] the root system Φ_{E_8} of type E_8 consists of the vectors

$$\begin{aligned}
\Phi_{E_8} &= \{\alpha \in I'' \mid (\alpha, \alpha) = 2\} \\
&= \{\pm(\epsilon_i \pm \epsilon_j),\ \frac{1}{2}\sum_{i=1}^{8}(-1)^{k(i)}\epsilon_i \mid i \neq j,\ k(i) = 0, 1 \text{ and } \sum_{i=1}^{8} k(i) \in 2\,\mathbb{Z}\}.
\end{aligned}$$

As a basis of Φ_{E_8} we can take the 8 vectors

$$\begin{aligned}
\Delta_{E_8} = \{\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8 - (\epsilon_2 + ... + \epsilon_7)),\ \alpha_2 = \epsilon_1 + \epsilon_2,\ \alpha_3 = \epsilon_2 - \epsilon_1,\ \alpha_4 = \epsilon_3 - \epsilon_2, \\
\alpha_5 = \epsilon_4 - \epsilon_3,\ \alpha_6 = \epsilon_5 - \epsilon_4,\ \alpha_7 = \epsilon_6 - \epsilon_5,\ \alpha_8 = \epsilon_7 - \epsilon_6\}
\end{aligned}$$

where the ordering is chosen such that we can identify canonically a base of E_6 with a subset of Δ_{E_8}. Thus a basis of the root system of type E_6 consists of the vectors

$$\begin{aligned}
\Delta_{E_6} = \{\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8 - (\epsilon_2 + ... + \epsilon_7)),\ \alpha_2 = \epsilon_1 + \epsilon_2,\ \alpha_3 = \epsilon_2 - \epsilon_1,\ \alpha_4 = \epsilon_3 - \epsilon_2, \\
\alpha_5 = \epsilon_4 - \epsilon_3,\ \alpha_6 = \epsilon_5 - \epsilon_4\}.
\end{aligned}$$

We use reflections to construct all remaining positive roots of Φ. Therefore let $\alpha_j \in \Delta$ and let $\beta = \sum_{i=1}^6 c_i\alpha_i$ be a positive root of Φ. The image of the reflection

$$\sigma_{\alpha_j}(\beta) = \beta - \langle\beta, \alpha_j\rangle\alpha_j = \beta - (\sum_{i=1}^{6} c_i\langle\alpha_i, \alpha_j\rangle)\alpha_j$$

is a root of Φ and is determined by the Cartan integers $\langle \alpha_i, \alpha_j \rangle = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$. Note that $\langle \alpha, \beta \rangle$ is only linear in the first variable for $\alpha, \beta \in \Phi$. The Cartan integers $\langle \alpha_i, \alpha_j \rangle$ are give at position i, j in the Cartan matrix which has in the case of E_6 the shape

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

We start our construction with the simple roots, i.e., we apply reflections σ_{α_j} ($\alpha_j \in \Delta$) to the simple roots $\alpha_i \in \Delta$. We obtain roots where we are only interested in the not yet known positive roots of height greater one. We continue our construction by applying the reflections σ_{α_j} to those roots. From [Hum72, Section 12.2, Table 1] we know that the number of positive roots of Φ is 36. We repeat this process until we get all 36 positive roots. The result of this computation is presented below where we numbered the positive roots of a given height k by $\beta_{h,k}$ with $h \in \mathbb{N}$:

$$\begin{aligned}
&\sigma_{\alpha_1}(\alpha_3) = \alpha_1 + \alpha_3 =: \beta_{1,2},\\
&\sigma_{\alpha_2}(\alpha_4) = \alpha_2 + \alpha_4 =: \beta_{2,2},\\
&\sigma_{\alpha_3}(\alpha_4) = \alpha_3 + \alpha_4 =: \beta_{3,2},\\
&\sigma_{\alpha_4}(\alpha_5) = \alpha_4 + \alpha_5 =: \beta_{4,2},\\
&\sigma_{\alpha_5}(\alpha_6) = \alpha_5 + \alpha_6 =: \beta_{5,2},\\
&\sigma_{\alpha_4}(\alpha_1 + \alpha_3) = \alpha_1 + \alpha_3 + \alpha_4 =: \beta_{1,3},\\
&\sigma_{\alpha_3}(\alpha_2 + \alpha_4) = \alpha_2 + \alpha_3 + \alpha_4 =: \beta_{2,3},\\
&\sigma_{\alpha_5}(\alpha_2 + \alpha_4) = \alpha_2 + \alpha_4 + \alpha_5 =: \beta_{3,3},\\
&\sigma_{\alpha_5}(\alpha_3 + \alpha_4) = \alpha_3 + \alpha_4 + \alpha_5 =: \beta_{4,3},\\
&\sigma_{\alpha_6}(\alpha_4 + \alpha_5) = \alpha_4 + \alpha_5 + \alpha_6 =: \beta_{5,3},\\
&\sigma_{\alpha_2}(\alpha_1 + \alpha_3 + \alpha_4) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 =: \beta_{1,4},\\
&\sigma_{\alpha_5}(\alpha_1 + \alpha_3 + \alpha_4) = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 =: \beta_{2,4},\\
&\sigma_{\alpha_5}(\alpha_2 + \alpha_3 + \alpha_4) = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 =: \beta_{3,4},\\
&\sigma_{\alpha_6}(\alpha_2 + \alpha_4 + \alpha_5) = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 =: \beta_{4,4},\\
&\sigma_{\alpha_6}(\alpha_3 + \alpha_4 + \alpha_5) = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 =: \beta_{5,4},\\
&\sigma_{\alpha_5}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 =: \beta_{1,5},\\
&\sigma_{\alpha_6}(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5) = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 =: \beta_{2,5},\\
&\sigma_{\alpha_6}(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 =: \beta_{3,5},\\
&\sigma_{\alpha_4}(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 =: \beta_{4,5},\\
&\sigma_{\alpha_6}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 =: \beta_{1,6},\\
&\sigma_{\alpha_4}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 =: \beta_{2,6},\\
&\sigma_{\alpha_6}(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5) = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 =: \beta_{3,6},
\end{aligned}$$

$$\sigma_{\alpha_4}(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6) = \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6 =: \beta_{1,7},$$
$$\sigma_{\alpha_3}(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5) = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5 =: \beta_{2,7},$$
$$\sigma_{\alpha_5}(\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6) = \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6 =: \beta_{3,7},$$
$$\sigma_{\alpha_5}(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6) = \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6 =: \beta_{1,8},$$
$$\sigma_{\alpha_3}(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6) = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6 =: \beta_{2,8},$$
$$\sigma_{\alpha_5}(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6) = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6 =: \beta_{1,9},$$
$$\sigma_{\alpha_4}(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6) = \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6 =: \beta_{1,10},$$
$$\sigma_{\alpha_2}(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6) = \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6 =: \beta_{1,11}.$$

Now suppose we have a representation of the Lie algebra of type E_6 to $\mathrm{Lie}(\mathrm{GL}_n(F))$. Let us denote the image of this representation by $\mathbf{L} < \mathrm{Lie}(\mathrm{GL}_n(F))$. Let a Cartan subalgebra $\mathbf{H}$ of $\mathbf{L}$ be given and let $\mathbf{L} = \mathbf{H} \oplus \bigoplus_{\alpha\in\Phi} \mathbf{L}_\alpha$ be the Cartan decomposition respective $\mathbf{H}$. Then we can choose for each $\alpha \in \Phi$ a nonzero element X_α of $\mathbf{L}_\alpha$ together with $H_\alpha = [X_\alpha, X_\alpha]$ such that the set $\{X_\alpha,\ H_\alpha \mid \alpha \in \Phi\}$ is a Chevalley basis. Then from the Chevalley construction we obtain a representation of the group $\mathcal{G}_{E_6}$ of type E_6 and the root subgroups $\mathcal{U}_\beta$. We denote a parametrized element of $\mathcal{U}_\beta$ by $U_\beta(\zeta)$ where $\zeta \in F$. For a root $\alpha \in \Phi$ the adjoint action of $U_\beta(\zeta)$ on X_α is

$$\mathrm{Ad}(U_\beta(\zeta))(X_\alpha) = \sum_{i\geq 0} m_{\alpha+i\beta}\cdot\zeta^i\cdot X_{\alpha+i\beta}. \tag{10.1}$$

For the proof of the transformation lemma it is necessary to know the image of the adjoint action for some specific roots α and β, since it is based on differential conjugation. In the case of interested, α is a simple positive root and β is a negative root of height greater than or equal to 2, i.e., β is by the above notation one of the roots $-\beta_{h,k}$ with $k \geq 2$. We analyse for each $-\beta_{h,k} \in \Phi^-$ and $\alpha_j \in \Delta$ if

$$\hat{\beta}_j := \alpha_j + (-\beta_{h,k}) \tag{10.2}$$

is a root of Φ, i.e., we determine if the term $m_{\alpha_j+i(-\beta_{h,k})}\cdot\zeta^i\cdot X_{\alpha_j+i(-\beta_{h,k})}$ of equation (10.1) for $i=1$ is zero or not. The results can be found in the tables (10.1)-(10.9). In the first row the roots $-\beta_{h,k}$ of a given height k are listed and in the first column we find the simple roots $\alpha_1,...,\alpha_6$. Then at position j', h' the root $\hat{\beta}_{j'}$ for $-\beta_{h',k}$ is given. If this position is empty then $\hat{\beta}_{j'} = \alpha_{j'} + (-\beta_{h',k})$ is not a root.

	$-\beta_{1,2}$	$-\beta_{2,2}$	$-\beta_{3,2}$	$-\beta_{4,2}$	$-\beta_{5,2}$
α_1	$-\alpha_3$				
α_2		$-\alpha_4$			
α_3	$-\alpha_1$		$-\alpha_4$		
α_4		$-\alpha_2$	$-\alpha_3$	$-\alpha_5$	
α_5				$-\alpha_4$	$-\alpha_6$
α_6					$-\alpha_5$

Table (10.1)

In table (10.1) and (10.2) we handle the roots of height 2 and 3 respectively. In table (10.3)

	$-\beta_{1,3}$	$-\beta_{2,3}$	$-\beta_{3,3}$	$-\beta_{4,3}$	$-\beta_{5,3}$
α_1	$-\alpha_3-\alpha_4$				
α_2		$-\alpha_3-\alpha_4$	$-\alpha_4-\alpha_5$		
α_3		$-\alpha_2-\alpha_4$		$-\alpha_4-\alpha_5$	
α_4	$-\alpha_1-\alpha_3$				$-\alpha_5-\alpha_6$
α_5			$-\alpha_2-\alpha_4$	$-\alpha_3-\alpha_4$	
α_6					$-\alpha_4-\alpha_5$

Table (10.2)

and (10.4) we find the analysis of the roots of height 4 and 5. Note that there are 5 roots of height 4 and only 4 roots of height 5. The results for the roots of height 6 and 7 can be

	$-\beta_{1,4}$	$-\beta_{2,4}$	$-\beta_{3,4}$	$-\beta_{4,4}$	$-\beta_{5,4}$
α_1	$-\alpha_2-\alpha_3-\alpha_4$	$-\alpha_3-\alpha_4-\alpha_5$			
α_2	$-\alpha_1-\alpha_3-\alpha_4$		$-\alpha_3-\alpha_4-\alpha_5$	$-\alpha_4-\alpha_5-\alpha_6$	
α_3			$-\alpha_2-\alpha_4-\alpha_5$		$-\alpha_4-\alpha_5-\alpha_6$
α_4					
α_5		$-\alpha_1-\alpha_3-\alpha_4$	$-\alpha_2-\alpha_3-\alpha_4$		
α_6				$-\alpha_2-\alpha_4-\alpha_5$	$-\alpha_3-\alpha_4-\alpha_5$

Table (10.3)

	$-\beta_{1,5}$	$-\beta_{2,5}$	$-\beta_{3,5}$	$-\beta_{4,5}$
α_1	$-\alpha_2-\alpha_3-\alpha_4-\alpha_5$	$-\alpha_3-\alpha_4-\alpha_5-\alpha_6$		
α_2	$-\alpha_1-\alpha_3-\alpha_4-\alpha_5$		$-\alpha_3-\alpha_4-\alpha_5-\alpha_6$	
α_3			$-\alpha_2-\alpha_4-\alpha_5-\alpha_6$	
α_4				$-\alpha_2-\alpha_3-\alpha_4-\alpha_5$
α_5	$-\alpha_1-\alpha_2-\alpha_3-\alpha_4$			
α_6		$-\alpha_1-\alpha_3-\alpha_4-\alpha_5$	$-\alpha_2-\alpha_3-\alpha_4-\alpha_5$	

Table (10.4)

found in the table (10.5) and (10.6). Note that there are four roots of height 5 and three roots of height 6. There are also three roots of height 7, i.e., the number of roots of height

	$-\beta_{1,6}$	$-\beta_{2,6}$	$-\beta_{3,6}$
α_1	$-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6$	$-\alpha_2-\alpha_3-2\alpha_4-\alpha_5$	
α_2	$-\alpha_1-\alpha_3-\alpha_4-\alpha_5-\alpha_6$		
α_3			
α_4		$-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5$	$-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6$
α_5			
α_6	$-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5$		$-\alpha_2-\alpha_3-2\alpha_4-\alpha_5$

Table (10.5)

	$-\beta_{1,7}$	$-\beta_{2,7}$	$-\beta_{3,7}$
α_1	$-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6$		
α_2			
α_3	$-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6$	$-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5$	
α_4			
α_5			$-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6$
α_6	$-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5$		

Table (10.6)

7 is equal to the number of roots of height 6. In table (10.7) we find the analysis of the two roots of height 8. Note that the number of roots of height 8 is less one than the number of roots of height 7.

	$-\beta_{1,8}$	$-\beta_{2,8}$
α_1	$-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6$	
α_2		
α_3		$-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6$
α_4		
α_5	$-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6$	
α_6		$-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5$

Table (10.7)

The results for the roots of height 9 and 10 are listed together in table (10.8).

	$-\beta_{1,9}$	$-\beta_{1,10}$
α_1		
α_2		
α_3	$-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6$	
α_4		$-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-\alpha_6$
α_5	$-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6$	
α_6		

Table (10.8)

In table (10.9) we analysed the negative root of maximal height $-\beta_{1,11}$. This root has shape $-\beta_{1,11} = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6$.

	$-\beta_{1,11}$
α_1	
α_2	$-\alpha_1-\alpha_2-2\alpha_3-3\alpha_4-2\alpha_5-\alpha_6$
α_3	
α_4	
α_5	
α_6	

Table (10.9)

10.2 The transformation lemma for E_6

We are going to prove the transformation lemma for the group of type E_6. We make use of the elaboration of the adjoint action and the root system done in the previous section. Therefore we keep all the notations done there. Further let (F, ∂) denote a differential field with field of constants C and let us define Ω as the set of the 6 negative roots

$$\begin{aligned}\Omega = \{&\gamma_1 = -\alpha_1,\ \gamma_2 = -\alpha_2-\alpha_4-\alpha_5-\alpha_6,\ \gamma_3 = -\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6,\\ &\gamma_4 = -\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6,\ \gamma_5 = -\alpha_1-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6,\\ &\gamma_6 = -\alpha_1-2\alpha_2-2\alpha_3-3\alpha_4-2\alpha_5-\alpha_6\ \}.\end{aligned}$$

Lemma 10.1. *Let* $A \in \sum_{i=1}^{6} X_{\alpha_i} + \mathbf{H}(F) + \sum_{\beta\in\Phi^-} \mathbf{L}_\beta(F)$. *Then there exists* $U \in \mathcal{U}^-$ *such that*

$$UAU^{-1} + \partial(U)U^{-1} \quad \in \quad \sum_{i=1}^{6} X_{\alpha_i} + \sum_{\gamma_i\in\Omega} \mathbf{L}_{\gamma_i}(F).$$

Proof. The element A can be written with respect to a Chevalley basis $\{X_\alpha,\ H_i \mid \alpha \in \Phi,\ 1 \le i \le 6\}$ and for suitable elements $a_{0,i}, a_{0,\beta} \in F$ as

$$A = \sum_{i=1}^{6} X_{\alpha_i} + a_{0,i}H_i + \sum_{\beta\in\Phi^-} a_{0,\beta}X_\beta.$$

In the first step we delete the terms $a_{0,i} \cdot H_i$ for $i = 1, ..., 6$, i.e., we delete the part of A lying in the Cartan subalgebra $\mathbf{H}$. To achieve this, we differentially conjugate A by the root group elements $U_{-\alpha_i}(\zeta_i)$ where $\alpha_i \in \Delta$ for $i = 1, ..., 6$. We have

$$\begin{aligned}\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(A) + l\delta(U_{-\alpha_i}(\zeta_i)) &= \sum_{j=1}^{6} \mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(X_{\alpha_j}) + a_j\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(H_j)\\ &+ \sum_{\beta\in\Phi^-} a_{1,\beta}\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(X_\beta) + l\delta(U_{-\alpha_i}(\zeta_i)).\end{aligned}$$

The term $l\delta(U_{-\alpha_i}(\zeta_i))$ lies by Proposition 3.5 in the root space $\mathbf{L}_{-\alpha_i}$. Since the signs of the roots $-\alpha_i$ and $\beta \in \Phi^-$ are negative, we deduce that the term $\sum_{\beta\in\Phi^-} a_{1,\beta}\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(X_\beta)$ is an element of the subspace $\sum_{\beta\in\Phi^-} \mathbf{L}_\beta$. Now we analyse the terms $\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(X_{\alpha_j})$. For $j \neq i$ we obtain

$$\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(X_{\alpha_j}) = X_{\alpha_j} + \sum_{l\ge1} m_{-\alpha_i,\alpha_j}\zeta^l X_{\alpha_j+l(-\alpha_i)} = X_{\alpha_j}.$$

In the case $j = i$ the term $\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(X_{\alpha_j})$ is

$$\begin{aligned}\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(X_{\alpha_j}) &= X_{\alpha_i} + \sum_{l\ge1} m_{-\alpha_i,\alpha_i}\zeta^l X_{\alpha_i+l(-\alpha_i)}\\ &\in X_{\alpha_i} + m_{-\alpha_i,\alpha_i,1}\zeta_i H_i + \mathbf{L}_{-\alpha_i}.\end{aligned}$$

Moreover, for $H_0 := \sum_{i=1}^{6} a_{0,i}H_i$ we have that $\mathrm{Ad}(U_{-\alpha_i}(\zeta_i))(H_0) = H_0 + \zeta_i\,[X_{-\alpha_i}, H_0]$ is an element of the subspace $H_0 + \mathrm{L}_{-\alpha_i}$.
We put now all of those results together. This yields

$$\begin{aligned}&\mathrm{Ad}(U_{-\alpha_6}(\zeta_6) \cdot ... \cdot U_{-\alpha_1}(\zeta_1))(A) + l\delta(U_{-\alpha_6}(\zeta_6) \cdot ... \cdot U_{-\alpha_1}(\zeta_1)) =\\ &\sum_{i=1}^{6} X_{\alpha_i} + (a_{0,i} + m_{-\alpha_i,\alpha_i,1}\zeta_i)H_i + \sum_{\beta\in\Phi^-} a_{1,\beta}X_\beta =: A_1.\end{aligned}$$

We define $\zeta_i = -\frac{a_{0,i}}{m_{-\alpha_i,\alpha_i,1}}$ for $i = 1, ..., 6$. Then the matrix A_1 becomes

$$A_1 = \sum_{i=1}^{6} X_{\alpha_i} + \sum_{\gamma\in\Phi^-} a_{1,\gamma}X_\gamma.$$

The next step is to delete all terms $a_{1,\gamma}X_\gamma$ of A_1 which correspond to the negative roots $\gamma \in \Phi^-$ except to the roots of

$$\begin{aligned}\Omega = \{&\gamma_1 = -\alpha_1,\ \gamma_2 = -\alpha_2 - \alpha_4 - \alpha_5 - \alpha_6,\ \gamma_3 = -\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6,\\ &\gamma_4 = -\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6,\ \gamma_5 = -\alpha_1 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6,\\ &\gamma_6 = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6\ \}.\end{aligned}$$

This transformation will be done for all negative roots of a given height k where $k = 1, ..., 10$. Thus, in each step there are some repeating arguments. To get rid of terms in the decomposition of the matrix A_k in step k which correspond to negative roots of height k, we differentially conjugate A_k with root subgroup elements $U_{-\beta_{h,k+1}}(\zeta_h)$ belonging to the roots $-\beta_{h,k+1}$ of height $k+1$. Then Proposition 3.5 yields that the logarithmic derivate $l\delta(U_{-\beta_{h,k+1}})$ lies in the root space $\mathbf{L}_{-\beta_{h,k+1}}$, i.e., in a root space corresponding to a root of height $k+1$. Similarly, for any negative root $\gamma \in \Phi^-$ the element $\mathrm{Ad}(U_{-\beta_{h,k+1}}(\zeta_h))(a_{k,\gamma}X_\gamma)$ has shape

$$\mathrm{Ad}(U_{-\beta_{h,k+1}}(\zeta_h))(a_{k,\gamma}X_\gamma) = a_{k,\gamma}X_\gamma + \sum_{\bar\gamma\in\Phi^-,\mathrm{ht}(\bar\gamma)\geq \mathrm{ht}(\gamma+(-\beta_{h,k+1}))} \mathbf{L}_{\bar\gamma},$$

i.e., we generate new entries in root spaces of height greater than $k+1$. To avoid that the proof becomes needlessly long we do not refer in each step of the transformation to those arguments.

In the first step we delete five of the six negative roots of height one. Note that there are five roots of height 2. Hence we have five parameters available for the first transformation. However, for the differential conjugation of A_1 by the element $U_{-\beta_{5,2}}(\zeta_5) \cdot ... \cdot U_{-\beta_{1,2}}(\zeta_1)$ we obtain with the help of table (10.1)

$$\begin{aligned}&\mathrm{Ad}(U_{-\beta_{5,2}}(\zeta_5) \cdot ... \cdot U_{-\beta_{1,2}}(\zeta_1))(A_1) + l\delta(U_{-\beta_{5,2}}(\zeta_5) \cdot ... \cdot U_{-\beta_{1,2}}(\zeta_1)) =\\ &\sum_{i=1}^{6} X_{\alpha_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 2} a_{2,\gamma}X_\gamma + X_{-\alpha_1}(a_{1,-\alpha_1} + m_{-\beta_{1,2},\alpha_3}\zeta_1)\\ &+X_{-\alpha_2}(a_{1,-\alpha_2} + m_{-\beta_{2,2},\alpha_4}\zeta_2) + X_{-\alpha_3}(a_{1,-\alpha_3} + m_{-\beta_{1,2},\alpha_1}\zeta_1 + m_{-\beta_{3,2},\alpha_4}\zeta_3)\\ &+X_{-\alpha_4}(a_{1,-\alpha_4} + m_{-\beta_{2,2},\alpha_2}\zeta_2 + m_{-\beta_{3,2},\alpha_3}\zeta_3 + m_{-\beta_{4,2},\alpha_5}\zeta_4)\\ &+X_{-\alpha_5}(a_{1,-\alpha_5} + m_{-\beta_{4,2},\alpha_4}\zeta_4 + m_{-\beta_{5,2},\alpha_6}\zeta_5) + X_{-\alpha_6}(a_{1,-\alpha_6} + m_{-\beta_{5,2},\alpha_5}\zeta_5)\\ &= \sum_{i=1}^{6} X_{\alpha_i} + a_{2,\gamma_1}X_{\gamma_1} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 2} a_{2,\gamma}X_\gamma =: A_2\end{aligned}$$

where $\gamma_1 = -\alpha_1$ and $a_{2,\gamma} \in F$ for $\gamma \in \Phi^-$ with $\mathrm{ht}(\gamma) \geq 2$. The last equation is obtained by defining

$$\begin{aligned}\zeta_1 &:= -\frac{1}{m_{-\beta_{1,2}\alpha_1}}(a_{1,-\alpha_3} + m_{-\beta_{3,2}\alpha_4}\zeta_3), \quad \zeta_2 := \frac{-a_{1,-\alpha_2}}{m_{-\beta_{2,2}\alpha_4}},\\ \zeta_3 &:= -\frac{1}{m_{-\beta_{3,2}\alpha_3}}(a_{1,-\alpha_4} + m_{-\beta_{2,2}\alpha_2}\zeta_2 + m_{-\beta_{4,2}\alpha_5}\zeta_4),\\ \zeta_4 &:= -\frac{1}{m_{-\beta_{4,2}\alpha_4}}(a_{1,-\alpha_5} + m_{-\beta_{5,2}\alpha_6}\zeta_5) \quad \text{and} \quad \zeta_5 := \frac{-a_{1,-\alpha_6}}{m_{-\beta_{5,2}\alpha_5}}.\end{aligned}$$

We use table (10.2) to delete all negative roots of height two. More precisely, we get

$$\begin{gathered}\mathrm{Ad}(U_{-\beta_{5,3}}(\zeta_5)\cdot\ldots\cdot U_{-\beta_{1,3}}(\zeta_1))(A_2)+l\delta(U_{-\beta_{5,3}}(\zeta_5)\cdot\ldots\cdot U_{-\beta_{1,3}}(\zeta_1))=\\ \sum_{i=1}^{6}X_{\alpha_i}+a_{3,\epsilon_1}X_{\epsilon_1}+\sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 3}a_{3,\gamma}X_\gamma+\\ X_{-\alpha_1-\alpha_3}(a_{2,-\alpha_1-\alpha_3}+m_{-\beta_{1,3},\alpha_4}\zeta_1)+\\ X_{-\alpha_2-\alpha_4}(a_{2,-\alpha_2-\alpha_4}+m_{-\beta_{2,3},\alpha_3}\zeta_2+m_{-\beta_{3,3},\alpha_5}\zeta_3)+\\ X_{-\alpha_3-\alpha_4}(a_{2,-\alpha_3-\alpha_4}+m_{-\beta_{2,3},\alpha_2}\zeta_2+m_{-\beta_{4,3},\alpha_5}\zeta_4)+\\ X_{-\alpha_4-\alpha_5}(a_{2,-\alpha_4-\alpha_5}+m_{-\beta_{3,3},\alpha_2}\zeta_3+m_{-\beta_{4,3},\alpha_3}\zeta_4+m_{-\beta_{5,3},\alpha_6}\zeta_5)+\\ X_{-\alpha_5-\alpha_6}(a_{2,-\alpha_5-\alpha_6}+m_{-\beta_{5,3},\alpha_4}\zeta_5)=:A_3\end{gathered}$$

with $a_{3,\gamma}\in F$ for $\gamma\in\Phi^-$ with $\mathrm{ht}(\gamma)\geq 3$. If we set

$$\begin{aligned}&\zeta_1:=\frac{-a_{2,-\alpha_1-\alpha_3}}{m_{-\beta_{1,3}\alpha_4}},\quad \zeta_2:=-\frac{1}{m_{-\beta_{2,3}\alpha_3}}(a_{2,-\alpha_2-\alpha_4}+m_{-\beta_{3,3}\alpha_5}\zeta_3),\\ &\zeta_3:=-\frac{1}{m_{-\beta_{3,3}\alpha_2}}(a_{2,-\alpha_4-\alpha_5}+m_{-\beta_{4,3}\alpha_3}\zeta_4+m_{-\beta_{5,3}\alpha_6}\zeta_5),\\ &\zeta_4:=-\frac{1}{m_{-\beta_{4,3}\alpha_5}}(a_{2,-\alpha_3-\alpha_4}+m_{-\beta_{2,3}\alpha_2}\zeta_2)\quad\text{and}\quad\zeta_5:=\frac{-a_{2,-\alpha_5-\alpha_6}}{m_{-\beta_{5,3}\alpha_4}},\end{aligned}$$

then A_3 becomes $A_3=\sum_{i=1}^6 X_{\alpha_i}+a_{3,\gamma_1}X_{\gamma_1}+\sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 3}a_{3,\gamma}X_\gamma$.
In the next step we get rid of all roots of height three. The definition of

$$\begin{gathered}\zeta_1=-\frac{1}{m_{-\beta_{1,4},\alpha_2}}(a_{3,-\beta_{1,3}}+m_{-\beta_{2,4},\alpha_5}\zeta_2),\qquad \zeta_3=-\frac{1}{m_{-\beta_{3,4},\alpha_5}}(a_{3,-\beta_{2,3}}+m_{-\beta_{1,4},\alpha_1}\zeta_1),\\ \zeta_2=-\frac{1}{m_{-\beta_{2,4},\alpha_1}}(a_{3,-\beta_{4,3}}+m_{-\beta_{3,4},\alpha_2}\zeta_3+m_{-\beta_{5,4},\alpha_6}\zeta_5),\\ \zeta_4=-\frac{1}{m_{-\beta_{4,4},\alpha_6}}(a_{3,-\beta_{3,3}}+m_{-\beta_{3,4},\alpha_3}\zeta_3)\quad\text{and}\quad\zeta_5=\frac{1}{m_{-\beta_{5,4},\alpha_3}}(a_{3,-\beta_{5,3}}+m_{-\beta_{4,4},\alpha_2}\zeta_4)\end{gathered}$$

together with table (10.3) yields

$$\begin{gathered}\mathrm{Ad}(U_{-\beta_{5,4}}(\zeta_5)\cdot\ldots\cdot U_{-\beta_{1,4}}(\zeta_1))(A_3)+l\delta(U_{-\beta_{5,4}}(\zeta_5)\cdot\ldots\cdot U_{-\beta_{1,4}}(\zeta_1))=\\ \sum_{i=1}^{6}X_{\alpha_i}+a_{4,\gamma_1}X_{\gamma_1}+\sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 4}a_{4,\gamma}X_\gamma+\\ X_{-\beta_{1,3}}(a_{3,-\beta_{1,3}}+m_{-\beta_{1,4},\alpha_2}\zeta_1+m_{-\beta_{2,4},\alpha_5}\zeta_2)+\\ X_{-\beta_{2,3}}(a_{3,-\beta_{2,3}}+m_{-\beta_{1,4},\alpha_1}\zeta_1+m_{-\beta_{3,4},\alpha_5}\zeta_3)+\\ X_{-\beta_{3,3}}(a_{3,-\beta_{3,3}}+m_{-\beta_{3,4},\alpha_3}\zeta_3+m_{-\beta_{4,4},\alpha_6}\zeta_4)+\\ X_{-\beta_{4,3}}(a_{3,-\beta_{4,3}}+m_{-\beta_{2,4},\alpha_1}\zeta_2+m_{-\beta_{3,4},\alpha_2}\zeta_3+m_{-\beta_{5,4},\alpha_6}\zeta_5)+\\ X_{-\beta_{5,3}}(a_{3,-\beta_{5,3}}+m_{-\beta_{4,4},\alpha_2}\zeta_4+m_{-\beta_{5,4},\alpha_3}\zeta_5)=\\ \sum_{i=1}^{6}X_{\alpha_i}+a_{4,\gamma_1}X_{\gamma_1}+\sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 4}a_{4,\gamma}X_\gamma:=A_4.\end{gathered}$$

There are five negative roots of height 4 and four negative roots of height 5. Thus we have only four parameters available to delete the coefficients in A_4 corresponding to those five negative roots. We choose the parameters $\zeta_1, ..., \zeta_4$ as

$$\zeta_1 = \frac{-a_{4,-\beta_{1,4}}}{m_{-\beta_{1,5},\alpha_5}}, \quad \zeta_2 = -\frac{1}{m_{-\beta_{2,5},\alpha_6}}(a_{4,-\beta_{2,4}} + m_{-\beta_{1,5},\alpha_2}\zeta_1),$$
$$\zeta_3 = -\frac{1}{m_{-\beta_{3,5},\alpha_2}}(a_{4,-\beta_{5,4}} + m_{-\beta_{2,5},\alpha_1}\zeta_2),$$
$$\zeta_4 = -\frac{1}{m_{-\beta_{4,5},\alpha_4}}(a_{4,-\beta_{3,4}} + m_{-\beta_{1,5},\alpha_1}\zeta_1 + m_{-\beta_{3,5},\alpha_6}\zeta_3)$$

and obtain with the help of table (10.4)

$$\begin{gathered}
\mathrm{Ad}(U_{-\beta_{4,5}}(\zeta_4)\cdot ... \cdot U_{-\beta_{1,5}}(\zeta_1))(A_4) + l\delta(U_{-\beta_{4,5}}(\zeta_4)\cdot ... \cdot U_{-\beta_{1,5}}(\zeta_1)) = \\
\sum_{i=1}^{6} X_{\alpha_i} + a_{5,\gamma_1}X_{\gamma_1} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 5} a_{5,\gamma}X_\gamma + \\
X_{-\beta_{1,4}}(a_{4,-\beta_{1,4}} + m_{-\beta_{1,5},\alpha_5}\zeta_1) + \\
X_{-\beta_{2,4}}(a_{4,-\beta_{2,4}} + m_{-\beta_{1,5},\alpha_2}\zeta_1 + m_{-\beta_{2,5},\alpha_6}\zeta_2) + \\
X_{-\beta_{3,4}}(a_{4,-\beta_{3,4}} + m_{-\beta_{1,5},\alpha_1}\zeta_1 + m_{-\beta_{3,5},\alpha_6}\zeta_3 + m_{-\beta_{4,5},\alpha_4}\zeta_4) + \\
X_{-\beta_{4,4}}(a_{4,-\beta_{4,4}} + m_{-\beta_{3,5},\alpha_3}\zeta_3) + \\
X_{-\beta_{5,4}}(a_{4,-\beta_{5,4}} + m_{-\beta_{2,5},\alpha_1}\zeta_2 + m_{-\beta_{3,5},\alpha_2}\zeta_3) + \\
\sum_{i=1}^{6} X_{\alpha_i} + a_{5,\gamma_1}X_{\gamma_1} + a_{5,\gamma_2}X_{\gamma_2} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 4} a_{5,\gamma}X_\gamma =: A_5
\end{gathered}$$

where $\gamma_2 = -\alpha_2 - \alpha_4 - \alpha_5 - \alpha_6$ and $a_{5,\gamma} \in F$ for $\gamma \in \Phi^-$ with $\mathrm{ht}(\gamma) \geq 4$.
Since there are only three negative roots of height 6 we have three parameters available for the transformation of the four negative roots of height 5, which can be calculated with the help of table (10.5) as

$$\begin{gathered}
\mathrm{Ad}(U_{-\beta_{3,6}}(\zeta_3)U_{-\beta_{2,6}}(\zeta_2)U_{-\beta_{1,6}}(\zeta_1))(A_5) + l\delta(U_{-\beta_{3,6}}(\zeta_3)U_{-\beta_{2,6}}(\zeta_2)U_{-\beta_{1,6}}(\zeta_1)) = \\
\sum_{i=1}^{6} X_{\alpha_i} + a_{6,\gamma_1}X_{\gamma_1} + a_{6,\gamma_2}X_{\gamma_2} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 6} a_{6,\gamma}X_\gamma + \\
X_{-\beta_{1,5}}(a_{5,-\beta_{1,5}} + m_{-\beta_{1,6},\alpha_6}\zeta_1 + m_{-\beta_{2,6},\alpha_2}\zeta_2) + \\
X_{-\beta_{2,5}}(a_{5,-\beta_{2,5}} + m_{-\beta_{1,6},\alpha_2}\zeta_1) + \\
X_{-\beta_{3,5}}(a_{5,-\beta_{3,5}} + m_{-\beta_{1,6},\alpha_1}\zeta_1 + m_{-\beta_{3,6},\alpha_4}\zeta_3) + \\
X_{-\beta_{4,5}}(a_{5,-\beta_{4,5}} + m_{-\beta_{2,6},\alpha_1}\zeta_2 + m_{-\beta_{3,6},\alpha_6}\zeta_3) =: A_6
\end{gathered}$$

with suitable new coefficients $a_{6,\gamma} \in F$. We define

$$\zeta_1 = \frac{-a_{5,-\beta_{2,5}}}{m_{-\beta_{1,6},\alpha_2}}, \quad \zeta_2 = -\frac{1}{m_{-\beta_{2,6},\alpha_2}}(a_{5,-\beta_{1,5}} + m_{-\beta_{1,6},\alpha_6}\zeta_1) \quad \text{and}$$
$$\zeta_3 = -\frac{1}{m_{-\beta_{3,6},\alpha_6}}(a_{5,-\beta_{4,5}} + m_{-\beta_{2,6},\alpha_1}\zeta_2).$$

This leads to $A_6 = \sum_{i=1}^{6} X_{\alpha_i} + \sum_{i=1}^{3} a_{6,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 6} a_{6,\gamma} X_\gamma$.
In the next step we delete all roots of height 6. Therefore we define the three parameters ζ_1, ζ_2 and ζ_3 as

$$\zeta_1 := \frac{-a_{6,-\beta_{1,6}}}{m_{-\beta_{1,7},\alpha_3}}, \quad \zeta_2 := -\frac{1}{m_{-\beta_{2,7},\alpha_3}}(a_{6,-\beta_{2,6}} + m_{-\beta_{1,7},\alpha_6}\zeta_1) \quad \text{and}$$
$$\zeta_3 := -\frac{1}{m_{-\beta_{3,7},\alpha_5}}(a_{6,-\beta_{3,6}} + m_{-\beta_{1,7},\alpha_1}\zeta_1).$$

Then we obtain together with table (10.6)

$$\begin{gathered}
\mathrm{Ad}(U_{-\beta_{3,7}}(\zeta_3)U_{-\beta_{2,7}}(\zeta_2)U_{-\beta_{1,7}}(\zeta_1))(A_6) + l\delta(U_{-\beta_{3,7}}(\zeta_3)U_{-\beta_{2,7}}(\zeta_2)U_{-\beta_{1,7}}(\zeta_1)) = \\
\sum_{i=1}^{6} X_{\alpha_i} + \sum_{i=1}^{3} a_{7,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 7} a_{7,\gamma} X_\gamma + \\
X_{-\beta_{1,6}}(a_{6,-\beta_{1,6}} + m_{-\beta_{1,7},\alpha_3}\zeta_1) + \\
X_{-\beta_{2,6}}(a_{6,-\beta_{2,6}} + m_{-\beta_{1,7},\alpha_6}\zeta_1 + m_{-\beta_{2,7},\alpha_3}\zeta_2) + \\
X_{-\beta_{3,6}}(a_{6,-\beta_{3,6}} + m_{-\beta_{1,7},\alpha_1}\zeta_1 + m_{-\beta_{3,7},\alpha_5}\zeta_3) = \\
\sum_{i=1}^{6} X_{\alpha_i} + \sum_{i=1}^{3} a_{7,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 7} a_{7,\gamma} X_\gamma =: A_7
\end{gathered}$$

with suitable new elements $a_{7,\gamma} \in F$. Since the number of roots of height 8 decreases on 2 from the number of roots of height 7, we are not able to delete all terms in the decomposition of A_7 which belong to those roots of height 7. With the help of table (10.7) we compute

$$\begin{gathered}
\mathrm{Ad}(U_{-\beta_{2,8}}(\zeta_2)U_{-\beta_{1,8}}(\zeta_1))(A_7) + l\delta(U_{-\beta_{2,8}}(\zeta_2)U_{-\beta_{1,8}}(\zeta_1)) = \\
\sum_{i=1}^{6} X_{\alpha_i} + \sum_{i=1} s a_{8,\epsilon_i} X_{\epsilon_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 8} a_{8,\gamma} X_\gamma + \\
X_{-\beta_{1,7}}(a_{7,-\beta_{1,7}} + m_{-\beta_{1,8},\alpha_5}\zeta_1 + m_{-\beta_{2,8},\alpha_3}\zeta_2) + \\
X_{-\beta_{2,7}}(a_{7,-\beta_{2,7}} + m_{-\beta_{2,8},\alpha_6}\zeta_2) + X_{-\beta_{3,7}}(a_{7,-\beta_{3,7}} + m_{-\beta_{1,8},\alpha_1}\zeta_1) =: A_8.
\end{gathered}$$

We choose for ζ_1 and ζ_2 the values

$$\zeta_1 = -\frac{1}{m_{-\beta_{1,8},\alpha_5}}(a_{7,-\beta_{1,7}} + m_{-\beta_{2,8},\alpha_3}\zeta_2) \quad \text{and} \quad \zeta_2 = \frac{-a_{7,-\beta_{2,7}}}{m_{-\beta_{2,8},\alpha_6}}.$$

Then A_8 becomes $A_8 = \sum_{i=1}^{6} X_{\alpha_i} + \sum_{i=1}^{4} a_{8,\gamma_i} X_{\gamma_i} + \sum_{\gamma\in\Phi^-,\mathrm{ht}(\gamma)\geq 8} a_{8,\gamma} X_\gamma$. For the transformation of the roots of height 8 we have only one parameter ζ_1 available. We define

$\zeta_1 = \frac{-a_{8,-\beta_{2,8}}}{m_{-\beta_{1,9},\alpha_5}}$. Then the first column of table (10.8) yields

$$\begin{gathered}
\mathrm{Ad}(U_{-\beta_{1,9}}(\zeta_1))(A_8) + l\delta(U_{-\beta_{1,9}}(\zeta_1)) = \\
\sum_{i=1}^{6} X_{\alpha_i} + \sum_{i=1}^{4} a_{9,\gamma_i} X_{\gamma_i} + \sum_{\gamma \in \Phi^-, \mathrm{ht}(\gamma) \geq 9} a_{9,\gamma} X_\gamma + \\
X_{-\beta_{1,8}}(a_{8,-\beta_{1,8}} + m_{-\beta_{1,9},\alpha_3}\zeta_1) + X_{-\beta_{2,8}}(a_{8,-\beta_{2,8}} + m_{-\beta_{1,9},\alpha_5}\zeta_1) = \\
\sum_{i=1}^{6} X_{\alpha_i} + \sum_{i=1}^{5} a_{9,\gamma_i} X_{\gamma_i} + \sum_{\gamma \in \Phi^-, \mathrm{ht}(\gamma) \geq 9} a_{9,\gamma} X_\gamma =: A_9.
\end{gathered}$$

We are going to delete the term $a_{9,-\beta_{1,9}} X_{-\beta_{1,9}}$ of A_9. Therefore let $\zeta_1 := \frac{-a_{9,-\beta_{1,9}}}{m_{-\beta_{1,10},\alpha_4}}$. We deduce with the help of the second column of table (10.8)

$$\begin{gathered}
\mathrm{Ad}(U_{-\beta_{1,10}}(\zeta_1))(A_9) + l\delta(U_{-\beta_{1,10}}(\zeta_1)) = \\
\sum_{i=1}^{6} X_{\alpha_i} + \sum_{i=1}^{5} a_{10,\gamma_i} X_{\gamma_i} + \sum_{\gamma \in \Phi^-, \mathrm{ht}(\gamma) \geq 10} a_{10,\gamma} X_\gamma + \\
X_{-\beta_{1,9}}(a_{9,-\beta_{1,9}} + m_{-\beta_{1,10},\alpha_4}\zeta_1) = \sum_{i=1}^{6} X_{\alpha_i} + \sum_{i=1}^{5} a_{10,\gamma_i} X_{\gamma_i} + \sum_{\gamma \in \Phi^-, \mathrm{ht}(\gamma) \geq 10} a_{10,\gamma} X_\gamma =: A_{10}.
\end{gathered}$$

In the last step we differentially conjugate A_{10} with the root group element $U_{-\beta_{1,11}}(\zeta_1)$ which corresponds to the negative root of maximal height $-\beta_{1,11}$. By table (10.9) this is

$$\begin{gathered}
\mathrm{Ad}(U_{-\beta_{1,11}}(\zeta_1))(A_{10}) + l\delta(U_{-\beta_{1,11}}(\zeta_1)) = \\
\sum_{i=1}^{6} X_{\alpha_i} + \sum_{i=1}^{5} a_{11,\gamma_i} X_{\gamma_i} + a_{11,\beta_{1,11}} X_{\beta_{1,11}} + \\
X_{-\beta_{1,10}}(a_{10,-\beta_{1,10}} + m_{-\beta_{1,11},\alpha_2}\zeta_1) =: A_{11}.
\end{gathered}$$

Thus, if we define $\zeta_1 = \frac{-a_{10,-\beta_{1,10}}}{m_{-\beta_{1,11},\alpha_2}}$ then A_{11} becomes $A_{11} = \sum_{i=1}^{6} X_{\alpha_i} + \sum_{i=1}^{6} a_{11,\gamma_i} X_{\gamma_i}$. Hence, the lemma follows. $\square$

10.3 The equation for the group of type E_6

In [How01] the authors R. Howlett, L. Rylands and D Taylor present a method to construct matrix generators for the exceptional groups of Lie type. Moreover, they computed in the same paper explicit matrix generators for the Lie algebras of exceptional type. For the Lie algebra of type E_6 the matrices representing the simple roots $\alpha_i \in \Delta =$

$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ are

$$\begin{aligned}
X_{\alpha_1} &= E_{1,2} + E_{11,13} + E_{14,16} + E_{17,18} + E_{19,20} + E_{21,22}, \\
X_{\alpha_2} &= E_{4,5} + E_{6,7} + E_{8,10} + E_{19,21} + E_{20,22} + E_{23,24}, \\
X_{\alpha_3} &= E_{2,3} + E_{9,11} + E_{12,14} + E_{15,17} + E_{20,23} + E_{22,24}, \\
X_{\alpha_4} &= E_{3,4} + E_{7,9} + E_{10,12} + E_{17,19} + E_{18,20} + E_{24,25}, \\
X_{\alpha_5} &= E_{4,6} + E_{5,7} + E_{12,15} + E_{14,17} + E_{16,18} + E_{25,26} \quad \text{and} \\
X_{\alpha_6} &= E_{6,8} + E_{7,10} + E_{9,12} + E_{11,14} + E_{13,16} + E_{26,27}.
\end{aligned}$$

The matrix representing the negative simple root $-\alpha_i$ for $\alpha_i \in \Delta$ is the transpose of the matrix X_{α_i} and the elements $\{X_{\pm\alpha_i} \mid \alpha_i \in \Delta\}$ generate the Lie algebra of type E_6 which we denote by $\mathbf{L}_{E_6}$. In addition to the representation of the matrices for the positive and negative simple roots we compute with the help of an computer algebra system the shape of the matrices which represent the roots

$$\begin{aligned}
\Omega = \{\gamma_1 = -\alpha_1,\ &\gamma_2 = -\alpha_2 - \alpha_4 - \alpha_5 - \alpha_6,\ \gamma_3 = -\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6, \\
&\gamma_4 = -\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6,\ \gamma_5 = -\alpha_1 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6, \\
&\gamma_6 = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6\ \}.
\end{aligned}$$

Those matrices are

$$\begin{aligned}
X_{\gamma_2} &= -X_{10,3} - X_{12,4} - X_{15,6} - X_{21,11} - X_{22,13} + X_{27,23}, \\
X_{\gamma_3} &= -X_{10,2} + X_{14,4} + X_{17,6} - X_{21,9} + X_{24,13} + X_{27,20}, \\
X_{\gamma_4} &= -X_{15,2} - X_{17,3} - X_{19,4} - X_{21,5} + X_{26,13} + X_{27,16}, \\
X_{\gamma_5} &= -X_{15,1} + X_{18,3} + X_{20,4} + X_{22,5} + X_{26,11} + X_{27,14} \quad \text{and} \\
X_{\gamma_6} &= X_{21,1} + X_{22,2} + X_{24,3} + X_{25,4} + X_{26,6} + X_{27,8}.
\end{aligned}$$

Now denote by $F := C\langle t_1, ..., t_6\rangle$ the differential field generated by the differential indeterminates $\boldsymbol{t} = (t_1, ..., t_6)$ over C. Let $\boldsymbol{y}$ be the vector $\boldsymbol{y} := (y_1, ..., y_{27})^T$ and define the matrix differential equation $\partial(\boldsymbol{y}) = A_{E_6}(\boldsymbol{t}) \cdot \boldsymbol{y}$ over F by

$$A_{E_6}(\boldsymbol{t}) := \sum_{i=1}^{6} X_{\alpha_i} + \sum_{\gamma_i \in \Omega} t_i X_{\gamma_i}.$$

The shape of the matrix $A_{E_6}(\boldsymbol{t})$ which is determined by the above representation is presented on the next page.
The next step is the computation of a linear differential equation for the matrix equation $\partial(\boldsymbol{y}) = A_{E_6}(\boldsymbol{t}) \cdot \boldsymbol{y}$. As in the cases of the other groups we can choose y_1 as a cyclic vector. Unfortunatley, y_1 does not lead to a nice differential equation which can be written down. Trying other cyclic vectors yield simular results. However, we guess that y_1 is the most easiest cyclic vector. Since the matrix differential equation $\partial(\boldsymbol{y}) = A_{E_6}(\boldsymbol{t}) \cdot \boldsymbol{y}$ already has an easy and nice shape, it does not make sense to compute an enormous linear differential equation for the group of type E_6. We continue our proof with the matrix equation $\partial(\boldsymbol{y}) = A_{E_6}(\boldsymbol{t}) \cdot \boldsymbol{y}$.

$$
A_{E_6}(\boldsymbol{t}) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -t_3 & -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-t_5 & -t_4 & 0 & 0 & 0 & -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -t_4 & 0 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & t_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -t_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
t_6 & 0 & 0 & 0 & -t_4 & 0 & 0 & 0 & -t_3 & 0 & -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & t_6 & 0 & 0 & t_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & t_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & t_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & t_6 & 0 & 0 & 0 & 0 & t_5 & 0 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t_6 & 0 & 0 & 0 & 0 & 0 & t_5 & 0 & t_4 & 0 & 0 & 0 & t_3 & 0 & 0 & t_2 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

From the usual Chevalley construction for $\{X_\alpha \mid \alpha \in \Phi\}$ we obtain the group $\mathcal{G}_{E_6}$ of type E_6 with Lie algebra $\mathbf{L}_{E_6}$. The next step to prove that the differential Galois group of the matrix differential equation $\partial(\boldsymbol{y}) = A_{E_6}\boldsymbol{y}$ over F is $\mathcal{G}_{E_6}$ is to combine the results of Lemma 10.1 and Corollary 3.12 in Corollary 10.2. Therefore let us denote by $\bar{F} := (C(z), \partial = \frac{d}{dz})$ the rational function field with standard derivation as in Section 3.4.

Corollary 10.2. *Apply Corollary 3.12 to the group $\mathcal{G}_{E_6}$ and the Cartan decomposition of $\mathbf{L}_{E_6}$ and let us denote by $A_{E_6}^{M\&S} \in \mathbf{L}_{E_6}(\bar{F})$ the matrix satisfying the stated conditions. Then there exists $U \in \mathcal{U}^-(\bar{F}) \subset \mathcal{G}_{E_6}(\bar{F})$ such that*

$$\bar{A}_{E_6} := UA_{E_6}^{M\&S}U^{-1} + \partial(U)U^{-1} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in\Omega} f_i X_{\gamma_i} \tag{10.3}$$

with at least one $f_i \in C[z] \setminus C$ and the differential Galois group of the matrix equation $\partial(\boldsymbol{y}) = \bar{A}_{E_6}\boldsymbol{y}$ over $\bar{F}$ is $\mathcal{G}_{E_6}(C)$.

Proof. Lemma 10.1 implies the existence of an element $U \in \mathcal{U}_0^- \subset \mathcal{G}_{E_6}$ such that equation (10.3) holds. Since differential conjugation defines a differential isomorphism, we deduce with Corollary 3.12 that the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{E_6}\boldsymbol{y}$ is again $\mathcal{G}_{E_6}(C)$ over $\bar{F}$. We still need to show the existence of $f_i \in C[z] \setminus C$ for some $\gamma_i \in \Omega$. Suppose $\bar{A}_{E_6} = \sum_{\alpha\in\Delta} X_\alpha + \sum_{\gamma_i\in\Omega} f_i X_{\gamma_i} \in \mathrm{Lie}(\mathcal{G}_{E_6})(C)$. Then the corresponding differential equation $L(y) \in C\{y\}$ has coefficients in C. But then by [Mag94, Corollary 3.28] the differential Galois group is abelian. Thus, we obtain $\bar{A}_{E_6} \in \mathrm{Lie}(\mathcal{G}_{E_6})(\bar{F}) \setminus \mathrm{Lie}(\mathcal{G}_{E_6})(C)$. Since $0 \neq A_1 \in \mathbf{H}(C)$ and $A = (z^2A_1 + A_0)$ in Corollary 3.12, we start our transformation with at least one coefficient lying in $C[z] \setminus C$. In each step the application of $\mathrm{Ad}(U_\beta(\zeta))$ generates at most new entries which are polynomials in ζ. Moreover, the logarithmic derivative is the product of the two matrices $\partial(U_\beta(\zeta))$ and $U_\beta(\zeta)^{-1} = U_\beta(-\zeta)$. In the proof of Lemma 10.1 we choose the parameter ζ to be one of the coefficients. Hence, we get $f_i \in C[z] \setminus C$. □

Theorem 10.3. *The matrix differential equation*

$$\partial(\boldsymbol{y}) = A_{E_6}(\boldsymbol{t})\boldsymbol{y}$$

has E_6 as differential Galois group over $C\langle t_1, ..., t_6\rangle$. Moreover, let $\hat{F}$ be a differential field with field of constants equal to C. Let $\hat{E}$ be a Picard-Vessiot extension over $\hat{F}$ with differential Galois group $\mathcal{G}_{E_6}(C)$ and suppose the defining matrix differential equation $\partial(\boldsymbol{y}) = \hat{A}\boldsymbol{y}$ satisfies $\hat{A} \in \sum_{\alpha_i\in\Delta} X_{\alpha_i} + \sum_{\alpha\in\Phi^-} \mathbf{L}_\alpha$. Then there is a specialization $\partial(\boldsymbol{y}) = A_{E_6}(\hat{t}_1, ..., \hat{t}_6)\boldsymbol{y}$ with $\hat{t}_i \in \hat{F}$ such that $\partial(\boldsymbol{y}) = A_{E_6}(\hat{t}_1, ..., \hat{t}_6)\boldsymbol{y}$ gives rise to the extension $\hat{E}$ over $\hat{F}$.

Proof. Let E be a Picard-Vessiot extension for the equation $\partial(\boldsymbol{y}) = A_{E_6}(\boldsymbol{t})\boldsymbol{y}$ over F and denote by $\mathcal{G}$ the differential Galois group. Since for our matrix differential equation $\partial(\boldsymbol{y}) = A_{E_6}(\boldsymbol{t})\boldsymbol{y}$ holds $A_{E_6}(\boldsymbol{t}) \in \mathrm{Lie}(\mathcal{G}_{E_6})(F)$, Proposition 2.1 yields $\mathcal{G}(C) \leq \mathcal{G}_{E_6}(C)$. By Corollary 10.2 there exists a specialization $\sigma : (t_1, ..., t_6) \to (f_1, ..., f_6)$ with $f_i \in C[z]$ such that $\sigma(A_{E_6}(t_1, ..., t_6)) = \bar{A}_{E_6}$ and the differential Galois group of $\partial(\boldsymbol{y}) = \bar{A}_{E_6}\boldsymbol{y}$ is $\mathcal{G}_{E_6}(C)$. Moreover, we have $C\{f_1, ..., f_6\} = C[z]$. Thus we can apply Corollary 2.15. This yields $\mathcal{G}_{E_6}(C) \leq \mathcal{G}(C)$. Hence, it holds $\mathcal{G}(C) = \mathcal{G}_{E_6}(C)$.

Since the defining matrix $\hat{A}$ satisfies $\hat{A} \in \sum_{\alpha_i \in \Delta} X_{\alpha_i} + \sum_{\alpha \in \Phi^-} \mathrm{Lie}(\mathcal{G}_{E_6})_\alpha$, Lemma 10.1 provides that $\hat{A}$ is differentially equivalent to a matrix $\tilde{A} = \sum_{\alpha_i \in \Delta} X_{\alpha_i} + \sum_{\gamma_i \in \Omega} \hat{a}_i X_{\gamma_i}$ with suitable $\hat{a}_i \in \hat{F}$. Obviously the specialization

$$\hat{\sigma} : (t_1, ..., t_6) \mapsto (\hat{a}_1, ..., \hat{a}_6)$$

does the required. $\square$

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