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Derivation of Analytical, Closed-form Formulas for the Calculations of Instantaneous Screw Axes of Arbitrary Rigid 3D Multi-Body Systems

Jonas Boungard[1,*] and **Jens Wackerfuß**[1]

[1] Institute of Structural Analysis, University of Kassel

A method to calculate the parameters for each instantaneous and relative instantaneous screw axis in an arbitrary rigid multi-body system is presented. At first a kinematic analysis is performed which calculates the displacements of all nodes and the rotation of each body. In the next step the displacements of the nodes and the rotation of the bodies are used to calculate the instantaneous screw axis of each body. With the information on the instantaneous screw axes the instantaneous relative screw axes can be computed too. An example of the calculation and the visualization of the instantaneous screw axes is given.

1 Introduction

Any spatial rigid body motion of a point p on a body b_i can be expressed as a combination of a rotation $\boldsymbol{\psi}^{b_i}$ and a translation $\mathbf{u}^{b_i}_{\text{trans.}}$ along a screw axis with a distance to the point $\mathbf{r}^{b_i}_p$. The screw axis is defined by the direction vector $\mathbf{n}^{b_i}$, one position vector $\mathbf{X}_{\mathrm{S}_{b_i}}$ and the pitch h^{b_i}. The relative motion of two bodies can be expressed similarly with the relative screw axis.

$$\mathbf{u}_p = (\boldsymbol{\psi}^{b_i} \times \mathbf{r}^{b_i}_p) + \mathbf{u}^{b_i}_{\text{trans.}} \qquad h^{b_i} = \frac{\|\mathbf{u}^{b_i}_{\text{trans.}}\|}{\|\boldsymbol{\psi}^{b_i}\|} \tag{1}$$

2 Kinematic Method

With the kinematic method presented in [4] arbitrary three-dimensional multi-body-systems can be analyzed with respect to their mobility. The constraints to which the system has to comply with in terms of displacements $\mathbf{u}$ are assembled to the matrix $\mathbf{C}$, which leads in general to a rectangular linear system. For each kinematic mode k of the structure the displacements $\bar{\mathbf{u}}^{(k)}$ can be calculated and then used for the calculation of the screw axes.

$$\mathbf{C} \cdot \mathbf{u} = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} \bar{\mathbf{C}}_1 & \bar{\mathbf{C}}_2 \end{bmatrix} \cdot \begin{bmatrix} \bar{\mathbf{u}}_1 \\ \bar{\mathbf{u}}_2 \end{bmatrix} = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} \bar{\mathbf{C}}_1 & \bar{\mathbf{C}}_2 \end{bmatrix} \cdot \begin{bmatrix} \bar{\mathbf{u}}_1 \\ \bar{\mathbf{e}}_k \end{bmatrix} = \mathbf{0} \quad \Rightarrow \quad \bar{\mathbf{u}}^{(k)} = \begin{bmatrix} -\bar{\mathbf{C}}_1^{-1} \cdot \bar{\mathbf{C}}_2 \cdot \bar{\mathbf{e}}_k \\ \bar{\mathbf{e}}_k \end{bmatrix} \tag{2}$$

3 Calculation of the Instantaneous Screw Axis

With the displacements at three distinct points $\mathbf{u}^{b_i}_{1,2,\alpha}$ of each body b_i the rotation $\boldsymbol{\psi}^{b_i}$ of each body and thus the direction of the screw axis $\mathbf{n}^{b_i}$ can be computed using equation $(3)_1$. This is done with the help of geometric helper matrices $\mathbf{T}^{b_i}$ and $\mathbf{R}^{b_i}$ which have already been constructed in [4] to generate the constraint matrix $\mathbf{C}$. This approach has the advantage to other formulas found in the literature, e.g. [2], that the three displacements vectors can be linearly dependent. In the next step one point on the screw axis $\mathbf{X}_{\mathrm{S}_{b_i}}$ is computed using equation $(3)_2$. Finally the pitch is computed using equation $(3)_3$.

$$\boldsymbol{\psi}^{b_i} = \mathbf{T}^{b_i} \cdot \mathbf{R}^{b_i} \cdot \mathbf{u}^{b_i}_{1,2,\alpha} \;\Rightarrow\; \mathbf{n}^{b_i} = \frac{\boldsymbol{\psi}^{b_i}}{\|\boldsymbol{\psi}^{b_i}\|}; \qquad \mathbf{X}_{\mathrm{S}_{b_i}} = \mathbf{X}^{b_i}_1 + \frac{\mathbf{n}^{b_i} \times \mathbf{u}^{b_i}_1}{\|\boldsymbol{\psi}^{b_i}\|}; \qquad h^{b_i} = \mathbf{n}^{b_i} \cdot \mathbf{u}^{b_i}_1 \tag{3}$$

4 Calculation of the Relative Instantaneous Screw Axis

Although the theorem of three axes by Phillips & Hunt [1] was derived to obtain a geometrical relationship between the relative screw axes of three bodies, it can also be used to compute the relative instantaneous screw axis of two bodies b_i and b_j based on the information on the instantaneous screw axes of both bodies. Due to the different objectives, the formulas are slightly modified. At first the rotation of the relative screw axis $\boldsymbol{\psi}^{b_i,b_j}$ and thus the direction vector $\mathbf{n}^{b_i,b_j}$ is calculated.

$$\boldsymbol{\psi}^{b_i,b_j} = -\boldsymbol{\psi}^{b_i} + \boldsymbol{\psi}^{b_j} \quad \Rightarrow \quad \mathbf{n}^{b_i,b_j} = \frac{\boldsymbol{\psi}^{b_i,b_j}}{\|\boldsymbol{\psi}^{b_i,b_j}\|} \tag{4}$$

At second a position of an arbitrary point on the relative screw axis $\mathbf{X}_{\mathrm{S}_{b_i,b_j}}$ is computed. Finally the pitch h^{b_i,b_j} of the relative screw axis is computed. This is done with several auxiliary variables whose calculation is only partially shown in the original paper [1]. Therefore their calculation is also presented here for completeness:

* Corresponding author: e-mail boungard@uni-kassel.de, phone +49 561 804 2253

- Angle γ between the two screw axes with respect to the shortest common perpendicular:

$$\gamma = \arccos\left(-\mathbf{n}^{b_i} \cdot \mathbf{n}^{b_j}\right) \tag{5}$$

- Angle δ between the first screw axis and the relative screw axis with respect to the shortest common perpendicular:

$$\delta = \arctan \frac{\|\boldsymbol{\psi}^{b_j}\| \sin\gamma}{\|\boldsymbol{\psi}^{b_i}\| + \|\boldsymbol{\psi}^{b_j}\| \cos\gamma} \tag{6}$$

- Distance between the two screw axes Δ^{b_i,b_j} along the shortest common perpendicular:

$$\begin{bmatrix} \mathbf{n}^{b_i} & \mathbf{n}^{b_j} & \frac{\mathbf{n}^{b_i} \times \mathbf{n}^{b_j}}{\|\mathbf{n}^{b_i} \times \mathbf{n}^{b_j}\|} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \Delta^{b_i,b_j} \end{bmatrix} = \mathbf{X}_{S_{b_j}} - \mathbf{X}_{S_{b_i}} \tag{7}$$

- Distance between the first screw axis and the relative screw axis z^{b_i,b_j} along the common perpendicular:

$$z^{b_i,b_j} = (\Delta^{b_i,b_j} \cot\gamma - (h^{b_i} - h^{b_j})) \sin\delta\cos\delta + (\Delta^{b_i,b_j} + (h^{b_i} - h^{b_j}) \cot\gamma) \sin^2\delta \tag{8}$$

- Pitch of the relative screw axis h^{b_i,b_j}:

$$h^{b_i,b_j} = h^{b_i} + (\Delta^{b_i,b_j} \cot\gamma - (h^{b_i} - h^{b_j})) \sin^2\delta - (\Delta^{b_i,b_j} + (h^{b_i} - h^{b_j}) \cot\gamma) \sin\delta\cos\delta \tag{9}$$

- Position vector of the relative screw axis $\mathbf{X}_{S_{b_i,b_j}}$:

$$\mathbf{X}_{S_{b_i,b_j}} = \mathbf{X}_{S_{b_i}} + \alpha\,\mathbf{n}^{b_i} + z^{b_i,b_j} \frac{\mathbf{n}^{b_i} \times \mathbf{n}^{b_j}}{\|\mathbf{n}^{b_i} \times \mathbf{n}^{b_j}\|} \tag{10}$$

In the special case that the instantaneous screw axes of two bodies are parallel to each other, the relative instantaneous screw has to be calculated in the same way as the relative center of rotation for plane problems.

5 Visualization

The screw axes are visualized via lines and additional triangles that connect the axes with the corresponding bodies. Thus the screw axes can be attributed clearly to the bodies in the resulting images. The size of the triangles give information on the pitch of the screw axis. The example shows the motion of the so called "Evolution Door" and the instantaneous screw axes.

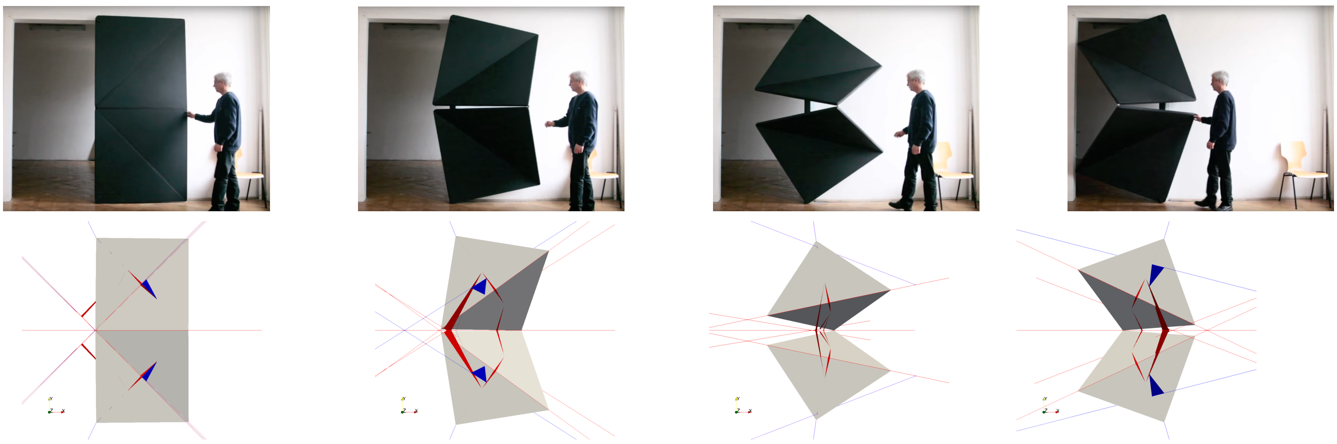

Fig. 1: top: Motion of the "Evolution Door" [3], bottom: Numerical simulation of the motion of the "Evolution Door", blue: screw axes, red: relative screw axes, please zoom

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