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A Study of Extensions of Classical Summation Theorems for the Series ${}_3F_2$ and ${}_4F_3$ with Applications

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Abstract. Very recently, Masjed-Jamei & Koepf [Some summation theorems for generalized hypergeometric functions, Axioms, 2018, 7, 38, 10.3390/axioms 7020038] established some summation theorems for the generalized hypergeometric functions. The aim of this paper is to establish extensions of some of their summation theorems in the most general form. As an application, several Eulerian-type and Laplace-type integrals have also been given. Results earlier obtained by Jun et al. and Koepf et al. follow special cases of our main findings.

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1. Introduction

The well-known and useful Pochhammer symbol (or the shifted or the raised factorial, since $(1)_n = n!$) denoted by $(a)_n$ for any complex number a is defined by

$$(a)_n = \begin{cases} a(a+1)\ldots(a+n-1) \; ; & (n \in \mathbb{N} \text{ and } a \in \mathbb{C}) \\ 1 & ; \; (n = 0 \text{ and } a \in \mathbb{C} \setminus \{0\}) \end{cases} \tag{1.1}$$

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In terms of the Gamma function, it is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{1.2}$$

where $\Gamma(x)$ is the well-known Gamma function defined by

$$\Gamma(x) = \int_0^\infty e^{-z} z^{x-1} dz$$

for $Re(x) > 0$. Thus, we may define the generalized hypergeometric function ${}_pF_q$ with p numerator parameters and q denominator parameters as follows [1,4,15,17–19].

$$\begin{aligned} {}_pF_q\left[a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x\right] &= {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ \\ b_1, \ldots, b_q \end{matrix}; x\right] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{x^n}{n!}. \end{aligned} \tag{1.3}$$

By the well-known ratio test [2], it can be easily verified that the series defined by (1.3) is convergent for all $p \leq q$. Also, the series (1.3) converges in $\mid z \mid< 1$ for $p = q + 1$ and converges everywhere for $p < q + 1$ and converges nowhere $(z \neq 0)$ for $p > q+1$. Further, for $p = q+1$, the series (1.3) converges absolutely for $|z| = 1$ provided $Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0$, and is converges conditionally for $\mid z \mid= 1$ and $z \neq 1$ if $-1 < Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) \leq 0$ and diverges for $|z| = 1$ if $Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) \leq -1$. In this regard, for more details about this function, we refer to the standard text [15].

For $p = 2, q = 1$ and $p = 1, q = 1$, we get two very important series, known in the literature as the hypergeometric function and the confluent hypergeometric function, respectively. For applications, we refer to [2,13].

In the theory of hypergeometric series, the following classical summation theorems play a key role.

Gauss summation theorem [6]

$${}_2F_1\left[\begin{matrix} a, b \\ \\ c \end{matrix}; 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \tag{1.4}$$

provided $Re(c - a - b) > 0$.

Gauss second summation theorem [6]

$$ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+1) & \end{matrix} ; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \tag{1.5} $$

Kummer summation theorem [6]

$$ {}_2F_1\left[\begin{matrix} a, & b \\ 1+a-b & \end{matrix} ; -1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)} \tag{1.6} $$

Bailey summation theorem [6]

$$ {}_2F_1\left[\begin{matrix} a,\ 1-a \\ b \end{matrix} ; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2}\right)} \tag{1.7} $$

Remark.

1. For interesting results by employing the above mentioned classical summation theorems, we refer to a paper by Bailey [3].
2. For generalizations of the above mentioned classical summation theorems (1.5), (1.6) and (1.7), we refer to research papers by Lavoie, et al. [9–11] and Rakha and Rathie [16].

In 2010, Kim et al. [8] extended the above mentioned classical summation theorems in the following form.

Extended Gauss summation theorem

$$ {}_3F_2\left[\begin{matrix} a, & b,\ d+1 \\ c+1, & d \end{matrix} ; 1\right] = \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a+1)\Gamma(c-b+1)}\left[(c-a-b)+\frac{ab}{d}\right] \tag{1.8} $$

provided $Re(c-a-b)>0$.

For $d=c$, it reduces to Gauss summation theorem (1.4).

Extended Gauss second summation theorem

$$ {}_3F_2\left[\begin{matrix} a, & b,\ d+1 \\ \frac{1}{2}(a+b+3), & d \end{matrix} ; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{3}{2}\right)} $$
$$ \times\left\{\frac{\left(\frac{1}{2}(a+b-1)-\frac{ab}{d}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}+\frac{\left(\frac{1}{d}(a+b+1)-2\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\right\}. \tag{1.9} $$

For $d=\frac{1}{2}(a+b+1)$, it reduces to Gauss second summation theorem (1.5).

Extended Kummer summation theorem

$$ {}_3F_2\left[\begin{matrix} a, & b,\ d+1 \\ 2+a-b, & d \end{matrix};-1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(2+a-b\right)}{2^a(1-b)}\left\{\frac{\left(\frac{1}{d}(1+a-b)-1\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b+\frac{3}{2}\right)} + \frac{\left(1-\frac{a}{d}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}a-b\right)}\right\}. \tag{1.10} $$

For $d = 1 + a - b$, it reduces to Kummer summation theorem (1.6).

Extended Bailey summation theorem

$$ {}_3F_2\left[\begin{matrix} a, & 1-a,\ d+1 \\ b+1, & d \end{matrix};\frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(b+1\right)}{2^b}\left\{\frac{\left(\frac{2}{d}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2}\right)} + \frac{\left(1-\frac{b}{d}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+1\right)}\right\}. \tag{1.11} $$

For $d = b$, it reduces to Bailey summation theorem (1.7).

Very recently, Masjed-Jamei and Koepf [12] generalized the classical summation theorems (1.4) to (1.7) in the following form for $m \in \mathbb{N}$.

$$ {}_3F_2\left[\begin{matrix} a,\ b,\ 1 \\ c,\ m \end{matrix};1\right] = \frac{\Gamma\left(m\right)\Gamma\left(c\right)\Gamma\left(1+a-m\right)\Gamma\left(1+b-m\right)}{\Gamma\left(a\right)\Gamma\left(b\right)\Gamma\left(1+c-m\right)} \times\left\{\frac{\Gamma\left(1+c-m\right)\Gamma\left(c-a-b+m-1\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)} - {}_2^{(m-2)}F_1\left[\begin{matrix} 1+a-m,\ 1+b-m \\ 1+c-m \end{matrix};1\right]\right\}. \tag{1.12} $$

For $m = 1$, it reduces to Gauss' summation theorem (1.4).

$$ {}_3F_2\left[\begin{matrix} a, & b,\ 1 \\ \frac{1}{2}(a+b+1), & m \end{matrix};\frac{1}{2}\right] = \frac{2^{m-1}\Gamma\left(m\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(a\right)\Gamma\left(b\right)} \times\frac{\Gamma\left(1+a-m\right)\Gamma\left(1+b-m\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}-m\right)}\left\{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}-m\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}m+1\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}m+1\right)} - {}_2^{(m-2)}F_1\left[\begin{matrix} 1+a-m, & 1+b-m \\ \frac{1}{2}(a+b+3)-m & \end{matrix};\frac{1}{2}\right]\right\}. \tag{1.13} $$

For $m = 1$, it reduces to Gauss' second summation theorem (1.5).

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & b, \; 1 \\ m+a-b, & m \end{matrix}\; ; -1\right] = \frac{(-1)^{m-1}\Gamma(m)\,\Gamma(m+a-b)}{\Gamma(a)\,\Gamma(b)} \\
&\times \frac{\Gamma(1+a-m)\,\Gamma(1+b-m)}{\Gamma(1+a-b)} \left\{ \frac{\Gamma(1+a-b)\,\Gamma\left(\frac{3}{2}+\frac{1}{2}a-\frac{1}{2}m\right)}{\Gamma(2+a-m)\,\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}m+\frac{1}{2}\right)} \right. \\
&\left. - \; {}^{(m-2)}_{\quad 2}F_1\left[\begin{matrix} 1+a-m, \; 1+b-m \\ 1+a-b \end{matrix}\; ; -1\right] \right\}. \qquad (1.14)
\end{aligned}
$$

For $m = 1$, it reduces to Kummer's summation theorem (1.6).

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, \; 2m-a-1, \; 1 \\ b, \quad m \end{matrix}\; ; \tfrac{1}{2}\right] = \frac{2^{m-1}\Gamma(m)\,\Gamma(b)}{\Gamma(a)\,\Gamma(2m-a-1)} \\
&\times \frac{\Gamma(1+a-m)\,\Gamma(m-a)}{\Gamma(1+b-m)} \left\{ \frac{\Gamma\left(\frac{1}{2}b-\frac{1}{2}m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}m+1\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}b-m+1\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2}\right)} \right. \\
&\left. - \; {}^{(m-2)}_{\quad 2}F_1\left[\begin{matrix} 1+a-m, \; m-a \\ 1+b-m \end{matrix}\; ; \tfrac{1}{2}\right] \right\}. \qquad (1.15)
\end{aligned}
$$

For $m = 1$, it reduces to Bailey's summation theorem (1.7).

Remark 1.1. For interesting applications of the results (1.12) to (1.15) in the evaluations of Laplace-type integrals and Eulerian-type integrals, we refer to recent papers by Jun et al. [5], Koepf et al. [7].

The paper is organised as follows. In Sects. 2, 3, 4 and 5, we shall establish the extensions of the summation theorems (1.12) to (1.15) due to Masjed-Jamei and Koepf [12] together with their derivations and special cases (known and unknown as well). As an applications, in Sect. 6, we evaluate Eulerian-type integrals involving generalized hypergeometric function, while Sect. 7, deals with Laplace-type integrals. Results obtained earlier by Jun et al. [5] and Koepf et al. [7] follow special cases of our main findings. For this, we shall require the following general result recorded in [14]:

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix} a_1, \ldots, a_{p-1}, \; 1 \\ b_1, \ldots, b_{q-1}, \; m \end{matrix}\; ; z\right] \\
&= \frac{\Gamma(b_1)\ldots\Gamma(b_{q-1})}{\Gamma(a_1)\ldots\Gamma(a_{p-1})}\,\frac{\Gamma(1+a_1-m)\ldots\Gamma(1+a_{p-1}-m)}{\Gamma(1+b_1-m)\ldots\Gamma(1+b_{q-1}-m)}\,\frac{(m-1)!}{z^{m-1}}
\end{aligned}
$$

$$\times \left\{ {}_{p-1}F_{q-1}\left[\begin{matrix} 1+a_1-m, \ldots, 1+a_{p-1}-m, \\ \\ 1+b_1-m, \ldots, 1+b_{q-1}-m, \end{matrix} ; z\right] - {}_{p-1}\overset{(m-2)}{F}_{q-1}\left[\begin{matrix} 1+a_1-m, \ldots, 1+a_{p-1}-m, \\ \\ 1+b_1-m, \ldots, 1+b_{q-1}-m, \end{matrix} ; z\right]\right\} \tag{1.16}$$

where ${}_p\overset{(m)}{F}_q$ is the finite sum of the hypergeometric series defined by

$${}_p\overset{(m)}{F}_q\left[\begin{matrix} a_1, \ldots, a_p, \\ \\ b_1, \ldots, b_q, \end{matrix} ; z\right] = \sum_{n=0}^{m} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}, \tag{1.17}$$

where for instance ${}_p\overset{(-1)}{F}_q[z] = 0$, ${}_p\overset{(0)}{F}_q[z] = 1$, ${}_p\overset{(1)}{F}_q[z] = 1 + \frac{a_1 \ldots a_p}{b_1 \ldots b_q} z$.

2. Extension of Gauss' Summation Theorem

In this section, we shall establish the extension of Gauss summation theorem (1.4) asserted in the following theorem.

Theorem 2.1. *For $m \in \mathbb{N}$ and $Re(c-a-b+m-1) > 0$, the following result holds true.*

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} a, \quad b, \ d+1, \ 1 \\ \\ c+1, \ d, \quad m \end{matrix} ; 1\right] \\ &= \frac{\Gamma(m)\Gamma(c+1)\Gamma(d)\Gamma(1+a-m)\Gamma(1+b-m)\Gamma(d-m+2)}{\Gamma(a)\Gamma(b)\Gamma(d+1)\Gamma(c-m+2)\Gamma(d-m+1)} \\ &\quad \times \left\{ \frac{\Gamma(c-m+2)\Gamma(c-a-b-1+m)}{\Gamma(c-a+1)\Gamma(c-b+1)} \right. \\ &\quad \times \left[(c-a-b-1+m) + \frac{(a-m+1)(b-m+1)}{(d-m+1)}\right] \\ &\quad \left. - {}_3\overset{(m-2)}{F}_2\left[\begin{matrix} a-m+1, b-m+1, d-m+2 \\ \\ c-m+2, \ d-m+1 \end{matrix} ; 1\right]\right\} \\ &= \Omega_1. \end{aligned} \tag{2.1}$$

Proof. In (1.16), set $p=4$, $q=3$, $a_1=a$, $a_2=b$, $a_3=d+1$, $b_1=c+1$, $b_2=d$, $z=1$, we have

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, & b, \; d+1, \; 1 \\ & & ;1 \\ c+1, \; d, & m \end{matrix}\right] \\
&= \frac{\Gamma(m)\Gamma(c+1)\Gamma(d)\Gamma(1+a-m)\Gamma(1+b-m)\Gamma(d-m+2)}{\Gamma(a)\Gamma(b)\Gamma(d+1)\Gamma(c-m+2)\Gamma(d-m+1)} \\
&\quad\times\left\{ {}_3F_2\left[\begin{matrix} a-m+1, \; b-m+1, \; d-m+2 \\ ;1 \\ c-m+2, \; d-m+1 \end{matrix}\right]\right. \\
&\quad\left. - {}_3F_2^{(m-2)}\left[\begin{matrix} a-m+1, \; b-m+1, \; d-m+2 \\ ;1 \\ c-m+2, \; d-m+1 \end{matrix}\right]\right\}.
\end{aligned}
$$

We now observe that the first ${}_3F_2$ appearing on the right-hand side can be evaluated with the help of the extended Gauss summation theorem (1.8), and we easily arrive at the right-hand side of (2.1). This completes the proof of (2.1). □

Remark. For $d=c$, result (2.1) reduces to the result (1.12).

Corollary 2.2. *(a) For $m=1$, the result (2.1) exactly gives the extended Gauss summation theorem (1.8).*

(b) In (2.1), if we take $m=2,\; 3$, we get the following results

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix} a, & b, \; d+1, \; 1 \\ & ;1 \\ c+1, \; d, & 2 \end{matrix}\right] &= \frac{c(d-1)}{d(a-1)(b-1)}\left[\frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-a+1)\Gamma(c-b+1)}\right. \\
&\quad\left.\left\{(c-a-b+1)+\frac{(a-1)(b-1)}{(d-1)}\right\}-1\right], \qquad (2.2)
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, & b, \; d+1, \; 1 \\ & ;1 \\ c+1, \; d, & 3 \end{matrix}\right] = \frac{2c(c-1)(d-2)}{d(a-1)(a-2)(b-1)(b-2)} \\
&\left[\frac{\Gamma(c-1)\Gamma(c-a-b+2)}{\Gamma(c-a+1)\Gamma(c-b+1)}\left\{(c-a-b+2)+\frac{(a-2)(b-2)}{(d-2)}\right\}\right. \\
&\left.-\left\{1+\frac{(a-2)(b-2)(d-1)}{(c-1)(d-2)}\right\}\right]. \qquad (2.3)
\end{aligned}
$$

In particular, in (2.2) and (2.3), if we take $d=c$, we recover known results due to Masjed-Jamei and Koepf [12]. Similarly, other results can be obtained.

3. Extension of Gauss' Second Summation Theorem

In this section, we shall establish the extension of Gauss' second summation theorem (1.5) asserted in the following theorem.

Theorem 3.1. *For $m \in \mathbb{N}$, the following result holds true.*

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, & b, \; d+1, \; 1 \\ \frac{1}{2}(a+b+3), \; d, & m \end{matrix}; \frac{1}{2}\right] \\
&= \frac{2^{m-1}\Gamma(m)\Gamma(d)\Gamma(1+a-m)\Gamma(1+b-m)\Gamma(d-m+2)\Gamma\left(\frac{1}{2}(a+b+3)\right)}{\Gamma(a)\Gamma(b)\Gamma(d+1)\Gamma\left(\frac{1}{2}(a+b+5)-m\right)\Gamma(d-m+1)} \\
&\times \left\{ k - {}^{(m-2)}_{\;\;\;3}F_2\left[\begin{matrix} a-m+1, & b-m+1, \; d-m+2 \\ \frac{1}{2}(a+b+5)-m, \; d-m+1 & \end{matrix}; \frac{1}{2}\right]\right\} \\
&= \Omega_2
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
k = &\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+5)-m\right)\Gamma\left(\frac{1}{2}(a-b-1)\right)}{\Gamma\left(\frac{1}{2}(a-b+3)\right)} \\
&\left\{ \frac{\left[\frac{1}{2}(a+b-2m+1) - \frac{(a-m+1)(b-m+1)}{(d-m+1)}\right]}{\Gamma\left(\frac{1}{2}a+1-\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}b+1-\frac{1}{2}m\right)} \right. \\
&\left. + \frac{\left[\frac{(a+b-2m+3)}{(d-m+1)} - 2\right]}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}-\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}-\frac{1}{2}m\right)} \right\}.
\end{aligned}$$

Proof. In (1.16), set $p = 4$, $q = 3$, $a_1 = a$, $a_2 = b$, $a_3 = d+1$, $b_1 = \frac{1}{2}(a+b+3)$, $b_2 = d$, $z = \frac{1}{2}$, we have

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, & b, \; d+1, \; 1 \\ \frac{1}{2}(a+b+3), \; d, & m \end{matrix}; \frac{1}{2}\right] \\
&= \frac{2^{m-1}\Gamma(m)\Gamma(d)\Gamma(1+a-m)\Gamma(1+b-m)\Gamma(d-m+2)\Gamma\left(\frac{1}{2}(a+b+3)\right)}{\Gamma(a)\Gamma(b)\Gamma(d+1)\Gamma\left(\frac{1}{2}(a+b+5)-m\right)\Gamma(d-m+1)} \\
&\times \left\{ {}_3F_2\left[\begin{matrix} a-m+1, & b-m+1, \; d-m+2 \\ \frac{1}{2}(a+b+5)-m, \; d-m+1 & \end{matrix}; \frac{1}{2}\right] \right. \\
&\left. - {}^{(m-2)}_{\;\;\;3}F_2\left[\begin{matrix} a-m+1, & b-m+1, \; d-m+2 \\ \frac{1}{2}(a+b+5)-m, \; d-m+1 & \end{matrix}; \frac{1}{2}\right]\right\}
\end{aligned}$$

We now observe that the first ${}_3F_2$ appearing on the right-hand side can be evaluated with the help of the extended Gauss' second summation theorem

(1.9), and we easily arrive at the right-hand side of (3.1). This completes the proof of (3.1). □

Remark. For $d = \frac{1}{2}(a+b+1)$, result (3.1) reduces to the result (1.13).

Corollary 3.2. *(a) For $m = 1$, the result (3.1) exactly gives the extended Gauss' second summation theorem (1.9).*

(b) In (3.1), if we take $m = 2,\ 3$; we get the following results

$$
{}_4F_3\left[\begin{matrix} a, & b,\ d+1,\ 1 \\ \frac{1}{2}(a+b+3),\ d, & 2 \end{matrix};\ \frac{1}{2}\right] = \frac{(d-1)(a+b+1)}{d(a-1)(b-1)}
\times \left\{ \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma\left(\frac{1}{2}(a-b-1)\right)}{\Gamma\left(\frac{1}{2}(a-b+3)\right)} \left[\frac{\left\{\frac{1}{2}(a+b-3) - \frac{(a-1)(b-1)}{(d-1)}\right\}}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)} + \frac{\left\{\frac{(a+b-1)}{(d-1)} - 2\right\}}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}\right)} \right] - 1 \right\}. \tag{3.2}
$$

and

$$
{}_4F_3\left[\begin{matrix} a, & b,\ d+1,\ 1 \\ \frac{1}{2}(a+b+3),\ d, & 3 \end{matrix};\ \frac{1}{2}\right] = \frac{2(d-2)(a+b+1)(a+b-1)}{d(a-2)_2(b-2)_2}
\times \left\{ k_1 - \left[1 + \frac{(d-1)(a-2)(b-2))}{(d-2)(a+b-1)}\right] \right\}. \tag{3.3}
$$

where

$$
k_1 = \frac{4\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b-1)\right)}{(a-b+1)(a-b-1)}
\times \left\{ \frac{\left[\frac{1}{2}(a+b-5) - \frac{(a-2)(b-2)}{(d-2)}\right]}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}\right)} + \frac{\left[\frac{a+b-3}{d-2} - 2\right]}{\Gamma\left(\frac{1}{2}a-1\right)\Gamma\left(\frac{1}{2}b-1\right)} \right\}
$$

In particular, in (3.2) and (3.3), if we take $d = \frac{1}{2}(a+b+1)$, we recover known results due to Masjed-Jamei and Koepf [12]. Similarly, other results can be obtained.

4. Extension of Kummer's Summation Theorem

In this section, we shall establish the extension of Kummer's summation theorem (1.6) asserted in the following theorem.

Theorem 4.1. *For $m \in \mathbb{N}$, the following result holds true.*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, & b,\ d+1,\ 1 \\ 1+a-b+m,\ d, & m \end{matrix} ;-1\right] \\
&= \frac{(-1)^{m-1}\Gamma(m)\Gamma(d)\Gamma(1+a-m)\Gamma(1+b-m)\Gamma(d-m+2)\Gamma\left(1+a-b+m\right)}{\Gamma(a)\Gamma(b)\Gamma(d+1)\Gamma\left(2+a-b\right)\Gamma(d-m+1)} \\
&\quad \times \left\{ k - {}^{(m-2)}_{\ \ 3}F_2\left[\begin{matrix} a-m+1,\ b-m+1,\ d-m+2 \\ 2+a-b,\ \ d-m+1 \end{matrix} ;-1\right]\right\} \\
&= \Omega_3
\end{aligned} \tag{4.1}
$$

where

$$
\begin{aligned}
k = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(2+a-b\right)}{2^{a-m+1}(m-b)} &\left[\frac{\left(\frac{a-b-d+m}{1+d-m}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}-\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}m+1\right)}\right. \\
&\left. +\frac{\left(\frac{d-a}{1+d-m}\right)}{\Gamma\left(\frac{1}{2}a+1-\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}m+\frac{1}{2}\right)}\right].
\end{aligned}
$$

Proof. In (1.16), set $p=4$, $q=3$, $a_1=a$, $a_2=b$, $a_3=d+1$, $b_1=1+a-b+m$, $b_2=d$, $z=-1$, we have

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, & b,\ d+1,\ 1 \\ 1+a-b+m,\ d, & m \end{matrix} ;-1\right] \\
&= \frac{(-1)^{m-1}\Gamma(m)\Gamma(d)\Gamma(1+a-m)\Gamma(1+b-m)\Gamma(d-m+2)\Gamma\left(1+a-b+m\right)}{\Gamma(a)\Gamma(b)\Gamma(d+1)\Gamma\left(2+a-b\right)\Gamma(d-m+1)} \\
&\quad \times \left\{ {}_3F_2\left[\begin{matrix} a-m+1,\ b-m+1,\ d-m+2 \\ 2+a-b,\ \ d-m+1 \end{matrix} ;-1\right]\right. \\
&\quad \left. - {}^{(m-2)}_{\ \ 3}F_2\left[\begin{matrix} a-m+1,\ b-m+1,\ d-m+2 \\ 2+a-b,\ \ d-m+1 \end{matrix} ;-1\right]\right\}.
\end{aligned}
$$

We now observe that the first ${}_3F_2$ appearing on the right-hand side can be evaluated with the help of the extended Kummer's summation theorem (1.10), and we easily arrive at the right-hand side of (4.1). This completes the proof of (4.1). □

Remark. For $d=a-b+m$, result (4.1) reduces to the result (1.14).

Corollary 4.2. *(a) For $m=1$, the result (4.1) exactly gives the extended Kummer summation theorem (1.10).*

(b) In (4.1), if we take $m = 2,\ 3$, we get the following results

$$
{}_4F_3\left[\begin{matrix} a, & b,\ d+1,\ 1 \\ 3+a-b,\ d, & 2 \end{matrix};-1\right] = \frac{(d-1)(2+a-b)}{d(a-1)(b-1)}
\times\left\{1-\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(2+a-b\right)}{2^{a-1}(2-b)}\left[\frac{\left(\frac{2+a-b-d}{d-1}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}\right)\Gamma\left(2+\frac{1}{2}a-b\right)}
+\frac{\left(\frac{d-a}{d-1}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}a-b\right)}\right]\right\} \tag{4.2}
$$

and

$$
{}_4F_3\left[\begin{matrix} a, & b,\ d+1,\ 1 \\ 4+a-b,\ d, & 3 \end{matrix};-1\right] = \frac{2(2+a-b)(3+a-b)(d-2)}{d(a-1)(a-2)(b-1)(b-2)}
\times\left\{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(2+a-b\right)}{2^{a-2}(3-b)}\left[\frac{\left(\frac{3+a-b-d}{d-2}\right)}{\Gamma\left(\frac{1}{2}a-1\right)\Gamma\left(\frac{1}{2}a-b+\frac{5}{2}\right)}
+\frac{\left(\frac{d-a}{d-2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+2\right)}\right]-\left[1-\frac{(a-2)(b-2)(d-1)}{(d-2)(2+a-b)}\right]\right\}. \tag{4.3}
$$

In particular, in (4.2) and (4.3), if we take $d = 2 + a - b$, we recover known results due to Masjed-Jamei and Koepf [12]. Similarly, other results can be obtained.

5. Extension of Bailey's Summation Theorem

In this section, we shall establish the extension of Bailey's summation theorem (1.7) asserted in the following theorem.

Theorem 5.1. *For $m \in \mathbb{N}$, the following result holds true.*

$$
{}_4F_3\left[\begin{matrix} a, & 2m-a-1,\ d+1,\ 1 \\ b+1, & d, & m \end{matrix};\frac{1}{2}\right]
= \frac{2^{m-1}\Gamma(m)\Gamma(b+1)\Gamma(d)\Gamma(m-a)\Gamma(1+a-m)\Gamma(d-m+2)}{\Gamma(a)\Gamma(d+1)\Gamma\left(2m-a-1\right)\Gamma(2+b-m)\Gamma(1+d-m)}
\times\left\{k-{}^{(m-2)}{}_3F_2\left[\begin{matrix} a-m+1, & m-a, & 2+d-m \\ 2+b-m, & 1+d-m \end{matrix};\frac{1}{2}\right]\right\}
= \Omega_4 \tag{5.1}
$$

where

$$k = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(2+b-m\right)}{2^{b-m+1}}\left[\frac{\left(\frac{2}{1+d-m}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}a-m+1\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2}\right)} + \frac{\left(1-\frac{1+b-m}{1+d-m}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}a-m+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+1\right)}\right].$$

Proof. In (1.16), set $p = 4$, $q = 3$, $a_1 = a$, $a_2 = 2m - a + 1$, $a_3 = d + 1$, $b_1 = b + 1$, $b_2 = d$, $z = \frac{1}{2}$, we have

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, & 2m-a-1, d+1, 1 \\ b+1, & d, \quad m \end{matrix};\frac{1}{2}\right] \\
&= \frac{2^{m-1}\Gamma(m)\Gamma(b+1)\Gamma(d)\Gamma(m-a)\Gamma(1+a-m)\Gamma(d-m+2)}{\Gamma(a)\Gamma(d+1)\Gamma\left(2m-a-1\right)\Gamma(2+b-m)\Gamma(1+d-m)} \\
&\quad \times \left\{ {}_3F_2\left[\begin{matrix} a-m+1, & m-a, & 2+d-m \\ 2+b-m, & 1+d-m \end{matrix};\frac{1}{2}\right] - {}_3F_2^{(m-2)}\left[\begin{matrix} a-m+1, & m-a, & 2+d-m \\ 2+b-m, & 1+d-m \end{matrix};\frac{1}{2}\right]\right\}.
\end{aligned}$$

We now observe that the first ${}_3F_2$ appearing on the right-hand side can be evaluated with the help of the extended Bailey's summation theorem (1.11), and we easily arrive at the right-hand side of (5.1). This completes the proof of (5.1). □

Remark. For $d = b$, result (5.1) reduces to the result (1.15).

Corollary 5.2. *(a) For $m = 1$, the result (5.1) exactly gives the extended Bailey's summation theorem (1.11).*

(b) In (5.1), if we take $m = 2,\ 3$, we get the following results

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, & 3-a, d+1, 1 \\ b+1, & d, \quad 2 \end{matrix};\frac{1}{2}\right] = \frac{2b(1-d)}{d(1-a)(2-a)} \\
&\quad \times \left\{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(b\right)}{2^{b-1}}\left[\frac{\left(\frac{2}{d-1}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}a-1\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2}\right)} + \frac{\left(\frac{d-b}{d-1}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}a-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+1\right)}\right] - 1\right\}
\end{aligned} \tag{5.2}$$

and

$$
{}_4F_3\left[\begin{matrix} a, & 5-a,\, d+1,\, 1 \\ & & ;\tfrac{1}{2} \\ b+1, & d, \quad 3 \end{matrix}\right] = \frac{8b(b-1)(d-2)}{d(a-4)(a-3)(a-2)(a-1)}
$$

$$
\times \left\{ \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(b-1\right)}{2^{b-2}} \left[\frac{\left(\frac{2}{d-2}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}a-2\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2}\right)} \right.\right.
$$

$$
\left.\left. + \frac{\left(\frac{d-b}{d-2}\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}a-\frac{3}{2}\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+1\right)} \right] - \left[1-\frac{(a-2)(a-3)(d-1)}{2(b-1)(d-2)}\right] \right\}. \tag{5.3}
$$

In particular, in (5.2) and (5.3), if we take $d = b$, we recover known results due to Masjed-Jamei and Koepf [12]. Similarly, other results can be obtained.

6. Eulerian-type Single Integrals

As an application of the results established in sections 2 to 5, this section deals with a new class of Eulerian-type integrals involving generalized hypergeometric functions.

First Integral For $m \in \mathbb{N}$, $Re(b) > 0$, $Re(c-b) > -1$ and $Re(c-a-b-d+m) > 0$, the following result holds true.

$$
\int_0^1 x^{b-1}(1-x)^{c-b}\ {}_3F_2\left[\begin{matrix} a,\, d+1,\, 1 \\ & ;x \\ d, \quad m \end{matrix}\right] dx = \frac{\Gamma(b)\Gamma(c-b+1)}{\Gamma(c+1)}\Omega_1 \tag{6.1}
$$

where Ω_1 is the same as given in (2.1).

Second Integral For $m \in \mathbb{N}$, $Re(b) > 0$ and $Re(a-b+3) > 0$, the following result holds true.

$$
\int_0^1 x^{b-1}(1-x)^{\frac{1}{2}(a-b+1)}\ {}_3F_2\left[\begin{matrix} a,\, d+1,\, 1 \\ & ;\tfrac{1}{2}x \\ d, \quad m \end{matrix}\right] dx
$$

$$
= \frac{\Gamma(b)\Gamma\left(\frac{1}{2}(a-b+3)\right)}{\Gamma\left(\frac{1}{2}(a+b+3)\right)}\Omega_2 \tag{6.2}
$$

where Ω_2 is the same as given in (3.1).

Third Integral For $m \in \mathbb{N}$, $Re(b) > 0$ and $Re(a-2b+m) > -1$, the following result holds true.

$$\int_0^1 x^{b-1}(1-x)^{a-2b+m} \; {}_3F_2\left[\begin{matrix} a, \; d+1, \; 1 \\ d, \quad m \end{matrix} ; -x\right] dx = \frac{\Gamma(b)\Gamma(1+a-2b+m)}{\Gamma(1+a-b+m)}\Omega_3 \tag{6.3}$$

where Ω_3 is the same as given in (4.1).

Fourth Integral For $m \in \mathbb{N}$, $Re(a) > 0$ and $Re(b-a) > -1$, the following result holds true.

$$\int_0^1 x^{a-1}(1-x)^{b-a} \; {}_3F_2\left[\begin{matrix} 2m-a-1, \; d+1, \; 1 \\ d, \quad m \end{matrix} ; \tfrac{1}{2}x\right] dx = \frac{\Gamma(a)\Gamma(1+b-a)}{\Gamma(1+b)}\Omega_4 \tag{6.4}$$

where Ω_4 is the same as given in (5.1).

Proof. In order to evaluate the integral (6.1), we proceed as follows. Denoting the left-hand side of (6.1) by I, we have

$$I = \int_0^1 x^{b-1}(1-x)^{c-b} \; {}_3F_2\left[\begin{matrix} a, \; d+1, \; 1 \\ d, \quad m \end{matrix} ; x\right] dx.$$

Now, expressing ${}_3F_2$ as a series and changing the order of integration and summation which is easily seen to be justified due to the uniform convergence of the series involved in the process, we have

$$I = \sum_{n=0}^{\infty} \frac{(a)_n(d+1)_n(1)_n}{(d)_n(m)_n n!} \int_0^1 x^{b+n-1}(1-x)^{c-b} dx.$$

Evaluating the beta integral and using the result (1.2) we get, after some simplification

$$I = \frac{\Gamma(b)\Gamma(c-b+1)}{\Gamma(c+1)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(d+1)_n(1)_n}{(c+1)_n(d)_n(m)_n n!}.$$

Summing up the series, we have

$$I = \frac{\Gamma(b)\Gamma(c-b+1)}{\Gamma(c+1)} \; {}_4F_3\left[\begin{matrix} a, \quad b, \; d+1, \; 1 \\ c+1, \; d, \quad m \end{matrix} ; 1\right].$$

Finally, using the summation theorem (2.1), we easily arrive at the right-hand side of (6.1). This completes the proof of (6.1). □

In exactly the same manner, the integrals (6.2) to (6.4) can be evaluated with the help of the summation theorems (3.1), (4.1) and (5.1), respectively. We leave this as an exercise to the interested reader.

6.1. Special Cases

(1) In (6.1), if we take $d = e$, we get a known result due to Jun et al. [5].
(2) In (6.2), if we take $d = \frac{1}{2}(a + b + 1)$, we get a known result due to Jun et al. [5].
(3) In (6.3), if we take $d = a - b + m$, we get a known result due to Jun et al. [5].
(4) In (6.4), if we take $d = b$, we get a known result due to Jun et al. [5].

We conclude this section by remarking that the integrals (6.1) to (6.4) are of very general nature because of the presence of $m \in \mathbb{N}$. So by giving values to m, we can obtain a large number of integrals, which may be potentially useful.

7. Laplace-type Integrals

In this section, we shall establish a new class of Laplace-type integrals involving generalized hypergeometric functions.

First Integral For $m \in \mathbb{N},\ Re(s) > 0,\ Re(b) > 0$ and $Re(c-a-b+m) > 1$, the following result holds true.

$$\int_0^\infty e^{-st} t^{b-1}\ {}_3F_3\left[\begin{matrix} a, & d+1, & 1 \\ c+1, & d, & m \end{matrix}; st\right] dt = \Gamma(b) s^{-b} \Omega_1 \tag{7.1}$$

where Ω_1 is the same as given in (2.1).

Second Integral For $m \in \mathbb{N},\ Re(s) > 0$ and $Re(a) > 1$, the following result holds true.

$$\int_0^\infty e^{-st} t^{a-1}\ {}_3F_3\left[\begin{matrix} b, & d+1, & 1 \\ \frac{1}{2}(a+b+3), & d, & m \end{matrix}; \tfrac{1}{2}st\right] dt = \Gamma(a) s^{-a} \Omega_2 \tag{7.2}$$

where Ω_2 is the same as given in (3.1).

Third Integral For $m \in \mathbb{N},\ Re(s) > 0$ and $Re(b) > 0$, the following result holds true.

$$\int_0^\infty e^{-st} t^{b-1}\ {}_3F_3\left[\begin{matrix} a, & d+1, & 1 \\ 1+a-b+m, & d, & m \end{matrix}; -st\right] dt = \Gamma(b) s^{-b} \Omega_3 \tag{7.3}$$

where Ω_3 is the same as given in (4.1).

Fourth Integral For $m \in \mathbb{N},\ Re(s) > 0$ and $Re(a) > 1$, the following result holds true.

$$\int_0^\infty e^{-st} t^{a-1}\ {}_3F_3\left[\begin{matrix} 2m-a-1, & d+1, & 1 \\ b+1, & d, & m \end{matrix}; \tfrac{1}{2}st\right] dt = \Gamma(a) s^{-a} \Omega_4 \tag{7.4}$$

where Ω_4 is the same as given in (5.1).

Proof. In order to evaluate the integral (7.1), we proceed as follows. Denoting the left-hand side of (7.1) by I, we have

$$I = \int_0^\infty e^{-st} t^{b-1} \, {}_3F_3 \left[\begin{matrix} a, & d+1, & 1 \\ c+1, & d, & m \end{matrix} ; st \right] dt.$$

Now, express ${}_3F_2$ as a series, change the order of integration and summation which is justified due to the uniform convergence of the series, we have

$$I = \sum_{n=0}^{\infty} \frac{(a)_n (d+1)_n (1)_n}{(c+1)_n (d)_n (m)_n n!} \int_0^\infty e^{-st} t^{b+n-1} dt.$$

Evaluating the Gamma integral and using the result (1.2) we have

$$I = \frac{\Gamma(b)}{s^b} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (d+1)_n (1)_n}{(c+1)_n (d)_n (m)_n n!}.$$

Summing up the series, we have

$$I = \frac{\Gamma(b)}{s^b} \, {}_4F_3 \left[\begin{matrix} a, & b, \, d+1, \, 1 \\ c+1, \, d, & m \end{matrix} ; 1 \right].$$

Finally, using the summation theorem (2.1), we easily arrive at the right-hand side of (7.1). This completes the proof of (7.1). □

In exactly the same manner, the integrals (7.2) to (7.4) can be evaluated with the help of the summation theorems (3.1), (4.1) and (5.1) respectively.

7.1. Special Cases

(1) In (7.1), if we take $d = c$, we get a known result due to Koepf et al. [7].
(2) In (7.2), if we take $d = \frac{1}{2}(a + b + 1)$, we get a known result due to Koepf et al. [7].
(3) In (7.3), if we take $d = a - b + m$, we get a known result due to Koepf et al. [7].
(4) In (7.4), if we take $d = b$, we get a known result due to Koepf et al. [7].

The Laplace-type integrals (7.1) to (7.4) established in this section are of very general nature because of the presence of $m \in \mathbb{N}$. So by giving values to m, we can obtain a large number of integrals in terms of Gamma functions, which may be useful in application point of view. We however prefer to omit the details.

We conclude this paper by remarking that the results established in this paper (including special cases) have been verified numerically using *MATHEMATICA*.

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