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RESEARCH ARTICLE

A note on dual prehomomorphisms from a group into the Margolis–Meakin expansion of a group

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Abstract

We give a category-free order theoretic variant of a key result in Auinger and Szendrei (J Pure Appl Algebra 204(3):493–506, 2006) and illustrate how it might be used to compute whether a finite X-generated group H admits a canonical dual prehomomorphism into the Margolis–Meakin expansion $M(G)$ of a finite X-generated group G. We show that for G the Klein four-group a suitable H must be of exponent 6 at least and recapture a result of Szakács.

Keywords Margolis–Meakin expansion · E-unitary inverse monoid · Dual prehomomorphism

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1 Introduction

The following note considers canonical, i.e. generator preserving dual prehomomorphisms from an X-generated group H into the Margolis–Meakin expansion $M(G)$ of an X-generated group G. It was shown by Auinger and Szendrei [1] that such mappings play an important role in constructing (finite) F-inverse covers for (finite) inverse monoids. We give a necessary and sufficient order theoretic condition for $M(G)$ to admit a canonical dual prehomomorphism from an X-generated group H. It can be seen as a variant of the key statement Lemma 3.1 in [1] and might be applicable on a computer. The idea is to represent the elements of both $M(G)$ and H as congruence classes of words in the free monoid with involution $(X \cup X^{-1})^*$. This enables us to handle the elements of H in relation to $M(G)$ by systematically going through the words in $(X \cup X^{-1})^*$. We use the slightly different view of $M(G)$, introduced in [2], to show how already known positive examples fit into the picture. Further, for G the Klein four-group, we prove that a suitable group H must be of exponent 6 at least and recapture a result of Szakács [6]. It should be noted that in our construction the groups H, we consider as possible candidates for admitting a canonical dual prehomomorphism into $M(G)$, may be arbitrarily X-generated extensions by G. This is in contrast to [1], where H is assumed to be an X-generated subgroup of a semidirect product of a relatively free group by G.

2 Preliminaries and notations

For all undefined notions and notations, the reader is referred to [3, 5]. Let X be a nonempty set and let G be an X-generated group with respect to an injection $\varepsilon_G : X \to G \setminus \{1_G\}$. Note that the mapping ε_G can be uniquely extended to a homomorphism $\varphi_G : (X \cup X^{-1})^* \to G$, where $(X \cup X^{-1})^*$ is the free monoid with involution on X. For $w \in (X \cup X^{-1})^*$ we denote $w\varphi_G$ by $\overline{w}$. By the Cayley graph $\Gamma(G)$ with respect to ε_G, we mean the directed graph whose vertex set $V(\Gamma(G))$ is G and whose edge set $E(\Gamma(G))$ is $G \times X$, where for each $g \in G, x \in X, (g, x)$ denotes an edge with initial vertex g and terminal vertex $g\overline{x}$. Put

$$\begin{aligned} M(G) =&\{(\Gamma, g) : \Gamma \text{ is a finite connected subgraph of } \Gamma(G) \text{ with at least one edge} \\ &\text{and } 1_G, g \in V(\Gamma)\} \cup \{(\emptyset, 1_G)\}. \end{aligned}$$

There is a natural action of G on the semilattice of all subgraphs of $\Gamma(G)$ with operation the set theoretic union, defined as follows: Put $g\emptyset = \emptyset$, and for each nonempty subgraph Γ of $\Gamma(G)$ and $g \in G$, let $g\Gamma$ be the subgraph of $\Gamma(G)$ with $V(g\Gamma) = \{gh : h \in V(\Gamma)\}$ and $E(g\Gamma) = \{(gh, x) : (h, x) \in E(\Gamma)\}$. The graphs we consider do not have isolated vertices, whence they are solely determined by their edge sets, and we conveniently may regard them as (possibly empty) subsets of $X \times G$.

The following theorem was essentially proved in [4].

Theorem 2.1 [4] *$M(G)$ is an E-unitary inverse monoid with respect to the multiplication $(\Gamma, g)(\Gamma', h) = (\Gamma \cup g\Gamma', gh)$ with identity element $(\emptyset, 1_G)$ and maximal group homomorphic image G. Further, $M(G)$ is X-generated as inverse monoid via the injection $\varepsilon_{M(G)} : x \mapsto (\{(1_G, x)\}, \overline{x})$.*

We often represent the elements of $M(G)$ by their corresponding images $\langle w \rangle$ in $(X \cup X^{-1})^* / \ker \varphi_{M(G)}$, where $\varphi_{M(G)}$ denotes the unique extension of $\varepsilon_{M(G)}$ to a homomorphism from $(X \cup X^{-1})^*$ onto $M(G)$. Then obviously $\langle \emptyset \rangle$ corresponds to $(\emptyset, 1_G)$. Let $\emptyset \neq w = \prod_{i=1}^{n} x_i^{\eta_i}$, $\eta_i \in \{-1, 1\}$, be a word in $(X \cup X^{-1})^*$. To w we associate a word $w' = h_1 x_1 h_2 x_2 \cdots h_n x_n h_{n+1}$ in the free product $X^* * G$, where X^* is the free monoid on X, by replacing each $x_i^{\eta_i}$ in w by $g_i x_i g_i$, where

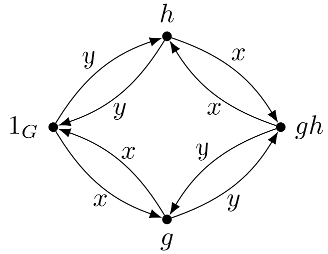
$$g_i = \begin{cases} 1_G & \text{if } \eta_i = 1, \\ \overline{x_i}^{-1} & \text{if } \eta_i = -1. \end{cases}$$

Then $\langle w \rangle$ corresponds to $(\Gamma(\langle w \rangle), \overline{w}) \in M(G)$, in symbols $\langle w \rangle \widehat{=} (\Gamma(\langle w \rangle), \overline{w})$, where $E(\Gamma(\langle w \rangle)) = \{(h_1, x_1), (h_1 \overline{x_1} h_2, x_2), \ldots, (h_1 \overline{x_1} h_2 \overline{x_2} \cdots h_n, x_n)\}$ and $\overline{w} = \prod_{i=1}^{n} \overline{x_i}^{\eta_i}$. Conversely, for each $(\Gamma, g) \in M(G)$ there is a unique $\langle w \rangle$ with $\langle w \rangle \widehat{=} (\Gamma, g)$ for some $w \in (X \cup X^{-1})^*$. For details we refer to [2]. We illustrate the situation by the following example.

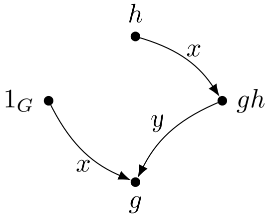
Example 2.1 Let $X = \{x, y\}$ and let $G = \{1_G, g, h, gh\}$ be the X-generated Klein four-group with $\overline{x} := g$ and $\overline{y} := h$. Then $\Gamma(G) = \{(1_G, x), (1_G, y), (g, x), (g, y), (h, x), (h, y), (gh, x), (gh, y)\}$. Now, let e.g. $w = xy^{-1}x^{-1} \in (X \cup X^{-1})^*$. We get $w' = x\overline{y}^{-1} y \overline{y}^{-1} \overline{x}^{-1} x \overline{x}^{-1} = xhyghxg$, whence $\langle w \rangle$ corresponds to

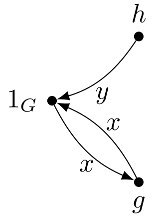
$$(\{(1_G, x), (gh, y), (h, x)\}, h) \in M(G).$$

On the other hand e.g. $(\{(1_G, x), (g, x), (h, y)\}, g) \in M(G)$ corresponds to $\left\langle x^2 y^{-1} y x^{-1} \right\rangle$, being equal to e.g. $\left\langle x^2 y^{-1} y x \right\rangle$.



$\Gamma(G)$





$\Gamma(\langle xy^{-1}x^{-1}\rangle)$ $\Gamma(\langle x^2y^{-1}yx^{-1}\rangle) = \Gamma(\langle x^2y^{-1}yx\rangle)$

3 Canonical dual prehomomorphisms into *M*(*G*)

In $M(G)$ the natural partial order is given by $\langle v\rangle \le \langle w\rangle$ if and only if $\overline{v} = \overline{w}$ and $\Gamma(\langle w\rangle) \subseteq \Gamma(\langle v\rangle)$. The following order theoretic statements are straightforward.

Proposition 3.1 *The least upper bound* $\vee_{i\in I}\langle w_i\rangle$ *with respect to* $\le$ *exists in* $M(G)$ *if and only if all* $\overline{w_i}$ *are equal to a given* $\overline{w}$, *say, and*

(1) $\overline{w} \neq 1_G$ *and* $\overline{w}$ *is a vertex of the* 1_G *containing connected part of* $\cap_{i\in I}\Gamma(\langle w_i\rangle)$, *denoted by* $\mathrm{cp}(\cap_{i\in I}\Gamma(\langle w_i\rangle))$, *in which case* $\vee_{i\in I}\langle w_i\rangle \,\widehat{=}\, (\mathrm{cp}(\cap_{i\in I}\Gamma(\langle w_i\rangle)), \overline{w})$ or
(2) $\overline{w} = 1_G$ *in which case* $\vee_{i\in I}\langle w_i\rangle \,\widehat{=}\, (\mathrm{cp}(\cap_{i\in I}\Gamma(\langle w_i\rangle)), 1_G)$, *if the latter exists, and* $\vee_{i\in I}\langle w_i\rangle \,\widehat{=}\, (\emptyset, 1_G) = 1_{M(G)}$ *otherwise.*

Note that the greatest lower bound $\wedge_{i\in I}\langle w_i\rangle$ exists in $M(G)$ for each finite set I if and only if all $\overline{w_i}$ are equal to a given $\overline{w}$, say, in which case $\wedge_{i\in I}\langle w_i\rangle \,\widehat{=}\, (\cup_{i\in I}\Gamma(\langle w_i\rangle)), \overline{w})$. Note further that $\vee_{i\in I}\langle w_i\rangle$ exists if and only if the set $\{\langle w_i\rangle, i\in I\}$ has an upper bound in $M(G)$.

Let H be an X- generated group via an injection $\varepsilon_H : X \to H \setminus \{1_H\}$. Like with $M(G)$ we may represent the elements of H by their corresponding images $[w]$ in $(X \cup X^{-1})^*/\ker \varphi_H$, where φ_H denotes the unique extension of ε_H to a homomorphism from $(X \cup X^{-1})^*$ onto H. A mapping $\psi : H \to M(G)$ is called a dual prehomomorphism if $([v][w])\psi \geq ([v])\psi([w])\psi$ and $([v]^{-1})\psi = ([v]\psi)^{-1}$ for all $[v], [w] \in H$, see [5]. According to [1], we call ψ canonical if $([x])\psi = \langle x \rangle$ for all $x \in X$. Note that a canonical dual prehomomorphism $\psi : H \to M(G)$ always induces a generator respecting homomorphism from H onto G, given by $[w] \mapsto \overline{w}$, which follows from the fact that in $M(G)$ we have that $(\Gamma(\langle v \rangle), \overline{v}) \leq (\Gamma(\langle w \rangle), \overline{w})$ implies $\overline{v} = \overline{w}$ and ψ respects generators. Thus H necessarily must be an extension by G. Further $(1_H)\psi = 1_{M(G)}$ since

$$(1_H)\psi = ([xx^{-1}])\psi \geq ([x])\psi([x^{-1}])\psi = ([x])\psi(([x])\psi)^{-1} = \langle x \rangle \langle x^{-1} \rangle$$

which corresponds to $(\Gamma(\langle x \rangle), 1_G) = (\{(1_G, x)\}, 1_G)$ and on the other hand $(1_H)\psi = ([x^{-1}x])\psi \geq \langle x^{-1} \rangle \langle x \rangle$ is corresponding to $(\{(\overline{x}^{-1}, x)\}, 1_G)$. Consequently $\Gamma((1_H)\psi) \subseteq \{(1_G, x)\} \cap \{(\overline{x}^{-1}, x)\} = \emptyset$ implying $(1_H)\psi = 1_{M(G)}$.

In what follows we give a necessary and sufficient condition for $M(G)$ to admit a canonical dual prehomomorphism $\psi : H \to M(G)$. Our condition is of an order theoretic form.

Theorem 3.2 *Let G and H be groups as defined above. Then H admits a canonical dual prehomomorphism $\psi : H \to M(G)$ if and only if the following sequence of least upper bounds exists for each $[w] \in H$:*

$$P_0([w]) := \vee_{[v]=[w]} \langle v \rangle$$
$$P_n([w]) := \vee_{[w_1][w_2]=[w]} P_{n-1}([w_1])P_{n-1}([w_2]), \quad n \in \mathbb{N}.$$

Proof Necessity: Let $\psi : H \to M(G)$ be a canonical dual prehomomorphism. Let $[w] \in H$, for some $w \in (X \cup X^{-1})^*$. Since ψ is canonical we obtain $([w])\psi \geq \langle v \rangle$ for all $v \in (X \cup X^{-1})^*$ with $[v] = [w]$. Consequently $P_0([w]) = \vee_{[v]=[w]} \langle v \rangle$ exists and $([w])\psi \geq P_0([w])$. Let now $[u], [v] \in H$ with $[u][v] = [w]$. Then $([w])\psi = ([u][v])\psi \geq ([u])\psi([v])\psi \geq P_0([u])P_0([v])$. Consequently $P_1([w]) = \vee_{[u][v]=[w]}(P_0([u])P_0([v]))$ exists and $([w])\psi \geq P_1([w])$. Continuing this process we see that all $P_n([w]), n \in \mathbb{N}_0$ exist.

Sufficiency: Let the condition in the assumption of Theorem 3.2 be satisfied. Note that $\{P_n([w])\}_{n \in \mathbb{N}_0}$ is increasing and will be constant after a finite number of steps, for each $[w] \in H$, since all occurring graphs are finite. Let $P([w]) := \lim_{n \to \infty} P_n([w]), [w] \in H$. We show that the mapping $\psi : [w] \mapsto P([w])$ defines a canonical dual prehomomorphism. Let $[u], [v] \in H$. It follows $P_1([uv]) \geq P_0([u])P_0([v]), P_2([uv]) \geq P_1([u])P_1([v])$, $\ldots$, $P_n([uv]) \geq P_{n-1}([u])P_{n-1}([v])$, $\cdots$ which after a finite number of steps gives $P([uv]) \geq P([u])P([v])$. Further $P([w]^{-1}) = (P([w]))^{-1}$, since $\langle u \rangle \vee \langle v \rangle$ exists if and only if $\langle u \rangle^{-1} \vee \langle v \rangle^{-1}$ exists in which case $\langle u \rangle^{-1} \vee \langle v \rangle^{-1} = (\langle u \rangle \vee \langle v \rangle)^{-1}$. This fact holds in any inverse semigroup S and easily follows from $s \leq t \Leftrightarrow s^{-1} \leq t^{-1}$, $s, t \in S$. Finally

ψ is canonical since from $\Gamma(P([x])) \subseteq \Gamma(\langle x\rangle)$ we infer $\Gamma(P([x])) = \Gamma(\langle x\rangle)$, whence $P([x]) = \langle x\rangle$. □

Note that the above defined mapping P is the least possible canonical dual prehomomorphism with respect to the pointwise order of mappings, since in the necessity proof of Theorem 3.2 we have $([w])\psi \geq P([w]), [w] \in H$.

Corollary 3.3 *In case* $P_0([w]) \mathrel{\hat{=}} (\cap_{[u]=[w]}\Gamma\langle u\rangle, \overline{w}) \in M(G)$, *for all* $[w] \in H$, *it follows* $P_0([w]) = P_n([w])$, *for all* $n \in \mathbb{N}$, *whence* $([w])\psi = P_0([w])$ *defines a canonical dual prehomomorphism* $\psi : H \to M(G)$.

Proof Under the assumptions we obtain for arbitrary $[w_1], [w_2] \in H$ with $[w_1][w_2] = [w]$

$$\begin{aligned} P_0([w_1])P_0([w_2]) &\mathrel{\hat{=}} (\cap_{[u_1]=[w_1]}\Gamma(\langle u_1\rangle) \cup \overline{w_1} \cap_{[u_2]=[w_2]} \Gamma(\langle u_2\rangle), \overline{w}) \\ &\leq (\cap_{[u]=[w]}\Gamma(\langle u\rangle), \overline{w}) \\ &\mathrel{\hat{=}} P_0([w]), \end{aligned}$$

since $\cap_{[u_1]=[w_1]}\Gamma(\langle u_1\rangle) \cup \overline{w_1} \cap_{[u_2]=[w_2]} \Gamma(\langle u_2\rangle) \supseteq \cap_{[u]=[w]}\Gamma(\langle u\rangle)$. Thus we have

$$P_1([w]) = \vee_{[w_1][w_2]=[w]}(P_0([w_1])P_0([w_2])) \leq P_0([w]) \leq P_1([w]),$$

whence $P_1([w]) = P_0([w])$ follows. We conclude by induction

$$P_0([w]) = P_1([w]) = P_2([w]) = \cdots = P([w]),$$

proving the assertion. □

Example 3.1 Let G be any X-generated group and let H be the free group on X. Then for any $[w] \in H$ we have $P_0([w]) \mathrel{\hat{=}} (\Gamma(\langle r(w)\rangle), \overline{w}) \in M(G)$ where $r(w)$ is the reduced word associated to $[w]$.

Example 3.2 Let G be the $\{x\}$-generated cyclic group of order n and let H be the $\{x\}$ -generated cyclic group of order $2n$. Inspecting $\Gamma(G)$ which is an n-cycle, we directly see

$$\cap_{[w]=[x^k]}\Gamma(\langle w\rangle) = \Gamma(\langle x^k\rangle), \quad 1 \leq k \leq n$$

and

$$\cap_{[w]=[x^l]}\Gamma(\langle w\rangle) = \Gamma(\langle x^{l-2n}\rangle), \quad n \leq l \leq 2n.$$

In particular we have $\cap_{[w]=[x^{2n}]}\Gamma(\langle w\rangle) = \emptyset$, since $[\emptyset] = [x^{2n}]$ corresponds to 1_H. Hence $\psi : H \to M(G)$ may be defined by $([x^k])\psi = \langle x^k\rangle, 1 \leq k \leq n, ([x^l])\psi = \langle x^{l-2n}\rangle$, $n < l < 2n$, and $([x^{2n}])\psi = \langle\emptyset\rangle \mathrel{\hat{=}} (\emptyset, 1_G) = 1_{M(G)}$, cf. ([2], Theorem 19).

To check whether a given extension H by a group G satisfies the condition of Theorem 3.2 it is crucial to determine $\cap_{[v]=[w]}(\Gamma(\langle v\rangle))$ for any $[w] \in H$. In what follows we describe a way of doing that for finite H and G which might be implemented on a computer. We start to determine a finite subset T of $(X \cup X^{-1})^*$ satisfying the following property: For each $w \in (X \cup X^{-1})^*$ there is $v \in T$ such that $[w] = [v]$ and $\Gamma(\langle v \rangle) \subseteq \Gamma(\langle w \rangle)$. To compute such a set T we describe a simple algorithm which directly implements the defining property of T.

(0) Put the identity element of $(X \cup X^{-1})^*$ into T.
(1) For T, constructed so far, construct a superset T' of T in the following way: Put all elements of T into T'. List the elements of $T \times X \times \{-1, 1\}$ and check for each (w, x, ε) in $T \times X \times \{-1, 1\}$ if there is $u \in T$ such that $[u] = [wx^\varepsilon]$ and $\Gamma(\langle u \rangle) \subseteq \Gamma(\langle wx^\varepsilon \rangle)$. If the answer for a given (w, x, ε) is yes, go to the next triple in the list. If the answer is no, put wx^ε into T' and go to the next triple in the list.
(2) If T is a proper subset of T', as constructed in (1), take T' as new T and start (1) again. If $T = T'$ the algorithm stops.

Note that since H and $M(G)$ are finite, the computation stops after a finite number of steps. To see that in the end T has the required property, we note that if a word w' is dropped in (1) of the above algorithm because $[w'] = [u]$ with $\Gamma(\langle u \rangle) \subseteq \Gamma(\langle w' \rangle)$ for some $u \in T$, then for each word $w'v, v \in (X \cup X^{-1})^*$ we have $[w'v] = [uv]$ with $\Gamma(\langle uv \rangle) \subseteq \Gamma(\langle w'v \rangle)$, where uv is in T or has been dropped earlier in (1), i.e. there is some $u' \in T$ such that $[uv] = [u']$ and $\Gamma(\langle u' \rangle) \subseteq \Gamma(\langle uv \rangle)$, whence $[w'v] = [u']$ and $\Gamma(\langle u' \rangle) \subseteq \Gamma(\langle w'v \rangle)$. Consequently the final set T satisfies the property that for each word w in $(X \cup X^{-1})^*$ there is a word u in T such that $[w] = [u]$ and $\Gamma(\langle u \rangle) \subseteq \Gamma(\langle w \rangle)$. Now for a given $[w] \in H$ we get

$$\cap_{[v]=[w]}\Gamma(\langle v \rangle) = \cap_{[u]=[w]}\Gamma(\langle u \rangle),$$

where $v \in (X \cup X^{-1})^*, u \in T$, and in case $\overline{w} \neq 1_G$ we have to check whether the right hand intersection contains a connected subgraph with vertices 1_G and $\overline{w}$, to seewhether $P_0([w])$ exists. Note that in case $\overline{w} = 1_G$, $P_0([w])$ always exists. If for some $[w] \in H$, $P_0([w])$ does not exist, the algorithm stops. If all $P_0([w]), [w] \in H$ exist, we check whether for each $[w] \in H$ $P_1([w]) = \vee_{[w_1][w_2]=[w]}(P_0([w_1])P_0([w_2]))$ exists, by going through all $|H|$ factorisations of $[w]$. If $P_1([w])$ does not exist for some $[w] \in H$, the algorithm stops. In the other case we continue, checking whether $P_2([w])$ exists, and so on. After a finite number of computations we end up with $n_0 \in \mathbb{N}$ such that either $P_{n_0}([w])$ does not exist for some $[w] \in H$, in which case H does not satisfy the conditions of Theorem 3.2 , or $P_{n_0}([w]) = P_{n_0+1}([w])$ for all $[w] \in H$. The latter must be the case since for each $[w] \in H$ the sequence $\{P_n([w])\}_{n \in \mathbb{N}_0}$ is decreasing whence eventually constant, since all occurring graphs are finite. Further H is finite. We then have $P_{n_0}([w]) = P_k([w])$ for all $k \geq n_0$, $[w] \in H$. Thus H satisfies the conditions of Theorem 3.2.

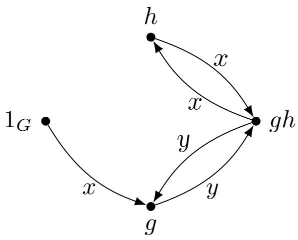
Even for a small finite noncyclic X-generated group G, an X-generated group H admitting a canonical dual prehomomorphism $\psi : H \to M(G)$ might be large. The following theorem points into this direction.

Theorem 3.4 *Let $G = \{1_G, g, h, gh\}$ be the $\{x, y\}$-generated Klein four-group with respect to $\overline{x} = g, \overline{y} = h$. Then any X-generated group H which admits a canonical dual prehomomorphism $\psi : H \to M(G)$ must be of exponent 6 at least.*

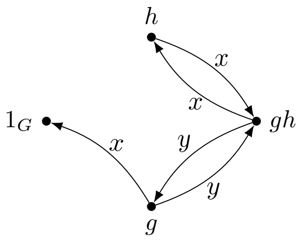
Proof We show that the $\{x, y\}$-generated Burnside group of exponent 4, $B(2; 4)$, does not admit a suitable $\psi : B(2;4) \to M(G)$. Assume that ψ exists. Note first that in $B(2; 4)$ we have $[xyx^2yx^{-1}] = [x^{-1}y^{-1}x^2y^{-1}x]$, since

$$\begin{aligned}[xyx^2yx^{-1}] &= [xyx^2yx^3]\\ &= [(xyx^2yx^2)x]\\ &= [x^{-1}y^{-1}yx^2yx^2yx^2x]\\ &= [x^{-1}y^{-1}(yx^2)^{-1}x]\\ &= [x^{-1}y^{-1}x^{-2}y^{-1}x]\\ &= [x^{-1}y^{-1}x^2y^{-1}x] =: [u].\end{aligned}$$

We get $\langle xyx^2yx^{-1}\rangle \leq ([xyx^2yx^{-1}])\psi = ([x^{-1}y^{-1}x^2y^{-1}x])\psi \geq \langle x^{-1}y^{-1}x^2y^{-1}x\rangle$, whence $\Gamma([u]\psi) \subseteq \Gamma(\langle xyx^2yx^{-1}\rangle) \cap \Gamma(\langle x^{-1}y^{-1}x^2y^{-1}x\rangle)$.



$\Gamma(\langle xyx^2yx^{-1}\rangle)$



$\Gamma(\langle x^{-1}y^{-1}x^2y^{-1}x\rangle)$

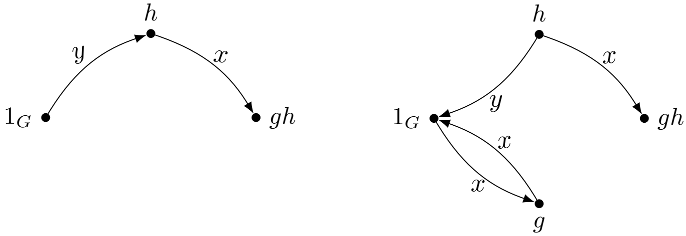
Since the intersection of both graphs does not contain a connected subgraph having at least one edge and vertex 1_G, we conclude that $\Gamma([u]\psi) = \emptyset$, whence $([u])\psi = 1_{M(G)}$. We infer

$$\begin{aligned}([x^2y^{-1}x])\psi &= ([yxx^{-1}y^{-1}x^2y^{-1}x])\psi\\ &\geq ([yx])\psi \underbrace{([x^{-1}y^{-1}x^2y^{-1}x])\psi}_{1_{M(G)}} = ([yx])\psi \geq \langle yx\rangle,\end{aligned}$$

and on the other hand $([x^2y^{-1}x])\psi \geq \langle x^2y^{-1}x\rangle$ which means

$$\Gamma(([x^2y^{-1}x])\psi) \subseteq \Gamma(\langle yx\rangle) \cap \Gamma(\langle x^2y^{-1}x\rangle)$$

with contradiction, since the intersection on the right hand side does not contain a connected subgraph with vertices 1_G and $x^2y^{-1}x = gh$.



$\Gamma(\langle yx \rangle)$ $\Gamma(\langle x^2y^{-1}x \rangle)$

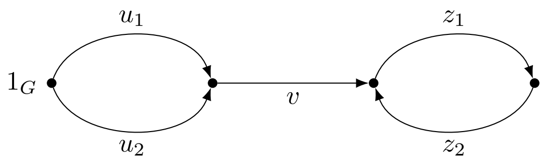
□

It is an open question whether the finite group $B(2;6)$ admits a canonical dual prehomomorphism into $M(G)$ with G the Klein four-group, or a contradiction can be achieved following the pattern in the proof of Theorem 3.4. It is also an open question whether the group G^U, as defined in [1], with U the variety of all groups of exponent $n = 3$, respectively $n = 4$, admits a suitable mapping $\psi : G^U \to M(G)$ in this case. In our setting G^U may be represented by $FG(\{x, y\})_{/\equiv}$, where $\equiv$ is the congruence on the free group $FG(\{x, y\})$ generated by the relators $w^3 = 1$, respectively $w^4 = 1$, where $\overline{w} = 1_G$, $w \in FG(\{x, y\})$. Since, by construction in [1], G^U is a subgroup of a semidirect product of the finite groups $B(8;3)$, respectively $B(8;4)$ by G, it is finite. Obviously $B(2;4)$ is a homomorphic image of G^U in case $n = 4$. However $B(2;4)$ itself is not of the form G^V for some group variety V, since the only possible choice of such V would be the variety of elementary Abelian 2-groups. Only if V has exponent 2, the group G^V has exponent $2 \cdot 2 = 4$. But in this case G^V is a subgroup of a semidirect product of the free elementary Abelian 2-group of rank 8 by G whence $|G^V| < 2^8 \cdot 2^2 = 2^{10} < 2^{12} = |B(2;4)|$. Note in particular that G^U has exponent 6 in case $n = 3$, and exponent 8 in case $n = 4$. Anyway it follows from [1], Proposition 4.4., referring to a remark of V. Guba, that $\psi : G^U \to M(G)$ exists if U is the variety of all groups of sufficiently large odd exponent n.

We continue our considerations with a theorem which also follows from a result of Szakács [6]. For sake of completeness we give an elementary direct proof.

Theorem 3.5 *Let G be an X-generated noncyclic group, and let H be a generator respecting X-generated extension by G such that the homomorphism $H \to G$, defined by $[w] \mapsto \overline{w}$ has a nontrivial Abelian kernel K. Then there is no canonical dual prehomomorphism $\psi : H \to M(G)$.*

Proof We show first that under the assumptions $\Gamma(G)$ contains a subgraph consisting of two disjoint cycles connected by a path, of the form



,

where u_1, u_2, v, z_1, z_2 are nonempty words in $(X \cup X^{-1})^*$, labeling the respective paths.

Assume first that there is $y \in X$ such that $\overline{y}$ has finite order $m \geq 2$. Since G is noncyclic there is $x \in X$ such that $\overline{x} \neq \overline{y}^n$, for all $n \in \mathbb{N}$. Consequently, by use of the words $u_1 = z_1 = y, u_2 = y^{1-m}, v = x, z_2 = y^{m-1}$, we may define a graph which consists of two cycles with vertex sets $A = \{1_G, \overline{y}, \ldots, \overline{y}^{m-1}\}$ and $B = \{\overline{yx}, \overline{yxy}, \ldots, \overline{yx}\,\overline{y}^{m-1}\}$ connected by the edge $(\overline{y}, x)$. Since A is a subgroup of G and $B = \overline{yx}A$, with $\overline{yx} \notin A$ by assumption, we obtain $A \cap B = \emptyset$.

Assume now that there is $x \in X$, such that $\overline{x}$ has infinite order. Since K is nontrivial there is a nonempty reduced word $w = y_1 \ldots y_m$, $m \geq 2$, with $y_i \in X \cup X^{-1}$, $1 \leq i \leq m$, such that $\overline{w} = 1_G$, and $\Gamma(\langle w \rangle)$ forms a cycle. Let $u_1 = y_1 = z_1, u_2 = (y_2 \cdots y_m)^{-1}$, $z_2 = y_2 \cdots y_m$, and $v' = x^n$, where n is such that $\overline{y_1}\,\overline{x}^n a \neq b$ for all a, b in the set $A = \{1_G, \overline{y_1}, \ldots, \overline{y_1 \cdots y_{m-1}}\}$. Such n exists, since the equality $\overline{y_1}\,\overline{x}^k a = b$ can only hold for at most one $k \in \mathbb{N}$ by the assumption that $\overline{x}$ has infinite order, and the set A, whence $A \times A$, is finite. We may use u_1, u_2, v', z_1, z_2 to define a graph which consists of the two disjoint cycles with vertex sets $A = \{1_G, \overline{y_1}, \ldots, \overline{y_1 \cdots y_{m-1}}\}$ and $B = \{\overline{y_1}\,\overline{x}^n, \overline{y_1}\,\overline{x}^n\overline{y_1}, \ldots, \overline{y_1}\,\overline{x}^n\overline{y_1 \cdots y_{m-1}}\}$, connected by the path with initial vertex $\overline{y_1}$ labeled by $v' = x^n$. Let $p, q \in \{1, \ldots, n\}$ such that p is the least element with $\overline{y_1}\,\overline{x}^p \notin A$ for all k, $p \leq k \leq n$, and such that q is the least element with $\overline{y_1}\,\overline{x}^q \in B$. Then the path with initial vertex $\overline{y_1}\,\overline{x}^{p-1}$ and label $v = x^{q-p+1}$ connects the cycles with vertex sets A and B precisely as shown in the graph above. We conclude

$$\langle u_1 v z_1 z_2 v^{-1} u_1^{-1} \rangle \vee \langle u_2 v z_1 z_2 v^{-1} u_2^{-1} \rangle = 1_{M(G)}.$$

On the other hand we obtain

$$\begin{aligned}[u_1 v z_1 z_2 v^{-1} u_1^{-1}] &= [u_1 v z_1 z_2 v^{-1} u_1^{-1} u_2 u_2^{-1}] \\ &= [u_1 u_1^{-1} u_2 v z_1 z_2 v^{-1} u_2^{-1}], \text{ since } [u_1^{-1} u_2], [v z_1 z_2 v^{-1}] \in K \\ &= [u_2 v z_1 z_2 v^{-1} u_2^{-1}].\end{aligned}$$

Hence for any canonical dual prehomomorphism $\psi : H \to M(G)$ we get

$$\begin{aligned}\big([u_1 v z_1 z_2 v^{-1} u_1^{-1}]\big)\psi &= \big([u_2 v z_1 z_2 v^{-1} u_2^{-1}]\big)\psi \\ &\geq \langle u_1 v z_1 z_2 v^{-1} u_1^{-1} \rangle \vee \langle u_2 v z_1 z_2 v^{-1} u_2^{-1} \rangle = 1_{M(G)},\end{aligned}$$

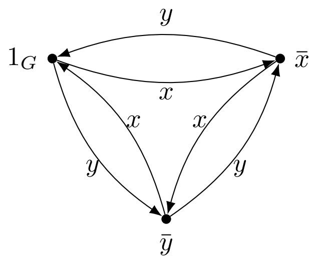
whence $\big([u_1 v z_1 z_2 v^{-1} u_1^{-1}]\big)\psi = 1_{M(G)}$.

By the rule $([w_1w_2])\psi = 1_{M(G)} \Rightarrow ([w_2])\psi \geq ([w_1^{-1}])\psi \Rightarrow ([w_2^{-1}])\psi \geq ([w_1])\psi$, since ψ respects inverses, we obtain with $[w_1] = [u_1vz_1]$ and $[w_2] = [z_2v^{-1}u_1^{-1}]$ that $([u_1vz_2^{-1}])\psi \geq ([u_1vz_1])\psi \geq \langle u_1vz_1\rangle$, which together with $([u_1vz_2^{-1}])\psi \geq \langle u_1vz_2^{-1}\rangle$ leads to a contradiction. Note in particular that $\langle u_1vz_1\rangle \vee \langle u_1vz_2^{-1}\rangle$ does not exist. □

Note that in case K is trivial in Theorem 3.5, i.e. $K = \{1_H\}$, we have that H is isomorphic to G via the homomorphism induced by the mapping $[x] \mapsto \overline{x}$.

It is shown in [2], see also Example 3.2, that for any $\{x\}$-generated cyclic group G of order n there is a canonical dual prehomomorphism ψ from the $\{x\}$-generated cyclic group H of order $2n$ into $M(G)$. Clearly the homomorphism $[w] \mapsto \overline{w}$ has Abelian kernel. If we regard, however, G as an e.g. $\{x, y\}$-generated group where $\overline{y} = \overline{x}^2$, say, $n \geq 3$, then the assertion of Theorem 3.5 remains true, although G is cyclic, as the following example shows for $n = 3$.

Example 3.3 Let $G = \{1_G, g, g^2\}$ be the three element cyclic group generated by $X = \{x, y\}$ with respect to $\overline{x} = g, \overline{y} = g^2$. The Cayley graph $\Gamma(G)$ looks as follows:

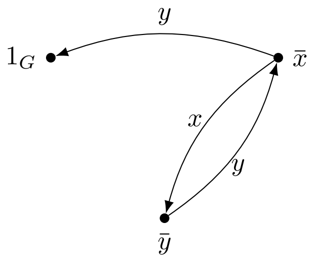


$\Gamma(\langle G\rangle)$

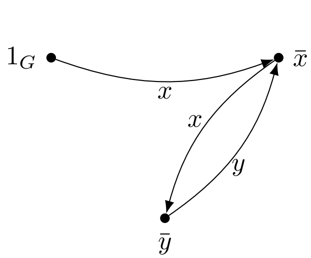
Assume ψ exists for some $\{x, y\}$-generated generator preserving group extension H by G , where $[w] \to \overline{w}$ has Abelian kernel. It follows

$$\begin{aligned}[y^{-1}xy^2] &= [x(x^{-1}y^{-1})(xy)y] \\ &= [x(xy)(x^{-1}y^{-1})y], \text{ since } \overline{xy} = \overline{x^{-1}y^{-1}} = 1_G \\ &= [x^2yx^{-1}].\end{aligned}$$

We obtain $([y^{-1}xy^2])\psi \geq \langle y^{-1}xy^2\rangle, \langle x^2yx^{-1}\rangle$
$\Rightarrow \Gamma(([y^{-1}xy^2])\psi) \subseteq \Gamma(\langle y^{-1}xy^2\rangle) \cap \Gamma(\langle x^2yx^{-1}\rangle)$
$\Rightarrow \Gamma(([y^{-1}xy^2])\psi) = \emptyset \Rightarrow ([y^{-1}xy^2])\psi = 1_{M(G)}$.



$\Gamma(\langle y^{-1}xy^2\rangle)$



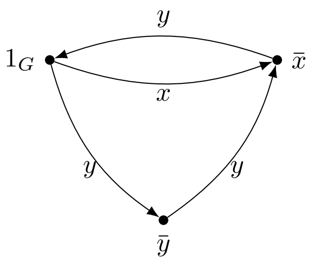
$\Gamma(\langle x^2yx^{-1}\rangle)$

We infer

$$\begin{aligned}([x])\psi &= \big([y(y^{-1}xy^2)y^{-2}]\big)\psi \\ &\geq ([y])\psi\,\underbrace{\big([y^{-1}xy^2]\big)\psi}_{1_{M(G)}}\,\big([y^{-2}]\big)\psi \\ &= ([y])\psi\big([y^{-2}]\big)\psi \geq \langle yy^{-1}y^{-1}\rangle,\end{aligned}$$

which means together with $([x])\psi = \langle x\rangle$ a contradiction.

Note that $\Gamma(G)$ contains a forbidden minor in the sense of Szakács, namely



In particular $\Gamma(\langle x\rangle)$ is a breaking path in her terminology.

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