Dissertation zur Erlangung des akademischen Grades Doktor der Natutwissenschaften (Dr. rer. nat.) am Fachbereich Mathematik und Naturwissenschaften der Universität Kassel

Automatic computation of continued fraction representations as solutions of explicit differential equations

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16. Oktober 2019

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Date of disputation
12.02.2020

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1. Introduction

Continued fractions, expressions of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}},$$

are not exactly a new field of mathematical study. They were already known and used by Euler, for example, to prove the irrationality of e. More recently, the study of continued fractions gained prominence, starting with Oscar Perron in 1913, continuing through Wall to Jones and Thron to Lorentzen and Waadeland. Despite that, continued fractions were sparsely found in collections and handbooks of special functions, until the release of the *Handbook of Continued Fractions for Special Functions* in 2008 [CBV$^+$08], which collected all known continued fraction representations of most special constants and functions into a single reference work.

The main focus of this thesis is to present a variation of an algorithm first presented by Maulat and Salvy, with which it is possible to algorithmically guess as well as prove continued fraction expansions of analytical expressions with the help of ordinary differential equations.

To that end, Chapter 1, which you are reading now, gives an overview of this thesis and its contents. Chapter 2 will lay the groundwork, giving definitions and basic properties in relation to continued fractions, as well as tools with which to check continued fractions for convergence. It also contains a short excursion to the Riemann zeta function ζ and, more specifically, continued fraction expansions of $\zeta(3)$, both previously known as well as new ones, the latter derived from known continued fraction expansions of the tetragamma function. The final stretch contains basic definitions from the field of hypergeometric summation that are relevant to the changes to the algorithm of Maulat and Salvy, as well as a basic overview of the approaches of both Petkovšek and van Hoeij to the problem of finding all hypergeometric term solutions of a given holonomic recurrence equation.

Chapter 3 will first present the theoretical underpinnings of the guess and prove method. The main changes compared to the work of Maulat and Salvy is support for differential equations of order higher than one as well as applying van Hoeij's algorithm instead of a second guessing step. The van Hoeij algorithm is extended and used, since in the verification step of the guess and prove algorithm two-term right factors of a holonomic recurrence satisfied by some sequence H_n are of interest. A two-term right factor of order m corresponds to an m-fold hypergeometric term solution. As it turns out, it suffices to consider two-term right factors of a holonomic recurrence satisfied by *any* subsequence H_{ln+i}. Because of this, the method used in this thesis does not produce true m-fold hypergeometric term solutions of a given holonomic recurrence. The chosen approach can however easily be generalized to do just that.

After the presentation of the main algorithm follow detailed demonstrations for $\tan x$ and $\exp x$, as well as less detailed examples mostly from the *Handbook of Continued Fractions for Special Functions*. This also includes two new continued fraction representations of $\exp x$ derived from the generating function of the Euler polynomials. It follows a section concentrating on how to find differential equations satisfied by a given expression, which of course has applications for the presented algorithm, but can be of interest elsewhere. This section also contains some examples concerning implicit differential equations.
Finally, the fully automated algorithm presented in Chapter 3 was implemented using *Maple 18* in the package `guessandprove.mpl`, which is an integral part of this thesis, hence the appendix contains instructions and examples for the use of this package.

2. Basics

In this chapter, some basic definitions and properties will be given, beginning in the first section with the topic of continued fractions. This is followed by a section on the Riemann zeta function ζ, as introduced by Riemann in [Rie60], culminating in the presentation of three families of continued fraction representations of $\zeta(3)$ not mentioned in [BC]. The chapter is finished by a section on hypergeometric terms and series, as well as a general overview of the algorithmic approaches of both Petkovšek and van Hoeij to the problem of finding all hypergeometric term solutions of a given holonomic recurrence equation. Because of its efficiency, van Hoeij's algorithm is used in the verification step of the main algorithm presented in this thesis in Chapter 3. None of these sections are meant to be exhaustive; for a more complete treatment of continued fractions, see [CBV+08], [LW92], [Per13] or [JT80]; for a more complete treatment of hypergeometric summation, see [Koe14].

2.1. Continued fractions

Definition 2.1.1. [LW92, p. 7] A *continued fraction* is an ordered pair $\left(\left((a_n)_{n\geq 1}, (b_n)_{n\geq 0}\right), (f_n)_{n\geq 0}\right)$, where (a_n), (b_n) are sequences of complex numbers with $a_n \neq 0$, (f_n) is a sequence of extended complex numbers, and (a_n), (b_n) give rise to complex functions $s_n(\omega)$, $S_n(\omega)$ with

$$\begin{aligned} S_0(\omega) &= s_0(\omega),\ S_n(\omega) = S_{n-1}(s_n(\omega)) && \text{for } n \geq 1, \\ s_0(\omega) &= b_0 + \omega,\ s_n(\omega) = \frac{a_n}{b_n + \omega} && \text{for } n \geq 1, \end{aligned}$$

such that

$$f_n = S_n(0) \text{ for } n \geq 0.$$

The complex numbers a_n and b_n are called *n-th partial numerators* and *n-th partial denominators*, respectively. Without distinguishing the partial numerators and denominators, they are also called the *elements* of the continued fraction.
The extended complex number f_n is called the *n-th approximant* of the continued fraction.

A continued fraction is denoted by any of the expressions

$$\begin{aligned} b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \\ = b_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n} \\ = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \dots . \end{aligned}$$

Analogously the n-th approximant f_n of a continued fraction is expressed by

$$\begin{aligned} f_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{\ddots + \cfrac{a_n}{b_n}}}}} \\ = b_0 + \mathop{\mathrm{K}}_{k=1}^{n} \frac{a_k}{b_k} \\ = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \dots + \frac{a_n|}{|b_n}; \end{aligned}$$

an expression of this form is also called a *finite continued fraction.*

Definition 2.1.2. [CBV⁺08, p. 12] A continued fraction is said to *converge* to the extended complex number f, if and only if its sequence of approximants (f_n) converges to f. In this case, the notations introduced in Definition 2.1.1 also denote the value f.

Definition 2.1.3. [CBV⁺08, p. 23] For a given continued fraction $b_0 + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k}{b_k}$, t_n denotes its *n-th tail*, given by

$$t_n = \mathop{\mathrm{K}}_{k=n+1}^{\infty} \frac{a_k}{b_k} \text{ for } n \geq 0.$$

From this it follows that t_n satisfies

$$S_n(t_n) = b_0 + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k}{b_k} \text{ for } n \geq 0.$$

It is easy to see that convergence of a continued fraction implies convergence of all its tails (which are continued fractions in their own right) and conversely convergence of at least one of its tails implies convergence of a continued fraction.

Theorem 2.1.4. *[LW92, p. 8] Let $(A_n)_{n\geq -1}$, $(B_n)_{n\geq -1}$ be sequences of complex numbers satisfying the recurrence relations*

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = b_n \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} + a_n \begin{bmatrix} A_{n-2} \\ B_{n-2} \end{bmatrix} \textit{ for } n \geq 1$$

with initial conditions

$$A_{-1} = B_0 = 1,\ A_0 = b_0,\ B_{-1} = 0,$$

where the sequences (a_n), (b_n) *are given by the continued fraction* $b_0 + \operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n}$. *Then the sequences* (A_n) *and* (B_n) *satisfy*

$$S_n(\omega) = \frac{A_n + A_{n-1}\omega}{B_n + B_{n-1}\omega} \quad \text{for } n \geq 0.$$

Proof. The claim follows by induction. For $n = 0$ one has

$$S_0(\omega) = s_0(\omega) = b_0 + \omega = \frac{b_0 + 1 \cdot \omega}{1 + 0 \cdot \omega}.$$

Assuming $S_n(\omega) = \frac{A_n + A_{n-1}\omega}{B_n + B_{n-1}\omega}$ holds for some $n \in \mathbb{N}_{\geq 0}$, one obtains

$$\begin{aligned} S_{n+1}(\omega) = S_n(s_{n+1}(\omega)) &= \frac{A_n + A_{n-1}\dfrac{a_{n+1}}{b_{n+1} + \omega}}{B_n + B_{n-1}\dfrac{a_{n+1}}{b_{n+1} + \omega}} \\ &= \frac{b_{n+1}A_n + a_{n+1}A_{n-1} + A_n\omega}{b_{n+1}B_n + a_{n+1}B_{n-1} + \omega B_n} \\ &= \frac{A_{n+1} + A_n\omega}{B_{n+1} + B_n\omega}. \end{aligned}$$

□

Definition 2.1.5. [LW92, p. 9] For a given continued fraction $b_0 + \operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n}$ the recurrence relations and initial conditions given in Theorem 2.1.4 *define* sequences (A_n) and (B_n) satisfying

$$f_n = S_n(0) = \frac{A_n}{B_n} \text{ for } n \geq 0.$$

In this case the complex numbers A_n and B_n are called the *n-th (canonical) numerator* and *n-th (canonical) denominator*, respectively, of the continued fraction.

Theorem 2.1.6. *[CBV⁺08, p. 14] The canonical numerators and denominators* A_n *and* B_n *of a continued fraction* $b_0 + \operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n}$ *satisfy the* determinant formula

$$\begin{vmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{vmatrix} = A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1} \prod_{k=1}^{n} a_k$$

for $n \geq 0$.

Proof. The claim follows by induction. For $n = 0$ one has

$$A_0B_{-1} - A_0B_{-1} = -1 = (-1)^{-1} \prod_{k=1}^{0} a_k.$$

Assuming the determinant formula holds for some $n \in \mathbb{N}_{\geq 0}$, it follows from Theorem 2.1.4 that

$$\begin{aligned} A_{n+1}B_n - A_nB_{n+1} &= b_{n+1}A_nB_n + a_{n+1}A_{n-1}B_n - b_{n+1}A_nB_n - a_{n+1}A_nB_{n-1} \\ &= -a_{n+1}(A_nB_{n+1} - A_{n+1}B_n) \\ &= -a_{n+1}(-1)^{n-1}\prod_{k=1}^{n} a_k \\ &= (-1)^n \prod_{k=1}^{n+1} a_k. \end{aligned}$$

□

Definition 2.1.7. [LW92, p. 72] Two continued fractions $b_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{b_n}$ and $b'_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a'_n}{b'_n}$ are equivalent, denoted by

$$b_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{b_n} \equiv b'_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a'_n}{b'_n},$$

if and only if their approximants f_n respectively f'_n satisfy

$$f_n = f'_n \text{ for } n \geq 0.$$

Theorem 2.1.8. *[LW92, p. 73] Two continued fractions* $b_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{b_n}$ *and* $b'_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a'_n}{b'_n}$ *are equivalent, if and only if there exists a sequence of complex numbers* $(r_n)_{n\geq 0}$ *with* $r_0 = 1$ *and* $r_n \neq 0$ *for* $n \geq 1$ *satisfying*

$$a'_n = r_{n-1}r_na_n,\ b'_0 = b_0,\ b'_n = r_nb_n \text{ for } n \geq 1.$$

Proof. Given a sequence of complex numbers $(r_n)_{n\geq 0}$ with $r_0 = 1$ and $r_n \neq 0$ for $n \geq 1$, by simplifying one easily obtains

$$f'_n = r_0b_0 + r_0\cfrac{r_1a_1}{r_1b_1 + r_1\cfrac{r_2a_2}{r_2b_2 + r_2\cfrac{r_3a_3}{\ddots + r_{n-1}\cfrac{r_na_n}{r_nb_n}}}} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n}}}} = f_n$$

for $n \geq 0$.
Conversely, if $f_n = f'_n$ for $n \geq 0$ is given, it follows from

$$b_0 = f_0 = f'_0 = b'_0 \text{ and } B'_0 = 1$$

that

$$b'_0 = r_0b_0,\ A'_0 = r_0A_0,\ B'_0 = r_0B_0, \text{ where } r_0 := 1.$$

Assuming for some non-negative integer N one has $r_0, \ldots, r_N$ satisfying

$$r_0 = 1,\ r_n \neq 0,\ a'_n = r_{n-1}r_na_n,\ b'_0 = b_0,\ b'_n = r_nb_n \text{ for } 1 \leq n \leq N$$

as well as

$$A'_n = \left(\prod_{k=0}^{n} r_k\right) A_n,\ B'_n = \left(\prod_{k=0}^{n} r_k\right) B_n \text{ for } 0 \le n \le N,$$

then it follows that

$$\begin{aligned}
\frac{A_{N+1}}{B_{N+1}} = f_{N+1} = f'_{N+1} &= \frac{A'_{N+1}}{B'_{N+1}} \\
&= \frac{b'_{N+1}A'_N + a'_{N+1}A'_{N-1}}{b'_{N+1}B'_N + a'_{N+1}B'_{N-1}} \\
&= \frac{b'_{N+1}\left(\prod_{k=0}^{N} r_k\right) A_N + a'_{N+1}\left(\prod_{k=0}^{N-1} r_k\right) A_{N-1}}{b'_{N+1}\left(\prod_{k=0}^{N} r_k\right) B_N + a'_{N+1}\left(\prod_{k=0}^{N-1} r_k\right) B_{N-1}} \\
&= \frac{b'_{N+1}r_N A_N + a'_{N+1}A_{N-1}}{b'_{N+1}r_N B_N + a'_{N+1}B_{N-1}} \\
&= \frac{\left(\frac{r_N a_{N+1}}{a'_{N+1}}\right) b'_{N+1}A_N + a_{N+1}A_{N-1}}{\left(\frac{r_N a_{N+1}}{a'_{N+1}}\right) b'_{N+1}B_N + a_{N+1}B_{N-1}}.
\end{aligned}$$

From this one obtains

$$b'_{N+1} = \frac{a'_{N+1}}{r_N a_{N+1}} b_{N+1}.$$

Setting

$$r_{N+1} := \frac{a'_{N+1}}{r_N a_{N+1}},$$

it follows that

$$r_{N+1} \neq 0,\ a'_{N+1} = r_N r_{N+1} a_{N+1},\ b'_{N+1} = r_{N+1} b_{N+1}$$

and

$$A'_{N+1} = \left(\prod_{k=0}^{N+1} r_k\right) A_{N+1},\ B'_{N+1} = \left(\prod_{k=0}^{N+1} r_k\right) B_{N+1}.$$

□

Theorem 2.1.9. *[LW92, p. 69] Two sequences of complex numbers $(A_n)_{n\ge -1}$ and $(B_n)_{n\ge -1}$ are the canonical numerators and denominators of a continued fraction $b_0 + \operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n}$, if and only if*

$$A_{-1} = B_0 = 1,\ B_{-1} = 0$$

and

$$A_n B_{n-1} - A_{n-1} B_n \neq 0 \text{ for } n \ge 1.$$

In this case the continued fraction is uniquely determined by

$$b_0 = A_0,\ b_1 = B_1,\ a_1 = A_1 - A_0 B_1$$

and

$$a_n = -\frac{A_nB_{n-1} - A_{n-1}B_n}{A_{n-1}B_{n-2} - A_{n-2}B_{n-1}},\ b_n = \frac{A_nB_{n-2} - A_{n-2}B_n}{A_{n-1}B_{n-2} - A_{n-2}B_{n-1}} \text{ for } n \geq 2.$$

Proof. If the continued fraction is given, the canonical numerators and denominators satisfy

$$A_{-1} = B_0 = 1,\ B_{-1} = 0,\ A_nB_{n-1} - A_{n-1}B_n \neq 0 \text{ for } n \geq 1$$

by Theorem 2.1.4 and Theorem 2.1.6.
Conversely, if (A_n) and (B_n) satisfying

$$A_{-1} = B_0 = 1,\ B_{-1} = 0,\ A_nB_{n-1} - A_{n-1}B_n \neq 0 \text{ for } n \geq 1$$

are given, the linear systems

$$A_n = b_nA_{n-1} + a_nA_{n-2}$$
$$B_n = b_nB_{n-1} + a_nB_{n-2}$$

have the unique solutions a_n and b_n for $n \geq 1$, given by

$$b_1 = B_1,\ a_1 = A_1 - A_0B_1$$

and

$$a_n = -\frac{A_nB_{n-1} - A_{n-1}B_n}{A_{n-1}B_{n-2} - A_{n-2}B_{n-1}},\ b_n = \frac{A_nB_{n-2} - A_{n-2}B_n}{A_{n-1}B_{n-2} - A_{n-2}B_{n-1}} \text{ for } n \geq 2.$$

By additionally setting $b_0 = A_0$, (A_n) and (B_n) then satisfy the recurrence formulas and initial conditions given in Theorem 2.1.4. Thus by Definition 2.1.5 (A_n) and (B_n) are the canonical numerators and denominators of the continued fraction $b_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n}$, completing the proof. □

Proposition 2.1.10. *[CBV⁺08, p. 19] Given a formal series* $\sum_{k=0}^{\infty} c_k$ *with* $c_k \in \mathbb{C} \setminus \{0\}$ *and partial sums* f_n*, there exists a continued fraction* $b_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n}$ *such that its* n*-th approximant equals* f_n *for* $n \geq 0$*.*

Proof. Setting

$$A_{-1} = 1,\ B_{-1} = 0,\ A_n = f_n,\ B_n = 1 \text{ for } n \geq 0$$

one has

$$A_nB_{n-1} - A_{n-1}B_n = f_n - f_{n-1} = c_n \neq 0 \text{ for } n \geq 1.$$

Utilizing Theorem 2.1.9 one obtains the continued fraction $b_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n}$ with its elements given by

$$b_0 = c_0,\ b_1 = 1,\ a_1 = c_1$$

and

$$a_n = -\frac{c_n}{c_{n-1}},\ b_n = 1 + \frac{c_n}{c_{n-1}} \text{ for } n \geq 1.$$

□

Definition 2.1.11. [CBV+08, p. 35] A continued fraction of the form

$$b_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n z^{\alpha_n}}{1}$$

with $a_n \in \mathbb{C} \setminus \{0\}$ and $\alpha_n \in \mathbb{N}$ for $n \in \mathbb{N}$ is called a *C-fraction.* In the case of $\alpha_n = 1$ for all $n \in \mathbb{N}$, the continued fraction is called a *regular C-fraction.*

The name C-fraction refers to the fact that there is a unique one-to-one *correspondence* between the set of (possibly finite) C-fractions and the set of power series $\sum\limits_{n=0}^{\infty} c_n z^n$, as shown for example in [CBV+08, p. 39] and [LW92, p. 253].

Definition 2.1.12. [CBV+08, pp. 59ff.] Let $f(z)$ be a complex function. A rational function

$$R_{m,n}(z) = \frac{P_{m,n}}{Q_{m,n}} = \frac{\sum\limits_{k=0}^{m} c_k z^k}{1 + \sum\limits_{k=1}^{n} d_k z^k}$$

is called a *Padé approximant* of f of order $[m,n]$ for some $m, n \in \mathbb{N}_{\geq 0}$, if and only if

$$f^{(i)}(0) = R^{(i)}_{m,n}(0), \ i = 0, \ldots, m+n,$$

or equivalently the coefficients of the Taylor expansions of f and $R_{m,n}$ in $z = 0$ agree up to inclusively $m+n$-th degree.
The Padé approximants $R_{m,n}$ are arranged in the *Padé table*, where m denotes the row and n denotes the column of the entry $R_{m,n}$. The Padé approximant $R_{m,n}$ is called *normal*, if and only if its occurrence in the Padé table is unique in the sense that there are no $\hat{n}, \hat{m} \in \mathbb{N}_{\geq 0}$ with $(n,m) \neq (\hat{n}, \hat{m})$ and $R_{n,m} = R_{\hat{n},\hat{m}}$. In the same vein, the Padé table being normal is equivalent to all Padé approximants being normal.

Regular C-fractions especially are closely connected to Padé approximants: If the Padé table of a given power series $S(z)$ is normal, then the descending staircase

$$(R_{0,0}, R_{1,0}, R_{1,1}, \ldots)$$

is the sequence of approximants of a regular C-fraction corresponding to $S(z)$ [CBV+08, p. 65].
Even with full knowledge of the elements of a continued fraction, it is usually not immediately obvious wether the continued fraction in question converges or not. As is the case with series, there are a multitude of convergence theorems to decide the question of convergence of a given continued fraction, some examples of which are covered below.
The following convergence theorem was first formulated and proven by Śleszyński in [Sle89b] and [Sle89a].

Theorem 2.1.13 (Śleszyński-Pringsheim's Theorem)**.** *[LW92, p. 30] Let $|b_n| \geq |a_n| + 1$ for all $n \in \mathbb{N}$. Then the continued fraction $\mathop{\mathrm{K}}\limits_{n=1}^{\infty} \frac{a_n}{b_n}$ converges to a value f with $|f| \leq 1$ and its approximants f_n satisfy $|f_n| < 1$ for all $n \in \mathbb{N}$.*

Proof. Let $|b_n| \geq |a_n| + 1$ for all $n \in \mathbb{N}$, then

$$\left|\frac{a_n}{b_n}\right| \leq \frac{|a_n|}{|a_n| + 1} < 1$$

holds, in particular $|f_1| < 1$.
Consider for some k with $1 \leq k < n$ and $n \geq 2$ that

$$\left|f_n^{(k)}\right| = \left|\frac{a_{k+1}|}{|b_{k+1}} + \ldots + \frac{a_n|}{|b_n}\right| < 1$$

holds, then

$$\left|f_n^{(k-1)}\right| = \left|\frac{a_k}{b_k + f_n^{(k)}}\right| \leq \frac{|a_k|}{|b_k| - \left|f_n^{(k)}\right|} \leq \frac{|a_k|}{|a_k| + 1 - \left|f_n^{(k)}\right|} < 1.$$

Iterating on k gives

$$|f_n| = \left|f_n^{(0)}\right| < 1.$$

By Theorem 2.1.6 for $n \in \mathbb{N}$ one has

$$f_n = \frac{A_n}{B_n} = \sum_{k=1}^{n} \left(\frac{A_k}{B_k} - \frac{A_{k-1}}{B_{k-1}}\right) = \sum_{k=1}^{n} \frac{(-1)^{k-1} \prod_{i=1}^{k} a_i}{B_k B_{k-1}},$$

so convergence of f_n is equivalent to convergence of the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \prod_{i=1}^{k} a_i}{B_k B_{k-1}}.$$

For $k \in \mathbb{N}$ it follows from the recurrence formulas given by Theorem 2.1.4 that

$$\begin{aligned} |B_k| = |b_k B_{k-1} + a_k B_{k-2}| &\geq |b_k|\,|B_{k-1}| - |a_k|\,|B_{k-2}| \\ &\geq (|a_k| + 1)\,|B_{k-1}| - |a_k|\,|B_{k-2}|\,. \end{aligned}$$

Thus

$$|B_k| - |B_{k-1}| \geq |a_k|\,(|B_{k-1}| - |B_{k-2}|)$$

holds for $k \in \mathbb{N}$. Iterating gives

$$|B_k| - |B_{k-1}| \geq \prod_{i=1}^{k} |a_i|$$

and hence

$$\left|\frac{(-1)^{k-1} \prod_{i=1}^{k} a_i}{B_k B_{k-1}}\right| \leq \frac{1}{|B_{k-1}|} - \frac{1}{|B_k|}.$$

From this it follows that

$$\sum_{k=1}^{n} \left|\frac{(-1)^{k-1} \prod_{i=1}^{k} a_i}{B_k B_{k-1}}\right| \leq \frac{1}{|B_0|} - \frac{1}{|B_k|} = 1 - \frac{1}{|B_k|} < 1.$$

So the series $f = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \prod_{i=1}^{k} a_i}{B_k B_{k-1}}$ converges absolutely and thus converges. Since $|f_n| < 1$, one has $|f| \leq 1$. □

With this, the next convergence theorem going back to Worpitzky in [Wor65] can be proven.

Theorem 2.1.14 (Worpitzky's Theorem)**.** *[LW92, p. 35] Let* $|a_n| \leq 1/4$ *for all* $n \in \mathbb{N}$*, then the continued fraction* $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{1}$ *converges to* f *with* $|f| \leq 1/2$ *and the approximants* f_n *satisfy* $|f_n| < 1/2$ *for all* $n \in \mathbb{N}$*.*

Proof. Let $|a_n| \leq 1/4$ and $r_0 = 1$, $r_n = 2$ for $n \in \mathbb{N}$. Then by Theorem 2.1.8 one has the following equivalence between continued fractions

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{1} \equiv \frac{2a_1|}{|2} + \underset{n=2}{\overset{\infty}{\mathrm{K}}} \frac{4a_n}{2}.$$

By Theorem 2.1.13 the right-hand side converges, since it follows from $|a_n| \leq 1/4$ for $n \geq 1$ that

$$|2a_1| + 1 \leq 2 \text{ and } |4a_n| + 1 \leq 2 \text{ for all } n \geq 2,$$

and the approximants all have an absolute value strictly smaller than one. This upper bound can be improved by considering that Theorem 2.1.13 is still applicable after multiplying $\frac{2a_1|}{|2} + \underset{n=2}{\overset{\infty}{\mathrm{K}}} \frac{4a_n}{2}$ by 2, yielding the continued fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{4a_n}{2}$. Since the approximants of $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{4a_n}{2}$ all have an absolute value strictly smaller than one, the approximants of $\frac{2a_1|}{|2} + \underset{n=2}{\overset{\infty}{\mathrm{K}}} \frac{4a_n}{2}$ all have an absolute value strictly smaller than 1/2. Since

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{1} \equiv \frac{2a_1|}{|2} + \underset{n=2}{\overset{\infty}{\mathrm{K}}} \frac{4a_n}{2},$$

by using Definition 2.1.7 one obtains that the original continued fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{1} = f$ converges with approximants f_n with $|f_n| < 1/2$ and thus $|f| \leq 1/2$. □

The following trio of convergence theorems were unified and extended by Beardon and Short in 2010 [BS10]. The Stern-Stolz Theorem goes back to Stern in [Ste60] and Stolz in [Sto86], the Seidel-Stern Theorem to Stern in [Ste48] and Seidel [Sei46]. Van Vleck's Theorem was first published in [VV01].

Theorem 2.1.15 (Stern-Stolz Theorem)**.** *[LW92, p.94] The continued fraction* $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$ *diverges, if*

$$\sum_{n=1}^{\infty} |b_n| < \infty.$$

Theorem 2.1.16 (Seidel-Stern Theorem)**.** *[LW92, p.98] Let* $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$ *be a continued fraction with* $b_n > 0$ *for all* $n \in \mathbb{N}$*. Then the continued fraction converges, if and only if*

$$\sum_{n=1}^{\infty} b_n = \infty.$$

Theorem 2.1.17 (Van Vleck's Theorem). *[LW92, p. 32] Let* $0 < \varepsilon < \pi/2$ *and*

$$-\frac{\pi}{2} + \varepsilon < \arg b_n < \frac{\pi}{2}$$

for all $n \in \mathbb{N}$*. Then the approximants* f_n *of the continued fraction* $\operatorname*{K}_{n=1}^{\infty} \frac{1}{b_n}$ *satisfy*

$$|f_n| < \infty \text{ and } -\frac{\pi}{2} + \varepsilon < \arg f_n < \frac{\pi}{2}.$$

Furthermore the sequences f_{2n} *and* f_{2n+1} *converge with*

$$\lim_{n\to\infty} |f_{2n}| < \infty \text{ and } \lim_{n\to\infty} |f_{2n+1}| < \infty$$

and lastly the continued fraction $\operatorname*{K}_{n=1}^{\infty} \frac{1}{b_n}$ *converges, if and only if*

$$\sum_{n=1}^{\infty} |b_n| = \infty.$$

2.2. The Riemann zeta function and related functions

2.2.1. Definitions and basic properties

Definition 2.2.1. [Rie60] For $z \in \mathbb{C}$, $\operatorname{Re} z > 1$ the *Riemann zeta function* $\zeta(z)$ is defined by

$$\zeta(z) := \sum_{n=1}^{\infty} n^{-z}.$$

Proposition 2.2.2. *[Eul44, p. 174] Let* $\mathbb{P} \subset \mathbb{N}$ *be the set of all prime numbers. Then for* $z \in \mathbb{C}$, $\operatorname{Re} z > 1$*, the Riemann zeta function can also be written as*

$$\zeta(z) = \prod_{p\in\mathbb{P}} \frac{1}{1 - p^{-z}}.$$

Definition 2.2.3. [AS84, p. 76] For $z \in \mathbb{C}$, $\operatorname{Re} z > 0$ the *Gamma function* $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} \, \mathrm{d}x.$$

Proposition 2.2.4. *[AS84, pp. 76f.] The Gamma function* Γ *satisfies both* $\Gamma(1) = 1$ *and the functional equation*

$$\Gamma(z + 1) = z\Gamma(z).$$

Utilizing the identity $\Gamma(z) = \Gamma(z + 1)/z$ *iteratively, the Gamma function can then be uniquely extended to a meromorphic function* $\Gamma : \mathbb{C} \to \hat{\mathbb{C}}$ *with simple poles on the set* $\mathbb{Z}_{\leq 0}$.

Proof. For $z = 1$ one obtains from Definition 2.2.3

$$\Gamma(1) = \int_0^\infty e^{-x}\,\mathrm{d}x = 0 - (-1) = 1.$$

Integration by parts on the formula for $\Gamma(z+1)$ given by Definition 2.2.3 yields

$$\Gamma(z+1) = \int_0^\infty x^z e^{-x}\,\mathrm{d}x = 0 - (-z)\int_0^\infty x^{z-1}e^{-x}\,\mathrm{d}x = z\Gamma(z).$$

□

Proposition 2.2.5. *[AS84, p. 77] For $z \in \mathbb{C}\setminus\mathbb{Z}$ the Gamma function Γ satisfies the reflection formula*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Proposition 2.2.6. *[Rie60] Let C be a curve starting at $+\infty$, circling the origin once in positive direction without enclosing any other integer multiple of $2\pi i$ and returning back towards $+\infty$. Then the Riemann zeta function satisfies*

$$2\sin(\pi z)\,\Gamma(z)\zeta(z) = i\oint_C \frac{(-t)^{z-1}}{e^t - 1}\,\mathrm{d}t$$

for $z \in \mathbb{C}\setminus\{1\}$.
This identity can be used to construct the analytic continuation of $\zeta(z)$ for all complex $z \neq 1$ with a simple pole at $z = 1$.

Proposition 2.2.7. *[Rie60] For $z \in \mathbb{C}\setminus\{0,1\}$ the Riemann zeta function ζ satisfies the reflection formulas*

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\zeta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\zeta(1-z)$$

and

$$\zeta(z) = 2(2\pi)^{z-1}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z).$$

Definition 2.2.8. [Rie60] The *Riemann Xi functions* $\xi(z)$ and $\Xi(z) = \xi(1/2 + zi)$ are defined by

$$\xi(z) = \Gamma\left(\frac{z}{2}\right)\frac{z(z-1)}{2}\pi^{-\frac{z}{2}}\zeta(z).$$

By Proposition 2.2.7 they satisfy the simple reflection formulas

$$\Xi(z) = \Xi(-z) \text{ and } \xi(z) = \xi(1-z).$$

Proposition 2.2.9. *[Rie60] Outside the* critical strip $0 \le \operatorname{Re} z \le 1$ *the Riemann zeta function ζ has only the* trivial zeroes $\zeta(-2n)$, $n \in \mathbb{N}$.

Proof. By Proposition 2.2.2 it is easy to see that $\zeta(z) \neq 0$ for $\operatorname{Re} z > 1$. Since Γ has no zeroes for $\operatorname{Re} z > 1$ as well, by Proposition 2.2.7, the zeroes of ζ for $\operatorname{Re} z < 0$ are exactly the zeroes of $\sin(\pi z/2)$, that is $z = -2n$, $n \in \mathbb{N}$. □

The well-known *Riemann hypothesis* [Rie60] states that for all non-trivial zeroes z_0 of ζ one has $\text{Re}(z_0) = \frac{1}{2}$. Proving or disproving the hypothesis would have far reaching consequences for many different branches of mathematics.

Proposition 2.2.10. *[AS84, p. 361][Nør24, p. 66] For $n \in \mathbb{N}$, the values $\zeta(2n)$ can be expressed as*

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|,$$

and the values $\zeta(2n+1)$ can be expressed as

$$\zeta(2n+1) = \frac{(-1)^{n+1}(2\pi)^{2n+1}}{2(2n+1)!} \int_0^1 B_{2n+1}(x) \cot(\pi x) \, \mathrm{d}x,$$

where $B_n(x)$ are the Bernoulli polynomials *and $B_n = B_n(0)$ are the* Bernoulli numbers *defined by the generating function*

$$\frac{z \exp(xz)}{\exp z - 1} = \sum_{k=0}^{\infty} \frac{z^k}{k!} B_k(x), \; |x| < 2\pi.$$

Since the Bernoulli numbers are rational and thus $\zeta(2n)/\pi^{2n}$ is rational, it is easily deduced that $\zeta(2n)$ is transcendental for all $n \in \mathbb{N}$. For $\zeta(2n+1)$, the case is far less clear cut, and in fact Kohnen conjectures in [Koh89] that $\zeta(2n+1)/\pi^{2n+1}$ is transcendental.

Definition 2.2.11. [AS84, pp. 79ff.] The digamma function ψ is defined as the logarithmic derivative of the Gamma function Γ, that is

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{\mathrm{d}}{\mathrm{d}z} \ln \Gamma(z).$$

The polygamma function ψ_n of order n is defined as the n-th derivative of the digamma function ψ, that is

$$\psi_n(z) = \frac{\mathrm{d}^n}{\mathrm{d}z} \psi(z) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z} \ln \Gamma(z).$$

Proposition 2.2.12. *[CBV+08, p. 229] For $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ the polygamma functions have the series representations*

$$\psi(z) = -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{1+k} - \frac{1}{z+k} \right),$$

$$\psi_n(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}, \; n \geq 1,$$

where γ is the Euler-Mascheroni constant

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=0}^{n} \frac{1}{k} - \ln n \right).$$

From this it is easy to see that the polygamma functions are related to the ζ-function by

$$\psi_n(k+1) = (-1)^{n+1} n! \left(\zeta(n+1) - \sum_{m=1}^{k} \frac{1}{m^{n+1}} \right).$$

2.2.2. Continued fraction representations

In [CK97], Cvijović and Klinowski give the continued fraction representation

$$\zeta(n) = \frac{1}{1-2^{1-n}} \operatorname*{K}_{k=1}^{\infty} \frac{a_k(n)}{1}$$

for $n \in \mathbb{N}$, $n \geq 2$, where

$$a_1(n) = 1,\ a_{2k}(n) = -\frac{D_{1,k}(n)D_{0,k-1}(n)}{D_{0,k}(n)D_{1,k-1}(n)},\ a_{2k+1}(n) = -\frac{D_{1,k-1}(n)D_{0,k+1}(n)}{D_{0,k}(n)D_{1,k}(n)} \text{ for } k \geq 1$$

and

$$D_{r,k}(n) = \det \begin{pmatrix} d_{1,1}(n) & \cdots & d_{1,k}(n) \\ \vdots & \ddots & \vdots \\ d_{k,1}(n) & \cdots & d_{k,k}(n) \end{pmatrix},\ d_{i,j}(n) = \frac{(-1)^{i+j+r}}{(r+i+j-1)^n},$$

although they noted that their proof can be extended to any real $n > 1$. Unfortunately there is no known closed form for the $D_{r,k}(n)$, so the elements of the continued fraction cannot be given explicitly.

There are known explicit continued fraction representations for $\zeta(2)$ and $\zeta(3)$, perhaps most famously the continued fraction representation

$$\zeta(3) = \operatorname*{K}_{n=1}^{\infty} \frac{a_n}{b_n},$$

with

$$a_1 = 6,\ a_n = (n-1)^6 \text{ for } n \geq 2,$$
$$b_n = 34n^3 + 51n^2 + 27n + 5 \text{ for } n \geq 1,$$

given by Apéry in [Apé79], which he used to prove the irrationality of $\zeta(3)$. A more detailed version of this proof was given by Cohen in [Coh78]. See also [vdP79].

Alternative approaches to derive this continued fraction were given by Batut and Olivier in [BO79] as well as Prévost in [Pré96]. Unfortunately both approaches fail to give an explicit continued fraction representation of $\zeta(5)$.

Proposition 2.2.13. *[CBV+08, pp. 235 - 236] The tetragamma function $\psi_2(z)$ has the following three continued fraction representations:*

$$\psi_2(z) = -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^2} \operatorname*{K}_{k=1}^{\infty} \frac{a_k/z^2}{1},\ |\arg(z)| < \frac{\pi}{2}$$

with

$$a_1 = \frac{1}{2},\ a_{2k} = \frac{k^2(k+1)}{2(2k+1)},\ a_{2k+1} = \frac{k(k+1)^2}{2(2k+1)},\ k \geq 1;$$

$$\psi_2(z) = \operatorname*{K}_{k=1}^{\infty} \frac{a_k/z(z-1)}{k},\ \operatorname{Re} z > \frac{1}{2},\ z \notin \left]\frac{1}{2}, 1\right],$$

where

$$a_1 = -1,\ a_{2k} = a_{2k+1} = k^4,\ k \geq 1;$$

and

$$\psi_2(z) = -\frac{1}{z} + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k/z}{1}, \ \operatorname{Re} z > 1,$$

with

$$a_1 = 1, \ a_{4k-2} = \frac{k^2 - 2k + 2}{2k - 1}, \ a_{4k-1} = \frac{(k-1)(k-3)}{2k-1},$$
$$a_{4k} = \frac{k^3}{2(k^2+1)}, \ a_{4k+1} = -\frac{k^3}{2(k^2+1)}, \ k \geq 1.$$

Note that even though the form of these continued fractions might lead to such an assumption, they cannot be obtained with the algorithm presented and implemented later in this thesis, as it requires a differential equation of specific type satisfied by the expression in question.

Corollary 2.2.14. $\zeta(3)$ *has three families of continued fraction representations*

$$\zeta(3) = \sum_{m=1}^{k} \frac{1}{m^3} - \frac{1}{2}\psi_2(k+1), \ k \geq 1$$

where $\psi_2(k+1)$ *is expressed using one of the three continued fraction representations given by Proposition 2.2.13.*

Proof. This result is a simple consequence of rearranging the formula expressing the relation between the polygamma functions and the ζ-function given in Proposition 2.2.12 for $n = 2$ and then applying Proposition 2.2.13. □

Substituting $\psi_2(z)$ with the first representation given in Proposition 2.2.13, the formula given in Corollary 2.2.14 actually holds for $k = 0$ as well, yielding the continued fraction

$$\zeta(3) = -\frac{1}{2}\psi_2(1) = 1 + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{a_j}{1},$$

where

$$a_1 = \frac{1}{4}, \ a_{2j} = \frac{j^2(j+1)}{2(2j+1)}, \ a_{2j+1} = \frac{j(j+1)^2}{2(2j+1)}, \ j \geq 1;$$

that is

$$\zeta(3) = 1 + \cfrac{1/4}{1 + \cfrac{1/3}{1 + \cfrac{2/3}{1 + \cfrac{6/5}{1 + \cfrac{9/5}{1 + \dots}}}}}.$$

2.3. Hypergeometric terms and series

Definition 2.3.1. [Koe14, p. 12] A series S of the form

$$S = \sum_{k=-\infty}^{\infty} c_k$$

is called a *hypergeometric series* if and only if the quotient c_{k+1}/c_k can be expressed as a rational function in k. In that case c_k is called a *hypergeometric term.*

Definition 2.3.2. [Koe14, p. 3] Let $z \in \mathbb{C}$ and $n \in \mathbb{N}_{\geq 0}$, then the *rising factorial* or *Pochhammer symbol* $z^{\overline{n}}$ is defined by

$$z^{\overline{n}} = \prod_{k=0}^{n-1}(z+k) = \frac{\Gamma(z+n)}{\Gamma(z)}.$$

The right-hand side allows to extend the definition to arbitrary $n \in \mathbb{C}$.

Definition 2.3.3. [Koe14, pp. 12f.] The *generalized hypergeometric function* ${}_pF_q$ is given by

$${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{k=0}^{\infty} c_k z^k = \sum_{k=0}^{\infty} \frac{\alpha_1^{\overline{k}} \cdot \ldots \cdot \alpha_p^{\overline{k}}}{\beta_1^{\overline{k}} \cdot \ldots \cdot \beta_q^{\overline{k}}} \frac{z^k}{k!}$$

with $\beta_i \notin \mathbb{Z}_{\leq 0}$ for $1 \leq i \leq q$. The α_i are called *upper parameters*, the β_i *lower parameters.*

Note that if any $\alpha_i \in \mathbb{Z}_{\leq 0}$, then ${}_pF_q$ is a polynomial in z. In general ${}_pF_q$ is a convergent series, if $p \leq q$, or $p = q+1$ and $|z| < 1$.
By Definition 2.3.2 it is easy to see that the term ratio

$$\frac{c_{k+1}z^{k+1}}{c_k z^k} = \frac{(\alpha_1+k) \cdot \ldots \cdot (\alpha_p+k)}{(\beta_1+k) \cdot \ldots \cdot (\beta_q+k)} \frac{z}{k+1}$$

is rational in k and that every rational function with known zeros and poles has such a representation.
Study of generalized hypergeometric functions is of interest for the fact that many special functions can be expressed in terms of a generalized hypergeometric function, for example the exponential function

$$\exp z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = {}_0F_0(z)$$

and geometric series

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k = {}_1F_0(1; -; z)$$

as seen in [Koe14, p.14]. A more complex example is

$$\operatorname{erf} z = \frac{2z}{\sqrt{\pi}} {}_1F_1(\frac{1}{2}; \frac{3}{2}; -z^2) = \frac{2z}{\sqrt{\pi}} \exp(-z^2) {}_1F_1(1; \frac{3}{2}; z^2)$$

where erf is the error function [AS84, p.85].
Representing special functions in terms of generalized hypergeometric functions can also be useful regarding continued fraction representations. Consider for example the *Legendre function* $P_\lambda(z)$. In [Dav74] David presents an iterative approach to construct a continued fraction representation of $\frac{P'_\lambda(z)}{P_\lambda(z)}$ as follows:
$P_\lambda(z)$ satisfies the differential equation

$$\left(1-z^2\right) Y'' - 2zY' + \lambda(\lambda+1)Y = 0,$$

which can be rearranged to yield

$$\frac{Y'}{Y} = \frac{\lambda(\lambda+1)}{2z - (1-z^2)\,\dfrac{Y''}{Y'}}.$$

The term $\dfrac{Y''}{Y'}$ can be substituted by differentiating the differential equation once and rearranging the result to obtain

$$\frac{Y''}{Y'} = \frac{\lambda(\lambda+1) - 2}{4z - (1-z^2)\,\dfrac{Y'''}{Y''}}.$$

Iterating this process leads to the continued fraction

$$\frac{P_\lambda'(z)}{P_\lambda(z)} = -\frac{1}{1-z^2} \operatorname*{K}_{k=0}^{\infty} \frac{(1-z^2)\,(\lambda(\lambda+1) - k(k+1))}{2(k+1)z},$$

although this identity is only formal and one does not know the domain on which the right-hand side converges. The exception are the values $\lambda \in \mathbb{N}_{\geq 0}$, in which case the continued fraction is finite and corresponds to a logarithmic derivative of a *Legendre polynomial*. Now consider instead the *Nörlund fraction* given by [CBV+08, p.300]

$$\frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a+1,b+1;c+1;z)} = \frac{c-(a+b+1)z}{c} + \frac{1}{c} \operatorname*{K}_{k=1}^{\infty} \frac{c_k\,(z-z^2)}{d_k + e_k z}$$

with $\operatorname{Re} z < 1/2$, $a, b \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, where

$$c_k = (a+k)(b+k),\ d_k = c+k,\ e_k = -(a+b+2k+1),\ k \geq 1.$$

Applying this identity to $\dfrac{P_\lambda'(z)}{P_\lambda(z)}$ with $P_\lambda(z) = {}_2F_1\left(-\lambda, \lambda+1; 1; \dfrac{1-z}{2}\right)$ [AS84, p.94] and $P_\lambda'(z) = \dfrac{\lambda(\lambda+1)}{2} {}_2F_1\left(-\lambda+1, \lambda+2; 2; \dfrac{1-z}{2}\right)$ yields

$$\begin{aligned}\frac{P_\lambda'(z)}{P_\lambda(z)} &= \frac{\lambda(\lambda+1)}{2} \frac{1}{\left(\dfrac{{}_2F_1(-\lambda,\lambda+1;1,(1-z)/2)}{{}_2F_1(-\lambda+1,\lambda+2;2;(1-z)/2)}\right)} \\ &= -\frac{1}{1-z^2} \frac{-(1-z^2)\,(\lambda(\lambda+1) - 0\cdot 1)/2}{(0+1)z + \operatorname*{K}_{k=1}^{\infty} \dfrac{-(1-z^2)\,(\lambda(\lambda+1)-k(k+1))/4}{(k+1)z}},\end{aligned}$$

which is equivalent to the continued fraction given by David by Theorem 2.1.8 with $r_1 = 1$, $r_k = 2$ for $k \geq 0$. By considering the restrictions on the Nörlund fraction, one can see that this continued fraction converges for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$.

2.3.1. Hypergeometric term solutions of holonomic recurrence equations

The presentation in this section of both Petkovšek's and van Hoeij's approaches to the computation of all hypergeometric term solutions of a given holonomic recurrence equation

$$0 = \sum_{j=0}^{J} c_j(n) H_{n+j}$$

follows [Koe14, Ch. 9].
Petkovšek presented an algorithm to solve the stated problem in [Pet92]. His approach works in two parts. The first part is an algorithm to compute all polynomial solutions of a given holonomic recurrence equation (Algorithm 1, as presented here found in [Koe14, p. 177]).

Input : A holonomic recurrence equation $0 = \sum_{j=0}^{J} c_j(n) H_{n+j}$ with polynomial coefficients $c_j(n) = \sum_{l=0}^{M} \alpha_{ij} n^{M-l} \in \mathbb{Q}[n]$ and $M = \max \deg c_j(n)$
Output: The set of all polynomial solutions of the given holonomic recurrence equation

for $m = 0, 1, \ldots$ **do**
  for $l = 0, \ldots, m$ **do**
    $b_{lm} \leftarrow \sum_{j=0}^{J} j^l \alpha_{j,m-l}$
  end
  if $b_{lm} \neq 0$ *for at least one* $l \in \{0, \ldots, m\}$ **then**
    break
  end
end
$\mathcal{N} \leftarrow$ the set of nonnegative integer roots $N \in \mathbb{N}_{\geq 0}$ of the polynomial $\sum_{l=0}^{m} \binom{N}{l} b_{lm}$
if $\mathcal{N} = \emptyset$ **then**
  return $\emptyset$
end
$N \leftarrow \max \mathcal{N}$
$p \leftarrow$ generic polynomial of degree N
equate coefficients of $0 = \sum_{j=0}^{J} c_j(n) p(n+j)$
return *solutions of the resulting linear system*

Algorithm 1: Polynomial solutions of holonomic recurrences

The second part requires the following Lemma.

Lemma 2.3.4. *[Koe14, pp. 177ff.] Any rational function* $t_n \in \mathbb{Q}(n) \setminus \{0\}$ *can be expressed uniquely in the form*

$$t_n = C \frac{p_{n+1}}{p_n} \frac{q_{n+1}}{r_{n+1}},$$

where p_n, q_n, r_n *are rational polynomials in* n *with leading coefficient* 1*,* C *is a rational number and the following divisibility properties hold:*

(i) $\gcd(q_n, r_{n+j}) = 1$ *for all* $j \in \mathbb{Z}_{\geq 0}$*,*

(ii) $\gcd(p_n, q_{n+1}) = 1$*,*

(iii) $\gcd(p_n, r_n) = 1$.

Now note that if a solution H_n of the holonomic recurrence equation

$$0 = \sum_{j=0}^{J} c_j(n) H_{n+j}$$

is assumed to be a hypergeometric term with the term ratio $H_{n+1}/H_n = t_n$, by dividing both sides by H_n the recurrence equation can be transformed to

$$0 = \sum_{j=0}^{J} c_j(n) \prod_{i=0}^{j-1} t_{n+i}.$$

Expressing t_n in the form

$$t_n = C \frac{p_{n+1}}{p_n} \frac{q_{n+1}}{r_{n+1}}$$

as in Lemma 2.3.4 and multiplying the equation with $p_n r_{n+1} \ldots r_{n+J}$ yields

$$0 = \sum_{j=0}^{J} c_j(n) C^j p_{n+j} \left(\prod_{i=1}^{j} q_{n+i} \right) \left(\prod_{i=j+1}^{J} r_{n+i} \right).$$

From the divisibility properties stated in Lemma 2.3.4 it follows that q_{n+1} is relatively prime to each of $p_n, r_{n+1}, \ldots, r_{n+J}$. Thus dividing the equation by q_{n+1} shows that q_{n+1} must be a factor of $c_0(n)$. It follows analogously that r_{n+J} is a factor of $c_J(n)$. With an index shift this can instead be stated in the form that q_n is a factor of $c_0(n-1)$ and r_n is a factor of $c_J(n-J)$. Since both q_n and r_n have leading coefficient 1 by Lemma 2.3.4, there are only finitely many possible choices for each.

Do note though that the number of possible choices of pairs (q_n, r_n), though finite, can still be exceedingly large. Let d_0 and d_J be the degree of $c_0(n-1)$ and $c_J(n-J)$ in n, respectively. In the worst case scenario of both $c_0(n-1)$ and $c_J(n-J)$ having the maximum possible amount of distinct monic linear factors, there are $2^{d_0+d_J}$ possible choices for the pair (q_n, r_n).

To determine the constant C, consider the leading coefficient of

$$\sum_{j=0}^{J} c_j(n) C^j p_{n+j} \left(\prod_{i=1}^{j} q_{n+i} \right) \left(\prod_{i=j+1}^{J} r_{n+i} \right),$$

which is a polynomial of degree at most J in C, yielding at most J possible choices for the constant C.

For any fixed choice of the triple (q_n, r_n, C), Algorithm 1 can be used to check

$$0 = \sum_{j=0}^{J} c_j(n) C^j p_{n+j} \left(\prod_{i=1}^{j} q_{n+i} \right) \left(\prod_{i=j+1}^{J} r_{n+i} \right)$$

for nonzero polynomial solutions p_n. Each such solution found gives a hypergeometric term solution of

$$0 = \sum_{j=0}^{J} c_j(n) H_{n+j}.$$

Input : A holonomic recurrence equation $0 = \sum_{j=0}^{J} c_j(n) H_{n+j}$ with polynomial coefficients $c_j(n) \in \mathbb{Q}[n]$
Output: The set L of term ratios of all hypergeometric term solutions of the given holonomic recurrence equation

$L \leftarrow \emptyset$
for *all monic factors* q_n *of* $c_0(n-1)$ *and* r_n *of* $c_J(n-J)$ **do**
  for $j = 0, \ldots, J$ **do**
    $h_j(n) \leftarrow c_j(n) \prod_{l=1}^{j} q_{n+l} \prod_{l=j+1}^{J} r_{n+l}$
  end
  $M \leftarrow \max_j \deg h_j(n)$
  for $j = 0, \ldots, J$ **do**
    $\alpha_j \leftarrow$ coefficient of n^M in $h_j(n)$
  end
  for *solutions* C *of* $0 = \sum_{j=0}^{J} \alpha_j C^j$ **do**
    $P \leftarrow$ the result of applying Algorithm 1 to the recurrence equation $0 = \sum_{j=0}^{J} C^j h_j(n) p_{n+j}$
    for $p_n \in P$ **do**
      add the term ratio $t_n = C \frac{p_{n+1}}{p_n} \frac{q_{n+1}}{r_{n+1}}$ to the set L
    end
  end
end
return L

Algorithm 2: Hypergeometric term solutions of holonomic recurrences

Conversely, each hypergeometric term solution will be found using this approach. This algorithm is summarized in Algorithm 2 [Koe14, p. 187].

Next the main ideas behind the approach of van Hoeij to find the hypergeometric term solutions of a holonomic recurrence equation ([vH99] and [CvH06]) will be presented. The approach bears some similarity to Petkovšek's method. Again a unique representation of the term ratio H_{n+1}/H_n is needed first.

Lemma 2.3.5. *[Koe14, pp. 190f.] Let H_n be a hypergeometric term, then it can be expressed in the form*

$$H_n = R(n) \cdot z^n \cdot \prod_{j=1}^{J} \Gamma(n - \gamma_j)^{e_j}, \; R(n) \in \mathbb{Q}(n), \; \gamma_j \in \mathbb{C}, \; e_j \in \mathbb{Z} \setminus \{0\},$$

where all $\operatorname{Re} \gamma_j \in [m, m+1[$ *for some integer* m. *This representation is unique up to the choice of* m.

Proof. As a hypergeometric term, H_n can be written as

$$H_n = \frac{\alpha_1^{\overline{n}} \cdot \ldots \cdot \alpha_p^{\overline{n}}}{\beta_1^{\overline{n}} \cdot \ldots \cdot \beta_q^{\overline{n}}} \frac{z^n}{n!}$$

as in Definition 2.3.3. Using Definition 2.3.2 this can be rewritten in terms of Gamma functions

$$H_n = C \frac{\Gamma(n+\alpha_1) \ldots \Gamma(n+\alpha_p)}{\Gamma(n+\beta_1) \ldots \Gamma(n+\beta_q) \cdot \Gamma(n+1)} z^n,$$

where C is a complex constant. With the functional equation of the Gamma function (Proposition 2.2.4) the Gamma factors can be rewritten such that $\operatorname{Re} \alpha_k, \operatorname{Re} \beta_k \in \,]m, m+1]$ for some integer m, yielding

$$H_n = R(n) \frac{\Gamma(n+\alpha_1) \ldots \Gamma(n+\alpha_p)}{\Gamma(n+\beta_1) \ldots \Gamma(n+\beta_q) \cdot \Gamma(n+1)} z^n$$

for some rational function $R(n) \in \mathbb{Q}(n)$. This representation is unique up to the choice of m. Since some of the α_k and β_k might coincide at this point, one ultimately obtains

$$H_n = R(n) \cdot z^n \cdot \prod_{j=1}^{J} \Gamma(n - \gamma_j)^{e_j}, \;\; R(n) \in \mathbb{Q}(n), \;\; \gamma_j \in \mathbb{C}, \;\; e_j \in \mathbb{Z} \setminus \{0\}.$$

□

Definition 2.3.6. [Koe14, p. 191] Let H_n be a hypergeometric term uniquely expressed as in Lemma 2.3.5, then the *rational certificate* $\operatorname{cert}(H_n)$ of H_n is defined by

$$\operatorname{cert}(H_n) = \frac{H_{n+1}}{H_n} = \frac{R(n+1)}{R(n)} \cdot z \cdot \prod_{j=1}^{J} (n - \gamma_j)^{e_j} \in Q(n).$$

Each of the Gamma factors in the representation given by Lemma 2.3.5 creates a distinct infinite number of zeroes or poles, but $R(n)$ has only finitely many zeroes or poles. The idea now is to check possible solutions H_n by investigating their zeroes and poles. By identifying the so-called *singularity structure*, the solutions of a holonomic recurrence equation can be found. This can be achieved by considering the zeroes of the leading and the trailing coefficient of the underlying recurrence equation, where the zeroes of the leading coefficient give the candidates for Gamma factors in the denominator and the zeroes of the trailing coefficient give the candidates for Gamma factors in the numerator of the representation given in Lemma 2.3.5. This can be seen by applying the holonomic recurrence equation to compute the values of H_n in a forward or backward manner, respectively.

Definition 2.3.7. [Koe14, pp. 191f.] Let H_n be a hypergeometric term uniquely expressed as in Lemma 2.3.5, then the *singularity structure* of H_n at its finite singularities γ_j is given by the the set of pairs

$$\operatorname{Sing}(H_n) = \{(\gamma_j, e_j) \mid j = 1, \ldots, J\},$$

where the pairs (γ_j, e_j) are called the *local types* of H_n at its finite singularities γ_j. Substituting $n = 1/t$ in the rational certificate and taking the asymptotic expansion, one obtains

$$\operatorname{cert}(H_n)\left(\frac{1}{t}\right) = ct^{-v}\left(1 + dt + \mathcal{O}\left(t^2\right)\right) = cn^v\left(1 + \frac{d}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right).$$

The uniquely determined triple (c, v, d) is called the *local type* of H_n at ∞.

The number of considered cases can further be reduced by exploiting the following properties of the local type at ∞:

Theorem 2.3.8 (Fuchs relations)**.** *[Koe14, p. 192] Let H_n be a hypergeometric term expressed as in Lemma 2.3.5, where $R(n) = p(n)/q(n)$ with $p(n), q(n) \in \mathbb{Q}[n]$, and let (c, v, d) be the local type of H_n at ∞. Then the following relations hold:*

(i)

$$v = \sum_{j=1}^{J} e_j,$$

(ii)

$$d = -\sum_{j=1}^{J} \gamma_j e_j + \deg(p(n)) - \deg(q(n)),$$

(iii)

$$c = z.$$

Proof. Expanding the rational certificate of H_n yields for $n \to \infty$

$$\begin{aligned}\operatorname{cert}(H_n) &= \frac{p(n)q(n+1)}{p(n+1)q(n)} z \prod_{j=1}^{J} (n-\gamma_j)^{e_j} \\ &= z\left(n^{\sum_{j=1}^{J} e_j} - \sum_{j=1}^{J} \gamma_j e_j n^{\sum_{j=1}^{J} e_j - 1} + (\deg(p(n)) - \deg(q(n)))\, n^{\sum_{j=1}^{J} e_j - 1} + \dots\right).\end{aligned}$$

From this the properties (i)-(iii) can be directly read off. □

It turns out that van Hoeij's approach is vastly more efficient than the approach of Petkovšek, since it has to cosider fewer cases. Koepf demonstrates this with a comparative example, where Petkovšek's algorithm has to consider 15360 cases [Koe14, pp. 188ff.]. Van Hoeij's algorithm on the other hand only has to check 2304 possible solutions, which by applying the Fuchs relations in Theorem 2.3.8 can be further reduced to an impressively low 22 cases [Koe14, pp. 194ff.]. This efficiency is the motivation for using van Hoeij's algorithm for the algorithm presented in Chapter 3.

3. Continued fraction solutions of differential equations

In 2015 Sébastien Maulat and Bruno Salvy [MS15] presented an algorithmic approach to construct general formulas for the elements of continued fraction solutions of explicit non-linear differential equations with initial conditions. Their strategy given for the first order case is roughly as follows:
Given an explicit non-linear differential operator $\mathcal{D}$ and the initial condition $Y(0) = 0$, take the unique power series solution of $\mathcal{D}\,Y = 0$ and compute the first few partial numerators $a_k(x)$ of its corresponding C-fraction. On this basis a general formula for $a_k(x)$ can be algorithmically guessed. This conjectured formula can then be proven by showing that $\lim_{k\to\infty} \operatorname{val} \mathcal{D}\,f_k = \infty$, where the f_k are the convergents of the guessed C-fraction. To do this, a linear recurrence for the numerator H_k of $\mathcal{D}\,f_k$ is generated by linear algebra, since $\operatorname{val} H_k = \operatorname{val} \mathcal{D}\,f_k$.
In general, this recurrence will be too complex to directly check the increase in valuation of H_k, so a simpler right factor of the recurrence operator is searched for. This is again done by computing some initial values of H_k and based on this guessing a simpler recurrence. Afterwards the numerator of the greatest common right divisor of the recurrence operators is computed and checked for satisfying H_k.
In this chapter, I will present a modified version of Maulat's and Salvy's approach. The changes consist of extending the algorithm to be applicable to differential equations of order higher than one and replacing the second guessing step with an application of Mark van Hoeij's algorithm for computing a basis of hypergeometric term solutions of a linear recurrence equation, presented in [vH99] and [CvH06] (see also [Koe14]).

3.1. The guess and prove method by Maulat and Salvy

Proposition 3.1.1. *In the ring of formal power series* $\mathbb{K}[[X]]$*, the valuation of a formal power series* $S = \sum_{n=0}^{\infty} c_n X^n$ *is given by*

$$\operatorname{val} S = \min\{n \geq 0 \mid c_n \neq 0\}$$

with the convention $\operatorname{val} 0 = \infty$*; that is* val *has the following properties for all* $S, T \in \mathbb{K}[[X]]$*:*

(i) $\operatorname{val} S = \infty \Leftrightarrow S = 0$,

(ii) $\operatorname{val}(S \cdot T) = \operatorname{val} S + \operatorname{val} T$,

(iii) $\operatorname{val}(S + T) \geq \min(\operatorname{val} S, \operatorname{val} T)$.

This valuation induces a metric dist *on* $\mathbb{K}[[X]]$ *given by*

$$\operatorname{dist}(S, T) = 2^{-\operatorname{val}(S-T)}.$$

Both concepts are easily extended to include formal Laurent series $S = \sum_{n=-k}^{\infty} c_n X^n$ *with* $k \in \mathbb{N}$ *to allow for negative valuations.*

Proposition 3.1.2. *Given a function* $F \in \mathbb{C}(X)[Y, Y', \ldots, Y^{(m-1)}]$ *that is not singular in* $X = 0$*, the explicit differential equation* $Y^{(m)} = F(X, Y, Y', \ldots, Y^{(m-1)})$ *with initial conditions* $Y^{(i)}(0) = y_0^i$ *for* $i = 0, \ldots, m-1$ *has a unique power series solution* $S(X)$*.*

Proof. The value of $Y^{(m)}(0)$ can be computed from the given equation

$$Y^{(m)} = F(X, Y, Y', \ldots, Y^{(m-1)})$$

by substituting the initial conditions $Y^{(i)}(0) = y_0^i$, $i = 0, \ldots, m-1$. Differentiating both sides of the equation and substituting the values of $Y(0), \ldots, Y^{(m)}(0)$ allows one to compute $Y^{(m+1)}(0)$. By iterating this process the value of $Y^{(n)}(0)$ is uniquely determined for all $n \geq 0$.
As a result a power series $S(X) = \sum_{n=0}^{\infty} c_n X^n$ is a solution of the differential equation if and only if for all $n \geq 0$

$$n! c_n = S^{(n)}(0) = Y^{(n)}(0).$$

It follows that the power series

$$S(X) = \sum_{n=0}^{\infty} \frac{Y^{(n)}(0)}{n!} X^n$$

is the uniquely determined power series solution of the differential equation $Y^{(m)} = F(X, Y, Y', \ldots, Y^{(m-1)})$ with initial conditions $Y^{(i)}(0) = y_0^i$, $i = 0, \ldots, m-1$. □

Proposition 3.1.3. *[MS15, p. 278] Let* $F \in \mathbb{C}(X)[Y, Y', \ldots, Y^{(m-1)}]$ *be a function not singular in* $X = 0$ *and* $(f_n(X))_{n \geq 1}$ *a sequence of power series in* $\mathbb{C}[[X]]$*. Furthermore, let* $S(X)$ *be the unique power series solution of the explicit differential equation*

$$Y^{(m)} = F(X, Y, Y', \ldots, Y^{(m-1)})$$

with initial conditions $Y^{(i)}(0) = y_0^i$, $i = 0, \ldots, m-1$ *given by Proposition 3.1.2. Then* $f_n(X)$ *converges to* $S(X)$ *if and only if*

$$\operatorname{val}\left(f_n^{(m)}(X) - F(X, f_n(X), f_n'(X), \ldots, f_n^{(m-1)}(X))\right) \to \infty$$

and $f_n^{(i)}(0) = y_0^i$, $i = 0, \ldots, m-1$ *for sufficiently large* n*.*

Proof. Let $\mathcal{I} : G(X) \mapsto \int G(X)\,\mathrm{d}x$ and $\mathcal{F} : Y \mapsto \mathcal{I}^m F(X, Y, Y', \ldots, Y^{(m-1)})$, then this operator satisfies the inequality

$$
\begin{aligned}
& \operatorname{val}(\mathcal{F}(Y_1) - \mathcal{F}(Y_2)) \\
= \; & \operatorname{val}\left(\mathcal{I}^m\left(F\left(X, Y_1, Y_1', \ldots, Y_1^{(m-1)}\right) - F\left(X, Y_2, Y_2', \ldots, Y_2^{(m-1)}\right)\right)\right) \\
= \; & \operatorname{val}\Bigg(\mathcal{I}^m\Bigg(F\left(X, Y_2, Y_2', \ldots, Y_2^{(m-1)}\right) \\
& + \sum_{i=0}^{m-1}\left(\frac{\partial F}{\partial Y_1^{(i)}}\left(X, Y_2, \ldots, Y_2^{(i)}, Y_1^{(i+1)}, \ldots, Y_1^{(m-1)}\right)\left(Y_1^{(i)} - Y_2^{(i)}\right) + \mathcal{O}\left(\left(Y_1^{(i)} - Y_2^{(i)}\right)^2\right)\right) \\
& - F\left(X, Y_2, Y_2', \ldots, Y_2^{(m-1)}\right)\Bigg)\Bigg) \\
= \; & \operatorname{val}\left(\mathcal{I}^m \sum_{i=0}^{m-1}\left(\left(\frac{\partial F}{\partial Y_1^{(i)}}\left(X, Y_2, \ldots, Y_2^{(i-1)}, Y_1^{(i)}, \ldots, Y_1^{(m-1)}\right) + \mathcal{O}\left(Y_1^{(i)} - Y_2^{(i)}\right)\right)\left(Y_1^{(i)} - Y_2^{(i)}\right)\right)\right) \\
\geq \; & \min_i\left(\operatorname{val}\left(\mathcal{I}^m\left(\frac{\partial F}{\partial Y_1^{(i)}}\left(X, Y_2, \ldots, Y_2^{(i-1)}, Y_1^{(i)}, \ldots, Y_1^{(m-1)}\right) + \mathcal{O}\left(Y_1^{(i)} - Y_2^{(i)}\right)\right)\left(Y_1^{(i)} - Y_2^{(i)}\right)\right)\right) \\
\geq \; & \min_i\left(\operatorname{val}\left(\mathcal{I}^m\left(Y_1^{(i)} - Y_2^{(i)}\right)\right)\right) \\
> \; & \operatorname{val}(Y_1 - Y_2)
\end{aligned}
$$

Now let $f_n^{(i)}(0) = y_0^i$, $i = 0, \ldots, m-1$ for $n \geq N$ for some $N \in \mathbb{N}$ and $\operatorname{val}(f_n^{(m)} - F(X, f_n, f_n', \ldots, f_n^{(m-1)})) \to \infty$.
If $\operatorname{val}(f_n^{(m)} - F(X, f_n, f_n', \ldots, f_n^{(m-1)})) = k$ for some $n \geq N$, then

$$
\begin{aligned}
S - f_n &= (S(0) - f_n(0)) + (S'(0) - f_n'(0))X + \ldots + (S^{(m-1)}(0) - f_n^{(m-1)}(0))X^{m-1} \\
&\quad + \mathcal{I}^m(S^{(m)} - f_n^{(m)}) \\
&= \mathcal{I}^m\left(F(X, S, S', \ldots, S^{(m-1)}) - (F(X, f_n, f_n', \ldots, f_n^{(m)}) + \mathcal{O}(X^k))\right) \\
&= \mathcal{I}^m\left(F(X, S, S', \ldots, S^{(m-1)}) - (F(X, f_n, f_n', \ldots, f_n^{(m)})\right) + \mathcal{O}(X^{k+m}) \\
&= (\mathcal{F}(S) - \mathcal{F}(f_n)) + \mathcal{O}(X^{k+m}).
\end{aligned}
$$

It follows that

$$\operatorname{val}(S - f_n) \geq \min(\operatorname{val}(\mathcal{F}(S) - \mathcal{F}(f_n)), k + m)$$

and since $\operatorname{val}(S - f_n) < \operatorname{val}(\mathcal{F}(S) - \mathcal{F}(f_n))$

$$\operatorname{val}(S - f_n) \geq k + m > \operatorname{val}(f_n^{(m)} - F(X, f_n, f_n', \ldots, f_n^{(m-1)}))$$

holds. From $\operatorname{val}(f_n^{(m)} - F(X, f_n, f_n', \ldots, f_n^{(m-1)})) \to \infty$ it can be concluded that

$$\operatorname{val}(S - f_n) \to \infty.$$

Conversely, if $f_n \to S$ or that is to say $\operatorname{val}(S - f_n) \to \infty$, then there exists some $N \in \mathbb{N}$, such that $\operatorname{val}(S - f_n) \geq m$ for $n \geq N$. In other words S and f_n agree up to m-th degree for $n \geq N$, so $f_n^{(i)}(0) = y_0^i$, $i = 0, \ldots, m-1$ for $n \geq N$. Additionally, since the operator

$\mathcal{D} : Y \mapsto Y^{(m)} - F(X, Y, Y', \ldots, Y^{(m-1)})$ is continuous, it follows from $f_n \to S$ and $\mathcal{D}S = 0$ that

$$\text{val}(\mathcal{D}f_n) \to \text{val}(\mathcal{D}S) = \infty.$$

□

Even in the case that $F \in \mathbb{C}(X)[Y, Y', \ldots, Y^{(m-1)}]$ is singular in $X = 0$, Proposition 3.1.3 is still applicable, as long as the inequality

$$\text{val}(\mathcal{F}(Y_1) - \mathcal{F}(Y_2)) > \text{val}(Y_1 - Y_2)$$

can still be proven to hold and the differential equation

$$Y^{(m)} \;=\; F(X, Y, Y', \ldots, Y^{(m-1)})$$

with initial conditions $Y^{(i)}(0) = y_0^i$, $i = 0, \ldots, m-1$ has at least one power series solution $S(X)$.
The uniqueness of this solution follows like this: Let $T(X)$ be a power series solution of the differential equation. Since T is a solution, its partial sums T_n satisfy

$$\text{val}\left(T_n^{(m)}(X) - F(X, T_n(X), T_n'(X), \ldots, T_n^{(m-1)}(X))\right) \to \infty$$

and $T_n^{(i)}(0) = y_0^i$, $i = 0, \ldots, m-1$ for sufficiently large n. But since T_n satisfies both conditions, it follows that $T_n \to S$. Hence $S = T$.

Theorem 3.1.4. *[MS15, p. 278] Let (A_n) and (B_n) be holonomic sequences of rational functions in X and let $F \in \mathbb{C}(X)\left[Y, Y', \ldots, Y^{(m-1)}\right]$ be a polynomial in Y and its derivatives up to order $m-1$ with degree $d > 0$. Further let H_n be the numerator of the expression*

$$\left(\frac{A_n}{B_n}\right)^{(m)} - F\left(X, \left(\frac{A_n}{B_n}\right), \left(\frac{A_n}{B_n}\right)', \ldots, \left(\frac{A_n}{B_n}\right)^{(m-1)}\right),$$

then the sequence (H_n) satisfies a linear recurrence with coefficients in $\mathbb{C}(n, X)$.

Proof. Let M be the order of recurrence satisfied by (A_n), then all A_{n+l}, $l \in \mathbb{N}$ can be expressed as linear combinations of A_{n+i}, $i = 0, \ldots, M-1$ with coefficients in $\mathbb{C}(n, X)$. Further the derivatives $A_{n+l}^{(r)}$ with $r = 1, \ldots, m$ can be expressed as linear combinations of the A_{n+i} and their derivatives up to order r for $i = 0, \ldots, M-1$ simply by differentiating the corresponding expression for A_{n+l}.
Let $\hat{M}$ be the order of the recurrence satisfied by (B_n), then an analogous argument applies for expressing the B_{n+l} and their derivatives as linear combinations.
By definition H_n is a polynomial of degree at most $\hat{d} := \max(m+1, dm)$ in A_n, B_n and their derivatives up to order m. Hence, all H_{n+l} can be rewritten as linear combinations of monomials of degree at most $\hat{d}$ in A_{n+i}, B_{n+j} for $i = 0, \ldots, M-1$ and $j = 0, \ldots, \hat{M}-1$ and their respective derivatives. There are only finitely many such monomials, at most $N = ((m+1)(M+\hat{M}))^{\hat{d}}$. Thus a linear dependency between $H_n, \ldots, H_{n+N}$, that is to say a linear recurrence of order N with coefficients in $\mathbb{C}(n, X)$, can be found by linear algebra. □

Proposition 3.1.5. *[MS15, p. 280] Let* $F \in \mathbb{C}(X)\left[Y, Y', \dots, Y^{(m-1)}\right]$ *be a function not singular in* $X = 0$ *and* $S(X)$ *the unique power series solution of the explicit differential equation* $Y^{(m)} = F(X, Y, Y', \dots, Y^{(m-1)})$ *with initial conditions* $Y^{(i)}(0) = y_0^i$, $i = 0, \dots, m-1$ *given by Proposition 3.1.2. Let* a_n *be a rational function in* X *and* n *with positive valuation in* X*. Let* A_n *and* B_n *be sequences satisfying the recurrences*

$$A_n = A_{n-1} + a_n A_{n-2} \text{ and } B_n = B_{n-1} + a_n B_{n-2} \text{ for } n \geq 1$$

with initial conditions $A_{-1} = B_0 = 1$ *and* $A_0 = B_{-1} = 0$*. Finally let* H_n *be defined as in Theorem 3.1.4.*
Then, if for some $o, p \in \mathbb{N}_{\geq 0}$, $o > p$, *one has* $\operatorname{val} H_{no+p} \to \infty$ *as* $n \to \infty$ *and* $f_n^{(i)}(0) = y_0^i$, $i = 0, \dots, m-1$ *where* $f_n = \dfrac{A_{no+p}}{B_{no+p}}$ *for sufficiently large* n*, the continued fraction* $\mathrm{K}_{n=1}^{\infty} \dfrac{a_n}{1}$ *is the continued fraction solution of the differential equation* $Y^{(m)} = F(X, Y, Y', \dots, Y^{(m-1)})$ *with initial conditions* $Y^{(i)}(0) = 0$, $i = 0, \dots, m-1$*.*

Proof. Since $\operatorname{val} a_n > 0$ for all $n \geq 1$, the C-fraction $\mathrm{K}_{n=1}^{\infty} \dfrac{a_n}{1}$ with canonical numerators A_n and canonical denominators B_n corresponds to a power series $G(X)$. Let $o, p \in \mathbb{N}_{\geq 0}$, $o > p$. If the subsequence $f_n = \dfrac{A_{no+p}}{B_{ko+p}}$ converges to $S(X)$, it follows that $G(X) = S(X)$ and thus the continued fraction corresponds to the power series solution of the given differential equation.
Since $a_n(0) = 0$ for $n \geq 1$, iterating over the corresponding recurrence relation shows that $B_n(0) = 1$ for all $n \geq 0$. It follows that $\operatorname{val}(B_n) = 0$ and since

$$H_n = \left(\left(\frac{A_n}{B_n}\right)^{(m)} - F\left(X, \left(\frac{A_n}{B_n}\right), \left(\frac{A_n}{B_n}\right)', \dots, \left(\frac{A_n}{B_n}\right)^{(m-1)}\right)\right) B_n^s$$

for some $s \in \mathbb{N}$, one obtains $\operatorname{val} H_{no+p} = \operatorname{val}\left(f_n^{(m)} - F\left((X, f_n, f_n', \dots, f_n^{(m-1)}\right)\right)$ for $n \geq 0$. Assuming $f_n^{(i)}(0) = y_0^i$, $i = 0, \dots, m-1$ for sufficiently large n and $\operatorname{val} H_{no+p} \to \infty$ as $n \to \infty$, it finally follows by Proposition 3.1.3 that f_n converges to $S(X)$. □

The typical setting for this approach would be that some analytic expression f is given and a continued fraction expansion of f is wanted. A general algorithmic approach can be outlined as follows:

Given the expression f, first compute an explicit differential equation $\mathcal{D}$ satisfied by f. One possible method is described by Algorithm 6 in Section 3.3.
Next, compute the partial sum S_N of the power series expansion of f for some $N \in \mathbb{N}$ and convert it to a finite C-fraction $\mathrm{K}_{n=1}^{\hat{N}} \dfrac{a_n}{1}$. Note that it is possible that $\hat{N} < N$. In *Maple 18*, this conversion is possible with the commands `Term` and `ContinuedFraction` in the `NumberTheory` package. To ensure the condition $f_n^{(i)}(0) = f^{(i)}(0) = y_0^i$, $i = 0, \dots, m-1$ is met later on in the process, N should be chosen to be at least the order of $\mathcal{D}$, although in practical terms N tends to be larger anyway.
By way of rational interpolation guess a general formula for a_n based on $a_1, \dots, a_{\hat{N}}$. In the accompanying *Maple 18* implementation this is done by using the `RationalInterpolation`

function in the `CurveFitting` package. Since the rational interpolation step is always successful, it is advisable to choose a suitable stepsize s and guess the general formula of a_n based on S_{N+is} and $S_{N+(i+1)s}$, $i \geq 0$ until both guesses coincide. The general formula of a_n is not neccessarily represented by a single rational function, but can instead exhibit an l-fold symmetry; that is, there exist l rational functions a_n^i, $i = 0, \ldots, l-1$, such that $a_n = a_n^{(n \bmod l)}$ for $n \geq 1$. The case $l = 2$ is an especially common case, see Example 3.2.2 for a demonstration in the case of $\exp x$.
Define A_n and B_n as in Proposition 3.1.5 to obtain H_n from Proposition 3.1.4. Since $f_n^{(i)}(0) = f^{(i)}(0) = y_0^i$, $i = 0, \ldots, m-1$, is ensured due to choice of N, it is sufficient by Proposition 3.1.5 to show $\operatorname{val} H_{no+p} \to \infty$ for some $o, p \in \mathbb{N}_{\geq 0}$, $o > p$ as $n \to \infty$, to prove that the guessed formula holds. One way to show this is to take a look at the recurrence satisfied by H_n, the existence and construction of which are provided by Proposition 3.1.4. In the case that a_n exhibits an l-fold symmetry, it is advisable to instead look at the subsequence H_{nl} to ensure the recurrence has a single explicit form.
If a two-term right factor $H_{(n+j)l} - r_{nl} H_{nl}$ can be found for some $j \in \mathbb{N}$, such that

$$\operatorname{val} \frac{H_{(n+j)l}}{H_{nl}} = \operatorname{val} r_{nl} \geq 1,$$

it easily follows that $\operatorname{val} H_{nl} \to \infty$ as $n \to \infty$. Finding a two-term right factor of the recurrence satisfied by H_{nl} is equivalent to searching for a j-fold hypergeometric term solution.
Since Proposition 3.1.5 allows restriction to subsequences of H_n and thus subsequences of H_{nl}, it is actually sufficient to find hypergeometric term solutions of the holonomic recurrence satisfied by H_{nlj+lp} for some $j, p \in \mathbb{N}_{\geq 0}$, $j > p$. This can be achieved with Mark van Hoeij's algorithm [vH99] (see also [Koe14]), implemented in *Maple 18* under the name `hypergeomsols` in the `LRETools` package. This is the approach chosen in this thesis.
Alternatively one could also directly search for j-fold hypergeometric term solutions, see [HKS12] and [PS93].

3.2. Detailed examples and further results

Example 3.2.1. Starting from the expression $\tan x$ it is both well known and easy to see that $\tan x$ satisfies the differential equation

$$0 = \mathcal{D}Y := \frac{\mathrm{d}}{\mathrm{d}x} Y(x) - Y(x)^2 - 1, \; Y(0) = 0.$$

By Proposition 3.1.2 the power series expansion of $\tan x$ is the unique power series solution of this differential equation. Truncating the expansion at order $\mathcal{O}(x^{15})$ gives

$$x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1382}{155925}x^{11} + \frac{21844}{6081075}x^{13}$$

and converting this into a finite continued fraction of the form $\mathop{\mathrm{K}}_{n=1}^{\hat{N}} \frac{a_n}{1}$ yields

$$\frac{x}{1} + \frac{-x^2/3}{1} + \frac{-x^2/15}{1} + \frac{-x^2/35}{1} + \frac{-x^2/63}{1} + \frac{-x^2/99}{1} + \frac{-x^2/143}{1}.$$

Input : An analytic expression $f(x)$, natural numbers N, L, and s
Output: A conjectural continued fraction representation $b_0 + \mathrm{K}_{n=1}^{\infty} \frac{a_n}{1}$ of f in the form of an expression b_0, a list of expressions $a_1, \ldots, a_{\hat{n}}$ and a list of functions $a_n^1, \ldots, a_n^{l-1}, a_n^0$ rational in n, where $a_n = a_n^{(n \bmod l)}$ for $n > \hat{n}$ or **FAIL** if none is found

for $l = 1, \ldots, L$ **do**
  for $ord = sl, 2sl, \ldots$ *and* $ord \leq Nl$ **do**
    $newS \leftarrow ord$-th partial sum $S_{ord} = \sum_{k=0}^{ord} c_k x^k$ of the power series expansion of f
    if $newS = oldS$ **then**
      next
    end
    $C \leftarrow$ finite C-fraction $b_0 + \mathop{\mathrm{K}}_{n=1}^{\hat{N}} \frac{a_n}{1}$ corresponding to $newS$
    for $j = 1, \ldots, l$ **do**
      $a_n^j \leftarrow$ guess a general formula by rational interpolation on a_{il+j}, where $i \in \mathbb{N}$ and $il + j \leq \hat{N}$
      $a_n^0 \leftarrow a_n^l$
    end
    if $oldguess(j) = a_n^j$ *for* $j = 0, \ldots, l-1$ **then**
      $\hat{n} \leftarrow$ maximal index $n \leq \hat{N}$ such that $a_n \neq a_n^{(n \bmod l)}$
      return b_0*, a list containing* $a_1, \ldots, a_{\hat{n}}$*, and a list containing* $a_n^1, \ldots, a_n^{l-1}, a_n^0$
    else
      $oldS \leftarrow newS$
      for $j = 0, \ldots, l-1$ **do**
        $oldguess(j) \leftarrow a_n^j$
      end
    end
  end
end
return FAIL

Algorithm 3: guessCfracFromExpr

Input : An explicit differential equation $0 = \mathcal{D}Y(x)$ of order d, a continued fraction $b_0 + \mathrm{K}_{n=1}^{\infty} \frac{a_n}{1}$ in the form of an expression b_0, a list of expressions $a_1, \ldots, a_{\hat{n}}$ and a list of functions $a_n^1, \ldots, a_n^{l-1}, a_n^0$ rational in n, where $a_n = a_n^{(n \bmod l)}$ for $n > \hat{n}$

Output: A corresponding holonomic recurrence $0 = \mathcal{R}H_{ln}(x)$ of order i with initial values $H_0(x), H_l(x), \ldots, H_{li}(x)$, with H_n defined as in Proposition 3.1.4

compute a linear recurrence $A_{l(n+2)} = r_{n+2}A_{l(n+1)} + s_{n+2}A_{ln}$ from $A_{n+2} = A_{n+1} + a_{n+2}A_n$ with linear algebra (also satisfied by B_{ln})

compute initial values A_0, A_l, B_0, B_l from the recurrences $A_{n+2} = A_{n+1} + a_{n+2}A_n$, $B_{n+2} = B_{n+1} + a_{n+2}B_n$ with initial values $A_0 = b_0, A_1 = b_0 + a_1, B_0 = 1, B_1 = 1$

$T_0(n) \leftarrow \mathcal{D}\frac{A_{ln}}{B_{ln}}$

for $i = 1, 2, \ldots$ **do**

  $T_i(n) \leftarrow T_{i-1}(n+1)$ rewritten in terms of $A_{l(n+1)}, A_{ln}, B_{l(n+1)}, B_{ln}$ and their derivatives up to order d

  if *the linear equation* $T_i(n) + \sum_{k=0}^{i-1} c_k T_k(n)$ *has a solution in the unknowns* $c_0, \ldots, c_{i-1}$ **then**

    $H_0(x), H_l(x), \ldots, H_{li}(x) \leftarrow T_0(0), T_1(0), \ldots, T_i(0)$

    return $H_{l(n+i)}(x) + \sum_{k=0}^{i-1} c_k H_{l(n+k)}(x)$ *and* $H_0(x), H_l(x), \ldots, H_{li}(x)$

  end

end

Algorithm 4: searchCorrRec

From these initial elements a general formula for the elements of the C-fraction corresponding to $\tan x$ can be guessed by rational interpolation, namely

$$a_1 = x, \;\; a_n = -\frac{x^2}{(2n-1)(2n-3)}.$$

Let $f_n = \frac{A_n}{B_n}$ be the sequence of approximants of the conjectured continued fraction. Using Proposition 3.1.3 the conjectured formula can be proven to hold: The condition $f_n(0) = 0$ is obviously true for all $n \geq 0$ by nature of the construction. To show $\operatorname{val} \mathcal{D}f_n \to \infty$ let

$$H_n := A_n' B_n - B_n^2 - A_n^2 - A_n B_n'$$

as in Theorem 3.1.4. As canonical numerators and denominators of the continued fraction $\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{1}$ both A_n and B_n satisfy the recurrences

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} + a_n \begin{bmatrix} A_{n-2} \\ B_{n-2} \end{bmatrix} \text{ for } n \geq 1$$

with initial conditions

$$A_{-1} = B_0 = 1, \;\; A_0 = B_{-1} = 0.$$

Input : A holonomic recurrence $0 = \mathcal{R}H_n(x) = H_{n+ord}(x) + \sum_{k=0}^{ord-1} c_k(x)H_{n+k}(x)$, its initial values $H_0(x), \ldots, H_{ord-1}(x)$, a natural number L

Output: The term ratio corresponding to a two-term right factor of the recurrence proving the increase of $\operatorname{val} H_n(x)$ as $n \to \infty$ or **FAIL** if none is found

try to compute a hypergeometric term solution of the recurrence $0 = \mathcal{R}H_n(x)$ with initial conditions $H_0(x), \ldots, H_{ord-1}(x)$ with the van Hoeij algorithm
if *a hypergeometric term solution is found* **then**
  $sol(n, x) \leftarrow$ hypergeometric term solution
  $v \leftarrow \operatorname{val}\left(\dfrac{sol(n+1, x)}{sol(n, x)}\right)$
  if $v > 0$ **then**
    return $H_{n+1}(x) = \dfrac{sol(n+1, x)}{sol(n, x)} H_n(x)$
  end
end
for $l = 2, \ldots, L$ **do**
  compute initial values $H_{(l-1)ord}(x), \ldots, H_{l \cdot ord-1}(x)$ from $H_0(x), \ldots, H_{(l-1)ord-1}(x)$ and $0 = \mathcal{R}H_n(x)$
  for $m = 0, \ldots, l-1$ **do**
    construct a holonomic recurrence $0 = \mathcal{R}_{l,m}H_{ln+m}n(x) = H_{l(n+ord)+m}(x) + \sum_{k=0}^{ord-1} c_k(x)H_{l(n+k)+m}(x)$ from the recurrence $0 = \mathcal{R}H_n(x)$ with linear algebra
    try to compute a hypergeometric term solution of the recurrence $0 = \mathcal{R}_{l,m}H_{ln+m}(x)$ with initial conditions $H_{l \cdot 0+m}(x), \ldots, H_{l(ord-1)+m}(x)$ with the van Hoeij algorithm
    if *a hypergeometric term solution is found* **then**
      $sol(n, x) \leftarrow$ hypergeometric term solution
      $v \leftarrow \operatorname{val}\left(\dfrac{sol(n+1, x)}{sol(n, x)}\right)$
      if $v > 0$ **then**
        return $H_{l(n+1)+m}(x) = \dfrac{sol(n+1, x)}{sol(n, x)} H_{ln+m}(x)$
      end
    end
  end
end
return FAIL

Algorithm 5: checkValIncrease

From $a_n(0) = 0$ for all $n \geq 1$ it follows that $B_n(0) = 1$ for all $n \geq 0$ by iteratively applying the recurrence formula for B_n. Thus $\operatorname{val} B_n = 0$ and therefore $\operatorname{val} H_n = \operatorname{val} \mathcal{D} f_n$.
As described in Theorem 3.1.4 any H_{n+l}, $l \geq 0$ can be rewritten as a linear combination in terms of

$$A'_{n+i}B_{n+j}, A_{n+i}B'_{n+j}, A_{n+i}A_{n+j}, B_{n+i}B_{n+j}$$

with $i, j \in \{0, 1\}$ by applying the recurrence formulas for A_n and B_n. There are only 16 such terms, so $(H_n, \ldots, H_{n+16})$ must be linearly dependent. As such there must exist a linear recurrence of order at most 16. Searching for a recurrence

$$H_{n+l} = \sum_{i=0}^{l-1} c_{n+i} H_{n+i}, \; 1 \leq l \leq 16$$

with linear algebra yields the fourth order recurrence relation

$$\begin{aligned} H_{n+4} =& H(n+3) - \frac{x^2(4n^2 - x^2 + 20n + 21)}{(2n+3)(2n+5)(2n+7)^2} H_{n+2} + \frac{x^4}{(2n+3)(2n+5)^2(2n+7)} H_{n+1} \\ & - \frac{x^8}{(2n+1)^2(2n+3)^3(2n+5)^2(2n+7)} H_n, \end{aligned}$$

with initial conditions

$$H_0 = -1, \; H(1) = -x^2, \; H(2) = -\frac{x^4}{9}, \; H(3) = -\frac{x^6}{225}$$

obtained by substituting the corresponding values of the sequences A_n and B_n.
From this, van Hoeij's algorithm yields the hypergeometric term solution

$$H_n = -\frac{\pi (x^2/4)^n}{\Gamma(n + 1/2)^2},$$

that is the term ratio

$$H_{n+1} = \frac{x^2}{(2n+1)^2} H_n$$

corresponding to the right factor

$$\mathcal{R} H_n = H_{n+1} - \frac{x^2}{(2n+1)^2} H_n.$$

From this it is evident that $\operatorname{val} H_n \to \infty$ as $n \to \infty$, so the conjectured continued fraction indeed corresponds to the power series expansion of $\tan x$. In other words

$$\tan x = \frac{x}{1 + \operatorname*{K}_{n=2}^{\infty} \frac{a_n}{1}}, \; a_n = -\frac{x^2}{(2n-1)(2n-3)}, \; n \geq 2.$$

This continued fraction is equivalent to the one Lambert used in his proof of the irrationality of π, see [Lam61] and [CBV+08, p. 202].

Example 3.2.2. Starting from the expression $\exp(x)$ one easily obtains the differential equation

$$0 = \mathcal{D}Y := \frac{\mathrm{d}}{\mathrm{d}x}Y(x) - Y(x),\ Y(0) = 1.$$

The power series expansion of $\exp(x)$ is the unique power series solution of this differential equation by Proposition 3.1.2. Truncating the expansion at order $\mathcal{O}(x^{10})$ one obtains

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9.$$

A conversion to a finite C-fraction $1 + \mathop{\mathrm{K}}_{n=1}^{\hat{N}} \frac{a_n}{1}$ yields

$$1 + \frac{x}{1} + \frac{-x/2}{1} + \frac{x/6}{1} + \frac{-x/6}{1} + \frac{x/10}{1} + \frac{-x/10}{1} + \frac{x/14}{1} + \frac{-x/14}{1} + \frac{x/18}{1}.$$

Guessing a general formula from these initial elements by rational interpolation leads to

$$a_1 = x,\ a_{2n} = -\frac{x}{2(2n-1)},\ a_{2n+1} = \frac{x}{2(2n+1)}.$$

This guessed formula agrees with the one given in [CBV+08, p. 194].
Let $f_n = \frac{A_n}{B_n}$ be the sequence of approximants of the conjectured continued fraction. As in Example 3.2.1 the conjectured formula can be proven to hold by considering H_n as defined in Theorem 3.1.4. Since a_n exhibits 2-fold symmetry, the recurrences

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} + a_n \begin{bmatrix} A_{n-2} \\ B_{n-2} \end{bmatrix} \quad \text{for } n \geq 1$$

with initial conditions

$$A_{-1} = B_0 = 1,\ A_0 = B_{-1} = 0$$

are not holonomic. Nonetheless, the process described in Theorem 3.1.4 can be applied to H_n defined as the numerator of the expression

$$\left(\frac{A_n}{B_n}\right)^{(m)} - F\left(X, \left(\frac{A_n}{B_n}\right), \left(\frac{A_n}{B_n}\right)', \ldots, \left(\frac{A_n}{B_n}\right)^{(m-1)}\right)$$

to yield the recurrence

$$\begin{aligned} H_{n+4} = &-\frac{a_{n+3}a'_{n+4} - a'_{n+3}(a_{n+4}+1)}{a'_{n+3}}H_{n+3} + \frac{(a_{n+3}^2 + a_{n+3})a'_{n+4} + a'_{n+3}(a_{n+4}^2 + a_{n+4})}{a'_{n+3}}H_{n+2} \\ &+ \frac{a_{n+3}^2\left(a_{n+3}a'_{n+4} - a'_{n+3}a_{n+4} + a'_{n+4}\right)}{a'_{n+3}}H_{n+1} + \frac{a_{n+2}^2 a_{n+3}^2 a'_{n+4}}{a'_{n+3}}H_n \end{aligned}$$

for H_n. Owing to the 2-fold symmetry of a_n, this recurrence has no general explicit form. By Proposition 3.1.5 it is sufficient to instead consider a recurrence relation for H_{2n} though, which can be obtained by constructing holonomic recurrence formulas for A_{2n} and B_{2n} from the respective recurrence formulas for A_n and B_n and only then applying Theorem 3.1.4. To this end consider that both $A_{2(n+1)}$ and $A_{2(n+2)}$ can be rewritten as a linear combination in terms of A_{2n} and A_{2n+1}, so $A_{2(n+2)}$ can also be rewritten as a linear combination in terms of A_{2n} and $A_{2(n+1)}$, namely

$$A_{2(n+2)} = (1 + a_{2n+3} + a_{2n+4})A_{2(n+1)} - a_{2n+3}a_{2n+4}A_{2n}.$$

A holonomic recurrence formula for B_{2k} can be obtained in the same way. The initial conditions

$$A_0 = 0,\ A_2 = a_1 = x,\ B_0 = 1,\ B_2 = 1 + a_2 = 1 - x/2$$

can be computed from the original recurrence.
Applying Theorem 3.1.4 to A_{2n} and B_{2n} yields the recurrence

$$\begin{aligned} H_{2(n+4)} = H_{2(n+3)} &+ \frac{x^2(16n^2 + x^2 + 80n + 84)}{8(2n+3)(2n+5)(2n+7)^2} H_{2(n+2)} \\ &+ \frac{x^4}{16(2n+3)(2n+5)^2(2n+7)} H_{2(n+1)} \\ &- \frac{x^8}{256(2n+1)^2(2n+3)^3(2n+5)^2(2n+7)} H_{2n} \end{aligned}$$

satisfied by H_{2n} with initial conditions

$$H_0 = -1,\ H_2 = \frac{x^2}{4},\ H_4 = -\frac{x^4}{144},\ H_6 = \frac{x^6}{14400}.$$

Applying van Hoeij's algorithm to $\hat{H}_n = H_{2n}$ yields the hypergeometric term solution

$$H_{2n} = \hat{H}_n = -\frac{\pi(x^2/16)^n}{\Gamma(n+1/2)^2},$$

that is the term ratio

$$H_{2(n+1)} = -\frac{x^2}{4(2n+1)^2} H_{2n}$$

corresponding to a two-term right factor. Note that this approach does not necessarily yield a 2-fold hypergeometric term solution of the original recurrence equation for H_n, since H_{2n+1} wasn't considered at all. But for the purposes of the algorithm this is sufficient to show that $\operatorname{val} H_{2n} \to \infty$ as $n \to \infty$. In that case by Proposition 3.1.5 the conjectured continued fraction indeed corresponds to the power series expansion of $\exp(x)$, that is

$$\exp x = 1 + \cfrac{x}{1 + \underset{n=2}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{1}},\ a_{2n} = -\frac{x}{2(2n-1)},\ a_{2n+1} = \frac{x}{2(2n+1)},\ n \geq 1.$$

3.2.1. The exponential and logarithm function

Besides the continued fraction given in Example 3.2.2 $\exp x$ can also be expressed by the continued fraction (see [CBV+08, p. 194])

$$\exp x = 1 + \frac{2x}{2-x} + \frac{x^2/6}{1} + \underset{n=3}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{1},\ a_n = \frac{1}{4(2n-3)(2n-1)}.$$

This representation is easily guessed by considering the series expansion of the expression

$$\frac{2x}{\exp(x)-1} - (2-x)$$

to ensure the specific form of the first partial numerator and denominator. Unfortunately, the corresponding differential equation

$$0 = \mathcal{D}Y := 2x\frac{\mathrm{d}}{\mathrm{d}x}Y(x) + Y(x)^2 + 2Y(x) - x^2,\ Y(0) = 0$$

is singular in $x = 0$, preventing Proposition 3.1.3 from being applicable. Nevertheless, computing H_n and the corresponding recurrence formula yields

$$\begin{aligned} H_{n+4} = H_{n+3} &+ \frac{x^2(16n^2 + x^2 + 112n + 180)}{8(2n+5)(2n+7)(2n+9)^2}H_{n+2} \\ &+ \frac{x^4}{16(2n+5)(2n+7)^2(2n+9)}H_{n+1} \\ &- \frac{x^8}{256(2n+3)^2(2n+5)^3(2n+7)^2(2n+9)}H_n \end{aligned}$$

with initial conditions

$$H_0 = -x^2,\ H_1 = \frac{x^4}{36},\ H_2 = -\frac{x^6}{3600},\ H_3 = \frac{x^8}{705600}.$$

This recurrence has a right factor with the corresponding term ratio

$$H_{n+1} = -\frac{x^2}{4(2n+3)^2}H_n,$$

showing the increase of valuation of H_n. This suggests there may be a way to extend the applicability of Proposition 3.1.3 to some cases where the differential equation is singular in $x = 0$.

Moving on, the logarithm function $\ln(1+x)$ has a power series representation in $x = 0$ and satisfies the differential equation

$$0 = \mathcal{D}Y := (1+x)\frac{\mathrm{d}}{\mathrm{d}x}Y(x) - 1,\ Y(0) = 0.$$

Guessing the continued fraction representation (see [CBV+08, p. 196])

$$\ln(1+x) = \frac{x|}{|1} + \underset{n=2}{\overset{\infty}{\mathrm{K}}}\frac{a_n x}{1},\ a_{2k} = \frac{n}{2(2n-1)},\ a_{2n+1} = \frac{n}{2(2n+1)},$$

it is proven by computing the recurrence

$$\begin{aligned} H_{2(n+4)} = &\frac{(x+2)^2}{4}H_{2(n+3)} \\ &- \frac{x^2(n+3)^2}{16(2n+3)(2n+5)(2n+7)^2} \\ &\cdot (6n^2x^2 + 32n^2x + 30nx^2 + 32n^2 + 160xn + 31x^2 + 160n + 168x + 168)H_{2(n+2)} \\ &+ \frac{x^4(x+2)^2(n+2)^2(n+3)^2}{64(2n+3)(2n+5)^2(2n+7)}H_{2(n+1)} \\ &- \frac{x^8(n+1)^4(n+2)^2(n+3)^2}{256(2n+1)^2(2n+3)^3(2n+5)^2(2n+7)}H_{2n} \end{aligned}$$

with initial values

$$H_0 = -1,\ H_2 = -\frac{x^2}{4},\ H_4 = -\frac{x^4}{36},\ H_6 = -\frac{x^6}{400}$$

and term ratio

$$H_{2(n+1)} = \frac{x^2(n+1)^2}{4(2n+1)^2} H_{2n}$$

corresponding to a two-term right factor, which shows the increase in valuation of H_{2n}. The expression $\ln\left(\frac{1+x}{1-x}\right)$ satisfies the differential equation

$$0 = \mathcal{D}Y := (x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}Y(x) + 2,\ Y(0) = 0.$$

Guessing a C-fraction representation based on the inital terms of its power series expansion yields (see [CBV+08, p. 196])

$$\ln\left(\frac{1+x}{1-x}\right) = \frac{2x|}{|1} + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n x^2}{1},\ a_n = \frac{-n^2}{(2n-1)(2n+1)}.$$

The corresponding recurrence formula is

$$\begin{aligned} H_{n+4} = H_{n+3} &+ \frac{x^2(n+3)^2(2n^2x^2+10nx^2-8n^2+11x^2-40n-42)}{(2n+3)(2n+5)(2n+7)^2} H_{n+2} \\ &+ \frac{x^4(n+2)^2(n+3)^2}{(2n+3)(2n+5)^2(2n+7)} H_{n+1} \\ &- \frac{x^8(n+1)^4(n+2)^2(n+3)^2}{(2n+1)^2(2n+3)^3(2n+5)^2(2n+7)} H_n \end{aligned}$$

with inital values

$$H_0 = 2,\ H_1 = 2x^2,\ H_2 = \frac{8}{9}x^4,\ H_3 = \frac{8}{25}x^6.$$

The right factor with corresponding term ratio

$$H_{n+1} = \frac{x^2(n+1)^2}{(2n+1)^2} H_n$$

proves the validity of the guessed continued fraction formula.

3.2.2. Trigonometric functions and inverse trigonometric functions

A continued fraction expression for $\tan x$ was proven in Example 3.2.1. For $\arctan x$ the C-fraction representation (see[CBV+08, p. 207])

$$\arctan x = \frac{x|}{|1} + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n x^2}{1},\ a_n = \frac{n^2}{(2n-1)(2n+1)},$$

can either be computed with the guess and prove algorithm or it can be obtained from the continued fraction formula for $\ln\left(\frac{1+x}{1-x}\right)$ presented in the previous section, since it is easily seen that

$$\arctan x = \frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right),$$

as both sides agree in $x=0$ and have identical derivatives. Note that all occurences of x in the continued fraction formula for $\ln\left(\frac{1+x}{1-x}\right)$ are quadratic except for the first partial numerator, where it occurs linearly. Because of this, upon substituting ix for x all complex units simplify to real factors with exception of the first one, which gets canceled by the factor $1/2i$. Hence all elements of the given continued fraction expansion of $\arctan x$ have real coefficients.
Guessing a continued fraction representation of the expression $\frac{\arcsin x}{\sqrt{1-x^2}}$ yields (see [CBV+08, p. 205])

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \frac{x\rfloor}{\lceil 1} + \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{a_n x^2}{1},\ a_{2n} = -\frac{2n(2n-1)}{(4n-1)(4n-3)},\ a_{2n+1} = -\frac{2n(2n-1)}{(4n+1)(4n-1)}.$$

Taking the differential equation

$$0 = \mathcal{D}Y := (x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}Y(x) + xY(x) + 1,\ Y(0) = 0$$

satisfied by $\frac{\arcsin x}{\sqrt{1-x^2}}$ leads to a corresponding recurrence formula of fourth order for H_{2n} omitted due to length for H_{2n} with initial values

$$H_0 = 1,\ H_2 = \frac{4}{9}x^4,\ H_4 = \frac{64}{1225}x^8,\ H_6 = \frac{256}{53361}x^{12}$$

and right factor with corresponding term ratio

$$H_{2(n+1)} = \frac{4x^4(n+1)^2(2n+1)^2}{(4n+3)^2(4n+1)^2}H_{2n}.$$

3.2.3. Hyperbolic functions and inverse hyperbolic functions

Conjecturing a continued fraction representation of $\tanh x$ based on its power series expansion leads to the C-fraction (see [CBV+08, p. 211])

$$\tanh x = \frac{x\rfloor}{\lceil 1} + \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{a_n x^2}{1},\ a_n = \frac{1}{(2n-1)(2n-3)}.$$

Taking the differential equation

$$0 = \mathcal{D}Y := \frac{\mathrm{d}}{\mathrm{d}x}Y(x) + Y(x)^2 - 1,\ Y(0) = 0,$$

which holds for $Y = \tanh$, yields the corresponding recurrence formula

$$\begin{aligned} H_{n+4} = H_{n+3} &+ \frac{2x^2(4n^2+x^2+20n+21)}{8(2n+3)(2n+5)(2n+7)^2}H_{n+2} \\ &+ \frac{x^4}{16(2n+3)(2n+5)^2(2n+7)}H_{n+1} \\ &- \frac{x^8}{(2n+1)^2(2n+3)^3(2n+5)^2(2k+7)}H_n \end{aligned}$$

with initial values

$$H_0 = -1,\ H_1 = x^2,\ H_2 = -\frac{x^4}{9},\ H_3 = \frac{x^6}{225}.$$

This recurrence has a two-term right factor with corresponding term ratio

$$H_{n+1} = -\frac{x^2}{(2n+1)^2} H_n,$$

showing that the guessed continued fraction formula is correct.
The continued fraction expansion (see [CBV+08, p. 214])

$$\operatorname{Asinh} x = \frac{x\sqrt{1+x^2}}{1} + \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{a_n x^2}{1},\ a_{2n} = \frac{2n(2n-1)}{(4n-1)(4n-3)},\ a_{2n+1} = \frac{2n(2n-1)}{(4n+1)(4n-1)}.$$

can be obtained either by applying the guess and prove algorithm to the expression $\frac{\operatorname{Asinh} x}{\sqrt{1+x^2}}$ or by utilizing the relation

$$\operatorname{Asinh} x = i \arcsin\left(\frac{x}{i}\right)$$

and the C-fraction expansion of $\arcsin x$ shown in the previous section.
The same holds true for the C-fraction expansion (see [CBV+08, p. 216])

$$\operatorname{Atanh} x = \frac{x}{1} + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n x^2}{1},\ a_n = -\frac{n^2}{(2n-1)(2n+1)}$$

and the relation

$$\operatorname{Atanh} x = i \arctan\left(\frac{x}{i}\right).$$

3.2.4. Power functions

For the power function $(1+x)^\alpha$ three known continued fraction representations due to Perron can be found in [Per13, p. 348] and [CBV+08, p. 218]. The first continued fraction

$$(1+x)^\alpha = 1 + \frac{\alpha x}{1} + \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{a_n x}{1},\ a_{2n} = \frac{(n-\alpha)}{2(2n-1)},\ a_{2n+1} = \frac{(n+\alpha)}{2(2n+1)}$$

can be obtained by directly applying the guess and prove algorithm to the expression $(1+x)^\alpha$. This expression satisfies the differential equation

$$0 = \mathcal{D}Y := (1+x)\frac{\mathrm{d}}{\mathrm{d}x}Y(x) - \alpha Y(x), Y(0) = 1,$$

yielding the corresponding recurrence relation

$$\begin{aligned}
H_{2(n+4)} =& \frac{(x+2)^2}{4} H_{2(n+3)} \\
&+ (2\alpha^2 x^2 + 6n^2 x^2 + 32n^2 x + 30nx^2 + 32n^2 + 160xn + 31x^2 + 160n + 168x + 168) \\
&\cdot \frac{x^2(\alpha+n+3)(\alpha-n-3)}{16(2n+3)(2n+5)(2n+7)^2} H_{2(n+2)} \\
&+ \frac{x^4(x+2)^2(\alpha+n+2)(\alpha-n-2)(\alpha+n+3)(\alpha-n-3)}{64(2n+3)(2n+5)^2(2n+7)} H_{2(n+1)} \\
&- \frac{x^8(\alpha+n+1)^2(\alpha-n-1)^2(\alpha+n+2)(\alpha-n-2)(\alpha+n+3)(\alpha-n-3)}{256(2n+1)^2(2n+3)^3(2n+5)^2(2n+7)} H_{2n}
\end{aligned}$$

for H_{2n} with the initial values

$$H_0 = -\alpha,\ H_2 = \frac{\alpha^3 - b}{4}x^2,\ H_4 = -\frac{\alpha^5 - 5\alpha^3 + 4\alpha}{144}x^4,\ H_6 = \frac{\alpha^7 - 14\alpha^5 + 49\alpha^3 - 36\alpha}{14400}x^6.$$

The term ratio

$$H_{2(n+1)} = -\frac{x^2(\alpha+n+1)(\alpha-n-1)}{4(2n+1)^2}H_{2n}$$

corresponding to a right factor proves the conjectured formula.
Note that just like $((1+x)^\alpha - 1)/\alpha \to \ln(1+x)$ as $\alpha \to 0$, the previously shown continued fraction expansion of $\ln(1+x)$ can be obtained by substituting the above continued fraction representation of $(1+x)^\alpha$ into $((1+x)^\alpha - 1)/\alpha$ and taking the limit for $\alpha \to 0$.
A second continued fraction representation of $(1+x)^\alpha$ is

$$(1+x)^\alpha = \frac{1|}{|1} + \frac{-\alpha x|}{|1} + \mathop{\mathrm{K}}_{n=3} \frac{a_n x}{1},\ a_{2n} = \frac{n-1-\alpha}{2(2n-1)},\ a_{2n+1} = \frac{n+\alpha}{2(2n-1)}.$$

It is obtained by applying the guess and prove algorithm to the expression $\frac{1}{(1+x)^\alpha}$ and rearranging the result. It also follows directly by utilizing the relation

$$(1+x)^\alpha = \frac{1}{(1+x)^{-\alpha}}.$$

The third continued fraction expansion of $(1+x)^\alpha$ is

$$(1+x)^\alpha = \frac{1|}{|1} + \frac{-\alpha x/(1+x)|}{|1} + \mathop{\mathrm{K}}_{n=3}^{\infty} \frac{a_n x/(1+x)}{1},\ a_{2n} = \frac{-\alpha-n+1}{2(2n-1)},\ a_{2n+1} = \frac{\alpha-n}{2(2n-1)}.$$

To obtain this continued fraction with the guess and prove algorithm, substitute $x = z/(1-z)$, consider the expression $\frac{1}{(1+z/(1-z))^\alpha}$, rearrange the result, and re-substitute $z = x/(1+x)$. Using the same substitutions for x and z, the continued fraction also follows directly from the relation

$$(1+x)^\alpha = \left(1 + \frac{z}{1-z}\right)^\alpha = \frac{1}{(1+(-z))^\alpha}$$

by rewriting $(1+(-z))^\alpha$ in terms of the first given continued fraction expansion.
The following continued fraction representation for $\left(\frac{1+x}{1-x}\right)^\alpha$ due to Perron can be found in [Per13, p. 350] and [CBV$^+$08, p. 220]:

$$\left(\frac{1+x}{1-x}\right)^\alpha = 1 + \frac{2\alpha x|}{|1-\alpha x} + \mathop{\mathrm{K}}_{n=2}^{\infty} \frac{a_n x^2}{1},\ a_n = \frac{(\alpha-n+1)(\alpha+n-1)}{(2n-3)(2n-1)}.$$

To search for this representation with the guess and prove algorithm consider the expression

$$\frac{2\alpha x}{\left(\frac{1+x}{1-x}\right)^\alpha - 1} - (1-\alpha x),$$

which tends to 0 as $x \to 0$. Unfortunately, the differential equation

$$0 = \mathcal{D}Y := x(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}Y(x) - Y(x)^2 - (x^2+1)Y(x) + x^2(\alpha^2-1)$$

found for this expression is singular in $x = 0$, conflicting with the applicability of Proposition 3.1.3. Similar to the previous case where this occured with $\exp x$ all further steps would still be successful though, again suggesting there may be a way to extend the applicability of Proposition 3.1.3 to further cases.

3.2.5. Airy functions

In [MS15, p. 281] Maulat and Salvy give the following C-fraction involving the Airy function Ai:

$$x\frac{\mathrm{Ai}'}{\mathrm{Ai}}\left(\frac{1}{x^2}\right) = -1 - \frac{x^3/4|}{|1} + \mathop{\mathrm{K}}_{n=2}^{\infty}\frac{a_n x^3}{1},\ a_{2n} = \frac{6n-1}{8},\ a_{2n+1} = \frac{6n+1}{8}.$$

Due to the fact that along the real axis the left- and right-handed limit in $x = 0$ of the considered expression differ, being 2 and 0 respectively, restrict the domain to the positive real numbers. In this case the C-fraction is easily guessed from the resulting series representation.
The corresponding differential equation

$$0 = \mathcal{D}Y := x^4\frac{\mathrm{d}}{\mathrm{d}x}Y(x) - 2Y(x)^2 - x^3Y(x) + 2$$

is unfortunately again singular in $x = 0$. But just as is the case for $\exp x$ and $\left(\frac{1+x}{1-x}\right)^{\alpha}$, all further steps would be successful, leading to a corresponding linear recurrence for H_{2n} with initial values

$$H_0 = x^3,\ H_2 = \frac{35}{64}x^9,\ H_4 = -\frac{5005}{4096}x^{15},\ H_6 = \frac{1616615}{262144}x^{21}$$

and right factor exhibiting the desired increase in valuation, as seen in the corresponding term ratio

$$H_{2(n+1)} = \frac{x^6(6n+7)(6n+5)}{64}H_{2n}.$$

3.2.6. New results

Noting in Subsection 3.2.1 that the expression $\frac{2x}{\exp(x)-1} - (2-x)$, considered in order to find a continued fraction representation of $\exp x$, contains the generating function

$$\frac{x\exp(xt)}{\exp(x)-1} = \sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!}$$

of the Bernoulli polynomials for $t = 0$ (or in other words the Bernoulli numbers), it may be worthwhile instead to try applying the algorithm to an expression involving the generating function for the related Euler polynomials

$$\frac{2\exp(xt)}{\exp(x)+1} = \sum_{n=0}^{\infty} E_n(t)\frac{x^n}{n!},$$

both found, for example, in [AS84, p. 358]. And indeed, this leads to two new continued fraction expansions of $\exp x$, as follows.
Consider the expression $\dfrac{2}{\exp(x)+1}$, satisfying the differential equation

$$0 = \mathcal{D}Y := 2\frac{\mathrm{d}}{\mathrm{d}x}Y(x) - Y(x)^2 + 2Y(x),\ Y(0) = 1,$$

which is not singular in $x = 0$. The C-fraction representation

$$\frac{2}{\exp(x)+1} = 1 - \frac{x/2}{1} + \operatorname*{K}_{n=2}^{\infty} \frac{a_n x^2}{1},\ a_n = \frac{1}{4(2n-1)(2n-3)}$$

can be guessed by rational interpolation. The corresponding linear recurrence is

$$\begin{aligned} H_{n+4} = H_{n+3} &+ \frac{x^2(16n^2+x^2+80n+84)}{8(2n+3)(2n+5)(2n+7)^2} H_{n+2} \\ &+ \frac{x^4}{16(2n+3)(2n+5)^2(2n+7)} H_{n+1} \\ &- \frac{x^8}{256(2n+1)^2(2n+3)^3(2n+5)^2(2n+7)} H_n \end{aligned}$$

with initial values

$$H_0 = 1,\ H_1 = -\frac{x^2}{4},\ H_2 = \frac{x^4}{144},\ H_3 = -\frac{x^6}{14400}.$$

The right factor with corresponding term ratio

$$H_{n+1} = \frac{x^2}{4(2n+1)^2} H_n$$

of this recurrence proves the conjectured C-fraction representation. Furthermore, rearranging the representation formula easily yields the following continued fraction expansion of $\exp x$:

$$\exp x = -1 + \frac{2}{1} + \frac{-x/2}{1} + \operatorname*{K}_{n=2}^{\infty} \frac{a_n x^2}{1},\ a_n = \frac{1}{4(2n-1)(2n-3)}.$$

Another continued fraction can be found and proven by considering $t = 1$ instead of $t = 0$ in the generating function of the Euler polynomials. The continued fraction representation

$$\frac{2\exp(x)}{\exp(x)+1} = 1 + \frac{x/2}{1} + \operatorname*{K}_{n=2}^{\infty} \frac{a_n x^2}{1},\ a_n = \frac{1}{4(2n-1)(2n-3)}$$

can be proven by applying the guess and prove algorithm, but it also follows directly from the relation

$$\frac{2\exp(x)}{\exp(x)+1} - 1 = \frac{2\exp(x)}{\exp(x)+1} - 2 + 1 = -\frac{2}{\exp(x)+1} + 1 = -\left(\frac{2}{\exp(x)+1} - 1\right).$$

This representation formula can again be rearranged to obtain a continued fraction representation of $\exp x$:

$$\exp x = \frac{-1}{1} + \frac{-2}{1} + \frac{x/2}{1} + \operatorname*{K}_{n=2}^{\infty} \frac{a_n x^2}{1},\ a_n = \frac{1}{4(2n-1)(2n-3)}.$$

Both continued fractions converge for all $x \in \mathbb{C}$. To prove this, it suffices to show that the continued fraction $\mathop{\mathrm{K}}\limits_{n=2}^{\infty} \frac{a_n x^2}{1}$ with $a_n = \frac{1}{4(2n-1)(2n-3)}$ converges in $\mathbb{C}$, since it appears as a tail of both as defined in Definition 2.1.3. By Worpitzky's Theorem (Theorem 2.1.14) $\mathop{\mathrm{K}}\limits_{n=2}^{\infty} \frac{a_n x^2}{1}$ converges, if $|a_n x^2| \leq 1/4$, that is

$$\left|\frac{x^2}{4(2n-1)(2n-3)}\right| \leq \frac{1}{4}$$

or equivalently

$$|x| \leq \sqrt{|(2n-1)(2n-3)|},$$

for all $n \geq 2$.
Now let $x \in \mathbb{C}$, then there is some $N \in \mathbb{N}$, such that $|x| \leq \sqrt{|(2n-1)(2n-3)|}$ for all $n \geq N$, since the right-hand side is unbounded and monotonously increasing. In other words, by Worpitzky's Theorem the tail $\mathop{\mathrm{K}}\limits_{n=N}^{\infty} \frac{a_n x^2}{1}$ is convergent and thus $\mathop{\mathrm{K}}\limits_{n=2}^{\infty} \frac{a_n x^2}{1}$ converges. Hence both given continued fraction representations of $\exp x$ converge for the given $x \in \mathbb{C}$. Since the choice of x is arbitrary, it follows that both continued fractions converge on all of $\mathbb{C}$.

3.3. Constructing differential equations satisfied by a given expression

As the algorithm presented in the previous section requires knowledge of an explicit differential equation satisfied by the given analytic expression, an algorithm to construct such a differential equation from an expression is desirable. The algorithm presented in the following is a generalisation of the algorithm `FindDE` contained in the *Maple* package `FPS.mpl` [GMK], which tries to find a linear differential equation with rational coefficients satisfied by a given expression. The algorithm can be outlined as follows:
Starting with some expression $f(x)$ and upper bounds $o, d \in \mathbb{N}$ for the order and degree of the desired differential equation respectively, compute the derivatives $f', \ldots, f^{(o)}$. Let

$$0 = \mathcal{D}Y := Y^{(o)} + \sum_{|\alpha|=0}^{d} c_\alpha \prod_{i=0}^{o-1} \left(Y^{(i)}\right)^{\alpha_i},$$

where $\alpha = (\alpha_0, \ldots, \alpha_{o-1})$ with $\alpha_i \geq 0$ for all i and $|\alpha| = \sum_{i=0}^{o-1} \alpha_i$. Substitute $Y^{(i)} = f^{(i)}$ for all i, expand the result and then collect those terms that are rational multiples of each other. Setting each of these grouped summands to 0 leads to a set of equations that can be solved for the coefficients c_α. If no solution can be found, the algorithm fails under the given restrictions, but increasing o, d or both may yield a positive result. Otherwise $0 = \mathcal{D}Y$ is an explicit differential equation with initial conditions $Y^{(i)}(0) = f^{(i)}(0)$ for $i = 0, \ldots, o-1$ satisfied by f. Since in general coefficients that are rational in x are preferable, check the c_α for rationality in x and reject the result in the case of irrationality.

Example 3.3.1. Starting from the expression $\tan x$ the well known differential equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}Y(x) - Y(x)^2 - 1,\ Y(0) = 0$$

is obtained as follows:
Set $o = 1$, $d = 2$ and

$$0 = \mathcal{D}Y := \frac{\mathrm{d}}{\mathrm{d}x}Y(x) + c_2 Y(x)^2 + c_1 Y(x) + c_0.$$

Substituting $\tan x$ for $Y(x)$ and expanding results in

$$0 = (1 + \tan(x)^2) + c_2 \tan(x)^2 + c_1 \tan(x) + c_0 = (1 + c_2)\tan(x)^2 + c_1 \tan(x) + (1 + c_0).$$

Setting each summand equal to 0 leads to a system of equations

$$\begin{aligned} 0 &= (1 + c_2)\tan(x)^2 \\ 0 &= c_1 \tan(x) \\ 0 &= 1 + c_0 \end{aligned}$$

with the solution $c_0 = c_2 = -1$ and $c_1 = 0$, yielding the differential equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}Y(x) - Y(x)^2 - 1$$

with initial condition $Y(0) = \tan(0) = 0$.

In practical applications, a higher order and degree usually increase computational costs in relation to the differential equation. To ensure that the resulting order and degree are reasonably low, start by setting $o = d = 1$ and increasing them iteratively up to some upper bounds O and D, until a differential equation is found. For the purposes of this thesis, the implementation of this algorithm prioritizes a small order over a small degree. It should be mentioned that the result of this algorithm for special functions depends on the representation of the derivatives $f^{(i)}(x)$ of the given expression f. If the derivatives are not represented in terms of the original expression f, the algorithm will usually result in differential equations of higher order and degree than necessary or fail altogether. A prime example for this using the accompanying implementation of this algorithm are the various types of generalized hypergeometric functions. Applying the process presented here to some ${}_pF_q$ will usually result in a recurrence that is not (yet) automatically recognized as true by *Maple* and as such the algorithm fails, even though a differential equation under the given restrictions may exist. To counteract this, one would have to extend the algorithm with specific suitable derivative rules to support each type of special function one is intereseted in. This problem is also discussed by Gruntz und Koepf in [GK95, p. 4] in the context of a *Maple* algorithm to find linear differential equations and the Airy wave function.

3.3.1. Further results

While for the purposes of the guess and prove algorithm mainly explicit differential equations are of interest, the algorithm to search differential equations is easily modified to

```
Input  : An expression f(x), a name Y, natural numbers O and D
Output: An explicit differential equation 0 = 𝒟Y satisfied by f of order at most O
        and degree at most D with coefficients rational in x or FAIL if none is
        found

F(0) ← f
for o = 1, ..., O do
    F(o) ← f^(o)(x)
    for d = 1, ..., D do
        mon ← list of monomials in F(0), ..., F(o − 1) of degree at most d
        deq ← F(o) + Σ_{i=1}^{|mon|} c_i · mon(i)
        deq ← substitute F(i) = f^(i)(x) for i = 0, ..., o in deq and expand the result
        deq ← collect summands that are rational multiples of each other in x
        terms ← list of summands of deq
        if the system of equations (terms(1) = 0, ..., terms(|terms|) = 0) in the
         unknowns c_1, ..., c_|mon| has a solution rational in x then
            deq ← F(o) + Σ_{i=1}^{|mon|} c_i · mon(i)
            deq ← substitute F(i) = Y^(i)(x) in deq
            return deq
        end
    end
end
return FAIL
```

Algorithm 6: searchODE

allow for implicit differential equations as well by considering

$$0 = \mathcal{D}Y := \sum_{|\alpha|=0}^{d} c_\alpha \prod_{i=0}^{o} \left(Y^{(i)}\right)^{\alpha_i},$$

instead, where $\alpha = (\alpha_0, \dots, \alpha_o)$ with $\alpha_i \geq 0$ for all i and $|\alpha| = \sum_{i=0}^{o} \alpha_i$.
A neat side effect of allowing implicit differential equations is that for many expressions involving n-th powers of elementary functions, it allows to find simple forms of differential equations.

For example, for both $\sin(x)^n$ and $\cos(x)^n$ one finds the differential equation

$$0 = n\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}Y(x)\right)Y(x) - (n-1)\left(\frac{\mathrm{d}}{\mathrm{d}x}Y(x)\right)^2 + n^2 Y(x)^2.$$

For both their sum $\sin(x)^n + \cos(x)^n$ and difference $\sin(x)^n - \cos(x)^n$ one instead obtains

$$\begin{aligned}0 =&(n-1)\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}Y(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}Y(x)\right) - (2-n)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}Y(x)\right)^2\\ &+ (n^3-5n^2+6n)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}Y(x)\right)Y(x) - (n^3-7n^2+10n-4)\left(\frac{\mathrm{d}}{\mathrm{d}x}Y(x)\right)^2\\ &+ (n^4-4n^3+4n^2)Y(x)^2.\end{aligned}$$

Taking their quotients, for both $\tan(x)^n$ and $\cot(x)^n$ one gets

$$\begin{aligned}0 =& - n^2\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}Y(x)\right)^2 Y(x)^2 + 2n^2\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}Y(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}Y(x)\right)^2 Y(x) - (n^2-1)\left(\frac{\mathrm{d}}{\mathrm{d}x}Y(x)\right)^4\\ &- 4n^2\left(\frac{\mathrm{d}}{\mathrm{d}x}Y(x)\right)^2 Y(x)^2.\end{aligned}$$

For $\sec x$ one obtains

$$0 = n\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}Y(x)\right)Y(x) - (n+1)\left(\frac{\mathrm{d}}{\mathrm{d}x}Y(x)\right)^2 - n^2Y(x)^2.$$

Considering the generating functions of the Bernoulli and Euler polynomials, one finds

$$\begin{aligned}0 =&nx\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}Y(x)\right) - x(n+1)\left(\frac{\mathrm{d}}{\mathrm{d}x}Y(x)\right)^2 + n(2tx-x+2)\left(\frac{\mathrm{d}}{\mathrm{d}x}Y(x)\right)Y(x)\\ &-n^2\left(t^2x-tx+2t-1\right)Y(x)^2\end{aligned}$$

for $\left(\frac{x\exp(xt)}{\exp(x)-1}\right)^n$ and

$$\begin{aligned}0 =&n\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}Y(x)\right)Y(x) - (n+1)\left(\frac{\mathrm{d}}{\mathrm{d}x}Y(x)\right)^2 + n(2t-1)\left(\frac{\mathrm{d}}{\mathrm{d}x}Y(x)\right)Y(x)\\ &- n^2t(t-1)Y(x)^2\end{aligned}$$

for $\left(\frac{2\exp(xt)}{\exp(x)+1}\right)^n$.

Similar results can be produced for the inverse trigonometric functions, hyperbolic functions, inverse hyperbolic functions, and many more cases.

3.4. Conclusion

It should be mentioned that the guess and prove algorithm only results in a formal identity and makes no statement with regard to questions of convergence. Having obtained a formal continued fraction representation, to answer questions regarding its convergence one has to fall back on other means, namely convergence criteria like Worpitzky's Theorem (Theorem 2.1.14).

Furthermore it would be remiss not to mention that even though theoretically the guess and prove algorithm as presented here has the capability to handle cases with differential equations of order higher than one, the author has yet to find a working example of such a case. Just experimenting with cases where the differential equation is of order 2, a common occurence is that these differential equations are typically singular in $x = 0$.
For example trying to retrieve the continued fraction representation given in [CBV⁺08, p. 206] of $\arccos x$

$$\arccos x = \frac{x\sqrt{1-x^2}}{1} + \operatorname*{K}_{n=2}^{\infty} \frac{-a_n(1-x^2)}{1}, \; a_{2n} = \frac{2n(2n-1)}{(4n-1)(4n-3)}, \; a_{2n+1} = \frac{2n(2n-1)}{(4n+1)(4n-1)},$$

one substitutes $x = \sqrt{1-z^2}$ and considers the expression

$$\frac{z}{\frac{\arccos(\sqrt{1-z^2})}{\sqrt{1-z^2}}},$$

but this yields the second order differential equation

$$\begin{aligned} 0 = \mathcal{D}Y := &\left(x^6 - 2x^4 + x^2\right) \frac{\mathrm{d}^2}{\mathrm{d}x^2} Y(x) - \left(3x^5 - 5x^3 + 2x\right) \frac{\mathrm{d}}{\mathrm{d}x} Y(x) - 2Y(x)^3 \\ &+ \left(4x^4 - 4x^2 - 4\right) Y(x), \end{aligned}$$

which is singular in $x = 0$. Unlike the similiar cases in the previous sections, in this particular case looking for a right factor of H_{2n} does not succeed as well, despite the fact the initial values $H_0, H_2, \dots, H_{14}$ exhibit the desired increase in valuation, following the formula $\operatorname{val} H_{2n} = 2n + 2$. It is important to note that this does not disprove the existence of a right factor in general, but only in the scope of reasonable parameters.
Another example is the still conjectural continued fraction identity

$$\frac{\sin(x)}{\cos(x)-1} = -\frac{2}{x} + \frac{x/6|}{|1} + \operatorname*{K}_{n=2}^{\infty} \frac{a_n}{1}, \; a_n = -\frac{x^2}{4(2n-1)(2n+1)}.$$

Considering the expression $-\frac{x}{2}\frac{\sin x}{\cos(x)-1} - 1$ leads to the second order differential equation

$$\begin{aligned} 0 = \mathcal{D}Y := &x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} Y(x) + 4x \frac{\mathrm{d}}{\mathrm{d}x} Y(x) + 6x \left(\frac{\mathrm{d}}{\mathrm{d}x} Y(x)\right) Y(x) \\ &+ 4Y(x)^3 + 6Y(x)^2 + \left(x^2 + 2\right) Y(x) + x^2, \end{aligned}$$

which is singular in $x = 0$. As in the case before, searching for a right factor of H_n proves not successful, although the initial values $H_0, H_1, \dots, H_7$ exhibit the increase in valuation, following the formula $\operatorname{val} H_n = 2n + 2$. A very similar case with the same limitations is $\frac{\sinh(x)}{\cosh(x)-1}$.

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A. The Maple-package `guessandprove.mpl`

The algorithms presented in Chapter 3 were implemented in the package `guessandprove.mpl` using *Maple 18*. It depends on the built-in packages `NumberTheory` and `LREtools` as well as the external package `hsum17.mpl` by Koepf and Sprenger [KS]. This appendix contains information regarding the concrete implementation and instructions for using the package.
To start using the package, first load the package `hsum17.mpl`, followed by `guessanprove.mpl`, with the `read` command.

A.1. `searchODE`

Algorithm 6, `searchODE`, is based on the algorithm `FindDE` from the package `FPS.mpl` [GMK], which searches for linear ordinary differential equations satisfied by a given expression. The procedure `searchODE` takes the arguments `expr`, an analytical expression for which a differential equation is desired, and `func`, a name of the form `Y(x)`. Furthermore, `searchODE` can take additional optional arguments with their respective default values. First is `params = alpha`, a name used in the case that the differential equation contains additional parameters, which can occur when looking for implicit differential equations; if they occur, they are called `params(1)`, `params(2)`, and so on. The second and third optional arguments are `maxOrder = 4` and `maxDegree = 4`, defining upper bounds for the polynomial degree and derivative order in Y of the sought differential equation to ensure termination of the algorithm. The final optional parameter ist `explicit = true`, a binary flag setting wether the algorithm searches for an explicit or implicit differential equation. If a differential equation in $Y(x)$ satisfied by the given expression is found, the algorithm returns the whole equation in such a form that its right-hand side is zero.
As an example, to obtain the explicit differential equations satisfied by $\tan x$ and $\exp(x)$ as in Example 3.2.1 and Example 3.2.2, call

$deq := searchODE\left(\tan(x), Y(x)\right)$

$$deq := \frac{\mathrm{d}}{\mathrm{d}x} Y(x) - 1 - (Y(x))^2 = 0$$

and

$deq2 := searchODE\left(\mathrm{e}^x, Y(x)\right)$

$$deq2 := \frac{\mathrm{d}}{\mathrm{d}x} Y(x) - Y(x) = 0$$

respectively.
Note that `searchODE` prioritizes a small order over a small degree and to that end passes through the whole range of possible degrees each time the sought order is increased. In practical terms this means that depending on the values of `maxOrder` and `maxDegree` the result might be a differential equation of for example order 1 and degree 100, even though there exists another differential equation of order and degree 2.

Since Riccati differential equations, that is differential equations of first order and second degree, are ubiquitous in practical applications, the package also contains a procedure `searchRiccatiDE` for ease of use, which takes the arguments `expr` and `func`, as well as the optional argument `params = alpha`, which work just like the identically named arguments of `searchODE`.

A.2. guessCfracFromExpr

Algorithm 3, `guessCfracFromExpr`, takes the arguments `expr` and `partnum`. Here, `expr` is the expression from which a continued fraction is to be constructed and `partnum` is a name of the form `a(n,x)`, where `a` is the name of the partial numerators, `n` is the index and `x` is the indeterminate of `expr`. In addition, `guessCfracFromExpr` takes the optional arguments `lbound = 20`, `ubound = 50`, `stepsize = 10`, which are respectively lower bound, upper bound and stepsize when iterating over the order of the partial sum on which the guess is based, as well as `symlbound = 1` and `symubound = 4`, which are respectively the lower and upper bound for the parameter l, such that only l-fold symmetries of the partial numerators are admissible.
The result is returned in the form `[b,inits,pnums]`, where `b` is the value of `expr` at $x = 0$, `pnums` is a list of general formulas for the partial numerators of length l, such that the i-th entry of `pnums` corresponds to a_n^i and `inits` is the list of initial partial numerators $a_1, \ldots, a_N$, where N is the largest index, such that $a_N^{N \bmod l} \neq a_N$.
The algorithm uses the procedures `Term` and `ContinuedFraction` from the built-in package `NumberTheory` to construct finite continued fractions from the partial sums of `expr`, and the procedure `RationalInterpolation` to guess the general formula(s).
If `infolevel[guessandprove]` is set to at least 4, the algorithm additionally prints the initial terms of the series on which the guess is based, as well as the corresponding finite C-fraction.
To continue with the examples of $\tan x$ and $\exp(x)$, the corresponding call to guess the continued fraction from Example 3.2.1 would look like

$$pnum := guessCfracFromExpr\left(\tan(x), a(k,x), lbound = 3, stepsize = 1\right)$$

guessCfracFromExpr: guess based on initial series terms

$$x + 1/3\,x^3 + 2/15\,x^5 + \frac{17\,x^7}{315} + \frac{62\,x^9}{2835} + \frac{1382\,x^{11}}{155925} + \frac{21844\,x^{13}}{6081075} + O\left(x^{15}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0, [x, 1], [-1/3\,x^2, 1], [-1/15\,x^2, 1], [-1/35\,x^2, 1], [-\frac{x^2}{63}, 1], [-\frac{x^2}{99}, 1], [-\frac{x^2}{143}, 1]]$$

$$pnum := [0, [x], [-\frac{x^2}{(2\,k-1)\,(2\,k-3)}]]$$

and for the continued fraction from Example 3.2.2 the call is

$$pnum2 := guessCfracFromExpr\left(e^x, a(k,x), lbound = 3, stepsize = 1\right)$$

guessCfracFromExpr: guess based on initial series terms

$$1 + x + 1/2\,x^2 + 1/6\,x^3 + 1/24\,x^4 + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + O\left(x^{10}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[1, [x, 1], [-x/2, 1], [x/6, 1], [-x/6, 1], [x/10, 1], [-x/10, 1], [x/14, 1], [-x/14, 1], [x/18, 1]]$$

$$pnum2 := [1,[x],[\frac{x}{4\,k-2},-\frac{x}{4\,k-2}]]$$

A.3. `searchCorrRec`

Algorithm 4, `searchCorrRec`, takes the arguments `deq`, `func`, `recname`, `partnum` and `deqOrder`, where `deq` is an explicit ordinary differential equation of order `deqOrder` formatted as in the output of `searchODE`, `func` and `recname` are names of the form `Y(x)` and `H(n)`, corresponding to $Y(x)$ and H_n respectively, and `partnum` is a continued fraction expressed and formatted as in the output of `guessCfracFromExpr`.
The result is returned in the form `[inits,rec]`, where `rec` is the sought holonomic recurrence and `inits` is the list of initial values of `rec`.

Considering again the examples of $\tan x$ and $\exp(x)$, with `deq`, `deq2`, `pnum` and `pnum2` carried over from the preceding two subsections, the corresponding calls are

$$rec := searchCorrRec\,(deq, Y\,(x)\,, H\,(k)\,, pnum, 1)$$

$$rec := [[H\,(0) = -1, H\,(1) = -x^2, H\,(2) = -1/9\,x^4, H\,(3) = -\frac{x^6}{225}], H\,(k+4) = H\,(k+3) -2\,\frac{x^2\,(4\,k^2-x^2+20\,k+21)\,H\,(k+2)}{(2\,k+5)\,(2\,k+3)\,(2\,k+7)^2} + \frac{x^4H\,(k+1)}{(2\,k+7)\,(2\,k+3)\,(2\,k+5)^2} - \frac{x^8H\,(k)}{(2\,k+7)\,(2\,k+5)^2\,(2\,k+1)^2\,(2\,k+3)^3}]$$

and

$$rec2 := searchCorrRec\,(deq2, Y\,(x)\,, H\,(k)\,, pnum2, 1)$$

$$rec2 := [[H\,(0) = -1, H\,(2) = \frac{x^2}{4}, H\,(4) = -\frac{x^4}{144}, H\,(6) = \frac{x^6}{14400}], H\,(2\,k+8) = H\,(2\,k+6) + \frac{x^2\,(16\,k^2+x^2+80\,k+84)\,H\,(2\,k+4)}{8\,(2\,k+5)\,(2\,k+3)\,(2\,k+7)^2} + \frac{x^4H\,(2\,k+2)}{16\,(2\,k+7)\,(2\,k+3)\,(2\,k+5)^2} - \frac{x^8H\,(2\,k)}{256\,(2\,k+7)\,(2\,k+5)^2\,(2\,k+1)^2\,(2\,k+3)^3}]$$

respectively.

This algorithm can also be used to search for general corresponding recurrences by giving `partnum` in the form `[0,[],a(n,x)]`, where `a(n,x)` is a name corresponding to $a_n(x)$. For example, to obtain the general form of recurrences corresponding to Riccati differential equations already seen in Example 3.2.2, one would call

$$riccdeq := \frac{\mathrm{d}}{\mathrm{d}x}Y\,(x) + f\,(x)\,(Y\,(x))^2 + g\,(x)\,Y\,(x) + h\,(x) = 0$$

$$searchCorrRec\,(riccdeq, Y\,(x)\,, H\,(k)\,, [0,[],[a\,(k,x)]], 1)$$

$$[[H(0) = h(x), H(1) = f(x)(a(1,x))^2 + g(x)a(1,x) + h(x) + D_2(a)(1,x),$$
$$H(2) = f(x)(a(1,x))^2 + g(x)a(1,x)a(2,x) + h(x)(a(2,x))^2 + g(x)a(1,x)$$
$$+ 2h(x)a(2,x) - a(1,x)D_2(a)(2,x) + D_2(a)(1,x)a(2,x)$$
$$+ h(x) + D_2(a)(1,x),$$
$$H(3) = (a(3,x))^2 f(x)(a(1,x))^2 + (a(3,x))^2 g(x)a(1,x)$$
$$+ 2a(3,x)f(x)(a(1,x))^2 + a(3,x)g(x)a(1,x)a(2,x) + (a(3,x))^2 h(x)$$
$$+ (a(3,x))^2 D_2(a)(1,x) + 2a(3,x)g(x)a(1,x) + 2a(3,x)h(x)a(2,x)$$
$$- a(3,x)a(1,x)D_2(a)(2,x) + a(3,x)D_2(a)(1,x)a(2,x)$$
$$+ D_2(a)(3,x)a(1,x)a(2,x) + f(x)(a(1,x))^2 + g(x)a(1,x)a(2,x)$$
$$+ h(x)(a(2,x))^2 + 2a(3,x)h(x) + 2a(3,x)D_2(a)(1,x) + g(x)a(1,x)$$
$$+ 2h(x)a(2,x) - a(1,x)D_2(a)(2,x) + D_2(a)(1,x)a(2,x)$$
$$+ h(x) + D_2(a)(1,x)],$$
$$H(k+4) = -\frac{(a(k+3,x)D_2(a)(k+4,x) - D_2(a)(k+3,x)a(k+4,x) - D_2(a)(k+3,x))H(k+3)}{D_2(a)(k+3,x)}$$
$$+ \frac{\left((a(k+3,x))^2 D_2(a)(k+4,x) + (a(k+4,x))^2 D_2(a)(k+3,x) + a(k+3,x)D_2(a)(k+4,x) + D_2(a)(k+3,x)a(k+4,x)\right)H(k+2)}{D_2(a)(k+3,x)}$$
$$+ \frac{(a(k+3,x))^2(a(k+3,x)D_2(a)(k+4,x) - D_2(a)(k+3,x)a(k+4,x) + D_2(a)(k+4,x))H(k+1)}{D_2(a)(k+3,x)}$$
$$- \frac{(a(k+2,x))^2(a(k+3,x))^2 D_2(a)(k+4,x)H(k)}{D_2(a)(k+3,x)}]$$

Interestingly, except for the initial values this recurrence depends only on the partial numerators, not on the coefficient functions of the differential equation. Unfortunately the size of these general corresponding recurrences increases very swiftly; for differential equations of order 1 and degree 3 the corresponding recurrence already spans multiple pages, despite only increasing the degree of the differential equation by 1.

A.4. `checkValIncrease`

Algorithm 5, `checkValIncrease`, takes the arguments `rec`, `recname`, `inits` and `indet`, where `rec` is a holonomic recurrence formatted as in the output of `searchCorrRec`, `recname` is a name of the form `H(n)` corresponding to H_n, `inits` is the list of initial values of `rec` formatted as in the output of `searchCorrRec` and `indet` is the name of the indeterminate. In addition `checkValIncrease` accepts the optional argument `symbound = 4`, giving an upper bound for the parameter l when looking for hypergeometric term solutions of subrecurrences H_{ln+m}, $0 \leq m < l$ to ensure that the algorithm terminates.
To search for hypergeometric term solutions, this algorithm uses the van Hoeij algorithm, in *Maple 18* implemented in the built-in package `LREtools` under the name `hypergeomsols`.
To compute the ratio of the solution to check for the increase in valuation, the algorithm uses the procedure `ratio` from the package `hsum17.mpl` by Koepf and Sprenger [KS].
If the check was successful, the algorithm returns the term ratio of the corresponding right factor of the given recurrence in the form `H(l(n+1)+m) = r H(ln+m)`, where `r` is the ratio corresponding to the found hypergeometric term solution; should the check not have been successful, the algorithm returns `FAIL`.
If `infolevel[guessandprove]` is set to at least 5, the algorithm additionally prints the

hypergeometric term solution.
To finish with the examples of $\tan x$ and $\exp(x)$, the corresponding calls are

$$checkValIncrease\left(op\left(2, rec\right), H\left(k\right), op\left(1, rec\right), x\right)$$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$-\frac{\pi\left(1/4\,x^2\right)^k}{\left(\Gamma\left(k+1/2\right)\right)^2}$$

$$H\left(k+1\right)=\frac{x^2H\left(k\right)}{\left(2\,k+1\right)^2}$$

and

$$checkValIncrease\left(op\left(2, rec2\right), H\left(k\right), op\left(1, rec2\right), x\right)$$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$-\frac{\pi\left(-1/16\,x^2\right)^k}{\left(\Gamma\left(k+1/2\right)\right)^2}$$

$$H\left(k+1\right)=-1/4\,\frac{x^2H\left(k\right)}{\left(2\,k+1\right)^2}$$

A.5. gapCfrac

This is a simple wrapper function for the preceding algorithms. It takes the argument `expr`, an expression a C-fraction expansion is desired of. The optional arguments `funcname = Y(x)`, `pnumname = a(k,x)` and `recname = H(k)` name the variables used in the process. The optional arguments `paramname`, `maxDiffOrder` and `maxPolDegree` correspond in function in default value to the optional arguments `params`, `maxOrder` and `maxDegree` of `searchODE` respectively. The optional arguments `serieslbound`, `seriesubound`, `seriesstepsize`, `pnumsymlbound`, `pnumsymubound` correspond in function and default value to the optional arguments `lbound`, `ubound`, `stepsize`, `symlbound`, `symubound` of `guessCfracFromExpr`. The optional argument `recsymbound` correspond in function and from to the optional argument `symbound` of `checkValIncrease`.
In case of success, this algorithm returns the now proven C-fraction expansion of the given expression as computed by the subalgorithm `guessCfracFromExpr`, otherwise an error message will signify the point of failure.
If `infolevel[guessandprove]` is set to at least 3, the algorithm additionally prints the results of the preceding subalgorithms during computation.

A.6. Examples from Chapter 3 in *Maple 18*

For all of the following calls, `infolevel[guessandprove]` was set to 5. In the case of 2-fold symmetries of the partial numerators, recurrence equations for H_{2n} are rewritten as recurrences for H_n by substituting H with F, where $F(k) := H(k/2)$, and evaluating the result. This is necessary so that van Hoeij's algorithm `hypergeomsols` can be applied to the recurrence equations in question. The output of the substitution calls has been omitted. Some (parts of) outputs have been omitted if they are too large to be reasonably readable.

$infolevel_{guessandprove} := 5$

$$infolevel_{guessandprove} := 5$$

$F(k) := H(k/2)$

$$F := k \to H(k/2)$$

A.6.1. Examples from Section 3.2.1

exp(x)
A continued fraction representation of exp(x) is obtained by rearranging the result for 2*x/(exp(x)-1) - (2-x).
Problem: The differential equation is singular in x=0, so uniqueness of the power series solution is not assured.

$deq3 := searchODE\left(\frac{2\cdot x}{e^x-1} + (x-2), Y(x)\right)$

$$deq3 := 2\left(\frac{d}{dx}Y(x)\right)x + (Y(x))^2 - x^2 + 2Y(x) = 0$$

$pnum3 := guessCfracFromExpr\left(\frac{2\cdot x}{e^x-1} + (x-2), a(k,x), lbound = 4, stepsize = 2\right)$

guessCfracFromExpr: guess based on initial series terms

$$1/6\,x^2 - \frac{x^4}{360} + \frac{x^6}{15120} - \frac{x^8}{604800} + \frac{x^{10}}{23950080} - \frac{691\,x^{12}}{653837184000} + \frac{x^{14}}{37362124800} + O\left(x^{15}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0, [1/6\,x^2, 1], [\frac{x^2}{60}, 1], [\frac{x^2}{140}, 1], [\frac{x^2}{252}, 1], [\frac{x^2}{396}, 1], [\frac{x^2}{572}, 1], [\frac{x^2}{780}, 1]]$$

$$pnum3 := [0, [1/6\,x^2], [1/4\,\frac{x^2}{(2k-1)(2k+1)}]]$$

$rec3 := searchCorrRec(deq3, Y(x), H(k), pnum3, 1)$

$$rec3 := [[H(0) = -x^2, H(1) = 1/36\,x^4, H(2) = -\frac{x^6}{3600}, H(3) = \frac{x^8}{705600}], H(k+4) = H(k+3) + 1/8\,\frac{x^2\left(16k^2 + x^2 + 112k + 180\right)H(k+2)}{(2k+7)(2k+5)(2k+9)^2} + 1/16\,\frac{x^4 H(k+1)}{(2k+9)(2k+5)(2k+7)^2} - \frac{x^8 H(k)}{(512k+2304)(2k+7)^2(2k+3)^2(2k+5)^3}]$$

$checkValIncrease(op(2, rec3), H(k), op(1, rec3), x)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$-1/4\,\frac{x^2\pi\left(-1/16\,x^2\right)^k}{(\Gamma(k+3/2))^2}$$

$$H(k+1) = -1/4\,\frac{x^2 H(k)}{(2k+3)^2}$$

ln(1+x)

$deq4 := searchODE(\ln(1+x), Y(x))$

$$deq4 := -1 + (1+x)\frac{d}{dx}Y(x) = 0$$

$pnum4 := guessCfracFromExpr\left(\ln(1+x), a(k,x), lbound = 3, stepsize = 1\right)$

guessCfracFromExpr: guess based on initial series terms

$$x - 1/2\,x^2 + 1/3\,x^3 - 1/4\,x^4 + 1/5\,x^5 - 1/6\,x^6 + 1/7\,x^7 - 1/8\,x^8 + 1/9\,x^9 - 1/10\,x^{10} + 1/11\,x^{11} + O\left(x^{12}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0, [x, 1], [x/2, 1], [x/6, 1], [x/3, 1], [x/5, 1], [3/10\,x, 1], [3/14\,x, 1], [2/7\,x, 1], [2/9\,x, 1], [\frac{5\,x}{18}, 1], [\frac{5\,x}{22}, 1]]$$

$$pnum4 := [0, [x], [\frac{x\,(k-1)}{4\,k-2}, \frac{xk}{4\,k-2}]]$$

$rec4 := searchCorrRec\left(deq4, Y(x), H(k), pnum4, 1\right)$

$$rec4 := [[H(0) = -1, H(2) = -1/4\,x^2, H(4) = -1/36\,x^4, H(6) = -\frac{x^6}{400}], H(2\,k+8) = 1/4\,(x+2)^2\,H(2\,k+6) - 1/16\,\frac{x^2(k+3)^2\left(6\,k^2x^2+32\,xk^2+30\,kx^2+32\,k^2+160\,xk+31\,x^2+160\,k+168\,x+168\right)H(2\,k+4)}{(2\,k+5)(2\,k+3)(2\,k+7)^2} + \frac{x^4(x+2)^2(k+3)^2(k+2)^2H(2\,k+2)}{(128\,k+448)(2\,k+3)(2\,k+5)^2} - \frac{x^8(k+3)^2(k+2)^2(k+1)^4H(2\,k)}{(512\,k+1792)(2\,k+5)^2(2\,k+1)^2(2\,k+3)^3}]$$

$rec4 := eval\left(subs\left(H = F, rec4\right)\right)$

$checkValIncrease\left(op(2, rec4), H(k), op(1, rec4), x\right)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$-\frac{\pi\,\left(1/16\,x^2\right)^k\,(\Gamma(k+1))^2}{(\Gamma(k+1/2))^2}$$

$$H(k+1) = 1/4\,\frac{x^2\,(k+1)^2\,H(k)}{(2\,k+1)^2}$$

ln((1+x)/(1-x))

$deq5 := searchODE\left(\ln\left(\frac{1+x}{1-x}\right), Y(x)\right)$

$$deq5 := 2 + \left(x^2-1\right)\frac{\mathrm{d}}{\mathrm{d}x}Y(x) = 0$$

$pnum5 := guessCfracFromExpr\left(\ln\left(\frac{1+x}{1-x}\right), a(k,x), lbound = 3, stepsize = 1\right)$

guessCfracFromExpr: guess based on initial series terms

$$2\,x + 2/3\,x^3 + 2/5\,x^5 + 2/7\,x^7 + 2/9\,x^9 + 2/11\,x^{11} + 2/13\,x^{13} + 2/15\,x^{15} + O\left(x^{17}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0, [2\,x, 1], [-1/3\,x^2, 1], [-\frac{4\,x^2}{15}, 1], [-\frac{9\,x^2}{35}, 1], [-\frac{16\,x^2}{63}, 1], [-\frac{25\,x^2}{99}, 1], [-\frac{36\,x^2}{143}, 1], [-\frac{49\,x^2}{195}, 1]]$$

$$pnum5 := [0, [2\,x], [-\frac{x^2\,(k-1)^2}{(2\,k-1)\,(2\,k-3)}]]$$

$rec5 := searchCorrRec\left(deq5, Y(x), H(k), pnum5, 1\right)$

$$rec5 := [[H(0)=2, H(1)=2x^2, H(2)=\frac{8x^4}{9}, H(3)=\frac{8x^6}{25}], H(k+4)=H(k+3) + \frac{x^2(k+3)^2(2k^2x^2+10kx^2-8k^2+11x^2-40k-42)H(k+2)}{(2k+5)(2k+3)(2k+7)^2} + \frac{(k+3)^2(k+2)^2x^4H(k+1)}{(2k+7)(2k+3)(2k+5)^2} - \frac{x^8(k+3)^2(k+2)^2(k+1)^4H(k)}{(2k+7)(2k+5)^2(2k+1)^2(2k+3)^3}]$$

$checkValIncrease(op(2, rec5), H(k), op(1, rec5), x)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$2\frac{\pi\left(1/4x^2\right)^k(\Gamma(k+1))^2}{(\Gamma(k+1/2))^2}$$

$$H(k+1)=\frac{x^2(k+1)^2H(k)}{(2k+1)^2}$$

A.6.2. Examples from Section 3.2.2

arctan(x)

$deq6 := searchODE(\arctan(x), Y(x))$

$$deq6 := -1+\left(x^2+1\right)\frac{\mathrm{d}}{\mathrm{d}x}Y(x)=0$$

$pnum6 := guessCfracFromExpr(\arctan(x), a(k,x), lbound=3, stepsize=1)$

guessCfracFromExpr: guess based on initial series terms

$$x-1/3x^3+1/5x^5-1/7x^7+1/9x^9-1/11x^{11}+1/13x^{13}-1/15x^{15}+O\left(x^{17}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0,[x,1],[1/3x^2,1],[\frac{4x^2}{15},1],[\frac{9x^2}{35},1],[\frac{16x^2}{63},1],[\frac{25x^2}{99},1],[\frac{36x^2}{143},1],[\frac{49x^2}{195},1]]$$

$$pnum6 := [0,[x],[\frac{x^2(k-1)^2}{(2k-1)(2k-3)}]]$$

$rec6 := searchCorrRec(deq6, Y(x), H(k), pnum6, 1)$

$$rec6 := [[H(0)=-1, H(1)=x^2, H(2)=-4/9x^4, H(3)=\frac{4x^6}{25}], H(k+4)=H(k+3) + \frac{x^2(k+3)^2(2k^2x^2+10kx^2+8k^2+11x^2+40k+42)H(k+2)}{(2k+5)(2k+3)(2k+7)^2} + \frac{(k+3)^2(k+2)^2x^4H(k+1)}{(2k+7)(2k+3)(2k+5)^2} - \frac{x^8(k+3)^2(k+2)^2(k+1)^4H(k)}{(2k+7)(2k+5)^2(2k+1)^2(2k+3)^3}]$$

$checkValIncrease(op(2, rec6), H(k), op(1, rec6), x)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$-\frac{\pi\left(-1/4x^2\right)^k(\Gamma(k+1))^2}{(\Gamma(k+1/2))^2}$$

$$H(k+1) = -\frac{x^2(k+1)^2 H(k)}{(2k+1)^2}$$

Asinh(x)

$deq7 := searchODE\left(\frac{\arcsin(x)}{\sqrt{-x^2+1}}, Y(x)\right)$

$$deq7 := 1 + \left(x^2-1\right)\frac{d}{dx}Y(x) + xY(x) = 0$$

$pnum7 := guessCfracFromExpr\left(\frac{\arcsin(x)}{\sqrt{-x^2+1}}, a(k,x), lbound = 3, stepsize = 2\right)$

guessCfracFromExpr: guess based on initial series terms

$$x + 2/3\,x^3 + \frac{8\,x^5}{15} + \frac{16\,x^7}{35} + \frac{128\,x^9}{315} + \frac{256\,x^{11}}{693} + \frac{1024\,x^{13}}{3003} + \frac{2048\,x^{15}}{6435} + \frac{32768\,x^{17}}{109395} + \frac{65536\,x^{19}}{230945} + \frac{262144\,x^{21}}{969969} + \frac{524288\,x^{23}}{2028117} + \frac{4194304\,x^{25}}{16900975} + \frac{8388608\,x^{27}}{35102025} + \frac{33554432\,x^{29}}{145422675} + O\left(x^{31}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0,[x,1],[-2/3\,x^2,1],[-2/15\,x^2,1],[-\frac{12\,x^2}{35},1],[-\frac{4\,x^2}{21},1],[-\frac{10\,x^2}{33},1],[-\frac{30\,x^2}{143},1],[-\frac{56\,x^2}{195},1],[-\frac{56\,x^2}{255},1],[-\frac{90\,x^2}{323},1],[-\frac{30\,x^2}{133},1],[-\frac{44\,x^2}{161},1],[-\frac{132\,x^2}{575},1],[-\frac{182\,x^2}{675},1],[-\frac{182\,x^2}{783},1]]$$

$$pnum7 := [0,[x],[-2\,\frac{x^2(k-1)(2k-3)}{(4k-3)(4k-5)}, -2\,\frac{kx^2(2k-1)}{(4k-1)(4k-3)}]]$$

$rec7 := searchCorrRec(deq7, Y(x), H(k), pnum7, 1)$

$rec7 := eval(subs(H = F, rec7))$

$checkValIncrease(op(2, rec7), H(k), op(1, rec7), x)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$2\,\frac{\pi\,\left(1/16\,x^4\right)^k(\Gamma(k+1))^2(\Gamma(k+1/2))^2}{(\Gamma(k+1/4))^2(\Gamma(k+3/4))^2}$$

$$H(k+1) = 4\,\frac{x^4(k+1)^2(2k+1)^2 H(k)}{(4k+3)^2(4k+1)^2}$$

A.6.3. Examples from Section 3.2.3

tanh(x)

$deq8 := searchODE(\tanh(x), Y(x))$

$$deq8 := \frac{d}{dx}Y(x) - 1 + (Y(x))^2 = 0$$

$pnum8 := guessCfracFromExpr(\tanh(x), a(k,x), lbound = 3, stepsize = 1)$

guessCfracFromExpr: guess based on initial series terms

$$x - 1/3\,x^3 + 2/15\,x^5 - \frac{17\,x^7}{315} + \frac{62\,x^9}{2835} - \frac{1382\,x^{11}}{155925} + \frac{21844\,x^{13}}{6081075} + O\left(x^{15}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0, [x, 1], [1/3\,x^2, 1], [1/15\,x^2, 1], [1/35\,x^2, 1], [\frac{x^2}{63}, 1], [\frac{x^2}{99}, 1], [\frac{x^2}{143}, 1]]$$

$$pnum8 := [0, [x], [\frac{x^2}{(2\,k-1)\,(2\,k-3)}]]$$

$rec8 := searchCorrRec\,(deq8, Y\,(x)\,, H\,(k)\,, pnum8, 1)$

$$rec8 := [[H\,(0) = -1, H\,(1) = x^2, H\,(2) = -1/9\,x^4, H\,(3) = \frac{x^6}{225}], H\,(k+4) = H\,(k+3) + 2\,\frac{x^2\,(4\,k^2 + x^2 + 20\,k + 21)\,H\,(k+2)}{(2\,k+5)\,(2\,k+3)\,(2\,k+7)^2} + \frac{x^4 H\,(k+1)}{(2\,k+7)\,(2\,k+3)\,(2\,k+5)^2} - \frac{x^8 H\,(k)}{(2\,k+7)\,(2\,k+5)^2\,(2\,k+1)^2\,(2\,k+3)^3}]\}$$

$checkValIncrease\,(op\,(2, rec8)\,, H\,(k)\,, op\,(1, rec8)\,, x)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$-\frac{\pi\,\left(-1/4\,x^2\right)^k}{\left(\Gamma\,(k+1/2)\right)^2}$$

$$H\,(k+1) = -\frac{x^2 H\,(k)}{(2\,k+1)^2}$$

Asinh(x)

$deq9 := searchODE\left(\frac{\operatorname{arcsinh}(x)}{\sqrt{x^2+1}}, Y\,(x)\right)$

$$deq9 := -1 + \left(x^2+1\right)\frac{\mathrm{d}}{\mathrm{d}x}Y\,(x) + xY\,(x) = 0$$

$pnum9 := guessCfracFromExpr\left(\frac{\operatorname{arcsinh}(x)}{\sqrt{x^2+1}}, a\,(k, x)\,, lbound = 3, stepsize = 2\right)$

guessCfracFromExpr: guess based on initial series terms

$$x - 2/3\,x^3 + \frac{8\,x^5}{15} - \frac{16\,x^7}{35} + \frac{128\,x^9}{315} - \frac{256\,x^{11}}{693} + \frac{1024\,x^{13}}{3003} - \frac{2048\,x^{15}}{6435} + \frac{32768\,x^{17}}{109395} - \frac{65536\,x^{19}}{230945} + \frac{262144\,x^{21}}{969969} - \frac{524288\,x^{23}}{2028117} + \frac{4194304\,x^{25}}{16900975} - \frac{8388608\,x^{27}}{35102025} + \frac{33554432\,x^{29}}{145422675} + O\left(x^{31}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0, [x, 1], [2/3\,x^2, 1], [2/15\,x^2, 1], [\frac{12\,x^2}{35}, 1], [\frac{4\,x^2}{21}, 1], [\frac{10\,x^2}{33}, 1], [\frac{30\,x^2}{143}, 1], [\frac{56\,x^2}{195}, 1], [\frac{56\,x^2}{255}, 1], [\frac{90\,x^2}{323}, 1], [\frac{30\,x^2}{133}, 1], [\frac{44\,x^2}{161}, 1], [\frac{132\,x^2}{575}, 1], [\frac{182\,x^2}{675}, 1], [\frac{182\,x^2}{783}, 1]]$$

$$pnum9 := [0,[x],[2\,\frac{x^2\,(k-1)\,(2\,k-3)}{(4\,k-3)\,(4\,k-5)},2\,\frac{kx^2\,(2\,k-1)}{(4\,k-1)\,(4\,k-3)}]]$$

$rec9 := searchCorrRec\,(deq9, Y\,(x)\,, H\,(k)\,, pnum9, 1)$

$rec9 := eval\,(subs\,(H = F, rec9))$

$checkValIncrease\,(op\,(2, rec9)\,, H\,(k)\,, op\,(1, rec9)\,, x)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$-2\,\frac{\pi\,\left(1/16\,x^4\right)^k(\Gamma\,(k+1))^2\,(\Gamma\,(k+1/2))^2}{(\Gamma\,(k+1/4))^2\,(\Gamma\,(k+3/4))^2}$$

$$H\,(k+1) = 4\,\frac{x^4\,(k+1)^2\,(2\,k+1)^2\,H\,(k)}{(4\,k+3)^2\,(4\,k+1)^2}$$

Atanh(x)

$deq10 := searchODE\,(\mathrm{arctanh}\,(x)\,, Y\,(x)\,)$

$$deq10 := 1 + \left(x^2-1\right)\frac{\mathrm{d}}{\mathrm{d}x}Y\,(x) = 0$$

$pnum10 := guessCfracFromExpr\,(\mathrm{arctanh}\,(x)\,, a\,(k, x)\,, lbound = 3, stepsize = 1)$

guessCfracFromExpr: guess based on initial series terms

$$x + 1/3\,x^3 + 1/5\,x^5 + 1/7\,x^7 + 1/9\,x^9 + 1/11\,x^{11} + 1/13\,x^{13} + 1/15\,x^{15} + O\left(x^{17}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0,[x,1],[-1/3\,x^2,1],[-\frac{4\,x^2}{15},1],[-\frac{9\,x^2}{35},1],[-\frac{16\,x^2}{63},1],[-\frac{25\,x^2}{99},1],[-\frac{36\,x^2}{143},1],[-\frac{49\,x^2}{195},1]]$$

$$pnum10 := [0,[x],[-\frac{x^2\,(k-1)^2}{(2\,k-1)\,(2\,k-3)}]]$$

$rec10 := searchCorrRec\,(deq10, Y\,(x)\,, H\,(k)\,, pnum10, 1)$

$$rec10 := [[H\,(0) = 1, H\,(1) = x^2, H\,(2) = 4/9\,x^4, H\,(3) = \frac{4\,x^6}{25}], H\,(k+4) = H\,(k+3)$$
$$+\frac{x^2\,(k+3)^2\left(2\,k^2x^2 + 10\,kx^2 - 8\,k^2 + 11\,x^2 - 40\,k - 42\right)H\,(k+2)}{(2\,k+5)\,(2\,k+3)\,(2\,k+7)^2}$$
$$+\frac{(k+3)^2\,(k+2)^2\,x^4H\,(k+1)}{(2\,k+7)\,(2\,k+3)\,(2\,k+5)^2} - \frac{x^8\,(k+3)^2\,(k+2)^2\,(k+1)^4\,H\,(k)}{(2\,k+7)\,(2\,k+5)^2\,(2\,k+1)^2\,(2\,k+3)^3}]$$

$checkValIncrease\,(op\,(2, rec10)\,, H\,(k)\,, op\,(1, rec10)\,, x)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$\frac{\pi\,\left(1/4\,x^2\right)^k(\Gamma\,(k+1))^2}{(\Gamma\,(k+1/2))^2}$$

$$H\,(k+1) = \frac{x^2\,(k+1)^2\,H\,(k)}{(2\,k+1)^2}$$

A.6.4. Examples from Section 3.2.4

(1+x)^a

$deq11 := searchODE\,((1+x)^a\,, Y\,(x))$

$$deq11 := (1+x)\frac{d}{dx}Y(x) - aY(x) = 0$$

$pnum11 := guessCfracFromExpr\left((1+x)^a, a(k,x), lbound = 3, stepsize = 1\right)$

guessCfracFromExpr: guess based on initial series terms

$$1 + ax + \left(1/2\,a^2 - a/2\right)x^2 + \left(a/3 - 1/2\,a^2 + 1/6\,a^3\right)x^3$$
$$+\left(-a/4 + \tfrac{11\,a^2}{24} - 1/4\,a^3 + 1/24\,a^4\right)x^4$$
$$+\left(a/5 - \tfrac{5\,a^2}{12} + \tfrac{7\,a^3}{24} - 1/12\,a^4 + \tfrac{a^5}{120}\right)x^5$$
$$+\left(-a/6 + \tfrac{137\,a^2}{360} - \tfrac{5\,a^3}{16} + \tfrac{17\,a^4}{144} - 1/48\,a^5 + \tfrac{a^6}{720}\right)x^6$$
$$+\left(a/7 - \tfrac{7\,a^2}{20} + \tfrac{29\,a^3}{90} - \tfrac{7\,a^4}{48} + \tfrac{5\,a^5}{144} - \tfrac{a^6}{240} + \tfrac{a^7}{5040}\right)x^7$$
$$+\left(\tfrac{363\,a^2}{1120} - \tfrac{469\,a^3}{1440} + \tfrac{967\,a^4}{5760} - \tfrac{7\,a^5}{144} + \tfrac{23\,a^6}{2880} - \tfrac{a^7}{1440} + \tfrac{a^8}{40320} - a/8\right)x^8$$
$$+\left(-\tfrac{761\,a^2}{2520} + \tfrac{29531\,a^3}{90720} - \tfrac{89\,a^4}{480} + \tfrac{1069\,a^5}{17280} - \tfrac{a^6}{80} + \tfrac{13\,a^7}{8640} - \tfrac{a^8}{10080} + \tfrac{a^9}{362880} + a/9\right)x^9$$
$$+\left(\tfrac{7129\,a^2}{25200} - \tfrac{1303\,a^3}{4032} + \tfrac{4523\,a^4}{22680} - \tfrac{19\,a^5}{256} + \tfrac{3013\,a^6}{172800} - \tfrac{a^7}{384} + \tfrac{29\,a^8}{120960} - \tfrac{a^9}{80640} + \tfrac{a^{10}}{3628800} - a/10\right)x^{10}$$
$$+\Big(-\tfrac{671\,a^2}{2520} + \tfrac{16103\,a^3}{50400} - \tfrac{7645\,a^4}{36288} + \tfrac{31063\,a^5}{362880} - \tfrac{781\,a^6}{34560} + \tfrac{683\,a^7}{172800} - \tfrac{11\,a^8}{24192} + \tfrac{a^9}{30240} - \tfrac{a^{10}}{725760}$$
$$+\tfrac{a^{11}}{39916800} + a/11\Big)x^{11} + O\left(x^{12}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[1,[ax,1],[-1/2\ (a-1)\,x,1],[1/6\ (a+1)\,x,1],[-1/6\ (a-2)\,x,1],$$
$$[1/10\ (a+2)\,x,1],[-1/10\ (a-3)\,x,1],[1/14\ (a+3)\,x,1],$$
$$[-1/14$$
$$mbox\ (a-4)\,x,1],[1/18\ (a+4)\,x,1],[-1/18\ (a-5)\,x,1],[1/22\ (a+5)\,x,1]]$$

$$pnum11 := [1,[ax],[\frac{x\,(a+k-1)}{4\,k-2},-\frac{x\,(a-k)}{4\,k-2}]]$$

$rec11 := searchCorrRec\left(deq11, Y(x), H(k), pnum11, 1\right)$

$$rec11 := [[H(0) = -a, H(2) = \left(1/4\,a^3 - a/4\right)x^2, H(4) = \left(\frac{5\,a^3}{144} - a/36 - \frac{a^5}{144}\right)x^4,$$
$$H(6) = \left(\tfrac{a^7}{14400} - \tfrac{a}{400} + \tfrac{49\,a^3}{14400} - \tfrac{7\,a^5}{7200}\right)x^6], H(2\,k+8) = 1/4\ (x+2)^2\,H(2\,k+6)$$
$$+1/16\,\frac{x^2(a+3+k)(a-3-k)\left(2\,a^2x^2+6\,k^2x^2+32\,k^2x+30\,kx^2+32\,k^2+160\,xk+31\,x^2+160\,k+168\,x+168\right)H(2\,k+4)}{(2\,k+5)(2\,k+3)(2\,k+7)^2}$$
$$+\frac{x^4(x+2)^2(a+k+2)(a-k-2)(a+3+k)(a-3-k)H(2\,k+2)}{(128\,k+448)(2\,k+3)(2\,k+5)^2}$$
$$-\frac{x^8(a+k+2)(a-k-2)(a+3+k)(a-3-k)(a+k+1)^2(a-k-1)^2H(2\,k)}{(512\,k+1792)(2\,k+5)^2(2\,k+1)^2(2\,k+3)^3}]$$

$rec11 := eval\left(subs\left(H = F, rec11\right)\right)$

$checkValIncrease\left(op\left(2, rec11\right), H(k), op\left(1, rec11\right), x\right)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$-\frac{a\pi\ \left(1/16\,x^2\right)^k \Gamma\left(a+k+1\right)\Gamma\left(k-a+1\right)}{\Gamma\left(a+1\right)\quad\Gamma\left(-a+1\right)\left(\Gamma\left(k+1/2\right)\right)^2}$$

$$H\left(k+1\right)=-1/4\,\frac{x^2\left(a+k+1\right)\left(a-k-1\right)H\left(k\right)}{\left(2\,k+1\right)^2}$$

1/(1+x)^a

$deq12\ :=\ searchODE\left(\left(\left(1+x\right)^a\right)^{-1},Y\left(x\right)\right)$

$$deq12\ :=\ \left(1+x\right)\frac{\mathrm{d}}{\mathrm{d}x}Y\left(x\right)+aY\left(x\right)=0$$

$pnum12\ :=\ guessCfracFromExpr\left(\left(\left(1+x\right)^a\right)^{-1},a\left(k,x\right),lbound=3,stepsize=1\right)$

guessCfracFromExpr: corresponding finite C-fraction

$$[1,[-ax,1],[1/2\ (a+1)\,x,1],[-1/6\ (a-1)\,x,1],[1/6\ (a+2)\,x,1],$$
$$[-1/10\ (a-2)\,x,1],[1/10\ (a+3)\,x,1],[-1/14\ (a-3)\,x,1],$$
$$[1/14\ (a+4)\,x,1],[-1/18\ (a-4)\,x,1],[1/18\ (a+5)\,x,1],$$
$$[-1/22\ (a-5)\,x,1]]$$

$$pnum12\ :=\ [1,[-ax],[-\frac{x\left(a-k+1\right)}{4\,k-2},\frac{x\left(a+k\right)}{4\,k-2}]]$$

$rec12\ :=\ searchCorrRec\left(deq12,Y\left(x\right),H\left(k\right),pnum12,1\right)$

$$rec12\ :=\ [[H\left(0\right)=a,H\left(2\right)=\left(-1/4\,a^3+a/4\right)x^2,H\left(4\right)=\left(-\frac{5\,a^3}{144}+a/36+\frac{a^5}{144}\right)x^4,$$
$$H\left(6\right)=\left(-\tfrac{a^7}{14400}+\tfrac{a}{400}-\tfrac{49\,a^3}{14400}+\tfrac{7\,a^5}{7200}\right)x^6],H\left(2\,k+8\right)=1/4\ \left(x+2\right)^2H\left(2\,k+6\right)$$
$$+1/16\,\frac{x^2(a+3+k)(a-3-k)\left(2\,a^2x^2+6\,k^2x^2+32\,k^2x+30\,kx^2+32\,k^2+160\,xk+31\,x^2+160\,k+168\,x+168\right)H(2\,k+4)}{(2\,k+5)(2\,k+3)(2\,k+7)^2}$$
$$+\frac{x^4(x+2)^2(a+k+2)(a-k-2)(a+3+k)(a-3-k)H(2\,k+2)}{(128\,k+448)(2\,k+3)(2\,k+5)^2}$$
$$-\frac{x^8(a+k+2)(a-k-2)(a+3+k)(a-3-k)(a+k+1)^2(a-k-1)^2H(2\,k)}{(512\,k+1792)(2\,k+5)^2(2\,k+1)^2(2\,k+3)^3}]$$

$rec12\ :=\ eval\left(subs\left(H=F,rec12\right)\right)$

$checkValIncrease\left(op\left(2,rec12\right),H\left(k\right),op\left(1,rec12\right),x\right)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$\frac{a\pi\ \left(1/16\,x^2\right)^k \Gamma\left(a+k+1\right)\Gamma\left(k-a+1\right)}{\Gamma\left(a+1\right)\quad\Gamma\left(-a+1\right)\left(\Gamma\left(k+1/2\right)\right)^2}$$

$$H\left(k+1\right)=-1/4\,\frac{x^2\left(a+k+1\right)\left(a-k-1\right)H\left(k\right)}{\left(2\,k+1\right)^2}$$

1/(1+x/(1-x))^a

$deq13\ :=\ searchODE\left(\left(\left(1+\frac{x}{1-x}\right)^a\right)^{-1},Y\left(x\right)\right)$

$$deq13\ :=\ \left(-1+x\right)\frac{\mathrm{d}}{\mathrm{d}x}Y\left(x\right)-aY\left(x\right)=0$$

$pnum13\ :=\ guessCfracFromExpr\left(\left(\left(1+\frac{x}{1-x}\right)^a\right)^{-1},a\left(k,x\right),lbound=3,stepsize=1\right)$

guessCfracFromExpr: corresponding finite C-fraction

$$[1,[-ax,1],[1/2\ (a-1)\,x,1],[-1/6\ (a+1)\,x,1],[1/6\ (a-2)\,x,1],$$
$$[-1/10\ (a+2)\,x,1],[1/10\ (a-3)\,x,1],[-1/14\ (a+3)\,x,1],$$
$$[1/14\ (a-4)\,x,1],[-1/18\ (a+4)\,x,1],[1/18\ (a-5)\,x,1],$$
$$[-1/22\ (a+5)\,x,1]]$$

$$pnum13\ :=\ [1,[-ax],[-\frac{x\,(a+k-1)}{4\,k-2},\frac{x\,(a-k)}{4\,k-2}]]$$

$$rec13\ :=\ searchCorrRec\,(deq13,Y\,(x)\,,H\,(k)\,,pnum13,1)$$

$$rec13\ :=\ [[H\,(0)=-a,H\,(2)=\left(1/4\,a^3-a/4\right)x^2,H\,(4)=\left(\frac{5\,a^3}{144}-a/36-\frac{a^5}{144}\right)$$
$$x^4,H\,(6)=\left(\tfrac{a^7}{14400}-\tfrac{a}{400}+\tfrac{49\,a^3}{14400}-\tfrac{7\,a^5}{7200}\right)x^6],H\,(2\,k+8)=1/4\ (x-2)^2\,H\,(2\,k+6)$$
$$+1/16\,\frac{x^2(a+3+k)(a-3-k)\left(2\,a^2x^2+6\,k^2x^2-32\,k^2x+30\,kx^2+32\,k^2-160\,xk+31\,x^2+160\,k-168\,x+168\right)H(2\,k+4)}{(2\,k+5)(2\,k+3)(2\,k+7)^2}$$
$$+\frac{x^4(x-2)^2(a+k+2)(a-k-2)(a+3+k)(a-3-k)H(2\,k+2)}{(128\,k+448)(2\,k+3)(2\,k+5)^2}$$
$$-\frac{x^8(a+k+2)(a-k-2)(a+3+k)(a-3-k)(a+k+1)^2(a-k-1)^2H(2\,k)}{(512\,k+1792)(2\,k+5)^2(2\,k+1)^2(2\,k+3)^3}]$$

$$rec13\ :=\ eval\,(subs\,(H=F,rec13))$$

$$checkValIncrease\,(op\,(2,rec13)\,,H\,(k)\,,op\,(1,rec13)\,,x)$$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$-\frac{a\pi\ \left(1/16\,x^2\right)^k\Gamma\,(a+k+1)\,\Gamma\,(k-a+1)}{\Gamma\,(a+1)\quad\Gamma\,(-a+1)\,(\Gamma\,(k+1/2))^2}$$
$$H\,(k+1)=-1/4\,\frac{x^2\,(a+k+1)\,(a-k-1)\,H\,(k)}{(2\,k+1)^2}$$

((1+x)/(1-x))^a
A continued fraction representation for ((1+x)/(1-x))^a can be obtained by rearranging the result for (2*a*x)/(((1+x)/(1-x))^a-1)-(1-b*x).
Problem: The differential equation is singular in x=0, so uniqueness of the power series solution is not assured.

$$deq14\ :=\ searchODE\left(2\,ax\left(\left(\tfrac{1+x}{1-x}\right)^a-1\right)^{-1}+ax-1,Y\,(x)\right)$$

$$deq14\ :=\ \left(x^3-x\right)\frac{\mathrm{d}}{\mathrm{d}x}Y\,(x)-(Y\,(x))^2+\left(-x^2-1\right)Y\,(x)+a^2x^2-x^2=0$$

$$pnum14\ :=\ guessCfracFromExpr\Big(2\,ax\left(\left(\tfrac{1+x}{1-x}\right)^a-1\right)^{-1}+ax-1,a\,(k,x)\,,$$
$$lbound=4,stepsize=2\Big)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0, [1/3\left(a^2-1\right)x^2, 1], [1/15\left(a^2-4\right)x^2, 1], [1/35\left(a^2-9\right)x^2, 1], [\frac{(a^2-16)\,x^2}{63}, 1], [\frac{(a^2-25)\,x^2}{99}, 1], [\frac{(a^2-36)\,x^2}{143}, 1]]$$

$$pnum14 := [0, [], [\frac{x^2\,(a-k)\,(a+k)}{(2\,k-1)\,(2\,k+1)}]]$$

$rec14 := searchCorrRec\,(deq14, Y\,(x), H\,(k), pnum14, 1)$

$$rec14 := [[H\,(0) = \left(a^2-1\right)x^2, H\,(1) = \left(5/9\,a^2 - 4/9 - 1/9\,a^4\right)x^4, H\,(2) = \left(-\frac{4}{25} + \frac{49\,a^2}{225} + \frac{a^6}{225} - \frac{14\,a^4}{225}\right)x^6, H\,(3) = \left(-\frac{a^8}{11025} - \frac{64}{1225} + \frac{2\,a^6}{735} - \frac{13\,a^4}{525} + \frac{164\,a^2}{2205}\right)x^8], H\,(k+4) = H\,(k+3) + \frac{x^2\,(a+4+k)\,(a-4-k)\,(2\,a^2x^2 - 2\,k^2x^2 - 14\,kx^2 + 8\,k^2 - 23\,x^2 + 56\,k + 90)\,H\,(k+2)}{(2\,k+7)\,(2\,k+5)\,(2\,k+9)^2} + \frac{(a+3+k)\,(a-3-k)\,(a+4+k)\,(a-4-k)\,x^4H\,(k+1)}{(2\,k+9)\,(2\,k+5)\,(2\,k+7)^2} - \frac{x^8\,(a+3+k)\,(a-3-k)\,(a+4+k)\,(a-4-k)\,(a+k+2)^2\,(a-k-2)^2\,H\,(k)}{(2\,k+9)\,(2\,k+7)^2\,(2\,k+3)^2\,(2\,k+5)^3}]$$

$checkValIncrease\,(op\,(2, rec14), H\,(k), op\,(1, rec14), x)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$1/4\,\frac{x^2\pi\,(a^2-1)\,(1/4\,x^2)^k\,\Gamma\,(a+k+2)\,\Gamma\,(k-a+2)}{\Gamma\,(a+2)\,\Gamma\,(-a+2)\,(\Gamma\,(k+3/2))^2}$$

$$H\,(k+1) = -\frac{x^2\,(a-k-2)\,(a+k+2)\,H\,(k)}{(2\,k+3)^2}$$

A.6.5. Examples from Section 3.2.5

x*(AiryAi'/AiryAi)(1/x^2)

$$deq15 := searchODE\left(\frac{x\mathrm{Ai}^{(1)}(x^{-2})}{\mathrm{Ai}(x^{-2})}, Y\,(x)\right)$$

$$deq15 := \left(\frac{\mathrm{d}}{\mathrm{d}x}Y\,(x)\right)x^4 - Y\,(x)\,x^3 - 2\,(Y\,(x))^2 + 2 = 0$$

$$pnum15 := assuming\left([guessCfracFromExpr\left(\frac{x\mathrm{Ai}^{(1)}(x^{-2})}{\mathrm{Ai}(x^{-2})}, a\,(k,x)\right)], [0 \le x]\right)$$

guessCfracFromExpr: guess based on initial series terms

$$-1 - 1/4\,x^3 + \frac{5\,x^6}{32} - \frac{15\,x^9}{64} + \frac{1105\,x^{12}}{2048} - \frac{1695\,x^{15}}{1024} + \frac{414125\,x^{18}}{65536} - \frac{59025\,x^{21}}{2048} + \frac{1282031525\,x^{24}}{8388608} - \frac{242183775\,x^{27}}{262144} + \frac{1683480621875\,x^{30}}{268435456} - \frac{198147676875\,x^{33}}{4194304} + \frac{6718940277925125\,x^{36}}{17179869184} - \frac{59217351295125\,x^{39}}{16777216} + \frac{18962375127249928125\,x^{42}}{549755813888} - \frac{97404235669134375\,x^{45}}{268435456} + \frac{575440151532675686278125\,x^{48}}{140737488355328} - \frac{844097335215098919375\,x^{51}}{17179869184} + \frac{28246507470894255861524843 75\,x^{54}}{4503599627370496} - \frac{232989647110235013820312 5\,x^{57}}{274877906944} + O\left(x^{60}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[-1, [-1/4\,x^3, 1], [5/8\,x^3, 1], [\frac{7\,x^3}{8}, 1], [\frac{11\,x^3}{8}, 1], [\frac{13\,x^3}{8}, 1], [\frac{17\,x^3}{8}, 1], [\frac{19\,x^3}{8}, 1], [\frac{23\,x^3}{8}, 1], [\frac{25\,x^3}{8}, 1], [\frac{29\,x^3}{8}, 1], [\frac{31\,x^3}{8}, 1], [\frac{35\,x^3}{8}, 1], [\frac{37\,x^3}{8}, 1], [\frac{41\,x^3}{8}, 1], [\frac{43\,x^3}{8}, 1], [\frac{47\,x^3}{8}, 1], [\frac{49\,x^3}{8}, 1], [\frac{53\,x^3}{8}, 1], [\frac{55\,x^3}{8}, 1]]$$

$$pnum15 := [-1, [-1/4\,x^3], [1/8\,x^3\,(6\,k-5)\,, 1/8\,x^3\,(6\,k-1)]]$$

$rec15 := searchCorrRec\,(deq15, Y\,(x)\,, H\,(k)\,, pnum15, 1)$

$rec15 := eval\,(subs\,(H = F, rec15))$

$checkValIncrease\,(op\,(2, rec15)\,, H\,(k)\,, op\,(1, rec15)\,, x)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$3\,\frac{x^3\Gamma\,(k+5/6)\,\Gamma\,(k+7/6)}{\pi}\left(\frac{9\,x^6}{16}\right)^k$$

$$H\,(k+1) = \frac{x^6\,(6\,k+7)\,(6\,k+5)\,H\,(k)}{64}$$

A.6.6. Examples from Section 3.2.6

2/(exp(x)+1)

$deq16 := searchODE\left(2\,\left(e^x+1\right)^{-1}, Y\,(x)\right)$

$$deq16 := -\left(Y\,(x)\right)^2 + 2\,\frac{d}{dx}Y\,(x) + 2\,Y\,(x) = 0$$

$pnum16 := guessCfracFromExpr\left(2\,\left(e^x+1\right)^{-1}, a\,(k,x)\right)$

guessCfracFromExpr: guess based on initial series terms

$$1 - x/2 + 1/24\,x^3 - \frac{x^5}{240} + \frac{17\,x^7}{40320} - \frac{31\,x^9}{725760} + \frac{691\,x^{11}}{159667200} - \frac{5461\,x^{13}}{12454041600} + \frac{929569\,x^{15}}{20922789888000} - \frac{3202291\,x^{17}}{711374856192000} + \frac{221930581\,x^{19}}{486580401635328000} - \frac{4722116521\,x^{21}}{102181884343418880000} + \frac{56963745931\,x^{23}}{12165654935945871360000} - \frac{14717667114151\,x^{25}}{31022420086661971968000000} + \frac{2093660879252671\,x^{27}}{43555477801673408643072000000} - \frac{86125672563201181\,x^{29}}{17683523987479403909087232000000} + O\left(x^{30}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[1, [-x/2, 1], [1/12\,x^2, 1], [\frac{x^2}{60}, 1], [\frac{x^2}{140}, 1], [\frac{x^2}{252}, 1], [\frac{x^2}{396}, 1], [\frac{x^2}{572}, 1], [\frac{x^2}{780}, 1], [\frac{x^2}{1020}, 1], [\frac{x^2}{1292}, 1], [\frac{x^2}{1596}, 1], [\frac{x^2}{1932}, 1], [\frac{x^2}{2300}, 1], [\frac{x^2}{2700}, 1], [\frac{x^2}{3132}, 1]]$$

$$pnum16 := [1, [-x/2], [1/4\,\frac{x^2}{(2\,k-1)\,(2\,k-3)}]]$$

$rec16 := searchCorrRec\,(deq16, Y\,(x)\,, H\,(k)\,, pnum16, 1)$

$$rec16 := [[H\,(0) = 1, H\,(1) = -1/4\,x^2, H\,(2) = \frac{x^4}{144}, H\,(3) = -\frac{x^6}{14400}], H\,(k+4) = H\,(k+3) + 1/8\,\frac{x^2\,(16\,k^2 + x^2 + 80\,k + 84)\,H\,(k+2)}{(2\,k+5)\,(2\,k+3)\,(2\,k+7)^2} + 1/16\,\frac{x^4 H\,(k+1)}{(2\,k+7)\,(2\,k+3)\,(2\,k+5)^2} - \frac{x^8 H\,(k)}{(512\,k+1792)\,(2\,k+5)^2\,(2\,k+1)^2\,(2\,k+3)^3}]$$

$checkValIncrease\,(op\,(2, rec16)\,, H\,(k)\,, op\,(1, rec16)\,, x)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$\frac{\pi\,\left(-1/16\,x^2\right)^k}{(\Gamma\,(k+1/2))^2}$$

$$H\,(k+1) = -1/4\,\frac{x^2 H\,(k)}{(2\,k+1)^2}$$

2*exp(x)/(exp(x)+1)

$deq17 := searchODE\left(2\,\frac{e^x}{e^x+1}, Y\,(x)\right)$

$$deq17 := (Y\,(x))^2 + 2\,\frac{d}{dx} Y\,(x) - 2\,Y\,(x) = 0$$

$pnum17 := guessCfracFromExpr\left(2\,\frac{e^x}{e^x+1}, a\,(k, x)\right)$

guessCfracFromExpr: guess based on initial series terms

$$1+x/2-1/24\,x^3+\frac{x^5}{240}-\frac{17\,x^7}{40320}+\frac{31\,x^9}{725760}-\frac{691\,x^{11}}{159667200}+\frac{5461\,x^{13}}{12454041600}-\frac{929569\,x^{15}}{20922789888000}+\frac{3202291\,x^{17}}{711374856192000}-\frac{221930581\,x^{19}}{486580401635328000}+\frac{4722116521\,x^{21}}{102181884343418880000}-\frac{56963745931\,x^{23}}{12165654935945871360000}+\frac{14717667114151\,x^{25}}{31022420086661971968000000}-\frac{2093660879252671\,x^{27}}{4355547780167340864307200000}+\frac{86125672563201181\,x^{29}}{176835239874794039090872320000000}+O\left(x^{30}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[1,[x/2,1],[1/12\,x^2,1],[\frac{x^2}{60},1],[\frac{x^2}{140},1],[\frac{x^2}{252},1],[\frac{x^2}{396},1],[\frac{x^2}{572},1],[\frac{x^2}{780},1],[\frac{x^2}{1020},1],[\frac{x^2}{1292},1],[\frac{x^2}{1596},1],[\frac{x^2}{1932},1],[\frac{x^2}{2300},1],[\frac{x^2}{2700},1],[\frac{x^2}{3132},1]]$$

$$pnum17 := [1,[x/2],[1/4\,\frac{x^2}{(2\,k-1)\,(2\,k-3)}]]$$

$rec17 := searchCorrRec\left(deq17, Y\left(x\right), H\left(k\right), pnum17, 1\right)$

$$rec17 := [[H(0)=-1, H(1)=1/4\,x^2, H(2)=-\frac{x^4}{144}, H(3)=\frac{x^6}{14400}], H(k+4)=H(k+3)+1/8\,\frac{x^2\left(16\,k^2+x^2+80\,k+84\right)H(k+2)}{(2\,k+5)\,(2\,k+3)\,(2\,k+7)^2}+1/16\,\frac{x^4H(k+1)}{(2\,k+7)\,(2\,k+3)\,(2\,k+5)^2}-\frac{x^8H(k)}{(512\,k+1792)\,(2\,k+5)^2\,(2\,k+1)^2\,(2\,k+3)^3}]$$

$checkValIncrease\left(op\left(2, rec17\right), H\left(k\right), op\left(1, rec17\right), x\right)$

checkValIncrease: found hypergeometric term solution with increasing valuation

$$-\frac{\pi\,\left(-1/16\,x^2\right)^k}{\left(\Gamma\left(k+1/2\right)\right)^2}$$

$$H(k+1)=-1/4\,\frac{x^2H(k)}{(2\,k+1)^2}$$

A.6.7. Examples from Section 3.3.1

$searchODE\left(\left(\sin\left(x\right)\right)^n, Y\left(x\right), explicit=false\right)$

$$n^2\left(Y(x)\right)^2+\left(\frac{d^2}{dx^2}Y(x)\right)Y(x)\,n+(-n+1)\left(\frac{d}{dx}Y(x)\right)^2=0$$

$$searchODE\left((\cos(x))^n, Y(x), explicit = false\right)$$

$$n^2 (Y(x))^2 + \left(\frac{d^2}{dx^2} Y(x)\right) Y(x)\, n + (-n+1) \left(\frac{d}{dx} Y(x)\right)^2 = 0$$

$$searchODE\left((\sin(x))^n + (\cos(x))^n, Y(x), explicit = false\right)$$

$$\left(n^4 - 4n^3 + 4n^2\right) (Y(x))^2 + \left(n^3 - 5n^2 + 6n\right) \left(\frac{d^2}{dx^2} Y(x)\right) Y(x) + \left(-n^3 + 7n^2 - 10n + 4\right) \left(\frac{d}{dx} Y(x)\right)^2 + (n-1) \left(\frac{d^3}{dx^3} Y(x)\right) \frac{d}{dx} Y(x) + (-n+2) \left(\frac{d^2}{dx^2} Y(x)\right)^2 = 0$$

$$searchODE\left((\sin(x))^n - (\cos(x))^n, Y(x), explicit = false\right)$$

$$\left(n^4 - 4n^3 + 4n^2\right) (Y(x))^2 + \left(n^3 - 5n^2 + 6n\right) \left(\frac{d^2}{dx^2} Y(x)\right) Y(x) + \left(-n^3 + 7n^2 - 10n + 4\right) \left(\frac{d}{dx} Y(x)\right)^2 + (n-1) \left(\frac{d^3}{dx^3} Y(x)\right) \frac{d}{dx} Y(x) + (-n+2) \left(\frac{d^2}{dx^2} Y(x)\right)^2 = 0$$

$$searchODE\left((\tan(x))^n, Y(x), explicit = false\right)$$

$$\left(-n^2 + 1\right) \left(\frac{d}{dx} Y(x)\right)^4 - 4 \left(\frac{d}{dx} Y(x)\right)^2 (Y(x))^2 n^2 + 2 \left(\frac{d^2}{dx^2} Y(x)\right) \left(\frac{d}{dx} Y(x)\right)^2 Y(x)\, n^2 - \left(\frac{d^2}{dx^2} Y(x)\right)^2 (Y(x))^2 n^2 = 0$$

$$searchODE\left((\cot(x))^n, Y(x), explicit = false\right)$$

$$\left(-n^2 + 1\right) \left(\frac{d}{dx} Y(x)\right)^4 - 4 \left(\frac{d}{dx} Y(x)\right)^2 (Y(x))^2 n^2 + 2 \left(\frac{d^2}{dx^2} Y(x)\right) \left(\frac{d}{dx} Y(x)\right)^2 Y(x)\, n^2 - \left(\frac{d^2}{dx^2} Y(x)\right)^2 (Y(x))^2 n^2 = 0$$

$$searchODE\left((\sec(x))^n, Y(x), explicit = false\right)$$

$$-n^2 (Y(x))^2 + \left(\frac{d^2}{dx^2} Y(x)\right) Y(x)\, n + (-n-1) \left(\frac{d}{dx} Y(x)\right)^2 = 0$$

$$searchODE\left(\left(\frac{x e^{xt}}{-1 + e^x}\right)^n, Y(x), explicit = false\right)$$

$$\left(-n^2t^2x + n^2tx - 2\,n^2t + n^2\right)(Y(x))^2 + (2\,ntx - nx + 2\,n)\left(\frac{d}{dx}Y(x)\right)Y(x) + \left(\frac{d^2}{dx^2}Y(x)\right)Y(x)\,nx + (-nx - x)\left(\frac{d}{dx}Y(x)\right)^2 = 0$$

$$\mathit{searchODE}\left(\left(2\,\frac{e^{xt}}{e^x+1}\right)^n, Y(x), \mathit{explicit} = \mathit{false}\right)$$

$$\left(-n^2t^2 + n^2t\right)(Y(x))^2 + (2\,nt - n)\left(\frac{d}{dx}Y(x)\right)Y(x) + \left(\frac{d^2}{dx^2}Y(x)\right)Y(x)\,n + (-n - 1)\left(\frac{d}{dx}Y(x)\right)^2 = 0$$

A.6.8. Examples from Section 3.4

x*sqrt(-x^2+1)/arccos(sqrt(-x^2+1))-1
A continued fraction representation for arccos(x) can be obtained by rearranging the result and substituting x = sqrt(1-z^2).
Problems: The differential equation is singular in x=0, so uniqueness of the power series solution is not assured. Checking for the increase in valuation of H(k) does not terminate in a reasonable timeframe, though looking at the initial values indicates it is indeed increasing.

$$\mathit{deq18} := \mathit{searchODE}\left(\frac{x\sqrt{-x^2+1}}{\arccos\left(\sqrt{-x^2+1}\right)}, Y(x)\right)$$

$$\mathit{deq18} := \left(x^6 - 2\,x^4 + x^2\right)\frac{d^2}{dx^2}Y(x) - 2\,(Y(x))^3 + \left(4\,x^4 - 4\,x^2 + 2\right)Y(x) + \left(-3\,x^5 + 5\,x^3 - 2\,x\right)\frac{d}{dx}Y(x) = 0$$

$$\mathit{pnum18} := \mathit{assuming}\left(\left[\mathit{guessCfracFromExpr}\left(\frac{x\sqrt{-x^2+1}}{\arccos\left(\sqrt{-x^2+1}\right)}, a(k,x), \mathit{lbound} = 4, \mathit{stepsize} = 2\right)\right], [0 < \mathrm{Re}(x)]\right)$$

guessCfracFromExpr: guess based on initial series terms

$$1 - 2/3\,x^2 - \frac{4\,x^4}{45} - \frac{8\,x^6}{189} - \frac{368\,x^8}{14175} - \frac{8416\,x^{10}}{467775} - \frac{8562368\,x^{12}}{638512875} - \frac{20097152\,x^{14}}{1915538625} - \frac{4151058176\,x^{16}}{488462349375} - \frac{1377000432128\,x^{18}}{194896477400625} - \frac{27538553375744\,x^{20}}{4593988395871875} - \frac{11470339948890112\,x^{22}}{2218896395206115625} - \frac{1368320620961476198\,4\,x^{24}}{302879357945634782812\,5} - \frac{72552185592821432\,32\,x^{26}}{18172761476738086968\,75} + O\left(x^{28}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[1,[-2/3\,x^2,1],[-2/15\,x^2,1],[-\frac{12\,x^2}{35},1],[-\frac{4\,x^2}{21},1],$$
$$[-\frac{10\,x^2}{33},1],[-\frac{30\,x^2}{143},1],[-\frac{56\,x^2}{195},1],[-\frac{56\,x^2}{255},1],$$
$$[-\frac{90\,x^2}{323},1],[-\frac{30\,x^2}{133},1],[-\frac{44\,x^2}{161},1],[-\frac{132\,x^2}{575},1],$$
$$[-\frac{182\,x^2}{675},1]]$$

$$pnum18 := [1,[],[-2\,\frac{kx^2\,(2\,k-1)}{(4\,k-1)\,(4\,k-3)},-2\,\frac{kx^2\,(2\,k-1)}{(4\,k-1)\,(4\,k+1)}]]$$

$op(1,rec18), map\,(term \rightarrow ldegree\,(rhs\,(term))\,, op\,(1,rec18))$

$$[2,6,10,14,18,22,26,30]$$

-(1/2)*x*sin(x)/(cos(x)-1)-1
Problems: The differential equation is singular in x=0, so uniqueness of the power series solution is not assured. Checking for the increase in valuation of H(k) does not terminate in a reasonable timeframe, though looking at the initial values indicates it is indeed increasing.

$deq19 := searchODE\left(-1/2\,\frac{x\sin(x)}{\cos(x)-1}-1, Y\,(x)\right)$

$$deq19 := \left(\frac{d^2}{dx^2}Y\,(x)\right)x^2+4\,(Y\,(x))^3+6\,(Y\,(x))^2+6\,\left(\frac{d}{dx}Y\,(x)\right)Y\,(x)\,x$$
$$+\left(x^2+2\right)Y\,(x)\,+4\,\left(\frac{d}{dx}Y\,(x)\right)x+x^2=0$$

$pnum19 := guessCfracFromExpr\left(-1/2\,\frac{x\sin(x)}{\cos(x)-1}-1, a\,(k,x)\,, lbound=5, stepsize=1\right)$

guessCfracFromExpr: guess based on initial series terms

$$-1/12\,x^2-\frac{x^4}{720}-\frac{x^6}{30240}-\frac{x^8}{1209600}-\frac{x^{10}}{47900160}+O\left(x^{12}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0,[-1/12\,x^2,1],[-\frac{x^2}{60},1],[-\frac{x^2}{140},1],[-\frac{x^2}{252},1],[-\frac{x^2}{396},1]]$$

$$pnum19 := [0,[],[-1/4\,\frac{x^2}{(2\,k-1)\,(2\,k+1)}]]$$

$rec19 := searchCorrRec\,(deq19, Y\,(x)\,, H\,(k)\,, pnum19, 2)$

$map\,(term \rightarrow ldegree\,(rhs\,(term))\,, op\,(1,rec19))$

$$[2,4,6,8,10,12,14,16]$$

-(1/2)*x*sinh(x)/(cosh(x)-1)-1
Problems: The differential equation is singular in x=0, so uniqueness of the power series solution is not assured. Checking for the increase in valuation of H(k) does not terminate in a reasonable timeframe, though looking at the initial values indicates it is indeed increasing.

$deq20 := searchODE\left(1/2\,\frac{\sinh(x)x}{\cosh(x)-1} - 1, Y(x)\right)$

$$deq20 := \left(\frac{d^2}{dx^2}Y(x)\right)x^2 + 4\,(Y(x))^3 + 6\,(Y(x))^2 + 6\left(\frac{d}{dx}Y(x)\right)Y(x)\,x + \left(-x^2+2\right)Y(x) + 4\left(\frac{d}{dx}Y(x)\right)x - x^2 = 0$$

$pnum20 := guessCfracFromExpr\left(1/2\,\frac{\sinh(x)x}{\cosh(x)-1} - 1, a(k,x), lbound = 5, stepsize = 1\right)$

guessCfracFromExpr: guess based on initial series terms

$$1/12\,x^2 - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \frac{x^{10}}{47900160} + O\left(x^{12}\right)$$

guessCfracFromExpr: corresponding finite C-fraction

$$[0, [1/12\,x^2, 1], [\frac{x^2}{60}, 1], [\frac{x^2}{140}, 1], [\frac{x^2}{252}, 1], [\frac{x^2}{396}, 1]]$$

$$pnum20 := [0, [], [1/4\,\frac{x^2}{(2\,k-1)(2\,k+1)}]]$$

$rec20 := searchCorrRec\left(deq20, Y(x), H(k), pnum20, 2\right)$

$map\left(term \to ldegree\left(rhs\left(term\right)\right), op\left(1, rec20\right)\right)$

$$[2, 4, 6, 8, 10, 12, 14, 16]$$