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IMEX-DG Schemes for Advection-Diffusion Problems using Term-based or Domain-based IMEX Splitting

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IMEX partitioning may be based on a decomposition of the computational domain into two subgrids by implicitly discretizing the unknowns on the smallest cells while explicit time stepping is applied to moderately sized elements. Alternatively, advection-diffusion IMEX splitting applies implicit time discretization only to the viscous terms while inviscid terms are discretized explicitly. Analytical investigations have shown that a careful choice of both the specific IMEX scheme and the particular DG approach yield an additional stability property of such schemes resulting in time step restrictions independent of grid refinement. In this contribution, we will compare the domain-based IMEX partitioning to advection-diffusion splitting with respect to time step restrictions and efficiency.

1 Introduction

For advection-diffusion equations discretized in space by high-order methods such as discontinuous Galerkin (DG) schemes, explicit time discretization may lead to severe time step restrictions due to stability requirements. In particular for locally refined grids, the smallest grid cells often dictate the time step size scaling quadratically with respect to the cell size with a scaling factor depending on the order of the DG scheme. Reducing the computational effort of fully implicit schemes, IMEX approaches are an interesting alternative. The right-hand side of the semi-discrete ODE system is thereby split into two terms, one of which is discretized implicitly in time and the other one explicitly. IMEX partitioning may be based on a decomposition of the computational domain into two subgrids. Typically, the unknowns on the smallest cells are thereby discretized implicitly while explicit time stepping is applied to moderately sized cells. Alternatively, advection-diffusion IMEX splitting implicitly discretizes viscous terms while inviscid terms are discretized explicitly. For specific combinations of both the specific IMEX scheme and the particular DG approach, a favorable stability property of this type of splitting results in time step restrictions independent of grid refinement, $\Delta t = \mathcal{O}(d/a^2)$, where a and d denote the advection and diffusion coefficient, respectively, see for instance [1, 2]. On the other hand, domain-based IMEX splitting is subject to a CFL-type stability condition with time step restriction depending on the cell sizes of the explicit region. A priori, it is often not clear which type of splitting is more efficient. In this contribution, we compare domain-based and advection-diffusion IMEX splitting for DG discretizations of practically relevant non-linear advection-diffusion problems in terms of allowable time step sizes and efficiency.

2 IMEX Splitting for DG-discretized Advection-Diffusion Problems

For the linear advection-diffusion problem given by $U_t + aU_x = dU_{xx},\ a, d > 0$, on a domain $\Omega = (x_L, x_R)$ supplemented by periodic boundary conditions, the DG scheme can compactly be written as $(u_t, v)_j = a\mathcal{H}_j(u, v) + d\mathcal{L}_j(u, v)\ \forall v \in V_h$. Herein, Ω is decomposed into cells $I_j = (x_j, x_{j+1})$, we set $(u, v)_j = \int_{x_j}^{x_{j+1}} uv\,dx$ and $V_h = \{v \in L^2(\Omega) \mid v|_{I_j} \in \mathcal{P}_N(I_j)\,\forall j\}$. We employs upwind fluxes for advection, $\mathcal{H}_j(u, v) = (u, v_x)_j - u_{j+1}^- v_{j+1}^- + u_j^- v_j^+$, and the (σ, μ)-scheme for diffusion,

$$\mathcal{L}_j(u, v) = -(u_x, v_x)_j + \{u_x\}_{j+1} v_{j+1}^- - \{u_x\}_j v_j^+ + \frac{\sigma}{2}\left((v_x^-[u])\,|_{j+1} + (v_x^+[u])\,|_j\right) + \frac{\mu}{\Delta x_{j,j+1}}\left([u]v^-\right)_{j+1} - \frac{\mu}{\Delta x_{j-1,j}}\left([u]v^+\right)_j,$$

where $[u] = u^+ - u^-,\ \{u_x\} = \frac{1}{2}(u_x^+ + u_x^-)$ and $\Delta x_{j,j+1} = \frac{2\Delta x_j \Delta x_{j+1}}{\Delta x_j + \Delta x_{j+1}}$. Considering classical DG diffusion discretizations, the BR1 scheme corresponds to $\sigma = -1, \mu = 1$ and a BR2-type scheme is obtained by setting $\sigma = -1, \mu = 3$. For the BR1 scheme, grid-independent stability of advection-diffusion IMEX splitting does not hold as shown in [2].

We now study the second-order DG scheme on Lobatto nodes with second order IMEX time integration as in [1, 2] for the linear advection-diffusion problem with $d = 0.1, a = 0.1$. Initial conditions are given by L^2-projection of $U_0(x) = \sin(x - at)$ to the DG space V_h. Tables 1 and 2 compare the maximum L^2-stable time steps with non-increasing L^2-norm of numerical solution for advection-diffusion and domain-based IMEX splitting. Herein, the initial grid in Fig. 1 is uniformly refined in case of the results in Table 1 and refined only in the implicit region in case of Table 2. We observe that the maximum allowable time steps are much smaller for the domain-based setting even if refinement is only applied to the implicit region.

Next, we study the application of the IMEX-DG scheme with BR2 diffusion fluxes to the 1D chemo-taxis problem from [3, Sect. I.1.4] describing tumour angiogenesis. Considering the concentration ρ of endothelial cells in the blood vessel, the DG($N = 1$) solution on $K = 165$ cells using advection-diffusion IMEX splitting is given in Fig. 2. For the domain-based IMEX splitting, the implicit region is indicated in Fig. 2 by the red cells. The corresponding IMEX-DG solution is visually

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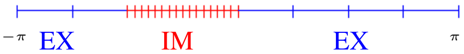

Fig. 1: Base grid used for linear advection-diffusion test case.

Table 1: Values of $\tau = \frac{a^2}{d}\Delta t_{max}$ for $\frac{\Delta x_{min}}{\Delta x_{max}} = \frac{1}{8}$.

K	Adv-diff IMEX		Domain-based IMEX		
	BR2 ($\eta = 3$)	$\sigma = 1/4$ $\mu = 9/4$	BR2 ($\eta = 3$)	BR1	$\sigma = 1/4$ $\mu = 9/4$
11	1.9	1.7	3.6e-01	8.3e-01	3.9e-01
22	1.9	1.7	9.5e-02	2.6e-01	1.1e-01
44	1.6	1.5	2.4e-02	6.0e-02	2.9e-02
88	1.5	1.4	6.2e-03	1.5e-02	7.4e-03
176	1.5	1.4	1.5e-03	3.7e-03	1.9e-03

Table 2: Values of $\tau = \frac{a^2}{d}\Delta t_{max}$ for $\frac{\Delta x_{min}}{\Delta x_{max}} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$.

K	Adv-diff IMEX		Domain-based IMEX		
	BR2 ($\eta = 3$)	$\sigma = 1/4$ $\mu = 9/4$	BR2 ($\eta = 3$)	BR1	$\sigma = 1/4$ $\mu = 9/4$
20	1.7	1.4	2.4e-02	7.1e-02	2.9e-02
28	1.6	1.4	2.4e-02	7.0e-02	2.9e-02
44	1.6	1.5	2.4e-02	6.0e-02	2.9e-02
76	1.6	1.5	2.4e-02	5.3e-02	2.9e-02
140	1.6	1.5	2.4e-02	5.2e-02	2.9e-02

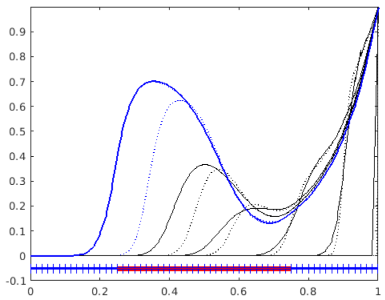

Fig. 2: 1D chemo-taxis. Solid lines: Numerical solution ρ on $K = 165$ cells at times $t = 0.1, 0.3, 0.5, 0.6, 0.7$. Dotted lines: Reference solution.

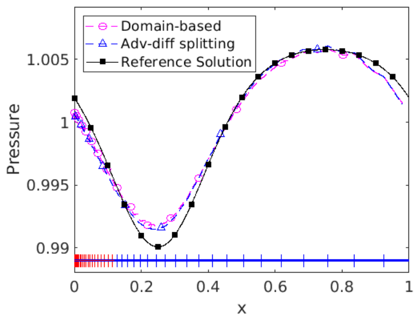

Fig. 3: 1D compressible viscous fluid flow problem from [4] on stretched grid. DG pressure solutions for domain-based and advection-diffusion IMEX splitting.

Table 3: Time step sizes and ratio $r = \frac{CPU_{ad}}{CPU_{dom}}$.

K	N	Avg. Δt adv-diff	Avg. Δt domain-based	r
20	1	2.5e-03	8.4e-03	2
40	1	1.3e-03	2.0e-03	1.3
80	1	6.2e-04	4.6e-04	0.8
10	2	3.1e-03	1.2e-02	2.1
20	2	1.4e-03	2.5e-03	1.4
40	2	7.4e-04	5.3e-04	0.7
10	3	2.0e-03	6.0e-03	2.1
20	3	9.9e-04	1.0e-03	0.9

indistinguishable from the one using advection-diffusion splitting. While the DG scheme with advection-diffusion IMEX splitting may use a stable time of $\Delta t = 10^{-4}$ for this test case and the implicit treatment only requires the solution of linear systems, domain-based IMEX splitting is only stable for $\Delta t = 2 \cdot 10^{-5}$ and requires a nonlinear solver. This suggests that non-linear advection-diffusion with linear diffusion terms is more efficiently solved by advection-diffusion IMEX splitting.

On the other hand, we may consider viscous fluid flow as an advection-diffusion problem. However, diffusion is not present in the continuity equation. Thus, grid-independent stability as in [1, 2] does not apply to this situation. As an example, we carry out a long-time simulation based on the 1D compressible Navier-Stokes equations using initial conditions $\rho(x, 0) = 1$, $v(x, 0) = 1$, $p(x, 0) = 1 + 0.1 \sin(2\pi x)$ on $\Omega = [0, 1]$ as in [4]. Fig. 3 shows the numerical results at time $t = 20$ for a second order DG scheme on Lagrange nodes for a stretched grid of 40 cells with $x_j = \frac{c^{\tilde{x}_j} - 1}{c - 1}$ for $c = 50$, $\tilde{x}_j = j\Delta x$, $j = 0, \ldots, 40$. The IMEX-DG solutions with either advection-diffusion or domain-based IMEX splitting are of similar quality. Again, the implicit region for domain-based splitting is given by the red cells in Fig. 3. Table 3 provides the allowable time step sizes and the ratio of required CPU times between advection-diffusion and domain-based IMEX splitting for different values of the number of cells K and the polynomial degree N. These results imply a turning point for the comparative efficiency between the two splitting variants. The advection-diffusion IMEX splitting is more efficient on relatively fine grids with small cells also in the explicit regions of the domain-based approach. On the other hand, on course grids, the lack of grid-independent stability enforces time step restrictions based on the smallest cells due to the explicit discretization of advection terms whereas these cells are found in the implicit region of the domain-based IMEX approach. In such situations, the development of suitable splitting-type indicators is desired, for instance based on approximated Gershgorin circles or error estimators.

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