Generalized Involutive Bases and Their Induced Free Resolutions

Dissertation zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften (Dr. rer. nat.)

Vorgelegt im Fachbereich Mathematik und Naturwissenschaften
der Universität Kassel
von

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Kassel, im Mai 2022

Tag der mündlichen Prüfung
27. Juli 2022

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Acknowledgements

First of all, I want to say a heartfelt thank you to my advisor Werner Seiler, whose guidance, not only in scientific matters, has been invaluable for me over the past years. By teaching me algebra in my undergraduate days, he gave me the necessary mathematical toolkit needed for this project. By expressing his confidence in my abilities also in phases of doubt on my part, he encouraged me to overcome difficulties and finish this thesis.

I thank also very much my second advisor Amir Hashemi. From his visit to Kassel in the summer of 2019 onwards, his steady support and attention to detail have been a great inspiration for me. This thesis has profited immensely from our frequent discussions.

I thank Cristina Bertone and Francesca Cioffi for two very enjoyable weeks of mathematical discussions in Turin, one each in the autumn of 2020 and 2021. These visits not only provided the foundations for one important building block of this thesis, but also allowed me to widen my horizons.

Thanks to all my colleagues at Universität Kassel. I would like to thank especially Matthias Fetzer, Matthias Seiß, and Merlin Mouafo Wouodjie; moreover, many thanks to Martina Syborg, Maxim Urich, and Marcus Aichmayr. Of course there are many more to whom I am grateful for the smaller and larger things that make each working day so much more enjoyable.

Lastly, but most importantly, I thank my family for their love and support.

Chapter 1

Introduction

This thesis is focused on investigating the interplay of generalized types of involutive divisions and bases with factor ring structures of commutative polynomial rings. Quotient rings of commutative polynomial rings arise by factorizing the ring modulo a given ideal. We will work with involutive bases *relative* to this ideal. Moreover, we introduce the notion of *involutive-like* bases and also investigate their application to factor structures. A common property of all these types of bases is that they induce free resolutions for the ideals they generate. We study these free resolutions. Finally, we introduce the concept of *relative marked bases* for ideals in factor rings, generalizing the concept of marked bases for polynomial ideals associated to Pommaret bases of monomial ideals.

The main object of study in this work, involutive bases, are a form of generating set of an ideal in a polynomial ring $\mathcal{R}$ of finitely many indeterminates over a field. More specifically, they are a special kind of Gröbner bases with additional combinatorial properties. Gröbner bases, introduced by Buchberger in his PhD thesis [19], allow to decide the ideal membership problem and to perform many common operations on polynomial ideals. By Buchberger's algorithm [19], a Gröbner basis can be computed for any given ideal in a finite number of steps. The optimization of this algorithm and, in general, the development of fast algorithms for computing Gröbner bases, was initiated already by Buchberger [17] and continues to be an active research topic [40, 38, 39]. Gröbner bases exist and can be computed also for submodules of finitely generated free $\mathcal{R}$-modules. Moreover, each Gröbner basis G is associated to the module of algebraic relations among the elements of G, its syzygy module. A Gröbner basis of this syzygy module is induced by G in a natural way [94]. A Gröbner basis generates a free module exactly when only the zero algebraic relation exists among its elements. Thus, iterating the computation of syzygy modules by Gröbner bases, one obtains a sequence of modules which measures the defect in freeness of the original basis. This sequence, or complex, of modules can be interpreted as a free resolution [10, 69, 91]. Complexes and resolutions of modules are more generally studied in homological algebra [91]. The literature on Gröbner bases is vast and is best accessible by consulting one of the textbooks devoted to their discussion [9, 3, 31].

Involutive bases have their origin in the works by Janet on the analysis of systems of (linear) partial differential equations [66, 67]. As in Gröbner basis theory, Janet used monomial, and thus combinatorial, structures as a tool by the means of which more complex (differential) algebraic structures can be analysed. Inspired by Janet's—and also Pommaret's [85]—works, Zharkov and Blinkov developed involutive bases for polynomial ideals [108]. Gerdt and Blinkov studied different types of involutive bases, introducing the framework of involutive divisions in the process [47]. The most well-known involutive divisions—the Janet and Pommaret divisions—go back to Janet's works. Further involutive divisions have been studied; see, e.g., [99, 59]. As Gröbner bases, involutive bases induce free resolutions of the ideals they generate. For some types of involutive divisions, the syzygy modules in this resolution are generated by involutive bases of the same type [96]. In the case of the Pommaret division, homological invariants like projective dimension and Castelnuovo-Mumford regularity can be read off directly from the original Pommaret basis. Not every monomial ideal possesses a finite Pommaret basis; those that do are termed *quasi-stable*. For the resolution induced by the Pommaret basis of a quasi-stable monomial ideal, an explicit formula is known [96]; however, this resolution is not necessarily minimal. This formula generalizes the well-known resolution formula found by Eliahou and Kervaire [37], which only applies when the Pommaret basis coincides with the minimal generating set of the ideal. A polynomial ideal is said to be in *quasi-stable position* when it possesses a finite Pommaret basis for the given coordinates; moreover, this position is a generic one [95]. For a comprehensive study and applications of the theory of involutive bases to commutative algebra and the geometric theory of partial differential equations, we refer to [97].

Our discussion has been restricted up to now to ideals $\mathcal{I}$ in the polynomial ring $\mathcal{R}$. However, the concept of Gröbner basis applies also to ideals in and finitely generated modules over the quotient ring $\mathcal{R}/\mathcal{I}$ [79, 69, 9]. Historically, the necessary concepts were already deveoped by Spear [102] and Zacharias [107]. Also for modules over these rings, Gröbner bases induce free resolutions [69]. These resolutions are in general infinite in length, i.e., infinite sequences of modules [73, 8, 69]. Finite parts of these resolutions can be efficiently computed and there exist implementations for this task, e.g. in Macaulay2 [54].

The type of marked bases that we consider are bases marked over quasi-stable monomial submodules of finitely generated free $\mathcal{R}$-modules [26, 28, 12]. In contrast to Gröbner bases, marked bases do not use term orders to ensure termination of the polynomial reduction processes associated to them. Instead, each basis polynomial is marked on one element of the minimal Pommaret basis of the quasi-stable monomial submodule used; these marked terms are then used to define a Noetherian and confluent reduction linked to the basis. For each given quasi-stable monomial submodule, one can consider the *marked family* of all bases marked on this specific module; membership in this marked family is described by an ideal of algebraic relations. Moreover, this construction is functorial [70]. Marked families can be used as a computational approach to the structural analysis of Hilbert and Quot schemes [6]. Moreover, they allow for a quasi-stable open covering of such schemes [6].

Our first contributions address the problem of *Gröbner redundancy* of involutive bases: For general polynomial ideals, minimal involutive bases are of larger cardinality than the corresponding minimal Gröbner bases. With the concept of Janet-like bases, Gerdt and Blinkov made the first step towards remedying this problem [50, 49]. Janet-like bases are in general more compact than the corresponding Janet bases, while preserving the good algorithmic properties of Janet bases [49]. We investigate this relation of algorithmic properties by introducing Janet-like trees. This tree structure is a recursive with respect to the variables of the polynomial ring. We exploit this structure first by applying Janet-like trees to the computation of complementary decompositions of monomial ideals. Further, we connect it to a construction by Hironaka [65] for complementary decompositions and derive an alternative characterization of quasi-stability from it. Secondly, we exploit the tree structure to develop new efficient algorithms for several tasks in involutive basis theory: the minimization of Janet bases, the recursive construction of both Janet and Pommaret bases, and a novel criterion for detecting obstructions to quasi-stability, which we apply to give a new deterministic algorithm to find a linear change of coordinates transforming a homogeneous polynomial ideal to quasi-stable position.

We proceed to introduce the new concept of involutive-like divisions and bases, generalizing the Janet-like division. We define the Pommaret-like division and thoroughly analyse its relation to the Janet-like division. Moreover, we show that both these involutive-like divisions are well-behaved with respect to syzygy constructions: they induce involutive-like bases of same type for their syzygy modules.

The second part of our contributions concerns the generalization of the concepts of involutive and involutive-like bases to ideals in quotient rings of the form $\mathcal{R}/\mathcal{I}$. We begin with a thorough analysis of Gröbner bases for ideals in such rings, for which we use the terminology *relative* Gröbner bases. We give algorithms for their constructions which are based on computations in the original polynomial ring $\mathcal{R}$. We continue with relative involutive and involutive-like bases; for syzygies, we establish that in particular relative Pommaret bases are adapted to the analysis of syzygy modules. However, such bases only exist in relative quasi-stable position. We examine this new notion of genericity and present algorithms to find linear coordinate changes to reach this position.

Our third main part of contributions concerns the use of relative involutive-like bases for the computation and analysis of free resolutions. For this, we focus on (relative) Pommaret and Pommaret-like bases. While Pommaret bases capture many homological properties of ideals in quasi-stable position [97], the resolutions induced by them need not be minimal, because already the basis of the ideal is not a minimal generating system. We show that Pommaret-like bases represent a significant improvement in this respect. Another aspect we investigate is the application to monomial ideals. For these, we are able to identify different classes of (relatively) quasi-stable ideals for which Pommaret-like bases induce the minimal free resolution. Even for other cases, the induced resolution has useful properties like Gröbner-reducedness in all higher syzygy modules. For a class of quasi-stable monomial ideals relative to a Clements-Lindström ideal [29], we obtain closed formulas for the differential of the induced resolution, thereby significantly generalizing the formula by Eliahou–Kervaire [37] and also a resolution of squarefree Borel ideals found by Gasharov et al. [42].

Our last block of contributions concerns marked bases, which present a framework for computationally analysing the properties of Hilbert and Quot schemes over the polynomial ring $\mathcal{R}$ [6]. The properties of Hilbert schemes over quotient rings are known for some types of rings, e.g. for Clements-Lindström rings [81]. Clements-Lindström rings arise by factorizing by irreducible quasi-stable monomial ideals satisfying additional degree assumptions. We introduce the novel concept of relative marked bases and give algorithms for the computation of relative marked families with respect to quotient rings defined by quasi-stable monomial ideals. Then we proceed to consider two classes of quasi-stable monomial ideals: Those that are Cohen-Macaulay and saturated, and those that are saturated Macaulay-Lex. For the first class, we obtain a quasi-stable open covering of the corresponding Hilbert scheme; for the second class, we analyse the lex-points of the scheme. Both classes include, but are not limited to, Clements-Lindström rings of positive Krull dimension.

The thesis is organized as follows. In Chapter 2, we recall some definitions and results that are fundamental to the chapters that follow. We will give pointers to the literature for these basic definitions and results. Moreover, this chapter serves as a means to introduce the notation that we will use in the sequel.

In Chapter 3, we first investigate tree structures for Janet bases and introduce Janet-like trees. Then we apply Janet-like trees for the analysis of complementary decompositions of monomial ideals. After this, we investigate recursive structures for involutive bases and use them to give a novel approach to find linear coordinate transformations that transform homogeneous polynomial ideals to quasi-stable position. Lastly, we introduce the concept of involutive-like bases, define the Pommaret-like division and discuss its properties.

Chapter 4 is devoted to the generalization of involutive and involutive-like bases to ideals in quotient rings. To this end, we first investigate thoroughly Gröbner bases for ideals in such rings, together with their syzygy theory. Then we proceed with the definition and analysis of the novel concepts of involutive and involutive-like bases for ideals in such rings. For each of these new types of bases, we also give their syzygy theory. Moreover we introduce the concept of relative quasi-stable position and present, in the homogeneous case, algorithms for determining linear coordinate transformations which transform ideals to this position.

Chapter 5 builds on the two previous chapters. We define the free reolution induced by the relative Pommaret basis of an ideal in relative quasi-stable position. In the case of Pommaret-like bases, we first work in the ordinary polynomial ring and analyse the free resolution induced by these bases. For the special case of monomial ideals, we state conditions under which the induced resolution is minimal. For relative Pommaret-like bases, we restrict our attention to quotient rings defined by Clements-Lindström ideals. We give a description of the module basis of the induced free resolution and apply it to give closed formulas for the Betti numbers and Poincaré series in case the resolution is minimal. Finally, for some classes of monomial ideals, we present a closed formula for the differential of the Pommaret-like induced resolution.

Chapter 6 is devoted to relative marked bases and can be read independently of the other chapters. After recalling the definition and basic properties of marked bases, we proceed to define relative marked bases and their associated functors. For the algorithmic part, we restrict to relative marked bases in quotient rings defined by quasi-stable monomial ideals and give an algorithm to compute the ideal representing the associated marked functor. For Cohen-Macaulay quotient rings defined by quasi-stable monomial ideals, we introduce a novel open covering of the Hilbert scheme defined by an admissible Hilbert polynomial. For Macaulay-Lex quotient rings defined by quasi-stable monomial ideals, we use our algorithmic decription of the associated marked functor to investigate the lex-points and we give examples for which this lex-point is singular.

Finally, in Chapter 7, we conclude and mention open questions arising from our work and possible further research directions.

Documents Used and Coauthors. This thesis incorporates material taken and partially adapted from documents originating from collaboration with several people. Chapters 2 to 4 include material from the articles [57, 58, 56], being joint work with Amir Hashemi and Werner Seiler. Chapter 5 has profited much from discussions with Amir Hashemi and Werner Seiler. Chapter 6 is joint work with Cristina Bertone, Francesca Cioffi, and Werner Seiler [13].

Let us move from chapters to sections. In Chapter 2, Section 2.2 is taken mostly from the introductory sections of [57, 58, 56]. Sections 2.1 and 2.3 are not found in the articles just cited. Section 3.1 is adapted from the preliminary sections of [58, 56]. Section 3.2 is largely equal to the main contents of [58]. The same is true for Sections 3.3 up to 3.5 and the article [56]. The last section of Chapter 3 is adapted from the conclusions sections of [58, 56]. Chapter 4 is largely equal to the contents of [57], with the exception of Section 4.4, which is not found in the articles cited. The origin of the material of Chapters 5 and 6 has already been stated in the preceding paragraph.

Finally, it should be noted that the results on the arithmetic complexity of algorithms presented in Chapter 3 are to the largest part due to Amir Hashemi, while the computational examples presented in Section 6.7 are due to Francesca Cioffi.

Chapter 2

Preliminaries

This chapter collects basic definitions and statements that we will need in the subsequent chapters. Furthermore, we fix notations that will be used often. Nowhere in this chapter is there any originality claimed. Moreover, not all basic results about the objects and structures defined will be mentioned in this chapter; some well-known constructions and results will be introduced at appropriate places in later chapters.

Note that we will use the symbol $\mathbb{N}$ for the set of all positive integers $\mathbb{Z}_{>0}$; we will write $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$.

2.1 Ideals, Modules, and Free Resolutions

Let $\mathbb{K}$ be a field. For most parts of this work, no further assumptions are necessary for $\mathbb{K}$; sometimes, we will require $\mathbb{K}$ to be infinite, but of arbitrary characteristic. We work with the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, \dots, x_n] = \mathbb{K}[X]$ over $\mathbb{K}$ in $n \geq 1$ variables.

Terms. Monomials. Degrees. As a $\mathbb{K}$-vector space, $\mathcal{R}$ has the basis $\mathcal{T} = \{x_1^{\mu_1} \cdots x_n^{\mu_n} \mid \mu_1, \dots, \mu_n \in \mathbb{N}_0\}$ of *terms*, which are products of non-negative integer powers of the variables. If $Y = \{y_1, \dots, y_k\} \subseteq X$ is a subset of variables, then we denote by $\mathcal{T}_Y = \{y_1^{\mu_1} \cdots y_k^{\mu_k} \mid \mu_i \in \mathbb{N}_0, 1 \leq i \leq k\}$ the monoid of all terms in $\mathcal{R}$ depending only on the variables in Y. To each term $t = \mathbf{x}^\mu = x_1^{\mu} \cdots x_n^{\mu_n} \in \mathcal{T}$ we associate its *total degree* $\deg(t) = \sum_{i=1}^n \mu_i$ and its *exponent vector*, *multidegree*, or *multiindex* $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}_0^n$. For an integer $d \geq 0$, we collect the subset of all terms of degree d in the set $\mathcal{T}_d \subset \mathcal{T}$. $\mathcal{T}_d$ generates the finite dimensional $\mathbb{K}$-vector space $\mathcal{R}_d$ of polynomials *homogeneous of degree* d: $\mathcal{R}_d := \langle \mathcal{T}_d \rangle_{\mathbb{K}}$. (Note that the zero polynomial is homogeneous of any degree.) Obviously, we have the direct sum of $\mathbb{K}$-vector spaces $\mathcal{R} = \oplus_{d \geq 0} \mathcal{R}_d$, and $\mathcal{R}_d \cdot \mathcal{R}_e \subseteq \mathcal{R}_{d+e}$ for all $d, e \geq 0$. For a given multidegree $\mu = (\mu_1, \dots, \mu_n)$, we write $\mathcal{R}_\mu = \langle \mathbf{x}^\mu \rangle_{\mathbb{K}}$ for the one-dimensional $\mathbb{K}$-vector space of *monomials* supported on the term $\mathbf{x}^\mu$. Also for this grading, we obtain the direct sum of vector spaces $\mathcal{R} = \oplus_{\mu \in \mathbb{N}_0^n} \mathcal{R}_\mu$; and as before, $\mathcal{R}_\mu \cdot \mathcal{R}_\nu \subseteq \mathcal{R}_{\mu+\nu}$ for all $\mu, \nu \in \mathbb{N}_0^n$.

Ideals. Homogeneity and Monomial Ideals. For ideals in the ring $\mathcal{R}$, we will use the letters $\mathcal{I}, \mathcal{J}, \ldots$, and we will write $\mathcal{I} \trianglelefteq \mathcal{R}$ to state that $\mathcal{I}$ is an ideal in the ring $\mathcal{R}$. Sometimes we also use normal font and inclusion symbols: $I \subseteq \mathcal{R}$, when no confusion can arise. If $G \subseteq \mathcal{R}$ is a subset, we write $\langle G\rangle_{\mathcal{R}}$ or simply $\langle G\rangle$ for the ideal generated by the elements of G in the ring $\mathcal{R}$. We call G a *generating system* of the ideal $\langle G\rangle$ generated by it. An ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is *homogeneous* if $\mathcal{I} = \oplus_{d\geq 0}(\mathcal{I} \cap \mathcal{R}_d)$. Note that an ideal is homogeneous if and only if there exists a generating system of it consisting of homogeneous polynomials. An ideal $\mathcal{I}$ is *multihomogeneous* or *monomial*, if $\mathcal{I} = \oplus_{\mu\in\mathbb{N}_0^n}(\mathcal{I}\cap\mathcal{R}_\mu)$. Note that $\mathcal{I}\cap\mathcal{R}_\mu \in \{\{0\}, \mathcal{R}_\mu\}$ for all multidegrees $\mu \in \mathbb{N}_0^n$; hence we may write a monomial ideal as $\mathcal{I} = \oplus_{\mathbf{x}^\mu\in\mathcal{I}}\mathcal{R}_\mu$ and thus the term set $\mathcal{T} \cap \mathcal{I}$ is a basis for $\mathcal{I}$ as a $\mathbb{K}$-vector space. There is a unique minimal finite generating system of $\mathcal{I}$ consisting of terms. It consists exactly of those terms in $\mathcal{T} \cap \mathcal{I}$ which are not divisible by any other terms in $\mathcal{T} \cap \mathcal{I}$. We write $\mathrm{Min}(\mathcal{I})$ for this minimal generating set of the monomial ideal $\mathcal{I}$. A monomial ideal $\mathcal{I}$ is called *irreducible* if $\mathrm{Min}(\mathcal{I})$ consists of pure variable powers; it is a prime ideal if and only if it is generated by variables.

Order Ideals. An *order ideal* is a subset $\mathcal{O} \subseteq \mathcal{T}$ such that for each term $x^\mu \in \mathcal{O}$, all divisors of x^μ are also contained in $\mathcal{O}$. In other words, $\mathcal{O}$ is an order ideal, if and only if there exists a monomial ideal $\mathcal{I}$ such that $\mathcal{T} \setminus \mathcal{O} = \mathcal{I} \cap \mathcal{T}$.

Quotient Rings. Each ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ induces the quotient ring $\mathcal{R}/\mathcal{I}$. Let $f \in \mathcal{R}$ be a polynomial. We write its equivalence class modulo $\mathcal{I}$ as either $f + \mathcal{I}$, $[f]_{\mathcal{I}}$, or simply $[f]$ if no misunderstandings can occur. In case each element of $\mathcal{R}/\mathcal{I}$ has a unique normal form, we even just write f. We will apply this short notation when $\mathcal{I}$ is a monomial ideal, and the set of terms $\mathcal{T} \setminus \mathcal{I}$ is a $\mathbb{K}$-vector space basis of $\mathcal{R}/\mathcal{I}$. We then represent elements of $\mathcal{R}/\mathcal{I}$ by polynomials supported on $\mathcal{T} \setminus \mathcal{I}$.

Modules. Free Modules and Degree Shifts. The polynomial ring $\mathcal{R}$ and its quotient rings $\mathcal{R}/\mathcal{I}$ are examples for $\mathcal{R}$-modules. $\mathcal{R}$ acts on $\mathcal{R}/\mathcal{I}$ via $g \cdot [f]_{\mathcal{I}} = [gf]_{\mathcal{I}}$. More generally, we work with finitely generated free $\mathcal{R}$-modules and their submodules. For $r \in \mathbb{N}$ consider the free $\mathcal{R}$-module $\mathcal{R}^r$ of *rank* r; we write its elements as column vectors $\mathbf{f} = (f_1, \ldots, f_r)^T$. For $i \in \{1, \ldots, r\}$ we write $\mathbf{e}_i$ for the vector whose ith component is 1 and whose other components are 0. The *terms* of $\mathcal{R}^r$ are $\mathbf{x}^\mu \cdot \mathbf{e}_i$ with $\mathbf{x}^\mu \in \mathcal{T}$ and $i \in \{1, \ldots, r\}$. $\mathbf{x}^\mu$ is the *polynomial part* of the term $\mathbf{x}^\mu \cdot \mathbf{e}_i$. We also use finitely generated modules $(\mathcal{R}/\mathcal{I})^r$; its elements are of the form $([f_1], \ldots, [f_r])^T$.

We will use free modules with degree shifts for free resolutions. Let $e \in \mathbb{N}$. We write $\mathcal{R}(-e)$ for a copy of the polynomial ring with shifted total degrees: precisely, $\mathcal{R}(-e)_{d+e} = \mathcal{R}_d$ for all $d \in \mathbb{N}_0$. Thus, the unit element 1 has degree $e > 0$ in $\mathcal{R}(-e)$. Degree shifts can also be applied to the $\mathcal{R}$-module $\mathcal{R}/\mathcal{I}$ when $\mathcal{I}$ is homogeneous. Degree shifts by multidegrees are also possible, for instance the unit element 1 has multidegree $(1, 2)$ in $\mathbb{K}[x_1, x_2](-(1, 2))$. For $\mathcal{R}/\mathcal{I}$, shifts by multidegrees are only possible when $\mathcal{I}$ is monomial.

Syzygies. Let $F = \{\mathbf{f}_1, \dots, \mathbf{f}_r\} \subseteq \mathcal{R}^s$ be an enumerated finite subset of a finitely generated free module $\mathcal{R}^s$. The *syzygy module* of F is a submodule of $\mathcal{R}^r$ defined by

$$\mathrm{Syz}(F) = \left\{ (g_1, \dots, g_r)^T \in \mathcal{R}^r \mid \sum_{i=1}^{r} g_i \mathbf{f}_i = 0 \right\}.$$

The elements of $\mathrm{Syz}(F)$ are called *syzygies*, and if F consists of homogeneous elements of degrees $d_1, \dots, d_r$, then $\mathrm{Syz}(F)$ is generated by homogeneous elements in $\mathcal{R}(-d_1) \oplus \cdots \oplus \mathcal{R}(-d_r)$. Also here, we can use multidegrees instead of total degrees. For subsets $F \subseteq (\mathcal{R}/\mathcal{I})^s$, we write $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}(F)$ to emphasize that we are working over the quotient ring.

Free Resolutions. We use syzygies to construct free resolutions of homogeneous ideals $\mathcal{I} \trianglelefteq \mathcal{R}$. A free resolution $\mathbf{F}$ of $\mathcal{I}$ is given by finitely generated free $\mathcal{R}$-modules $F_0, F_1, \dots$ and homogeneous $\mathcal{R}$-linear maps $\delta_0, \delta_1, \delta_2, \dots$ as in the following diagram

$$\mathbf{F}: \quad \cdots \xrightarrow{\delta_{m+2}} F_{m+1} \xrightarrow{\delta_{m+1}} F_m \xrightarrow{\delta_m} F_{m-1} \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} \mathcal{I} \to 0,$$

such that $\mathrm{im}(\delta_0) = \mathcal{I}$ and $\mathrm{im}(\delta_{m+1}) = \ker(\delta_m)$ for all $m \in \mathbb{N}_0$. The collection $\{\delta_m\}_{m \geq 0}$ of maps is called the *differential* of the resolution. Leaving aside degree shifts, we can write $F_m = \mathcal{R}^{r_m}$ for $m \geq 0$. Each map δ_m is completely described by the images $\delta(\mathbf{e}_i)$, $i \in \{1, \dots, r_m\}$; equivalently, δ_m is represented by a matrix $D_m \in \mathcal{R}^{r_{m-1} \times r_m}$, whose ith column is exactly $\delta_m(\mathbf{e}_i)$. (Note that we interpret the module $\mathcal{I}$ as a submodule of $\mathcal{R}^1$, so the matrix D_0 describing δ_0 is of format $(r_0 \times 1)$.) Moreover, $D_m \cdot D_{m+1} = 0$ for all m.

The definitions of syzygy modules and free resolutions now imply the following: $G := \{\delta_0(\mathbf{e}_1), \dots, \delta_0(\mathbf{e}_{r_0})\}$ is a homogeneous generating set of $\mathcal{I}$ and the columns of D_1 form a homogeneous generating set G_1 of $\mathrm{Syz}(G)$. Generally, the set G_m of columns of D_m is a homogeneous generating set of the iterated syzygy module $\mathrm{Syz}^m(G)$.

We equip all free modules F_m with appropriate degree shifts for their components, such that the columns of each matrix D_m are homogeneous in F_{m-1}. Thus, $F_m = \oplus_{d \geq 0} \mathcal{R}(-d)^{\beta_{m,d}}$, with $\beta_{m,d} \in \mathbb{N}_0$ and $\sum_{d \geq 0} \beta_{m,d} < \infty$ for all m. Analogously, we can apply multigraded shifts if the resolved ideal $\mathcal{I}$ is monomial.

Finally, we note that also for ideals $\mathcal{J} \trianglelefteq \mathcal{R}/\mathcal{I}$ in a quotient ring over a homogeneous ideal $\mathcal{I}$, resolutions by finitely generated free $\mathcal{R}/\mathcal{I}$-modules exist. For these resolutions, $\delta_0(F_0) = \mathcal{J}/\mathcal{I}$ and all modules F_m, $m \geq 0$, are direct sums of copies of $\mathcal{R}/\mathcal{I}$. Otherwise, the terminology is the same.

Minimal Free Resolutions. Betti Numbers and Tables. Since we work with homogeneous ideals $\mathcal{I}$, the matrices in any free resolution of $\mathcal{I}$ have homogeneous polynomials as entries. A free resolution is *minimal* if all entries in the matrices are either 0 or of positive degree. Up to isomorphism, there is exactly one minimal free resolution for each ideal $\mathcal{I}$. Since the ranks of the involved free modules F_m in a minimal free resolution are invariant under isomorphisms, they are a homological invariant of $\mathcal{I}$. They are called *(bigraded) Betti numbers* of $\mathcal{I}$.

Assume that in a minimal free resolution $\mathbf{F}$ of $\mathcal{I}$, $F_m = \oplus_{d\geq 0}\mathcal{R}(-d)^{\beta_{m,d}}$ for $m > 1$; then the numbers $\beta_{m,d} = \beta_{m,d}(\mathcal{I})$ are the Betti numbers of $\mathcal{I}$. By Hilbert's syzygy theorem, the minimal free resolution of $\mathcal{I} \trianglelefteq \mathcal{R}$ is of finite length. Thus, the collection $\{\beta_{m,d}(\mathcal{I})\}_{m,d\geq 0}$ of non-zero Betti numbers of $\mathcal{I}$ is finite. By minimality of $\mathbf{F}$, the sequence $(\min\{d \geq 0 \mid \beta_{m,d}(\mathcal{I}) > 0\})_{m\geq 0}$ is increasing; thus we can present the non-zero Betti numbers in a matrix $(b_{d,m})_{0\leq d\leq r, 0\leq m\leq s} = (\beta_{m,d+m}(\mathcal{I})) \in \mathbb{N}_0^{r\times s}$ for some positive integers $s = s(\mathcal{I}), r = r(\mathcal{I})$, such that there are neither trailing zero rows nor trailing zero columns. For further information see [34, Sec. 1B]

Example 2.1.1. *Consider the monomial ideal* $\mathcal{I} = \langle x^5, x^2y^2, y^3, z\rangle \trianglelefteq \mathcal{R} = \mathbb{K}[x, y, z]$. *It has a finite minimal free resolution*

$$\mathbf{F}: \quad 0 \longrightarrow F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} \mathcal{I} \to 0,$$

where $F_0 = \mathcal{R}(-5) \oplus \mathcal{R}(-4) \oplus \mathcal{R}(-3) \oplus \mathcal{R}(-1)$, $F_1 = \mathcal{R}(-7) \oplus \mathcal{R}(-6) \oplus \mathcal{R}(-5)^2 \oplus \mathcal{R}(-4)$, *and* $F_2 = \mathcal{R}(-8) \oplus \mathcal{R}(-6)$. *We collect the Betti numbers in the following Betti table:*

	0	*1*	*2*
0	*0*	*0*	*0*
1	*1*	*0*	*0*
2	*0*	*0*	*0*
3	*1*	*1*	*0*
4	*1*	*2*	*1*
5	*1*	*1*	*0*
6	*0*	*1*	*1*

Regularity and Projective Dimension. Consider a homogeneous ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ and a minimal free resolution of it, yielding the numbers $r(\mathcal{I})$ and $s(\mathcal{I})$ of rows and columns in its Betti table. Then $\operatorname{reg}(\mathcal{I}) := r(\mathcal{I})$ is the *Castelnuovo-Mumford regularity*, or simply *regularity*, of $\mathcal{I}$, [34, Sec. 4A] and $\operatorname{projdim}(\mathcal{I}) := s(\mathcal{I})$ is its *projective dimension* [34, p.206].

Poincaré Series for Infinite Free Resolutions. The minimal $\mathcal{R}/\mathcal{I}$-free resolutions of homogeneous ideals $\mathcal{J} \trianglelefteq \mathcal{R}/\mathcal{I}$ are in general infinite in the sense that inifinitely many non-zero Betti numbers exist. Thus, ideals in $\mathcal{R}/\mathcal{I}$ in general do not have finite regularity or projective dimension. As a succinct way of writing the infinitely many Betti numbers, we use *Poincaré series*. They are formal power series in two independent variables—u and s, say—such that the coefficient of a term $u^m s^d$ is given by the Betti number $\beta_{m,d}(\mathcal{J})$. See also [73, p. 15].

2.2 Gröbner and Involutive Bases

Term Orderings. A *term ordering* is a well-ordering $\prec$ on $\mathcal{T}$ which respects the multiplication of monomials, that is, $1 \prec x^\mu$ for all $x^\mu \neq 1$ and if $x^\mu \prec x^\nu$, then $x^\mu \cdot x^\rho \prec x^\nu \cdot x^\rho$ for all $x^\rho \in \mathcal{T}$. Given a monomial ordering $\prec$ and a polynomial $f \in \mathcal{R} \setminus \{0\}$, we denote the *leading term of* f by $\operatorname{lt}(f) := \max_\prec\{\operatorname{supp}(f)\}$ where $\operatorname{supp}(f)$ stands for the set of all terms appearing in f. Also, we write $\operatorname{lc}(f)$ for the coefficient of $\operatorname{lt}(f)$ in f. The leading monomial is then written as $\operatorname{lm}(f) := \operatorname{lc}(f)\operatorname{lt}(f)$. For each subset $F \subset \mathcal{R}$, we denote by $\operatorname{lt}(F)$ the set $\{\operatorname{lt}(f) \mid f \in F\}$.

We will assume throughout that $x_1 \prec \cdots \prec x_n$ for any used term ordering $\prec$, unless otherwise stated. In most cases, we will work either with the *lexicographic* ordering (shortly: *lex ordering*), or the *degree reverse lexicographic ordering* (shortly: *degrevlex ordering*).

Definition 2.2.1. *Let $x^\mu \neq x^\nu \in \mathcal{T}$ with $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$.*

- *With respect to the lex ordering, $x^\mu \prec x^\nu$ if and only if the last non-zero entry of $\mu - \nu$ is negative.*
- *With respect to the degrevlex ordering, $x^\mu \prec x^\nu$ if either* $\deg(x^\mu) < \deg(x^\nu)$ *or* $\deg(x^\mu) = \deg(x^\nu)$ *and the first non-zero entry of $\mu - \nu$ is positive.*

Gröbner Bases. For each ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ and each term ordering $\prec$, there exists a finite subset $G \subseteq \mathcal{I}$ such that $\operatorname{lt}(G)$ generates the monomial ideal $\operatorname{lt}(\mathcal{I}) := \langle \operatorname{lt}(f) \mid f \in \mathcal{I}\rangle_{\mathcal{R}}$. Such a subset G of $\mathcal{I}$ is called a *Gröbner basis* of $\mathcal{I}$ for the term ordering $\prec$. Note that every Gröbner basis of $\mathcal{I}$ is in particular also a generating set of $\mathcal{I}$. Gröbner bases are not unique, but every ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ has for each term ordering $\prec$ a unique *reduced Gröbner basis* which satisfies additionally that for each $g \in G$, $\operatorname{lc}(g) = 1$, and no monomial of g lies in $\langle \operatorname{lt}(G \setminus \{g\})\rangle$. The terms in the order ideal $\mathcal{T} \setminus \operatorname{lt}(\mathcal{I})$ form a $\mathbb{K}$-linear basis of the quotient ring $\mathcal{R}/\mathcal{I}$ and each polynomial $f \in \mathcal{P}$ has a unique *normal form* with respect to the Gröbner basis G which is a linear combination of terms of $\mathcal{T} \setminus \operatorname{lt}(\mathcal{I})$. We denote this normal form by $\operatorname{NF}_G(f)$. If $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis of the ideal $\mathcal{I}$ for $\prec$, then there exists for each ideal member $f \in \mathcal{I}$ a representation $f = q_1 g_1 + \cdots + q_t g_t$ with $q_i \in \mathcal{R}$ and where for each index i with $q_i \neq 0$ we have $\operatorname{lt}(q_i g_i) \preceq \operatorname{lt}(f)$. Such a representation is called a *standard representation* with respect to G for f; it is generally not unique.

We state explicitly, for later reference, the following result due to Macaulay.

Proposition 2.2.2 ([31, Prop. 4, pp. 250])**.** *Let G be a Gröbner basis of the ideal $\mathcal{I} \triangleleft \mathcal{R}$. Then, the factor ring $\mathcal{R}/\mathcal{I}$ is isomorphic as a $\mathbb{K}$-linear space to the space generated by all terms $t \notin \langle \operatorname{lt}(G)\rangle$.*

Gröbner bases and the first algorithm to compute them were introduced by Buchberger in his PhD thesis [18]. For a description of the division algorithm, Buchberger's algorithm to compute Gröbner bases and further details on their theory, we refer to standard textbooks like [9, 31, 30].

Module Gröbner Bases. The concept of term orderings and Gröbner bases can straightforwardly be extended to submodules of free $\mathcal{R}$-modules. Let $s \geq 1$ be a positive integer and let $\{\mathbf{e}_1, \dots, \mathbf{e}_s\}$ be the standard basis of $\mathcal{R}^s$. Then, every vector $\mathbf{f} \in \mathcal{R}^s$ is a finite $\mathbb{K}$-linear combination of module terms $x^\mu \mathbf{e}_i$ with $x^\mu \in \mathcal{T}$ and $i \in \{1, \dots, s\}$. A *module term ordering* $\prec$ is a total ordering and well-ordering on the set of all module terms such that, for all i and for all $x^\mu, x^\nu \in \mathcal{T}$, we have $1\mathbf{e}_i \prec x^\mu \mathbf{e}_i$ and if $x^\mu \mathbf{e}_i \prec x^\nu \mathbf{e}_j$, then $x^\mu x^\rho \mathbf{e_i} \prec x^\nu x^\rho \mathbf{e}_j$ for all $x^\rho \in \mathcal{T}$. Similar to the polynomial case, any element $\mathbf{f} \in \mathcal{R}^s$ can be written as a linear combination of module terms and one is able to define the notions of module leading coefficient, module leading term and module leading monomial for $\mathbf{f}$ which are denoted by $\mathbf{lc}(\mathbf{f})$, $\mathbf{lt}(\mathbf{f})$ and $\mathbf{lm}(\mathbf{f})$, respectively. If $\mathbf{u} := x^\mu \mathbf{e}_i$ and $\mathbf{v} := x^\nu \mathbf{e}_j$ are two module terms in $\mathcal{R}^s$, then we say that $\mathbf{u}$ divides $\mathbf{v}$, and write $\mathbf{u} \mid \mathbf{v}$ if $i = j$ and x^μ divides x^ν in $\mathcal{T}$. If $\mathbf{u}$ divides $\mathbf{v}$, then the quotient $\mathbf{v}/\mathbf{u}$ is defined to be $x^\nu / x^\mu \in \mathcal{T}$. Based on these definitions, one is able to build a theory of Gröbner bases for submodules of $\mathcal{R}^s$ analogous to the one for ideals in $\mathcal{R}$.

The following construction due to Schreyer [94] yields, for a given Gröbner basis G, a module Gröbner basis of the syzygy module $\operatorname{Syz}(G) \subseteq \mathcal{R}^{|G|}$. The module term ordering is derived from the term ordering used in $\mathcal{R}$ together with the set $\operatorname{lt}(G)$.

Construction 2.2.3. *Let $G = \{g_1, \dots, g_s\}$ be a Gröbner basis of the ideal $\mathcal{I} \trianglelefteq \mathcal{R}$. We write* $\operatorname{Syz}(G) = \operatorname{Syz}(g_1, \dots, g_s)$*; recall that* $(p_1, \dots, p_s) \in \operatorname{Syz}(G)$ *if and only if* $\sum_{i=1}^s p_i g_i = 0$. *On the module $\mathcal{R}^s$, define the* Schreyer module term ordering *$\prec_S$ by*

$$x^\mu \mathbf{e}_i \prec_S x^\nu \mathbf{e}_j \iff x^\mu \operatorname{lt}(g_i) \prec x^\nu \operatorname{lt}(g_j) \vee \big(x^\mu \operatorname{lt}(g_i) = x^\nu \operatorname{lt}(g_j) \wedge j < i\big). \tag{2.1}$$

*For $1 \leq i < j \leq s$, the S-*polynomial *of the generators g_i and g_j is defined to be* $\operatorname{S}(g_i, g_j) := \frac{\operatorname{lcm}(\operatorname{lt}(g_i),\operatorname{lt}(g_j))}{\operatorname{lm}(g_i)} g_i - \frac{\operatorname{lcm}(\operatorname{lt}(g_i),\operatorname{lt}(g_j))}{\operatorname{lm}(g_j)} g_j$. *By Buchberger's criterion,* $\operatorname{S}(g_i, g_j)$ *reduces to zero with respect to G for each i, j, which entails that it has a standard representation $\sum_{\ell=1}^s q_\ell g_\ell$, where the polynomials $q_\ell \in \mathcal{R}$ are such that* $\operatorname{lt}(q_\ell)\operatorname{lt}(g_\ell) \preceq \operatorname{lt}(\operatorname{S}(g_i, g_j))$ *for all $\ell \in \{1, \dots, s\}$ with $q_\ell \neq 0$. By definition of the Schreyer ordering, the leading module term of the resulting syzygy*

$$\mathbf{S}_{ij} := \frac{\operatorname{lcm}(\operatorname{lt}(g_i), \operatorname{lt}(g_j))}{\operatorname{lm}(g_i)} \mathbf{e}_i - \frac{\operatorname{lcm}(\operatorname{lt}(g_i), \operatorname{lt}(g_j))}{\operatorname{lm}(g_j)} \mathbf{e}_j - \sum_{\ell=1}^s q_\ell \mathbf{e}_\ell \tag{2.2}$$

is $\frac{\operatorname{lcm}(\operatorname{lt}(g_i),\operatorname{lt}(g_j))}{\operatorname{lm}(g_i)} \mathbf{e}_i$, *and one can show that the set $\{\mathbf{S}_{ij} \mid 1 \leq i < j \leq s\}$ is a Gröbner basis of* $\operatorname{Syz}(G)$ *with respect to the Schreyer module term ordering adapted to G. We refer to $\mathbf{S}_{ij}$ as the S-*syzygy *corresponding to g_i and g_j.*

2.2.1 Involutive Bases

Involutive bases are a central topic of this thesis. Hence, we give a more detailed presentation and present their basic properties in a form allowing for easy reference in later chapters. For a more complete overvew of the theory of involutive bases, see [97].

The main idea of involutive divisions is that to each generator h in a basis H a subset $\mathrm{M}_{\mathcal{L}}(h,H) \subseteq X$ of multiplicative variables is assigned and that one considers only linear combinations of the generators where each generator $h \in H$ is multiplied by a coefficient depending only on the variables in $\mathrm{M}_{\mathcal{L}}(h,H)$. In contrast to Gröbner bases, not every monomial basis of a monomial ideal is automatically an involutive basis.

Involutive Divisions. The rule for the assignment of the multiplicative variables is called an *involutive division*.

Definition 2.2.4. *An* involutive division $\mathcal{L}$ *on* $\mathcal{T} \subset \mathcal{R}$ *associates to any finite set* $U \subset \mathcal{T}$ *of terms and any term* $u \in U$ *a set of* $\mathcal{L}$-non-multipliers $\bar{\mathcal{L}}(u,U)$ *given by the terms contained in a prime monomial ideal. The variables generating this prime ideal are called the* non-multiplicative variables $\mathrm{NM}_{\mathcal{L}}(u,U) \subseteq X$ *of* $u \in U$*. The set of* $\mathcal{L}$-multipliers $\mathcal{L}(u,U)$ *is given by the order ideal* $\mathcal{T} \setminus \bar{\mathcal{L}}(u,U)$*; it is a subring of* $\mathcal{R}$ *generated by the set of* multiplicative variables $\mathrm{M}_{\mathcal{L}}(u,U) = X \setminus \mathrm{NM}_{\mathcal{L}}(u,U)$*. For any term* $u \in U$*, its* involutive cone *is defined as* $\mathcal{C}_{\mathcal{L}}(u,U) = u \cdot \mathcal{L}(u,U)$*. For an involutive division, the involutive cones must satisfy the following conditions:*

(i) For two terms $v \neq u \in U$ *with* $\mathcal{C}_{\mathcal{L}}(u,U) \cap \mathcal{C}_{\mathcal{L}}(v,U) \neq \emptyset$*, we have* $u \in \mathcal{C}_{\mathcal{L}}(v,U)$ *or* $v \in \mathcal{C}_{\mathcal{L}}(u,U)$*.*
(ii) If a term $v \in U$ *lies in an involutive cone* $\mathcal{C}_{\mathcal{L}}(u,U)$*, then* $\mathcal{L}(v,U) \subset \mathcal{L}(u,U)$*.*
(iii) For any term u *in a subset* $V \subset U$*, we have* $\mathcal{L}(u,U) \subseteq \mathcal{L}(u,V)$*.*

We write $u \mid_{\mathcal{L}} w$ *for a term* $u \in U$ *and an arbitrary term* $w \in \mathcal{T}$*, if* $w \in \mathcal{C}_{\mathcal{L}}(u,U)$*. In this case,* u *is called an* $\mathcal{L}$-involutive divisor *of* w *and* w *an* $\mathcal{L}$-involutive multiple *of* u*.*

Conditions (i) and (ii) ensure that involutive cones can intersect only trivially. Condition (iii) is often called the *filter axiom*. Obviously, it suffices for defining an involutive division to say what are the (non-)multiplicative variables for each term u in a finite set U. Note that involutive divisibility $u \mid_{\mathcal{L}} w$ implies ordinary divisibility, but not vice versa.

Monomial Involutive Bases. As with Gröbner bases, involutive bases are defined via monomial structures. For monomial ideals, we define involutive bases as follows.

Definition 2.2.5. *For a finite set of terms* $U \subset \mathcal{T}$ *and an involutive division* $\mathcal{L}$ *on* $\mathcal{T}$*, the* involutive span *of* U *is the union* $\mathcal{C}_{\mathcal{L}}(U) = \bigcup_{u \in U} \mathcal{C}_{\mathcal{L}}(u,U)$*. The set* U *is* involutively complete *or a* weak involutive basis *of the ideal generated by* U*, if* $\mathcal{C}_{\mathcal{L}}(U) = \mathcal{T} \cdot U$*. For a* (strong) involutive basis *the union must be disjoint, i. e. every term in* $\mathcal{C}_{\mathcal{L}}(U)$ *has a unique involutive divisor. An involutive division is* Noetherian*, if every monomial ideal in* $\mathcal{R}$ *possesses an involutive basis. The* $\mathcal{L}$*-involutive basis* H *is* minimal*, if any other* $\mathcal{L}$*-involutive basis* H' *of* $\mathcal{I}$ *contains* H *as subset.*

For involutive divisions that are *continuous* (see [97, Def. 4.1.3]) or even *constructive* (see [97, Def. 4.1.7]), the following useful properties hold:

Proposition 2.2.6. *[97, Prop. 4.1.4] For a continuous involutive division $\mathcal{L}$, a finite set of terms $U \subset \mathcal{T}$ is an $\mathcal{L}$-involutive basis of the monomial ideal $\langle U \rangle$ if and only if, for each $u \in U$ and $x \in \mathrm{NM}_{\mathcal{L}}(u, U)$, we have $xu \in \mathcal{C}_{\mathcal{L}}(U)$.*

We call the criterion implied by Proposition 2.2.6 the criterion of *local involutivity*.

Proposition 2.2.7. *For a constructive Noetherian involutive division $\mathcal{L}$, every monomial ideal has a unique minimal $\mathcal{L}$-involutive basis.*

Polynomial Involutive Bases. Given a finite set H of polynomials, a term ordering $\prec$ and an involutive division $\mathcal{L}$, we call H a *weak $\mathcal{L}$-involutive basis* of the ideal $\mathcal{I} = \langle H \rangle$, if $\mathrm{lt}(H)$ is a weak $\mathcal{L}$-involutive basis of $\mathrm{lt}(\mathcal{I})$. For a *(strong) $\mathcal{L}$-involutive basis*, we require in addition that $\mathrm{lt}(H)$ is a strong $\mathcal{L}$-involutive basis and that all generators $h \in H$ have pairwise distinct leading terms. We assign to each polynomial $h \in H$ the multiplicative variables $\mathrm{M}_{\mathcal{L}}(\mathrm{lt}\,(h), \mathrm{lt}\,(H))$ and define the involutive cone $\mathcal{C}_{L,H,\prec}(h) := h\mathbb{K}[\mathrm{M}_{\mathcal{L}}(\mathrm{lt}\,(h), \mathrm{lt}\,(H))]$. A strong involutive basis H of an ideal $\mathcal{I}$ induces then a disjoint decomposition $\mathcal{I} = \bigoplus_{h \in H} \mathcal{C}_{\mathcal{L},H,\prec}(h)$ as $\mathbb{K}$-linear spaces. H is a *minimal* $\mathcal{L}$-involutive basis of $\mathcal{I}$, if $\mathrm{lt}(H)$ is a minimal $\mathcal{L}$-involutive basis of $\mathrm{lt}(\mathcal{I})$. If G is an involutive basis of the polynomial ideal $\mathcal{I}$, then $\mathrm{lt}\,(G)$ is an involutive basis of the monomial ideal $\mathrm{lt}\,(\mathcal{I})$. Thus any involutive basis is also a Gröbner basis.

For most applications, we will require involutive bases to be *involutively head autoreduced*, in the following sense:

Definition 2.2.8. *The set $F \subset \mathcal{R}$ of polynomials is* involutively head autoreduced *for the involutive division $\mathcal{L}$ and the term ordering $\prec$, if for no $f \in F$ there exists an $h \in F \setminus \{f\}$ with $\mathrm{lt}(h) \mid_{\mathcal{L}} \mathrm{lt}(f)$. Let $\mathcal{I} \subset \mathcal{R}$ be an ideal. An $\mathcal{L}$-involutively head autoreduced subset $G \subset \mathcal{I}$ is an involutive basis of $\mathcal{I}$ for $\mathcal{L}$ and $\prec$, if for any ideal element $f \in \mathcal{I}$ there exists a generator $g \in G$ such that $\mathrm{lt}(g) \mid_{\mathcal{L}} \mathrm{lt}(f)$.*

The Janet and Pommaret Divisions. For most purposes, two involutive divisions are particularly important: The Janet and Pommaret divisions.

The *Janet division* was, like the Pommaret division, already introduced by Janet [67, pp. 16-17]. Let $U \subset \mathcal{T}$ be a finite set of terms. For each sequence $d_1, \ldots, d_n$ of non-negative integers and for each index $1 \leq i \leq n$, we introduce the corresponding *Janet class* as the subset

$$U_{[d_i,\ldots,d_n]} = \left\{u \in U \mid \deg_j(u) = d_j,\ i \leq j \leq n\right\} \subseteq U\,. \tag{2.3}$$

The variable x_n is *Janet multiplicative* (or shorter $\mathcal{J}$-multiplicative) for the term $u \in U$, if $\deg_n(u) = \max\{\deg_n(v) \mid v \in U\}$. For $i < n$ the variable x_i is Janet multiplicative for $u \in U_{[d_{i+1},\ldots,d_n]}$, if $\deg_i(u) = \max\{\deg_i(v) \mid v \in U_{[d_{i+1},\ldots,d_n]}\}$. The Janet division is Noetherian, continuous, and constructive. We sometimes write $\mathrm{MinJB}(\mathcal{I})$ for the minimal Janet basis of a given monomial ideal $\mathcal{I}$. We write $\mathrm{M}_{\mathcal{J}}(u, U)$ for the set of Janet multiplicative variables of a term $u \in U$, and by $\mathrm{NM}_{\mathcal{J}}(u, U)$ we denote the non-multiplicative variables.

Example 2.2.9. *In the polynomial ring $\mathbb{K}[x_1, x_2, x_3]$, consider the monomial ideal $\mathcal{I} = \langle x_1x_3^2,\ x_2x_3,\ x_1^2x_3\rangle$. The given minimal generating set is not a Janet basis of $\mathcal{I}$, but if we extend it to the set $\{x_1x_3^2,\ x_2x_3,\ x_1^2x_3,\ x_2x_3^2\}$, then we obtain one.*

We now proceed to the Pommaret division. The *class* of a term $1 \neq x^\mu \in \mathcal{T}$ with $\mu = (\mu_1, \ldots, \mu_n)$ is defined as the index $\operatorname{cls}(x^\mu) = \min\{i \mid \mu_i \neq 0\}$. A variable x_i is Pommaret multiplicative for x^μ, if $i \leq k$. All variables are Pommaret multiplicative for the trivial term 1. Note that the thus defined *Pommaret division* is global, i. e. the assignment of multiplicative variables is independent of any finite set $U \subset \mathcal{T}$. In contrast to the Janet division, the Pommaret division is not Noetherian, as e. g. the ideal $\mathcal{I} = \langle x_1x_2\rangle$ does not possess a finite Pommaret basis (it does not contain an element of class 2). Nevertheless, the Pommaret division is continuous and constructive. If a monomial ideal $\mathcal{I}$ possesses a Pommaret basis, we sometimes write $\operatorname{MinPB}(\mathcal{I})$ for its minimal Pommaret basis. We write $\mathrm{M}_{\mathcal{P}}(u)$ for the set of Pommaret multiplicative variables of a term $u \in \mathcal{T}$, and by $\mathrm{NM}_{\mathcal{P}}(u)$ we denote the non-multiplicative variables.

Quasi-Stable Ideals and Quasi-Stable Position. For sufficiently large fields $\mathbb{K}$, this non-Noetherianity of the Pommaret division is only a problem of the used coordinates. After a generic linear change of variables any ideal $\mathcal{I} \subseteq \mathcal{R}$ admits a finite Pommaret basis [97, Thm. 4.3.15]. In this case, $\mathcal{I}$ is said to be in *quasi-stable position*. An in-depth study of this question can be found in [60] together with a deterministic algorithm for the explicit construction of "good" coordinates for any given ideal $\mathcal{I} \subset \mathcal{R}$. For Pommaret bases, we will always consider the degree reverse lexicographical ordering $\prec$ with $x_1 \prec \cdots \prec x_n$, as it is the only class-respecting term ordering [97, Lem. A.1.8]. As generally a monomial ideal does not remain monomial after a linear change of variables, Pommaret bases exist only for a special class of monomial ideals.

Definition 2.2.10. *A monomial ideal $\mathcal{I}$ is called* quasi-stable*, if for any term $x^\mu \in \mathcal{I}$ and for any index i with $\operatorname{cls}(x^\mu) < i \leq n$ an exponent $s \geq 0$ exists such that $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \in \mathcal{I}$. A polynomial ideal $\mathcal{I}$ is in* quasi-stable position*, if $\operatorname{lt}(\mathcal{I})$ is quasi-stable.*

One easily verifies that it suffices to consider in the definition of a quasi-stable ideal $\mathcal{I}$ only the terms x^μ in an arbitrary finite monomial generating set of $\mathcal{I}$. Quasi-stable ideals appear in many places (and are known under many different names like *ideals of Borel type*, *ideals of nested type* or *weakly stable ideals*). Besides the above combinatorial definition, they can be characterised by many algebraic properties. For our purposes, the following characterisation is relevant.

Proposition 2.2.11 ([97, Prop. 5.3.4])**.** *A monomial ideal $\mathcal{I}$ possesses a finite Pommaret basis, if and only if it is quasi-stable.*

We now define stable monomial ideals and describe their relation to the Janet and Pommaret involutive divisions.

Definition 2.2.12. *A monomial ideal $\mathcal{I} \subset \mathcal{R}$ is called* stable*, if for any term $x^\mu \in \mathcal{I}$ and for any index $k = \operatorname{cls}(x^\mu) < i \leq n$, we have $x_i x^\mu / x_k \in \mathcal{I}$.*

Remark 2.2.13. *It is clear from the definitions that stable ideals are quasi-stable.*

While quasi-stable ideals are characterized by the existence of a finite Pommaret basis for them, we can characterize stable ideals as follows:

Proposition 2.2.14. *A monomial ideal $\mathcal{I} \subseteq \mathcal{R}$ is stable if and only if its minimal generating set is also a Pommaret basis of $\mathcal{I}$. Moreover, this basis coincides also with the minimal Janet basis of the ideal.*

Graphs and Resolutions Induced by Involutive Bases. Consider an $\mathcal{L}$-involutive basis $H \subset \mathcal{R}$ of a polynomial ideal $\mathcal{I} = \langle H \rangle$ with respect to a continuous involutive division $\mathcal{L}$. The set $\operatorname{lt}(H)$ is a strong $\mathcal{L}$-involutive basis of the leading ideal $\operatorname{lt}(\mathcal{I})$. One can construct an acyclic directed graph, the *$\mathcal{L}$-graph*, with node set $\operatorname{lt}(H)$ and arrows from $\operatorname{lt}(h_i)$ to $\operatorname{lt}(h_j)$ whenever there is a non-multiplicative variable $x \in \mathrm{NM}_{\mathcal{L}}(\operatorname{lt}(h_i), \operatorname{lt}(H))$ such that $\operatorname{lt}(h_j)$ is an $\mathcal{L}$-divisor of $x \operatorname{lt}(h_i)$ [97, Lem. 5.4.5]. Now consider the following method of enumerating $\operatorname{lt}(H)$: As first element $\operatorname{lt}(h_1)$, take any leading term whose node in the $\mathcal{L}$-graph is not the target of any arrow. Deleting $\operatorname{lt}(h_1)$ and its associated arrows from the graph, we obtain another acyclic graph, and as the second element $\operatorname{lt}(h_2)$ in the enumeration we take a leading term whose node is not the target of any arrow in the modified graph. Continuing in this manner, we obtain an *$\mathcal{L}$-ordering* of $\operatorname{lt}(H)$.

Using a Construction 2.2.3, one can use the $\mathcal{L}$-involutive homogeneous basis H, ordered according to an $\mathcal{L}$-ordering, to construct a Gröbner basis G_{Syz} of $\operatorname{Syz}(H)$ that has as leading terms exactly the module terms $x\mathbf{e_i}$, where $x \in \mathrm{NM}_{\mathcal{L}}(\operatorname{lt}(h_i), \operatorname{lt}(H))$. If $\mathcal{L}$ is of *Schreyer type* [97, Def. 5.4.8], then G_{Syz} is again an $\mathcal{L}$-involutive basis, and the construction can be iterated to yield a linear, but generally non-minimal, free resolution of $\langle H \rangle$. The Pommaret and Janet divisions are of Schreyer type [97, Lem. 5.4.9]. We will use Schreyer-type constructions in Chapters 3, 4 and 5.

The resolution induced by the Pommaret basis of a homogeneous ideal $\mathcal{I}$ in quasi-stable position can be used to determine the Castelnuovo-Mumford regularity and projective dimension of $\mathcal{I}$ without knowing a minimal free resolution. The Castelnuovo-Mumord regularity is simply the largest degree of a generator in the Pommaret basis; the projective dimension is the maximal number of non-multiplicative variables that an element of the Pommaret basis can have. For further details, see [97, Sec. 5.5].

For a quasi-stable monomial ideal $\mathcal{I}$, we refer to [97, Thm. 5.4.18] for an explicit formula for the differential of the resolution induced by the monomial Pommaret basis. It is immediate from [97, Eq. (5.53)] that the resolution is minimal if and only if $\mathcal{I}$ is stable. The formula can be read off from the *weighted $\mathcal{P}$-Graph* of the basis, which includes for each arrow $h_i \to h_j$ not only the variable $x \in \mathrm{NM}_{\mathcal{P}}(h_i)$ with $xh_i \in \mathcal{C}_{\mathcal{P}}(h_j)$, but also the cofactor $t \in \mathbb{K}[\mathrm{M}_{\mathcal{P}}(h_j)]$ such that $xh_i = th_j$. We will generalize this differential formula in Chapter 5, but as we will use slightly different notation, we do not repeat the results of [97] verbatim here.

2.3 Hilbert Schemes

In this section, we give a brief overview over the notions one needs to define Hilbert schemes. For our purposes, we view Hilbert schemes as sets of polynomial ideals that share the same Hilbert polynomial—sets that can be themselves given the structure of an algebraic variety in a natural way. A description of the full background of the notion of Hilbert schemes being schemes that parameterize the Hilbert functor, and the notions from category theory needed to define this functor, are out of the scope of this work.

Hilbert Functions and Hilbert Polynomials. Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a homogeneous ideal. The *Hilbert function* of $\mathcal{I}$ is defined as follows:

$$\mathrm{HF}_{\mathcal{I}} : \mathbb{N}_0 \to \mathbb{N}_0,\ d \mapsto \dim_{\mathbb{K}}(\mathcal{R}_d/(\mathcal{R}_d \cap \mathcal{I})). \tag{2.4}$$

For a degree respecting term order $\prec$ such as degrevlex, $\mathrm{HF}_{\mathcal{I}} = \mathrm{HF}_{\mathrm{lt}_{\prec}(\mathcal{I})}$. Thus the analysis of Hilbert functions is essentially a combinatorial study of monomial ideals. For monomial $\mathcal{I}$, $\mathrm{HF}_{\mathcal{I}}(d)$ is the number of terms of degree d that are not contained in $\mathcal{I}$. For sufficiently large $d > 0$, the values $\mathrm{HF}_{\mathcal{I}}(d)$ are equal to the values $\mathrm{HP}_{\mathcal{I}}(d)$ of the *Hilbert polynomial* of $\mathcal{I}$, $\mathrm{HP}_{\mathcal{I}} \in \mathbb{Q}[x]$, and this Hilbert polynomial is unique for $\mathcal{I}$. The *Hilbert series* of $\mathcal{I}$ is a formal power series that has the values of $\mathrm{HF}_{\mathcal{I}}$ as coefficients:

$$\mathrm{HS}_{\mathcal{I}} = \sum_{d=0}^{\infty} \mathrm{HF}_{\mathcal{I}}(d) z^d. \tag{2.5}$$

Lexicographic Ideals. Gotzmann Number. Not every formal power series in one variable with non-negative integer coefficients occurs as the Hilbert series of a monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$. One can show that for every Hilbert series $\sum a_d z^d$ ($a_d \in \mathbb{N}_0$) that does occur, taking for each $d \geq 0$ the a_d lexicographically largest terms in $\mathcal{T}_d$ and taking the union, one obtains the $\mathbb{K}$-basis of a monomial ideal $\mathcal{L}$ attaining this Hilbert series. We call $\mathcal{L}$ the *lex-ideal* associated to the Hilbert series. Several Hilbert series can lead to the same Hilbert polynomial, so there are several lex-ideals attaining a given Hilbert polynomial $p \in \mathbb{Q}[x]$. But there is exactly one saturated lex-ideal $\mathcal{L}(p)$ realizing p; moreover, among all lex-ideals realizing p, $\mathcal{L}(p)$ has, coefficient-wise, the smallest Hilbert series.

In is not hard to show that each lex-ideal $\mathcal{L}$ is quasi-stable (even stable). Thus, the regularity of $\mathcal{L}$ is exactly the highest total degree of an element of $\mathrm{Min}(\mathcal{L})$, because this is also its minimal Pommaret basis. Gotzmann's regularity theorem [53] states that, given a Hilbert polynomial p, the *Gotzmann number* $D(p) := \mathrm{reg}(\mathcal{L}(p))$ is a (sharp) upper bound for the regularity of any homogeneous ideal with Hilbert polynomial p. In particular, any such ideal can be generated in degrees $\leq D(p)$.

Construction of Hilbert Schemes. Given a valid Hilbert polynomial p, the broad idea underlying the Hilbert scheme associated to this Hilbert polynomial is to equip the set of all saturated homogeneous ideals $\mathcal{I} \trianglelefteq \mathcal{R}$ with $\mathrm{HP}_{\mathcal{I}} = p$ with a topological structure by representing this set of ideals as an algebraic variety in some projective space over $\mathbb{K}$—possibly of very large dimension.

Let $D = D(p)$ be the Gotzmann number of the Hilbert polynomial p. Two homogeneous saturated ideals $\mathcal{I}, \mathcal{J} \trianglelefteq \mathcal{R}$ are equal if and only if their truncation ideals $\mathcal{I}_{\geq D} = \oplus_{d \geq D}(\mathcal{I} \cap \mathcal{R}_d)$ and $\mathcal{J}_{\geq D} = \oplus_{d \geq D}(\mathcal{J} \cap \mathcal{R}_d)$ are equal. Combining this with Gotzmann's regularity theorem, we can work with ideals generated in degree D. By Gotzmann's persistence theorem [53], if $\mathcal{R}_D/(\mathcal{I} \cap \mathcal{R}_D)$ has $\mathbb{K}$-dimension $p(D)$, then $\mathcal{I}_{\geq D}$ has Hilbert polynomial p if and only if also in the next degree, $D+1$, we have $\dim_{\mathbb{K}}(\mathcal{R}_{D+1}/(\mathcal{I} \cap \mathcal{R}_{D+1})) = p(D+1)$.

Thus, we can interpret the Hilbert scheme associated to the polynomial p as a subscheme of the Grassmannian of $(\dim_{\mathbb{K}}(\mathcal{R}_D) - p(D))$-dimensional $\mathbb{K}$-subspaces of $\mathcal{R}_D$—the condition such a subscheme has to fulfil in order to belong to the Hilbert scheme can be described by algebraic equations. The Grassmannian, in its turn, can be interpreted as a subscheme of a higher-dimensional projective space via the Plücker embedding. Thus, all in all, the Hilbert scheme can be viewed as an algebraic variety in a projective space of high dimension.

For more information on Hilbert schemes and their category theoretical definitions, see e.g. [52, Ch. 14], [35, VI.2.2].

Chapter 3

Recursive Structures and Involutive-like Bases

In this chapter, we work with tree representations of sets of terms, especially of involutive bases. The tree representations encode recursive structures of the involutive bases we consider; we use these structures to obtain new results on complementary decompositions of monomial ideals, on the minimization of Janet bases and on the effective determination of coordinates in which a given homogeneous ideal is in quasi-stable position.

A second focus of the chapter is on the Janet-like division introduced by Gerdt and Blinkov. This generalization of the concept of Janet division provides algorithmic advantages like sparseness of the induced Janet-like bases, while profiting from the same type of recursive structure, which can also be expressed in the form of a tree.

In Section 3.1, we will give Gerdt and Blinkov's definition of the Janet-like division and introduce the tree representations that we will use subsequently. In Section 3.2 we will use tree structures, and especially the Janet-like tree to obtain new results on complementary decompositions of monomial ideals. In Section 3.3, we will exploit the recursive structures to obtain a new algorithm for the determination of minimal Janet bases and for the deterministic construction of quasi-stable coordinates for a homogeneous ideal. In Section 3.4, we will introduce the new concept of involutive-like divisions. We focus on the Pommaret-like and Janet-like divisions, their properties, and their relation to each other. Finally, we will develop the syzygy theory of Janet-like and Pommaret-like bases in Section 3.5.

3.1 Janet-like Division and Tree

Gerdt [46] proposed an efficient algorithm for the construction of involutive bases by a completion process where the products of elements of the current basis by non-multiplicative variables are reduced with respect to the basis. This process terminates for any Noetherian division in finitely many steps. To further improve the computation of Gröbner bases for ideals where the Janet basis is much larger than the reduced Gröbner basis (toric ideals are a prototypical example), Gerdt and Blinkov introduced [50, 49] a generalisation of Janet bases, the so-called Janet-like bases, where not only non-multiplicative variables but also non-multiplicative powers are considered in the completion process.

Definition 3.1.1. *Let $U \subset \mathcal{T}$ be a finite set of terms. For any term $u \in U$ and any index $1 \leq i \leq n$, we set*

$$h_i(u, U) = \max\left\{\deg_i(v) \mid u, v \in U_{[d_{i+1},\dots,d_n]}\right\} - \deg_i(u)\,.$$

If $h_i(u, U) > 0$, the power $x_i^{k_i}$ with

$$k_i = \min\left\{\deg_i(v) - \deg_i(u) \mid v, u \in U_{[d_{i+1},\dots,d_n]}, \deg_i(v) > \deg_i(u)\right\}$$

is called a non-multiplicative *power of u for the* Janet-like division. *The set of all non-multiplicative powers of $u \in U$ is denoted by* $\mathrm{NMP}(u, U)$. *The elements of the set*

$$\mathrm{NM}(u, U) = \{v \in \mathcal{T} \mid \exists w \in \mathrm{NMP}(u, U) : w \mid v\}$$

*are called the J-*non-multipliers *for $u \in U$. The terms outside of it are the J-*multipliers *for u. An element $u \in U$ will be called a* Janet-like divisor *of $w \in \mathcal{T}$, if $w = u \cdot v$ with v a J-multiplier for u.*

Although the Janet-like division is not an involutive division, it preserves all algorithmic properties of the Janet division and allows for the construction of Janet-like bases and in turn Gröbner bases. Indeed, the main algorithmic idea for the construction of Janet-like bases is similar to that of Janet bases, instead of multiplying with non-multiplicative variables one now multiplies with non-multiplicative powers. One can show that any ideal has a Janet-like basis which is a subset of its Janet basis.

The lattice of the Janet classes together with the set theoretic inclusion relation possesses a natural tree structure for any finite set $U \subset \mathcal{T}$ of terms. Following Gerdt et al. [51], we call this tree the *Janet tree* of U, although our tree is not the same as theirs. As their main concern was efficiency, they presented immediately a representation as binary tree which somewhat obscures the very natural underlying mathematical structure. Our presentation follows [97, Addendum §3.1] adapted to our purposes here. One should note that the bar codes of Ceria encode essentially the same information in a different manner [23].

Janet trees allow us to perform many operations relevant for Janet bases – like determining multiplicative variables or finding an involutive divisor – in a very efficient manner. We will show later that one can read off a complementary decomposition without any further computations by simply traversing the Janet tree. Each node in the Janet tree corresponds to a non-empty Janet class and the edges represent inclusions. It turns out to be convenient to represent the Janet class $U_{[d_i,\dots,d_n]}$ by the term $x_i^{d_i} \cdots x_n^{d_n}$ (although this term is not necessarily contained in the class!). Furthermore, we store in each node a list of variables which are multiplicative for any term contained in the class so that each node is a pair (x^ν, V) consisting of a term and a subset of the variables X.

Assume that $U = \{x^{\mu_1}, \ldots, x^{\mu_m}\}$ where $\mu_i = (\mu_{i1}, \ldots, \mu_{in})$ for each i and $x^{\mu_1} \prec_{lex} \cdots \prec_{lex} x^{\mu_m}$ with $x_1 \prec_{lex} \cdots \prec_{lex} x_n$. We divide the tree into $n+1$ levels with the root being at level $n+1$ and all leaves at level 1. The root contains the term 1 (corresponding to the Janet class $U_{[]} = U$) and the empty set. Its children correspond to the non-empty classes $U_{[d_n]}$ with $0 \leq d_n \leq \mu_{mn}$ and each contains the term $x_n^{d_n}$ and the empty set except for $d_n = \mu_{mn}$ which contains the set $\{x_n\}$, as x_n is multiplicative for all terms in this Janet class. Then we continue recursively. Assume that we have a node (x^ν, V) at level $i+1$ with $i < n$, i. e. $\operatorname{cls} x^\nu \geq i+1$. Then its children correspond to the non-empty Janet classes represent by terms of the form $x_i^a x^\nu$ and they all contain the same set V except for the one with the maximal value of a where x_i is added to V. We sort the children according to increasing values of a, so that it is always the rightmost child which obtains the additional multiplicative variable x_i. The nodes at level 1 contain then in lexicographic order the terms in U together with their Janet multiplicative variables. Figure 3.1 shows the Janet tree of the set $U = \{x_1^2x_3^3,\, x_2^4x_3^3,\, x_1^2x_3^5,\, x_2^2x_3^5\} \subset \mathbb{K}[x_1, x_2, x_3]$.

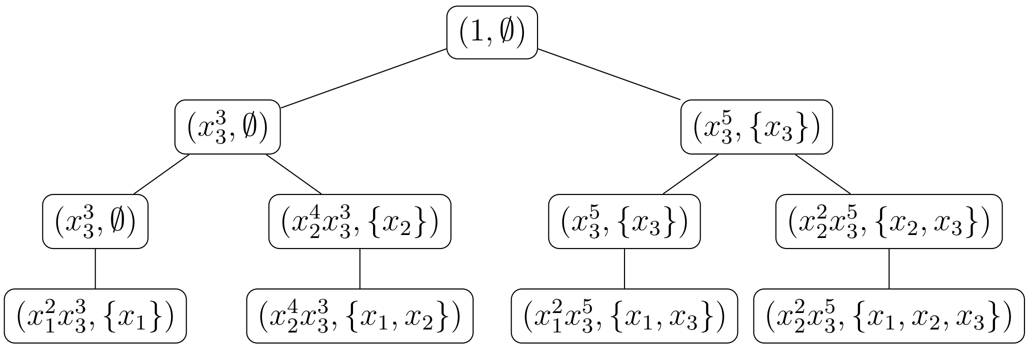

Figure 3.1: Janet tree of $U = \{x_1^2x_3^3,\, x_2^4x_3^3,\, x_1^2x_3^5,\, x_2^2x_3^5\} \subset \mathbb{K}[x_1, x_2, x_3]$.

To adapt this tree representation to the Janet-like division, we add to each node a set M of Janet non-multipliers and obtain what we call the *Janet-like tree* of U. Assume as above that at level $i+1$ we have the node (x^ν, V, M) and that it has ℓ children represented by the terms $x_i^{a_j}x^\nu$ where $a_1 < a_2 < \cdots < a_\ell$. Then the first $\ell - 1$ children are given by the nodes $(x_i^{a_j}x^\nu, V, M \cup \{x_i^{a_{j+1}-a_j}\})$ and the last child is $(x_i^{a_\ell}x^\nu, V \cup \{x_i\}, M)$. We find then again at level 1 the terms of U in lexicographic order together with their multiplicative variables and their non-multiplicative powers. Figure 3.2 contains the Janet-like tree of the set $U = \{x_1^2x_2x_3,\, x_2^3x_3,\, x_3^3\} \subset \mathbb{K}[x_1, x_2, x_3]$.

While the Janet and especially the Janet-like tree decribed above have the advantage that not only the needed projections of multiindices, Janet multiplicative variables, and non-multiplicative powers can be read off directly, they use up a considerable amount of space on paper even for small examples. In later sections, we will work with slightly larger examples, and for these we use tree notation that only encodes the minimum of information needed to reconstruct the term set represented

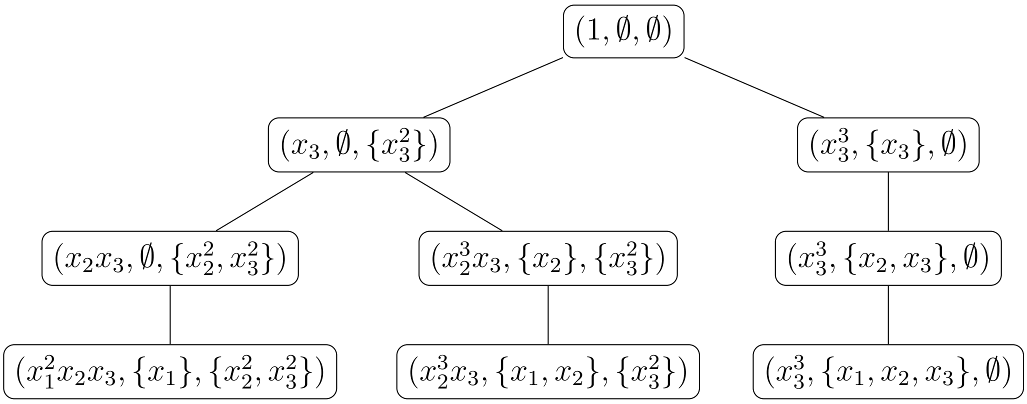

Figure 3.2: Janet-like tree of $U = \{x_1^2x_2x_3,\ x_2^3x_3,\ x_3^3\} \subset \mathbb{K}[x_1, x_2, x_3]$.

by the tree. If $x^\mu \in \mathcal{T}$ is a term with $\mu = (\mu_1, \ldots, \mu_n)$ represented in a tree, then exactly one leaf will correspond to it which is tagged with the x_1-degree μ_1 of x^μ; one can reconstruct μ by following the path leading from this leaf to the root, where at each level i one can read off the x_i-degree μ_i of x^μ.

As a simple example, in the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$ let us consider the set $U = \{x_1^2x_3^3,\ x_2^4x_3^3,\ x_1^2x_3^5,\ x_2^2x_3^5\}$. Its tree representation is shown in Figure 3.3.

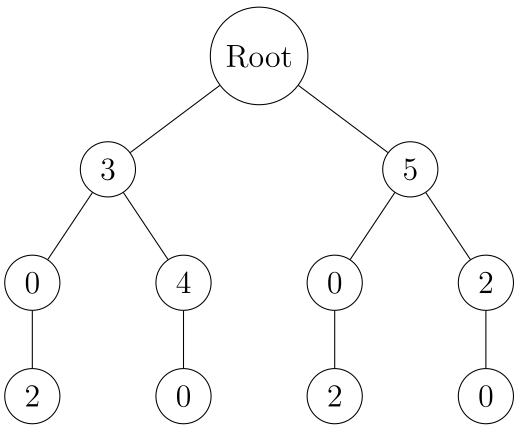

Figure 3.3: Tree representation of $U = \{x_1^2x_3^3,\ x_2^4x_3^3,\ x_1^2x_3^5,\ x_2^2x_3^5\}$.

The level i in this representation corresponds to the variable x_i and one can read off the Janet multiplicative variables for any node. For example, for $x_1^2x_3^5 \in U$ the variables x_1 and x_3 are multiplicative, since the path from the root to the corresponding leave uses the respective last branch at the levels 1 and 3. However, this is not the case at level 2 and so x_2 is non-multiplicative.

The Janet trees introduced in [51] correspond to a transformation of the above described tree into a binary tree. They are extensively used for the fast construction of Janet bases, as many necessary operations like searching for a Janet divisor can be performed very efficiently with them. The bar codes introduced in [22] provide a similar representation of a set of terms using a two-dimensional diagram; the relation between the two representations is studied in [23]. We refer to [72] for the complexity of constructing a tree representation.

3.2 Complementary Decompositions

Combinatorial decompositions of polynomial ideals often appear in commutative algebra, as they are very useful for many theoretical considerations [106]. Actually, they were first prominently used by Riquier [89] and Janet [67] in their works on general systems of partial differential equations and Janet also provided effective algorithms for their determination. Within commutative algebra, such decompositions were studied with an emphasis on complementary ones only much later: first by Rees [86] in a generic situation and then generally by Stanley [103] in the context of Hilbert function computations. Algorithms for the construction of complementary decompositions were presented e. g. by Sturmfels and White [105]. A particular combinatorial decomposition was also crucial for Dubé's analysis of the complexity of Gröbner bases without genericity assumptions [32].

Involutive bases are closely related to combinatorial decompositions. Any strong involutive basis induces a direct sum decomposition of the ideal as linear space and thus allows for the immediate construction of the volume function of the ideal (and indirectly of its Hilbert function). Complementary decompositions, i. e. decompositions of the factor ring, are a bit harder to get, but the two types of involutive bases most often used in practise, Janet and Pommaret bases, also induce such decompositions. We will show that Janet-like bases provide a more efficient algorithm for the construction of a (condensed) complementary decomposition than Janet's original algorithm.

In his work on idealistic exponents, Hironaka [65] constructed a complementary decomposition for monomial ideals in generic position. We will show that his construction terminates with a finite decomposition, if and only if the monomial ideal is quasi-stable, i. e. we find here a by now well studied genericity condition. This observation also implies that Hironaka essentially just rediscovered Rees' decomposition. We will show furthermore that Janet's algorithm presented almost 50 years before Hironaka's work constructs the same decomposition more efficiently.

This section is structured as follows. Subsection 3.2.1 discusses a classical recursive algorithm for the construction of a complementary decomposition. We determine its complexity and describe an iterative variant of it. The construction of complementary decomposition following Janet's ideas is the topic of Subsection 3.2.2. We formulate his algorithm in a graph theoretical language showing that it corresponds to a simple breadth-first traversal of the Janet tree associated with the given monomial ideal. This observation immediately gives us its complexity. We then show how Janet-like bases can be used for obtaining a more efficient version of the algorithm. In Subsection 3.2.3, we recall Hironaka's construction and relate it to Pommaret bases and thus quasi-stable ideals. The topic of Subsection 3.2.4 is the relation of complementary decompositions with primary and irreducible decompositions. We show that Hironaka's construction yields as a by-product a primary decomposition and explain how an irreducible decomposition can be extracted from a complementary one.

3.2.1 Decompositions from Arbitrary Generating Sets

In this section, we recall the definition of complementary decompositions and give a more efficient variant of a well-known recursive approach for their construction for the special class of monomial ideals, for more details see [97, Sec. 5.1].

Definition 3.2.1. *A* cone *is a set of the form* $\mathcal{C}_Y(t) = \mathcal{T}_Y \cdot t$ *for some term* $t \in \mathcal{T}$*, its* vertex*, and some set of* multiplicative variables $Y \subseteq X$*. A* cone decomposition *of an arbitrary subset* $\mathcal{S} \subseteq \mathcal{T}$ *is a representation as a disjoint finite union of cones:* $\mathcal{S} = \bigsqcup_{(t,Y)\in\mathcal{D}} \mathcal{C}_Y(t)$ *for some finite set* $\mathcal{D}$ *of pairs* $(t \in \mathcal{T}, Y \subseteq X)$*.*

In the above definition, we consider arbitrary subsets $\mathcal{S} \subseteq \mathcal{T}$. In practice, two cases are particularly relevant: $\mathcal{S} = \mathcal{I} \cap \mathcal{T}$ consists of the terms contained in a monomial ideal $\mathcal{I}$ or $\mathcal{S}$ is the complement of such a set, i. e. $\mathcal{S}$ consists of the terms contained in an order ideal. In this work, we are mainly concerned with the second case which we call a *complementary decomposition* of the monomial ideal $\mathcal{I}$. By Proposition 2.2.2, a complementary decomposition corresponds to a decomposition of the factor ring $\mathcal{R}/\mathcal{I}$ as a $\mathbb{K}$-linear space. Complementary decompositions are often called *Stanley decomposition*, as Stanley [103] used them for computing the Hilbert function of an ideal $\mathcal{I}$ (actually, this approach to Hilbert functions goes back already to Janet [67]). Rees [86] considered already earlier the special case where all sets Y of multiplicative variables are of the special form $Y = \{x_i, x_{i+1}, \dots, x_n\}$ for some index i; one then speaks of a *Rees decomposition*.

Given any finite complementary decomposition D of an ideal $\mathcal{I}$, it is indeed straightforward to read off the Hilbert series and the Hilbert polynomial of $\mathcal{I}$. Given a cone $(t, Y) \in \mathcal{D}$, we write $q_t = \deg(t)$ for the degree of its vertex and $k_t = |Y|$ for its dimension. The Hilbert series of $\mathcal{I}$ is then given by

$$\mathrm{HS}_{\mathcal{I}}(\lambda) = \sum_{(t,Y)\in\mathcal{D}} \frac{\lambda^{q_t}}{(1-\lambda)^{k_t}}$$

and the Hilbert polynomial by

$$\mathrm{HP}_{\mathcal{I}}(q) = \sum_{\substack{(t,Y)\in\mathcal{D}\\ k_t>0}} \binom{q - q_t + k_t - 1}{k_t - 1}.$$

This follows immediately from the disjointness required from a cone decomposition and from the fact that the above binomial coefficient gives the number of terms of degree q in the cone (t, Y) (for degrees $q \geq q_t$). For the Hilbert function, one must enforce that the cone (t, Y) contributes nothing for any degree $q \leq q_t$. Hence, using the Kronecker-Iverson symbol $[\cdot]$ which yields 1 if the condition in the bracket is satisfied and 0 otherwise, we can write

$$\mathrm{HF}_{\mathcal{I}}(q) = \sum_{\substack{(t,Y)\in\mathcal{D}\\ k_t>0}} [q \geq q_t]\binom{q-q_t+k_t-1}{k_t-1} + \sum_{\substack{(t,Y)\in\mathcal{D}\\ k_t=0}} [q = q_t]\,. \tag{3.1}$$

Thus complementary decompositions provide us with an elementary proof of the fact that the Hilbert function of any ideal is of polynomial type and the maximal value of q_t bounds the Hilbert regularity. In [16], one can find a number of further results on Hilbert series that can be derived via complementary decompositions stemming from Pommaret bases.

Remark 3.2.2. *By definition, any involutive basis of a monomial ideal $\mathcal{I}$ induces a cone decomposition of $\mathcal{I}$. As we will discuss in more details in the subsequent sections, Janet and Pommaret bases also induce complementary decompositions. In the case of a Pommaret basis, both the decomposition of $\mathcal{I}$ and the complementary decomposition are Rees decompositions, see [97].*

The subject of computing complementary decompositions for monomial ideals has a long tradition, see e. g. [103, 104, 105, 97]. The recursive Algorithm 1 represents a slightly optimised form of an approach which seems to be folklore. It can be found implicitly in [31] or explicitly in [105] (see also [97, Alg. 5.1] or [90] for variants). However, it seems that its complexity has never been studied.

Theorem 3.2.3. *Algorithm 1 terminates in finitely many steps and is correct. Its arithmetic complexity is $O\big((\lambda m)^n\big)$ where*

$$\lambda = \max\left\{\lambda_0^{(i)}, \lambda_1^{(i)} - \lambda_0^{(i)}, \dots, \lambda_\ell^{(i)} - \lambda_{\ell-1}^{(i)} \mid i = 1, \dots, n\right\}$$

with $\lambda_0^{(i)} < \lambda_1^{(i)} < \cdots < \lambda_\ell^{(i)}$ the sequence of the x_i-degrees of the terms t_j used to generate the ideal $\mathcal{I}$.

Proof. The termination and the correctness follow from [97, Prop. 5.1.3]. To prove the complexity bound, we first note that one can construct the Janet tree of U using $O(m^2 + nm)$ comparisons, see [72, Thm. 4.2]. Now, it suffices to show (by an induction over the number n of variables) that at each iteration the number of constructed cones in $\mathcal{D}$ is $O\big((\lambda m)^n\big)$. Here, following the notations used in the algorithm, we may assume without loss of generality that the elements of U are distributed uniformly and thus U'_{λ_i} contains $m/(\ell+1)$ elements for each i. If $n = 1$, then there is nothing to prove. Assume now that the assertion holds for $n-1$. Then the **for**-loop is repeated $\ell+1$ times and in each iteration (by the lines 17 and 19) the set $\bigcup_{j=0}^{i} U'_{\lambda_j}$ has $(i+1)m/(\ell+1)$ elements. Thus the number of cones in $\mathcal{D}$ is $\lambda \times \sum_{i=0}^{\ell} O\big((\lambda(i+1)m/(\ell+1))^{n-1}\big) \sim O\big(\lambda(\ell+1)(\lambda m)^{n-1}\big)$. It follows from the fact that $\ell+1 \leq m$ that the total number of elements added to $\mathcal{D}$ is $O\big((\lambda m)^{n-1}\big)$. Finally, we may assume that $m, n \geq 2$ and therefore $(\lambda m)^{n-1}$ in the dominant factor in the complexity $O\big((\lambda m)^{n-1} + m^2 + nm\big)$ and this ends the proof. $\square$

Algorithm 1: RecursiveComplementaryDecomposition

Data: Generating set $U = \{t_1, \dots, t_m\}$ of monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$
Result: Finite complementary decomposition $\mathcal{D}$ of $\mathcal{I}$
begin
 $(\lambda_0, \lambda_1, \dots, \lambda_\ell) \longleftarrow$ sequence of x_1-degrees of terms t_i with $\lambda_0 < \lambda_1 < \cdots < \lambda_\ell$
 if $U = \emptyset$ **then**
  return $\big\{(1, \{x_1, \dots, x_n\})\big\}$
 else if $n = 1$ and $\lambda_0 = 0$ **then**
  return $\emptyset$
 else if $n = 1$ and $\lambda_0 \neq 0$ **then**
  return $\big\{(1, \emptyset), (x_1, \emptyset), \dots, (x_1^{\lambda_0 - 1}, \emptyset)\big\}$
 $\mathcal{D} \longleftarrow \emptyset$
 if $\lambda_0 \neq 0$ **then**
  $\mathcal{D} \longleftarrow \big\{(x_1^\lambda, \{x_2, \dots, x_n\}) \mid \lambda = 0, \dots, \lambda_0 - 1\big\}$
 for i **from** 0 **to** ℓ **do**
  $U'_{\lambda_i} \longleftarrow \{t \in \mathbb{K}[x_2, \dots, x_n] \mid t \cdot x_1^{\lambda_i} \in U\}$
  $D'_{\lambda_i} \longleftarrow$ `RecursiveComplementaryDecomposition`$(\bigcup_{j=0}^{i} U'_{\lambda_j})$
  if $i = \ell$ **then**
   $\mathcal{D} \longleftarrow \mathcal{D} \cup \big\{(u[1] \cdot x_1^{\lambda_\ell}, u[2] \cup \{x_1\}) \mid u \in \mathcal{D}'_{\lambda_i}\big\}$
  else
   $\mathcal{D} \longleftarrow \mathcal{D} \cup \big\{(u[1] \cdot x_1^{\lambda}, u[2]) \mid u \in \mathcal{D}'_{\lambda_i}, \lambda = \lambda_i, \dots, \lambda_{i+1} - 1\big\}$
 return $\mathcal{D}$

Example 3.2.4. *Let us consider the ideal $\mathcal{I} = \langle x_1^2 x_2 x_3,\ x_2^3 x_3,\ x_3^3 \rangle$ in the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$. By considering x_1 as the main variable, we have $\lambda_0 = 0$, $\lambda_1 = 2$, $U'_0 = \{x_2^3 x_3,\ x_3^3\}$ and $U'_2 = \{x_2 x_3\}$. By applying the algorithm to U'_0, we get*

$$\mathcal{D}'_0 = \big\{(1, \emptyset),\ (x_3, \emptyset),\ (x_3^2, \emptyset),\ (x_2, \emptyset),\ (x_2 x_3, \emptyset),\ (x_2 x_3^2, \emptyset),\ (x_2^2, \emptyset),\\ (x_2^2 x_3, \emptyset),\ (x_2^2 x_3^2, \emptyset),\ (x_2^3, \{x_2\})\big\}\ .$$

Thus, by multiplying the first component of the elements of this set by both 1 and x_1, we obtain the first version of $\mathcal{D}$. Now, we continue with λ_1. Here, one observes that

$$U'_{\lambda_0} \cup U'_{\lambda_1} = \{x_2 x_3,\ x_2^3 x_3,\ x_3^3\}$$

and the ideal generated by this set is $\langle x_2 x_3, x_3^3 \rangle$. Applying the algorithm to it, on obtains its complementary decomposition $\big\{(1, \emptyset),\ (x_3, \emptyset),\ (x_3^2, \emptyset),\ (x_2, \{x_2\})\big\}$ and finally

$$\mathcal{D} \cup \big\{(x_1^2, \{x_1\}),\ (x_1^2 x_3, \{x_1\}),\ (x_1^2 x_3^2, \{x_1\}),\ (x_1^2 x_2, \{x_1, x_2\})\big\}$$

defines a complementary decomposition for $\mathcal{I}$.

For a better understanding of the structure of the recursive Algorithm 1, we describe now an iterative variant of it: Algorithm 2. In it, we first order the set U according to the lexicographical order and then construct sets $\mathcal{D}_{i,j}$, where—with the notations used in the algorithm—$\mathcal{D}_{i,j}$ provides a complementary decomposition for the ideal $\langle t_j, \dots, t_m \rangle|_{x_1=\cdots=x_{i-1}=1}$. Thus, $\mathcal{D}_{1,1}$ defines the desired complementary decomposition for the given ideal $\mathcal{I}$.

Algorithm 2: ComplementaryDecomposition

Data: Generating set $U = \{t_1, \dots, t_m\}$ of monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$
Result: Finite complementary decomposition $\mathcal{D}$ of $\mathcal{I}$
begin
- Sort $U = \{t_1, \dots, t_m\}$ such that $t_m \prec_{lex} \cdots \prec_{lex} t_1$ with $x_n \prec \cdots \prec x_1$
- $(\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_m^{(i)}) \longleftarrow$ sequence of x_i-degrees of elements of U
- **for** j **from** 1 **to** m **do**
  - $\lambda \longleftarrow \min\left\{\lambda_j^{(n)}, \dots, \lambda_m^{(n)}\right\}$
  - $\mathcal{D}_{n,j} \longleftarrow \left\{(1, \emptyset), (x_n, \emptyset), \dots, (x_n^{\lambda-1}, \emptyset)\right\}$
- **for** i **from** $n-1$ **to** 1 **do**
  - $\mathcal{D}_{i,m} \longleftarrow \left\{\left(x_i^{\lambda}, \{x_{i+1}, \dots, x_n\}\right) \mid \lambda = 0, \dots, \lambda_m^{(i)} - 1\right\} \cup \left\{\left(u[1]x_i^{\lambda_m^{(i)}}, u[2] \cup \{x_i\}\right) \mid u \in \mathcal{D}_{i+1,m}\right\}$
  - Reorder U such that $\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \ldots \geq \lambda_m^{(i)}$
  - **for** j **from** $m-1$ **to** 1 **do**
    - $\mathcal{D}_{i,j} \longleftarrow \left\{\left(u[1]x_i^{\lambda_j^{(i)}}, u[2] \cup \{x_i\}\right) \mid u \in \mathcal{D}_{i+1,j}\right\} \cup \left\{\left(u[1]x_i^{s}, u[2]\right) \mid u \in \mathcal{D}_{i+1,\ell},\ \ell = j+1, \dots, m,\ s = \lambda_{\ell-1}^{(i)}, \dots, \lambda_\ell^{(i)} - 1\right\}$
    - **if** $\lambda_m^{(i)} \neq 0$ **then**
      - $\mathcal{D}_{i,j} \longleftarrow \mathcal{D}_{i,j} \cup \left\{\left(x_i^{\lambda}, \{x_{i+1}, \dots, x_n\}\right) \mid \lambda = 0, \dots, \lambda_m^{(i)} - 1\right\}$
- **return** $(\mathcal{D}_{1,1})$

Theorem 3.2.5. *Algorithm 2 terminates for any input and is correct. Its arithmetic complexity is $O\big(\gamma m(\lambda m)^{n-1}\big)$ where*

$$\lambda = \max\left\{\lambda_0^{(i)}, \lambda_1^{(i)} - \lambda_0^{(i)}, \dots, \lambda_\ell^{(i)} - \lambda_{\ell-1}^{(i)} \mid i = 1, \dots, n\right\}$$

with $\lambda_0^{(i)} < \lambda_1^{(i)} < \cdots < \lambda_\ell^{(i)}$ being the sequence of the x_i-degrees of the generators t_j and $\gamma = \lambda_\ell^{(n)}$.

Proof. Since this algorithm is a non-recursive variant of Algorithm 1, its finite termination and correctness follow from those of Algorithm 1. To prove the arithmetic complexity, we mainly follow the lines of the proof of Theorem 3.2.3. We proceed to find the number of cones in the decomposition $\cup_{j=1}^{m} \mathcal{D}_{i,j}$ for each index $i = n, \ldots, 1$ and the total number of constructed cones determines the complexity of the algorithm. We observe that for $i = n$ the number of constructed cones is $O(\gamma m)$. It follows from line 11 by applying a simple induction, that the number of cones in $\mathcal{D}_{i,j}$ is $O\big(\gamma(\lambda m)^{n-i}\big)$. Thus, the total number of constructed cones is $\gamma m + \gamma m(\lambda m) + \cdots \gamma m(\lambda m)^{n-1}$. We may assume that $\lambda m \geq 2$ and this shows the claim. $\square$

Remark 3.2.6. *One can see that arithmetic complexity of this algorithm is very close to that of Algorithm 1. However, at each iteration of Algorithm 2, we get complementary decompositions for the ideals $\langle t_j, \ldots, t_m \rangle|_{x_1=\cdots=x_{i-1}=1}$ which provide additional information about the input ideal. More precisely, for each $1 \leq i \leq n$ and $1 \leq j \leq m$, the set $\mathcal{D}_{i,j}$, constructed during the algorithm, forms a complementary decomposition for this ideal.*

The following example illustrates the steps of the algorithm.

Example 3.2.7. *Let us consider the ideal $\mathcal{I} = \langle U \rangle$ in the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$ with $U = \{x_1^2x_2x_3,\ x_2^3x_3,\ x_3^3\}$. Obviously, we have $\lambda_1^{(3)} = 1$, $\lambda_2^{(3)} = 1$, and $\lambda_3^{(3)} = 3$. We get $\mathcal{D}_{3,3} = \big\{(1, \emptyset), (x_3, \emptyset), (x_3^2, \emptyset)\big\}$ and $\mathcal{D}_{3,2} = \mathcal{D}_{3,1} = \big\{(1, \emptyset)\big\}$. Note that $\mathcal{D}_{3,2} = \mathcal{D}_{3,1}$ since $x_3 \in \langle x_3, x_3^3 \rangle$. Entering into the main* **for***-loop, we must consider in the iteration with $i = 2$ the three terms x_2x_3, $x_2^3x_3$, $x_3^3 \in \mathbb{K}[x_2, x_3]$ and hence obtain $\lambda_1^{(2)} = 1$, $\lambda_2^{(2)} = 3$, $\lambda_3^{(2)} = 0$. The algorithm then yields*

$$\begin{aligned}
\mathcal{D}_{2,3} &= \Big\{ \big(1, \{x_2\}\big), \big(x_3, \{x_2\}\big), \big(x_3^2, \{x_2\}\big) \Big\}\,, \\
\mathcal{D}_{2,2} &= \Big\{ (1, \emptyset), (x_3, \emptyset), (x_3^2, \emptyset), (x_2, \emptyset), (x_2x_3, \emptyset), (x_2x_3^2, \emptyset), (x_2^2, \emptyset), \\
&\qquad (x_2^2x_3, \emptyset), (x_2^2x_3^2, \emptyset), \big(x_2^3, \{x_2\}\big) \Big\}\,, \\
\mathcal{D}_{2,1} &= \Big\{ (1, \emptyset), (x_3, \emptyset), (x_3^2, \emptyset), \big(x_2, \{x_2\}\big) \Big\}\,.
\end{aligned}$$

It is worth noting that we may simplify the construction of $\mathcal{D}_{2,1}$ by removing the extra term $x_2^3x_3$ and considering only the two terms x_2x_3, x_3^3. For $i = 1$, we consider the given set U and obtain $\lambda_1^{(1)} = 2, \lambda_2^{(1)} = \lambda_3^{(1)} = 0$. Thus,

$$\begin{aligned}
\mathcal{D}_{1,3} &= \{1, \{x_1, x_2\}),\ (x_3, \{x_1, x_2\}),\ (x_3^2, \{x_1, x_2\})\}\,, \\
\mathcal{D}_{1,2} &= \{(1, \{x_1\}),\ (x_3, \{x_1\}),\ (x_3^2, \{x_1\}),\ (x_2, \{x_1\}),\ (x_2x_3, \{x_1\}),\ (x_2x_3^2, \{x_1\}), \\
&\qquad (x_2^2, \{x_1\}),\ (x_2^2x_3, \{x_1\}),\ (x_2^2x_3^2, \{x_1\}),\ (x_2^3, \{x_1, x_2\}), \}\,, \\
\mathcal{D}_{1,1} &= \Big\{ (1, \emptyset),\ (x_3, \emptyset),\ (x_3^2, \emptyset),\ (x_2, \emptyset),\ (x_2x_3, \emptyset),\ (x_2x_3^2, \emptyset),\ (x_2^2, \emptyset),\ (x_2^2x_3, \emptyset), \\
&\qquad (x_2^2x_3^2, \emptyset),\ \big(x_2^3, \{x_2\}\big),\ (x_1, \emptyset),\ (x_1x_3, \emptyset),\ (x_1x_3^2, \emptyset),\ (x_1x_2, \emptyset), \\
&\qquad (x_1x_2x_3, \emptyset),\ (x_1x_2x_3^2, \emptyset),\ (x_1x_2^2, \emptyset),\ (x_1x_2^2x_3, \emptyset),\ (x_1x_2^2x_3^2, \emptyset), \\
&\qquad \big(x_1x_2^3, \{x_2\}\big),\ \big(x_1^2, \{x_1\}\big),\ \big(x_1^2x_3, \{x_1\}\big),\ \big(x_1^2x_3^2, \{x_1\}\big),\ \big(x_1^2x_2, \{x_1, x_2\}\big) \Big\}\,.
\end{aligned}$$

$\mathcal{D} = \mathcal{D}_{1,1}$ is the constructed complementary decomposition for $\mathcal{I}$. Finally, we note that since the sequence of the x_1-degrees in U is $2, 0$, we do not need to construct $\mathcal{D}_{1,3}$.

3.2.2 Decompositions from Janet and Janet-like Bases

Janet [67, Sect. 15] presented an algorithm for the construction of a complementary decomposition from a Janet basis. Gerdt [44, Lem. 24] proposed a version that related the form of the cones to the Janet division. However, his proof is not completely correct. Algorithm 3 corresponds to the version appearing in [97, Alg. 5.2] (more precisely, an improved form contained in the errata to [97] obtainable at the web page of the author). It has been formulated in a manner that makes it apparent that this algorithm does nothing but a breadth-first transversal of the Janet tree associated to the given monomial ideal. Thus this algorithm does not need any real computations, but simply writes down a complementary decomposition. All computations have already taken place when the Janet tree was determined as an extended form of the Janet basis.

Algorithm 3: Complementary decomposition (from Janet basis)

```
Data: Janet tree JT of monomial ideal I ⊴ K[x_1, ..., x_n]
Result: Finite complementary decomposition D of I
begin
    D ⟵ ∅
    for k = n, ..., 1 do
        foreach node (x^ν, V) at level k+1 in JT do
            let (x_k^m x^ν, W) be the leftmost child of (x^ν, V)
            if m > 0 then
                N ⟵ {x_1, ..., x_{k-1}} ∪ V
                for i = 0, ..., m-1 do
                    D ⟵ D ∪ {(x_k^i x^ν, N)}
    return D
```

Theorem 3.2.8. *Let the ideal $\mathcal{I}$ be generated by the terms $t_1, \ldots, t_m$. Algorithm 3 terminates in finitely many steps and is correct. The algorithm has arithmetic complexity $O(nm^2\lambda^{2n})$, where*

$$\lambda = \max_{i,j} \deg_{x_i}(t_j) \ .$$

Proof. It follows from [97, Lem. 3.1.19] that the set

$$\{x^\mu t_i \mid i = 1, \ldots, m, x^\mu \mid \operatorname{lcm}(t_1, \ldots, t_m)/t_i\}$$

contains a Janet basis of $\mathcal{I}$. The maximal number of elements in this set is $m\lambda^n$. On the other hand, the construction of the Janet tree corresponding to the Janet basis of

$\mathcal{I}$ needs $O(m^2\lambda^{2n} + mn\lambda^n)$ comparisons, see [72, Thm. 4.2]. By the structure of the algorithm, it is seen that the number of constructed cones is at most $nm\lambda^{n+1}$. These arguments show that the arithmetic complexity of the algorithm is $O(nm^2\lambda^{2n})$. $\square$

Remark 3.2.9. *If we compare the algorithmic complexity of the Algorithms 1 and 2 on the one hand and of Algorithm 3 on the other hand (and in particular the number of constructed cones), then we see that in the algorithms from the last section the cardinality m of the generating set of $\mathcal{I}$ is the decisive factor, as it appears in the form m^n. For the algorithm using a Janet basis, the maximal degree λ in an individual variable is the dominant factor, whereas m plays only a minor role. Thus for ideals with a large number of generators it should be preferable. One should also note that the factor $m\lambda^n$ coming from the use of [97, Lem. 3.1.19] is generally much larger than the actual size of the Janet basis.*

If the monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is quasi-stable and H is its Pommaret basis, then H is also the minimal Janet basis of $\mathcal{I}$ by [45, Thm. 17]. In this situation, for each term $t \in H$, we have the equality $M_{\mathcal{J}}(t, H) = M_{\mathcal{P}}(t)$ of the sets of Janet and Pommaret multiplicative variables. This means that in order to compute a complementary decomposition for a quasi-stable ideal, we can apply Algorithm 3 to the minimal Pommaret basis of $\mathcal{I}$.

Remark 3.2.10. *Note that we can apply Algorithm 3 to any Janet basis H, i. e. also to non-minimal bases. Since the minimal Janet basis $H_{\min}$ is a subset of any other Janet basis H of the same ideal, each Janet class of the minimal basis $H_{\min}$ is also a Janet class of H. This observation implies that each leftmost child node chosen in Line 5 of Algorithm 3 when applied to $H_{\min}$ is also chosen when the algorithm is applied to H. Hence, in the complementary decomposition obtained from the basis H, we get at least as many cones as in the decomposition obtained from $H_{\min}$. Since a term $t \in H_{\min}$ possesses potentially less Janet-multiplicative variables when considered as element of the non-minimal basis H, some of these cones may be of smaller dimension than their counterparts in the decomposition obtained from $H_{\min}$. Hence the decomposition obtained from H will contain in general strictly more cones than the decomposition obtained from $H_{\min}$. So we may speak of the* minimal Janet complementary decomposition *which is obtained from the minimal Janet basis $H_{\min}$.*

In the remainder of this section, we will describe how we can obtain the Janet complementary decomposition of a monomial ideal $\mathcal{I}$ already from its minimal Janet-like basis, which is always a subset, and most often a proper subset, of the minimal Janet basis of $\mathcal{I}$. As a starting point, we recall the following result which explains how the minimal Janet-like basis is related to the minimal Janet basis.

Proposition 3.2.11. *Given a Janet-like basis $U \subset \mathcal{T}$ of the monomial ideal $\mathcal{I} = \langle U \rangle \trianglelefteq \mathcal{R}$, a Janet basis U' of the same ideal is defined by*

$$U' = \left\{ t \cdot x^{\mu} \mid t \in U \wedge x^{\mu} \mid \prod_{x_a^{p_a} \in \mathrm{NMP}(t,U)} x_a^{p_a - 1} \right\}.$$

Proof. The assertion follows immediately from elementary properties of the Janet and Janet-like division, respectively. See also [50, Thm. 3], a related, though not identical, statement. □

Algorithm 3 traverses the Janet tree of the minimal Janet basis and adds for each node which is the leftmost node of its parent certain cones to the complementary decomposition. The Janet-like tree of the same ideal can be regarded as a subtree of this tree. To be able to read off the Janet complementary decomposition already from this subtree, one needs to relate the minimal child nodes of the larger tree to certain nodes of the smaller tree. This is indeed possible with Proposition 3.2.11.

Proposition 3.2.12. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a monomial ideal with Janet basis U' and Janet-like basis U. We denote by JT and JLT its Janet and Janet-like tree, respectively. Moreover, for any tree S, we denote by $\mathrm{Left}_i(S)$ the set of nodes at level i which are the minimal children of their respective parent nodes. Then for each i with $1 \leq i \leq n$ the sets $\mathrm{Left}_i(JT)$, $\mathrm{Left}_i(JLT)$ are related by*

$$\mathrm{Left}_i(JT) = \left\{ (t, V) \mid \exists (s, V, M) \in \mathrm{Left}_i(JLT) : s \mid t \wedge (t/s) \mid \prod_{\substack{x_a^{h_a} \in M, \\ a > i}} x_a^{h_a - 1} \right\}. \quad (3.2)$$

Proof. Let $(t, V) \in \mathrm{Left}_i(JT)$ be a minimal child node at level i in the Janet tree. The term t is of the form $\tilde{t}|_{x_1 = \cdots = x_{i-1} = 1}$, with $\tilde{t} \in U'$. Then, by Proposition 3.2.11, there is a term $\tilde{s} \in JLB$ such that $\tilde{s} \mid \tilde{t}$. We can consider its projection $s := \tilde{s}|_{x_1 = \cdots = x_{i-1} = 1}$; this term is contained in a node $(s, V, M) \in JLT$. Moreover, again by Proposition 3.2.11, we have the degree conditions $\deg_j(s) \leq \deg_j(t) < \deg_j(s) + h_j$, where h_j is defined by $x_j^{h_j} \in M$, for all $j > i$ with $x_j \notin V$. Now, if $\deg_i(s) < \deg_i(t)$ were true, then $\tilde{s}$ would induce a term $x_i^{\deg_i(s)}(\tilde{t}/x_i^{\deg_i(t)})$ in JB, a contradiction to $(t, V) \in \mathrm{Left}_i(JT)$. Hence $\deg_i(s) = \deg_i(t)$. In addition, if (s, V, M) were not in $\mathrm{Left}_i(JLT)$, then there would be a node $(u, W, N) \in \mathrm{Left}_i(JLT)$ with $\deg_i(u) < \deg_i(s)$ but with $\deg_j(u) = \deg_j(s)$ for all $j > i$. In particular, we would have $N \cap K[x_{i+1}, \ldots, x_n] = M \cap K[x_{i+1}, \ldots, x_n]$. Hence it would induce a node $(v, W) \in JT$ with $\deg_i(v) = \deg_i(u)$ and $\deg_j(v) = \deg_j(t)$ for all $j > i$. This is again a contradiction to $(t, V) \in \mathrm{Left}_i(JT)$. Thus, we have shown the inclusion $\subseteq$ in equality (3.2).

Conversely, if $(s, V, M) \in \mathrm{Left}_i(JLT)$ and $(t, V) \in JT$ is a node in the Janet tree derived from it by the rules stated in equality (3.2), then, using a basic fact about the Janet-like division [50, Prop. 2], it is not hard to see that (t, V) is indeed in $\mathrm{Left}_i(JT)$, proving the inclusion $\supseteq$ and finishing the proof. □

The very technical Proposition 3.2.12 has the benefit that the Janet complementary decomposition of a monomial ideal $\mathcal{I}$ can, with its help, be read off already from its Janet-like tree. Moreover, the cones come in a natural grouping. This grouping helps to write the decomposition down in a much better readable way. The cones of the decompositions are sorted into groups. The cones in each group have the same multiplicative variables and there is one cone in the group whose vertex divides all other cone vertices of the group:

Corollary 3.2.13. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a monomial ideal and let JLT be its Janet-like tree. Let* $\mathrm{MinNodes}(\mathcal{I})$ *denote the set of nodes in JLT which are minimal children of their respective parent nodes. For each such node (x^ν, V, M), let k denote its level in the tree. Then, a complementary decomposition of $\mathcal{I}$ is given by*

$$\mathcal{D} = \bigcup_{\substack{(x^\nu,V,M)\in\mathrm{MinNodes}(\mathcal{I}) \\ \text{with parent } (x^\rho,V',M')}} \left\{ \left(x^\rho \cdot x^\mu, \{x_1,\ldots,x_{k-1}\} \cup V'\right) \mid \ x^\mu \mid x_k^{\nu_k-1} \prod_{x_a^{h_a}\in M'} x_a^{h_a-1} \right\}. \tag{3.3}$$

Proof. The assertion is an immediate consequence of Proposition 3.2.12 and the correctness of Algorithm 3. □

Corollary 3.2.13 induces Algorithm 4 computing a complementary decomposition of a monomial ideal from a Janet-like basis of it.

Algorithm 4: Complementary decomposition (from Janet-like basis)

Data: Janet-like tree JLT of monomial ideal $\mathcal{I} \trianglelefteq \mathbb{K}[x_1,\ldots,x_n]$
Result: Finite complementary decomposition $\mathcal{D}$ of $\mathcal{I}$
begin
  $\mathcal{D} \longleftarrow \emptyset$
  for $k = n,\ldots,1$ **do**
    foreach *node* (x^ρ, V', M') *at level* $k+1$ *in* JLT **do**
      let $(x_k^m x^\rho, V, M)$ be the leftmost child of (x^ρ, V', M')
      if $m > 0$ **then**
        $N \longleftarrow \{x_1,\ldots,x_{k-1}\} \cup V'$
        $\mathcal{D} \longleftarrow \mathcal{D} \cup \left\{ (x^\rho x^\mu, N) \mid \ x^\mu \mid x_k^{m-1} \prod_{x_a^{h_a}\in M'} x_a^{h_a-1} \right\}$
  return $\mathcal{D}$

The following example serves to illustrate how Proposition 3.2.12 can be applied.

Example 3.2.14. *Let $\mathcal{I} = \langle x_1^n, \ldots, x_n^n\rangle \subset \mathcal{R}$. Its minimal Janet basis has $1+n+\cdots+n^{n-1} = (n^n-1)/(n-1)$ elements and its Janet tree has, including the root, $\left(\sum_{k=0}^{n-2}\sum_{\ell=0}^{k} n^\ell\right) + 2(n^n-1)/(n-1) = O(n^{n-1})$ nodes. There are n^{n-1} nodes in the tree which contribute cones to the Janet complementary decomposition. They are all at the lowest level 1 of the tree, and they are of the form (t, V, M) with $\deg_1(t) = n$ and $V \subseteq \{x_1\}$. This means that each of them contributes exactly n zero-dimensional (one-element) cones to the complementary decomposition of $\mathcal{I}$.*

By contrast, the minimal Janet-like basis of $\mathcal{I}$ is equal to its minimal generating set, it has n elements and its Janet-like tree has (counting also the root) $n+(n^2+n)/2$ nodes, of which exactly one contributes cones to the complementary decomposition. It is the node $\left(x_1^n, \emptyset, \{x_2^n,\ldots,x_n^n\}\right)$ at level 1. It yields the complementary decomposition of $\mathcal{I}$ without any further computation:

$$\mathcal{D} = \left\{ (x^\mu, \emptyset) \mid \ x^\mu \mid x_1^{n-1}\cdots x_n^{n-1} \right\}.$$

If $\mathcal{I} \subset \mathcal{R}$ is a monomial ideal, then its complementary decomposition obtained from its Janet-like tree as described in (3.3) can be used to derive a representation of the Hilbert polynomial and function of $\mathcal{I}$ as linear combinations of binomial coefficients. By (3.3), the cones of the Janet complementary decomposition $\mathcal{D}$ of $\mathcal{I}$ can be collected into groups. Each group contains all cones of the form $(t \cdot s, Y_t)$ where (t, V, M) is a node in the Janet-like tree of $\mathcal{I}$ and the term s varies in the complement of a zero-dimensional irreducible monomial ideal $\mathcal{A}_t = \langle x_{a_1}^{\ell_1}, \dots, x_{a_r}^{\ell_r} \rangle \trianglelefteq \mathbb{K}[x_{a_1}, \dots, x_{a_r}] \subset \mathcal{R}$. Note that, according to (3.3), the generators of $\mathcal{A}_t$ are the non-multiplicative powers M of the node (t, V, M) together with $x_k^{\nu_k}$, where $(x^\nu, \tilde{V}, \tilde{M})$ is the minimal child node of (t, V, M) at level k. The main point here is that all cones in such a group have the same set of multiplicative variables Y_t. Define the *compressed decomposition* $\mathcal{D}^c \subseteq \mathcal{D}$ which contains for each such group only its minimal representative (t, Y_t) together with the irreducible ideal $\mathcal{A}_t$. The Hilbert function of $\mathcal{A}_t$ has non-zero values only for integers i in the range $0 \le i \le m_t := \sum_{j=1}^{r}(\ell_j - 1)$. Additionally, we write again $q_t = \deg(t)$ and $k_t = |Y_t|$. Summing up, we then obtain:

Proposition 3.2.15. *With the above notations and assumptions, the Hilbert function of the monomial ideal $\mathcal{I}$ is*

$$\begin{aligned} \mathrm{HF}_{\mathcal{I}}(q) = &\sum_{\substack{(t,Y_t,\mathcal{A}_t)\in\mathcal{D}^c\\ k_t>0}} \sum_{i=0}^{m_t} \big[q \ge q_t + i\big]\, \mathrm{HF}_{\mathcal{A}_t}(i) \binom{q-(q_t+i)+k_t-1}{k_t-1} \\ &+ \sum_{\substack{(t,Y_t,\mathcal{A}_t)\in\mathcal{D}^c\\ k_t=0}} \sum_{i=0}^{m_t} \big[q = q_t + i\big]\, \mathrm{HF}_{\mathcal{A}_t}(i)\,. \end{aligned} \tag{3.4}$$

Moreover, the Hilbert polynomial of $\mathcal{I}$ is obtained by simply dropping the contributions of zero-dimensional cones and all Kronecker-Iverson symbols:

$$\mathrm{HP}_{\mathcal{I}}(q) = \sum_{\substack{(t,Y_t,\mathcal{A}_t)\in\mathcal{D}^c\\ k_t>0}} \sum_{i=0}^{m_t} \mathrm{HF}_{\mathcal{A}_t}(i) \binom{q-(q_t+i)+k_t-1}{k_t-1}\,. \tag{3.5}$$

If we compare (3.4) with the expression obtained by applying (3.1) to the Janet complementary decomposition, then it will in general have much less summands. However, it is not fully explicit, as the numbers $\mathrm{HF}_{\mathcal{A}_t}(i)$ (the h-vectors of the zero-dimensional ideals $\mathcal{A}_t$) have to be computed for each vertex t. Thus, one may say that Prop. 3.2.15 reduces the problem of computing the Hilbert function of an arbitrary monomial ideal to the determination of the Hilbert function of zero-dimensional irreducible ideals. As these are very special ideals, it is not difficult to obtain the required values. For simplicity, we work with $\mathcal{A} = \langle x_1^{\ell_1}, \dots, x_r^{\ell_r} \rangle \trianglelefteq \mathbb{K}[x_1, \dots, x_r]$. It is easy to see that the Hilbert series of the ideal $\mathcal{A}$ is

$$\mathrm{HS}_{\mathcal{A}}(z) = \prod_{i=1}^{r} \sum_{j=0}^{\ell_i - 1} z^i\,.$$

Hence, the values to be computed are just the coefficients of z^k in $\mathrm{HS}_{\mathcal{A}}(z)$ for all $0 \leq k \leq \sum_{i=1}^{r}(\ell_i - 1)$. One way to proceed is to use known fast algorithms for polynomial multiplication, using fast Fourier transforms and related techniques.

However, irreducible monomial ideals are highly structured and possess a symmetry which can be exploited to achieve a lower complexity. Observe that the map

$$f \colon \mathcal{T} \setminus \mathcal{A} \to \mathcal{T} \setminus \mathcal{A}, \quad u \mapsto \Bigl(\prod_{i=1}^{r} x_i^{\ell_i - 1}\Bigr)/u$$

is a bijection with inverse $f^{-1} = f$. This implies $\mathrm{HF}_{\mathcal{A}}(k) = \mathrm{HF}_{\mathcal{A}}\Bigl(\bigl(\sum_{i=1}^{r}(\ell_i - 1)\bigr) - k\Bigr)$ for all integers $0 \leq k \leq \sum_{i=1}^{r}(\ell_i - 1)$. Hence the computation of the first half of the values $\mathrm{HF}_{\mathcal{A}}(k)$ suffices. Assume now that we have already expanded the Hilbert series of the "truncated" ideal $\tilde{\mathcal{A}} = \langle x_1^{\ell_1}, \dots, x_{r-1}^{\ell_{r-1}} \rangle \trianglelefteq \mathbb{K}[x_1, \dots, x_{r-1}]$ in one variable in the explicit form $\mathrm{HS}_{\tilde{\mathcal{A}}}(z) = \sum_{j=0}^{\tilde{d}} c_j z^j$ and want to compute now the Hilbert series $\mathrm{HS}_{\mathcal{A}}(z) = \sum_{j=0}^{d} d_j z^j$ of the original ideal $\mathcal{A}$. This is then easily achieved by multiplying $\mathrm{HS}_{\tilde{\mathcal{A}}}$ with $(1 + z + \cdots + z^{\ell_r - 1})$. The coefficients are $d_j = \sum_{k=0}^{j} c_k$ for $0 \leq j \leq \ell_r - 1$ and $d_j = \sum_{k=j-\ell_r+1}^{j} c_k$ for $\ell_r - 1 \leq j \leq \lceil d/2 \rceil$. This implies that all these new coefficients can be obtained by either one single addition or by an addition followed by a subtraction. The number of required additions and subtractions is $O(d)$. Overall, building up the Hilbert series $\mathrm{HS}_{\mathcal{A}}(z)$ step by step, we see that $O\bigl(r^2 \cdot \ell\bigr)$ additions are needed, where $\ell = \max\{\ell_i \mid i = 1, \dots, r\}$.

3.2.3 Hironaka's Construction

Using ideas of Hironaka [65, §4], one can design an algorithm for the computation of a complementary decomposition of a *quasi-stable* monomial ideal. Before giving the algorithm, let us first recall Hironaka's combinatorial definitions, leading to a description of a complementary decomposition via projection operators.

Construction 3.2.16 (Hironaka's construction)**.** *Let $\mathcal{I} \trianglelefteq \mathcal{R} = \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal and let $k \in \{0, 1, \dots, n\}$. Consider the projection*

$$\mathrm{pr}_k : \mathcal{T} \to \mathcal{T}, \; t \mapsto t|_{x_1 = \cdots = x_k = 1} \, .$$

For $t \in \mathcal{T}$, define the monomial cone $\mathcal{C}_k(t) = \mathcal{C}_{\{x_1, \dots, x_k\}}(t)$. Finally, let $\overline{N}_k(\mathcal{I}) := \mathcal{C}_{\{x_{k+1}\}}(\mathrm{pr}_{k+1}(\mathcal{I}) \cap \mathcal{T}) \setminus \mathrm{pr}_k(\mathcal{I})$. Then, a complementary decomposition of the ideal $\mathcal{I}$ is given by $\mathcal{T} \setminus \mathcal{I} = \bigsqcup_{k=0}^{n-1} \mathcal{C}_k(\overline{N}_k(\mathcal{I}))$, where $\mathcal{C}_k(\overline{N}_k(\mathcal{I})) = \bigcup_{s \in \overline{N}_k(\mathcal{I})} \mathcal{C}_k(s)$.

Construction 3.2.16 works for arbitrary monomial ideals $\mathcal{I}$, but the decompositions obtained by it can be infinite; more precisely, the set $\bigcup_{k=0}^{n-1} \overline{N}_k(\mathcal{I})$ can be infinite.

Example 3.2.17. *Consider first the monomial ideal $\mathcal{I} := \langle x_1 x_2 \rangle \trianglelefteq \mathbb{K}[x_1, x_2]$. Since no multiindex of class 2 is contained in $\mathcal{I}$, this ideal is not quasi-stable. Observe that*

$\mathrm{pr}_1(\mathcal{I}) \cap \mathcal{T} = \{x_2^\ell \mid \ell \geq 1\}$ *and* $\mathrm{pr}_2(\mathcal{I}) \cap \mathcal{T} = \{1\}$. *Hence, Hironaka's construction yields the two sets*

$$\overline{N}_0(\mathcal{I}) = \mathcal{C}_{x_1}\big(\{x_2^\ell \mid \ell \geq 1\}\big) \setminus \mathcal{I} = \big\{x_1^k x_2^\ell \mid k \geq 0, \ell \geq 1\big\} \setminus \mathcal{I} = \{x_2^\ell \mid \ell \geq 1\}\ ,$$
$$\overline{N}_1(\mathcal{I}) = \mathcal{C}_{x_2}\big(\{1\}\big) \setminus \mathrm{pr}_1(\mathcal{I}) = \big\{x_2^\ell \mid \ell \geq 0\big\} \setminus \big\{x_2^\ell \mid \ell \geq 1\big\} = \{1\}.$$

Thus we obtain the infinite complementary decomposition

$$\mathcal{T} \setminus \mathcal{I} = \Big(\bigsqcup_{n \in \mathbb{N}} \mathcal{C}_0(x_2^n)\Big) \sqcup \mathcal{C}_1(1)\ .$$

Consider now the quasi-stable ideal $\mathcal{J} = \langle x_1x_2,\, x_2^2\rangle = \langle \mathcal{I},\, x_2^2\rangle$. *As for* $\mathcal{I}$, *we have* $\mathrm{pr}_1(\mathcal{J}) \cap \mathcal{T} = \{x_2^\ell \mid \ell \geq 1\}$ *and* $\mathrm{pr}_2(\mathcal{J}) \cap \mathcal{T} = \{1\}$, *but this time Hironaka's construction yields the two sets*

$$\overline{N}_0(\mathcal{J}) = \mathcal{C}_{x_1}\big(\{x_2^\ell \mid \ell \geq 1\}\big) \setminus \mathcal{J} = \big\{x_1^k x_2^\ell \mid k \geq 0, \ell \geq 1\big\} \setminus \mathcal{J} = \{x_2\}\ ,$$
$$\overline{N}_1(\mathcal{J}) = \mathcal{C}_{x_2}\big(\{1\}\big) \setminus \mathrm{pr}_1(\mathcal{J}) = \big\{x_2^\ell \mid \ell \geq 0\big\} \setminus \big\{x_2^\ell \mid \ell \geq 1\big\} = \{1\}\ .$$

This time we obtain the finite complementary decomposition

$$\mathcal{T} \setminus \mathcal{J} = C_0(x_2) \sqcup \mathcal{C}_1(1)\ .$$

We now show that those monomial ideals for which Hironaka's construction yields a finite complementary decomposition are exactly the quasi-stable monomial ideals (or equivalently, by Proposition 2.2.11, the ideals with finite Pommaret bases).

Proposition 3.2.18. *Let* $\mathcal{I} \trianglelefteq \mathcal{R}$ *be a monomial ideal. Hironaka's construction yields a finite complementary decomposition of* $\mathcal{I}$, *if and only if* $\mathcal{I}$ *possesses a finite Pommaret basis.*

Proof. If $\mathcal{I}$ is quasi-stable, then Hironaka's construction can be realised by Algorithm 5 below, which obviously yields a finite output. The correctness proof of the algorithm (see Proposition 3.2.19) then finishes this direction of the proof.

If Hironaka's construction applied to $\mathcal{I}$ yields a finite decomposition, then it is a complementary decomposition of $\mathcal{I}$ of the form

$$\mathcal{T} \setminus \mathcal{I} = \bigsqcup_{k=0}^{n-1} \mathcal{C}_k\big(\overline{N}_k(\mathcal{I})\big). \tag{3.6}$$

Define the set $H := \bigcup_{k=0}^{n-1}\Big(\bigcup_{\ell > k}\big(x_\ell \cdot \overline{N}_k(\mathcal{I})\big)\Big) \cap \mathcal{I} \subset \mathcal{I}$. This set is obviously finite. We will now show that it is a Pommaret basis of $\mathcal{I}$.

Let $t \in \mathcal{T} \cap \mathcal{I}$ be an arbitrary term in $\mathcal{I}$ and set $a := \mathrm{cls}\,(t)$. There is a minimal integer $\ell \in \{a, a+1, \ldots, n\}$ such that $\mathrm{pr}_\ell(t) \notin \mathcal{I}$ (assume $\mathcal{I} \neq \mathcal{R}$; if $\mathcal{I} = \mathcal{R}$, then it is obviously quasi-stable) . Since $\mathrm{pr}_{\ell-1}(t) \in \mathcal{I}$, there is an integer d with $0 \leq d < \deg_\ell(t)$ such that $s := x_\ell^d\,\mathrm{pr}_\ell(t) \notin \mathcal{I}$ and $x_\ell \cdot s = x_\ell^{d+1}\,\mathrm{pr}_\ell(t) \in \mathcal{I}$. We claim that there is an integer $k \leq \ell$ such that $s \in \overline{N}_k(\mathcal{I})$.

Since $s \notin \mathcal{I}$, there is some integer $b \in \{0, 1, \ldots, n-1\}$ and a term $u \in \overline{N}_b(\mathcal{I})$ such that $s \in \mathcal{C}_b(u)$ by equality (3.6). If $b \le \ell$, then obviously $u = s$ and $s \in \overline{N}_b(\mathcal{I})$, and setting $k := b$ yields the claim. Otherwise, $b > \ell$. From definition of $\overline{N}_b(\mathcal{I})$, we know that $u \notin \operatorname{pr}_b(\mathcal{I})$, but $u \in \mathcal{C}_{\{b+1\}}(\operatorname{pr}_{b+1}(\mathcal{I}))$. In particular, $\operatorname{cls}(u) \ge b+1$. By definition of u, we must also have $\deg_r(u) = \deg_r(s)$ for all $r \ge b+1$. But now it is clear that u is exactly $\operatorname{pr}_b(s)$, and since $s = x_\ell^d \operatorname{pr}_\ell(t)$, we then have $u = \operatorname{pr}_b(s) = \operatorname{pr}_b\big(x_\ell^d \operatorname{pr}_\ell(t)\big) = \operatorname{pr}_b(t)$, contradicting $u \notin \operatorname{pr}_b(\mathcal{I})$. Thus we have proven the claim.

Now, obviously $x_\ell s = x_\ell^{d+1} \operatorname{pr}_\ell(t) \in H$ is a Pommaret divisor of t. Since t was an arbitrary element of $\mathcal{I}$, this observation finally proves that H is a finite Pommaret basis of $\mathcal{I}$ and we are done. □

Algorithm 5: Complementary decomposition à la Hironaka

Data: Minimal Pommaret basis H of the monomial ideal $\mathcal{I} = \langle H \rangle \trianglelefteq \mathcal{R}$
Result: Finite complementary decomposition $\mathcal{D}$ of $\mathcal{I}$

```
begin
    D ⟵ ∅
    H ⟵ H
    for k = 1, ..., n do
        A ⟵ {x^μ ∈ H | cls(x^μ) = k}
        B ⟵ {x^ν|_{x_k=1} | x^ν ∈ A}
        foreach x^ν ∈ A do
            for i = 0, ..., ν_k − 1 do
                D ⟵ D ∪ {( x_k^i · (x^ν|_{x_k=1}), {x_1, ..., x_{k−1}} )}
        H ⟵ PommaretAutoreduction((H \ A) ∪ B)
    return D
```

Proposition 3.2.19. *Given the minimal Pommaret basis of a quasi-stable monomial ideal as input, Algorithm 5 terminates and its output is exactly the decomposition from Hironaka's construction.*

Proof. The algorithm obviously terminates on its input. Let $k \in \{1, \ldots, n\}$. We will show iteratively that:

- At the start of the k^{th} iteration of the outer **for** loop, $\mathcal{H}$ is the minimal Pommaret basis of $\operatorname{pr}_{k-1}(\mathcal{I})$.
- During this iteration of the outer **for** loop, exactly the cones $\mathcal{C}_{k-1}\big(\overline{N}_{k-1}(\mathcal{I})\big)$ from Hironaka's decomposition of $\overline{\mathcal{I}}$ are added to $\mathcal{D}$.
- At the end of this iteration of the outer **for** loop, $\mathcal{H}$ is the minimal Pommaret basis of $\operatorname{pr}_k(\mathcal{I})$.

So let $k = 1$. Obviously, $\mathcal{H} = H$ is the minimal basis of the input ideal $\mathcal{I} = \operatorname{pr}_0(\mathcal{I})$ at the start of the first iteration of the outer **for** loop. The elements of class 1 are collected in A and the set $\operatorname{pr}_1(A)$ is assigned to B. Let $x^\mu \in \operatorname{pr}_1(\mathcal{I})$ be a term in the first projection ideal. Then x^μ is divisible by either an element of $B = \mathrm{pr}_1(A)$ or by an element of $\mathrm{pr}_1(\mathcal{H} \setminus A)$, but not by any element of A, proving that $(\mathcal{H} \setminus A) \cup B$ generates $\mathrm{pr}_1(\mathcal{I})$. Let $x^\nu \in \mathcal{I}$ with $\mathrm{pr}_1(x^\nu) = x^\mu$; then x^ν possesses a Pommaret divisor $x^\rho \in \mathcal{H}$. One can easily show that $\mathrm{pr}_1(x^\rho)$ is a Pommaret divisor of $\mathrm{pr}_1(x^\nu) = x^\mu$. Putting things together, $(\mathcal{H} \setminus A) \cup B$ is a Pommaret basis of $\mathrm{pr}_1(\mathcal{I})$ and its autoreduction then yields the minimal Pommaret basis of $\mathrm{pr}_1(\mathcal{I})$. Note that, by these arguments, we have shown that $\mathrm{pr}_1(\mathcal{I})$ is quasi-stable.

Now we consider the cones that are added to $\mathcal{D}$ in this iteration of the loop. It is obvious that their vertices are elements of $\mathcal{C}_{\{1\}}(\mathrm{pr}_1(\mathcal{I}))$, but not of $\mathrm{pr}_0(\mathcal{I}) = \mathcal{I}$. So all added cones are of the form $\mathcal{C}_0\big(\overline{N}_0(\mathcal{I})\big)$ from Hironaka's construction. Conversely, let $x^\zeta \in \overline{N}_0(\mathcal{I})$. Then $x^\zeta \notin \mathcal{I}$, but there exists an exponent $\ell > 0$ such that $x_1^\ell \cdot x^\zeta \in \mathcal{I}$. For the minimal ℓ with this property, we have that $x_1^\ell x^\zeta$ is an element of the minimal Pommaret basis of the input ideal $\mathcal{I}$, since otherwise $x_1^{\ell-1} \cdot x^\zeta \in \mathcal{I}$, contradicting the minimality of ℓ. Now, in order to see that $\mathcal{C}_0(x^\zeta)$ is added to $\mathcal{D}$ in this iteration of the loop, it only remains to be shown that $\mathrm{cls}\,(x_1^\ell \cdot x^\zeta) = 1$. But this is clear, since $\ell > 0$. So, this multiindex is of class 1 and an element of the minimal Pommaret basis of the input ideal, proving that x^ζ is a (zero-dimensional) cone added to $\mathcal{D}$ in this iteration of loop.

If $k > 1$, similar arguments lead to the desired result, since the **for** loop gets the input $\mathcal{H}$, which at this point of the algorithm is the minimal Pommaret basis of the quasi-stable ideal $\mathrm{pr}_{k-1}(\mathcal{I})$. □

Example 3.2.20. *We illustrate how Algorithm 5 works by applying it to the quasi-stable monomial ideal $\mathcal{I} = \langle H \rangle \trianglelefteq \mathbb{K}[x_1, x_2, x_3]$ generated by the minimal Pommaret basis*

$$H = \left\{x_3^3,\, x_2^3 x_3,\, x_2^3 x_3^2,\, x_1 x_2 x_3,\, x_1 x_2^2 x_3,\, x_1 x_2 x_3^2,\, x_1 x_2^2 x_3^2\right\}.$$

- *In the first iteration of the outer **for** loop, we have*

  $$A = \left\{x_1 x_2 x_3,\, x_1 x_2^2 x_3,\, x_1 x_2 x_3^2,\, x_1 x_2^2 x_3^2\right\}, \quad B = \left\{x_2 x_3,\, x_2^2 x_3,\, x_2 x_3^2,\, x_2^2 x_3^2\right\}$$

  and the cones $\mathcal{C}_0\big(\{x_2 x_3,\, x_2^2 x_3,\, x_2 x_3^2,\, x_2^2 x_3^2\}\big)$ are added to $\mathcal{D}$. The set

  $$(\mathcal{H} \setminus A) \cup B = \left\{x_3^3,\, x_2^3 x_3,\, x_2^3 x_3^2,\, x_2 x_3,\, x_2^2 x_3,\, x_2 x_3^2,\, x_2^2 x_3^2\right\}$$

  is involutively autoreduced to the minimal Pommaret basis $\left\{x_3^3,\, x_2 x_3,\, x_2 x_3^2\right\}$.
- *In the second iteration of the outer **for** loop, we have $A = \{x_2 x_3,\, x_2 x_3^2\}$ and $B = \{x_3,\, x_3^2\}$ and the cones $\mathcal{C}_1\big(\{x_3, x_3^2\}\big)$ are added to $\mathcal{D}$. The set $(\mathcal{H} \setminus A) \cup B = \{x_3^3,\, x_3,\, x_3^2\}$ is involutively autoreduced to the minimal Pommaret basis $\{x_3\}$.*
- *In the third iteration of the outer **for** loop, we have $A = \{x_3\}$ and $B = \{1\}$. The cone $\mathcal{C}_2\big(\{1\}\big)$ is added to $\mathcal{D}$. The set $(\mathcal{H} \setminus A) \cup B = \{1\}$ is computed. (This is a general property of the algorithm: In the last instance of the outer **for** loop, always the set $\{1\}$ is obtained.)*

Since we can apply Algorithm 3 to a Janet basis of an arbitrary monomial ideal, it also works for quasi-stable ideals where any Janet basis is simultaneously a Pommaret basis. Moreover, since the Janet algorithm only performs a traversal of the

Janet tree, it has a lower complexity than Algorithm 5 performing Pommaret autoreductions in intermediate steps. Hence, for the computation of a complementary decomposition, it is preferable to apply Algorithm 3 whenever possible. The following result states that in the considered situation the outputs of both algorithms are identical.

Theorem 3.2.21. *Given the minimal Pommaret basis H of the quasi-stable ideal $\mathcal{I}$ as input, Algorithms 3 and 5 produce the same output.*

Proof. Choose an index $k \in \{1, \dots, n\}$ and let $x^\mu \in H$ be a term contained in the Janet class $H_{[\mu_{k+1},\dots,\mu_n]}$ with μ_k minimal. Algorithm 3 adds all cones of the form

$$\big(x_k^i \cdot (x^\mu|_{x_1=\cdots=x_k=1}), \{1, \dots, k-1\}\big)$$

with $i \in \{0, \dots, \mu_k - 1\}$ to $\mathcal{D}$. If we can show that Algorithm 5 does the same, then we are done: under this assumption, the algorithm adds at least all cones to the complementary decomposition which are found by Algorithm 3, but then, by the disjointness of such decompositions, it cannot add any additional cones, meaning the decomposition produced by Algorithm 5 is exactly the same as the decomposition found by Algorithm 3. Moreover, we may assume that $\mu_k > 0$, since if $\mu_k = 0$, then x^μ does not contribute any monomial cones to $\mathcal{D}$ during Algorithm 3.

Define $\mathcal{I}_q$ as the ideal generated by $\mathcal{H}$ at the very end of the q^{th} iteration of the outer **for** loop of Algorithm 3. Certainly, $t_\mu := x^\mu|_{x_1=\cdots=x_{k-1}=1} \in \mathcal{I}_{k-1}$. We need to show that no strict Pommaret divisor of t_μ is in $\mathcal{I}_{k-1}$, since if this is the case, then t_μ belongs to $\mathcal{H}$ also at the beginning of the k^{th} iteration of the outer **for** loop of Algorithm 3 (that is, after Pommaret autoreduction) applied to H, and the desired monomial cones are then added to $\mathcal{D}$ in this loop iteration.

So let us suppose $t_\nu := x_k^{\nu_k} x_{k+1}^{\nu_{k+1}} \cdots x_n^{\nu_n} \in \mathcal{H}$ with $\nu_k < \mu_k$ and $t_\nu \mid_P t_\mu$. Observe that if $s > 0$, then $t_\nu|_{x_k=1} = t_\mu|_{x_k=1}$. Let $x^\rho \in \mathbb{K}[x_1, \dots, x_{k-1}]$ be a term such that $x^\nu := x^\rho \cdot t_\nu \in H$ — such a term must exist, because in order to construct $\mathcal{I}_{k-1}$ during Algorithm 3, coming from elements of H, one only divides out powers of the first $k-1$ variables or leaves out some superfluous terms during autoreductions. We must distinguish several cases:

- $\operatorname{cls}(x^\nu) \geq k$: This case cannot occur, since x^ν would be a strict Pommaret divisor of x^μ in this case: $x^\rho = 1$ and $x^\nu = t_\nu \mid_{\mathcal{P}} t_\mu \mid_{\mathcal{P}} x^\mu$. This is a contradiction to $x^\mu \in H$.
- $\operatorname{cls}(x^\nu) < k$ and $\nu_k > 0$: Then x^ν is in the Janet class $H_{[\mu_{k+1},\dots,\mu_n]}$ and $\nu_k < \mu_k$, a contradiction to the minimality of μ_k.
- $\operatorname{cls}(x^\nu) < k$ and $\nu_k = 0$: If also $t_\nu|_{x_k=1} = t_\mu|_{x_k=1}$, then x^ν is in the Janet class $H_{[d_{k+1},\dots,d_n]}$ and $\nu_k = 0$, which is a contradiction to the minimality of μ_k (recall that $\mu_k > 0$). If $t_\nu|_{x_k=1} \neq t_\mu|_{x_k=1}$, then the nontrivial Pommaret-nonmultiplicative prolongation $x^\rho \cdot (t_\mu|_{x_k=1})$ of x^ν possesses a unique Pommaret divisor x^τ in H. If $\operatorname{cls}(x^\tau) < k$, then x^τ is in the Janet class $H_{[\mu_{k+1},\dots,\mu_n]}$ and $\tau_k = 0$, again a contradiction to the minimality of μ_k. And finally, $\operatorname{cls}(x^\tau) \geq k$ cannot occur, since then x^τ would be a proper Pommaret divisor of x^μ, impossible because of the Pommaret autoreducedness of H. $\square$

Example 3.2.22. *Consider – as in Example 3.2.20 – the quasi-stable ideal* $\mathcal{I} = \langle x_3^3,\, x_2^3x_3\,, x_1x_2x_3\rangle$. *We follow the steps of Algorithm 3 for* $\mathcal{I}$ *to see that it produces indeed the same output as Algorithm 5. Note that* $\mathcal{I}$ *has the minimal Janet (and Pommaret) basis*

$$H = \left\{x_3^3,\, x_2^3x_3,\, x_2^3x_3^2,\, x_1x_2x_3,\, x_1x_2^2x_3,\, x_1x_2x_3^2,\, x_1x_2^2x_3^2\right\}\,.$$

- *For* $k = 3$, *we have the non-empty Janet class* $H_{[]} = H$ *at level* $k + 1 = 4$. *It corresponds to the root of the Janet tree and its leftmost child node is* $(x_3, \emptyset)$. *We add the cone* $\big(1, \{x_1, x_2\}\big)$ *to* $\mathcal{D}$.
- *For* $k = 2$, *we have the non-empty Janet classes* $H_{[3]}$, $H_{[2]}$, $H_{[1]}$ *at level* $k+1 = 3$ *and their leftmost child nodes are* $\big(x_3^3, \{x_3\}\big)$, $(x_2x_3^2, \emptyset)$, *and* $(x_2x_3, \emptyset)$. *For the first node, no cone is added, since the* x_2*-degree of its first entry is zero. The two cones that are added are* $\big(x_3^2, \{x_1\}\big)$ *and* $\big(x_3, \{x_1\}\big)$.
- *For* $k = 1$, *we have the non-empty Janet classes*

  $$H_{[0,3]},\, H_{[3,1]},\, H_{[3,2]},\, H_{[1,1]},\, H_{[2,1]},\, H_{[1,2]},\, H_{[2,2]}$$

  at level $k + 1 = 2$. *Only the last four classes yield leftmost child nodes that contribute cones to* $\mathcal{D}$, *namely* $\big(x_1x_2x_3, \{x_1\}\big)$, $\big(x_1x_2^2x_3, \{x_1\}\big)$, $\big(x_1x_2x_3^2, \{x_1\}\big)$ *and* $\big(x_1x_2^2x_3^2, \{x_1\}\big)$. *The added cones are, accordingly,*

  $$(x_2x_3, \emptyset), (x_2^2x_3, \emptyset), (x_2x_3^2, \emptyset), (x_2^2x_3^2, \emptyset)\,.$$

Remark 3.2.23. *As just mentioned, Algorithms 3 and 5 produce the same number of cones (see Theorem 3.2.8). However, since in the latter algorithm we apply a Pommaret autoreduction procedure, its complexity is not easily determined.*

3.2.4 Primary and Irreducible Decompositions

The cone decompositions of the complements of monomial ideals that we have studied so far are not the only way in which one can decompose such a complement. Primary and irreducible decompositions of monomial ideals are representations of such an ideal as the intersection of associated ideals with an easier structure. Dually, they can be interpreted as a representation of the complement of the decomposed ideal as a union of the complements of ideals that are easier to study. In this section, we review the definitions of these different types of decompositions, study how they are related to each other and finally give an algorithm to compute minimal primary decompositions of quasi-stable ideals using Pommaret bases.

Remark 3.2.24. *A monomial ideal* $\mathcal{I} \trianglelefteq \mathcal{R}$ *is prime if and only if it can be generated by a set of variables. A monomial ideal* $\mathcal{Q}$ *is primary to* $\mathcal{I} = \langle x_{i_1}, \ldots, x_{i_r}\rangle$, *if and only if it has a generating set that only depends on the variables generating* $\mathcal{I}$ *and that contains for each* x_{i_j} *a pure power* $x_{i_j}^{k_j}$. *The associated primes of a monomial ideal are again monomial ideals. A primary decomposition of a monomial ideal* $\mathcal{I}$ *is a representation* $\mathcal{I} = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_k$ *with each* $\mathcal{Q}_\ell$ *a primary monomial ideal. Such a decomposition is called* minimal, *if the associated primes* $\sqrt{\mathcal{Q}_\ell}$ *are pairwise different and none of the* $\mathcal{Q}_\ell$ *can be omitted in the representation.*

Definition 3.2.25. *A monomial ideal $\mathcal{Q} \trianglelefteq \mathcal{R}$ is called* irreducible, *if there is a term $x^\mu \in \mathcal{T}$ such that $\mathcal{Q} = \langle x_i^{\mu_i} \mid 1 \leq i \leq n,\ \mu_i > 0\rangle$. An* irreducible decomposition *of a monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is a decomposition $\mathcal{I} = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_k$ of $\mathcal{I}$ into irreducible monomial ideals $\mathcal{Q}_1, \dots, \mathcal{Q}_k$. Such a decomposition is called* irredundant, *if none of the ideals $\mathcal{Q}_\ell$ can be omitted in the decomposition.*

Remark 3.2.26. *Since irreducible monomial ideals are obviously primary, it is clear that one can obtain a minimal primary decomposition from an irredundant irreducible decomposition by simply collecting, for each appearing prime, the irreducible components primary to it. Hence, irreducible decompositions can be regarded as being finer than primary ones. In turn, if an irredundant irreducible decomposition $\mathcal{I} = \mathcal{Q}_1 \cap \cdots \cap \mathcal{Q}_k$ of the monomial ideal $\mathcal{I}$ is known, then obviously $\mathcal{T} \setminus \mathcal{I} = (\mathcal{T} \setminus \mathcal{Q}_1) \cup \cdots \cup (\mathcal{T} \setminus \mathcal{Q}_k)$ provides a decomposition of the complement of $\mathcal{I}$ into sets which are easy to describe. A term x^ν is in the complement $\mathcal{T} \setminus \mathcal{Q}$ of a monomial irreducible ideal $\mathcal{Q}$, if and only if $\nu_i < \mu_i$ for each variable x_i dividing the term x^μ describing $\mathcal{Q}$. This decomposition is of course not disjoint, unless $\mathcal{I}$ is itself irreducible. Since cone decompositions are disjoint, they can be regarded as an even finer type of decomposition than the irreducible ones.*

We now present a way of obtaining irreducible decompositions from cone decompositions. The key point is the following lemma.

Lemma 3.2.27. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a monomial ideal and $\{(t, X_t) \mid t \in U\}$ a complementary cone decomposition of it, where $U \subseteq \mathcal{T} \setminus \mathcal{I}$ is some subset. Then a (generally redundant) irreducible decomposition of $\mathcal{I}$ is given by:*

$$\mathcal{I} = \bigcap_{t \in U} \langle x_i^{\mu_i+1} \mid t = x^\mu, x_i \notin X_t\rangle\ . \tag{3.7}$$

Proof. We show the equivalent statement

$$\mathcal{T} \setminus \mathcal{I} = \bigcup_{t \in U} \bigl(\mathcal{T} \setminus \langle x_i^{\mu_i+1} \mid t = x^\mu, x_i \notin X_t\rangle\bigr)\ . \tag{3.8}$$

First, let $x^\nu \notin \mathcal{I}$ be a term from the complement. Since $\{(t, X_t) \mid t \in U\}$ decomposes $\mathcal{T} \setminus \mathcal{I}$, there is a term $t = x^\mu \in U$ and a term $x^\rho \in \mathbb{K}[X_t]$ such that $x^\nu = x^\mu x^\rho$. This implies, for each variable $x_i \notin X_t$, $\nu_i \leq \mu_i$, and hence $x^\nu \notin \langle x_i^{\mu_i+1} \mid x_i \notin X_t\rangle$. Thus, the inclusion $\subseteq$ in (3.8) follows.

Now, let $t = x^\mu \in U$ be a term appearing in the cone decomposition of $\mathcal{T} \setminus \mathcal{I}$ and let $x^\nu \notin \langle x_i^{\mu_i+1} \mid x_i \notin X_t\rangle$. An immediate consequence are the inequalities $\nu_i \leq \mu_i$ for each index i with $x_i \notin X_t$. Moreover, multiplying with powers of the variables from X_t, we find a term $tx^\rho \in t\mathbb{K}[X_t]$ such that x^ν divides tx^ρ. Observe that $tx^\rho \notin \mathcal{I}$ and hence also $x^\nu \notin \mathcal{I}$. This proves the inclusion $\supseteq$ in (3.8), finishing the proof. □

Once we have obtained an irreducible decomposition, we can always extract an irredundant decomposition by discarding redundant components. Note that, if the irreducible ideals $\mathcal{Q}_\mu, \mathcal{Q}_\nu$ with $\sqrt{\mathcal{Q}_\mu} = \sqrt{\mathcal{Q}_\nu}$ are described by the terms x^μ, x^ν, then

$\mathcal{Q}_\mu \subseteq \mathcal{Q}_\nu$ if and only if x^ν divides x^μ. In this case we can discard the larger one, $\mathcal{Q}_\nu$. This means we have to keep those components $\mathcal{Q}_\mu$ with x^μ maximal with respect to the partial order of divisibility. This amounts to a form of monomial autoreduction. It is desirable to be able to detect a number of redundant components already from the structure of the cone decomposition. This can be done via Janet-like bases.

Proposition 3.2.28. *Let $U \subseteq \mathcal{T}$ be the minimal Janet-like basis of the monomial ideal $\mathcal{I} = \langle U \rangle \trianglelefteq \mathcal{R}$. Let the partial multiindex $M = [\mu_k, \dots, \mu_n]$ belong to a minimal node in the Janet-like tree of V induced by the term $x^\mu \in U$ and contributing cones to the complementary decomposition of $\mathcal{I}$. Then at most one of the cones belonging to M leads to an irredundant component in the irreducible decomposition of $\mathcal{I}$. This component is given by the irreducible ideal*

$$\mathcal{Q} = \left\langle x_k^{\mu_k} \right\rangle + \left\langle x_\ell^{\mu_\ell + b_\ell} \mid \ell > k \wedge x_\ell^{b_\ell} \in \mathrm{NMP}(x^\mu, U) \right\rangle . \tag{3.9}$$

Proof. By Corollary 3.2.13, the cones contributed to the complementary decomposition by the partial multiindex M all have the same set of multiplicative variables. Moreover, they are supported on terms of the form $x_k^i x_{k+1}^{\mu_{k+1}+a_{k+1}} \cdots x_n^{\mu_n + a_n}$ with $1 \leq i < \mu_k$ and $x_{k+1}^{a_{k+1}} \cdots x_n^{a_n} \mid \prod_{x_\ell^{b_\ell} \in \mathrm{NMP}(x^\mu, U)} x_\ell^{b_\ell - 1}$. Among them, there is only one term which is maximal with respect to divisibility, namely

$$x_k^{\mu_k - 1} x_{k+1}^{\mu_{k+1}} \cdots x_n^{\mu_n} \cdot \prod_{\ell > k, x_\ell^{b_\ell} \in \mathrm{NMP}(x^\mu, U)} x_\ell^{b_\ell - 1} .$$

Hence, applying Lemma 3.2.27, we are done. □

Remark 3.2.29. *The algorithm for the computation of irreducible decompositions implied by Proposition 3.2.28 can be seen to be largely equivalent to an algorithm developed by Gao and Zhu [41, Alg. 1]. There, also tree structures of monomial bases are exploited. While the authors report good performance for highly non-generic monomial ideals [41, Sec. 7], their algorithm is not the fastest available. Roune's* slice algorithm *[92, 93] shows overall better performance. However, his algorithm contains Gao and Zhu's approach as a special case for some choices of splitting strategies [92, Sec. 5.2].*

With the results of the present section in mind, it is not surprising that algorithms specialised to computing the irreducible decomposition will show better performance than algorithms based on tree structures or Janet-like bases, because these give not only irreducible decompositions but also the finer disjoint complementary decompositions and hence compute more information.

Example 3.2.30. *Let us revisit the ideal of Examples 3.2.20 and 3.2.22 and compute an irredundant irreducible decomposition for it. One can easily check that $U = \{x_1^2 x_2 x_3,\ x_2^3 x_3,\ x_3^3\}$ is already the minimal Janet-like basis of the ideal $\mathcal{I} = \langle U \rangle$. The minimal nodes in the Janet tree are given by the partial multiindices $[1]$, $[1,1]$, and $[2,1,1]$. These are all induced by $x^\mu = x_1^2 x_2 x_3 \in U$ with $\mathrm{NMP}(x^\mu, U) = \{x_2^2,\ x_3^2\}$. Applying Proposition 3.2.28, we get the candidates $\mathcal{J}_1 = \langle x_3 \rangle$, $\mathcal{J}_2 = \langle x_2,\ x_3^3 \rangle$, and*

$\mathcal{J}_3 = \langle x_1^2, x_2^3, x_3^3\rangle$ for components of the irreducible decomposition. Their associated primes are pairwise different, so the decomposition cannot be reduced any further and we are done: $\mathcal{I} = \mathcal{J}_1 \cap \mathcal{J}_2 \cap \mathcal{J}_3$ is the desired irredundant irreducible decomposition.

We now specialise to quasi-stable monomial ideals. If we want to compute minimal primary decompositions for them, we can apply an extended version of Algorithm 5, which not only gives such a decomposition, but also Pommaret bases for each component, thus also proving that the components are quasi-stable, too.

Remark 3.2.31. *From the operations performed during the first iteration of the **for** loop in Algorithm 5, it is immediately clear that after it, $\mathcal{H}$ is the minimal Pommaret basis of the saturation $\mathcal{I} : \langle x_1\rangle^\infty$. By one of the many equivalent characterisations of quasi-stable ideals, see e. g. [97, Prop. 5.3.4(iii)], we know that for $\mathcal{I}$ the chain of inclusions*

$$\mathcal{I} : \langle x_1\rangle^\infty \subseteq \mathcal{I} : \langle x_2\rangle^\infty \subseteq \cdots \subseteq \mathcal{I} : \langle x_n\rangle^\infty \tag{3.10}$$

*holds. From it and by a well-known property of ideal quotients, we get for any $1 \leq i < n$ the inclusions $(\mathcal{I} : \langle x_i\rangle^\infty) : \langle x_{i+1}\rangle^\infty \subseteq \mathcal{I} : \langle x_{i+1}\rangle^\infty$. Since the saturations on the left hand side commute, we even get an equality. Hence Algorithm 5 computes in the kth iteration of its **for** loop a Pommaret basis of the saturation $\mathcal{I} : \langle x_k\rangle^\infty$.*

Now, let d be the smallest class of a term in the Pommaret basis H of $\mathcal{I}$ and let D be maximal among the indices j such that no pure power $x_j^{p_j}$ appears in H and for $d \leq j \leq D$ let s_j be the maximal exponent of the variable x_j appearing in any term of H. Then, by [97, Prop. 5.3.9], a minimal primary decomposition of $\mathcal{I}$ is given by

$$\bigcap_{\substack{d\leq j\leq D\\ j\in Q}} \mathcal{Q}_j\ ,$$

where $Q \subseteq \{d, d+1, \ldots, D\}$ is the subset of those indices $1 \leq j \leq n$ for which we have a proper inclusion $\mathcal{I} : \langle x_j\rangle^\infty \subsetneq \mathcal{I} : \langle x_{j+1}\rangle^\infty$ and where $\mathcal{Q}_j = \mathcal{I} : \langle x_j\rangle^\infty + \langle x_{j+1}^{s_{j+1}}, x_{j+2}^{s_{j+2}}, \ldots, x_D^{s_D}\rangle$. Note that $\mathcal{Q}_j$ is a $\langle x_{j+1}, x_{j+2}, \ldots, x_n\rangle$-primary ideal.

To test the condition $\mathcal{I} : \langle x_{j-1}\rangle^\infty \subsetneq \mathcal{I} : \langle x_j\rangle^\infty$ in Algorithm 5, we note that it is equivalent to the condition that at least one term of class j appears in the Pommaret basis of $\mathcal{I} : \langle x_{j-1}\rangle^\infty$. This means that in the current iteration the set $C \neq \emptyset$ is not empty. Whenever this condition is satisfied, we can add the primary ideal $\langle \mathcal{H}, x_j^{s_j}, x_{j+1}^{s_{j+1}}, \ldots, x_D^{s_D}\rangle$ to the primary decomposition to be computed. For a concrete computation, see Example 3.2.32.

From Remark 3.2.31, we get the following adapted version of Algorithm 5, which computes a minimal primary decomposition for a quasi-stable monomial ideal, instead of a complementary decomposition.

Example 3.2.32. *Consider, as in Example 3.2.20, the following quasi-stable ideal: $\mathcal{I} = \langle x_3^3, x_2^3x_3, x_1x_2x_3\rangle$. Note that $\mathcal{I}$ has the minimal Pommaret basis*

$$H = \{x_3^3, x_2^3x_3, x_2^3x_3^2, x_1x_2x_3, x_1x_2^2x_3, x_1x_2x_3^2, x_1x_2^2x_3^2\}\ .$$

Algorithm 6: Minimal Primary Decomposition from Pommaret Basis

Data: Minimal Pommaret basis H of the monomial ideal $\mathcal{I} = \langle H\rangle \trianglelefteq \mathcal{R}$
Result: Finite number of monomial primary ideals $\mathcal{Q}_j$ such that $\mathcal{I} = \bigcap_j \mathcal{Q}_j$
begin
  $d \longleftarrow \min\{\mathrm{cls}(t) \mid t \in H\}$
  $D \longleftarrow \min\{i \mid \exists k_i \in \mathbb{N}\, \exists x_i^{k_i} \in H\} - 1$
  for $\ell = d, \ldots, D$ **do**
    $s_\ell \longleftarrow \max\{\deg_\ell(t) \mid t \in H\}$
  $\mathcal{H} \longleftarrow H$
  for $k = d, \ldots, D$ **do**
    $A \longleftarrow \{x^\mu \in \mathcal{H} \mid \mathrm{cls}(x^\mu) = k\}$
    if $A \neq \emptyset$ **then**
      $B \longleftarrow \{x^\nu|_{x_k=1} \mid x^\nu \in A\}$
      $\mathcal{Q}_k \longleftarrow \langle \mathcal{H},\, x_k^{s_k}, \ldots,\, x_D^{s_D}\rangle$
      $\mathcal{H} \longleftarrow \texttt{PommaretAutoreduction}\big((\mathcal{H} \setminus A) \cup B\big)$
    else
      $\mathcal{Q}_k \longleftarrow \mathcal{P}$
  return $\mathcal{Q}_d, \ldots, \mathcal{Q}_D$

We show how to obtain a minimal primary decomposition of this ideal using the computations performed in Algorithm 5. We use the arguments described in Remark 3.2.31. First, note that 1 is the least class of an element of H and 2 is the maximal index such that no pure power of its corresponding variable appears in H. We have the maximal exponents $s_1 = 1$ and $s_2 = 3$. Since there is a term of class 1 in H, already in the first loop iteration we need to add $\langle H,\, x_1,\, x_2^3\rangle = \langle x_1,\, x_2^3,\, x_3^3\rangle$ to the minimal primary decomposition. The minimal Pommaret basis of the saturation $\mathcal{I} : \langle x_1\rangle^\infty$ is computed as $H_1 = \{x_3^3,\, x_2x_3,\, x_2x_3^2\}$ (cf. Example 3.2.20). It has a term of class 2. Hence, we need to add $\langle H_1,\, x_2^3\rangle = \langle x_2^3,\, x_2x_3,\, x_2x_3^2,\, x_3^3\rangle$ to the minimal primary decomposition as a new component. The minimal Pommaret basis of the saturation $\mathcal{I} : \langle x_2\rangle^\infty$ is computed as $H_2 = \{x_3\}$ (cf. again Example 3.2.20). It has a term of class 3 and hence we must add $\langle H_2\rangle = \langle x_3\rangle$ as a third component to the minimal primary decomposition. The algorithm stops after this loop iteration. Hence, all in all, we obtain the minimal primary decomposition $\mathcal{I} = \langle x_1,\, x_2^3,\, x_3^3\rangle \cap \langle x_2^3,\, x_2x_3,\, x_2x_3^2,\, x_3^3\rangle \cap \langle x_3\rangle$.

Example 3.2.33. *Consider the ideal $\mathcal{I} = \langle x_1x_3,\, x_2x_3,\, x_3^2\rangle \trianglelefteq \mathbb{K}[x_1, x_2, x_3]$, which is quasi-stable. We compute a minimal primary decomposition of $\mathcal{I}$ using Algorithm 5 and Remark 3.2.31. Note first that $\mathcal{I}$ is already given by its minimal Pommaret basis $H = \{x_1x_3,\, x_2x_3,\, x_3^2\}$. There is a term of class 1 in H, so $d = 1$, and the maximal index j such that no pure x_j-power appears in H is $D = 2$. Moreover, we have the maximal degrees $s_1 = 1$ for x_1 and $s_2 = 1$ for x_2. Since there is a term of class 1, we must immediately add the primary ideal $\langle H,\, x_1^{s_1},\, x_2^{s_2}\rangle = \langle x_1,\, x_2,\, x_3^2\rangle$ as a first* *component to the primary decomposition. Algorithm 5 now computes the minimal Pommaret basis $H_1 = \{x_3\}$ of $\mathcal{I} : \langle x_1 \rangle^\infty$. There is no term of class 2 in H_1, so no component is added to the primary decomposition in this loop iteration. In the next iteration, simply $\langle x_3 \rangle$ is added and the algorithm terminates. Thus, we get the minimal primary decomposition $\mathcal{I} = \langle x_1, x_2, x_3^2 \rangle \cap \langle x_3 \rangle$, in which no $\langle x_2, x_3 \rangle$-primary ideal appears.*

3.3 Recursive Structures for Involutive Bases

In this section, we are concerned with recursive structures in the theory of involutive bases where the recursion will mainly be over the number of variables in the underlying polynomial ring. We start in Subsection 3.3.1 with an old result by Janet providing a recursive criterion for a set of terms to form a Janet basis (Theorem 3.3.1). We will give a simpler proof for a slightly more general version of it (Theorem 3.3.4). As a first extension, we will prove a corresponding recursive criterion for *minimal* Janet bases (Theorem 3.3.10) and use it to provide an algorithm to minimise an arbitrary Janet basis (Algorithm 9). Currently, the main algorithm for computing a minimal Janet basis is the TQ-algorithm of [48] which determines the basis from scratch. While it is in principle possible to give this algorithm a Janet basis as input, it will not benefit from this—in fact, this is even bad input. By contrast, our novel algorithm efficiently minimises any given Janet basis.

Combining our recursive criteria with Algorithm 7, we develop in Subsection 3.3.2 novel recursive algorithms for the construction of monomial and polynomial Janet and Janet-like bases (Algorithms 11 and 13). We give a recursive criterion for a set to be a (minimal) Janet-like basis (Theorems 3.3.14 and 3.3.17).

In Subsection 3.3.3 we proceed to the construction of Pommaret bases where a key issue is to find "good" coordinates, i. e. to obtain a *quasi-stable position* for the given ideal (see [60] for an extensive discussion of this topic). We provide first recursive criteria both for Pommaret bases (Theorem 3.3.32) and for quasi-stability (Corollary 3.3.36) and then a deterministic algorithm for the construction of "good" coordinates (Algorithm 16). In contrast to the approach found in [60], the method we give here incorporates variable permutations as a means of keeping the coordinate changes sparse. Minor modifications of the underlying ideas lead to recursive criterion for Noether position (Proposition 3.3.45) which also translates immediately into a corresponding deterministic algorithm.

Instead of the Buchberger algorithm in its most basic form, in this section we refer to a variant of it as presented e.g. in [10]. First, we shall need a particular form of the division algorithm based on the enumeration of the divisors. Let $f_1, \dots, f_k \in \mathcal{R}$ be an ordered sequence of nonzero polynomials and $f \in \mathcal{R}$ a further polynomial. Then quotients $h_1, \dots, h_k \in \mathcal{R}$ and a unique remainder $r \in \mathcal{R}$ exist such that:

- $f = h_1 f_1 + \cdots + h_k f_k + r$,
- No term in $h_i \operatorname{lt}(f_i)$ is divisible by any $\operatorname{lt}(f_j)$ with $j < i$,
- No term in r is a multiple of $\operatorname{lt}(f_i)$ for any i.

Buchberger's criterion is stated in this setting as follows: An (ordered) finite set

$G = \{g_1, \ldots, g_m\} \subset \mathcal{R}$ is a Gröbner basis, if and only if for each index i and each term t in the minimal generating set of the monomial ideal $\big\langle \mathrm{lt}(g_1), \ldots, \mathrm{lt}(g_{i-1})\big\rangle : \mathrm{lt}\,(g_i)$, the division of tg_i by G yields zero as remainder. Based on this result, we can now describe a variant of Buchberger's algorithm to compute Gröbner bases. In Algorithm 7, $\texttt{Division}(f, [f_1, \ldots, f_k])$ returns the remainder of the division of f by the list $[f_1, \ldots, f_k]$ by applying the above procedure. In addition, $\mathrm{Min}(\mathcal{I})$ denotes the minimal generating set of the monomial ideal $\mathcal{I}$.

Algorithm 7: Berkesch–Schreyer Variant of Buchberger Algorithm

Data: A finite list of polynomials $F = [f_1, \ldots, f_k]$ and a term ordering $\prec$
Result: A Gröbner basis G of $\langle F\rangle$

```
begin
    G ⟵ F
    P ⟵ { x^μ f_i | x^μ ∈ Min(⟨lt(f_1), ..., lt(f_{i-1})⟩ : lt(f_i)), i = 2, ..., k }
    while P ≠ ∅ do
        Select and remove a product x^μ f_i from P
        r ⟵ Division(x^μ f_i, G)
        if r ≠ 0 then
            P ⟵ P ∪ { x^μ r | x^μ ∈ Min(⟨lt(G)⟩ : lt(r)) }
            G ⟵ append(G, r)
    return G
```

One of the advantages of this formulation of Buchberger's algorithm is that one can give a simpler proof of the Schreyer theorem [10, Cor. 1.11]: Keeping the above notations, there are $h_{ij} \in \mathcal{R}$ such that $tg_i = h_{i1}g_1 + \cdots + h_{im}g_m$. Then, the set of all syzygies $t\mathbf{e}_i - h_{i1}\mathbf{e}_1 - \cdots - h_{im}\mathbf{e}_m$ for each i and for any choice of t forms a Gröbner basis for the syzygy module of $g_1, \ldots, g_m$ with respect to the induced Schreyer ordering.

3.3.1 A Recursive Janet Basis Test

Janet reported [66, page 86] the following recursive criterion for a Janet basis as a consequence of a lengthy discussion of the properties of the Janet division (see also [21, Cor. 4.11] from where we learned of this result). We will provide below a new proof for an improved variant.

Theorem 3.3.1. *Let $U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$ be a finite set of terms. We define $t'_i = t_i|_{x_n=1}$ for all i and $U' = \{t'_1, \ldots, t'_m\} \subset \mathbb{K}[x_1, \ldots, x_{n-1}]$. If*

$$\alpha = \max\{\deg_n(t_1), \ldots, \deg_n(t_m)\},$$

then we introduce for each degree $\lambda \leq \alpha$ the sets $I_\lambda = \{i \mid \deg_n(t_i) = \lambda\}$ and $U'_\lambda = \{t'_i \mid i \in I_\lambda\}$. Then, U is a Janet basis, if and only if the following two conditions are satisfied:

(i) For each $\lambda \leq \alpha$ the set U'_λ is a Janet basis in $\mathbb{K}[x_1, \ldots, x_{n-1}]$.
(ii) Each term $t'_i \in U'_\lambda$ with $\lambda < \alpha$ lies in the Janet span of $U'_{\lambda+1}$.

Example 3.3.2. *In the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$, we consider the following set of terms $U = \{x_2x_3^3,\ x_1^2x_3^3,\ x_2x_3^2,\ x_2^2x_3,\ x_1^3x_2x_3\}$. One observes that:*

1. *$U'_3 = \{x_2, x_1^2\}$, $M(x_2, U'_3) = \{x_1, x_2\}$ and $M(x_1^2, U'_3) = \{x_1\}$,*
2. *$U'_2 = \{x_2\}$ and $M(x_2, U'_2) = \{x_1, x_2\}$,*
3. *$U'_1 = \{x_2^2, x_1^3x_2\}$, $M(x_2^2, U'_1) = \{x_1, x_2\}$ and $M(x_1^3x_2, U'_1) = \{x_1\}$.*

One readily checks that all sets U'_λ are Janet bases. In addition, we can see that $x_2^2, x_1^3x_2 \in U'_1$ belong to the Janet span of U'_2 and $x_2 \in U'_2$ lies in the Janet span of U'_3. Thus, the set U is a Janet basis by Janet's theorem.

We will improve Janet's Theorem 3.3.1 by a slight modification: instead of the Janet span as in Theorem 3.3.1 (ii), we use in Theorem 3.3.4 (ii) the ordinary span which makes the condition easier to verify. For its proof, we shall need the following lemma which follows immediately from the definition of the Janet division.

Lemma 3.3.3. *In the situation of Theorem 3.3.1, for each term $t_i = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in U$ and for each variable x_j we have*
(i) if $j = n$, then x_n is Janet non-multiplicative for t, if and only if $\alpha_n < \alpha$,
(ii) if $j < n$, then x_j is Janet non-multiplicative for $t_i \in U$, if and only if it is Janet non-multiplicative for $t'_i \in U'_\lambda$ with $\lambda = \alpha_n$.

Theorem 3.3.4. *In the situation of Theorem 3.3.1, let*

$$\beta = \min\{\deg_n(t_1), \ldots, \deg_n(t_m)\}.$$

Then, U is a Janet basis, if and only if the following conditions are satisfied:
(i) For each $\lambda \leq \alpha$, U'_λ is a Janet basis in $\mathbb{K}[x_1, \ldots, x_{n-1}]$.
(ii) For each $\beta \leq \lambda < \alpha$, we have $U'_\lambda \subset \langle U'_{\lambda+1} \rangle$.

Proof. It is easy to see that any Janet basis U satisfies the given conditions: (i) holds, as multiplying a term with a non-multiplicative variable x_j with $j < n$ does not lead outside the Janet span of U and (ii) holds, as the same is true for $j = n$.

For the converse, consider a term $t_i = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in U$. Assume that x_j is Janet non-multiplicative for t_i. We distinguish two cases. If $j = n$, then $\lambda := \alpha_n < \alpha$. Since by (i) $U'_{\lambda+1}$ is a Janet basis of the ideal it generates and $t'_i \in U'_\lambda$ lies in this ideal by (ii), we may conclude that x_nt_i lies in the Janet span of the set $\{tx_n^{\lambda+1} \mid t \in U'_{\lambda+1}\} \subseteq U$. By Lemma 3.3.3, x_nt_i is thus in the Janet span of U as required for a Janet basis. If $j < n$, then, by Lemma 3.3.3, we know that x_j remains Janet non-multiplicative for $t'_i \in U'_\lambda$. Since U'_λ is a Janet basis, $x_jt'_i$ has an involutive divisor in U'_λ implying again by Lemma 3.3.3 that x_jt_i lies in the Janet span of $\{tx_n^\lambda \mid t \in U'_\lambda\}$ and thus of U. □

Example 3.3.5. *We consider again the set U of Example 3.3.2. There we showed already that all sets U'_λ are Janet bases. One can see by direct inspection without determining any multiplicative variables that we have $U'_\lambda \subset \langle U'_{\lambda+1}\rangle$ for all $1 \leq \lambda < 3$ and this shows that U is a Janet basis.*

The criterion provided by Theorem 3.3.4 translate immediately into the simple recursive Algorithm 8 testing whether a monomial set is a Janet basis.

Algorithm 8: JanetTest

Data: A polynomial ring $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_n]$ with n variables and a finite set $U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$ of terms.
Result: True if U is a Janet basis for the ideal it generates in $\mathcal{R}$ and false otherwise.

```
begin
    α ⟵ max {deg_n(t_1), ..., deg_n(t_m)}
    if n = 1 then
        β ⟵ min {deg(t_1), ..., deg(t_m)}
        if ∃β < i < α with x_n^i ∉ U then
            return (false)
        else
            return (true)
    β ⟵ min {deg_n(t_1), ..., deg_n(t_m)}
    for i = β, ..., α do
        U'_i ⟵ {t ∈ K[x_1, ..., x_{n-1}] | t · x_n^i ∈ U}
    if JanetTest(K[x_1, ..., x_{n-1}], U'_α) ≠ true then
        return (false)
    for i = α − 1, ..., β do
        if JanetTest(K[x_1, ..., x_{n-1}], U'_i) ≠ true then
            return (false)
        else if ∃t ∈ U'_i \ ⟨U'_{i+1}⟩ then
            return (false)
    return (true)
```

Theorem 3.3.6. *Algorithm 8 terminates in finitely many steps and is correct. Moreover, its arithmetic complexity is $O(dnm^2)$ where $d \geq 2$ denotes the average of the differences between the maximal and minimal degrees of the elements of U with respect to each of the variables.*

Proof. The correctness follows directly from Theorem 3.3.4 and the termination is trivial. To prove the complexity bound, we first note that using [72, Thm. 4.2], one can construct the tree representation corresponding to the exponent vectors of the elements of U by using $O(m^2 + nm)$ comparisons. Suppose that $t_i = x^{\mu_i}$

with $\mu_i = (\mu_{i1}, \ldots, \mu_{in})$. Now, assume that we are given the tree representation of $\{\mu_1, \ldots, \mu_m\}$. Without loss of generality, we may assume that the cardinality of U'_i for each i is m/d. To check one inclusion $U'_\lambda \subset \langle U'_{\lambda+1}\rangle$, we need nm/d comparisons for the membership test of an element of U'_λ and thus in all we need nm^2/d^2 operations. It follows that to test the chain of inclusions $\langle U'_\beta\rangle \subseteq \cdots \subseteq \langle U'_\alpha\rangle$ we need $O(nm^2/d)$ operations. Therefore, by taking into account the fact that $d \geq 2$, the Janet test on U may be done within $O(nm^2/d + nm^2/d^2 + \cdots + nm^2/d^n) = O(dnm^2)$, which proves the claim. □

Remark 3.3.7. *It is worth noting that the naive Janet test for $U = \{t_1, \ldots, t_m\}$ needs $O(n^2m^2)$ comparisons. Indeed, the tree representation corresponding to U is constructed within $O(m^2 + nm)$ comparisons. Using this representation, one is able to read off the non-multiplicative variables for each term $t_i \in U$. Having at most n non-multiplicative variables for each terms, one needs to perform nm operations to test whether a non-multiplicative product has a Janet divisor. Thus, all in all, we need $O(n^2m^2)$ comparisons for the Janet test of U. This shows that in the case that $n \gg d$ Algorithm 8 is more efficient than the classic approach. Note that the case $d \gg n$ is e. g. typical for toric ideals and it is well-known that in this case involutive bases are generally highly redundant, i. e. much larger than reduced Gröbner bases, and therefore should be avoided anyway.*

Example 3.3.8. *We illustrate the steps of Algorithm 3.3.4 for the set*

$$U = \{x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3,\ x_2^2x_3\} \subset \mathbb{K}[x_1, x_2, x_3]\,.$$

One obtains at the first recursion level the following sets in $\mathbb{K}[x_1, x_2]$ and Janet multiplicative variables of their elements:

1. *$U'_3 = \{x_1^3, x_1^2\}$ with $M(x_1^3, U'_3) = \{x_1, x_2\}$ and $M(x_1^2, U'_3) = \{x_1\}$,*
2. *$U'_2 = \{x_1^2x_2^2\}$ with $M(x_1^2x_2^2, U'_2) = \{x_1, x_2\}$,*
3. *$U'_1 = \{x_1^2x_2^2, x_1x_2^2, x_2^2\}$ with $M(x_1^2x_2^2, U'_1) = \{x_1, x_2\}, M(x_1x_2^2, U'_1) = \{x_1\}$ and $M(x_2^2, U'_1) = \{x_1\}$.*

These multiplicative variables show that U'_1, U'_2 and U'_3 are Janet bases. On the other hand, since we have $x_2^2 \in U'_1 \setminus \langle U'_2\rangle$, the algorithm returns correctly false, as indeed the non-multiplicative product $x_3 \cdot x_2^2x_3$ does not lie in the Janet span of U.

Since for the Janet division any monomial set is involutively autoreduced, the notion of a minimal Janet basis is crucial for efficiency reasons. We now adapt Theorem 3.3.4 to a test whether or not a given Janet basis is minimal.

Definition 3.3.9. *An $\mathcal{L}$-involutive (or a Janet-like) basis $U \subset \mathcal{P}$ is called* minimal, *if no proper subset of U is an $\mathcal{L}$-involutive (or a Janet-like) basis of the ideal $\langle U\rangle$.*

Theorem 3.3.10. *With the notations of Theorem 3.3.1, let U be a Janet basis for the ideal it generates. Then, U is minimal, if and only if the following conditions are satisfied:*

(i) For each $\lambda \leq \alpha$, U'_λ is a minimal Janet basis.
(ii) We have $\langle U'_{\alpha-1}\rangle \neq \langle U'_\alpha\rangle$.

Proof. Suppose that U is a minimal Janet basis. Then U'_λ is trivially a minimal Janet basis for each $\lambda \leq \alpha$, cf. Theorem 3.3.4. Now, assume that $\langle U'_{\alpha-1}\rangle = \langle U'_\alpha\rangle$. Since $U'_{\alpha-1}$ is a Janet basis by Theorem 3.3.4, $U \setminus \{t_i \mid t'_i \in U'_\alpha\}$ remains a Janet basis for $\langle U\rangle$, contradicting the minimality of U.

Conversely, assume that the properties (i) and (ii) hold for U, but that there exists a proper subset $V \subset U$ defining a minimal Janet basis for $\langle U\rangle$. Let x^μ be any element of $U \setminus V$. Then there exists a term $x^\nu \in V$ which involutively divides x^μ; we write $x^\mu = x^\eta x^\nu$. Assume that x_ℓ is the largest variable appearing in x^η. This implies that the two terms x^μ and x^ν lie in the same Janet class $U_{[\nu_{\ell+1},\dots,\nu_n]}$ with $\nu = (\nu_1,\dots,\nu_n)$. For each index i, write $u_i = t_i|_{x_{\ell+1}=\cdots=x_n=1}$ and define the set $W = \{u_i \mid t_i \in U_{[\nu_{\ell+1},\dots,\nu_n]}\} \subset \mathbb{K}[x_1,\dots,x_\ell]$. Applying property (i) recursively $n-\ell$ times to U, we see that W is a minimal Janet basis. Let γ be the largest x_ℓ-degree of a term $u_i \in W$. Then, similar to the notations above, we introduce the sets $W', W'_0, \dots, W'_\gamma$ and find $x_1^{\nu_1}\cdots x_{\ell-1}^{\nu_{\ell-1}} \in W'_{\nu_\ell}$ and $\nu_\ell < \mu_\ell \leq \gamma$. Furthermore, V cannot contain any element whose image under the map $\phi(u) = u|_{x_\ell=\cdots=x_n=1}$ lies in one of the sets $W'_{\nu_\ell+1},\dots,W'_\gamma$, as otherwise x_ℓ could not be multiplicative for x^ν. This shows that $W'_{\nu_\ell+1},\dots,W'_\gamma \subset \langle W'_{\nu_\ell}\rangle$. On the other hand, Theorem 3.3.4 entails that $\langle W'_{\nu_\ell}\rangle \subset \cdots \subset \langle W'_\gamma\rangle$ and in particular we get $\langle W'_{\gamma-1}\rangle = \langle W'_\gamma\rangle$, showing that property (ii) does not hold for the minimal Janet basis W, a contradiction. Hence, no proper subset of U can be a Janet basis of $\langle U\rangle$ and U is minimal. □

Theorem 3.3.10 leads immediately to the recursive Algorithm 9 for turning an arbitrary Janet basis into the minimal one. To the best of our knowledge, it represents the first such minimisation algorithm, as alternative approaches like the TQ-algorithm of [48] determine a minimal Janet basis directly from an arbitrary generating set and cannot exploit the knowledge of a non-minimal basis. It suffices that we describe the algorithm for monomial ideals, as also for a polynomial Janet basis the minimisation process depends only on the leading terms.

Theorem 3.3.11. *Algorithm 9 terminates in finitely many steps and is correct. Its arithmetic complexity is $O(dnm^2)$ with $d \geq 2$ the average difference between the maximal and minimal degrees of the elements of U with respect to each of the variables.*

Proof. The correctness follows by Theorem 3.3.10 and the termination is obvious. The complexity bound is obtained similarly to the proof of Theorem 3.3.6. □

Remark 3.3.12. *The complexity bound presented in Remark 3.3.7 remains true for a naive algorithm to compute minimal Janet bases.*

Example 3.3.13. *We demonstrate the working of Algorithm 9 for the Janet basis*

$$U = \{x_1x_2^2x_3^3,\ x_2^2x_3^3,\ x_1^3x_2x_3^3,\ x_1^2x_2x_3^3,\ x_1^3x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^2,\ x_1x_2^2x_3^2,\ x_2^2x_3^2,\ x_1^2x_2^2x_3,\ x_1x_2^2x_3,\ x_2^2x_3\}\,. \tag{3.11}$$

Algorithm 9: MinimalJanetBasis

Data: A polynomial ring $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_n]$ with n variables and a Janet basis $U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$.
Result: The minimal Janet basis of the ideal $\langle U \rangle$.
begin
  $\alpha \longleftarrow \max\{\deg_n(t_1), \ldots, \deg_n(t_m)\}$
  if $n = 1$ **then**
    $\beta \longleftarrow \min\{\deg(t_1), \ldots, \deg(t_m)\}$
    return $(\{x_1^\beta\})$
  $V \longleftarrow \emptyset$
  $\beta \longleftarrow \min\{\deg_n(t_1), \ldots, \deg_n(t_m)\}$
  for $i = \beta, \ldots, \alpha$ **do**
    $U'_i \longleftarrow \{t \in \mathbb{K}[x_1, \ldots, x_{n-1}] \mid t \cdot x_n^i \in U\}$
    $U'_i \longleftarrow$ `MinimalJanetBasis`$\big(\mathbb{K}[x_1, \ldots, x_{n-1}], U'_i\big)$
    $V \longleftarrow V \cup \{tx_n^i \mid t \in U'_i\}$
  for $i = \alpha, \ldots, \beta + 1$ **do**
    if $\langle U'_i \rangle = \langle U'_{i-1} \rangle$ **then**
      $V \longleftarrow V \setminus \{tx_n^i \mid t \in U'_i\}$
    else
      return (V)
  return (V)

Its tree representation can be seen in Figure 3.3.13.

We consider the subset $V := U_3 = \{x_1x_2^2, x_2^2, x_1^3x_2, x_1^2x_2, x_1^3, x_1^2\}$. Then, we get $W := V_2 = \{1, x_1\}$. Finally, applying Theorem 3.3.10, we have $W_0 = W_1 = \{1\}$ which shows that W is not a minimal Janet basis. Following the structure of the algorithm, in order to minimise W, we must remove the branch W_1. This shows that we shall delete $x_1x_2^2$ from V and in turn $x_1x_2^2x_3^3$ from U. In the same way and by eliminating the extra terms from U, we see that $\{x_2^2x_3^3, x_1^2x_2x_3^3, x_1^2x_3^3, x_2^2x_3^2, x_2^2x_3\}$ is the minimal Janet basis of $\langle U \rangle$. Its tree representation shown in Figure 3.5 is obviously a subtree.

Janet's criterion can be generalised to Janet-like bases. First, we introduce some notations. If $U = \{t_1, \ldots, t_m\}$ is a set of terms, then there exist natural numbers $\beta \leq \alpha$ and a sequence of natural numbers $\lambda_0, \ldots, \lambda_\ell$ with ℓ depending on U such that each λ_i is the x_n-degree of some term $t_j \in U$ and such that conversely for each $t_j \in U$ there is a λ_i which is the x_n-degree of t_j.

Theorem 3.3.14. *Let $U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$ be a set of terms and let $\beta = \lambda_0 < \lambda_1 < \cdots < \lambda_\ell = \alpha$ be natural numbers encoding the x_n-degrees appearing in U. For each index $0 \leq i \leq \ell$, let $U_{\lambda_i} \subseteq U$ be the subset of terms of U having x_n-degree λ_i and set $U'_{\lambda_i} = \{t/x_n^{\lambda_i} \mid t \in U_{\lambda_i}\}$. Then U is a Janet-like basis of the ideal it generates, if and only if the following two conditions are satisfied:*

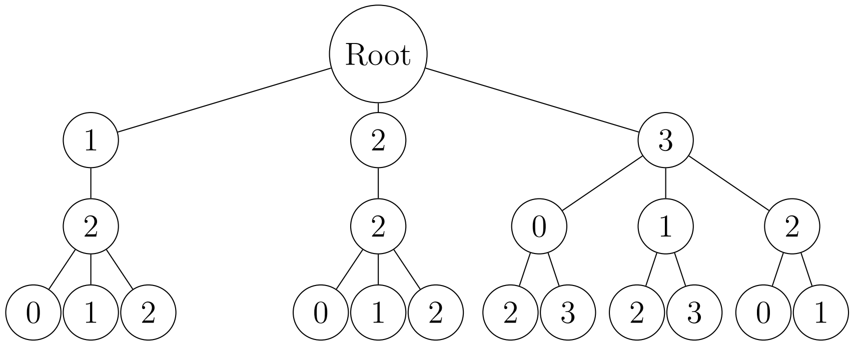

Figure 3.4: Tree representation of the Janet basis (3.11)

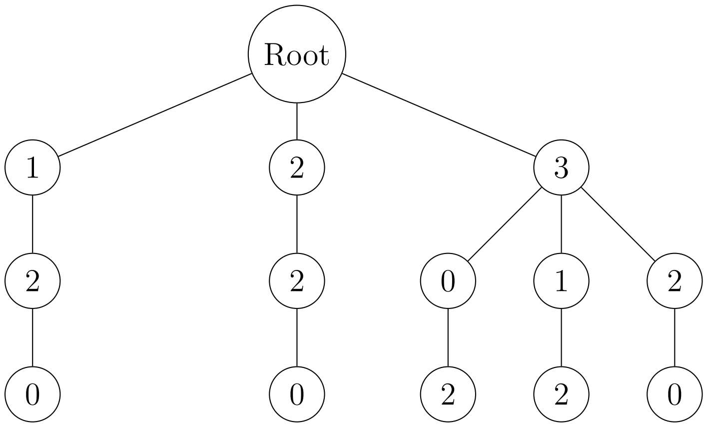

Figure 3.5: Tree representation of the minimal Janet basis

(i) Each U'_{λ_i} *is a Janet-like basis of the monomial ideal* $\langle U'_{\lambda_i}\rangle \subseteq \mathbb{K}[x_1,\dots,x_{n-1}]$.
(ii) For each $0 \le i < \ell$, *the inclusion* $U'_{\lambda_i} \subset \langle U'_{\lambda_{i+1}}\rangle$ *holds.*

Proof. The necessity of (i) follows from the observation that $x_k^{p_k}$ with $1 \le k < n$ and $p_k \ge 0$ is a Janet-like non-multiplicative power of $t \in U$, if and only if it is a Janet-like non-multiplicative power of $t' \in U'_{\deg_n(t)} \subset \mathbb{K}[x_1,\dots,x_{n-1}]$. Condition (ii) is entailed by the fact that for each $0 \le i < \ell$ the Janet-like non-multiplicative power of x_n of the term $t \in U_{\lambda_i}$ is exactly $x_n^{\lambda_{i+1}-\lambda_i}$ and that the product $x_n^{\lambda_{i+1}-\lambda_i}t$ can only be contained in the Janet-like span of U, if it lies in the Janet-like cone of a term $s \in U_{\lambda_{i+1}}$.

For the proof of the sufficiency of the two conditions (i) and (ii), a main ingredient is again the observation that $x_k^{p_k}$ with $1 \le k < n$ and $p_k \ge 0$ is a Janet-like non-multiplicative power of $t \in U$, if and only if it is a Janet-like non-multiplicative power of $t' \in U'_{\deg_n(t)} \subset \mathbb{K}[x_1,\dots,x_{n-1}]$. We must check that all products by non-multiplicative powers are contained in the Janet-like span of U. We first consider Janet-like non-multiplicative powers of the form above: $x_k^{p_k}$ with $1 \le k < n$. Let $x_k^{p_k}t$ be a product resulting from such a power. Then the x_n-degrees of t and its product are equal, say, to λ_i. Since we have in the polynomial subring with $n-1$ variables the relation $x_k^{p_k}t/x_n^{\lambda_i} \in \langle U'_{\lambda_i}\rangle$, we see by the first condition that $x_k^{p_k}t/x_n^{\lambda_i}$ is in the Janet-like span of U'_{λ_i}. But this implies easily that $x_k^{p_k}t$ is also in the Janet-like span of U in the polynomial ring with n variables.

We finally consider the Janet-like non-multiplicative powers of the form $x_n^{p_n}$. For them, there exists some index i with $0 \leq i < \ell$ such that $p_n = \lambda_{i+1} - \lambda_i$ and such that this non-multiplicative power belongs to a term $t \in U_{\lambda_i}$. By (ii), we have in the polynomial subring with $n-1$ variables the relation $x_n^{p_n}t/x_n^{\lambda_{i+1}} \in \langle U'_{\lambda_{i+1}} \rangle$. By (i), $x_n^{p_n}t/x_n^{\lambda_{i+1}}$ is in the Janet-like span of $U'_{\lambda_{i+1}}$. It is easy to see that then $x_n^{p_n}t$ is in the Janet-like span of U. $\square$

Example 3.3.15. *We consider the set $U = \{x_1^2x_3^3,\ x_1^2x_2^2x_3^3,\ x_2^4x_3^3,\ x_1^2x_3^5,\ x_2^2x_3^5\}$ in the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$. Evaluating the x_n-degrees appearing in U, we see that, in the terminology of Theorem 3.3.14, $\beta = 3 = \lambda_0 < \lambda_1 = 5 = \alpha$. We first check that the sets U'_{λ_i} are Janet-like complete:*

1. *$U'_5 = \{x_1^2, x_2^2\}$. Only one non-multiplicative power exists: $\mathrm{NMP}(x_1^2, U'_5) = \{x_2^2\}$. The product $x_1^2x_2^2$ is in the Janet-like cone of x_2^2 so that U'_5 is Janet-like complete.*

2. *$U'_3 = \{x_1^2, x_1^2x_2^2, x_2^4\}$. The term x_2^4 does not have non-multiplicative powers. Furthermore, $\mathrm{NMP}(x_1^2, U'_3) = \{x_2^2\}$. The corresponding product is already contained in U'_3. Finally, $\mathrm{NMP}(x_1^2x_2^2, U'_3) = x_2^2$ and the corresponding product is in the Janet-like cone of x_2^4. Hence, U'_3 is a Janet-like basis.*

In addition, we have $U'_3 = \{x_1^2, x_1^2x_2^2, x_2^4\} \subset \langle U'_5 \rangle = \langle x_1^2, x_2^2 \rangle$ and thus the given set U is a Janet-like basis by Theorem 3.3.14.

Example 3.3.16. *Consider the set $V = \{x_2^2x_3^5,\ x_2^4x_3^3,\ x_1^2x_3^3,\ x_1^2x_2^2x_3^3\}$ obtained by removing the term $x_1^2x_3^5$ from the set U of Example 3.3.15. We still find $\beta = \lambda_0 = 3 < 5 = \lambda_1 = \alpha$. The singleton set $V'_5 = \{x_2^2\}$ is obviously Janet-like complete; the set V'_3 equals the set U'_3 of Example 3.3.15 and thus is complete. However, we have $V'_3 \not\subseteq \langle V'_5 \rangle$, since $x_1^2 \notin \langle x_2^2 \rangle$. Thus V is not Janet-like complete by Theorem 3.3.14. Moreover, observe that $\mathrm{NMP}(x_1^2x_3^3, V) = \{x_3^2\}$ and that the corresponding product is the eliminated term $x_1^2x_3^5$, which is not contained in the Janet-like span of V. Thus, one can compute the Janet-like completion U of V by adding this product to V.*

Theorem 3.3.17. *Keeping the notations of Theorem 3.3.14, let U be a Janet-like basis for the ideal it generates. Then, U is minimal, if and only if the following conditions are satisfied:*

(i) For each $i \leq \ell$, U'_{λ_i} is a minimal Janet-like basis.
(ii) For each $i < \ell$, we have $\langle U'_{\lambda_i} \rangle \neq \langle U'_{\lambda_{i+1}} \rangle$.

Proof. The proof is similar to the one of Theorem 3.3.10. If U is a minimal Janet-like basis, then it is clear that for each $i \leq \ell$, U'_{λ_i} is a minimal Janet-like basis, see Theorem 3.3.14. To prove (ii), assume that $\langle U'_{\lambda_i} \rangle = \langle U'_{\lambda_{i+1}} \rangle$ for some $i < \ell$. Then, $U \setminus \{t'_i x_n^{\lambda_{i+1}} \mid t'_i \in U'_{\lambda_{i+1}}\}$ is a Janet-like basis as well, contradicting the minimality of U.

Conversely, assume that the properties (i) and (ii) hold for U, but that there exists a proper subset $V \subset U$ forming a minimal Janet-like basis for $\langle U \rangle$. Let x^μ be any element of $U \setminus V$. There must exist $x^\nu \in V$ which divides x^μ for the Janet-like division; we write $x^\mu = x^\eta x^\nu$. Assume that x_ℓ is the largest variable appearing in x^η. This implies that x^μ and x^ν lie in the same Janet class $U_{[\nu_{\ell+1},\dots,\nu_n]}$ with $\nu = (\nu_1, \dots, \nu_n)$. For each index i let $u_i = t_i|_{x_{\ell+1}=\cdots=x_n=1}$ and set $W = \{u_i \mid t_i \in U_{[\nu_{\ell+1},\dots,\nu_n]}\} \subset \mathbb{K}[x_1, \dots, x_\ell]$. Applying recursively property (i) $n - \ell$ times to U, we see that W is a minimal Janet-like basis. Let δ be the largest x_ℓ-degree of a term $u_i \in W$. Then, similar to the notations above, we can introduce the sets $W', W'_{\gamma_0}, \dots, W'_{\gamma_t}$ with $\gamma_t = \delta$. Thus $x_1^{\nu_1} \cdots x_{\ell-1}^{\nu_{\ell-1}} \in W'_{\nu_\ell}$ and $\nu_l < \delta$. We set $u = x_1^{\nu_1} \cdots x_{\ell-1}^{\nu_{\ell-1}}$ and $v = x_1^{\mu_1} \cdots x_{\ell-1}^{\mu_{\ell-1}}$ with $\mu = (\mu_1, \dots, \mu_n)$ and assume that $\gamma_i = \nu_\ell$. Then, two cases may occur. If $v \in W'_{\gamma_{i+1}}$, then we can remove the terms in U whose images lie in $W'_{\gamma_{i+1}}$ and this shows that $W'_{\gamma_i} = W'_{\gamma_{i+1}}$ which contradicts property (ii). Otherwise, v belongs to W'_{γ_j} with $j > i + 1$. By Theorem 3.3.14, we have $u \in \langle W'_{\gamma_j} \rangle$ and thus W'_{γ_j} is not minimal contradicting property (i). Therefore, no proper subset of U can be a Janet-like basis of $\langle U \rangle$ and U is minimal. $\square$

Example 3.3.18. *We consider the Janet-like basis U given in Example 3.3.15 and verify if it is a minimal Janet-like basis. The tree representation of U is shown in Figure 3.6. We observe that $V := U'_3 = \{x_1^2, x_1^2x_2^2, x_2^4\}$ and check whether it is minimal or not. We know that $V'_0 = \{x_1^2\}$, $V'_2 = \{x_1^2\}$ and $V'_4 = \{1\}$. Since $\langle V'_0 \rangle = \langle V'_2 \rangle$, U is not a minimal Janet-like basis. It follows that we get the minimal Janet-like basis, if we remove the useless branch $x_1^2x_2^2x_3^3$.*

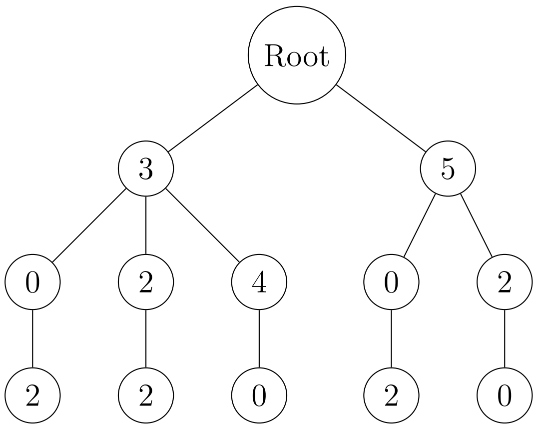

Figure 3.6: Tree representation of U

3.3.2 Janet Completion Procedure

We now show how our results from the previous section allow us to design a variant of the Berkesch–Schreyer algorithm for Gröbner bases [10] which can compute Janet(-like) bases. We take the lexicographic term ordering induced by $x_1 \prec_{lex} \cdots \prec_{lex} x_n$. Let $U := (t_1, \dots, t_m) \in \mathcal{T}^m$ be a sequence of terms such that we have $t_m \prec_{lex} \cdots \prec_{lex}$

t_1. We associate to U the following $(m-1)$-tuple $Q(U)$ of terms:

$$Q(U) := \big(t_{m-1} : t_m,\, \ldots\, , t_1 : t_m\big)\ .$$

Here $u : v = \operatorname{lcm}(u, v)/v$ for any two terms u and v. In addition, using the tree representation of U, one can compute $Q(U)$ efficiently. However, since such questions are not the main subject of this work, we do not give further details. The tuple $Q(U)$ is related to the Janet non-multiplicative variables of the term t_m. Assume that $Q(U) = (u_1, \ldots, u_{m-1})$. We know that there exist a positive integer r, indices $1 \leq a_1 < a_2 < \cdots < a_r \leq n$ and indices $1 = b_1 < b_2 < \cdots < b_r < b_{r+1} = m-1$ such that the highest variable dividing u_j is x_{a_ℓ} where $\ell \in \{1, \ldots, r\}$ and $b_\ell \leq j < b_{\ell+1}$. Moreover, for indices $b_\ell \leq j_1 < j_2 < b_{\ell+1}$, we have $\deg_{a_\ell}(u_{j_1}) \leq \deg_{a_\ell}(u_{j_2})$. With these notations, we obtain the following assertion.

Lemma 3.3.19. $\mathrm{NM}_{\mathcal{J}}\big(t_m, \{t_1, \ldots, t_m\}\big) = \big\{x_{a_1},\, \ldots,\, x_{a_r}\big\}$.

Proof. We first show that each x_{a_i} is Janet non-multiplicative for t_m. Since x_{a_i} is the highest variable appearing in the quotients $u_{b_i}, \ldots, u_{b_{i+1}}$, we have $t_m \in U_{[d_{a_i+1},\ldots,d_n]}$ where $d_j = \deg_j(t_m)$ and $\deg_{a_i}(t_m)$ is not maximal among the x_{a_i}-degree of the elements of this set. Thus, using the fact that the sequence of $t_1, \ldots, t_m$ is sorted in lexicographical order, x_{a_i} is Janet non-multiplicative for t_m. Conversely, assume that x_ℓ is Janet non-multiplicative for t_m. Then we have $t_m \in U_{[d_{\ell+1},\ldots,d_n]}$. We know that this set is non-empty and we can choose an element $t \neq t_m$ from this set. Hence, x_ℓ is the highest variable appearing in the quotient $t : t_m$ and this ends the proof. $\square$

This result induces a partition of $Q(U)$ into subsets $Q_{x_{a_\ell}}$ consisting for each Janet non-multiplicative variable x_{a_ℓ} of t_m of exactly those quotient terms u_j with x_{a_ℓ} as highest dividing variable.

Example 3.3.20. *In the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$, consider the set of terms $U = \big(x_2x_3^3,\, x_1^2x_3^3,\, x_2x_3^2,\, x_2^2x_3,\, x_1^3x_2x_3\big)$ which form a Janet basis of the ideal generated by them. We obtain for the quotients $Q(u) = \big(x_2,\, x_3,\, x_3^2,\, x_3^2\big)$. Now one sees that $r = 2$ and $NM_{\mathcal{J}}\big(x_1^3x_2x_3,\, U\big) = \{x_2, x_3\}$. In addition, $Q_{x_2} = \{x_2\}$ and $Q_{x_3} = \big\{x_3, x_3^2,\, x_3^2\big\}$.*

Lemma 3.3.21. *Keeping the above notations, U is a Janet basis, if and only if the set $\{t_1, \ldots, t_i\}$ is a Janet basis for each index $1 \leq i \leq m$.*

Proof. If $\{t_1, \ldots, t_i\}$ is a Janet basis for each index i, then this is in particular true for $U = \{t_1, \ldots, t_m\}$. Conversely, if $U = \{t_1, \ldots, t_m\}$ is a Janet basis, then consider for some index $1 < j < m$ the set $U_j := \{t_1, \ldots, t_j\}$. Let k be any index with $1 \leq k \leq j$ and let x_i be any Janet non-multiplicative variable for $t_k \in U_j$ with respect to the Janet division. The definition of Janet multiplicative variables implies that x_i is also not Janet multiplicative for $t_k \in U$. Since U is a Janet basis, there is some index $1 \leq \ell \leq m$ such that x_it_k is in the Janet cone of t_ℓ with respect to the set U. The index ℓ cannot be greater than j. Indeed, arguing by *reductio ad absurdum*, assume

$\ell > j$. Since by assumption $t_k \succ_{\text{lex}} t_\ell$, there is a variable x_a such that the x_a-power of t_ℓ is less than that of t_k and in turn x_a is Janet non-multiplicative for $t_\ell \in U$. This contradicts the fact that $x_i t_k$ lies in the Janet cone of t_ℓ. Thus we have shown that $x_i t_k$ lies in the Janet cone with respect to U of an element of U_j. By the filter axiom of involutive divisions, $x_i t_k$ must then also lie in the Janet cone of the same element with respect to U_j. This proves that U_j is a Janet basis and this finishes the proof. □

Theorem 3.3.22. *With the above notations, U is a Janet basis, if and only if for each $i > 1$ the following condition holds. If we write $Q(t_1, \ldots, t_i) = (u_1, \ldots, u_{i-1})$ and partition*

$$\{u_1, \ldots, u_{m-1}\} = \bigsqcup_{x \in \mathrm{NM}_{\mathcal{J}}(t_i, \{t_1, \ldots, t_i\})} Q_x \ ,$$

then there exists for each non-multiplicative variable $x_\ell \in \mathrm{NM}_{\mathcal{J}}\big(t_i, \{t_1, \ldots, t_i\}\big)$ a term $u \in Q_{x_\ell}$ such that $\deg_\ell(u) = 1$. Moreover, if in this situation $x_\ell \neq x_1$, then in the ring $\mathbb{K}[x_1, \ldots, x_{n-1}]$ we have the relation

$$t_i|_{x_\ell = \cdots = x_n = 1} \in \langle t_j|_{x_\ell = \cdots = x_n = 1} \mid t_j : t_i \in Q_{x_\ell},\ j < i,\ \deg_\ell(t_j : t_i) = 1 \rangle \ .$$

Proof. Let $U = \{t_1, \ldots, t_m\}$ be a Janet basis. By Lemma 3.3.21, $U_k = \{t_1, \ldots, t_k\}$ is also a Janet basis for all $1 \leq k \leq m-1$. Now, let $i \in \{2, \ldots, m\}$ be an arbitrary index; we need to show that $Q(t_1, \ldots, t_i)$ satisfies the conditions stated. Let x_ℓ be the highest variable which is Janet non-multiplicative for $t_i \in U_i$. Write $t_i = x^\nu$ with $\nu = (\nu_1, \ldots, \nu_n)$. It is easy to see that the Janet class $C = U_{[\nu_{\ell+1}, \ldots, \nu_n]}$ is itself a Janet basis of the ideal it generates. Applying Theorem 3.3.4 to C, we see that the subset $V = \big\{t'_j = t_j|_{x_{\ell+1} = \cdots = x_n = 1} \mid t_j \in C\big\}$ of the polynomial ring $\mathbb{K}[x_1, \ldots, x_\ell]$ is also a Janet basis. In addition, we can partition the set V into *non-empty* subsets V_λ where $\beta \leq \lambda \leq \alpha$, $\beta = \deg_\ell(t'_i)$ and $\alpha = \deg_\ell(t'_1)$. By Theorem 3.3.4, we know that $\langle V_\beta \rangle \subset \langle V_{\beta+1} \rangle$. This implies, for the variable x_ℓ, simultaneously the degree condition on the elements of $Q(t_1, \ldots, t_i)$ having highest variable x_ℓ and the containment of $t'_i|_{x_\ell = 1}$ in the ideal $\langle t'_j|_{x_\ell = 1} \mid t'_j \in V_{\beta+1} \rangle$.

We have thus verified the conditions for the highest Janet non-multiplicative variable. Now we apply again Theorem 3.3.4 to obtain the Janet basis $\overline{U} = \big\{t'_j|_{x_\ell = 1} \mid t'_j \in V_\beta\big\}$. By construction, the highest Janet non-multiplicative variable of $t'_i \in \overline{U}$ is equal to the second highest Janet non-multiplicative variable of $t_i \in U_i$ and exactly those terms $t_j \in U_i$ which yield quotients u_j in $Q(t_1, \ldots, t_i)$ with highest variable lower than x_ℓ contribute terms to $\overline{U}$ via the projection $t_j \mapsto t_j|_{x_\ell = \cdots = x_n = 1}$. Proceeding as in the case of x_ℓ and then iteratively going through all Janet non-multiplicative variables of $t_i \in U_i$, we arrive at our claim.

Let us now assume that for each $1 \leq i \leq m$ the conditions on the quotient list $Q(t_1, \ldots, t_i) = (u_1, \ldots, u_{i-1})$ are satisfied. We want to show that U is a Janet basis. By Lemma 3.3.21, it suffices to show that each set $U_i = \{t_1, \ldots, t_i\}$ is a Janet basis. Since it is clear that $\{t_1\}$ is a Janet basis, we may proceed by induction on i and assume that U_{i-1} is a Janet basis. We then need to show that $U_i = U_{i-1} \cup \{t_i\}$ is a Janet basis. For this, we verify the two conditions of Theorem 3.3.4 for U_i. We partition U_i into the sets $U_\beta, \ldots, U_\alpha$ according to the x_n-degrees of the elements of U_i with $\beta \leq \alpha$ for $\beta = \deg_n(t_i)$, and $\alpha = \deg_n(t_1)$. Firstly, we verify the inclusions $U'_\lambda \subset \langle U'_{\lambda+1}\rangle$ for any $\beta \leq \lambda < \alpha$. If $\beta = \alpha$, there is nothing to do. Otherwise, x_n is a Janet non-multiplicative variable for $t_i \in U_i$ and hence x_n appears as the highest variable of some quotient term u_k in the list $Q(t_1, \ldots, t_i)$. Furthermore, in at least one u_k it must appear with degree one. Moreover, the containment of the projection $t_i|_{x_n=1}$ in the $\mathbb{K}[x_1, \ldots, x_{n-1}]$-ideal defined by the projections of all t_j such that the x_n-degree of the quotient term $t_j : t_i$ is one must hold. All these terms t_j come from $U'_{\beta+1}$. This implies $t_i|_{x_n=1} \in \langle U'_{\beta+1}\rangle$. On the other hand, U_{i-1} is a Janet basis and by applying Theorem 3.3.4 on this set, it is clear that $U'_\beta \setminus \{t_i|_{x_n=1}\} \subset \langle U'_{\beta+1}\rangle \cdots \subset \langle U'_\alpha\rangle$. All these observations together imply that U'_λ is not empty for any λ, that $U'_\beta \subset \langle U'_{\beta+1}\rangle$ and that the inclusion conditions on the monomial ideals $\langle U'_\lambda\rangle$ are fulfilled by U_i.

We still have to show that each set U'_λ is a Janet basis. The sets U'_λ with $\lambda > \beta$ are Janet bases, as U_{i-1} is a Janet basis and thus fulfills the conditions of Theorem 3.3.4. If we have $U'_\beta = \{t_i|_{x_n=1}\}$, we are done. Otherwise, there exists some index $1 < a < i$ such that $U'_\beta = \{t_a|_{x_n=1}, \ldots, t_i|_{x_n=1}\}$. Removing the last element of this set, we obtain the set U'_β of the Janet basis U_{i-1}, which is again a Janet basis. And the quotient terms of the elements of this set by $t_i|_{x_n=1}$ inherit for the variables $x_1, \ldots, x_{n-1}$ all properties which hold for the original quotient terms u_k with respect to these variables. By an induction on the number of variables in the ambient polynomial ring, we are done (the case of a polynomial ring with one variable being trivial). Thus we have shown that the individual sets U'_λ are Janet bases and verified the conditions of Theorem 3.3.4 for the set U_i. This finishes the proof. □

Example 3.3.23. *Let $U = \big(x_1^3x_3^3,\, x_1^2x_3^3,\, x_1^2x_2^2x_3^2,\, x_1^2x_2^2x_3,\, x_1x_2^2x_3,\, x_2^2x_3\big)$ be a sequence of terms in $\mathcal{R} := \mathbb{K}[x_1, x_2, x_3]$. In the following, we show how we can apply the above result to compute a Janet basis for U.*

1. *Since $Q\big(x_1^3x_3^3,\, x_1^2x_3^3\big) = (x_1)$, the two first elements form a Janet basis.*

2. *We have $Q\big(x_1^3x_3^3,\, x_1^2x_3^3,\, x_1^2x_2^2x_3^2\big) = (x_3,\, x_1x_3)$. The x_3-degrees of x_3, x_1x_3 are both one and so we check only whether $x_1^2x_2^2 \in \langle x_1^3, x_1^2\rangle$. Since this is the case, the sequence of the first three elements forms a Janet basis.*

3. *Let us now consider $Q\big(x_1^3x_3^3,\, x_1^2x_3^3,\, x_1^2x_2^2x_3^2,\, x_1^2x_2^2x_3\big) = \big(x_3,\, x_3^2,\, x_1x_3^2\big)$. Since all elements in this quotient tuple contain x_3 as the highest variable, we shall consider only the first quotient which is linear. So, we check $x_1^2x_2^2 \in \langle x_1^2x_2^2\rangle$ which is true. Thus, the sequence the first four elements forms a Janet basis.*

4. *As next step we consider*
$$Q\big(x_1^3x_3^3,\, x_1^2x_3^3,\, x_1^2x_2^2x_3^2,\, x_1^2x_2^2x_3,\, x_1x_2^2x_3\big) = \big(x_1,\, x_1x_3,\, x_1x_3^2,\, x_1^2x_3^2\big)\,.$$
We ignore the first quotient and check whether $x_1x_2^2 \in \langle x_1^2x_2^2\rangle$. As this does not hold, we add $x_1x_2^2x_3^2$ to U and obtain
$$U_1 = \big(x_1^3x_3^3,\, x_1^2x_3^3,\, x_1^2x_2^2x_3^2,\, x_1x_2^2x_3^2,\, x_1^2x_2^2x_3,\, x_1x_2^2x_3, x_2^2x_3\big)\,.$$

5. *We now consider* $Q\big(x_1^3x_3^3,\, x_1^2x_3^3,\, x_1^2x_2^2x_3^2,\, x_1x_2^2x_3^2\big) = \big(x_1,\, x_1x_3,\, x_1^2x_3\big)$*. Since the* x_1*-degree of the first element is one, we ignore it. Hence, we must check whether* $x_1x_2^2 \in \langle x_1^2\rangle$*. As it does not hold, we add* $x_1x_2^2x_3^3$ *to* U_1 *and arrive at*
$$U_2 = \big(x_1x_2^2x_3^3,\, x_1^3x_3^3,\, x_1^2x_3^3,\, x_1^2x_2^2x_3^2,\, x_1x_2^2x_3^2,\, x_1^2x_2^2x_3,\, x_1x_2^2x_3,\, x_2^2x_3\big)\ .$$

6. *We next consider* $Q\big(x_1x_2^2x_3^3,\, x_1^3x_3^3\big) = \big(x_2^2\big)$*. Since the quotient is not linear, we must add* $x_1^3x_2x_3^3$ *to* U_2 *obtaining*
$$U_3 = \big(x_1x_2^2x_3^3,\, x_1^3x_2x_3^3,\, x_1^3x_3^3,\, x_1^2x_3^3,\, x_1^2x_2^2x_3^2,\, x_1x_2^2x_3^2,\, x_1^2x_2^2x_3,\, x_1x_2^2x_3,\, x_2^2x_3\big)\ .$$

7. *Next,* $Q\big(x_1x_2^2x_3^3,\, x_1^3x_2x_3^3\big) = (x_2)$ *and* $Q\big(x_1x_2^2x_3^3,\, x_1^3x_2x_3^3,\, x_1^3x_3^3\big) = \big(x_2,\, x_2^2\big)$*. Since* $x_1^3 \in \langle x_1^3\rangle$*, the first three terms of* U_3 *form a Janet basis.*

8. *Next,* $Q\big(x_1x_2^2x_3^3,\, x_1^3x_2x_3^3,\, x_1^3x_3^3,\, x_1^2x_3^3\big) = \big(x_1,\, x_1x_2,\, x_2^2\big)$*. Since* $x_1^2 \notin \langle x_1^3x_2\rangle$*, we add* $x_1^2x_2x_3^3$ *to* U_3 *obtaining*
$$U_4 = \big(x_1x_2^2x_3^3, x_1^3x_2x_3^3, x_1^2x_2x_3^3, x_1^3x_3^3, x_1^2x_3^3, x_1^2x_2^2x_3^2, x_1x_2^2x_3^2, x_1^2x_2^2x_3, x_1x_2^2x_3, x_2^2x_3\big).$$

9. *We find that* $Q(U_4) = \big(x_1,\, x_1^2,\, x_1x_3,\, x_1^2x_3,\, x_1^2x_3^2,\, x_1^3x_3^2,\, x_1^2x_3^2,\, x_1^3x_3^2,\, x_1x_3^2\big)$*. We ignore the first two quotients and check whether* $x_2^2 \in \big\langle x_1^2x_2^2,\, x_1x_2^2\big\rangle$*. As this does not hold, we add* $x_2^2x_3^2$ *to* U_4 *obtaining*
$$U_5 = \big(x_1x_2^2x_3^3,\, x_1^3x_2x_3^3,\, x_1^2x_2x_3^3,\, x_1^3x_3^3,\, x_1^2x_3^3,\\ x_1^2x_2^2x_3^2,\, x_1x_2^2x_3^2,\, x_2^2x_3^2,\, x_1^2x_2^2x_3,\, x_1x_2^2x_3,\, x_2^2x_3\big)\ .$$

10. *We consider next*
$$Q\big(x_1x_2^2x_3^3,\, x_1^3x_2x_3^3,\, x_1^2x_2x_3^3,\, x_1^3x_3^3,\, x_1^2x_3^3,\, x_1^2x_2^2x_3^2,\, x_1x_2^2x_3^2,\, x_2^2x_3^2\big) =\\ \big(x_1,\, x_1^2,\, x_1^2x_3,\, x_1^3x_3,\, x_1^2x_3,\, x_1^3x_3,\, x_1x_3\big)\ .$$
Since $x_2^2 \notin \big\langle x_1x_2^2,\, x_1^3x_2,\, x_2x_1^2,\, x_1^3, x_1^2\big\rangle$*, we add* $x_2^2x_3^3$ *to* U_5 *finally reaching the set*
$$U_6 = \big(x_1x_2^2x_3^3,\, x_2^2x_3^3,\, x_1^3x_2x_3^3,\, x_1^2x_2x_3^3,\, x_1^3x_3^3,\, x_1^2x_3^3,\\ x_1^2x_2^2x_3^2,\, x_1x_2^2x_3^2,\, x_2^2x_3^2,\, x_1^2x_2^2x_3,\, x_1x_2^2x_3,\; x_2^2x_3\big)$$
which satisfies all the condition of the above theorem and thus is a Janet basis of the ideal generated by U*.*

Theorem 3.3.22 translates straightforwardly into Algorithm 10 which checks whether a given monomial set is a Janet basis of the ideal generated by it. If the output is `false`, then the algorithm returns in addition an element which should be added. The correctness and the termination of the algorithm is obvious.

The strategy applied in Example 3.3.23 for completing a monomial set to a Janet basis can then be easily extended to the general monomial completion Algorithm 11.

Algorithm 10: JanetTest

Data: A finite set $U \subset \mathcal{T}$ of terms.
Result: True if U is a Janet basis for the ideal it generates and false otherwise.

```
begin
    flag ⟵ false
    while flag = false do
        flag ⟵ true;   (t_1, ..., t_m) ⟵ sort(U, ≺)
        for i from 2 to m do
            (u_1, ..., u_{i-1}) ⟵ Q(t_1, ..., t_i)
            if ∃j s.t. the highest variable x_ℓ in u_j is not linear then
                return (false, x_ℓ t_i)
            else if the highest variable in the quotients t_1 : t_i, ..., t_{is} : t_i is
             x_ℓ ≠ x_1 then
                if t_i|_{x_ℓ=···=x_n=1} ∉ ⟨t_{i_1}|_{x_ℓ=···=x_n=1}, ..., t_{i_s}|_{x_ℓ=···=x_n=1}⟩ then
                    return (false, x_ℓ t_i)
    return (true)
```

Theorem 3.3.24. *Algorithm 11 terminates in finitely many steps and is correct.*

Proof. The termination of this algorithm is a consequence of the fact that Janet division is Noetherian, see [47, Prop. 4.5]. Its correctness is a corollary to Theorem 3.3.22 and the constructivity and continuity of the Janet division, see [47]. □

We now proceed to the determination of a Janet basis for a given set of polynomials. Let $U = (t_1, \ldots, t_m)$ be a sequence of terms and $Q(U) = (u_1, \ldots, u_{m-1})$. By Lemma 3.3.19, we know that the highest variables in the u_i's are the Janet non-multiplicative variables for u_m as an element of U. Based on this observation and using the Janet polynomial completion algorithm (see e.g. [46, Sec. 4] or [95, Alg. 3]), we can describe a variant of the Berkesch–Schreyer algorithm which computes a Janet basis for a polynomial ideal.

Again we begin with the auxiliary Algorithm 12 determining in the described manner the Janet non-multiplicative variables of the last polynomial in an ordered finite set. Its correctness is an immediate consequence of Lemma 3.3.19.

Based on this algorithm, we obtain the polynomial completion Algorithm 13. In it, we denote for any ordered set X by $X[i..j]$ the ordered subset containing all elements from the i-th one to the j-th one.

Theorem 3.3.25. *Algorithm 13 terminates in finitely many steps and is correct.*

Proof. Since the structure of the algorithm is essentially that of [95, Alg. 3], its termination and correctness follow by [95, Thm. 7.4]. □

Algorithm 11: JanetMonomialCompletion

Data: A finite set $U \subset \mathcal{T}$ of terms.
Result: A Janet basis of $\langle U \rangle$.
begin
  $T \longleftarrow U$; $A \longleftarrow$ `JanetTest`(T)
  while $A = (false, t)$ **do**
    $T \longleftarrow T \cup \{t\}$
    $A \longleftarrow$ `JanetTest`(T)
  return (T)

Algorithm 12: JanetNonMultVar

Data: An ordered finite set $F \subset \mathcal{R}$ of polynomials and a term ordering $\prec$.
Result: Set of Janet non-multiplicative variables of last polynomial in F.
begin
  $(t_1, \ldots, t_m) \longleftarrow \operatorname{lt}(F)$; $(u_1, \ldots, u_{m-1}) \longleftarrow Q(t_1, \ldots, t_m)$
  return *(set of highest variables for* $\prec$ *appearing in the* u_i*'s)*

Remark 3.3.26. *If H is already a sorted list of polynomials, then one can use an efficient insertion sort algorithm for sorting $H \cup \{g\}$. For the special case of lists of terms, we refer to [71].*

We conclude this section by providing a similar approach for the construction of Janet-like bases. Let again $U := (t_1, \ldots, t_m) \in \mathcal{T}^m$ be a sequence of terms such that $t_1 \succ_{\text{lex}} \cdots \succ_{\text{lex}} t_m$ and $Q(U) = (u_1, \ldots, u_{m-1})$. Then, there exist a positive integer r, indices $1 \leq a_1 < a_2 < \cdots < a_r \leq n$ and indices $1 = b_1 < b_2 < \cdots < b_r < b_{r+1} = m-1$ such that the highest variable dividing u_j is x_{a_ℓ} for all $\ell \in \{1, \ldots, r\}$ and for all indices j with $b_\ell \leq j < b_{\ell+1}$. Furthermore, we denote by d_{a_ℓ} the x_{a_ℓ}-degree of u_{b_ℓ}. Keeping these notations, we obtain the next result analogous to Lemma 3.3.19.

Lemma 3.3.27. $\operatorname{NMP}\big(t_m, \{t_1, \ldots, t_m\}\big) = \Big\{x_{a_1}^{d_{a_1}}, \ldots, x_{a_r}^{d_{a_r}}\Big\}$.

Proof. From the proof of Lemma 3.3.19, we know that $\{x_{a_1}, \ldots, x_{a_r}\}$ is the set of all Janet non-multiplicative variables. On the other hand, from the underlying term ordering, we have that $\deg_{a_\ell}(u_{j_1}) \leq \deg_{a_\ell}(u_{j_2})$ for all indices $b_\ell \leq j_1 < j_2 < b_{\ell+1}$. It follows that $\deg_{a_\ell}(u_{b_\ell})$ has the minimal x_{a_ℓ}-degree among all elements $u_{b_\ell}, \ldots, u_{b_{\ell+1}-1}$. These observations imply the desired assertion. □

This lemma induces a partition of the set $\{u_1, \ldots, u_{m-1}\}$ into subsets $Q_{x_{a_\ell}}$ consisting for each Janet non-multiplicative variable x_{a_ℓ} of t_m exactly of those quotient terms u_j with x_{a_ℓ} as highest dividing variable.

Example 3.3.28. *The sequence $U = \big(x_2^2x_3^5, x_1^2x_3^5, x_2^4x_3^3, x_1^2x_2^2x_3^3, x_1^2x_3^3\big)$ forms, by Example 3.3.15, a Janet-like basis in the ring $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$. Here $Q(U) =$*

Algorithm 13: JanetPolynomialCompletion

```
Data: A finite set F ⊂ R of polynomials and a term ordering ≺.
Result: A Janet basis of ⟨F⟩.
begin
    H ⟵ sort(F, ≺) from the highest leading term to the lowest one
    while true do
        flag ⟵ false
        for i from 2 to |H|   while flag = false do
            A ⟵ JanetNonMultVar(H[1..i], ≺)
            foreach a ∈ A do
                g ⟵ an involutive normal form of a · H[i] with respect to H
                if g ≠ 0 then
                    H ⟵ sort(H ∪ {g}, ≺);   flag ⟵ true
        if flag = false then
            return (H)
```

$\left(x_2^2,\, x_2^4,\, x_3^2,\, x_2^2x_3^2\right)$ *and thus* $\mathrm{NMP}\left(x_1^2x_3^3,\, U\right) = \left\{x_2^2,\, x_3^2\right\}$. *Furthermore, we find the subsets* $Q_{x_2} = \left\{x_2^2,\, x_2^4\right\}$ *and* $Q_{x_3} = \left\{x_3^2,\, x_2^2x_3^2\right\}$.

Lemma 3.3.29. *With the above notations, U is a Janet-like basis, if and only if the subsets $\{t_1, \ldots, t_i\}$ are Janet-like bases for each index $1 \leq i \leq m$.*

The proof of this lemma is analogous to the one of Lemma 3.3.21 and thus omitted. Finally, we adapt Theorem 3.3.22 to the Janet-like division. Taking Theorem 3.3.14 into account, its proof is similar to to the one of Theorem 3.3.22 and hence also not detailed.

Theorem 3.3.30. *With the above notations, U is a Janet-like basis, if and only if for each $i > 1$ the following condition holds. If we write $Q(t_1, \ldots, t_i) = (u_1, \ldots, u_{i-1})$ and partition*

$$\left\{u_1,\, \ldots,\, u_{m-1}\right\} = \bigsqcup_{x \in \mathrm{NM}_{\mathcal{J}}(t_i, \{t_1, \ldots, t_i\})} Q_x\ ,$$

then there exists for each non-multiplicative power $x_\ell^{d_\ell} \in \mathrm{NMP}\left(t_i,\, \{t_1, \ldots, t_i\}\right)$ a term $u \in Q_{x_\ell}$ with minimal x_ℓ-degree d_ℓ. Moreover, if in this situation $x_\ell \neq x_1$, then in the ring $\mathbb{K}[x_1, \ldots, x_{n-1}]$ we have the relation

$$t_i|_{x_\ell=\cdots=x_n=1} \in \langle t_j|_{x_\ell=\cdots=x_n=1} \mid t_j : t_i \in Q_{x_\ell},\, j < i,\, \deg_\ell\left(t_j : t_i\right) = d_\ell\rangle\ .$$

Remark 3.3.31. *Based on these results, it is straightforward to provide also algorithms for computing Janet-like bases for both monomial and polynomial ideals by adapting Algorithms 11 and 13. We omit the obvious details.*

3.3.3 A Recursive Pommaret Basis Construction

So far, we have concentrated on Janet bases. We now provide a criterion similar to Theorem 3.3.4 for a finite set to be a Pommaret basis. As the existence of a finite Pommaret basis is equivalent to the ideal being quasi-stable, it is not surprising that the criterion can be extended to a recursive test of quasi-stability.

Theorem 3.3.32. *Let $U = \{t_1, \dots, t_m\}$ be a finite set of terms. We write $t'_i = t_i|_{x_n=1}$ for each index $1 \leq i \leq m$ and set $U' = \{t'_1, \dots, t'_m\}$ and*

$$\alpha = \max\left\{\deg_n(t_1), \dots, \deg_n(t_m)\right\}.$$

For each degree $\lambda \leq \alpha$, we introduce the index set $I_\lambda = \left\{i \mid \deg_n(t_i) = \lambda\right\}$ and the set $U'_\lambda = \{t'_i \mid i \in I_\lambda\}$. Then U is a Pommaret basis, if and only if the following three conditions are satisfied:

(i) For each degree $\lambda \leq \alpha$, the set U'_λ is a Pommaret basis.
(ii) For each degree $\lambda < \alpha$, we have the inclusion $U'_\lambda \subset \langle U'_{\lambda+1}\rangle$,
(iii) We have $U \cap \mathbb{K}[x_n] = x_n^\alpha$.

Proof. Assume first that U is a Pommaret basis. By Definition 2.2.8, U is autoreduced with respect to the Pommaret division. Let $\lambda \leq \alpha$ be a non-negative integer such that there exists a term $t \in U$ with $\deg_n(t) = \lambda$. We now show that U'_λ is a Pommaret basis of $\langle U'_\lambda\rangle \trianglelefteq \mathbb{K}[x_1, \dots, x_{n-1}]$. Note that U'_λ must be Pommaret autoreduced, too, as otherwise U could not be Pommaret autoreduced. Since the Pommaret division is continuous, we can check the involutivity of U'_λ by testing it for local involution. Choose a term $t'_i \in U'_\lambda$ and let x_k (with $k < n$) be a Pommaret non-multiplicative variable for it. Then, by definition of the Pommaret division, x_k is also not Pommaret multiplicative for $t_i \in U$. Since U is a Pommaret basis, there exists a Pommaret divisor $s \in U$ of $x_k \cdot t_i$. We claim that $s \in U_\lambda$. Indeed, $\deg_n(s) > \deg_n(t_i)$ is not possible because of $s \mid x_k \cdot t_i$. Also, $\deg_n(s) < \deg_n(t_i)$ is not possible because then the Pommaret divisibility of $x_k \cdot t_i$ by s implies that s is a pure power of x_n and hence a strict Pommaret divisor of t_i, in contradiction to the Pommaret autoreducedness of U. So, $s \in U_\lambda$ as claimed and s' is a Pommaret divisor of $x_k \cdot t'_i$ in the ring $\mathbb{K}[x_1, \dots, x_{n-1}]$.

That U satisfies Condition (ii) is easily seen: U is the unique Pommaret basis of the quasi-stable ideal $\langle U\rangle$ and hence also a Janet basis of $\langle U\rangle$. Condition (ii) now immediately follows by Theorem 3.3.4. Finally, Condition (iii) follows from the fact that the Pommaret autoreducedness of U implies that U contains exactly one pure x_n-power and this power must be x_n^α.

Now, we assume conversely that the set U satisfies Conditions (i) to (iii). We first show that U is Pommaret autoreduced. Arguing by *reductio ad absurdum*, suppose that there are terms $t_i, t_j \in U$ with $t_i \neq t_j$ and t_i is a Pommaret divisor of t_j. If $\deg_n(t_i) = \deg_n(t_j)$, then there is an integer λ such that $\{t'_i, t'_j\} \subseteq U'_\lambda$ and t'_i is a Pommaret divisor of t'_j in the ring $\mathbb{K}[x_1, \dots, x_{n-1}]$. This contradicts the Pommaret autoreducedness of U'_λ which is guaranteed by Condition (i). Otherwise, we have $\deg_n(t_i) < \deg_n(t_j)$ implying that t_i is a pure x_n-power. By Condition (iii), $t_i = x_n^\alpha$.

But now necessarily $\deg_n(t_j) > \alpha$, in contradiction to the definition of α as the maximal x_n-degree appearing in U.

We still need to show the involutivity of U, which we do again via local involution. Consider a term $t \in U$ with $\deg_n(t) = \lambda$ and let x_k be a Pommaret non-multiplicative variable of t. Now, if $k < n$, then x_k is also a Pommaret non-multiplicative variable of t' in the ring $\mathbb{K}[x_1, \dots, x_{n-1}]$. Since U'_λ is a Pommaret basis by Condition (i), there is a term $s' \in U'_\lambda$ and a term $x^\mu \in \mathbb{K}[x_1, \dots, x_{n-1}]$ Pommaret multiplicative for s' such that $t' = x^\mu \cdot s'$. This implies $t = x^\mu \cdot s$. It is easy to see that x^μ is also Pommaret multiplicative for $s \in \mathbb{K}[x_1, \dots, x_n]$. Thus, in the case $k < n$ we are done. Now, assume $k = n$. Recall that $t \in U_\lambda$. Since, by Condition (ii), $t' \in \langle U'_{\lambda+1} \rangle$ and, by Condition (i), $U'_{\lambda+1}$ is a Pommaret basis, there are terms $t_\ell \in U_{\lambda+1}$ and x^ν Pommaret multiplicative for t'_ℓ in the ring $\mathbb{K}[x_1, \dots, x_{n-1}]$ such that $t' = x^\nu \cdot t'_\ell$. This implies $x_n \cdot t = x^\nu \cdot t_\ell$. It is easy to see that x^ν is also Pommaret multiplicative for t_ℓ in the ring $\mathbb{K}[x_1, \dots, x_n]$. This finishes the proof of local involutivity of U, and we are done. □

We provide two examples for the application of Theorem 3.3.32, a positive one and a negative one.

Example 3.3.33. *In the trivariate polynomial ring $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$, we consider the set $U = \left\{x_3^3,\, x_2^2x_3^2,\, x_2^2x_3,\, x_1x_2x_3^2,\, x_1^2x_3^2,\, x_1^2x_2x_3,\, x_1^2x_3\right\}$. One observes that*

1. $\beta = 1 \leq 3 = \alpha$,
2. $U_3 = \left\{x_3^3\right\}$ *and* $U'_3 = \{1\}$, *which is obviously a Pommaret basis,*
3. $U'_2 = \left\{x_2^2,\, x_1x_2,\, x_1^2\right\}$, *which is also a Pommaret basis,*
4. $U'_1 = \left\{x_2^2,\, x_1^2x_2,\, x_1^2\right\}$, *which is also a Pommaret basis,*
5. $U'_1 \subset \langle U'_2 \rangle$, *and finally*
6. $U'_2 \subset \langle U'_3 \rangle$.

Hence, U is a Pommaret basis. Here, we have used that in two variables, one can identify Pommaret bases very easily. But in principle Theorem 3.3.32 requires to carry the recursion further, until it is only left to check subsets of $\mathbb{K}[x_1]$ for being a Pommaret basis, for which one applies Condition (iii), i. e., one must check whether one has a singleton set.

Example 3.3.34. *In the same polynomial ring $\mathcal{P} = \mathbb{K}[x_1, x_2, x_3]$, we consider now the set $U = \left\{x_3^3,\, x_2^2x_3^2,\, x_2^2x_3,\, x_1\right\}$. One observes that*

1. $\beta = 0 \leq 3 = \alpha$,
2. $U_3 = \left\{x_3^3\right\}$ *and* $U'_3 = \{1\}$, *which is obviously a Pommaret basis,*
3. $U'_2 = \left\{x_2^2\right\}$, *which is also a Pommaret basis,*

4. *$U'_1 = \{x_2^2\}$, which is also a Pommaret basis,*

5. *$U'_0 = \{x_1\}$, which is not a Pommaret basis, as $U'_0 \cap \mathbb{K}[x_2] = \emptyset$.*

Hence, U is not a Pommaret basis.

Remark 3.3.35. *Theorem 3.3.10 holds for Pommaret bases, too, if one replaces everywhere in it "Janet basis" by "Pommaret basis". This follows immediately from the fact that any Pommaret basis is also a Janet basis for the ideal it generates.*

Since quasi-stability is equivalent to the existence of a finite Pommaret basis by Proposition 2.2.11, we can use our results to derive also a recursive criterion for a monomial ideal to be quasi-stable using an arbitrary monomial generating set. This criterion, formulated in Corollary 3.3.36, translates directly into Algorithm 14 as an effective test for quasi-stability similar to Algorithm 10.

Algorithm 14: QuasiStableTest

Data: A finite set $U = \{t_1, \dots, t_m\} \subset \mathcal{T}$ of terms.
Result: True if $\langle U \rangle$ is quasi-stable and false otherwise.
begin
 $(\lambda_0, \lambda_1, \dots, \lambda_\ell) \longleftarrow$ the sequence of x_n-degrees of the terms t_i ordered such that $\lambda_0 < \lambda_1 < \dots < \lambda_\ell$
 if $n = 1$ **then**
  return $(true)$
 if $U \cap \mathbb{K}[x_n] = \emptyset$ **then**
  return $(false)$
 for i **from** 0 **to** ℓ **do**
  $U'_{\lambda_i} \longleftarrow \{\, t \in \mathbb{K}[x_1, \dots, x_{n-1}] \mid t \cdot x_n^{\lambda_i} \in U \,\}$
  if `QuasiStableTest`$(\bigcup_{j=0}^{i} U'_{\lambda_j}) = false$ **then**
   return $(false)$
 return $(true)$

Corollary 3.3.36. *Let $U = \{t_1, \dots, t_m\} \subset \mathcal{T}$ be a set of terms with $\lambda_0 < \lambda_1 < \dots < \lambda_\ell$ being the x_n-degrees of its elements. For each $0 \leq i \leq \ell$, we denote by $U_{\lambda_i} \subseteq U$ the subset containing those terms t with $\deg_n(t) = \lambda_i$ and we write $U'_{\lambda_i} = \{t|_{x_n=1} \mid t \in U_{\lambda_i}\}$. Then the monomial ideal $\langle U \rangle$ is quasi-stable, if and only if the following conditions hold:*
(i) For each $i \leq \ell$, the ideal $\langle \bigcup_{j=0}^{i} U'_{\lambda_j} \rangle \subseteq \mathbb{K}[x_1, \dots, x_{n-1}]$ is quasi-stable,
(ii) We have $U \cap \mathbb{K}[x_n] \neq \emptyset$.

Proof. Suppose first that $\langle U \rangle$ is quasi-stable. By Proposition 2.2.11, this ideal possesses thus a finite Pommaret basis H. By Theorem 3.3.32, H'_γ is a Pommaret basis for each $\gamma \leq \lambda_\ell$ and in addition $H'_\gamma \subset \langle H'_{\gamma+1} \rangle$ for each $\gamma < \lambda_\ell$. Since $\langle U \rangle = \langle H \rangle$, we have $\langle \bigcup_{j=0}^{i} U_{\lambda_i} \rangle = \langle \bigcup_{\gamma \le \lambda_i} H_\gamma \rangle$ for each $i < \ell$. Projecting to $\mathbb{K}[x_1, \dots, x_{n-1}]$ and using the inclusions $H'_\gamma \subset \langle H'_{\gamma+1} \rangle$, we get $\langle \bigcup_{j=0}^{i} U'_{\lambda_i} \rangle = \langle \bigcup_{\gamma \le \lambda_i} H'_\gamma \rangle = \langle H'_{\lambda_i} \rangle$. Thus H'_{λ_i} is a Pommaret basis for $\langle \bigcup_{j=0}^{i} U'_{\lambda_j} \rangle$ for each $i < \ell$. It follows from Proposition 2.2.11 that $\langle \bigcup_{j=0}^{i} U'_{\lambda_j} \rangle$ is quasi-stable and this proves (i). Item (ii) follows directly from the definition of quasi-stability, as U must contain a pure power of x_n.

Conversely, assume that conditions (i) and (ii) are satisfied and consider an arbitrary term $t = x_j^{\mu_j} \cdots x_n^{\mu_n} \in U$ with $\mu_j \neq 0$ for $j = \mathrm{cls}\,(t)$. A necessary condition for the quasi-stability of $\langle U \rangle$ is that there exists an exponent s such that $x_n^s t / x_j \in \langle U \rangle$. From (ii), we know that some power x_n^a lies in $\langle U \rangle$ and hence we can simply choose any $s \ge a$. However, as a sufficient condition for the quasi-stability of $\langle U \rangle$, we must also check the membership $x_k^s t / x_j \in \langle U \rangle$ for any index $n > k > j$ and sufficiently high exponent s. For this, we must recursively descend via (i).

Similar to (2.3), we introduce

$$U_{(d_s, \dots, d_n)} = \left\{ u|_{x_s = \dots = x_n = 1} \mid u \in U,\, \deg_i(u) \le d_i,\, i = s, \dots, n \right\}$$

and consider our term t as an element of the subset $V := U_{(\mu_{k+1}, \dots, \mu_n)}$. Let $\gamma_0 < \gamma_1 < \dots < \gamma_l$ be the x_k-degrees of the elements of V. By (i), $\cup_{j=0}^{\mu_k} V'_{\gamma_j}$ generates a quasi-stable ideal in $\mathbb{K}[x_1, \dots, x_k]$ which by (ii) must contain a term x_k^b for some exponent b. Hence, the original set U must contain a term $x_k^b x_{k+1}^{\nu_{k+1}} \cdots x_n^{\nu_n}$ with $\nu_i \le \mu_i$. Choosing $s \ge b$, this term is a divisor of $x_k^s t / x_j$ so that indeed $x_k^s t / x_j \in \langle U \rangle$ as required for the completion of the proof. □

Theorem 3.3.37. *Algorithm 14 terminates in finitely many steps and is correct.*

Proof. The termination of the algorithm is trivial due to the recursive structure of the algorithm and also the use of the **for** loops. The correctness of the algorithm is a consequence of Corollary 3.3.36.

□

Remark 3.3.38. *For alternative approaches to testing quasi-stability, we refer to [55, 98]. The algorithm presented in [55, Prop. 3.4] has complexity $O(m^2 n^2)$, as is shown there. The complexity of Algorithm 14 depends on its implementation, for instance the use of optimized algorithms for the construction of the Janet trees. We do not analyse the complexity here.*

Example 3.3.39. *We consider $U = \left\{x_4^3,\, x_3 x_4^2,\, x_2^2 x_4^2,\, x_1 x_2 x_4^2,\, x_1^3 x_4^2,\, x_3^2 x_4,\, x_3^3\right\}$ in the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3, x_4]$. To illustrate the application of our test, we need the tree representation of U shown in Figure 3.7. One observes that*

1. $\lambda_0 = 0, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$,
2. $U'_3 = \{1\}$,
3. $U'_2 = \left\{x_3,\, x_2^2,\, x_1 x_2,\, x_1^3\right\}$,
4. $U'_1 = \left\{x_3^2\right\}$,

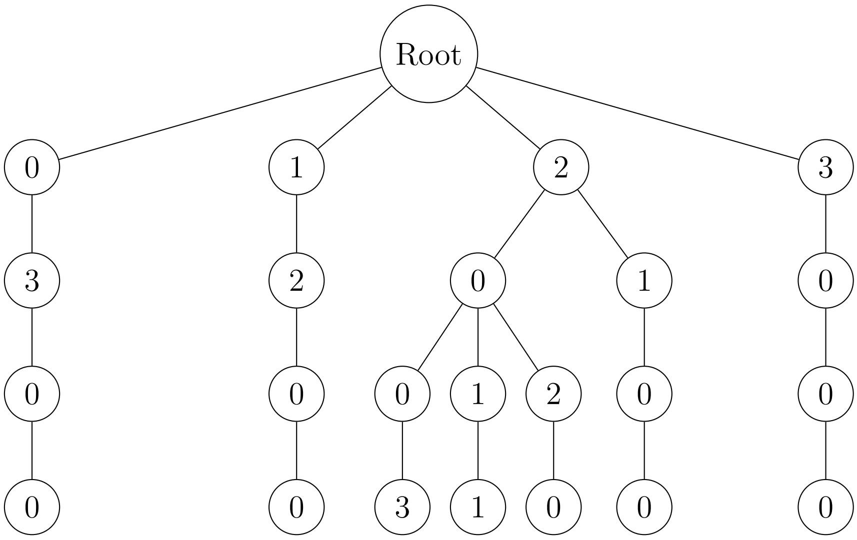

Figure 3.7: Tree representation of the set U in Example 3.3.39

5. $U'_0 = \{x_3^3\}$.

We have $x_3^3 \in U$ *which satisfies Condition (ii) in Corollary 3.3.36. In addition, each* $\langle \cup_{j=0}^{i} U'_{\lambda_j} \rangle$ *for each* i *is quasi-stable. So, the ideal generated by* U *is quasi-stable.*

Based on Corollary 3.3.36, we propose the simple Algorithm 16 to transform a given homogeneous ideal into quasi-stable position. For this purpose, we present a variant of Algorithm 14 which tests the quasi-stability of a given monomial ideal and in the case that the input ideal is not quasi-stable the algorithm returns a term representing an obstruction to quasi-stability.

We present now the new Algorithm 16 for transforming any polynomial ideal into quasi-stable position using our new recursive criterion. The basic idea of the algorithm is still the same as of the algorithm proposed in [60, Alg. 2]. The main novelty lies in the fact that the recursive criterion immediately suggests to combine several elementary moves into one transformation, whereas the old algorithm applied one elementary move after the other. Since we need to compute a new Gröbner basis after each transformation, this should represent a significant increase in efficiency.

The termination proof of the new algorithm follows the same strategy as presented in [60]. We introduce an ordering on finite lists of terms and show that we always ascend with respect to this ordering. This ordering is defined as follows [60, Def. 6.1].

Definition 3.3.40. *Let* $F \subset \mathcal{R}$ *be a finite set of polynomials with* $\operatorname{lt}(F) = \{t_1, \ldots, t_\ell\}$ *such that* $t_1 \succ_{revlex} \cdots \succ_{revlex} t_\ell$ *where* $\succ_{revlex}$ *refers to the pure reverse lexicographic ordering with* $x_1 \prec \cdots \prec x_n$. *Then we denote the ordered tuple of these leading terms by* $\mathcal{L}(F) = (t_1, \ldots, t_\ell)$. *If* $F, \tilde{F} \subset \mathcal{R}$ *are two finite sets of polynomials with* $\mathcal{L}(F) = (t_1, \ldots, t_\ell)$ *and* $\mathcal{L}(\tilde{F}) = (\tilde{t}_1, \ldots, \tilde{t}_{\tilde{\ell}})$, *then we define an ordering on the*

Algorithm 15: QuasiStableObstruction

Data: A finite set U of terms and an ordered set $\{x_1, \ldots, x_n\}$ of variables with $U \subset \mathbb{K}[x_1, \ldots, x_n]$.
Result: The empty list $[\,]$ if $\langle U \rangle$ is quasi-stable and a list containing a term along with a variable, otherwise.

```
begin
    (λ_0, λ_1, ..., λ_ℓ) ⟵ the sequence of x_n-degrees of the elements of U
        ordered such that λ_0 < λ_1 < ··· < λ_ℓ
    if n = 1 then
        return ([ ])
    if U ∩ K[x_n] = ∅ then
        choose a term t ∈ U with minimal number of variables
        return ([t, x_n])
    for i from 0 to ℓ − 1 do
        U'_{λ_i} ⟵ { t ∈ K[x_1, ..., x_{n−1}] | t · x_n^{λ_i} ∈ U }
        A := QuasiStableObstruction(⋃_{j=0}^{i} U'_{λ_j}, {x_1, ..., x_{n−1}})
        if A ≠ [ ] then
            return (A)
    return ([ ])
```

corresponding tuples of terms by

$$\mathcal{L}(F) \prec_{\mathcal{L}} \mathcal{L}(\tilde{F}) \iff \begin{cases} \exists\, j \leq \min(\ell, \tilde{\ell})\ \forall\, i < j : t_i = \tilde{t}_i \wedge t_j \prec_{revlex} \tilde{t}_j \quad \textit{or} \\ \forall\, j \leq \min(\ell, \tilde{\ell}) : t_j = \tilde{t}_j \wedge \ell < \tilde{\ell}\,. \end{cases}$$

The new algorithm furthermore tries to incorporate transpositions of the variables as a further measure to increase efficiency, since permutations do not affect the sparsity of the ideal generators. They are not necessary for the correctness of the algorithm, but represent only an optimisation in some situations. One should note that the number of possible transpositions is given by $n(n-1)/2$ and we will show that this does not harm the termination of the algorithm.

Theorem 3.3.41. *Assume that* $\mathbb{K}$ *is infinite. Then, Algorithm 16 terminates in finitely many steps and is correct.*

Proof. Let us deal first with the termination. Keeping the notations of the algorithm, if the `QuasiStableObstruction` algorithm returns a non-empty list, then from Corollary 3.3.36, we can see easily that there exists some term $t = x_{i_1}^{\mu_1} \cdots x_{i_k}^{\mu_k} \in U$ such that $x_i^p t / x_{i_1} \notin \langle U \rangle$ for each p and some $i > i_1$. Thus, we get an obstruction in the sense of the combinatorial definition of quasi-stability presented in [60].

Now, suppose that $G \subset \mathcal{P}$ is the reduced Gröbner basis for the ideal it generates. Let $i < s$ and ϕ_i be the homomorphism which sends x_i to $x_i + a x_s$ where $a \in \mathbb{K} \setminus \{0\}$ and leaves all other variables unchanged. From [60, Prop. 6.9], we know that for

Algorithm 16: QuasiStableLinChange

Data: A reduced and homogeneous Gröbner basis $F \subset \mathbb{K}[x_1, \ldots, x_n]$.
Result: A linear change Φ so that $\langle \Phi(F) \rangle$ is in quasi-stable position.

```
begin
    Φ ⟵ id;   G ⟵ F ;   U ⟵ lt(G)
    A ⟵ QuasiStableObstruction(U, {x_1, ..., x_n})
    T[i, j] ⟵ false for each 1 ≤ i < j ≤ n
    while A ≠ [ ] do
        t ⟵ A[1] = x_{i_1}^{μ_1} ··· x_{i_k}^{μ_k};   x_s ⟵ A[2]
        flag ⟵ false
        if t does not contain x_s then
            if T[i_k, s] = false then
                π ⟵ transposition x_{i_k} ⟷ x_s
                G̃ ⟵ reduced Gröbner basis of ⟨π(G)⟩
                Φ ⟵ π ∘ Φ
                flag ⟵ true
                T[i_k, s] ⟵ true
        if flag = false then
            φ ⟵ homomorphism with x_{i_j} ↦ x_{i_j} + x_s for j ≤ k, i_j ≠ s and
              all other variables unchanged
            G̃ ⟵ reduced Gröbner basis of ⟨φ(G)⟩
            Φ ⟵ φ ∘ Φ
            while L(G) ⪰_L L(G̃) do
                G̃ ⟵ reduced Gröbner basis of ⟨φ(G̃)⟩
                Φ ⟵ φ ∘ Φ
        G ⟵ G̃ ;   U ⟵ lt(G)
        A ⟵ QuasiStableObstruction(U, {x_1, ..., x_n})
    return (Φ)
```

any generic choice of a, it holds

$$\mathcal{L}(G) \prec_{\mathcal{L}} \mathcal{L}\big(\phi_i(G)^{\triangle}\big)$$

where $X^{\triangle}$ stands for the head autoreduced form of a finite set X. More precisely, the number of "bad" values of a is finite. Now, let us generalise this result by assuming that ϕ is the homomorphism which maps x_{i_j} to $x_{i_j}+ax_s$ where $a \in \mathbb{K}\setminus\{0\}$, $j \leq k$ with $i_j \neq s$ and keeps all other variables unchanged. An iterative use of the mentioned result shows that for any generic choice of a we have $\mathcal{L}(G) \prec_{\mathcal{L}} \mathcal{L}\big(\phi(G)^{\triangle}\big)$. Applying this result and taking into account the fact that there exists only finitely many possible leading term ideals under coordinate transformations (see the proof of [60, Thm. 6.11]), we deduce that the number of the homomorphism ϕ that increases the leading term ideal is finite as well. On the other hand, by the structure of the algorithm the number of performed transpositions is bounded by $(n^2-n)/2$. All these arguments proves that the algorithm terminates in finitely many steps.

The correctness of the algorithm is an obvious consequence of Corollary 3.3.36. $\square$

Example 3.3.42. *Consider the set* $U = \{x_3^3, x_2^2x_3^2, x_1\} \subset \mathbb{K}[x_1, x_2, x_3]$ *introduced in Example 3.3.34. Since no pure power of* x_2 *lies in* $U'_0 = \{x_1\} \subset \mathbb{K}[x_1, x_2]$*, the ideal generated by this set is not quasi-stable and in turn the ideal generated by* U *is not quasi-stable. Following Algorithm 16, we apply the permutation* $x_1 \longleftrightarrow x_2$ *to* $\mathcal{I}$ *and obtain the ideal* $\tilde{\mathcal{I}} = \langle x_3^3, x_1^2x_3^2, x_2\rangle$ *which is quasi-stable.*

Example 3.3.43. *We consider the ideal treated by [36, Sect. 2] (see also [98]) in their quest for the construction of optimal systems of parameters in the sense that they are as sparse as possible. Take*

$$F = \{\, x_5x_6,\, x_4x_6,\, x_4x_5,\, x_3x_5,\, x_2x_5,\, x_3x_4,\, x_2x_4,\, x_2x_3,\, x_1x_3,\, x_1x_2 \,\} \subset \mathbb{K}[x_1, \dots, x_6]\,.$$

Algorithm 16 performs first the linear change $x_5 \mapsto x_5 + x_6$ *which yields a new leading ideal generated by*

$$U := \{x_6^2,\, x_4x_6,\, x_3x_6,\, x_2x_6,\, x_4x_5,\, x_3x_4,\, x_2x_4,\, x_2x_3,\, x_1x_3,\, x_1x_2\}\,.$$

Then, from the set $U'_0 = \{x_4x_5, x_3x_4, x_2x_4, x_2x_3, x_1x_3, x_1x_2\} \subset \mathbb{K}[x_1, \dots, x_5]$*, it derives and performs the linear change* $x_4 \mapsto x_4 + x_5$*. This leads to a new leading ideal generated by*

$$V := \{x_6^2,\, x_5x_6,\, x_3x_6,\, x_2x_6,\, x_5^2,\, x_3x_5,\, x_2x_5,\, x_2x_3,\, x_1x_3,\, x_1x_2\}\,.$$

We have $W := V'_0 = \{x_5^2, x_3x_5, x_2x_5, x_2x_3, x_1x_3, x_1x_2\}$*. Since the ideal generated by* $W'_0 = \{x_2x_3, x_1x_3, x_1x_2\} \subset \mathbb{K}[x_1, \dots, x_4]$ *is not quasi-stable and since* W'_0 *does not contain* x_4*, Algorithm 16 proceeds with the linear change* $x_3 \longleftrightarrow x_4$ *and* $x_2 \mapsto x_2 + x_4$*. The new leading ideal is generated by*

$$Z := \{x_6^2,\, x_5x_6,\, x_4x_6,\, x_2x_6,\, x_5^2,\, x_4x_5,\, x_2x_5,\, x_4^2,\, x_1x_4,\, x_1x_2\}\,.$$

Set $T := Z'_0 = \{x_5^2, x_4x_5, x_2x_5, x_4^2, x_1x_4, x_1x_2\}$ *and* $R := T'_0 = \{x_4^2, x_1x_4, x_1x_2\}$*. Algorithm 16 considers now the set* $R'_0 = \{x_1x_2\} \subset \mathbb{K}[x_1, x_2, x_3]$*. Since no term in it contains* x_3*, it performs the linear change* $x_2 \longleftrightarrow x_3$ *and* $x_1 \mapsto x_1 + x_3$ *and obtains as new leading ideal*

$$\langle x_6^2,\, x_5x_6,\, x_4x_6,\, x_3x_6,\, x_5^2,\, x_4x_5,\, x_3x_5,\, x_4^2,\, x_3x_4,\, x_3^2\rangle$$

which is quasi-stable. One sees that the number of elementary linear changes applied is 4*, which is the same as for the transformation proposed in [98].*

Remark 3.3.44. *Consider the ideal* $\langle U\rangle$ *with generating set* $U = \{x_3^3, x_1^2x_3, x_2\} \subset \mathbb{K}[x_1, x_2, x_3]$*. We have* $U'_3 = \{1\}$*,* $U'_1 = \{x_1^2\}$ *and* $U'_0 = \{x_2\}$*. One can see that* $U'_0 \not\subset \langle U'_1\rangle$ *and therefore the second condition of Theorem 3.3.32 does not hold. Indeed, although the ideal is quasi-stable,* U *is not its Pommaret basis.*

We conclude this section by discussing a recursive test for being in Noether position. An ideal $\mathcal{I} \subset \mathcal{R}$ with the Krull dimension D is in *Noether position*, if the ring extension $\mathbb{K}[x_1, \dots, x_D] \hookrightarrow \mathcal{R}/\mathcal{I}$ is integral, i. e. the image in $\mathcal{R}/\mathcal{I}$ of x_i for any $i = D+1, \dots, n$ is a root of a polynomial of the form $X^s + g_1 X^{s-1} + \cdots + g_s = 0$ where s is an integer and $g_1, \dots, g_s \in \mathbb{K}[x_1, \dots, x_D]$ (see e. g. [33]). [11] proved that $\mathcal{I}$ is in Noether position, if and only if for each $i = D+1, \dots, n$ there exists r_i such that $x_i^{r_i}$ belongs to the leading ideal of $\mathcal{I}$ with respect to $\prec$. Furthermore, they showed that this is equivalent to the fact that $\mathcal{I} + \langle x_1, \dots, x_D \rangle$ is zero-dimensional. These observation show that $\mathcal{I}$ is in Noether position, if and only if $\operatorname{lt}(\mathcal{I})$ is as well. While Noether position is implied by quasi-stable position, the converse is not true. In the next proposition, we give a recursive test for being in Noether position using the minimal generating set of a monomial ideal.

Proposition 3.3.45. *Let $U = \{t_1, \dots, t_m\} \subset \mathcal{R}$ be a set of terms with $\lambda_0 < \lambda_1 < \cdots < \lambda_\ell$ the x_n-degrees of its elements. For each $0 \le i \le \ell$, we denote by $U_{\lambda_i} \subseteq U$ the subset of U containing the terms t with $\deg_n(t) = \lambda_i$ and set $U'_{\lambda_i} = \{ t|_{x_n=1} \mid t \in U_{\lambda_i} \}$. Then the monomial ideal $\langle U \rangle$ is in Noether position, if and only if the following conditions hold:*

(i) The ideal $\langle U'_{\lambda_0} \rangle \subseteq \mathbb{K}[x_1, \dots, x_{n-1}]$ is in Noether position,

(ii) $U \cap \mathbb{K}[x_n] \neq \emptyset$.

Proof. Suppose that the ideal $\langle U \rangle$ is in Noether position and has dimension D. Then, by [11, Lem 4.1], we know that $\langle U \rangle + \langle x_1, \dots, x_D \rangle$ is zero-dimensional and a pure power of x_n appears in U. Thus, $\langle U'_{\lambda_0} \rangle \subseteq \mathbb{K}[x_1, \dots, x_{n-1}]$ is an ideal of dimension D and $\langle U'_{\lambda_0} \rangle + \langle x_1, \dots, x_D \rangle$ is zero-dimensional, proving item (i). On the other hand, for any $i = D+1, \dots, n$ there exists r_i such that $x_i^{r_i} \in U$ and this proves item (ii).

Conversely, to prove that $\langle U \rangle$ is in Noether position, we note that a pure power of x_n belongs to U. It follows that $\langle U \rangle$ and $\langle U'_{\lambda_0} \rangle \subseteq \mathbb{K}[x_1, \dots, x_{n-1}]$ share the same dimension D. From the fact that $\langle U'_{\lambda_0} \rangle$ is in Noether position, we conclude that $\langle U'_{\lambda_0} \rangle + \langle x_1, \dots, x_D \rangle$ is zero-dimensional and hence that $\langle U \rangle + \langle x_1, \dots, x_D \rangle$ is zero-dimensional too, proving the claim. □

Example 3.3.46. *Consider the ideal $\mathcal{I} = \langle x_1^3,\, x_2x_3,\, x_3^2 \rangle \subset \mathbb{K}[x_1, x_2, x_3]$. With $U := \{x_1^2,\, x_2x_3,\, x_3^2\}$, one sees that the ideal $\langle U'_0 \rangle = \langle x_1^2 \rangle \subset \mathbb{K}[x_1, x_2]$ is not in Noether position and hence, by Proposition 3.3.45, $\mathcal{I}$ is also not in Noether position.*

Remark 3.3.47. *We can adapt Algorithm 16 to transform a given ideal into Noether position by simply performing the last* **for***-loop only for $m = 0$. If we consider the ideal presented in Example 3.3.43, then one finds the same linear change to transform the ideal into Noether position. As it has been mentioned, this approach allows us to perform permutations of the variables to get a sparser linear change.*

3.4 Involutive-like Divisions and Bases

While Gerdt and Blinkov extended solely the Janet division to the Janet-like division [50, 49], we will introduce the general concept of an involutive-like division

(Definition 3.4.1) and related notions like continuity or constructivity. Our main emphasis will be on Janet-like and Pommaret-like bases and how they are related to each other and to Janet and Pommaret bases, respectively (Propositions 3.4.8 and 3.4.16, Theorem 3.4.19).

Definition 3.4.1. *An* involutive-like division L *on* $\mathcal{T} \subset \mathcal{R}$ *associates to any finite set* $U \subset \mathcal{T}$ *of terms and any term* $u \in U$ *a set of* L-non-multipliers $\bar{L}(u,U)$ *given by the terms contained in an irreducible monomial ideal. The powers generating this irreducible ideal are called the* non-multiplicative powers $\mathrm{NMP}_L(u,U)$ *of* $u \in U$*. The set of* L-multipliers $L(u,U)$ *is given by the order ideal* $\mathcal{T} \setminus \bar{L}(u,U)$*. For any term* $u \in U$*, its* involutive cone *is defined as* $\mathcal{C}_L(u,U) = u \cdot L(u,U)$*. For an involutive division, the involutive cones must satisfy the following conditions:*

(i) For two terms $v \neq u \in U$ *with* $\mathcal{C}_L(u,U) \cap \mathcal{C}_L(v,U) \neq \emptyset$*, we have* $u \in \mathcal{C}_L(v,U)$ *or* $v \in \mathcal{C}_L(u,U)$*.*

(ii) If a term $v \in U$ *lies in an involutive cone* $\mathcal{C}_L(u,U)$*, then* $L(v,U) \subset L(u,U)$*.*

There are two differences between this definition and Definition 2.2.4 of an involutive division. Firstly, the non-multipliers are now only required to generate an irreducible ideal instead of a prime one. Therefore we must speak of non-multiplicative powers instead of non-multiplicative variables. Secondly, we have dropped the filter axiom 2.2.4 (iii), as we were not able to come up with a Pommaret-like division respecting it in its classical form. The filter axiom is relevant for completion algorithms for the Janet and closely related divisions and for the existence of a strong basis within each weak basis. As we will show below, all these applications are still possible within our framework.

Definition 3.4.2. *For a finite set of terms* $U \subset \mathcal{T}$ *and an involutive-like division* L *on* $\mathcal{T}$*, the* involutive span *of* U *is the union* $\mathcal{C}_L(U) = \bigcup_{u \in U} \mathcal{C}_L(u,U)$*. The set* U *is* involutively complete *or a* weak involutive basis*, if* $\mathcal{C}_L(U) = U \cdot \mathcal{T}$*. For a* (strong) involutive basis *the union is disjoint, i. e. every term in* $\mathcal{C}_L(U)$ *has a unique involutive divisor.*

Definition 3.4.3. *Let* L *be an involutive-like division on* $\mathcal{T}$ *and let* $U \subset \mathcal{T}$ *be a finite set of terms. The terms* $t \cdot \mathrm{NMP}_L(t,U)$ *with* $t \in U$ *are minimal among those terms of the monomial ideal* $\langle U \rangle$ *which are possibly not contained in the involutive span of* U*. Those terms which are indeed not contained in* $\mathcal{C}_L(U)$ *are called* L-obstructions *of* U *and we write*

$$\mathrm{Obstr}_L(U) = \Big(\bigcup_{t \in U} t \cdot \mathrm{NMP}(t,U)\Big) \setminus \mathcal{C}_L(U).$$

The set of minimal elements of $\mathrm{Obstr}_L(U)$ *with respect to divisibility is denoted by* $\mathrm{MinObstr}_L(U)$*.*

Example 3.4.4. *The* Janet-like division *assigns non-multiplicative powers to a term* x^μ *contained in a finite set* $U \subset \mathcal{T}$ *as follows:*

$$\mathrm{NMP}_J(x^\mu, U) = \left\{ x_a^{p(J,x^\mu,U,a)} \mid x_a \in \mathrm{NM}_{\mathcal{J}}(x^\mu, U) \right\},$$

where the exponents are given by

$$p(J, x^\mu, U, a) = \min\left\{\nu_a - \mu_a \mid x^\nu \in U_{[\mu_{a+1},\dots,\mu_n]} \wedge \nu_a > \mu_a\right\}.$$

Here, the letter J stands for the Janet-like division, while the classical involutive Janet division from which it is derived is denoted by the calligraphic letter $\mathcal{J}$. We will always use calligraphic letters to denote involutive divisions and roman letters to denote the involutive-like divisions derived from them.

We extend now important notions for involutive divisions like Noetherianity, continuity and constructivity to involutive-like divisions.

Definition 3.4.5. *The involutive-like division L is called*

(i) Noetherian, *if for every finite set of terms $U \subset \mathcal{T}$ there exists a finite set $\overline{U} \subset \mathcal{T}$ with $U \subseteq \overline{U}$ such that $\overline{U}$ is an L-basis of the monomial ideal $\langle U\rangle$; such a set $\overline{U}$ is called an L-*completion *of U;*

(ii) continuous, *if for every finite set $U \subset \mathcal{T}$ every sequence $(t_1, t_2, \dots, t_k) \in U^k$ such that $t_i \cdot \mathrm{NMP}_L(t_i, U) \cap \mathcal{C}_L(t_{i+1}, U) \neq \emptyset$ for each index $i \in \{1, \dots, k-1\}$ consists of k distinct terms, i. e. $t_i \neq t_j$ for all $1 \leq i < j \leq k$;*

(iii) constructive, *if it is continuous and if additionally for every finite set of terms $U \subset \mathcal{T}$ and for each term $s \in \mathrm{MinObstr}_L(U)$ no term $s' \in \mathcal{C}_L(U)$ exists such that $s \in \mathcal{C}_L\big(U \cup \{s'\}\big)$.*

Remark 3.4.6. *The above definitions of Noetherianity and continuity are straightforward generalisations of their classical counterparts. However, the definition of constructivity uses a more restrictive condition than in the classical theory. Because of the filter axiom 2.2.4 (iii), one only has to control there the involutive cone of the newly added term s'. In the involutive-like case without such an axiom, we must at the same time also control the involutive cones of all the other terms $t \in U$, as they might get larger when adding s' to U.*

The following property of an involutive-like division will serve us as a substitute for the missing filter axiom in some situations.

Definition 3.4.7. *Let L be an involutive-like division on the set of terms $\mathcal{T} \subset \mathcal{R}$. We say that L satisfies the* strong basis property *if for every weak L-basis $U \subset \mathcal{T}$ of the monomial ideal $\langle U\rangle$, there is a subset $\tilde{U} \subseteq U$ such that $\tilde{U}$ is a strong L-basis of the same monomial ideal.*

Proposition 3.4.8. *The Janet-like division is a Noetherian, continuous and constructive involutive-like division. Moreover, it satisfies the strong basis property.*

Proof. The first statement is due to [50, Prop. 2, Thms. 1–3]. For the strong basis property, we simply remark that every finite set of terms $H \subset \mathcal{T}$ is autoreduced with respect to the Janet-like division; this follows also from [50, Prop. 2]. □

Theorem 3.4.9. *For a continuous involutive-like division L, the finite set of terms $U \subset \mathcal{T}$ is a weak L-basis of the monomial ideal $\langle U\rangle$, if and only if* $\mathrm{MinObstr}_L(U) = \emptyset$.

Proof. The proof is a straightforward generalisation of the proof of the analogous result for involutive divisions. □

Proposition 3.4.10. *The Janet-like division is related to the Janet division as follows:*

(i) For each term t contained in a finite set $U \subset \mathcal{T}$, we have $\mathcal{C}_{\mathcal{J}}(t,U) \subseteq \mathcal{C}_J(t,U)$.

(ii) Every Janet basis of a monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is also a Janet-like basis.

(iii) From a Janet-like basis U of the monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$, one can obtain a Janet basis U' of the same ideal as follows:

$$U' = \left\{ t \cdot x^\mu \mid t \in U \wedge x^\mu \mid \Pi_{x_a^{p_a} \in \mathrm{NMP}_{\mathcal{J}}(t,U)} x_a^{p_a - 1} \right\}.$$

Proof. Item (i) follows directly from the definitions. Item (ii) is a direct consequence of (i).

Item (iii) holds, if we can prove that $\mathcal{C}_J(U) \subseteq \mathcal{C}_{\mathcal{J}}(U')$. Let $v \in \mathcal{C}_J(U)$ be an arbitrary term in the Janet-like span of U. Then there exists a term $t \in U$, a term x^μ dividing $\Pi_{x_a^{p_a} \in \mathrm{NMP}_{\mathcal{J}}(t,U)} x_a^{p_a-1}$ and a term $x^\rho \in \mathbb{K}[\mathrm{M}_{\mathcal{J}}(t,U)]$ such that $v = t \cdot x^\mu \cdot x^\rho$. By definition of U', we see that $t \cdot x^\mu \in U'$. It remains to show that $x^\rho \in \mathbb{K}[\mathrm{M}_{\mathcal{J}}(t \cdot x^\mu, U')]$. For this, it suffices to show that $\mathrm{M}_{\mathcal{J}}(t,U) \subseteq \mathrm{M}_{\mathcal{J}}(t \cdot x^\mu, U')$. We do this iteratively by ordering the set of variables $\mathrm{M}_{\mathcal{J}}(t,U)$ descendingly according to their indices and showing the containments $x_j \in \mathrm{M}_{\mathcal{J}}(t \cdot x^\mu, U')$ one after the other.

So let x_j be the variable with the highest index in $\mathrm{M}_{\mathcal{J}}(t,U)$. By definition of the Janet division, we have that $\deg_j(t)$ is maximal among the x_j-degrees of the Janet class $U_{[\deg_{j+1}(t),\dots,\deg_n(t)]}$. We know that $\deg_j(t \cdot x^\mu) = \deg_j(t)$ and we have to show that it is maximal among the x_j-degrees of the Janet class $U'_{[\deg_{j+1}(t \cdot x^\mu),\dots,\deg_n(t \cdot x^\mu)]}$. To see this, we now analyse which elements $s \in U$ induce elements $s \cdot x^\theta$ in this Janet class of U'. We consider first those terms $s \in U$ which are *not* in the same Janet class of U as t. If s is lexicographically smaller than t, then by analysing the highest variable index ℓ where s and t differ, we see, by definition of Janet-like non-multiplicative powers, that all terms $u = s \cdot x^\theta \in U'$ induced by s have $\deg_\ell(u) < \deg_\ell(t)$. However, $\deg_\ell(t \cdot x^\mu) \geq \deg_\ell(t)$. Hence, $s \cdot x^\theta$ and $t \cdot x^\mu$ are *not* in the same Janet class of U'. If s is lexicographically larger than t, then again by analysing the highest variable index ℓ where s and t differ, we see that $p(J,t,U,\ell) \leq \deg_\ell(s) - \deg_\ell(t)$ and hence, $\deg_\ell(t \cdot x^\mu) < \deg_\ell(s)$, whereas, obviously, $\deg_\ell(s \cdot x^\theta) \geq \deg_\ell(s)$ for all terms $s \cdot x^\theta$ induced by s in U'. Hence, $t \cdot x^\mu$ and $s \cdot x^\theta$ are *not* in the same Janet class of U'.

It remains to analyse the case of a term $s \in U$ which is in the same Janet class $U_{[\deg_{j+1}(t),\dots,\deg_n(t)]}$ as t. If $\deg_j(s) < \deg_j(t)$, then it is easy to see that also for the induced term $s \cdot x^\theta$, $\deg_j(s \cdot x^\theta) < \deg_j(t)$. If, on the other hand, $\deg_j(s) \geq \deg_j(t)$, then by the Janet multiplicativity of x_j for t, we have in fact $\deg_j(s) = \deg_j(t)$ and x_j is also Janet multiplicative for s. Moreover, the Janet-like powers of variables x_a with $a > j$ are the same for s and t. So s can induce terms $s \cdot x^\theta$ which are in the same Janet class of U' as $t \cdot x^\mu$, namely exactly for those x^θ which have the same projection on the subring $\mathbb{K}[x_{j+1}, x_{j+2}, \dots, x_n]$ as x^μ. But since x_j is Janet multiplicative for s, we must have $\deg_j(x^\theta) = 0$ by definition of U'. This proves that $\deg_j(t)$ is still maximal among all x_j-degrees of elements of the Janet class $U'_{[\deg_{j+1}(t\cdot x^\mu),\dots,\deg_n(t\cdot x^\mu)]}$.

Thus, we have shown that $x_j \in \mathrm{M}_{\mathcal{J}}(t\cdot x^\mu, U')$. The iteration over the variables of $\mathrm{M}_{\mathcal{J}}(t\cdot x^\mu, U')$ which have lower indices than j can now be performed using similar arguments, making use of the equality $\deg_j(t) = \deg_j(t\cdot x^\mu)$. □

We introduce now an involutive-like division based on the Pommaret division. Note that it is no longer a global division. This is not very surprising, as the very idea of involutive-like divisions consists of comparing different terms in a given set.

Definition 3.4.11. *The* Pommaret-like division P *assigns to each term* $t \in \mathcal{T}$ *contained in a finite set of terms* $U \subset \mathcal{T}$ *non-multiplicative powers as follows: For each* x_a *with* $a > \mathrm{cls}(t)$*, set*

$$p(P,t,U,a) = \begin{cases} 1, & \text{if } x_a \in \mathrm{M}_{\mathcal{J}}(t,U),\\ p(J,t,U,a), & \text{if } x_a \in \mathrm{NM}_{\mathcal{J}}(t,U).\end{cases}$$

Note that no non-multiplicative power is assigned to any variable x_b *with* $b \le \mathrm{cls}(t)$.

Proposition 3.4.12. *The Pommaret-like division is an involutive-like division.*

Proof. Let $U \subset \mathcal{T}$ be a finite set of terms. Let s and t be two terms in U whose Pommaret-like cones have a non-empty intersection: $\mathcal{C}_P(s,U) \cap \mathcal{C}_P(t,U) \neq \emptyset$. Without loss of generality, $\mathrm{cls}(s) \le \mathrm{cls}(t)$. Consider an arbitrary variable $x_j \in \mathrm{M}_{\mathcal{J}}(t,U)$ with $j > \mathrm{cls}(t)$. By definition of the Pommaret-like division, we have $\deg_j(u) = \deg_j(t)$ for all terms $u \in \mathcal{C}_P(t,U)$. Thus, if we pick a term $v \in \mathcal{C}_P(s,U) \cap \mathcal{C}_P(t,U)$, we also have $\deg_j(v) = \deg_j(t)$. By definition, s divides v and hence $\deg_j(s) \le \deg_j(t)$. If the strict inequality $\deg_j(s) < \deg_j(t)$ were true, then this would imply $x_j \in \mathrm{NM}_{\mathcal{J}}(s,U)$ and $\mathrm{NMP}_P(s,U) \le \deg_j(t) - \deg_j(s)$, in contradiction to $v \in \mathcal{C}_P(s,U)$. Hence, we can conclude that $\deg_j(s) = \deg_j(t)$.

Now let x_ℓ be a variable such that $\ell > \mathrm{cls}(t)$ and $\ell \in \mathrm{NM}_{\mathcal{J}}(t,U)$. A power of it is a Pommaret-like non-multiplicative power of t and we have that $\deg_\ell(t) \le \deg_\ell(u) < \deg_\ell(t) + p(P,t,U,\ell)$ for all terms $u \in \mathcal{C}_P(t,U)$. In particular, these inequalities hold for any term $v \in \mathcal{C}_P(s,U) \cap \mathcal{C}_P(t,U)$. Now let $\hat{\ell}$ be the greatest index of such a variable. Then, since by the first paragraph of this proof $s \in U_{[\deg_{\hat{\ell}+1}(t),\dots,\deg_n(t)]}$ and since $\deg_{\hat{\ell}}(s) \le \deg_{\hat{\ell}}(v) < \deg_\ell(t) + p(P,t,U,\ell)$ by the definition of Janet-like non-multiplicative powers, $\deg_{\hat{\ell}}(s) \le \deg_{\hat{\ell}}(t)$. But a strict inequality is not possible here (apply again the definition of Janet-like non-multiplicative powers). Hence, $\deg_{\hat{\ell}}(s) = \deg_{\hat{\ell}}(t)$. It is now possible to apply the same arguments to the next highest index ℓ and so on obtaining after finitely many steps that $\deg_j(s) = \deg_j(t)$ for all $j > \mathrm{cls}(t)$.

It now only remains to analyse the degrees at the variable $x_{\mathrm{cls}(t)}$. First, let us assume additionally that $\mathrm{cls}(s) < \mathrm{cls}(t)$. For any term $v \in \mathcal{C}_P(s,U) \cap \mathcal{C}_P(t,U)$, we have that $\deg_{\mathrm{cls}(t)}(v) \ge \deg_{\mathrm{cls}(t)}(t)$. If the strict inequality $\deg_{\mathrm{cls}(t)}(s) < \deg_{\mathrm{cls}(t)}(t)$ held, then, using the fact that $s \in U_{[\deg_{\mathrm{cls}(t)+1}(t),\dots,\deg_n(t)]}$ and the definition of Janet-like non-multiplicative powers, we would obtain the inequality $p(P,s,U,\mathrm{cls}(t)) \le$

$\deg_{\mathrm{cls}(t)}(t) - \deg_{\mathrm{cls}(t)}(s)$, in contradiction to the constraints on $\deg_{\mathrm{cls}(t)}(v)$ found above.

Finally, consider the case $\mathrm{cls}(s) = \mathrm{cls}(t)$. Then, by definition of the Pommaret-like division, there exists neither for s nor for t a non-multiplicative power of the variable $x_{\mathrm{cls}(t)}$ and, keeping in mind the conclusion of the second paragraph of this proof, we get that $\mathcal{C}_P(s, U) \subset \mathcal{C}_P(t, U)$ in the case that $\deg_{\mathrm{cls}(t)}(t) < \deg_{\mathrm{cls}(t)}(s)$ and $\mathcal{C}_P(t, U) \subset \mathcal{C}_P(s, U)$ in the case that $\deg_{\mathrm{cls}(t)}(s) < \deg_{\mathrm{cls}(t)}(t)$. This finishes the proof. □

Proposition 3.4.13. *The Pommaret-like division is not Noetherian.*

Proof. The monomial ideal $\mathcal{I} = \langle x_1 \rangle \trianglelefteq \mathbb{K}[x_1, x_2]$ does not possess a finite Pommaret-like basis. To see this, observe that for any finite set of terms $U \subset \mathcal{I}$ and for all terms $t \in U$ with $\deg_2(t) =: D$ maximal, $x_2 \in \mathrm{M}_{\mathcal{J}}(t, U)$, and hence $x_2 \in \mathrm{NMP}_P(t, U)$. For all terms $s \in U$ with the degree $\deg_2(s)$ non-maximal in U, we have $\deg_2(v) \leq D$ for all $v \in \mathcal{C}_P(s, U)$. So, for all terms $u \in \mathcal{I}$ with $\deg_2(u) > D$, we have $u \notin \mathcal{C}_P(U)$. □

Proposition 3.4.14. *The Pommaret-like division is continuous.*

Proof. The proof is a generalisation of the proof of the analogous result for the classical Pommaret division. Let $U \subset \mathcal{T}$ be a finite set of terms and $(t_1, \ldots, t_k) \in U^k$ a sequence of terms as in the definition of involutive-like continuity. Let $i \in \{1, \ldots, k-1\}$ be an arbitrary index and let $v_i \in t_i \cdot \mathrm{NMP}_P(t_i, U)$ be a prolongation with $v_i \in \mathcal{C}_P(t_{i+1}, U)$. We know that $v_i = t_i \cdot x_j^p$ for some $j > \mathrm{cls}(t_i)$. The divisibility of v_i by t_{i+1} implies that $\mathrm{cls}(t_{i+1}) \geq \mathrm{cls}(t_i)$ and if $\mathrm{cls}(t_{i+1}) = \mathrm{cls}(t_i)$, then $\deg_{\mathrm{cls}(t_i)}(t_{i+1}) \leq \deg_{\mathrm{cls}(t_i)}(t_i)$.

Finally, let us assume that $\mathrm{cls}(t_{i+1}) = \mathrm{cls}(t_i)$ and $\deg_{\mathrm{cls}(t_i)}(t_{i+1}) = \deg_{\mathrm{cls}(t_i)}(t_i)$. Then t_{i+1} is in the Janet class $U_{[\deg_{j+1}(t_i), \ldots, \deg_n(t_i)]}$. Indeed, for all indices $b > j$ the divisibility of v_i by t_{i+1} gives $\deg_b(t_{i+1}) \leq \deg_b(t_i)$ and if any of these inequalities were strict, then we would get a Pommaret-like non-multiplicative power for t_{i+1} at that index, in contradiction to $v_i \in \mathcal{C}_P(t_{i+1}, U)$. Moreover, a similar argument now gives that $\deg_j(t_{i+1}) = \deg_j(v_i)$. Thus, $t_{i+1} \succ_{\mathrm{lex}} t_i$. In conclusion, the sequence $(t_1, \ldots, t_k)$ must consist of pairwise distinct terms, finishing the proof. □

Proposition 3.4.15. *The Pommaret-like division is constructive.*

Proof. Let $U \subset \mathcal{T}$ be a finite set of terms, $t \in U$ a term in it and $s = t \cdot x_j^p$ a product of t with $x_j^p \in \mathrm{NMP}_P(t, U)$ such that $s \in \mathrm{MinObstr}_P(t, U)$. We must show that for no term $s' \in \mathcal{C}_P(U) \setminus U$ the relation $s \in \mathcal{C}_P(U \cup \{s'\})$ holds. Before coming to the main part of the proof, let us show that whenever $u \in \mathrm{Obstr}_P(U)$, we must have $u \in \mathcal{C}_P(v, U \cup \{v\})$ for any term $v \in \mathcal{C}_P(U)$ with $u \in \mathcal{C}_P(U \cup \{v\})$. To see this, first note that the only way how there can be a term $h \in U$ for which a term r exists with $r \in \mathcal{C}_P(h, U \cup \{v\}) \setminus \mathcal{C}_P(h, U)$ is if there exists an index $\ell > \mathrm{cls}(h)$ with $x_\ell \in \mathrm{M}_{\mathcal{J}}(h, U)$ such that $x_\ell \in \mathrm{NM}_{\mathcal{J}}(h, U \cup \{v\})$. This means that $\deg_\ell(h)$ is maximal among the x_j-degrees of the Janet class $U_{[\deg_{\ell+1}(h), \ldots, \deg_n(h)]}$, that there is the additional term v in the Janet class $(U \cup \{v\})_{[\deg_{\ell+1}(h), \ldots, \deg_n(h)]}$ and that $\deg_\ell(v) > \deg_\ell(h)$. There is a term $w \in U$ with $v \in \mathcal{C}_P(w, U)$. We now distinguish two cases. If $\mathrm{cls}(w) > \ell$, then, by definition of Pommaret-like non-multiplicative powers and by the fact that w divides the term $v \in (U \cup \{v\})_{[\deg_{\ell+1}(h),\ldots,\deg_n(h)]}$, we have that w is in the Janet class $U_{[\deg_{\mathrm{cls}\,(w)+1}(h),\ldots,\deg_n(h)]}$ and that $\deg_{\mathrm{cls}\,(w)}(w) \leq \deg_{\mathrm{cls}\,(w)}(h)$. But from this it follows in particular that $\mathrm{NMP}_P(w,U) = \mathrm{NMP}_P(h,U) \cap \mathbb{K}[x_{\mathrm{cls}\,(w)+1},\ldots,x_n]$. But, by construction, also

$$\mathrm{NMP}_P(h,U) \cap \mathbb{K}[x_{\mathrm{cls}\,(w)+1},\ldots,x_n] = \mathrm{NMP}_P(h,U\cup\{v\}) \cap \mathbb{K}[x_{\mathrm{cls}\,(w)+1},\ldots,x_n]\,.$$

We can conclude that $\mathcal{C}_P(h,U\cup\{v\}) \subseteq \mathcal{C}_P(w,U)$ and now the assumption $u \in \mathcal{C}_P(h,U\cup\{v\})$ would lead to the contradiction $u \in \mathcal{C}_P(w,U)$. Thus, it is not possible that $\mathrm{cls}\,(w) > \ell$. If, on the other hand, $\mathrm{cls}\,(w) \leq \ell$, then, arguing similarly as above, we get that w is in the Janet class $U_{[\deg_{\ell+1}(h),\ldots,\deg_n(h)]}$. Recall that v is in the Pommaret-like cone $\mathcal{C}_P(w,U)$ and that $\deg_\ell(v) > \deg_\ell(h)$. However, we also know that $\deg_\ell(h)$ is maximal among the x_j-degrees of the Janet class $U_{[\deg_{\ell+1}(h),\ldots,\deg_n(h)]}$. Thus $\deg_\ell(w) \leq \deg_\ell(h)$. But this implies that the exponent $p(P,w,U,\ell)$ of the Pommaret-like non-multiplicative power for w at x_ℓ is less or equal to $\deg_\ell(h) - \deg_\ell(w)$. Hence, no term in the Pommaret-like cone $\mathcal{C}_P(w,U)$ can have an x_ℓ-degree greater than $\deg_\ell(h)$. This contradicts $\deg_\ell(v) > \deg_\ell(h)$. Thus, we have shown that should it at all be possible to lift the minimal obstruction $s \in \mathrm{MinObstr}_P(U)$ by adding an element $s' \in \mathcal{C}_P(U)$ to U, we must have $s \in \mathcal{C}_P(s',U\cup\{s'\})$.

Let us return to the terms $s = t \cdot x_j^p$ and s'. We distinguish two cases, in accordance with the case distinction of the assignment of Pommaret non-multiplicative variables. First, assume that $x_j^p \in \mathrm{NMP}_J(t,U)$. By the definition of the Pommaret-like division, we know additionally that $j > \mathrm{cls}\,(t)$ and that $\mathrm{cls}\,(t) = \mathrm{cls}\,(s)$. Arguing by *reductio ad absurdum*, assume that there does exist a term $s' \in \mathcal{C}_P(U)$ with $s \in \mathcal{C}_P(U\cup\{s'\})$. Here again, we can distinguish two cases. First, let us assume that $\mathrm{cls}\,(s') \leq \mathrm{cls}\,(s)$. Then, by taking the projections of all the terms in U, of s and of s' to the subring $\mathbb{K}[x_{\mathrm{cls}\,(s)+1},\ldots,x_n]$, we obtain a configuration which is a counterexample to the constructivity of the Janet-like division. Indeed, denoting all projections by adding a bar on top of the symbols, we have then $\overline{s} \in \mathrm{MinObstr}_J(\overline{U})$, $\overline{s'} \in \mathcal{C}_J(\overline{U})$ and $\overline{s} \in \mathcal{C}_J(\overline{U}\cup\{\overline{s'}\})$. So we are left with the other case, i. e. with the case $\mathrm{cls}\,(s') > \mathrm{cls}\,(s)$. Note that by construction, s' is a proper divisor of s, as it must obviously be a divisor and the two terms cannot be equal since $s \in \mathrm{Obstr}_P(U)$. In particular, this implies $s' \prec_{\mathrm{lex}} s$. So there is a maximal index ℓ where the x_ℓ-degrees of s and s' differ and we have $\deg_\ell(s') < \deg_\ell(s)$. Again we must distinguish two cases. Firstly, let us assume that $\ell \leq \mathrm{cls}\,(s')$. An immediate consequence is $\ell \leq \mathrm{cls}\,(v)$ for any term v with $s' \in \mathcal{C}_P(v,U)$. But then also $s \in \mathcal{C}_P(v,U)$, which is not possible as $s \in \mathrm{Obstr}_P(U)$. Secondly, consider the case $\ell > \mathrm{cls}\,(s')$. Then $x_\ell \in \mathrm{NM}_{\mathcal{J}}(s',U\cup\{s'\})$, because either $\deg_\ell(s) = \deg_\ell(t)$ and then $t \in U$ causes x_ℓ to be Janet non-multiplicative for s' or $j = \ell$, $\deg_\ell(s) = \deg_\ell(t) + p(J,t,U,\ell)$ and the same term, say, $r \in U$, which causes the Janet-like non-multiplicative power for t at x_ℓ in U causes x_ℓ to be Janet non-multiplicative also for s' in $U \cup \{s'\}$. The exponent of the corresponding Janet-like non-multiplicative power then satisfies the inequality $p(J,s',U\cup\{s'\},\ell) \leq \deg_\ell(s) - \deg_\ell(s')$. This of course then also holds for the induced Pommaret-like non-multiplicative power. This contradicts $s \in \mathcal{C}_P(s', U \cup \{s'\})$. The analysis of the case $x_j^p \in \mathrm{NMP}_J(t, U)$ is now finished.

Let us turn to the analysis of the case $x_j \in \mathrm{M}_{\mathcal{J}}(t, U)$. Here $p = 1$ and $x_j^p = x_j$. So, $s = t \cdot x_j$. By the definition of the Pommaret-like division, we know additionally that $j > \mathrm{cls}\,(t)$ and that $\mathrm{cls}\,(t) = \mathrm{cls}\,(s)$. Arguing again by *reductio ad absurdum*, assume that there does exist a term $s' \in \mathcal{C}_P(U)$ with $s \in \mathcal{C}_P(U \cup \{s'\})$. Similarly to the situation in the last paragraph, we must have $s' \prec_{\mathrm{lex}} s$ and there is a maximal index ℓ where the x_ℓ-degrees of s' and s differ. We know then that $\deg_\ell(s') < \deg_\ell(s)$. We now distinguish several cases which reflect the relation of the indices j and ℓ.

The first main case is $\ell > j$. Then it follows that s' is in the Janet class $(U \cup \{s'\})_{[\deg_{\ell+1}(t),\dots,\deg_n(t)]}$ and that $\deg_\ell(t) = \deg_\ell(s) > \deg_\ell(s')$. Hence $x_\ell \in \mathrm{NM}_{\mathcal{J}}(s', U \cup \{s'\})$ and $p(J, s', U \cup \{s'\}, \ell) \leq \deg_\ell(s) - \deg_\ell(s')$ leading to a contradiction if this Janet-like non-multiplicative power is also a Pommaret-like non-multiplicative power for s'. Otherwise, we would have $\mathrm{cls}\,(s') \geq \ell$ and from this it is not hard to show that for the term $w \in U$ with $s' \in \mathcal{C}_P(w, U)$ we would also have $s \in \mathcal{C}_P(w, U)$, in contradiction to $s \in \mathrm{Obstr}_P(U)$.

The second main case is $\ell = j$. It follows that s' is in the Janet class $(U \cup \{s'\})_{[\deg_{j+1}(t),\dots,\deg_n(t)]}$. Again, if $\mathrm{cls}\,(s') \geq j$, then it is not hard to obtain a contradiction to $s \in \mathrm{Obstr}_P(U)$. So we may assume that $\mathrm{cls}\,(s') < j$. Since $\deg_j(s) = \deg_j(t) + 1$ and $s \in \mathcal{C}_P(s', U \cup \{s'\})$, we must have $\deg_j(s') = \deg_j(t)$. But this implies that $x_j \in \mathrm{M}_{\mathcal{J}}(s', U \cup \{s'\})$ and we get the Pommaret-like non-multiplicative power $x_j \in \mathrm{NMP}_P(s', U \cup \{s'\})$, in contradiction to $s \in \mathcal{C}_P(s', U \cup \{s'\})$.

The third main case is $\ell < j$. We then get that s' is in the Janet class $(U \cup \{s'\})_{[\deg_j(t)+1,\deg_{j+1}(t),\dots,\deg_n(t)]}$. Again, if $\mathrm{cls}\,(s') \geq j$, then it is not hard to obtain a contradiction to $s \in \mathrm{Obstr}_P(U)$. So we may assume that $\mathrm{cls}\,(s') < j$. But then again, we know that there is a term $w \in U$ with $s' \in \mathcal{C}_P(w, U)$ and since this term divides s', one can see quite easily that it must belong to the Janet class $U_{[\deg_{j+1}(t),\dots,\deg_n(t)]}$. If it has $\mathrm{cls}\,(w) \geq j$, then again it is not hard to obtain a contradiction to $s \in \mathrm{Obstr}_P(U)$, and if $\mathrm{cls}\,(w) < j$, then, via the fact that $x_j \in \mathrm{M}_{\mathcal{J}}(t, U)$, we get that $\deg_j(w) \leq \deg_j(t)$ and thus a contradiction to $s' \in \mathcal{C}_P(w, U)$. This finishes the proof. □

Proposition 3.4.16. *The Pommaret-like division is related to the Pommaret division as follows:*

(i) For each term $t \in \mathcal{T}$ in a finite set $U \subset \mathcal{T}$, we have $\mathcal{C}_{\mathcal{P}}(t, U) \subseteq \mathcal{C}_P(t, U)$.

(ii) Every Pommaret basis of a monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is also a Pommaret-like basis.

(iii) From a Pommaret-like basis U of the monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$, one can obtain a Pommaret basis U' of the same ideal as follows:

$$U' = \left\{ t \cdot x^\mu \mid t \in U \wedge x^\mu \mid \Pi_{x_a^{p_a} \in \mathrm{NMP}_{\mathcal{P}}(t,U)} x_a^{p_a - 1} \right\} .$$

(iv) A monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is quasi-stable, if and only if it possesses a finite Pommaret-like basis.

Proof. Item (i) follows directly from the definitions; item (ii) is an immediate consequence of it.

Item (iii) follows, if we can show that $\mathcal{C}_P(U) \subseteq \mathcal{C}_{\mathcal{P}}(U')$. Let $u \in \mathcal{C}_P(U)$ be an arbitrary term in the Pommaret-like span. Then there exists a term $t \in U$, a divisor x^μ of $\Pi_{x_a^{p_a} \in \mathrm{NMP}_{\mathcal{P}}(t,U)} x_a^{p_a-1}$ and a term $x^\rho \in \mathbb{K}[x_1, \dots, x_{\mathrm{cls}\,(t)}]$ such that $u = t \cdot x^\mu \cdot x^\rho$. We have to show that there is a Pommaret divisor of u in the set U'. We know that $t \cdot x^\mu \in U'$. It is clear that $\mathrm{cls}\,(t \cdot x^\mu) = \mathrm{cls}\,(t)$. Hence, $u \in \mathcal{C}_{\mathcal{P}}(t \cdot x^\mu)$ and we have proved (iii).

Item (iv) is a direct consequence of (ii) and (iii), as a monomial ideal is quasi-stable, if and only if it possesses a finite Pommaret basis. $\square$

Lemma 3.4.17. *A finite set of terms $U \subset \mathcal{T}$ is Pommaret-like autoreduced, if and only if it is Pommaret autoreduced.*

Proof. The only if direction is obvious. So let $U \subset \mathcal{T}$ be a finite set of terms which is Pommaret autoreduced. We want to show that it is also Pommaret-like autoreduced. We argue by *reductio ad absurdum.* Assume that U is Pommaret, but not Pommaret-like autoreduced. Then there exist two terms $s \neq t \in U$ such that $s \in \mathcal{C}_P(t, U)$. Let $k = \mathrm{cls}\,(t)$. We know that $\deg_\ell(s) \geq \deg_\ell(t)$ for each index $k < \ell \leq n$. There must exist an index $k < j \leq n$ with $\deg_j(s) > \deg_j(t)$, since otherwise t would be a Pommaret divisor of s, contradicting the assumed Pommaret autoreducedness. We pick the maximal such index j. Then there exists a Janet-like non-multiplicative power $x_j^{p(\mathcal{J},t,U,j)} \in \mathrm{NMP}_{\mathcal{J}}(t, U)$ with $1 \leq p(\mathcal{J}, t, U, j) \leq \deg_j(s) - \deg_j(t)$. This gives also a Pommaret-like non-multiplicative power for t at x_j with the same exponent. Hence, $s \notin \mathcal{C}_P(t, U)$ contradicting our assumptions. $\square$

Corollary 3.4.18. *The Pommaret-like division satisfies the strong basis property.*

Proof. Let the finite set of terms $U \subset \mathcal{T}$ be a weak Pommaret-like basis of the monomial ideal $\langle U \rangle$. If it is a strong basis, then we are done. Otherwise, it is not Pommaret-like autoreduced, and hence it is also not Pommaret autoreduced by Lemma 3.4.17. We claim that the Pommaret autoreduction $\tilde{U} \subset U$ is a strong Pommaret-like basis of $\langle U \rangle$. More precisely, we will show that $\mathcal{C}_P(u, \tilde{U}) = \mathcal{C}_P(u, U)$ for each term $u \in \tilde{U}$ which is equivalent to $\mathrm{NMP}_P(u, \tilde{U}) = \mathrm{NMP}_P(u, U)$. The latter statement can be reduced to an analysis of Janet-like non-multiplicative powers: We have to show that

$$\mathrm{NMP}_J(u, \tilde{U}) \cap \mathcal{K}[x_{\mathrm{cls}\,(u)+1}, \dots, x_n] = \mathrm{NMP}_J(u, U) \cap \mathcal{K}[x_{\mathrm{cls}\,(u)+1}, \dots, x_n]\ .$$

The set $\tilde{U}$ arises from U by removing elements which possess strict Pommaret divisors in U. It is clear that the removal of strict Pommaret multiples of u does not change the Janet-like non-multiplicative powers of u lying in $\mathbb{K}[x_{\mathrm{cls}\,(u)+1}, \dots, x_n]$. Let $v \in \tilde{U} \setminus \{u\}$ be any term for which a strict Pommaret multiple $t \in U \setminus \tilde{U}$ has been removed. If this removal would change a Janet-like non-multiplicative power of u lying in $\mathbb{K}[x_{\mathrm{cls}\,(u)+1}, \dots, x_n]$, then there would be some index $\ell > \mathrm{cls}\,(u)$ such that t lies in the Janet class $U_{[\deg_{\ell+1}(u), \dots, \deg_n(u)]}$. Since the removal of t changes the non-multiplicative power of u at x_ℓ and we know that $\deg_\ell(v) \leq \deg_\ell(t)$, we must have $\deg_\ell(v) \leq \deg_\ell(u) < \deg_\ell(t)$. Since v is a Pommaret divisor of t, it follows that v is in the same Janet-class of U as u and t and additionally, $\mathrm{cls}\,(v) \geq \ell$. This in turn shows that v is a strict Pommaret divisor of u, which is impossible, since u and v both survived the Pommaret autoreduction of U. □

Theorem 3.4.19. *The Pommaret-like and the Janet-like divisions are related as follows:*

(i) Let $U \subset \mathcal{T}$ be a finite set of terms which is autoreduced with respect to the Pommaret-like division. Then $\mathcal{C}_P(t,U) \subseteq \mathcal{C}_J(t,U)$ for each $t \in U$.

(ii) Let $U \subset \mathcal{T}$ be a Pommaret-like basis of the monomial ideal $\langle U \rangle$. Then U is also a Janet-like basis of the same ideal.

(iii) Any minimal Janet-like basis is Pommaret-like autoreduced.

(iv) The unique minimal Janet-like basis of a quasi-stable monomial ideal is also a Pommaret-like basis of the same ideal.

(v) In the situation of (ii), the set U is the unique minimal Pommaret-like basis of $\langle U \rangle$, if and only if it is the unique minimal Janet-like basis of this ideal.

Proof. For item (i), let U be a finite Pommaret-like autoreduced set of terms, $t \in U$ a term and $j \leq \mathrm{cls}\,(t)$ an index. Then x_j is Pommaret-like multiplicative for t and we must show that it is also Janet-like multiplicative. If not, then there exists a term $s \in U$ in the Janet class $U_{[\deg_{j+1}(t),\dots,\deg_n(t)]}$ with $\deg_j(s) > \deg_j(t)$. By the definition of Pommaret-like multiplicative powers, we see that $s \in \mathcal{C}_P(t,U) \setminus \{t\}$, in contradiction to the Pommaret-like autoreducedness of U. It only remains to observe that, by the definition of Pommaret-like multiplicative powers, it is clear that for indices $j > \mathrm{cls}\,(t)$ the Janet-like non-multiplicative powers of t are either identical to the Pommaret non-multiplicative powers or that the Pommaret non-multiplicative powers are linear while the Janet-like division does not pose any restriction on the given variable. This concludes the proof of the first item.

Item (ii) is a direct consequence of (i), because a Pommaret-like basis is by definition autoreduced with respect to the Pommaret-like division.

For item (iii), we only need to show that any minimal Janet-like basis is Pommaret autoreduced in view of Lemma 3.4.17. So for a given minimal Janet-like basis $U \subset \mathcal{T}$ of the monomial ideal $\mathcal{I} = \langle U \rangle$, we must show that the Pommaret autoreduction of U is still a Janet-like basis of $\mathcal{I}$. If U is already Pommaret autoreduced, there is nothing to prove. If not, then there exists a disjoint partition $U = U_1 \sqcup U_2 \sqcup \ldots \sqcup U_r$ such that, $\mathcal{C}_\mathcal{P}(s) \cap \mathcal{C}_\mathcal{P}(t) = \emptyset$ for any two indices $i \neq j \in \{1,\dots,r\}$ and any two terms $s \in U_i$ and $t \in U_j$ and for each $i \in \{1,\dots,r\}$ there exists a unique term $t_i \in U_i$ such that $U_i \subset \mathcal{C}_\mathcal{P}(t_i)$, i.e. t_i is a strict Pommaret divisor of every term $s \in U_i \setminus \{t_i\}$. We must show that $\{t_1,\dots,t_r\}$ is still a Janet-like basis of $\langle U \rangle$. It suffices to show that we have $\mathcal{C}_J(t_i,\{t_1,\dots,t_r\}) \supseteq \bigcup_{s \in U_i} \mathcal{C}_J(s,U)$ for each i. To this end, fix an index i and look at the Janet-like non-multiplicative powers of t_i for a variable x_ℓ with $\ell > \mathrm{cls}\,(t_i)$. We have $p(J,s,U,\ell) = p(J,t_i,U,\ell)$ for each $s \in U_i$, since $\deg_\ell(s) = \deg_\ell(t_i)$ by Pommaret divisibility. Hence, using [50, Prop. 3], $p(J,s,U,\ell) \leq p(J,t_i,\{t_1,\dots,t_r\},\ell)$ for all $s \in U_i$. Since $\{t_1,\dots,t_r\}$ is Pommaret autoreduced and hence also Pommaret-like autoreduced, using (i), we see that there are no Janet-like non-multiplicative powers for t_i in $\{t_1,\dots,t_r\}$ at any variable x_ℓ

with $\ell \leq \operatorname{cls}(t_i)$. Putting everything together, we get $\mathcal{C}_J(t_i, \{t_1, \dots, t_r\}) \supseteq \mathcal{C}_J(s, U)$ for all $s \in U_i$, which suffices to prove our claim.

For item (iv), let U be the minimal Janet-like basis of the quasi-stable monomial ideal $\langle U \rangle$. Then U is Pommaret autoreduced by (iii). By Proposition 3.4.10 (iii), we can construct a Janet basis $\overline{U} \supseteq U$ of $\langle U \rangle$. We claim that $\overline{U}$ is also Pommaret autoreduced. Indeed, let $\overline{s}, \overline{t} \in \overline{U}$ be two distinct terms which arise as multiples of the terms $s, t \in U$. If $s = t$, then it is not hard to show that $\mathcal{C}_{\mathcal{P}}(\overline{s}) \cap \mathcal{C}_{\mathcal{P}}(\overline{t}) = \emptyset$. So assume that $s \neq t$. Without loss of generality, let $\operatorname{cls}(s) \leq \operatorname{cls}(t) = k$. Then s cannot be in the Janet class $U_{[\deg_k(t), \dots, \deg_n(t)]}$, because otherwise t would be a strict Pommaret divisor of s, contradicting the Pommaret autoreducedness of U. Hence, there is a maximal index ℓ with $k \leq \ell \leq n$ where $\deg_\ell(s) \neq \deg_\ell(t)$.

If now $\ell = k = \operatorname{cls}(t)$, then we must have $\deg_\ell(s) < \deg_\ell(t)$ and $\operatorname{cls}(s) < k$, again since U is Pommaret autoreduced. Hence, s has a Janet-like non-multiplicative power with respect to the set U at x_ℓ with exponent $p(J, s, U, \ell) \leq \deg_\ell(t) - \deg_\ell(s)$. By construction of the Janet basis $\overline{U}$, this implies $\deg_\ell(\overline{s}) < \deg_\ell(t) \leq \deg_\ell(\overline{t})$ and, since $\operatorname{cls}(\overline{s}) = \operatorname{cls}(s)$, we get $\mathcal{C}_{\mathcal{P}}(\overline{s}) \cap \mathcal{C}_{\mathcal{P}}(\overline{t}) = \emptyset$. If $\ell > k = \operatorname{cls}(t)$, then $\ell > \max(\operatorname{cls}(s), \operatorname{cls}(t))$ and, similarly as in the last paragraph, we find that $\mathcal{C}_{\mathcal{P}}(\overline{s}) \cap \mathcal{C}_{\mathcal{P}}(\overline{t}) = \emptyset$. Hence, we have proved that the Janet basis $\overline{U}$ is Pommaret autoreduced.

Since the ideal $\langle U \rangle$ is quasi-stable, $\overline{U}$ must be the Pommaret basis of $\langle U \rangle$. This implies that for each term $\overline{t} \in \overline{U}$, its Janet multiplicative variables with respect to $\overline{U}$ agree with its Pommaret multiplicative variables [50]. On the other hand, it is not hard to show that the Janet non-multiplicative variables of $\overline{t} \in \overline{U}$ also agree with the Janet non-multiplicative variables of the term $t \in U$ used for the construction of $\overline{t}$. This finally shows that for all $u \in U$, $\mathcal{C}_P(u, U) = \mathcal{C}_J(u, U)$, which means that U is a Pommaret-like basis of $\langle U \rangle$.

Item (v) follows from (ii) and (iv). □

3.5 Syzygies of Involutive-like Bases

In the theory of involutive bases, it is well-known that from a given Pommaret or Janet basis, respectively, of a polynomial ideal, one can obtain a Pommaret or Janet basis, respectively, of the syzygy module of this basis with respect to a suitable module term ordering [97, Sec. 5.4]. The goal of this section is to generalise these results also to Pommaret-like and Janet-like involutive bases. We start with an analysis of the set of non-multiplicative powers associated to some term t contained in a finite set of terms U which is not assumed to be an involutive-like basis.

Lemma 3.5.1. *Let a term $t \in \mathcal{T}$ contained in a finite set of terms $U \subset \mathcal{T}$ be given. Then the set $\mathrm{NMP}_P(t, U)$ is a Pommaret-like basis of the monomial ideal $\langle \mathrm{NMP}_P(t, U) \rangle$ generated by it and the set $\mathrm{NMP}_J(t, U)$ is a Janet-like basis of the monomial ideal $\langle \mathrm{NMP}_J(t, U) \rangle$.*

Proof. Let us first consider the set of non-multiplicative powers $\mathrm{NMP}_P(t, U)$, which is of the form $\left\{x_a^{p(a)}, x_{a+1}^{p(a+1)}, \dots, x_n^{p(n)}\right\}$ where $a = \operatorname{cls}(t) + 1$ and $p(b) \in \mathbb{Z}_{>0}$ for all $a \le b \le n$. Let $x_j^p \in \mathrm{NMP}_P(t,U)$. Then one can easily see that $\mathrm{cls}\,(x_j^p) = j$, $\mathrm{NM}_{\mathcal{J}}\big(x_j^p, \mathrm{NMP}_P(t,U)\big) = \{x_{j+1}, x_{j+2}, \ldots, x_n\}$, and

$$\begin{aligned}\mathrm{NMP}_P\big(x_j^p, \mathrm{NMP}_P(t,U)\big) &= \mathrm{NMP}_J\big(x_j^p, \mathrm{NMP}_P(t,U)\big)\\ &= \big\{v \in \mathrm{NMP}_P(t,U) \mid \mathrm{cls}\,(v) > j\big\}\ .\end{aligned}$$

Trivially, x_j^p is Pommaret-like multiplicative for all terms in $v \in \mathrm{NMP}_P(t,U)$ with $\mathrm{cls}\,(v) > j$ and hence $x_j^p \cdot \mathrm{NMP}\big(x_j^p, \mathrm{NMP}_P(t,U)\big) \subset \mathcal{C}_P\big(\mathrm{NMP}_P(t,U)\big)$, proving the statement for the Pommaret-like division.

We now consider $\mathrm{NMP}_J(t,U)$, which is of the form $\big\{x_a^{p(a)} \mid x_a \in \mathrm{NM}_{\mathcal{J}}(t,U)\big\}$. Let $x_j^p \in \mathrm{NMP}_J(t,U)$ be a non-multiplicative power. One can see easily that

$$\begin{aligned}\mathrm{NM}_{\mathcal{J}}\big(x_j^p, \mathrm{NMP}_J(t,U)\big) &= \big\{x_b \in \mathrm{NM}_{\mathcal{J}}(t,U) \mid b > j\big\}\ ,\\ \mathrm{NMP}_J\big(x_j^p, \mathrm{NMP}_J(t,U)\big) &= \big\{x_b^{p(b)} \in \mathrm{NMP}_J(t,U) \mid b > j\big\}\ .\end{aligned}$$

Furthermore, x_j^p is Janet-like multiplicative for all terms $x_b^{p(b)} \in \mathrm{NMP}_J(t,U)$ with $b > j$ and hence $x_j^p \cdot \mathrm{NMP}_J\big(x_j^p, \mathrm{NMP}_J(t,U)\big) \subset \mathcal{C}_J\big(\mathrm{NMP}_J(t,U)\big)$, proving the statement for the Janet-like division. $\square$

Up to now, we have only considered involutive-like bases of monomial ideals. For general polynomial ideals in $\mathcal{P}$, we use the following definition.

Definition 3.5.2. *Let L be an involutive-like division, $\prec$ a term ordering on $\mathcal{T}$ and $\mathcal{I} \trianglelefteq \mathcal{R}$ a polynomial ideal. Then a finite set $H \subset \mathcal{I}\setminus\{0\}$ is called an L-*involutive-like basis *of $\mathcal{I}$ with respect to the term ordering $\prec$, if the set of leading terms $\mathrm{lt}(H)$ is a strong L-involutive-like basis of the leading ideal $\mathrm{lt}(\mathcal{I})$ and $|H| = |\,\mathrm{lt}(H)|$ (i. e. the leading terms of the elements of H are distinct).*

Remark 3.5.3. *Let $H \subset \mathcal{T}$ be an L-involutive like basis for the ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ with respect to some involutive-like division L and some term ordering $\prec$. Then H is also a Gröbner basis of $\mathcal{I}$ for $\prec$, since $\mathrm{lt}(H)$ is a generating system of $\mathrm{lt}(\mathcal{I})$. It is also straightforward to introduce the notion of an involutive-like standard representation for every polynomial in an ideal generated by an involutive-like basis and to show that it is unique.*

Recall Construction 2.2.3, which, given a Gröbner basis H of a polynomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$, yields a module term ordering $\prec_H$ on the free module $\mathcal{R}^{|H|}$ and a Gröbner basis of the syzygy module $\mathrm{Syz}(H) \subset \mathcal{R}^{|H|}$ for this ordering $\prec_H$.

Let us now consider the special case when H is an L-involutive-like basis of the polynomial ideal $\langle H\rangle$ for a continuous involutive-like division L. Analogously to the case of involutive bases, we can construct a directed graph with one node for each leading term $\mathrm{lt}(h_j) \in \mathrm{lt}(H)$ and a directed edge from $\mathrm{lt}(h_j)$ to $\mathrm{lt}(h_i)$ exactly when there is an L-non-multiplicative power $x_k^p \in \mathrm{NMP}_L\big(\mathrm{lt}(h_j), \mathrm{lt}(H)\big)$ such that $x_k^p \cdot \mathrm{lt}(h_j) \in \mathcal{C}_L\big(\mathrm{lt}(h_i), \mathrm{lt}(H)\big)$. We call it the *$L$-graph of* $\mathrm{lt}(H)$. Note that it is acyclic because of the continuity of L. This leads to the concept of L-orderings.

Definition 3.5.4. *Let $U \subset \mathcal{T}$ be a strong L-involutive-like basis for the monomial ideal $\langle U \rangle$ for a continuous involutive-like division L. Then an* L-ordering *of U is an enumeration $U = \{u_1, \dots, u_r\}$ for which $i < j$ whenever there exists a non-multiplicative power $x_k^p \in \mathrm{NMP}_L(u_i, U)$ such that $x_k^p \cdot u_i \in \mathcal{C}_L(u_j, U)$.*

The following proposition is immediate from the above discussion.

Proposition 3.5.5. *Let the involutive-like division L be continuous. Then for each strong L-involutive-like basis U there exists an L-ordering.*

We continue the analysis of the syzygies of an L-involutive-like basis H of the polynomial ideal $\langle H \rangle$ with respect to a continuous involutive-like division L. Assume that $H = \{h_1, \dots, h_r\}$ is enumerated according to an L-ordering. Let $x_k^p \in \mathrm{NMP}_L\big(\mathrm{lt}(h_i), \mathrm{lt}(H)\big)$ be a non-multiplicative power of a leading term $\mathrm{lt}(h_i) \in \mathrm{lt}(H)$. Then there exists a unique generator $h_j \in H \setminus \{h_i\}$ such that $x_k^p \in \mathcal{C}_L\big(\mathrm{lt}(h_j), \mathrm{lt}(H)\big)$. The polynomial $h_{ij} := x_k^p h_i - c\big(x_k^p \,\mathrm{lt}(h_i)/\,\mathrm{lt}(h_j)\big) h_j \in \mathcal{I}$ has, for a suitably chosen scalar $c \in \mathcal{K}$, the leading term $\mathrm{lt}(h_{ij}) \prec x_k^p \,\mathrm{lt}(h_i)$. Then, the standard representation obtained by involutive-like reduction $h_{ij} = \sum_{h_\alpha \in H} q_\alpha \cdot h_\alpha$ yields the syzygy $\mathbf{S}_{i;k} = x_k^p \mathbf{e}_i - c\big(x_k^p \,\mathrm{lt}(h_i)/\,\mathrm{lt}(h_j)\big)\mathbf{e}_j - \sum_{h_\alpha \in H} q_\alpha \mathbf{e}_\alpha$. We have $\mathrm{lt}(\mathbf{S}_{i;k}) = x_k^p \mathbf{e}_i$ with respect to the Schreyer ordering $\prec_H$. We now show that in the case that L is either the Pommaret-like or the Janet-like division the collection of the thus obtained syzygies is an involutive-like basis of the syzygy module $\mathrm{Syz}(H)$.

Theorem 3.5.6. *Let $H = \{h_1, \dots, h_r\} \subset \mathcal{R}$ be a strong Janet-like or Pommaret-like, respectively, basis of the polynomial ideal $\langle H \rangle$ enumerated according to a J- or P-ordering, respectively. Then the set*

$$H_{\mathrm{Syz}} = \Big\{ \mathbf{S}_{i;k} \mid 1 \le i \le r \,\wedge\, \exists x_k^p \in \mathrm{NMP}_L\big(\mathrm{lt}(h_i), \mathrm{lt}(H)\big) \Big\}$$

of syzygies induced by non-multiplicative powers is a Janet-like or Pommaret-like, respectively, basis of the syzygy module $\mathrm{Syz}(H)$ with respect to the Schreyer module term ordering $\prec_H$.

Proof. By construction, $H_{\mathrm{Syz}} \subseteq \mathrm{Syz}(H)$. Let $\mathbf{0} \neq \mathbf{S} = \sum_{i=1}^{|H|} s_i \mathbf{e}_i \in \mathrm{Syz}(H)$ be any non-zero syzygy. Then there exists a module term $x^\mu \mathbf{e}_\ell \in \mathrm{supp}(\mathbf{S})$ such that $x^\mu \,\mathrm{lt}(h_\ell) \notin \mathcal{C}_L\big(\mathrm{lt}(h_\ell), \mathrm{lt}(H)\big)$, as otherwise the leading terms of the summands $s_i h_i$ are distinct and the highest appearing term cannot cancel out. Thus, a non-multiplicative power $x_k^p \in \mathrm{NMP}_L\big(\mathrm{lt}(h_\ell), \mathrm{lt}(H)\big)$ exists such that x_k^p divides x^μ and thus $\mathrm{lt}(\mathbf{S}_{\ell;k})$ divides $x^\mu \mathbf{e}_\ell$. This means that any non-zero syzygy is reducible with respect to H_{Syz} which implies the existence of a standard representation of $\mathbf{S}$ with respect to H_{Syz}. Hence, H_{Syz} is a Gröbner basis of $\mathrm{Syz}(H)$ with respect to $\prec_H$. It is in fact an involutive-like basis because of Lemma 3.5.1. □

3.6 Notes

In this chapter, we discussed and compared different approaches to complementary decompositions, in particular with respect to their complexity. It turned out that the oldest approach, namely the one presented by Janet almost 100 years ago, is the most efficient one, in particular in the novel optimised form based on Janet-like bases presented here. As already remarked above, the presented complexity of the algorithms using Janet(-like) bases includes and is dominated by the cost for computing the basis; the decomposition itself requires essentially only the operations needed to write it down.

When Gerdt and Blinkov [50, 49] introduced Janet-like bases, they were mainly concerned with the algorithmic advantages of this "condensed" form of Janet bases. But they also noted that it is possible to read off Hilbert function and polynomial directly from a Janet-like basis. However, they did not use a complementary decomposition, but instead computed the Hilbert function as the difference of the volume functions of the full polynomial ring and the ideal, respectively (this approach is applicable for any involutive basis, as one needs only a cone decomposition of the ideal itself). Furthermore, they did not use any compressed form, but provided for the volume function of the ideal a sum containing as many summands as the Janet basis obtained by expanding the Janet-like basis contains elements. Thus strictly speaking, they did not really use the Janet-like basis, but derived exactly the same expression one obtains from the Janet basis. By contrast, Proposition 3.2.15 provides an expression for the Hilbert function that contains generally much less summands than the one by Gerdt and Blinkov. Although it is not completely explicit, we have shown that it can be evaluated very efficiently.

We did not consider specifically Rees decompositions [86], i. e. decompositions where the sets of the multiplicative variables are always of the form $\{x_1, x_2, \ldots, x_k\}$ for some index k depending on the vertex of the cone, which are of interest for some theoretical applications. Their construction requires some generic choices and an expensive algorithm was presented by Sturmfels and White [105]. It is much simpler to obtain Rees decompositions directly from a Pommaret basis using Janet's algorithm. The generic choices then appear in the construction of the Pommaret basis. A rather redundant Rees decomposition can be immediately written down in closed form from any Pommaret basis [97, Prop. 5.1.6]. However, the decompositions obtained by Janet's algorithm contain usually much less cones.

Our results on Hironaka's construction are mainly of theoretical interest. First of all, they clarify the meaning of his genericity condition (called Hironaka's box condition in [7] where it plays an important role) showing again that Hironaka worked implicitly with Pommaret bases. Then they show that Hironaka's construction actually provides more than just a complementary decomposition. As a by-product, it determines Pommaret bases of the chain of saturations associated with any quasi-stable ideal and this chain contains all information required to write down an irreducible primary decomposition of the ideal.

We studied a not much known recursive criterion for Janet bases already proven by Janet himself. We provided a slightly modified form of it with a novel proof and exploited this for the design of a novel algorithm for the construction of Janet bases. Right now, we cannot make any statements about the efficiency of this algorithm compared to the classical one. From a theoretical point of view it is interesting to note that the novel approach also leads to an algorithm for turning a given Janet basis into a minimal one. To the best of our knowledge, this is the first such algorithm; previous algorithms only permit the direct construction of minimal Janet bases and cannot really exploit the previous knowledge of a (non-minimal) Janet basis.

We extended the recursive approach also to Pommaret bases. In their construction, a crucial part is always to determine "good" coordinates. We showed that the novel recursive criterion also permits the effective construction of such coordinates. The basic strategy of the algorithm and of the termination proof is still the same as in [60]. It still remains to compare the relative performance of the old and the novel approach in practical computations. There are some indications that the novel approach might be more efficient, as Algorithm 16 e. g. naturally incorporates permutations (which help to preserve sparsity) and groups several elementary moves into one transformations (which reduces the number of required Gröbner bases computations). However, a decisive factor is how sparse the finally obtained linear change really is and this is difficult to predict for arbitrary inputs.

Another crucial factor is how the next linear change is chosen. In the pseudocode presented in [60] for the old approach simply the first detected obstruction is used. One could imagine here a number of heuristics which may lead to a better performance. In the novel approach, the choice is more strongly dictated by the form of our recursive criterion. But there is still some freedom. Corollary 3.3.36 essentially requires to completely traverse a tree of ideals with at most n levels where n is the number of variables in the original polynomial ring. At each level, the number of variables is reduced by one. The way Algorithm 15 proceeds through this tree corresponds to a depth-first traversal (this represents the easiest way to provide some pseudocode). However, it appears to be more in line of our understanding of the recursive criterion to perform a breadth-first traversal. This would require the explicit use of a queue data structure and thus was omitted here. We expect that it might actually be more efficient in practice, but again nothing precise can be said without actual benchmarks.

Chapter 4

Relative Gröbner and Involutive-like Bases

Let $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$. A well-known application of Gröbner and involutive bases is the construction of free resolutions of finitely generated $\mathcal{R}$-modules, see e. g. [10, 30, 69, 94, 96]. As already the title of Buchberger's thesis [18] indicates, Gröbner bases are also used for effective computations in the quotient ring $\mathcal{R}/\mathcal{I}$ where $\mathcal{I} \lhd \mathcal{R}$ is an ideal. We are interested in the construction of free resolutions of finitely generated $\mathcal{R}/\mathcal{I}$-modules. In this chapter, we extend the concept of Gröbner bases to ideals in and modules over $\mathcal{R}/\mathcal{I}$ introducing *relative Gröbner bases*. Such an extension is not new. It was already mentioned by Spear [102] and discussed by Zacharias [107]. Some ideas are treated in textbooks on Gröbner bases like [9]. La Scala and Stillman [69] sketched the necessary theoretical background and implemented procedures in Macaulay2 not only for computing Gröbner bases, but also for free resolutions.

Mora (partially with Ceria) [24, 79, 80] developed a very general framework for a theory of Gröbner bases in *effectively given rings* which includes besides quotient rings also polynomials over coefficient rings and even allows for non-commutativity. Within this framework, he also considered the construction of syzygies (which forms the foundation of computing resolutions) based on so-called Gebauer-Möller sets. While such a general framework is of great theoretical interest, as it unifies many different concepts and ideas, its downside is a considerable overhead of abstract constructions (see e. g. [79, Def. 1]) which probably makes an efficient implementation rather difficult. We therefore believe that, because of the great importance of the special case of $\mathcal{R}/\mathcal{I}$-modules for algebraic geometry, it is worth while considering it directly and not as part of a much more general theory. Indeed, for most of the algorithms proposed in this work, proof-of-concept implementations have been provided in Maple and their codes are freely available at the website `https://amirhashemi.iut.ac.ir/softwares`. These implementations are not intended for large scale computations and hence we do not include benchmarks, but we want to stress that one advantage of our approach is that the adaption of an optimised production code to the relative case along the ideas presented in this work would not require much work, whereas an implementation of the framework of Mora would have to start essentially from scratch.

For the here considered specific case of ideals in $\mathcal{R}/\mathcal{I}$ (the extension to modules is straightforward and not detailed), we present a variant of Buchberger's algorithm for the computation of relative Gröbner bases including some basic optimisations which may be interpreted as explicitly providing a smaller Gebauer-Möller set. However, our main novel contribution is the development of a theory of *relative involutive bases* which is different from the one recently presented by Mora and Ceria [25] for the above mentioned general framework. They leave the basic notions of involutive bases unchanged and generalise the proofs to their more complicated (possibly non-commutative) underlying arithmetics, whereas we adapt already the definition of an involutive division to the relative case. And while in [25] some questions remain open, we can provide a complete theory. We furthermore generalise the well-known combinatorial notion of a quasi-stable (monomial) ideal to *relative quasi-stable ideals* for the computation of relative Pommaret bases in $\mathcal{R}/\mathcal{I}$.

This chapter is structured as follows. In Section 4.1 we thoroughly investigate Gröbner bases for ideals in quotient rings $\mathcal{R}/\mathcal{I}$. We call these bases *relative Gröbner bases*. In Subsection 4.1.1, we introduce relative Gröbner bases and establish the "relative" analogue to the Schreyer construction for ideals in quotient rings. In Subsection 4.1.2, we provide the basics for the construction of relative Gröbner bases developing criteria analogous to Buchberger's criterion for a Gröbner basis and Buchberger's (first and second) criteria for avoiding superfluous reductions to zero. Section 4.2 is devoted to the study of relative involutive bases. In Subsection 4.2.1, we introduce the notion of a relative involutive division and study the basic properties of relative involutive bases. In Subsection 4.2.2, we investigate the required properties for the construction of relative involutive bases. Section 4.3 provides a study of some combinatorial properties of finite relative Pommaret bases. We introduce the new notion of a relative quasi-stable ideal and apply it to propose a deterministic algorithm for the construction of finite relative Pommaret bases. Finally, in Section 4.4, we introduce the notions of relative involutive-like divisions and bases and discuss their properties.

4.1 Relative Gröbner Bases

In this section, we define and investigate relative Gröbner bases for ideals in quotient rings $\mathcal{R}/\mathcal{I}$. Their basic theory is given in Subsection 4.1.1. A version of Schreyer's construction adapted to the quotient ring structure allows us to find for any relative Gröbner basis $G \subset \mathcal{R}/\mathcal{I}$ a relative Gröbner basis for its syzygy module over $\mathcal{R}/\mathcal{I}$. We define in this context the notion of A-polynomials as opposed to S-polynomials, a distinction which will be useful later for the analysis of relative involutive bases. In Subsection 4.1.2, we proceed to the analysis of the comutation of relative Gröbner bases. Using their syzygy theory, on the one hand we provide criteria that can be used for the optimization of these algorithms and on the other hand we give a number of equivalent conditions for a set to be a relative Gröbner basis.

4.1.1 Relative Gröbner Bases and Syzygies

A basic building block of the theory of Gröbner basis is polynomial division. Since we are interested in establishing an analogous theory for ideals in a quotient ring $\mathcal{R}/\mathcal{I}$, we need a division algorithm that takes the ideal $\mathcal{I}$ into account as well. Suppose we are given a reduced Gröbner basis G of an ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ with respect to a given term ordering $\prec$. Additionally, let $h_1, \ldots, h_r \in \mathcal{R}$ be polynomials which are reduced with respect to G, i. e. $\mathrm{NF}_G(h_i) = h_i$ for all $1 \leq i \leq r$. Finally, we are given a polynomial $f \in \mathcal{R}$ which we want to divide by the set $H = \{h_1, \ldots, h_r\}$ modulo $\mathcal{I}$. The result is then a polynomial $\tilde{f}$, reduced with respect to G and with no monomial in its support divisible by any monomial in $\mathrm{lt}(H)$. Algorithmically, this result can be achieved by repeatedly applying the normal form operation NF_G followed by a classical polynomial division step with respect to H. Algorithm 17 is a formalisation of this idea.

Algorithm 17: Relative Polynomial Division

Data: A monomial ordering $\prec$, an ideal $\mathcal{I} \trianglelefteq \mathcal{R}$, a Gröbner basis G of $\mathcal{I}$, a set of polynomials $H = \{h_1, \ldots, h_r\} \subset \mathcal{R}$ with $\mathrm{NF}_G(h_i) = h_i$ for all i and $f \in \mathcal{R}$

Result: A polynomial $p \in \mathcal{R}$ with support disjoint from $\langle \mathrm{lt}(\mathcal{I}), \mathrm{lt}(H)\rangle$, polynomials $q_1, \ldots, q_r \in \mathcal{R}$ with $f - p - \sum_{i=1}^{r} q_i h_i \in \mathcal{I}$

begin
- $\tilde{f} \longleftarrow f$; $p \longleftarrow 0$
- **for** $i = 1, \ldots, r$ **do**
  - $q_i \longleftarrow 0$
- **while** $\tilde{f} \neq 0$ **do**
  - **if** $\mathrm{lt}(\tilde{f}) \in \langle \mathrm{lt}(G)\rangle$ **then**
    - Choose $g \in G$ with $\mathrm{lt}(g) | \mathrm{lt}(\tilde{f})$
    - $\tilde{f} \longleftarrow \tilde{f} - \frac{\mathrm{lm}(\tilde{f})}{\mathrm{lm}(g)} g$
  - **else if** $\mathrm{lt}(\tilde{f}) \in \langle \mathrm{lt}(H)\rangle$ **then**
    - Choose $h_i \in H$ with $\mathrm{lt}(h_i) | \mathrm{lt}(\tilde{f})$
    - $q_i \longleftarrow q_i + \frac{\mathrm{lm}(\tilde{f})}{\mathrm{lm}(h_i)}$; $\tilde{f} \longleftarrow \tilde{f} - \frac{\mathrm{lm}(\tilde{f})}{\mathrm{lm}(h_i)} h_i$
  - **else**
    - $p \longleftarrow p + \mathrm{lm}(\tilde{f})$; $\tilde{f} \longleftarrow \tilde{f} - \mathrm{lt}(\tilde{f})$
- **return** $(p, q_1, \ldots, q_r)$

Remark 4.1.1. *The support of the quotient polynomial q_k belonging to h_k computed during the course of Algorithm 17 is contained in the order ideal $\mathcal{T} \setminus (\mathrm{lt}(\mathcal{I}) : \mathrm{lt}(h_k))$. Since $H \cup G$ need not be a Gröbner basis of $\langle H\rangle_{\mathcal{P}} + \mathcal{I}$, the polynomial p in the output of Algorithm 17 is not uniquely determined by the input, but depends on the chosen polynomials g and h_i, resp., in the various reduction steps.*

Definition 4.1.2. *If the polynomial p is a possible output of Algorithm 17 for input $f, H, \mathcal{I}, \prec$, then we write $f \longrightarrow^{*}_{H,\mathcal{I},\prec} p$ and say that f* reduces to p with respect to H modulo $\mathcal{I}$. *We omit the reference to the term ordering $\prec$ if no confusion can arise.*

Given an ideal $\mathcal{I} \lhd \mathcal{R}$, we are interested in defining something like Gröbner bases for ideals in the quotient ring $\mathcal{R}/\mathcal{I}$. As, without additional structure, it makes no sense to speak of terms in this ring, a direct approach does not appear meaningful. Instead, we exploit the well-known fact that any ideal in $\mathcal{R}/\mathcal{I}$ is of the form $\mathcal{J}/\mathcal{I}$ for an ideal $\mathcal{I} \subseteq \mathcal{J} \unlhd \mathcal{R}$. Therefore our basic idea is to determine suitable bases of $\mathcal{J}$ "relative" to $\mathcal{I}$ which may be interpreted as Gröbner bases of $\mathcal{J}/\mathcal{I}$.

From now on, we fix a term ordering $\prec$ on $\mathcal{R}$ and leading terms, monomials, coefficients, Gröbner bases etc. will always be determined with respect to it. In particular, the leading ideal $\operatorname{lt}(\mathcal{I})$ is thus fixed. Every coset $[f]_{\mathcal{I}} = f + \mathcal{I} \in \mathcal{R}/\mathcal{I}$ contains then a unique representative $\tilde{f} = \operatorname{NF}_{\mathcal{I}}(f)$ with $\operatorname{supp}(\tilde{f}) \cap \operatorname{lt}(\mathcal{I}) = \emptyset$ (it can be easily determined as the normal form of f with respect to an arbitrary Gröbner basis of $\mathcal{I}$). If not explicitly stated otherwise, we will in the sequel always assume that each coset $[f]$ is described by this unique representative. This allows us to define $\operatorname{lt}([f]) = \operatorname{lt}(f)$ and accordingly $\operatorname{lc}([f])$, $\operatorname{lm}([f])$. For an ideal $\mathcal{J}/\mathcal{I} \unlhd \mathcal{R}/\mathcal{I}$, we then find $\operatorname{lt}(\mathcal{J}/\mathcal{I}) = \operatorname{lt}(\mathcal{J}) \setminus \operatorname{lt}(\mathcal{I})$. Finally, we denote by π the canonical projection $\mathcal{R} \to \mathcal{R}/\mathcal{I}$.

Definition 4.1.3. *Let $\mathcal{I} \subseteq \mathcal{J} \lhd \mathcal{R}$ be ideals. The finite subset $H \subset \mathcal{J}$ is called a* Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$, *if $\langle \operatorname{lt}(H)\rangle + \operatorname{lt}(\mathcal{I}) = \operatorname{lt}(\mathcal{J})$. A finite subset $\hat{H} = \{[h_1], \dots, [h_r]\} \subset \hat{\mathcal{J}} = \mathcal{J}/\mathcal{I} \lhd \mathcal{R}/\mathcal{I}$ is a* Gröbner basis *of $\hat{\mathcal{J}}$, if $\{h_1, \dots, h_r\}$ is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$ or equivalently if $\langle \operatorname{lt}(\hat{H})\rangle + \operatorname{lt}(\mathcal{I}) = \operatorname{lt}(\mathcal{J})$.*

Relative Gröbner bases exist, since every Gröbner basis of $\mathcal{J}$ is also a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$. Given a relative Gröbner basis of $\mathcal{J}$ with respect to $\mathcal{I}$, we can extend it trivially to a Gröbner basis of $\mathcal{J}$. Relative Gröbner bases can be characterised similarly to the classical case.

Proposition 4.1.4. *Let $H = \{h_1, \dots, h_t\} \subset \mathcal{J}$ be a finite set and G a Gröbner basis of $\mathcal{I}$. Then the following statements are equivalent:*

- *H is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$.*
- *$H \cup G$ is a Gröbner basis of $\mathcal{J}$.*
- *For any $f \in \mathcal{J}$, we have $f \longrightarrow^{*}_{H,\mathcal{I},\prec} 0$.*
- *Any $f \in \mathcal{J}$ has a* relative standard representation *of the form $f = g + \sum_{i=1}^{t} q_i h_i$ where $g \in \mathcal{I}$ and $\operatorname{lt}(q_i h_i) \preceq \operatorname{lt}(f)$ for each i with $q_i \neq 0$.*

Proof. By definition of a relative Gröbner basis, we know that $\langle \operatorname{lt}(H)\rangle + \langle \operatorname{lt}(G)\rangle = \operatorname{lt}(\mathcal{J})$. Thus, if $f \in \mathcal{I}$, then $\operatorname{lt}(f)$ is divisible by some $\operatorname{lt}(g)$ with $g \in G$ and if $f \in \mathcal{J} \setminus \mathcal{I}$, then $\operatorname{lt}(f)$ is divisible by some $\operatorname{lt}(h)$ with $h \in H$. Thus, H being a relative Gröbner basis of $\mathcal{J}$ is equivalent to $H \cup G$ being a Gröbner basis of $\mathcal{J}$. The last two statements follow by classical properties of Gröbner bases. □

As a consequence, the classical Buchberger algorithm provides us already with a basic procedure to compute relative Gröbner bases. More precisely, we have the following observation.

Proposition 4.1.5. *Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$ be two polynomial ideals and F a finite generating set of $\mathcal{J}$. Let furthermore G be a Gröbner basis of $\mathcal{I}$ and call $H_{\text{Buchberger}}$ the Gröbner basis of $\mathcal{J}$ obtained by applying Buchberger's algorithm to the set $F \cup G$. Then $H := H_{\text{Buchberger}} \setminus \mathcal{I}$ is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$.*

Proof. Since $H_{\text{Buchberger}}$ is a Gröbner basis of $\langle F, G\rangle_{\mathcal{R}} = \mathcal{J}$, it is of course also a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$ and we can discard all elements belonging to $\mathcal{I}$, as their leading terms do not divide any term in $\operatorname{lt}(\mathcal{J}) \setminus \operatorname{lt}(\mathcal{I})$. □

Assume, again, that $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$ are polynomial ideals. If F generates $\mathcal{J}$, and G is a Gröbner basis of $\mathcal{I}$, then $\mathrm{NF}_G(F) \cup G$ also generates $\mathcal{J}$. Applying Proposition 4.1.5 and observing that each element that is added during the course of Buchberger's algorithm is reduced with respect to G, i. e. in normal form with respect to $\mathcal{I}$, we get a Gröbner basis H of $\mathcal{J}$ relative to $\mathcal{I}$ with $H = \mathrm{NF}_G(H)$. Iteratively discarding any element of H whose leading term is divisible by the leading term of another element of H, then performing a full auto-reduction and finally normalising leading monomials, we get a *reduced Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$*, that is, a set H with $\mathrm{NF}_G(H) = H$ with the additional properties

- $|H| = |\operatorname{Min}(\operatorname{lt}(\mathcal{J})) \setminus \operatorname{lt}(\mathcal{I})|$,
- $\{\operatorname{lt}(h) \mid h \in H\} = \operatorname{Min}(\operatorname{lt}(\mathcal{J})) \setminus \operatorname{lt}(\mathcal{I})$,
- $\forall h \in H : \operatorname{lc}(h) = 1$,
- $\forall h \in H : \operatorname{supp}(h - \operatorname{lm}(h)) \subseteq \mathcal{T} \setminus \operatorname{lt}(\mathcal{J})$.

Proposition 4.1.6. *Let H be a Gröbner basis (resp., the reduced Gröbner basis) of $\mathcal{J} \supseteq \mathcal{I}$ and let G be a Gröbner basis of $\mathcal{I}$. Then $\overline{H} := \mathrm{NF}_G(H)$ is a Gröbner basis (resp., the reduced Gröbner basis) of $\mathcal{J}$ relative to $\mathcal{I}$.*

Proof. Let $x^\mu \in \operatorname{Min}(\operatorname{lt}(\mathcal{J})) \setminus \operatorname{lt}(\mathcal{I})$. Then there exists a polynomial $h \in H$ with $\operatorname{lt}(h) = x^\mu$. Now, since the leading term of h cannot be reduced modulo $\mathcal{I}$ and since reduction modulo $\mathcal{I}$ (as, indeed, reduction modulo any set) does not introduce higher terms than the terms that are eliminated by the reduction, we have $\operatorname{lt}(\mathrm{NF}_G(h)) = x^\mu$. The claim follows. □

We may extend the above theory to modules. This requires a bit care with the used orderings. We continue to assume that we are given an ideal $\mathcal{I} \triangleleft \mathcal{R}$ and a term ordering $\prec$ defining the monomial ideal $\operatorname{lt}(\mathcal{I})$. Then we fix on the free module $\mathcal{R}^r$ with the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ a module term ordering $\prec$ which must be *compatible* to $\prec$ in the sense that if $x^\mu \prec x^\nu$ then also $x^\mu \mathbf{e}_i \prec x^\nu \mathbf{e}_i$ for any index $1 \le i \le r$ (obviously, any POT or TOP lift of $\prec$ will be compatible to $\prec$, but also any Schreyer ordering based on $\prec$).

If we write an element of $(\mathcal{R}/\mathcal{I})^r$ as a vector of cosets, then we use again the convention that for each coset the unique representative in normal form with respect to $\mathcal{I}$ has been chosen. As in the scalar case, this convention allows us to extend the notions of leading module term, leading coefficient etc. to vectors of cosets. Denoting again by π the extension of the canonical projection onto the cosets to the projection $\mathcal{R}^r \to (\mathcal{R}/\mathcal{I})^r$, we associate with any $\mathcal{R}/\mathcal{I}$-submodule $\widehat{\mathcal{N}} \subseteq (\mathcal{R}/\mathcal{I})^r$

the $\mathcal{R}$-submodule $\mathcal{N} = \pi^{-1}(\widehat{\mathcal{N}})$ and find then analogously to the scalar case that $\mathbf{lt}(\mathcal{N}) = \mathbf{lt}(\widehat{\mathcal{N}}) + \sum_{i=1}^{r} \mathrm{lt}(\mathcal{I})\mathbf{e}_i$.

Definition 4.1.7. *Let $\mathcal{I} \lhd \mathcal{R}$ be a polynomial ideal and $\mathcal{N} \subset \mathcal{R}^r$ a $\mathcal{R}$-submodule containing $\mathcal{I}^r$. A finite set $B \subset \mathcal{N}$ disjoint from $\mathcal{I}^r$ is called a* Gröbner basis of $\mathcal{N}$ relative to $\mathcal{I}$, *if $\langle \mathbf{lt}(B)\rangle + \sum_{i=1}^{r} \mathrm{lt}(\mathcal{I})\mathbf{e}_i = \mathbf{lt}(\mathcal{N})$. A finite subset $\{[\mathbf{h}_1], \dots, [\mathbf{h}_s]\}$ of a $\mathcal{R}/\mathcal{I}$-submodule $\widehat{\mathcal{N}} \subseteq (\mathcal{R}/\mathcal{I})^r$ is a* Gröbner basis *of $\widehat{\mathcal{N}}$, if $\{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ is a Gröbner basis of $\mathcal{N}$ relative to $\mathcal{I}$.*

Note that, if G is a Gröbner basis of $\mathcal{I}$, then B is a relative Gröbner basis of $\mathcal{N}$ with respect to $\mathcal{I}$ if and only if the set $B \cup \{g\mathbf{e}_i \mid g \in G, 1 \le i \le r\}$ is a Gröbner basis of $\mathcal{N}$ because of the assumed compatibility of the term orderings $\prec$ and $\prec$. With this definition, we can now analyse relative Gröbner bases of syzygy modules. Recall Schreyer's construction [94] (Construction 2.2.3), which allows us to compute a Gröbner basis of a syzygy module of a set of polynomials, if this set is a Gröbner basis of the ideal it generates. We now adapt it to the relative case.

Consider again two ideals $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$. Let $H = \{h_1, \dots, h_r\}$ be a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$ and let $G = \{g_1, \dots, g_s\}$ be a Gröbner basis of $\mathcal{I}$. We may as well assume both bases to be reduced, but this is not strictly necessary. But we will always assume that $\mathrm{NF}_G(H) = H$, i. e. H is given by polynomials in normal form with respect to the ideal $\mathcal{I}$ and the given term ordering. To apply Schreyer's construction, we need to choose an enumeration of the polynomials involved. We choose to give precedence to the polynomials in H over the polynomials in G. This will be useful later as a kind of elimination ordering when we look at the syzygies of H relative to $\mathcal{I}$.

In the Schreyer construction of $\mathrm{Syz}(H, G) := \mathrm{Syz}(h_1, \dots, h_r, g_1, \dots, g_s)$ – the set $H \cup G$ is a Gröbner basis of $\mathcal{J}$ by Proposition 4.1.4 – we have a certain degree of freedom in that for each S-polynomial, we may choose one of the various available standard representations with respect to $H \cup G$. Specifically, every time a term belonging to a term in $\mathrm{lt}(\mathcal{I})$ needs to be reduced, we can choose a reduction by an element of G; note that Algorithm 17 implements just this kind of reduction. This way, reductions with respect to H are only performed for monomials belonging to terms of the order ideal $\mathcal{T} \setminus \mathrm{lt}(\mathcal{I})$. This has the effect that the quotient polynomial q belonging to an element $h \in H$ is built up exclusively of monomials belonging to terms not in $\mathrm{lt}(\mathcal{I})$. Divisors of terms in $\mathcal{T} \setminus \mathrm{lt}(\mathcal{I})$ are again not in $\mathrm{lt}(\mathcal{I})$, since $\mathcal{T} \setminus \mathrm{lt}(\mathcal{I})$ is an order ideal.

The canonical projection π onto the cosets also induces a projection map from $\mathrm{Syz}(H, G)$ to $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}([h_1], \dots, [h_r])$ which we continue to call π. Let $\mathbf{p} \in \mathcal{R}^r$ and $\mathbf{q} \in \mathcal{R}^s$. Then we define

$$\pi : \mathrm{Syz}(h_1, \dots, h_r, g_1, \dots, g_s) \longrightarrow \mathrm{Syz}_{\mathcal{R}/I}([h_1], \dots, [h_r]), \quad (\mathbf{p}, \mathbf{q}) \longmapsto [\mathbf{p}], \tag{4.1}$$

where by $[\mathbf{p}]$ we denote the vector obtained from $\mathbf{p}$ by taking cosets in each component. Of course, we still need to prove that π has the properties that one would expect from a projection map. For this, we shall need the definition of an A-polynomial.

Definition 4.1.8. *With the above notations, an S-polynomial of a pair $(h_i, g_\alpha) \in H \times G$ is called an A-*polynomial *and is denoted by* $\mathrm{A}(h_i, g_\alpha)$*. As in Construction 2.2.3, we also introduce the corresponding notion of an A-*syzygy *denoted by* $\mathbf{A}_{i\alpha}$*.*

To justify the notations introduced in Definition 4.1.8, let us note that $\mathbf{S}_{ij}$ is the syzygy induced by the S-polynomial of two generators $h_i, h_j \in H$, whereas $\mathbf{A}_{i\alpha}$ is the syzygy induced by annihilating the leading term of h_i modulo $\mathrm{lt}(\mathcal{I})$. As it is well-known, the letter "S" is an abbreviation for "syzygy", whereas "A" refers to "annihilator" inspired by the work of Norton and Salagean [83] on Gröbner bases over principal ideal rings (see also [68]).

In the rest of this section, for the sake of simplicity, we will use the subindices $r+1, \ldots, r+s$ for the polynomials $g_1, \ldots, g_s$; i.e. we define $h_\alpha := g_{\alpha-r}$ for $\alpha = r+1, \ldots, r+s$. Thus, we will consider the set $\{h_1, \ldots, h_{r+s}\}$ and we will study the syzygy module $\mathrm{Syz}(H, G)$ of these polynomials in $\mathcal{R}^{r+s}$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_{r+s}\}$ be the standard basis of $\mathcal{R}^{r+s}$. By $\mathbf{S}_{ij}$ (respectively $\mathbf{A}_{i\alpha}$), we mean the syzygy module element corresponding to the S-polynomial $\mathrm{S}(h_i, h_j)$ with $1 \le i < j \le r$ (respectively to the A-polynomial $\mathrm{A}(h_i, h_j)$ with $1 \le i \le r$ and $r+1 \le \alpha \le r+s$) involving the module elements $\{\mathbf{e}_1, \ldots, \mathbf{e}_{r+s}\}$, see (2.2). However, when we write $\mathbf{S}_{\alpha\beta}$ we mean the syzygy module element corresponding to the S-polynomial $\mathrm{S}(h_\alpha, h_\beta)$ with $r+1 \le \alpha < \beta \le r+s$ containing merely the module elements $\{\mathbf{e}_{r+1}, \ldots, \mathbf{e}_{r+s}\}$. With these notation we can state the following proposition.

Proposition 4.1.9. *The map π defined in (4.1) is a $\mathcal{R}$-linear surjective map. The syzygy module $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}([h_1], \ldots, [h_r])$ is generated as a $\mathcal{R}/\mathcal{I}$-module by the image of the subset*

$$\{\mathbf{S}_{ij} \mid 1 \le i < j \le r\} \cup \{\mathbf{A}_{i\alpha} \mid 1 \le i \le r,\ r+1 \le \alpha \le r+s\} \subset \mathrm{Syz}(H, G)$$

under the map π.

Proof. We first show that π is well-defined. Let $(\mathbf{p}, \mathbf{q}) \in \mathrm{Syz}(H, G)$ with $\mathbf{p} = (p_1, \ldots, p_r)$ and $\mathbf{q} = (q_1, \ldots, q_s)$. Then $\sum_{i=1}^r p_i h_i + \sum_{\alpha=1}^s q_\alpha g_\alpha = 0$, so $\sum_{i=1}^r p_i h_i = -\sum_{\alpha=1}^s q_\alpha g_\alpha \in \mathcal{I}$, which implies $\sum_{i=1}^r [p_i][h_i] = [0]$ modulo $\mathcal{I}$. This means that $[\mathbf{p}] \in \mathrm{Syz}_{\mathcal{R}/I}([h_1], \ldots, [h_r])$. Hence, π is well-defined. The $\mathcal{R}$-linearity of π follows from the $\mathcal{R}$-linearity of the canonical projection $\mathcal{R} \to \mathcal{R}/\mathcal{I}$.

To show surjectivity, let $([p_1], \ldots, [p_r]) \in \mathrm{Syz}_{\mathcal{R}/\mathcal{I}}([h_1], \ldots, [h_r])$ be an arbitrary syzygy. Then $\sum_{i=1}^r [p_i][h_i] = [0]$ and thus $\sum_{i=1}^r p_i h_i \in \mathcal{I}$. A standard representation with respect to G yields polynomials $q_1, \ldots, q_s$ with $\sum_{i=1}^r p_i h_i + \sum_{\alpha=1}^s q_\alpha g_\alpha = 0$. Hence $(p_1, \ldots, p_r, q_1, \ldots, q_s) \in \mathrm{Syz}(h_1, \ldots, h_r, g_1, \ldots, g_s)$ is a preimage of the vector $([p_1], \ldots, [p_r])$ and π is surjective.

Since π is $\mathcal{R}$-linear and surjective, any generating set of $\mathrm{Syz}(H, G)$ is mapped to a generating set of the module $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}([h_1], \ldots, [h_r])$. By Schreyer's construction, we know that the syzygies $\mathbf{S}_{ij}, \mathbf{A}_{i\alpha}, \mathbf{S}_{\alpha\beta}$ form a generating set of $\mathrm{Syz}(H, G)$. The syzygies $\mathbf{S}_{\alpha\beta}$ with $r+1 \le \alpha < \beta \le r+s$ are mapped to 0 under π, as our special choice of a standard representation for $\mathrm{S}(h_\alpha, h_\beta)$ implies that $\mathbf{S}_{\alpha\beta}$ has its first r components equal to 0. Hence we can omit them and our claim follows. $\square$

This result motivates the introduction of some special notations to reflect the two somewhat different subsets making up the constructed generating set of $\mathrm{Syz}(H, G)$, in particular, as the two subsets will be treated quite differently at many places.

Definition 4.1.10. *Let $G = \{g_1, \dots, g_s\}$ be a Gröbner basis of the ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ and let $H = \{h_1, \dots, h_r\}$ be a set disjoint from $\mathcal{I}$ such that $\mathrm{NF}_G(H) = H$ and $H \cup G$ generates the ideal $\mathcal{J} \supseteq \mathcal{I}$. Keeping the above notations, we define the set of all S-syzygies of H relative to G by*

$$\mathcal{S}(H, G) = \{\mathbf{S}_{ij} \mid 1 \le i < j \le r\} \tag{4.2}$$

*and the set of all A-*syzygies *of H relative to G by*

$$\mathcal{A}(H, G) = \{\mathbf{A}_{i\alpha} \mid 1 \le i \le r,\ r + 1 \le \alpha \le r + s\}. \tag{4.3}$$

Construction 2.2.3 can be extended to $\mathcal{R}/\mathcal{I}$-submodules of $(\mathcal{R}/\mathcal{I})^r$. Thus, we are able to show that the generating set obtained in Proposition 4.1.9 is even a Gröbner basis.

Theorem 4.1.11. *Let $G = \{g_1, \dots, g_s\}$ be the reduced Gröbner basis of the ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ and let $H = \{h_1, \dots, h_r\}$ be a Gröbner basis of the ideal $\mathcal{J} \supseteq \mathcal{I}$ relative to $\mathcal{I}$ such that $\mathrm{NF}_G(H) = H$ and let π be the projection map of the corresponding syzygy modules defined in (4.1). Then the set $\pi(\mathcal{S}(H, G)) \cup \pi(\mathcal{A}(H, G))$ is a Gröbner basis of the syzygy module $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}([h_1], \dots, [h_r])$ for the Schreyer ordering $\prec_S$.*

Proof. From Proposition 4.1.9 above, we know already that the set $\pi(\mathcal{S}(H, G)) \cup \pi(\mathcal{A}(H, G))$ generates $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}([h_1], \dots, [h_r])$ as a $\mathcal{R}/\mathcal{I}$-module. Thus, we must only show that for each syzygy $[\mathbf{p}] \in \mathrm{Syz}_{\mathcal{R}/\mathcal{I}}([h_1], \dots, [h_r])$ there exists a generator $[\mathbf{q}] \in \pi(\mathcal{S}(H, G)) \cup \pi(\mathcal{A}(H, G))$ such that $\mathbf{lt}([\mathbf{q}])$ divides $\mathbf{lt}([\mathbf{p}])$ with the leading terms taken with respect to the Schreyer ordering $\prec_S$. Let $[\mathbf{p}] = ([p_1], \dots, [p_r])$. By the proof of Proposition 4.1.9, there exists a syzygy $\mathbf{S} = (p_1, \dots, p_r, q_1, \dots, q_s) \in \pi^{-1}([\mathbf{p}])$ which implies that the polynomial $g := \sum_{i=1}^{r} p_i h_i$ lies in the ideal $\mathcal{I}$. Since G is a Gröbner basis of $\mathcal{I}$, there exists a standard representation $g = \sum_{\alpha=1}^{s} q'_\alpha g_\alpha$ entailing that $\max_\prec\{\mathrm{lt}(q'_1 g_1), \dots, \mathrm{lt}(q'_s g_s)\} \preceq \mathrm{lt}(g)$. This shows that the preimage $\pi^{-1}([\mathbf{p}])$ also contains the syzygy $\mathbf{S}' = (p_1, \dots, p_r, q'_1, \dots, q'_s)$ which satisfies

$$\max_\prec \left\{\mathrm{lt}(q'_1 g_1), \dots, \mathrm{lt}(q'_s g_s)\right\} \preceq \max_\prec \left\{\mathrm{lt}(p_1 h_1), \dots, \mathrm{lt}(p_r h_r)\right\}.$$

Assume that $\mathrm{lt}(p_i h_i) = \max_\prec\{\mathrm{lt}(p_1 h_1), \dots, \mathrm{lt}(p_r h_r)\}$ where i is minimal with this property. Then $\mathrm{lt}(p_i)\mathbf{e}_i$ is the module leading term of $\mathbf{S}'$ with respect to $\prec_S$. Now, two cases may occur: If there exists $j \in \{1, \dots, r\} \setminus \{i\}$ such that $\mathrm{lt}(p_i h_i) = \mathrm{lt}(p_j h_j)$ then $\mathbf{lt}([\mathbf{p}])$ is divisible by $\mathbf{lt}(\pi(\mathbf{S}_{ij}))$. Otherwise, there exists $\alpha \in \{r + 1, \dots, r + s\}$ such that $\mathrm{lt}(p_i h_i) = \mathrm{lt}(q'_\alpha g_\alpha)$. It follows that $\mathbf{lt}([\mathbf{p}])$ is divisible by $\mathbf{lt}(\pi(\mathbf{A}_{i\alpha}))$ and this completes the proof. $\square$

4.1.2 Computation of Relative Gröbner Bases

In the previous section, we showed how to compute relative Gröbner bases using the most basic version of Buchberger's algorithm. In this section, we will develop "relative" criteria analogous to Buchberger's S-polynomial criterion for a Gröbner basis as well as Buchberger's (first and second) criteria for recognising unnecessary reductions. As starting point, we recall a result of Möller et al. [78] relating the computation of Gröbner bases of polynomial ideals to Gröbner bases of syzygy modules of sets of terms.

Theorem 4.1.12 ([78, Thm. 2.7]). *Let $G = \{g_1, \ldots, g_s\} \subset \mathcal{R}$ be a set of polynomials and B a Gröbner basis of the submodule $\operatorname{Syz}(\operatorname{lt}(g_1), \ldots, \operatorname{lt}(g_s)) \subset \mathcal{R}^s$. Then G is a Gröbner basis of the ideal it generates, if and only if for all $\mathbf{b} = (b_1, \ldots, b_r) \in B$ we have $\sum_{i=1}^{s} b_i g_i \longrightarrow_G^+ 0$.*

To obtain an analogous result in the context of relative Gröbner bases, we shall need the next proposition inspired by [83, Thm. 4.6].

Proposition 4.1.13. *Let $\mathcal{I} \lhd \mathcal{R}$ be a monomial ideal and $H = \{x^{\mu_1}, \ldots, x^{\mu_r}\} \subset \mathcal{T} \setminus \mathcal{I}$ a set of standard terms. With $x^{\mu_{ij}} := \operatorname{lcm}(x^{\mu_i}, x^{\mu_j})$, as a $\mathcal{R}/\mathcal{I}$-module, the syzygy module $\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}([x^{\mu_1}], \ldots, [x^{\mu_r}])$ is generated by*

$$B := \left\{ \left[\frac{x^{\mu_{ij}}}{x^{\mu_i}}\right] \mathbf{e}_i - \left[\frac{x^{\mu_{ij}}}{x^{\mu_j}}\right] \mathbf{e}_j \mid 1 \leq i < j \leq r \right\} \cup \bigcup_{i=1}^{r} [\operatorname{Min}(\mathcal{I} : x^{\mu_i})]\, \mathbf{e}_i \,.$$

Proof. Let $G = \{x^{\nu_1}, \ldots, x^{\nu_s}\}$ be the minimal generating set of $\mathcal{I}$. It is clear that G (resp. H) is a Gröbner basis for $\mathcal{I}$ (resp. the ideal it generates) with respect to any term ordering. Thus, applying Theorem 4.1.11 to the sets G and H, we see that the first subset in B consists of the projection of the S-polynomials between all pairs of elements of H and the second component comes from the projection of the A-polynomials between all pairs in $H \times G$.

To be more precise for the second component, we show that

$$\langle \pi(\mathcal{A}(H, G)) \rangle_{\mathcal{R}/\mathcal{I}} = \left\langle \bigcup_{i=1}^{r} [\operatorname{Min}(\mathcal{I} : x^{\mu_i})]\, \mathbf{e}_i \right\rangle_{\mathcal{R}/\mathcal{I}} .$$

By definition, $m x^{\mu_i} \in \mathcal{I}$ for any index i and any term $m \in \operatorname{Min}(\mathcal{I} : x^{\mu_i})$. Hence there exists an index α such that $x^{\nu_\alpha} \mid m x^{\mu_i}$. It follows that there exists a term $u \in \mathcal{R}$ such that $m x^{\mu_i} = u \operatorname{lcm}(x^{\mu_i}, x^{\nu_\alpha})$ and the module element $\mathbf{A}_{i\alpha} = \operatorname{lcm}(x^{\mu_i}, x^{\nu_\alpha})/x^{\mu_i} \mathbf{e}_i - \operatorname{lcm}(x^{\mu_i}, x^{\nu_\alpha})/x^{\nu_\alpha} \mathbf{e}_\alpha$ satisfies $u\mathbf{A}_{i\alpha} = m\mathbf{e}_i - m x^{\mu_i}/x^{\nu_\alpha} \mathbf{e}_\alpha$. We conclude that $[m\mathbf{e}_i] = [u]\pi(\mathbf{A}_{i\alpha}) \in \langle \pi(\mathcal{A}(H, G)) \rangle_{\mathcal{R}/\mathcal{I}}$ by the definition of π.

Conversely, let us consider the element $\pi(\mathbf{A}_{i\alpha}) \in \pi(\mathcal{A}(H, G))$ where $\mathbf{A}_{i\alpha}$ is the syzygy corresponding to the A-polynomial between x^{μ_i} and x^{ν_α}. Write $\mathbf{A}_{i\alpha} = \frac{x^\theta}{x^{\mu_i}} \mathbf{e}_i - \frac{x^\theta}{x^{\nu_\alpha}} \mathbf{e}_\alpha$ where $x^\theta = \operatorname{lcm}(x^{\mu_i}, x^{\nu_\alpha})$. By the definition of π, we have $\pi(\mathbf{A}_{i\alpha}) = [\frac{x^\theta}{x^{\mu_i}}]\mathbf{e}_i$. There may also exist $x^{\nu_\beta} \in G$ such that $\mathbf{lt}(\mathbf{A}_{i\beta})$ divides $\mathbf{lt}(\mathbf{A}_{i\alpha}) = \frac{x^\theta}{x^{\mu_i}} \mathbf{e}_i$. Without loss of generality, we may assume that $\mathbf{A}_{i\beta}$ is the minimal element satisfying this property. Let $x^\eta = \mathrm{lcm}(x^{\mu_i}, x^{\nu_\beta})$. Thus, $\pi(\mathbf{A}_{i\beta}) = [\frac{x^\eta}{x^{\mu_i}}]\mathbf{e}_i$. We have $x^\eta/x^{\mu_i} \in \mathcal{I} : x^{\mu_i}$. To finish the proof, it is enough to prove that x^η/x^{μ_i} belongs to the minimal generating set of $\mathcal{I} : x^{\mu_i}$. Suppose, by reductio ad absurdum, that $u \mid x^\eta/x^{\mu_i}$ where $u \in \mathrm{Min}(\mathcal{I} : x^{\mu_i})$ and $u \neq x^\eta/x^{\mu_i}$. Thus, $ux^{\mu_i} \in \mathcal{I}$ and $x^{\nu_\gamma} \mid ux^{\mu_i} \mid x^\eta$ for some γ. This entails that $\mathrm{lcm}(x^{\mu_i}, x^{\nu_\gamma})$ divides properly x^η, leading to a contradiction with the choice of $\mathbf{A}_{i\beta}$. Since $\pi(\mathbf{A}_{i\beta})$ divides $\mathbf{lt}(\mathbf{A}_{i\alpha})$, we must have $\mathbf{lt}(\mathbf{A}_{i\alpha}) \in \langle \bigcup_{i=1}^r [\mathrm{Min}(\mathcal{I} : x^{\mu_i})]\, \mathbf{e}_i \rangle_{\mathcal{R}/\mathcal{I}}$. □

Remark 4.1.14. *If $H = \{c_1x^{\mu_1}, \ldots, c_\mu x^{\mu_r}\}$ is a set of monomials, then Proposition 4.1.13 remains essentially true: B is a generating set for $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}([x^{\mu_1}], \ldots, [x^{\mu_r}])$ if in the first component of B, $\left[\frac{x^{\mu_{ij}}}{x^{\mu_i}}\right]\mathbf{e}_i - \left[\frac{x^{\mu_{ij}}}{x^{\mu_j}}\right]\mathbf{e}_j$ is replaced by $\left[\frac{x^{\mu_{ij}}}{c_ix^{\mu_i}}\right]\mathbf{e}_i - \left[\frac{x^{\mu_{ij}}}{c_jx^{\mu_j}}\right]\mathbf{e}_j$.*

Gröbner bases can be characterised using various properties, among them we mention, besides Buchberger's criterion, that a set G is a Gröbner basis, if and only if any polynomial in $\langle G\rangle$ has a standard representation. Furthermore, G is a Gröbner basis, if and only if any syzygy of $\mathrm{lm}(G)$ can be lifted to a syzygy of G and vice versa. We give below similar characterizations for relative Gröbner bases as in [78, Thm. 2.7].

Theorem 4.1.15. *Let $\prec$ be a term ordering on $\mathcal{T}$. Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$ be polynomial ideals, let $H = \{h_1, \ldots, h_r\} \subset \mathcal{J}$ be a relative generating set of $\mathcal{J}$, that is, $\langle H\rangle + \mathcal{I} = \mathcal{J}$. Then, the following statements are equivalent.*

(i) H is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$.

(ii) For all $\mathbf{b} = ([b_1], \ldots, [b_r]) \in \mathrm{Syz}_{\mathcal{R}/\mathrm{lt}(\mathcal{I})}(\mathrm{lm}(H))$, we have $\sum_{i=1}^r b_ih_i \longrightarrow^*_{H,\mathcal{I}} 0$.

(iii) For any generating set B of $\mathrm{Syz}_{\mathcal{R}/\mathrm{lt}(\mathcal{I})}(\mathrm{lm}(H))$ and any $\mathbf{b} = ([b_1], \ldots, [b_r]) \in B$, it holds $\sum_{i=1}^r b_ih_i \longrightarrow^*_{H,\mathcal{I}} 0$.

(iv) For all $h \in \mathcal{J}$, there exist polynomials $g \in \mathcal{I}$ and $q_i \in \langle \mathcal{T} \setminus (\mathrm{lt}(\mathcal{I}) : \mathrm{lt}(h_i))\rangle_{\mathbb{K}}$ for $1 \le i \le r$, such that $h = g + \sum_{i=1}^r q_ih_i$ and $\mathrm{lt}(q_ih_i) \preceq \mathrm{lt}(h)$ for all i with $q_i \neq 0$.

Proof. (i) $\Longrightarrow$ (ii). Let G be a Gröbner basis of $\mathcal{I}$. Since $\sum_{i=1}^r b_ih_i \in \mathcal{J}$ and $H \cup G$ is a Gröbner basis of $\mathcal{J}$, the claim follows from Proposition 4.1.4.

(ii) $\Longrightarrow$ (iii). This is obvious.

(iii) $\Longrightarrow$ (iv). Let $h \in \mathcal{J}$. We first claim that there exist $g \in \mathcal{I}$ and $q_1, \ldots, q_r \in \mathcal{R}$ such that $h = g + \sum_{i=1}^r q_ih_i$ and in addition $\mathrm{lt}(q_ih_i) \preceq \mathrm{lt}(h)$ for each i with $q_i \neq 0$. Arguing by reductio ad absurdum, suppose that for each choice of $g \in \mathcal{I}$ and $q_1, \ldots, q_r \in \mathcal{R}$ there exists i such that $\mathrm{lt}(q_ih_i) \succ \mathrm{lt}(h)$. Among all such representations of h, we pick a representation $h = g + \sum_{i=1}^r q_ih_i$ such that $X := \max\{\mathrm{lt}(q_1h_1), \ldots, \mathrm{lt}(q_rh_r)\}$ is minimal with respect to $\prec$. Without loss of generality, we may assume that $X = \mathrm{lt}(q_1h_1) = \cdots = \mathrm{lt}(q_kh_k)$ and $\mathrm{lt}(q_ih_i) \prec X$ for each $i > k$. In addition, since $h - \sum_{i=1}^r q_ih_i \in \mathcal{I}$, we have $\mathrm{lt}(g) \preceq X$. It follows that $\sum_{i=1}^k \mathrm{lm}(q_i)\,\mathrm{lm}(h_i) \in \mathrm{lt}(\mathcal{I})$ and in turn $([\mathrm{lm}(q_1)], \ldots, [\mathrm{lm}(q_k)], [0], \ldots, [0]) \in \mathrm{Syz}_{\mathcal{R}/\mathrm{lt}(\mathcal{I})}(\mathrm{lm}(H))$ can be written as a combination of the elements in B. From (3) and using the fact that the operation of computing remainders on division by a set is linear, we obtain $\sum_{i=1}^k \mathrm{lm}(q_i)h_i \longrightarrow^*_{H,\mathcal{I}}$

0. Thus, there exist $\tilde{q}_1,\dots,\tilde{q}_r \in \mathcal{R}$ such that $\sum_{i=1}^{k} \mathrm{lm}(q_i)h_i = \tilde{g} + \sum_{i=1}^{r} \tilde{q}_i h_i$ such that $\mathrm{lt}(\tilde{q}_i h_i) \prec X$ and $\mathrm{lt}(\tilde{g}) \preceq X$ with $\tilde{g} \in \mathcal{I}$. This yields a new representation for h of the form

$$g' + \sum_{i=1}^{r} q_i' h_i := g + \sum_{i=1}^{r} (q_i - \mathrm{lm}(q_i))h_i + \tilde{g} + \sum_{i=1}^{r} \tilde{q}_i h_i$$

with $g' \in \mathcal{I}$ and $\max_{\prec}\{\mathrm{lt}(q_1' h_1),\dots,\mathrm{lt}(q_r' h_r)\} \prec X$. As this contradicts our assumptions, our claim is proven. Thus, we are able to find a representation $g + \sum_{i=1}^{r} q_i h_i$ for h such that $\mathrm{lt}(q_i h_i) \preceq \mathrm{lt}(h)$ for each i. Now, if there exists i such that $\mathrm{lt}(q_i h_i)$ is reducible by G, then we can perform this reduction and in consequence we may assume that in the representation $h = g + \sum_{i=1}^{r} q_i h_i$ we have $q_i \in \langle \mathcal{T} \setminus (\mathrm{lt}(\mathcal{I}) : \mathrm{lt}(h_i))\rangle_{\mathbb{K}}$ for each i and this proves (4).

(iv) $\Longrightarrow$ *(i)*. Let $x^\mu \in \mathrm{lt}(\mathcal{J}) \setminus \mathrm{lt}(\mathcal{I})$. There exists an element $h \in \mathcal{J}$ with $\mathrm{lt}(h) = x^\mu$. From (4), write $h = g + \sum_{k=1}^{r} q_k h_k$. Since $h - \sum_{k=1}^{r} q_k h_k \in \mathcal{I}$, we may assume that $\mathrm{lt}(g) \preceq \mathrm{lt}(h)$. From the choice of x^μ, we conclude that $\mathrm{lt}(g) \prec \mathrm{lt}(h)$. Additionally, we know that for all i, $\mathrm{lt}(q_i h_i) \preceq \mathrm{lt}(h)$. Consequently, there exists i with $\mathrm{lt}(q_i h_i) = \mathrm{lt}(h)$ and this shows that H is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$. □

As a consequence of Propositions 4.1.4 and 4.1.13 and Theorem 4.1.15 (iii), we get the next theorem.

Theorem 4.1.16 (Relative Buchberger criterion). *Let $\prec$ be a term ordering on $\mathcal{T}$. Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$ be two polynomial ideals and $G = \{g_1,\dots,g_t\}$ a Gröbner basis of $\mathcal{I}$. Let $H = \{h_1,\dots,h_r\} \subset \mathcal{J}$ with $\langle H\rangle + \mathcal{I} = \mathcal{J}$. Then, H is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$ if and only if we have $\mathrm{A}(h_i, g_\alpha) \longrightarrow^*_{H,\mathcal{I}} 0$ and $\mathrm{S}(h_i, h_j) \longrightarrow^*_{H,\mathcal{I}} 0$ for all indices i, j, α.*

Based on this theorem, we are now able to provide the relative variant of Buchberger's algorithm to compute relative Gröbner bases, i. e. Algorithm 18. For making it more efficient, we recall first Buchberger's criteria which may be applied in Buchberger's algorithm to avoid some superfluous reductions in the course of Gröbner bases computation, for more details see [9, pages 222-225].

Lemma 4.1.17 (Buchberger's first criterion). *Let $f_i, f_j \in \mathcal{R}$ be two polynomials such that we have $\mathrm{lcm}(\mathrm{lt}(f_i), \mathrm{lt}(f_j)) = \mathrm{lt}(f_i)\,\mathrm{lt}(f_j)$. Then, $\mathrm{S}(f_i, f_j)$ is reduced to zero modulo $\{f_1, f_2\}$.*

Lemma 4.1.18 (Buchberger's second criterion). *Let $F \subset \mathcal{R}$ be finite and $p, f_i, f_j \in \mathcal{R}$ three polynomials such that the following conditions hold:*

- *$\mathrm{lt}(p)$ divides $\mathrm{lcm}(\mathrm{lt}(f_i), \mathrm{lt}(f_j))$,*
- *$\mathrm{S}(p, f_i)$ and $\mathrm{S}(p, f_j)$ have standard representations with respect to F.*

Then, $\mathrm{S}(f_i, f_j)$ has a standard representation with respect to F.

It is worth noting that these two criteria are applicable in the relative setting using the algorithm described in [9, pages 232]. To apply these criteria in Algorithm 18, we must use also the *relative normal selection strategy*. By this, we

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Algorithm 18: Relative Buchberger

Data: A term ordering $\prec$, a Gröbner basis $G = \{g_1, \ldots, g_t\}$ of $\mathcal{I} \trianglelefteq \mathcal{R}$, a finite set of polynomials $H = \{h_1, \ldots, h_r\} \subset \mathcal{R}$ with $\mathrm{NF}_G(h_i) = h_i$ for all i

Result: A Gröbner basis of $\langle H \rangle + \mathcal{I}$ relative to $\mathcal{I}$

begin

  $T \longleftarrow H$; $P \longleftarrow \{\{h_i, h_j\}, \{h_i, g\} \mid 1 \leq i < j \leq r, g \in G\}$

  while $P \neq \emptyset$ **do**

    Select and remove a critical pair $\{f_i, f_j\}$ from P

    Reduce $\mathrm{S}(f_i, f_j) \longrightarrow^*_{T,G} p$

    if $p \neq 0$ **then**

      $P := P \cup \{\{p, h\}, \{p, g\} \mid h \in T, g \in G\}$; $T := T \cup \{p\}$

  return T

---

mean that when we want to select a pair from P, we pick a pair $\{f_i, f_j\} \in P$ such that $\mathrm{lcm}(\mathrm{lt}(f_i), \mathrm{lt}(f_j))$ is as small as possible. In addition, if there are several pairs sharing the same least common divisor, we select a pair $\{f_i, f_j\} \in P$ such that $\{f_i, f_j\} \cap G \neq \emptyset$, if any. The main idea to prove Buchberger's second criterion is that using the mentioned conditions, one is able to write the S-syzygy corresponding to the pair $\{f_1, f_2\}$ as a combination of the S-syzygies corresponding to the pairs $\{p, f_1\}$ and $\{p, f_2\}$, see [9]. Applying this idea, and beside to the above criteria, we can state the next improvement applicable to the computation of relative Gröbner bases.

Proposition 4.1.19. *Assume that in Algorithm 18 the pair $\{f_i, f_j\}$ with $f_i, f_j \in T$ is considered. If $\mathrm{lcm}(\mathrm{lt}(f_i), \mathrm{lt}(f_j)) \in \mathrm{lt}(\mathcal{I})$, then this pair is superfluous.*

Proof. Let us first fix some notations. Let $\mathrm{lt}(f_\ell) = x^{\mu_\ell}$ for $\ell = i, j$ and $x^{\mu_{ij}} = \mathrm{lcm}(x^{\mu_i}, x^{\mu_j})$. Assume that $x^{\nu_\alpha} := \mathrm{lt}(g_\alpha) \mid x^{\mu_{ij}}$ for some $g_\alpha \in G$. By assumption, there exist terms x^γ and x^η such that $x^{\mu_{ij}} = x^\gamma \,\mathrm{lcm}(x^{\mu_i}, x^{\nu_\alpha})$ and $x^{\mu_{ij}} = x^\eta \,\mathrm{lcm}(x^{\mu_j}, x^{\nu_\alpha})$. Thus, we can write

$$\frac{x^{\mu_{ij}}}{\mathrm{lm}(f_i)}\mathbf{e}_i - \frac{x^{\mu_{ij}}}{\mathrm{lm}(f_j)}\mathbf{e}_j =$$
$$x^\gamma \left(\frac{\mathrm{lcm}(x^{\mu_i}, x^{\nu_\alpha})}{\mathrm{lm}(f_i)}\mathbf{e}_i - \frac{\mathrm{lcm}(x^{\mu_i}, x^{\nu_\alpha})}{\mathrm{lm}(g_\alpha)}\mathbf{e}_\alpha \right) - x^\eta \left(\frac{\mathrm{lcm}(x^{\mu_j}, x^{\nu_\alpha})}{\mathrm{lm}(f_j)}\mathbf{e}_j - \frac{\mathrm{lcm}(x^{\mu_j}, x^{\nu_\alpha})}{\mathrm{lm}(g_\alpha)}\mathbf{e}_\alpha \right).$$

Our selection strategy ensures that at the time we choose the pair $\{f_i, f_j\}$, the A-polynomials $\mathrm{A}(f_i, g_\alpha)$ and $\mathrm{A}(f_j, g_\alpha)$ have already relative standard representations and therefore the S-polynomial $\mathrm{S}(f_i, f_i)$ has a relative standard representation, too, which implies our claim. $\square$

Corollary 4.1.20. *In Proposition 4.1.13, one can replace B by*

$$B := \left\{ \left[\frac{x^{\mu_{ij}}}{x^{\mu_i}}\right] \mathbf{e}_i - \left[\frac{x^{\mu_{ij}}}{x^{\mu_j}}\right] \mathbf{e}_j \mid 1 \leq i < j \leq r \wedge x^{\mu_{ij}} \notin \mathcal{I} \right\} \cup \bigcup_{i=1}^r [\mathrm{Min}(\mathcal{I} : x^{\mu_i})]\, \mathbf{e}_i \,.$$

4.2 Relative Involutive Divisions and Bases

This section is divided into two parts. In Subsetion 4.2.1, we define the concept of an involutive division relative to a given monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$. This leads to the notion of relative involutive bases. We investigate their basic properties and develop their syzygy theory. In Subsection 4.2.2, we give algorithms to compute relative involutive bases provided that the relative involutive division one uses is induced by a constructive Noetherian classical involutive division. Together, this gives in particular a complete description of relative Janet bases and their computation.

4.2.1 Relative Involutive Bases

We adapt now the basic definitions from the theory of involutive bases to the situation that we work relative to an ideal $\mathcal{I}$. The basic idea is to require that the usual axioms hold only outside of $\mathcal{I}$. This yields the following extension of the definition of an involutive division which for $\mathcal{I} = 0$ coincides with the standard one. Note that "relative cones" are not necessarily cones in the usual sense, but cones parts of which have been removed.

Definition 4.2.1. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a monomial ideal with minimal generating set* $\mathrm{Min}(\mathcal{I})$*. An* involutive division $\mathcal{L}$ relative to $\mathcal{I}$ *is a rule which assigns to any term $x^\mu \in \mathcal{T} \setminus \mathcal{I}$ which is contained in a finite set $H \subset \mathcal{T} \setminus \mathcal{I}$ of terms a subset of variables $\mathrm{M}_{\mathcal{L}}(x^\mu, H)$, called* $\mathcal{L}$-multiplicative variables of $x^\mu \in H$*, such that the following conditions are satisfied for the* relative involutive cones $\mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\mu) := x^\mu \cdot \mathbb{K}[\mathrm{M}_{\mathcal{L}}(x^\mu, H)] \setminus \mathcal{I}$*:*

(i) If the set H contains two terms x^μ and x^ν such that $\mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\mu) \cap \mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\nu) \neq \emptyset$, then either $x^\mu \in \mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\nu)$ or $x^\nu \in \mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\mu)$.

(ii) If the set H contains two terms x^μ and x^ν such that $x^\mu \in \mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\nu)$, then $\mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\mu) \subseteq \mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\nu)$.

(iii) If $H_1 \subset H_2$ are two sets containing the term x^μ, then $\mathcal{C}_{\mathcal{L},H_2,\mathcal{I}}(x^\mu) \subseteq \mathcal{C}_{\mathcal{L},H_1,\mathcal{I}}(x^\mu)$.

Next we show that any classical involutive division induces a relative one (and thus provide many concrete instances of relative involutive divisions). The key question here is how one treats directions leading into $\mathcal{I}$, as different plausible possibilities exist. It turns out that the one chosen here is for many purposes the most convenient one.

Definition 4.2.2. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a monomial ideal with minimal generating set $\mathrm{Min}(\mathcal{I})$ and let $\mathcal{L}$ be an involutive division on $\mathcal{T}$. Then the following rule defines an associated relative division $\mathcal{L}_{\mathcal{I}}$ relative to $\mathcal{I}$: If a finite set of terms $H \subset \mathcal{T} \setminus \mathcal{I}$ is given, then for each variable x_i ($i \in \{1, \ldots, n\}$) and for each $x^\mu \in H$,*

$$x_i \in \mathcal{L}_{\mathcal{I}}(x^\mu, H) \quad \Longleftrightarrow \quad \big(x_i \in \mathcal{L}(x^\mu, H) \vee x_i x^\mu \in \mathcal{I}\big) \,. \tag{4.4}$$

Proposition 4.2.3. *If $\mathcal{L}$ is an involutive division on $\mathcal{T}$ and $\mathcal{I} \trianglelefteq \mathcal{R}$, then the rule $\mathcal{L}_{\mathcal{I}}$ defined by (4.4) is an involutive division relative to $\mathcal{I}$.*

Proof. For all terms $x^\mu \in H$, it is clear by definition that $\mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\mu) = \mathcal{C}_{\mathcal{L},H}(x^\mu) \setminus \mathcal{I}$. Now, if x^μ and x^ν are elements of H such that $\mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\mu) \cap \mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\nu) \neq \emptyset$, then also the classical involutive cones $\mathcal{C}_{\mathcal{L},H}(x^\mu)$ and $\mathcal{C}_{\mathcal{L},H}(x^\nu)$ intersect nontrivially, implying, without loss of generality, $x^\mu \in \mathcal{C}_{\mathcal{L},H}(x^\nu)$. But since $x^\mu \notin \mathcal{I}$, this implies $x^\mu \in \mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\nu)$, proving that the first defining property of relative involutive divisions is satisfied by the rule $\mathcal{L}_\mathcal{I}$. In the same situation, the inclusion $\mathcal{C}_{\mathcal{L},H}(x^\mu) \subseteq \mathcal{C}_{\mathcal{L},H}(x^\nu)$ must hold for the classical involutive cones, which immediately implies the same inclusion for the corresponding $\mathcal{L}_\mathcal{I}$-cones. If, finally, $H_1 \subset H_2$ are two sets of terms disjoint from $\mathcal{I}$ and if $x^\mu \in H_1$, then we have the inclusion $\mathcal{C}_{\mathcal{L},H_2}(x^\mu) \subseteq \mathcal{C}_{\mathcal{L},H_1}(x^\mu)$ for the classical involutive cones, which again immediately implies the same inclusion for the $\mathcal{L}_\mathcal{I}$-cones. □

Now we can define relative involutive bases. As in the classical case and as for relative Gröbner bases, we begin by considering the monomial case, before we proceed to general polynomial ideals.

Definition 4.2.4. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a monomial ideal and let $\mathcal{L}$ be an involutive division relative to $\mathcal{I}$. Let $H \subset \mathcal{T} \setminus \mathcal{I}$ be a finite set of terms disjoint from $\mathcal{I}$ and set $\mathcal{J} := \langle H \rangle + \mathcal{I}$. We call H a* weak $\mathcal{L}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, *if the $\mathbb{K}$-spans of the sets $\bigcup_{x^\mu \in H} \mathcal{C}_{\mathcal{L},\mathcal{I},H}(x^\mu)$ and $\mathcal{J} \setminus \mathcal{I}$ coincide. H is called* (strong) involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, *if it is a weak involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$ and the relative involutive cones $\mathcal{C}_{\mathcal{L},H,\mathcal{I}}(x^\mu)$ for $x^\mu \in H$ are pairwise disjoint.*

Example 4.2.5. *Let $\mathcal{R} = \mathbb{K}[x_1, x_2]$ be the polynomial ring in two variables, let the monomial ideal $\mathcal{I}$ be minimally generated by the set $\mathrm{Min}(\mathcal{I}) = \{x_2^3, x_1^2x_2^2, x_1^3\}$ and consider $H = \{x_1^2x_2, x_2, x_1\}$. We analyse this constellation of sets of terms first by using the relative involutive division induced by the Pommaret division and the ideal $\mathcal{I}$ and then by using the relative involutive division induced by the Janet division and the ideal $\mathcal{I}$.*

1. *For the Pommaret division $\mathcal{P}_\mathcal{I}$ relative to $\mathcal{I}$, we find that $\mathrm{M}_{\mathcal{P}_\mathcal{I}}(x_1^2x_2, H) = \{x_1, x_2\}$, as $\mathrm{cls}(x_1^2x_2) = 1$ and $x_2(x_1^2x_2) = x_1^2x_2^2 \in \mathcal{I}$. Moreover, $\mathrm{M}_{\mathcal{P}_\mathcal{I}}(x_2, H) = \{x_1, x_2\}$ as $\mathrm{cls}(x_2) = 2$ and $\mathrm{M}_{\mathcal{P}_\mathcal{I}}(x_1, H) = \{x_1\}$ as $\mathrm{cls}(x_1) = 1$ and $x_2(x_1) = x_1x_2 \notin \mathcal{I}$. One can now easily see that H is a weak Pommaret basis of $\mathcal{J} = \langle H \rangle + \mathcal{I}$ relative to $\mathcal{I}$. But it is not a strong relative Pommaret basis, because $\mathcal{C}_{\mathcal{P},H,\mathcal{I}}(x_1^2x_2) \subset \mathcal{C}_{\mathcal{P},H,\mathcal{I}}(x_2)$. But of course an autoreduction yields the strong relative Pommaret basis $H \setminus \{x_1^2x_2\} = \{x_1, x_2\}$.*

2. *For the Janet division $\mathcal{J}_\mathcal{I}$ relative to $\mathcal{I}$, we find that $\mathrm{M}_{\mathcal{J}_\mathcal{I}}(x_1^2x_2, H) = \{x_1, x_2\}$ and $\mathrm{M}_{\mathcal{J}_\mathcal{I}}(x_2, H) = \{x_2\}$ as both terms contain x_2 linearly and $\mathrm{M}_{\mathcal{J}_\mathcal{I}}(x_1, H) = \{x_1\}$. Since the term x_1x_2 does not lie in any of the three relative cones, H is not a weak Janet basis of $\mathcal{J}$ relative to $\mathcal{I}$. Nevertheless, one can easily see that $H \setminus \{x_1^2x_2\}$ is a strong Janet basis of $\mathcal{J}$ relative to $\mathcal{I}$.*

Definition 4.2.6. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a polynomial ideal, G a Gröbner basis of $\mathcal{I}$ and $\mathcal{L}$ an involutive division relative to $\mathcal{I}$. Let $H \subset \mathcal{R}$ be a finite set satisfying $\mathrm{NF}_G(H) = H$ and set $\mathcal{J} := \langle H \rangle + \mathcal{I}$. We call H a* weak $\mathcal{L}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, *if* $\operatorname{lt}(H)$ *is a weak involutive basis of* $\operatorname{lt}(\mathcal{J})$ *relative to* $\operatorname{lt}(\mathcal{I})$. *H is called a* (strong) $\mathcal{L}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, *if* $\operatorname{lt}(H)$ *is a strong involutive basis of* $\operatorname{lt}(\mathcal{J})$ *relative to* $\operatorname{lt}(\mathcal{I})$ *and the mapping* $h \mapsto \operatorname{lt}(h)$ *is a bijection from H to* $\operatorname{lt}(H)$.

Via a relative involutive polynomial division, any strong relative involutive basis of an ideal $\mathcal{J} \supseteq \mathcal{I}$ induces a finite direct sum decomposition of $\mathcal{J}/\mathcal{I}$ as a $\mathbb{K}$-linear space provided one uses the right definition of relative involutive cones in the polynomial case. To make this remark precise, we introduce first Algorithm 19 for the division and then define relative involutive cones using normal forms with respect to G.

Algorithm 19: Relative Involutive Division

Data: Ideal $\mathcal{I} \trianglelefteq \mathcal{R}$; Gröbner basis G of $\mathcal{I}$; set of polynomials $H = \{h_1, \dots, h_m\} \subset \mathcal{R}$ with $\operatorname{NF}_G(H) = H$; involutive division L relative to $\mathcal{I}$; polynomial $f \in \mathcal{R}$

Result: Polynomial $r \in \mathcal{R}$ with $\operatorname{supp}(r) \subseteq \mathcal{T} \setminus \big(\operatorname{lt}(\mathcal{I}) \cup \mathcal{C}_{\mathcal{L},\operatorname{lt}(H),\operatorname{lt}(\mathcal{I})}(\operatorname{lt}(H))\big)$, polynomials $q_1, \dots, q_m \in \mathcal{R}$ with $f - r - \sum_{k=1}^{m} q_k h_k \in \mathcal{I}$ and $q_k \in \mathbb{K}[\mathrm{M}_{\mathcal{L}}(\operatorname{lt}(h_k), \operatorname{lt}(H))]$

begin
- $\tilde{f} \longleftarrow f$; $\quad r \longleftarrow 0$
- **for** $k = 1, \dots, m$ **do**
  - $q_k \longleftarrow 0$
- **while** $\tilde{f} \neq 0$ **do**
  - **if** $\operatorname{lt}(\tilde{f}) \in \langle \operatorname{lt}(G) \rangle$ **then**
    - Choose $g \in G$ with $\operatorname{lt}(g) | \operatorname{lt}(\tilde{f})$
    - $\tilde{f} \longleftarrow \tilde{f} - \frac{\operatorname{lm}(\tilde{f})}{\operatorname{lm}(g)} g$
  - **else if** $\operatorname{lt}(\tilde{f}) \in \mathcal{C}_{\mathcal{L},\operatorname{lt}(H),\operatorname{lt}(\mathcal{I})}(\operatorname{lt}(H))$ **then**
    - Choose index k such that $\operatorname{lm}(\tilde{f}) \in \mathcal{C}_{\mathcal{L},\operatorname{lt}(H),\operatorname{lt}(\mathcal{I})}(\operatorname{lt}(h_k))$
    - $q_k \longleftarrow q_k + \frac{\operatorname{lm}(\tilde{f})}{\operatorname{lm}(h_k)}$; $\quad \tilde{f} \longleftarrow \tilde{f} - \frac{\operatorname{lm}(\tilde{f})}{\operatorname{lm}(h_k)} h_k$
  - **else**
    - $r \longleftarrow r + \operatorname{lm}(\tilde{f})$; $\quad \tilde{f} \longleftarrow \tilde{f} - \operatorname{lm}(\tilde{f})$
- **return** $(r, q_1, \dots, q_m)$

Definition 4.2.7. *Let* $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$ *be two polynomial ideals, G a Gröbner basis of* $\mathcal{I}$ *and* $\mathcal{L}$ *an involutive division relative to* $\operatorname{lt}(\mathcal{I})$. *Let* $H \subset \mathcal{J} \setminus \mathcal{I}$ *be a finite set satisfying* $\operatorname{NF}_G(H) = H$, *whose elements have pairwise distinct leading terms. For* $h \in H$ *define its* $\mathcal{L}$-involutive cone relative to $\mathcal{I}$ *to be the following* $\mathbb{K}$*-vector space:*

$$\mathcal{C}_{\mathcal{L},\operatorname{lt}(H),\operatorname{lt}(\mathcal{I})}(h) := \left\langle \operatorname{NF}_G(x^\rho h) \mid x^\rho \operatorname{lt}(h) \in \mathcal{C}_{\mathcal{L},\operatorname{lt}(H),\operatorname{lt}(\mathcal{I})}(\operatorname{lt}(h)) \right\rangle_{\mathbb{K}} . \tag{4.5}$$

Theorem 4.2.8. *Let* $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$ *be polynomial ideals,* $\mathcal{L}$ *an involutive division relative to* $\mathcal{I}$ *and* $H \subset \mathcal{J} \setminus \mathcal{I}$ *a strong* $\mathcal{L}$*-involutive basis of* $\mathcal{J}$ *relative to* $\mathcal{I}$. *Then* *we have the following finite direct sum decomposition of the ideal $\mathcal{J}$ as a $\mathbb{K}$-vector space*

$$\mathcal{J} = \left(\bigoplus_{h\in H} \mathcal{C}_{\mathcal{L},\mathrm{lt}(H),\mathrm{lt}(\mathcal{I})}(h)\right) \oplus \mathcal{I}\,. \tag{4.6}$$

Proof. Let us refer to the first summand in (4.6) as A. We show first that A is indeed a direct sum. For this it suffices to show that for any two distinct basis elements $h_1, h_2 \in H$ and any polynomials $f_1 \in \mathcal{C}_{\mathcal{L},\mathrm{lt}(H),\mathrm{lt}(\mathcal{I})}(h_1)$, $f_2 \in \mathcal{C}_{\mathcal{L},\mathrm{lt}(H),\mathrm{lt}(\mathcal{I})}(h_2)$ we have $\mathrm{lt}(f_1) \neq \mathrm{lt}(f_2)$. Indeed, if $h \in H$ is any basis element, then for each $f \in \mathcal{C}_{\mathcal{L},\mathrm{lt}(H),\mathrm{lt}(\mathcal{I})}(h)$ there exists a polynomial $p \in \mathbb{K}[\mathrm{M}_{\mathcal{L}}(\mathrm{lt}(h), \mathrm{lt}(H))]$ with $\mathrm{supp}(p) \subseteq \frac{1}{\mathrm{lt}(h)}\mathcal{C}_{\mathcal{L},\mathrm{lt}(H),\mathrm{lt}(\mathcal{I})}(\mathrm{lt}(h))$ such that $f = \mathrm{NF}_G(ph)$. But since the leading term $\mathrm{lt}(ph) = \mathrm{lt}(p)\,\mathrm{lt}(h) \notin \mathrm{lt}(\mathcal{I})$, we have $\mathrm{lt}(f) = \mathrm{lt}(ph) \in \mathcal{C}_{\mathcal{L},\mathrm{lt}(H),\mathrm{lt}(\mathcal{I})}(\mathrm{lt}(h))$. These relative monomial $\mathcal{L}$-cones are pairwise disjoint when h varies through H, because H is a strong relative $\mathcal{L}$-involutive basis. This proves that A is a direct sum. This argument also entails that $\mathrm{lt}(A) \cap \mathrm{lt}(\mathcal{I}) = \emptyset$, proving that $A \cap \mathcal{I} = \{0\}$.

Now we show that $A + \mathcal{I} = \mathcal{J}$. Let $f \in \mathcal{J}\setminus\{0\}$. Since H is a strong $\mathcal{L}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, Algorithm 19 applied to f yields the remainder $r = 0$ and we can write $f = g + \sum_{h\in H} q_h h$ with $g \in \mathcal{I}$ and $\mathrm{supp}(q_h) \subseteq \frac{1}{\mathrm{lt}(h)}\mathcal{C}_{\mathcal{L},\mathrm{lt}(H),\mathrm{lt}(\mathcal{I})}(\mathrm{lt}(h))$ for all h. Taking normal forms modulo $\mathcal{I}$ via a Gröbner basis, we get $\mathrm{NF}_{\mathcal{I}}(f) = \sum_{h\in H} \mathrm{NF}_{\mathcal{I}}(q_h h)$, and consequently $f = \tilde{g} + \sum_{h\in H} \mathrm{NF}_{\mathcal{I}}(q_h h)$ for some $\tilde{g} \in \mathcal{I}$. This finishes the proof. □

In the remainder of this section, we analyse relative syzygy modules $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}(H)$ where H is a strong $\mathcal{L}$-involutive basis of $\langle H\rangle + \mathcal{I}$ relative to $\mathcal{I}$ for some involutive division $\mathcal{L}$ relative to $\mathcal{I}$. The goal is to find relative involutive bases also for these syzygy modules. Since all relative involutive bases are *a fortiori* also relative Gröbner bases, we can build on the work done in previous sections. We need to describe carefully how the combinatorial structure of H carries over to the syzygy module. The distinction of S- and A-polynomials as building blocks of the syzygy modules will be the key for this. Let G be a Gröbner basis of $\mathcal{I}$. As in Proposition 4.1.9 and Theorem 4.1.11, we impose an ordering on $H \cup G$ where the elements of H get smaller indices than those of G. Additionally, we impose an $\mathcal{L}$-*ordering* on the elements of H, which means that if for some $h_1, h_2 \in H$ there exists a non-multiplicative variable $x_i \in \mathrm{NM}_{\mathcal{L}}(\mathrm{lt}(h_1), \mathrm{lt}(H))$ such that $x_i\,\mathrm{lt}(h_1) \in \mathcal{C}_{\mathcal{L},\mathrm{lt}(H),\mathcal{I}}(\mathrm{lt}(h_2))$, then h_1 precedes h_2 in the $\mathcal{L}$-ordering. The fact that a linear ordering of H can be achieved which is also an $\mathcal{L}$-ordering follows from the acyclicity of the $\mathcal{L}$-*graph* of H. This can be shown for relative involutive divisions induced by classical continuous divisions completely analogously to the case of classical involutive bases. For further details we refer to [97, Lem. 5.4.5] and the references therein. As a first step, we now analyse the S-polynomials $\mathcal{S}(H, G)$.

Proposition 4.2.9. *Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$ be polynomial ideals, G a Gröbner basis of $\mathcal{I}$, $\mathcal{L}_{\mathcal{I}}$ an involutive division relative to $\mathcal{I}$ induced by a continuous involutive division $\mathcal{L}$ on $\mathcal{T}$ and $H = \{h_1, \ldots, h_r\}$ a strong $\mathcal{L}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$ ordered* *according to an $\mathcal{L}_\mathcal{I}$-ordering. Then for each S-polynomial $\mathbf{S}_{ij} \in \mathcal{S}(H, G)$ satisfying* $\operatorname{lcm}(\operatorname{lt}(h_i), \operatorname{lt}(h_j)) \notin \operatorname{lt}(\mathcal{I})$*, we have* $\mathbf{lt}(\mathbf{S}_{ij}) \in \langle \mathrm{NM}_{\mathcal{L}_\mathcal{I}}(\operatorname{lt}(h), \operatorname{lt}(H))\rangle \mathbf{e}_i$.

Proof. Note that, by definition of S-polynomials, $1 \leq i < j \leq r$. Let us write $\operatorname{lt}(h_i) = x^{\mu_i}$, $\operatorname{lt}(h_j) = x^{\mu_j}$, and $\operatorname{lcm}(\operatorname{lt}(h_i), \operatorname{lt}(h_j)) = x^{\mu_{ij}}$. Again by definition of S-polynomials and by the proof of Theorem 4.1.11, we know that $\mathbf{lt}(\mathbf{S}_{ij}) = \frac{x^{\mu_{ij}}}{x^{\mu_i}}\mathbf{e}_i$. We have to show that the term $\frac{x^{\mu_{ij}}}{x^{\mu_i}} \notin \mathbb{K}[\mathrm{M}_{\mathcal{L}_\mathcal{I}}(x^{\mu_i}, \operatorname{lt}(H))]$. Assume this was the case. Since the $L_\mathcal{I}$-cones $\mathcal{C}_{\mathcal{L}_\mathcal{I}, \operatorname{lt}(H), \operatorname{lt}(\mathcal{I})}(\operatorname{lt}(h))$ for the generators $h \in H$ disjointly decompose $\operatorname{lt}(\mathcal{J}) \setminus \operatorname{lt}(\mathcal{I})$ by Theorem 4.2.8, it is then impossible that $\frac{x^{\mu_{ij}}}{x^{\mu_j}} \in \mathbb{K}[\mathrm{M}_{\mathcal{L}_\mathcal{I}}(x^{\mu_j}, \operatorname{lt}(H))]$, too. Thus there exists a non-multiplicative variable $x_a \in \mathrm{NM}_{\mathcal{L}_\mathcal{I}}(x^{\mu_j}, \operatorname{lt}(H))$ such that $x_a | \frac{x^{\mu_{ij}}}{x^{\mu_j}}$; and there exists a unique leading term $x^{\mu_{k_a}} = \operatorname{lt}(h_{k_a}) \in \operatorname{lt}(H)$ such that $x_a x^{\mu_j} \in \mathcal{C}_{\mathcal{L}_\mathcal{I}, \operatorname{lt}(H), \operatorname{lt}(\mathcal{I})}(x^{\mu_{k_a}})$. By the defining property of $\mathcal{L}_\mathcal{I}$-orderings, $j < k_a$. Now, if $\frac{x^{\mu_{ij}}}{x_a x^{\mu_j}} \in \mathbb{K}[\mathrm{M}_{\mathcal{L}_\mathcal{I}}(x^{\mu_{k_a}}, \operatorname{lt}(H))]$ were true, then $x^{\mu_{ij}} \in \mathcal{C}_{\mathcal{L}_\mathcal{I}, \operatorname{lt}(H), \operatorname{lt}(\mathcal{I})}(x^{\mu_{k_a}})$, which entails $h_i = h_{k_a}$, which is not possible since $i < j < k_a$. So there must necessarily exist a non-multiplicative variable $x_b \in \mathrm{NM}_{L_\mathcal{I}}(x^{\mu_{k_a}}, \operatorname{lt}(H))$ such that $x_b | \frac{x^{\mu_{ij}}}{x_a x^{\mu_j}}$. An iteration of this argument yields an infinite sequence of terms $x^{\mu_{k_a}}, x^{\mu_{k_b}}, x^{\mu_{k_c}}, \ldots$ in $\operatorname{lt}(H)$ belonging to basis elements $h_{k_a}, h_{k_b}, h_{k_c}, \ldots$ with indices strictly monotonically increasing, which is not possible. Consequently, the assumption $\frac{x^{\mu_{ij}}}{x^{\mu_i}} \in \mathbb{K}[\mathrm{M}_{\mathcal{L}_\mathcal{I}}(x^{\mu_i}, \operatorname{lt}(H))]$ was false and there must necessarily exist a non-multiplicative variable for x^{μ_i} dividing the polynomial part of the leading module term of the S-polynomial $\mathbf{S}_{ij}$. □

Proposition 4.2.9 helps to identify among the set of S-polynomials $\mathcal{S}(H, G)$ an irredundant subset $\mathcal{S}_{\mathcal{L}_\mathcal{I}}(H, G)$ of S-polynomials induced by non-multiplicative prolongations.

Lemma 4.2.10. *In the situation of Proposition 4.2.9, for each basis element $h_i \in H$ and for each non-multiplicative variable $x_k \in \mathrm{NM}_{\mathcal{L}_\mathcal{I}}(\operatorname{lt}(h_i), \operatorname{lt}(H))$, there exists an S-polynomial $\mathbf{S}_{ij} \in \mathcal{S}(H, G)$ such that* $\mathbf{lm}(\mathbf{S}_{ij}) = x_k \mathbf{e}_i$.

Proof. There is a unique basis element $h_j \in H$ such that $i < j$ and such that $x_k \operatorname{lt}(h_i)$ is in the $\mathcal{L}_\mathcal{I}$-cone $\mathcal{C}_{\mathcal{L}_\mathcal{I}, \operatorname{lt}(H), \operatorname{lt}(\mathcal{I})}(\operatorname{lt}(h_j))$. Also, trivially, $x_k \operatorname{lt}(h_i) \mid \operatorname{lcm}(\operatorname{lt}(h_i), \operatorname{lt}(h_j))$. But since $\operatorname{lt}(h_i) \neq \operatorname{lt}(h_j)$, it follows that $x_k \operatorname{lt}(h_i) = \operatorname{lcm}(\operatorname{lt}(h_i), \operatorname{lt}(h_j))$. This induces, by Construction 2.2.3, an S-polynomial $\mathbf{S}_{ij}$ with the desired properties. □

Definition 4.2.11. *In the situation of Proposition 4.2.9 and Lemma 4.2.10, denote by $\mathcal{S}_{\mathcal{L}_\mathcal{I}}(H, G)$ the set of all S-polynomials induced by non-multiplicative prolongations of elements from H.*

Having analysed the part of the syzygy module induced by the S-polynomials, we now turn to the A-polynomials. Since our goal is to obtain, in each module component of the relative syzygy module $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}(H)$, a relative involutive basis of the ideal in $\mathcal{R}/\mathcal{I}$ associated to this module component, we need an additional structure for the Gröbner basis G of $\mathcal{I}$. More concretely, we want to achieve that the leading terms of the A-polynomials $\mathbf{S}_{i\alpha}$ associated to the i-th module component of $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}(H)$ form part of an involutive basis of the leading ideal of the ideal associated to this i-th module component. To achieve this, a natural assumption on G is for it to be a (strong) $\mathcal{L}$-involutive basis of $\mathcal{I}$, where $\mathcal{L}$ is the continuous involutive division on $\mathcal{T}$ inducing the relative involutive division $\mathcal{L}_\mathcal{I}$.

Definition 4.2.12. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a polynomial ideal and $\mathcal{L}_\mathcal{I}$ an involutive division relative to $\operatorname{lt}(\mathcal{I})$ induced by a continuous involutive division $\mathcal{L}$ on $\mathcal{T}$. We say that $\mathcal{L}_\mathcal{I}$ is of* Schreyer type *if, whenever H is a strong $\mathcal{L}_\mathcal{I}$-involutive basis of $\langle H\rangle + \mathcal{I}$ relative to $\mathcal{I}$ and G is a strong $\mathcal{L}$-involutive basis of $\mathcal{I}$, we have that for all $x^\mu \in \operatorname{lt}(H)$ the set of terms*

$$B = \left(\left\{\frac{\operatorname{lcm}(x^\nu, x^\mu)}{x^\mu} \mid x^\nu \in \operatorname{lt}(G)\right\} \setminus \operatorname{lt}(\mathcal{I})\right) \cup \left(\operatorname{NM}_{\mathcal{L}_\mathcal{I}}\big(x^\mu, \operatorname{lt}(H)\big)\right) \tag{4.7}$$

is an $\mathcal{L}_{\operatorname{lt}(\mathcal{I})}$-involutive basis of the ideal $\langle B\rangle + \operatorname{lt}(\mathcal{I})$ relative to $\operatorname{lt}(\mathcal{I})$.

Theorem 4.2.13. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a polynomial ideal and $\mathcal{L}_\mathcal{I}$ an involutive division relative to $\operatorname{lt}(\mathcal{I})$ of Schreyer type. Furthermore, let G be a strong $\mathcal{L}$-involutive basis of $\mathcal{I}$, where $\mathcal{L}$ is the continuous involutive division on $\mathcal{T}$ inducing $\mathcal{L}_\mathcal{I}$, and H a strong $\mathcal{L}_\mathcal{I}$-involutive basis of $\langle H, \mathcal{I}\rangle$ relative to $\mathcal{I}$. Then, the set $\pi(\mathcal{A}(H,G)) \cup \pi(\mathcal{S}_{\mathcal{L}_\mathcal{I}}(H,G))$, where π is defined as in (4.1), is an $\mathcal{L}_\mathcal{I}$-involutive basis of the relative syzygy module $\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}(H)$.*

Proof. By Theorem 4.1.11, the set $\mathcal{A}(H,G) \cup \mathcal{S}(H,G)$ is mapped by π to a Gröbner basis of the relative syzygy module $\operatorname{Syz}_{\mathcal{R}/\mathcal{I}}(H)$. A closer inspection of the proof of Theorem 4.1.11 shows that in fact the subset of all A-polynomials and S-polynomials with a leading module term whose polynomial part does not belong to $\operatorname{lt}(\mathcal{I})$ suffices. Then, by Proposition 4.2.9, among the remaining S-polynomials, the subset $\mathcal{S}_{\mathcal{L}_\mathcal{I}}(H,G)$ suffices. Among the remaining A-polynomials $\mathbf{S}_{i\alpha}$, note that, if x^μ is the leading term of the basis element $h_i \in H$, then all terms from the set $\{\frac{\operatorname{lcm}(x^\nu, x^\mu)}{x^\mu} \mid x^\nu \in \operatorname{lt}(G)\} \setminus \operatorname{lt}(\mathcal{I})$ appear as polynomial part of $\operatorname{lm}(\mathbf{S}_{i\alpha})$ for some index α. And, since for all i, the minimal generators of the quotient ideal $\operatorname{lt}(\mathcal{I}) : x^\mu$ are included in this set, collecting the corresponding A-polynomials and the S-polynomials from $\mathcal{S}_{\mathcal{L}_\mathcal{I}}(H,G)$, we get a Gröbner basis of the relative syzygy module by projection via π. Since $\mathcal{L}_\mathcal{I}$ is of relative Schreyer type, it is even a relative involutive basis. $\square$

The notion of quasi-stability of monomial ideals is well-behaved with respect to standard ideal operations such as sum, intersection, and quotient of given ideals, see [97, Lem. 5.3.5]. Thus, one expects that the Pommaret division $\mathcal{P}$ induces a relative involutive division of Schreyer type with respect to the leading ideal $\operatorname{lt}(\mathcal{I})$ of an ideal $\mathcal{I}$ in quasi-stable position.

Proposition 4.2.14. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a polynomial ideal in quasi-stable position and $\mathcal{P}$ the Pommaret division on $\mathcal{T}$. Then the relative involutive division $\mathcal{P}_{\operatorname{lt}(\mathcal{I})}$ induced by $\mathcal{P}$ is of Schreyer type.*

Proof. Let G be a strong Pommaret basis of $\mathcal{I}$ and H a strong $\mathcal{P}_\mathcal{I}$-basis of the ideal $\mathcal{J} := \langle H\rangle + \mathcal{I}$ relative to $\mathcal{I}$. Let $x^\mu \in \operatorname{lt}(H)$ be the leading term of a generator $h_i \in H$ and analyse the set of terms $A := \big\{\frac{\operatorname{lcm}(x^\nu, x^\mu)}{x^\mu} \mid x^\nu \in \operatorname{lt}(G)\big\}$. Since A contains the minimal generators of the colon ideal $\mathrm{lt}(\mathcal{I}) : x^\mu$, it is a generating set of this ideal.

We now prove that A is an involutive set for the Pommaret division, i.e., it is a Pommaret basis of its polynomial span. Let $x^\rho \in \langle A \rangle$ be any term in the ideal generated by A. Then $x^\mu x^\rho \in \mathrm{lt}(\mathcal{I})$, so that there exists $x^\nu \in \mathrm{lt}(G)$ with $x^\mu x^\rho \in \mathcal{C}_\mathcal{P}(x^\nu)$. From this it follows that $\mathrm{lcm}(x^\nu, x^\mu) \mid x^\mu x^\rho$ and $\frac{x^\mu x^\rho}{\mathrm{lcm}(x^\nu,x^\mu)} \mid \frac{x^\mu x^\rho}{x^\nu} \in \mathbb{K}[\mathrm{M}_\mathcal{P}(x^\nu)]$. Note that $\mathrm{cls}(\frac{\mathrm{lcm}(x^\nu,x^\mu)}{x^\mu}) \geq \mathrm{cls}(x^\nu)$. In other words, every variable that is Pommaret multiplicative for x^ν is also Pommaret multiplicative for $\frac{\mathrm{lcm}(x^\nu,x^\mu)}{x^\mu}$. Hence, $x^\rho \in \mathcal{C}_\mathcal{P}(\frac{\mathrm{lcm}(x^\nu,x^\mu)}{x^\mu})$, proving the involutivity of the set A with respect to the Pommaret division.

We now turn to an analysis of the set $V := \mathrm{NM}_{\mathcal{P}_{\mathrm{lt}(\mathcal{I})}}(x^\mu)$. It contains exactly those variables x_j with index $j \geq \mathrm{cls}(x^\mu)$ for which additionally $x_j x^\mu \notin \mathrm{lt}(\mathcal{I})$. Let us take a closer look at the variables x_j for which $x_j x^\mu \in \mathrm{lt}(\mathcal{I})$. For such a variable, there necessarily exists a leading term $x^\nu \in \mathrm{lt}(G)$ such that $x_j x^\mu \in \mathcal{C}_\mathcal{P}(x^\nu)$. Since x^μ is an element of the order ideal $\mathcal{T} \setminus \mathrm{lt}(\mathcal{I})$, it follows immediately that $x_j x^\mu = \mathrm{lcm}(x^\mu, x^\nu)$, and so, $x_j = \frac{\mathrm{lcm}(x^\mu,x^\nu)}{x^\mu} \in V$. Consequently, $\langle A, V \rangle = \langle A \rangle + \langle \mathrm{NM}_\mathcal{P}(x^\mu) \rangle$, and since both A and $\mathrm{NM}_\mathcal{P}(x^\mu)$ are (weak) Pommaret bases of the monomial ideals they generate and the Pommaret division is global, by applying [97, Rem. 3.1.13], we have that $A \cup V$, which is equal to $A \cup (\mathrm{NM}_\mathcal{P}(x^\mu))$, is a weak Pommaret basis of $\langle A, V \rangle$.

Finally, by the equivalence (4.4) in Definition 4.2.2, the set of multiplicative variables $\mathrm{M}_{\mathcal{P}_{\mathrm{lt}(\mathcal{I})}}(x^\gamma)$ for any $x^\gamma \notin \mathrm{lt}(\mathcal{I})$ is a superset of $\mathrm{M}_\mathcal{P}(x^\gamma)$, the set of Pommaret multiplicative variables. This proves that $(A \cup V) \setminus \mathrm{lt}(\mathcal{I})$ is a weak $\mathcal{P}_{\mathrm{lt}(\mathcal{I})}$-involutive basis of $\langle (A \cup V) \setminus \mathrm{lt}(\mathcal{I}), \mathrm{lt}(\mathcal{I}) \rangle$ relative to $\mathrm{lt}(\mathcal{I})$. Since obviously $V \cap \mathrm{lt}(\mathcal{I}) = \emptyset$, we have proved that $\mathcal{P}_{\mathrm{lt}(\mathcal{I})}$ is of Schreyer type. □

The natural question is now whether the Janet division relative to a monomial ideal $\mathcal{I}$ is also of Schreyer type. It turns out that it is not; if one takes the minimal Janet basis for $\mathcal{I}$ and the minimal relative Janet basis of $\mathcal{J} \supset \mathcal{I}$ (for a definition, see Subsection 4.2.2), one cannot expect to obtain relative Janet bases when forming sets B defined as in (4.7). Here is a concrete counterexample.

Example 4.2.15. *Let the monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R} = \mathbb{K}[x, y, z]$ be minimally generated by $\mathrm{Min}(\mathcal{I}) = \{x^2y^2z\}$. Since $\mathcal{I}$ is a principal ideal, $\mathrm{Min}(\mathcal{I})$ is also the minimal Janet basis of $\mathcal{I}$. Let $\mathcal{J} = \langle x, y \rangle$; clearly, $\mathcal{J} \supset \mathcal{I}$. Moreover, $\{x, y\}$ is the minimal relative Janet basis of $\mathcal{J}$ with respect to $\mathcal{I}$. For y, every variable is $\mathcal{J}_\mathcal{I}$-multiplicative. For the generator x, only the variable y is non-multiplicative. Now, if one forms the set B as defined in Equation (4.7) for the generator x, one obtains $B = \{y, xy^2z\}$, whose first element is induced by the non-multiplicative variable, the second element being $\frac{\mathrm{lcm}(x^2y^2z,x)}{x}$. This set is autoreduced in the classical sense, so no subset of it is a basis of $\mathcal{J}$ relative to $\mathcal{I}$ in any sense – involutive or not. Furthermore, the variable z is $\mathcal{J}_\mathcal{I}$-non-multiplicative for y, and so the monomial yz is not contained in the relative Janet span of B. Hence, we need to perform an involutive completion on the set B to obtain a relative Janet basis. This example proves that the relative Janet division $\mathcal{J}_\mathcal{I}$ is not of Schreyer type.*

In Example 4.2.15, an important aspect is that we chose *minimal* Janet bases as generating sets. In a sense that will be made more precise in the following discussion, the minimal bases used in Example 4.2.15 are not enough adapted to one another. But one can find supersets of both sets which, joined together, form a Janet basis of the larger ideal $\mathcal{J}$ in the classical sense; moreover the sets B constructed as in (4.7) are then always relative Janet bases.

Lemma 4.2.16. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a monomial ideal generated by a set $G \subset \mathcal{T}$ and $x^\omega = \operatorname{lcm}(G)$ the least common multiple of all generators. Then $\mathcal{I}$ possesses a Janet basis $H \subset \mathcal{I}$ such that all basis elements $x^\mu \in H$ are divisors of x^ω and such that for all $x^\mu, x^\nu \in H$ we have $\operatorname{lcm}(x^\mu, x^\nu) \in H$, i.e., H is closed under the operation of least common multiple.*

Proof. The set $Z = \{x^\mu \in \mathcal{I} : x^\mu \mid x^\omega\}$ is a finite Janet basis of $\mathcal{I}$ (see for instance [47, Prop. 4.5]). Since $G \subseteq Z$, Z can be regarded as a completion of G. $\square$

Remark 4.2.17. *For a generating set G of $\mathcal{I}$, there may in some cases exist Janet bases of $\mathcal{I}$ closed under least common multiples and containing G which are smaller than the set Z introduced in the proof of Lemma 4.2.16. They can be constructed via a completion algorithm which alternates between the addition of non-multiplicative prolongations and the addition of new least common multiples. A termination proof of such a procedure can be obtained by noting that the sets constructed by these additions always remain subsets of the completion Z.*

Proposition 4.2.18. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{R}$ be two polynomial ideals and F a monomial Janet basis of $\operatorname{lt}(\mathcal{J})$ such that $\operatorname{Min}(\operatorname{lt}(\mathcal{J})) \cup \operatorname{Min}(\operatorname{lt}(\mathcal{I})) \subseteq F$ and F is closed under least common multiples. Then $\mathcal{J}$ possesses a strong $\mathcal{J}_{\operatorname{lt}(\mathcal{I})}$-involutive basis H relative to $\mathcal{I}$ such that $\operatorname{lt}(H) = F \setminus \mathcal{I}$. Moreover, if H is ordered according to a $\mathcal{J}_{\operatorname{lt}(\mathcal{I})}$-ordering, then for any $i \in \{1, \dots, |H|\}$, the ith component $\operatorname{lt}_i(\operatorname{Syz}_{\mathcal{R}/\mathcal{I}}(H))$ of the module of leading terms of the relative syzygy module of H has the set $B_i = \left\{\frac{x^\mu}{\operatorname{lt}(h_i)} : x^\mu \in F \wedge \operatorname{lt}(h_i) \mid x^\mu \wedge \operatorname{lt}(h_i) \neq x^\mu\right\}$ as a $\mathcal{J}_{\operatorname{lt}(\mathcal{I})}$-involutive basis.*

Proof. The assumption that F is a Janet basis of $\operatorname{lt}(\mathcal{J})$ implies trivially that the set $\tilde{H} := F \setminus \mathcal{I}$ is a $\mathcal{J}_{\operatorname{lt}(\mathcal{I})}$-involutive basis of $\operatorname{lt}(\mathcal{J})$ relative to $\operatorname{lt}(\mathcal{I})$. For each $x^\mu \in \tilde{H}$ choose a monic polynomial $h_\mu \in \mathcal{J}$ with $\operatorname{lm}(h) = x^\nu$; then $H := \{\operatorname{NF}_{\mathcal{I}}(h_\mu) \mid x^\mu \in \tilde{H}\}$ is a strong Janet basis of $\mathcal{J}$ relative to $\mathcal{I}$. For any index $i \in \{1, \dots, |H|\}$, the monomial ideal $\operatorname{lt}_i(\operatorname{Syz}_{\mathcal{R}/\mathcal{I}}(H))$ is generated by $\operatorname{Min}(\mathcal{I} : \operatorname{lt}(h_i))$ together with the non-multiplicative variables $x_k \in \operatorname{NM}_{\mathcal{J}_{\mathcal{I}}}(\operatorname{lt}(h_i), \operatorname{lt}(H))$. The set $\operatorname{Min}(\mathcal{I} : \operatorname{lt}(h_i)) = \left\{\frac{\operatorname{lcm}(x^\mu, \operatorname{lt}(h_i))}{\operatorname{lt}(h_i)} \mid x^\mu \in \operatorname{Min}(\mathcal{I})\right\}$ is contained in B_i, since F is closed under least common multiples. Moreover, for each non-multiplicative variable $x_k \in \operatorname{NM}_{\mathcal{J}_{\mathcal{I}}}(\operatorname{lt}(h_i), \operatorname{lt}(H))$, the prolongation $x_k \operatorname{lt}(h_i)$ is in the involutive cone of some other leading monomial $\operatorname{lt}(h_j)$ and hence $x_k \operatorname{lt}(h_i) = \operatorname{lcm}(\operatorname{lt}(h_i), \operatorname{lt}(h_j))$. This implies that $x_k \in B_i$. Thus, the set B_i is a basis of the ideal $\operatorname{lt}_i(\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}(H))$.

We still need to show that B_i is a $\mathcal{J}_{\mathcal{I}}$-involutive basis. For this it suffices to show that B_i is a Janet basis of $\langle B_i \rangle$ in the classical sense. By the homotheticity of the Janet division [99, p.265], we have for each $x^\nu \in B_i$ the equality

$\mathrm{M}_{\mathcal{J}}(x^\nu, B_i) = \mathrm{M}_{\mathcal{J}}(\mathrm{lt}(h_i)x^\nu, \mathrm{lt}(h_i)B_i)$. Since $\mathrm{lt}(h_i)B_i \subseteq F$, Axiom (3) of the definition of involutive divisions implies $\mathrm{M}_{\mathcal{J}}(x^\nu \,\mathrm{lt}(h_i), F) \subseteq \mathrm{M}_{\mathcal{J}}(\mathrm{lt}(h_i)x^\nu, \mathrm{lt}(h_i)B_i)$. We claim that this inclusion is in fact an equality. Indeed, let $x_k \in \mathrm{NM}_{\mathcal{J}}(x^\nu \,\mathrm{lt}(h_i), F)$ be any non-multiplicative variable of $x^\alpha := x^\nu \,\mathrm{lt}(h_i) \in F$. Then there exists some $x^\rho \in F$ with $\rho_k > \alpha_k$. But then also $x^\beta := \mathrm{lcm}(x^\rho, x^\alpha) \in F$, and obviously, $x^\beta \in B_i \,\mathrm{lt}(h_i)$. This implies that $x_k \in \mathrm{NM}_{\mathcal{J}}(\mathrm{lt}(h_i)x^\nu, \mathrm{lt}(h_i)B_i)$, because also x^β causes x_k to be non-multiplicative. Thus, we have shown that for each $x^\nu \in B_i$, $\mathrm{M}_{\mathcal{J}}(x^\nu, B_i) = \mathrm{M}_{\mathcal{J}}(x^\nu \,\mathrm{lt}(h_i), F)$. But it is easy to see that the Janet cones of $B_i \,\mathrm{lt}(h_i)$ with respect to F yield the whole ideal $\mathcal{J} \cap \langle \mathrm{lt}(h_i)\rangle$. This finishes the proof. $\square$

Example 4.2.19. *Let us take up again Example 4.2.15. An* lcm*-closed basis of* $\mathcal{J} = \langle x, y, x^2y^2z\rangle$ *is given by* $F = \{x, y, x^2y^2z, xy, xz, xyz, y^2z, xy^2z, yz\}$*. If we order the eight relative generators as* $H = \{x, y, xy, xz, yz, xyz, y^2z, xy^2z\}$ *(this is indeed a* $J_{\mathcal{I}}$*-ordering), we get the following relative Janet bases* B_i *for the ideals* $\mathrm{lt}_i(\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}(H))$*, where* $1 \le i \le 8$*:*

$$\begin{array}{lll} B_1 = \{y, z, yz, y^2z, xy^2z\}, & B_2 = \{x, xz, yz, xyz, z, x^2yz\}, & B_3 = \{z, yz, xyz\}, \\ B_4 = \{y, y^2, xy^2\}, & B_5 = \{x, y, xy, x^2y\}, & B_6 = \{y, xy\}, \\ B_7 = \{x, x^2\}, & B_8 = \{x\}. & \end{array}$$

4.2.2 Computation of Relative Involutive Bases

If one wants to compute a relative involutive basis for an ideal $\mathcal{J} \supseteq \mathcal{I}$ by going over to the respective leading ideals, one sees that a necessary condition is that $\mathrm{lt}(\mathcal{J})$ has a finite involutive basis relative to $\mathrm{lt}(\mathcal{I})$. If one chooses a Noetherian involutive division $\mathcal{L}$, every monomial ideal $\mathcal{Q} \trianglelefteq \mathcal{R}$ has a finite strong $\mathcal{L}$-involutive basis, see [97, Def. 3.1.18]. Thus, a natural choice of a relative involutive division for which one can expect to be able to obtain strong relative involutive bases is a relative division of the form $\mathcal{L}_{\mathcal{I}}$, where $\mathcal{I} \trianglelefteq \mathcal{R}$ is a monomial ideal and $\mathcal{L}$ is a classical Noetherian involutive division.

Lemma 4.2.20. *If* $\mathcal{L}$ *is a Noetherian involutive division,* $\mathcal{I} \trianglelefteq \mathcal{R}$ *a monomial ideal, and* $\mathcal{L}_{\mathcal{I}}$ *the involutive division induced by* $\mathcal{L}$ *relative to* $\mathcal{I}$*, then every monomial ideal* $\mathcal{J} \supseteq \mathcal{I}$ *possesses a strong* $\mathcal{L}_{\mathcal{I}}$*-involutive basis relative to* $\mathcal{I}$*.*

Proof. There exists a strong monomial $\mathcal{L}$-involutive basis $G \subset \mathcal{J}$ of $\mathcal{J}$. The $\mathcal{L}$-involutive span of the set $G \setminus \mathcal{I}$ is a superset of $\mathcal{J} \setminus \mathcal{I}$, since the $\mathcal{L}$-involutive span of G is a superset of $\mathcal{J}$ and by deletion of elements from G the remaining elements cannot lose multiplicative variables. Going over to $\mathcal{L}_{\mathcal{I}}$, the elements of $G \setminus \mathcal{I}$ may be assigned additional multiplicative variables, but no variable multiplicative with respect to $\mathcal{L}$ can become non-multiplicative. Consequently, $G \setminus \mathcal{I}$ is a weak involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$. Performing an $\mathcal{L}_{\mathcal{I}}$-involutive autoreduction, we arrive at a strong involutive basis $H \subseteq G \setminus \mathcal{I}$ of $\mathcal{J}$ relative to $\mathcal{I}$. $\square$

Next to the question of existence of a finite involutive basis, there is also the question whether there exists an algorithmic procedure to compute such a finite involutive basis in a finite number of steps. This is known to be true for classical involutive divisions that are constructive. We will not go into the details of this technical definition here, but rather recall a very important algorithmic property that constructive involutive divisions have (see the above cited literature on involutive bases for more details and proofs, in particular [47, Sec. 4], [97, Sec. 4.1]).

Definition 4.2.21. *Let $H \subset \mathcal{T}$ be a finite set of terms, $\mathcal{L}$ be any involutive division (possibly relative to some monomial ideal $\mathcal{I}$) and x^μ a term in H (in the relative case we assume $H \cap \mathcal{I} = \emptyset$). For any variable $x_i \in \mathrm{NM}_{\mathcal{L}}(x^\mu, H)$, the term $x_i x^\mu$ is called a* non-multiplicative prolongation *of x^μ with respect to the division $\mathcal{L}$ and the set H.*

Theorem 4.2.22. *Let $G \subset \mathcal{T}$ be a finite set of terms and $\mathcal{L}$ a constructive involutive division. Then G is a weak involutive basis of $\langle G \rangle$, if and only if all non-multiplicative prolongations of all elements of G possess an $\mathcal{L}$-involutive divisor in G. Moreover, given any finite monomial set $G \subset \mathcal{T}$, a weak $\mathcal{L}$-involutive basis $\overline{G} \supseteq G$ of $\langle G \rangle$ can be computed in a finite number of steps by adding to G non-multiplicative prolongations which do not possess involutive divisors.*

Now consider a monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ and a relative involutive division $\mathcal{L}_{\mathcal{I}}$ induced by a constructive Noetherian division $\mathcal{L}$. For obtaining an algorithm for the $\mathcal{L}_{\mathcal{I}}$-involutive completion of a set of terms $H \subset (\mathcal{T} \setminus \mathcal{I})$, we want to use a monomial completion algorithm for $\mathcal{L}$. For this, we need a relative version of local involution.

Proposition 4.2.23. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a monomial ideal and $H \subset (\mathcal{T} \setminus \mathcal{I})$ a finite set of terms disjoint from $\mathcal{I}$. Furthermore, let $\mathcal{L}$ be a Noetherian constructive involutive division and $\mathcal{L}_{\mathcal{I}}$ the relative involutive division induced by $\mathcal{L}$ and $\mathcal{I}$. Then H is a weak $\mathcal{L}_{\mathcal{I}}$-involutive basis of $\mathcal{J} := \langle H \rangle + \mathcal{I}$ relative to $\mathcal{I}$, if and only if for all $x^\mu \in H$ and all $x_k \in \mathrm{NM}_{\mathcal{L}_{\mathcal{I}}}(x^\mu, H)$ the non-multiplicative prolongation $x_k x^\mu$ possesses an $\mathcal{L}_{\mathcal{I}}$-involutive divisor in H. Moreover, this criterion of* local involution *translates into an algorithm which computes for any such set H a superset $H \subseteq \overline{H} \subset (\mathcal{T} \setminus \mathcal{I})$ such that $\overline{H}$ is a weak $\mathcal{L}_{\mathcal{I}}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$.*

Proof. Let H be such a finite set and set $\mathcal{J} := \langle H \rangle + \mathcal{I}$. Consider a term $x^\nu \in \mathcal{J} \setminus \mathcal{I}$. By definition of $\mathcal{L}_{\mathcal{I}}$, $x^\nu \in \mathcal{C}_{\mathcal{L}_{\mathcal{I}}}(H)$ if and only if $x^\nu \in \mathcal{C}_{\mathcal{L}}(H)$. This means that local involution of H with respect to $\mathcal{L}_{\mathcal{I}}$ implies that for all $x^\mu \in H$ and all $x_k \in \mathrm{NM}_{\mathcal{L}_{\mathcal{I}}}(x^\mu, H)$, the non-multiplicative prolongation $x_k x^\mu$ is an element of $\mathcal{C}_{\mathcal{L}}(H)$. Let now $x^\nu \in \mathcal{J} \setminus \mathcal{I}$ be any term not contained in $\mathcal{C}_{\mathcal{L}}(H)$. If, whenever $x^\mu \in H$ is a divisor of x^ν and $x_k \in \mathrm{NM}_{\mathcal{L}}(x^\mu, H)$ is a non-multiplicative variable, the non-multiplicative prolongation $x_k x^\mu$ is contained in $\mathcal{C}_{\mathcal{L}}(H)$, then one can construct – just as in the proof of [97, Prop. 4.1.4] – an infinite sequence of elements of H consisting of divisors of x^ν satisfying certain division properties, contradicting the assumption that $\mathcal{L}$ is continuous. Hence we can conclude that $x^\nu \in \mathcal{C}_{\mathcal{L}}(H)$, and so, *a fortiori*, also $x^\nu \in \mathcal{C}_{\mathcal{L}_{\mathcal{I}}}(H)$. In other words, H is a weak $\mathcal{L}_{\mathcal{I}}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$. But since $x^\nu \notin \mathcal{I}$ and $\mathcal{T} \setminus \mathcal{I}$ is an order ideal, the necessary containments of non-multiplicative prolongations of divisors of x^ν are indeed given under our assumptions.

To see that this relative local involution criterion translates to a completion algorithm, note that H is not locally involutive relative to $\mathcal{I}$, if and only if there is a classical $\mathcal{L}$-non-multiplicative prolongation which is contained in $\mathcal{J} \setminus \mathcal{I}$ but not in $\mathcal{C}_{\mathcal{L}}(H)$. Now, the existence of such an algorithm follows from the fact that in the classical monomial involutive completion algorithm, we are free to choose a selection strategy for the analysis of non-multiplicative prolongations, and so, we may give preference to those non-multiplicative prolongations which are not contained in $\mathcal{I}$. In other words, if we run the classical involutive monomial completion algorithm on the set H with this special selection strategy for the non-multiplicative prolongations, then a certain intermediate step will yield a weak involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$. And until this stage, no elements of $\mathcal{I}$ will have been added to the prospective involutive basis in the course of the algorithm at all. □

Proceeding to the more general case of two polynomial ideals $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$, an involutive completion algorithm becomes more complex, since one also has to consider the A-polynomials. But note that if the input set H which generates $\mathcal{J}$ relative to $\mathcal{I}$ is already a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$, then all A-polynomials reduce to zero, and an involutive completion procedure is now again largely equivalent to the combinatorial task of monomial relative involutive completion. To overcome the difficulties posed by inputs which are not relative Gröbner bases, it is useful to keep in mind that we are only interested in a combinatorial decomposition of the part of $\mathcal{J}$ that is disjoint from $\mathcal{I}$ and do not care about any decomposition of $\mathcal{I}$. This suggests to treat S-polynomials – in this context represented by non-multiplicative prolongations – differently than A-polynomials. Concretely, for the non-multiplicative prolongations, we use relative involutive reductions and for A-polynomials the usual relative reductions. The candidate set for an involutive basis will then only be enlarged by normal forms of A-polynomials, if they introduce a completely new leading monomial. Hence, in a suitable terminating completion algorithm, A-polynomials will cease to contribute new elements to the candidate set after a finite number of steps, and the algorithm will only add non-multiplicative prolongations to the candidate set from that point on. These considerations lead to Algorithm 20 which is adapted from [97, Algo. 4.5] which in turn is a slight reformulation of the algorithm originally introduced by Gerdt and Blinkov [47]. The two used reduction algorithms are identical with the division Algorithms 17 and 19 but return only the remainder. The subroutine of involutive head autoreduction is a straightforward adaption of [97, Algo. 4.2] to the relative situation.

Theorem 4.2.24. *Algorithm 20 is correct and terminates.*

Proof. Introduce the notation $H_0 := F$ and let H_k denote the set H after the kth time iteration of the **while** loop with $p \neq 0$. The set H_{k+1} is thus constructed from H_k by first adding a polynomial and then performing an involutive head autoreduction which implies that $\langle \mathrm{lt}(H_k)\rangle \subseteq \langle \mathrm{lt}(H_{k+1})\rangle$. Since the polynomial ring $\mathcal{R}$ is Noetherian, there exists an index ℓ such that $\langle \mathrm{lt}(H_k)\rangle = \langle \mathrm{lt}(H_\ell)\rangle$ for all $k \geq \ell$. Whenever H_{k+1} arises from H_k via the addition of the remainder r of an A-polynomial, $\mathrm{lt}(r)$ does not lie in $\langle \mathrm{lt}(H_k), \mathrm{lt}(\mathcal{I})\rangle$ and hence $\langle \mathrm{lt}(H_k)\rangle \subsetneq \langle \mathrm{lt}(H_{k+1})\rangle$. After the ℓth

Algorithm 20: Relative Involutive Basis

Data: Gröbner basis G of $\mathcal{I} := \langle G\rangle \trianglelefteq \mathcal{R}$, finite set $F \subset \mathcal{R}$ with $F \cap \mathcal{I} = \emptyset$, $\mathrm{NF}_G(F) = F$, constructive Noetherian involutive division $\mathcal{L}$ and its induced relative division $\mathcal{L}_{\mathcal{I}}$.
Result: $\mathcal{L}_{\mathcal{I}}$-involutive basis of $\mathcal{J} := \langle F\rangle + \mathcal{I}$ relative to $\mathcal{I}$

```
begin
    H ⟵ InvolutiveHeadAutoreduction(F, L_I)
    A ⟵ {x^α h | h ∈ H, x^α ∈ Min(lt(I) : lt(h))}
    S ⟵ {xh | h ∈ H, x ∈ NM_{L_I}(lt(h), lt(H))}
    while A ∪ S ≠ ∅ do
        if A ≠ ∅ then
            choose p ∈ A with lt(p) minimal in lt(A);   A ⟵ A \ {p}
            p ⟵ RelativeReduction(p, H, G)
        else
            choose p ∈ S with lt(p) minimal in lt(S);   S ⟵ S \ {p}
            p ⟵ RelativeInvolutiveReduction(p, H, G, L_I)
        if p ≠ 0 then
            H ⟵ InvolutiveHeadAutoreduction(H ∪ {p}, L_I)
            A ⟵ {x^α h | h ∈ H, x^α ∈ Min(lt(I) : lt(h))}
            S ⟵ {xh | h ∈ H, x ∈ NM_{L_I}(lt(h), lt(H))}
    return H
```

time the **while** loop has produced a further generator, therefore only remainders stemming from non-multiplicative prolongations are added and these remainders do not enlarge the leading ideal.

Let $p \in S$ be the non-multiplicative prolongation that is checked for the construction of H_{k+1} with $k \geq \ell$ and let $r \neq 0$ be its remainder after the relative involutive reduction. If $\mathrm{lt}(r) \neq \mathrm{lt}(p)$, then $\mathrm{lt}(r) \prec \mathrm{lt}(p)$. Since $\mathrm{lt}(r)$ is not $\mathcal{L}_{\mathcal{I}}$-involutively reducible by the current set $\mathrm{lt}(H_k)$ of leading monomials and also $\mathrm{lt}(r) \in \mathrm{lt}(H_k) = \mathrm{lt}(H_\ell)$, we see with an argument like in the proof of Proposition 4.2.23 that there must exist a generator $h \in H_k$ and a non-multiplicative variable $x_i \in \mathrm{NM}_{\mathcal{L}_{\mathcal{I}}}(\mathrm{lt}(h), \mathrm{lt}(H_k))$ such that $x_i\,\mathrm{lt}(h) \mid \mathrm{lt}(r)$ and $x_i\,\mathrm{lt}(h) \notin \mathcal{C}_{\mathcal{L}_{\mathcal{I}}}(\mathrm{lt}(H_k))$. Hence $x_i h$ cannot reduce to zero in a relative involutive reduction with respect to H_k and $\mathcal{I}$. But this contradicts the normal selection strategy used in Algorithm 20: the non-multiplicative prolongation $x_i h$ must have already been treated at this stage, since $\mathrm{lt}(x_i h) \prec \mathrm{lt}(p)$. Hence, $\mathrm{lt}(p) = \mathrm{lt}(r)$. This means that after the ℓth time the **while** loop has produced a new generator, the sets H_k are modified in such a way that the effect on the corresponding sets $\mathrm{lt}(H_k)$ is a monomial involutive completion with intercalated involutive autoreductions – a process which terminates, see [97, Rem. 4.2.2]. Hence, Algorithm 20 terminates on all inputs and we call the output set H.

We still have to prove the *correctness* of Algorithm 20. When the set H is returned, the sets S and A must be empty. Since S is empty, the set $\operatorname{lt}(H)$ is locally $\mathcal{L}_{\mathcal{I}}$-involutive and $\mathcal{L}_{\mathcal{I}}$-involutively autoreduced. Hence, it is a strong $\mathcal{L}_{\mathcal{I}}$-involutive basis of $\langle \operatorname{lt}(H)\rangle + \operatorname{lt}(\mathcal{I})$ relative to $\operatorname{lt}(\mathcal{I})$. At this point, however, we have not yet proven that $\langle \operatorname{lt}(H)\rangle + \operatorname{lt}(\mathcal{I}) = \operatorname{lt}(\mathcal{J})$. To this end, enumerate the sets H and $\operatorname{lt}(H)$ according to an $\mathcal{L}_{\mathcal{I}}$-ordering on $\operatorname{lt}(H)$. The term set $\operatorname{lt}(H) \subset \mathcal{T} \setminus \operatorname{lt}(\mathcal{I})$ constitutes a Gröbner basis of $\langle \operatorname{lt}(H)\rangle + \operatorname{lt}(\mathcal{I})$ relative to $\operatorname{lt}(\mathcal{I})$. By Theorem 4.1.11, the sets $\mathcal{S}(\operatorname{lt}(H), \operatorname{Min}(\operatorname{lt}(\mathcal{I})))$ and $\mathcal{A}(\operatorname{lt}(H), \operatorname{Min}(\operatorname{lt}(\mathcal{I})))$ of S-syzygies and A-syzygies induce a Gröbner basis of the relative syzygy module $\operatorname{Syz}_{\mathcal{R}/\operatorname{lt}(\mathcal{I})}(\operatorname{lt}(H))$ via the projection mapping π defined in (4.1). Applying Proposition 4.2.9 and Lemma 4.2.10 to $\operatorname{lt}(\mathcal{I}) \subseteq \langle \operatorname{lt}(H)\rangle + \operatorname{lt}(\mathcal{I})$, we see that by comparing module leading terms, we can replace the set $\mathcal{S}(\operatorname{lt}(H), \operatorname{Min}(\operatorname{lt}(\mathcal{I})))$ of all S-polynomials by the smaller set $\mathcal{S}_{\mathcal{L}_{\mathcal{I}}}(\operatorname{lt}(H), \operatorname{Min}(\operatorname{lt}(\mathcal{I})))$ introduced in Definition 4.2.11. Let $\mathbf{b} \in (\mathcal{R}/\operatorname{lt}(\mathcal{I}))^{|\operatorname{lt}(H)|}$ be a vector with entries b_i. If $\mathbf{b} \in \mathcal{S}_{\mathcal{L}_{\mathcal{I}}}(\operatorname{lt}(H), \operatorname{Min}(\operatorname{lt}(\mathcal{I})))$, then the fact that $S = \emptyset$ at the end of Algorithm 20 implies that $\sum_{i=1}^{|\operatorname{lt}(H)|} b_i \cdot h_i \longrightarrow^*_{H,\mathcal{I}} 0$. If $\mathbf{b} \in \mathcal{A}(\operatorname{lt}(H), \operatorname{Min}(\operatorname{lt}(\mathcal{I})))$, then it follows analgously from $A = \emptyset$ at the end of Algorithm 20 that $\sum_{i=1}^{|\operatorname{lt}(H)|} b_i \cdot h_i \longrightarrow^*_{H,\mathcal{I}} 0$. By Theorem 4.1.15, H is thus a Gröbner basis of $\langle H\rangle + \mathcal{I} = \langle F\rangle + \mathcal{I} = \mathcal{J}$ relative to $\mathcal{I}$. Together with the involutive head reducedness of H and the involutivity of $\operatorname{lt}(H)$, this implies that H is a strong $\mathcal{L}_{\mathcal{I}}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, finishing the proof of correctness of Algorithm 20. $\square$

Example 4.2.25. *Let $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$ be a polynomial ring in three variables endowed with the degree reverse lexicographical ordering $\prec$ with $x_3 \prec x_2 \prec x_1$. Let $\mathcal{I} = \langle g\rangle = \langle x_1x_2 + x_2^2\rangle$ be a principal ideal generated by the Gröbner basis $\{g\}$ and consider $\tilde{F} = \{x_1^2x_2 - x_2^3 + x_3^3,\ -x_1^3 + x_1x_2^2 + x_1x_2x_3\} \subset \mathcal{P}$. The elements of $\tilde{F}$ are not yet in normal form with respect to $\mathcal{I}$. Applying a normal form algorithm, we get the set $F = \{x_3^3,\ -x_1^3 - x_2^3 - x_2^2x_3\}$ with $\langle F\rangle + \mathcal{I} = \langle \tilde{F}\rangle + \mathcal{I}$.*

The data F, $\{g\}$, $\prec$ together with the Janet division $\mathcal{J}_{\operatorname{lt}(\mathcal{I})}$ relative to $\operatorname{lt}(\mathcal{I})$ form a valid input for Algorithm 20. For the Janet division every set of terms is Janet autoreduced. This property carries over to the relative Janet division $\mathcal{J}_{\operatorname{lt}(\mathcal{I})}$ and hence we can ignore the involutive head autoreductions in Algorithm 20. At first, set $H = \{h_1, h_2\}$ with $h_1 = x_3^3$ and $h_2 = -x_1^3 - x_2^3 - x_2^2x_3$. The A-polynomial with minimal leading term is $x_2 \cdot x_3^3$. It can be ignored, because $\operatorname{lcm}(\operatorname{lt}(g), \operatorname{lt}(x_3^3)) = \operatorname{lt}(g) \cdot \operatorname{lt}(x_3^3)$. At this stage, only the A-polynomial $x_2 \cdot (-x_1^3 - x_2^3 - x_2^2x_3)$ is left to check. Its normal form with respect to $\mathcal{I}$ is $-x_2^3x_3$, and this polynomial is reduced with respect to $H \cup \{g\}$. So it is added to H: $h_3 := -x_2^3x_3$. This yields the new A-polynomial $x_1 \cdot (h_3)$. Its normal form with respect to $\mathcal{I}$ is $x_2^4x_3$ and this is a multiple of h_3, so it reduces to zero.

This is the first time that no A-polynomials are left to check ($A = \emptyset$), so we turn to the $\mathcal{J}_{\operatorname{lt}(\mathcal{I})}$-nonmultiplicative prolongations. The variables x_1 and x_2 are multiplicative for all elements of $\operatorname{lt}(H)$, only x_3 is nonmultiplicative for $\operatorname{lt}(h_2)$ and $\operatorname{lt}(h_3)$. Our selection strategy is to choose the $\prec$-minimal prolongation. This is $x_3 \cdot h_3$, which is already involutively reduced and immediately yields the new element $h_4 := -x_2^3x_3^2$.

Its A-polynomial, $x_1 \cdot h_4$, has the normal form $x_2^4 x_3^2$ with respect to $\mathcal{I}$, which is again just a multiple of h_4 and thereby reduces to zero. Again, $A = \emptyset$, so we are asked to consider nonmultiplicative prolongations. The multiplicative variables of $\mathrm{lt}(h_1), \mathrm{lt}(h_2), \mathrm{lt}(h_3)$ are not altered by the addition of $\mathrm{lt}(h_4)$. This entails that we do not need to check $x_3 \cdot h_3$ again at this time. x_3 is the only nonmultiplicative variable of $\mathrm{lt}(h_4)$. This is also the $\prec$-minimal prolongation, so we check $x_3 \cdot h_4$. It reduces to zero involutively via h_1. There is only the prolongation $x_3 \cdot (h_2) = -x_1^3 x_3 - x_2^3 x_3 - x_2^2 x_3^2$ left to check. It reduces involutively to $h_5 := -x_1^3 x_3 - x_2^2 x_3^2$ with $\mathrm{lt}(h_5) = x_1^3 x_3$. The A-polynomial of h_5 is $x_2 \cdot h_5$ and its normal form with respect to $\mathcal{I}$ is $x_2^4 x_3 - x_2^3 x_3^2$. This reduces to zero via h_3 and h_4.

We are again asked to consider nonmultiplicative prolongations and since the multiplicative variables of $\mathrm{lt}(h_1), \dots, \mathrm{lt}(h_4)$ are not altered by adding $\mathrm{lt}(h_5)$, only the prolongation $x_3 \cdot h_5$ remains to be checked. (Note that $x_2 \in \mathrm{M}_{\mathcal{J}_{\mathrm{lt}(\mathcal{I})}}(\mathrm{lt}(h_5), \mathrm{lt}(H))$, because $x_2 \,\mathrm{lt}(h_5) \in \mathrm{lt}(\mathcal{I})$.) $x_3 \cdot h_5 = -x_1^3 x_3^2 - x_2^2 x_3^3$ reduces to $h_6 := -x_1^3 x_3^2$ involutively via h_1. The A-polynomial of h_6 is $x_2 \cdot h_6$ and its normal form with respect to $\mathcal{I}$ is $x_2^4 x_3^2$, which reduces to zero via h_3. So we are again asked to consider non-multiplicative prolongations, and this time no non-zero involutive remainders are computed. Therefore Algorithm 20 returns the strong relative $\mathcal{J}_{\mathrm{lt}(\mathcal{I})}$-involutive basis $H = \{x_3^3,\ -x_1^3 - x_2^3 - x_2^2 x_3,\ -x_2^3 x_3,\ -x_2^3 x_3^2,\ -x_1^3 x_3 - x_2^2 x_3^2,\ -x_1^3 x_3^2\}$.

From the theory of involutive bases in $\mathcal{R}$, it is known that for a given constructive Noetherian division L every monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ possesses a unique *minimal* $\mathcal{L}$-involutive basis, see [48], [97, Cor. 4.2.4]. The proof of this fact is algorithmic in the sense that one can show that the monomial completion algorithm using the addition of non-multiplicative prolongations, applied to the minimal generating set $\mathrm{Min}(\mathcal{I})$, always terminates with this unique minimal basis. This fact, in its turn, is proven by showing that each of the prolongations added during the course of the completion algorithm must necessarily be contained in *every* $\mathcal{L}$-involutive basis of $\mathcal{I}$. In view of Proposition 4.2.23, which shows that the monomial completion procedure can be adapted to the relative situation, this motivates the following definition, which generalises [48, Def. 4.2] to the relative case.

Definition 4.2.26. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{R}$ be two ideals and $\mathcal{L}$ a constructive Noetherian involutive division on $\mathcal{R}$. If $\mathcal{I}$ and $\mathcal{J}$ are monomial ideals and if $H \subseteq \mathcal{J} \setminus \mathcal{I}$ is an $\mathcal{L}_{\mathcal{I}}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, then we say that H is a* minimal relative $\mathcal{L}_{\mathcal{I}}$-involutive basis, *if $H \subseteq \tilde{H}$ for all $\mathcal{L}_{\mathcal{I}}$-involutive bases $\tilde{H}$ of $\mathcal{J}$ relative to $\mathcal{I}$. More generally, we say that a subset $H \subset \mathcal{J} \setminus \mathcal{I}$ is a minimal involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, if H is a strong $\mathcal{L}_{\mathrm{lt}(\mathcal{I})}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$ and $\mathrm{lt}(H)$ is a minimal $\mathcal{L}_{\mathrm{lt}(\mathcal{I})}$-involutive basis of $\mathrm{lt}(\mathcal{J})$ relative to $\mathrm{lt}(\mathcal{I})$.*

Proposition 4.2.27. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{R}$ be two ideals and $\mathcal{L}$ a constructive Noetherian involutive division on $\mathcal{T}$. Then there exists a unique $\mathcal{L}_{\mathrm{lt}(\mathcal{I})}$-involutively autoreduced minimal $\mathcal{L}_{\mathrm{lt}(\mathcal{I})}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$.*

Proof. The general case, for polynomial ideals, follows immediately from the monomial case. The monomial case can be proven by a straightforward adaption of the proofs for $\mathcal{L}$-involutive bases. We sketch here only the main argument which implies that the relevant proofs can be adapted to the relative case. A key point in the classical monomial completion algorithm is the selection strategy for non-multiplicative prolongations, which says that exactly those prolongations which do not possess a strict (non-involutive) divisor among the set of eligible prolongations are valid choices for the next element to be added. By Proposition 4.2.23, for monomial ideals $\mathcal{I}$ and $\mathcal{J}$, a relative $\mathcal{L}_{\mathcal{I}}$-involutive basis of $\mathcal{J}$ can be found by applying the $\mathcal{L}$-involutive completion algorithm to $\operatorname{Min}(\mathcal{J}) \setminus \mathcal{I}$, choosing prolongations which do not lie in $\mathcal{I}$ as long as possible. Now, if there exists any eligible non-multiplicative prolongation which does not lie in $\mathcal{I}$, then there obviously also exists a prolongation which does not lie in $\mathcal{I}$ and which possesses no strict (non-involutive) divisor among all eligible prolongations. This means the selection strategy can be adapted to the relative case, and the proof of existence and uniqueness of minimal relative $\mathcal{L}_{\mathcal{I}}$-involutive bases is thereby reduced to the respective results for $\mathcal{L}$-involutive bases. □

Algorithm 21 combines the ideas behind Algorithm 20 with the classical TQ algorithm for the construction of minimal involutive bases introduced by Gerdt and Blinkov [48] following the formulation given in [97, Alg. 4.6]. We omit an explicit proof of its termination and correctness, as it is obvious from the corresponding proofs for the two underlying algorithms.

4.3 Relative Quasi-Stable Position

It is well-known that Pommaret bases exist only in generic coordinates, more precisely, for ideals in quasi-stable position – see [96, 97]. In [62] a first algorithm for the deterministic construction of such coordinates was developed in the context of differential equations and in [61] it was extended to polynomial ideals. It was based on a comparison of the Janet and Pommaret multiplicative variables of the given basis. Later, an alternative approach to various kinds of stable position based on their combinatorial characterisations was presented in [60]. We will now extend some of these results to the relative setting.

Definition 4.3.1. *Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$ be two monomial ideals. We say that $\mathcal{J}$ is* quasi-stable relative to $\mathcal{I}$, *if for all terms $x^\mu \in \mathcal{J} \setminus \mathcal{I}$ and for all indices i with $\operatorname{cls}(x^\mu) < i \leq n$ there exists an exponent $s \geq 0$ such that either $x_i^s x^\mu \in \mathcal{I}$ or $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \in \mathcal{J}$.*

Remark 4.3.2. *Similar to [98, Lem. 3.4], one can show that it suffices to consider in Definition 4.3.1 the terms in $\operatorname{Min}(\mathcal{J}) \setminus \mathcal{I}$. Quasi-stability relative to $\mathcal{I} = \{0\}$ corresponds to the classical notion of quasi-stability. For $\mathcal{J} \supset \mathcal{I}$ to be quasi-stable relative to $\mathcal{I}$, neither $\mathcal{I}$ nor $\mathcal{J}$ need to be quasi-stable in the classical sense. As a simple example, consider in the ring $\mathcal{R} = \mathbb{K}[x_1, x_2]$ the ideals $\mathcal{I} = \langle x_1^2 x_2, x_1 x_2^2 \rangle$ and $\mathcal{J} = \langle x_1 x_2 \rangle$. One sees readily that $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$, however, neither $\mathcal{J}$ nor $\mathcal{I}$ contains a term of class 2, so both ideals are not quasi-stable.*

Algorithm 21: Minimal Relative Involutive Basis

Data: Gröbner basis G of $\mathcal{I} := \langle G \rangle \trianglelefteq \mathcal{R}$, finite set $F \subset \mathcal{R}$ with $F \cap \mathcal{I} = \emptyset$, $\mathrm{NF}_G(F) = F$, constructive Noetherian involutive division $\mathcal{L}$ and its induced relative division $\mathcal{L}_{\mathcal{I}}$. Also, $\mathrm{lt}(F)$ is $\mathcal{L}_{\mathcal{I}}$-involutively autoreduced.

Result: Minimal $\mathcal{L}_{\mathcal{I}}$-involutive basis of $\mathcal{J} := \langle F \rangle + \mathcal{I}$ relative to $\mathcal{I}$

```
begin
    H ⟵ ∅;   Q ⟵ F
    A ⟵ {x^α h | h ∈ H ∪ Q, x^α ∈ Min(lt(I) : lt(h))}
    while A ∪ Q ≠ ∅ do
        if A ≠ ∅ then
            choose p ∈ A with minimal lt(p) in A;   A ⟵ A \ {p}
            p ⟵ RelativeReduction(p, H ∪ Q, G)
            if p ≠ 0 then
                Q ⟵ Q ∪ {p};   A ⟵ A ∪ {x^α p | x^α ∈ Min(lt(I) : lt(p))}
        else
            choose q ∈ Q with lt(q) minimal in lt(Q);   Q ⟵ Q \ {q}
            q ⟵ RelativeInvolutiveReduction(q, H, G, L_I)
            if q ≠ 0 then
                H' ⟵ {h ∈ H | lt q ≺ lt h};   H ⟵ (H ∪ {q}) \ H'
                Q ⟵ Q ∪ H' ∪ {xh | h ∈ H, x ∈ NM_{L_I}(lt h, lt H)}
                A ⟵ {x^α h | h ∈ H ∪ Q, x^α ∈ Min(lt(I) : lt(h))}
    return H
```

However, we have the following result which is immediately implied by the definitions.

Lemma 4.3.3. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{R}$ be two monomial ideals. If $\mathcal{J}$ is quasi-stable, then $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$. If $\mathcal{I}$ is quasi-stable and $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$, then $\mathcal{J}$ is quasi-stable.*

Proposition 4.3.4. *Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$ be two monomial ideals. Then $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$, if and only if $\mathcal{J}$ possesses a finite Pommaret basis relative to $\mathcal{I}$.*

Proof. Suppose that $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$. Consider the set

$$H := \left\{ x^\rho \cdot x^\mu \mid x^\mu \in \mathrm{Min}(\mathcal{J}) \setminus \mathcal{I} \wedge x^\rho \in \mathbb{K}[x_{\mathrm{cls}(x^\mu)+1}, \ldots, x_n] \wedge \frac{x^\rho \cdot x^\mu}{x_{\mathrm{cls}(x^\mu)}} \notin \mathcal{J} \right\}. \tag{4.8}$$

By Definition 4.3.1, it is not difficult to see that H is finite. Thus, it suffices to show that H is a weak Pommaret basis for $\mathcal{J}$ relative to $\mathcal{I}$. Consider a term $x^\lambda \in \mathcal{J} \setminus \mathcal{I}$. We decompose it as $x^\lambda = x^\rho x^\sigma x^\mu$ where $x^\mu \in \mathrm{Min}(\mathcal{J}) \setminus \mathcal{I}$ is a minimal generator, x^σ

contains only multiplicative variables for x^μ with respect to the relative Pommaret division and x^ρ only non-multiplicative ones. If $x^\rho x^\mu \in H$, then we are done, as $\mathrm{cls}(x^\mu) = \mathrm{cls}(x^\rho x^\mu)$ and x^σ contains only multiplicative variables for $x^\rho x^\mu$. If $x^\rho x^\mu \notin H$, then we choose among all terms $x^\lambda \in \mathcal{J} \setminus \mathcal{I}$ with this property one having the same class and the smallest degree in $x_{\mathrm{cls}(x^\mu)}$. Without loss of generality, assume that our given x^λ is such an element. Therefore, from the definition of H, we conclude that $u := x^\rho \cdot x^\mu / x_{\mathrm{cls}(x^\mu)} \in \mathcal{J}$. Now, two cases may occur. If $u \in \mathcal{I}$, then $x^\lambda \in \mathcal{I}$ in contradiction to our assumptions. Otherwise, we have $u \in \mathcal{J} \setminus \mathcal{I}$ and the degree of u in $x_{\mathrm{cls}(x^\mu)}$ is less than that of x^λ. By our minimality assumption, there exists $v \in H$ which involutively divides u for the Pommaret division relative to $\mathcal{I}$. Thus v also involutively divides x^λ for this division, as $x_{\mathrm{cls}(x^\mu)} x^\sigma$ contains only multiplicative variables for v.

Conversely, suppose that $\mathcal{J}$ has a finite Pommaret basis H relative to $\mathcal{I}$. Arguing by reductio ad absurdum, suppose there exists a term $x^\mu \in \mathcal{J} \setminus \mathcal{I}$ with $\mathrm{cls}(x^\mu) < n$ and $j > \mathrm{cls}(x^\mu)$ such that $x_j^s x^\mu \notin \mathcal{I}$ and $x_j^s x^\mu / x_{\mathrm{cls}(x^\mu)} \notin \mathcal{J}$ for all $s \in \mathbb{N}$. Consider the set $\{x_j^s x^\mu \mid s \in \mathbb{N}\} \subset \mathcal{J} \setminus \mathcal{I}$. Since it is infinite and H is a finite Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$, there exists a generator $x^\nu \in H$ involutively dividing infinitely many of its elements for the Pommaret division relative to $\mathcal{I}$. Let us pick one of these elements, say $x_j^{s_0} x^\mu$. By the mentioned property, x_j must be multiplicative for x^ν and hence $\mathrm{cls}(x^\nu) > \mathrm{cls}(x^\mu)$. But then x^ν must divide $x_j^{s_0} x^\mu / x_{\mathrm{cls}(x^\mu)}$, leading to a contradiction. □

Corollary 4.3.5. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{R}$ be two monomial ideals. $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$, if and only if the term set*

$$P(\mathcal{I}, \mathcal{J}) := \left\{ x^\mu \in \mathcal{J} \setminus \mathcal{I} \mid \frac{x^\mu}{x_{\mathrm{cls}(x^\mu)}} \notin \mathcal{J} \right\}$$

is finite. In this case, $P(\mathcal{I}, \mathcal{J})$ is the unique minimal monomial Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$.

Proof. Suppose that $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$. One sees easily that the set H defined in (4.8) is equal to $P(\mathcal{I}, \mathcal{J})$ and thus it was already shown in the proof of Proposition 4.3.4, that (4.8) is a finite *weak* Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$. There only remains to show that it is in fact a strong basis. Assume that there exist two generators $x^\lambda, x^\mu \in P(\mathcal{I}, \mathcal{J})$ such that $x^\lambda \neq x^\mu$ and $x^\lambda = x^\sigma x^\mu$ where x^σ contains only multiplicative variables for x^μ for the relative Pommaret division. It follows that $\mathrm{cls}(x^\lambda) < \mathrm{cls}(x^\mu)$ and in turn $x^\lambda / x_{\mathrm{cls}(x^\lambda)} \in \mathcal{J}$, leading to a contradiction.

Conversely, suppose that $P(\mathcal{I}, \mathcal{J})$ is finite. Assume that for some term $x^\mu \in \mathcal{J} \setminus \mathcal{I}$, for some index $i > \mathrm{cls}(x^\mu)$ and for each exponent s we have $x_i^s x^\mu / x_{\mathrm{cls}(x^\mu)} \notin \mathcal{J}$ so that $\mathcal{J}$ is not quasi-stable for $\mathcal{I}$. Note that $x_i^s x^\mu$ and x^μ have the same class. Thus, by definition of $P(\mathcal{I}, \mathcal{J})$, for each s the term $x_i^s x^\mu / x_{\mathrm{cls}(x^\mu)}$ must lie in $P(\mathcal{I}, \mathcal{J})$ contradicting its finiteness. □

In the sequel, we use the degree reverse lexicographical ordering $\prec$ with $x_1 \prec \cdots \prec x_n$. The notion of ideals in quasi-stable position can be defined in the relative setting as follows.

Definition 4.3.6. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{R}$ be two polynomial ideals. We say that $\mathcal{J}$ is in* quasi-stable position relative to $\mathcal{I}$, *if* $\operatorname{lt}(\mathcal{J})$ *is quasi-stable relative to* $\operatorname{lt}(\mathcal{I})$.

As a consequence of [97, Thm. 4.3.15] and Lemma 4.3.3, we get the next result.

Proposition 4.3.7. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{R}$ be two homogeneous polynomial ideals. If $\mathbb{K}$ is an infinite field, then a generic linear change of variables transforms $\mathcal{J}$ into quasi-stable position relative to $\mathcal{I}$.*

Thus, given homogeneous ideals $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{R}$, $\mathcal{J}$ need not be in quasi-stable position relative to $\mathcal{I}$, but after a sufficiently general linear change of variables $\Phi : \mathcal{R} \to \mathcal{R}$, the ideal $\Phi(\mathcal{J}) \trianglelefteq \mathcal{R}$ will be in quasi-stable position relative to $\Phi(\mathcal{I})$. Under the assumption that the coefficient field $\mathbb{K}$ is large enough, [60, Alg. 2] describes a deterministic algorithm returning for a given homogeneous ideal a sparse linear change of variables such that the transformed ideal is in quasi-stable position. In our situation where two homogeneous ideals $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$ are given, we look for a linear change of variables Φ such that $\Phi(\mathcal{J})$ is in quasi-stable position relative to $\Phi(\mathcal{I})$.[1] For this, we extend the approach of [60] to the relative case. We use again the ordering $\prec_{\mathcal{L}}$ on ordered tuples of leading terms as stated in Definition 3.3.40.

The definition of quasi-stability relative to a monomial ideal leads immediately to a simple test realised in Algorithm 22. As we are not concerned with efficiency questions here, the test returns in the negative case simply the first obstruction detected.

Algorithm 22: Relative Quasi-Stable Test

Data: A monomial ideal $\mathcal{I}$ and the minimal generating B of the monomial ideal $\mathcal{J}$.
Result: True if $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$ and false otherwise.
begin
  for $x^\mu \in B$ **do**
    for i *from* $\operatorname{cls}(x^\mu) + 1$ *to* n **do**
      if *for each s we have $x_i^s x^\mu \notin \mathcal{I}$ and $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \notin \mathcal{J}$* **then**
        return $(false, x_i, x_{\operatorname{cls}(x^\mu)})$
  return *true*

Based on this test, it is straightforward to design a relative version of the algorithm in [60]. Algorithm 23 is based on the repeated determination of reduced Gröbner bases for our chosen ordering $\prec$ via the function `ReducedGrobnerBasis`(F). A key point is the inner **while** loop ensuring that in each iteration of the outer loop some progress is made – see the discussion in [60].

Theorem 4.3.8. *Algorithm 23 is correct and terminates in finitely many steps for a sufficiently large field $\mathbb{K}$.*

[1] It should be noted that, by Lemma 4.3.3, it follows that this change may be sparser than the change that we need to transform $\mathcal{J}$ into quasi-stable position.

Algorithm 23: Relative Quasi-Stable Position

Data: A homogeneous Gröbner basis G of $\mathcal{I} := \langle G \rangle \trianglelefteq \mathcal{R}$, finite set of homogeneous polynomials $F \subset \mathcal{R}$ with $\langle F \rangle + \mathcal{I} = \mathcal{J}$ and the term ordering $\prec$.
Result: A linear change Φ such that $\Phi(\mathcal{J})$ is in quasi-stable position relative to $\Phi(\mathcal{I})$.

```
begin
    Φ ⟵ The identity linear change
    K ⟵ ReducedGrobnerBasis(G)
    H ⟵ ReducedGrobnerBasis(F)
    A ⟵ RelativeQuasiStableTest(⟨lt(K)⟩, lt(H))
    while A ≠ true do
        φ ⟵ (A[3] ↦ A[3] + A[2]); Φ ⟵ φ ∘ Φ
        K̃ ⟵ ReducedGrobnerBasis(φ(K))
        H̃ ⟵ ReducedGrobnerBasis(φ(H))
        while L(H) ⪰_L L(H̃) do
            φ ⟵ (A[3] ↦ A[3] + A[2]); Φ ⟵ φ ∘ Φ
            K̃ ⟵ ReducedGrobnerBasis(φ(K̃))
            H̃ ⟵ ReducedGrobnerBasis(φ(H̃))
        K ⟵ K̃
        H ⟵ H̃
        A ⟵ RelativeQuasiStableTest(⟨lt(K)⟩, lt(H))
    return Φ
```

Proof. The main issue with this algorithm is its termination. Indeed, it is easy to see that upon termination the output satisfies the specification. Let $\mathcal{J}$ be an ideal which is not in quasi-stable position relative to $\mathcal{I}$, i. e. there exists a term $x^\mu \in \mathcal{J}$ with $x_i^s x^\mu / x_{\mathrm{cls}(x^\mu)} \notin \mathcal{J}$ for some index $i > \mathrm{cls}(x^\mu)$ and for all exponents s and Algorithm 22 will return x_i and $x_{\mathrm{cls}(x^\mu)}$. If we perform now a linear change of coordinates ϕ mapping $x_{\mathrm{cls}(x^\mu)} \mapsto x_{\mathrm{cls}(x^\mu)} + a x_i$ with a positive integer a and keeping all other variables unchanged, then, by [60, Prop. 6.9], $\mathcal{L}(H) \prec_{\mathcal{L}} \mathcal{L}(\tilde{H})$ where H is a Gröbner basis of $\mathcal{J}$ and $\tilde{H}$ is a Gröbner basis of $\phi(\mathcal{J})$. Finally, [60, Thm. 6.11] guarantees the termination of the algorithm in any characteristic for a sufficiently large field $\mathbb{K}$. □

Example 4.3.9. *For a better understanding of Algorithm 23, we illustrate its steps with a concrete example. Let $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$ and consider $\mathcal{I} = \langle x_1x_2 + x_2^2 \rangle$ and $\mathcal{J} = \langle x_1x_3, x_1x_2 + x_2^2 \rangle$. One sees that $\mathcal{J}$ is not in quasi-stable position relative to $\mathcal{I}$. Set $G = \{x_1x_2 + x_2^2\}$ and $H = \{x_1x_3, x_1x_2 + x_2^2\}$. Since $x_2^2 x_1 x_3 \in \langle \mathrm{lt}(G) \rangle$ and $x_3^s x_3 \notin \langle \mathrm{lt}(H) \rangle$ for any s, the algorithm* `RelativeQuasiStableTest` *returns $(false, x_3, x_1)$. Now, by performing the linear change $\phi := x_1 \mapsto x_1 + x_3$ on $\mathcal{I}$ and $\mathcal{J}$, we get $\tilde{G} = \{x_1x_2 + x_2^2 + x_2x_3\}$ and $\tilde{H} = \{x_1x_2 + x_2^2 + x_2x_3, x_1x_3 + x_3^2, x_1x_2^2 + x_2^3\}$.*

Therefore, we have $\mathcal{L}(H) = (x_1x_3, x_2^2) \prec_{\mathcal{L}} \mathcal{L}(\tilde{H}) = (x_3^2, x_3x_2, x_2^3)$*. It can be seen that* $\phi(\mathcal{J})$ *is in quasi-stable position relative to* $\phi(\mathcal{I})$ *and the algorithm terminates.*

As mentioned above, an alternative way to obtain quasi-stable position consists of comparing the Janet and the Pommaret multiplicative variables. We present a relative version of this approach. It is based on the following result (see [96, Prop. 4.3.6, Thm. 4.3.12] for more information).

Lemma 4.3.10. *Let* $\mathcal{J} \trianglelefteq \mathcal{R}$ *be a monomial ideal and* B *a Janet basis for* $\mathcal{J}$ *which is involutively autoreduced with respect to the Pommaret division. Then,* $\mathcal{J}$ *is quasi-stable, if and only if for each term* $x^\mu \in B$ *the sets of Janet respectively Pommaret multiplicative variables coincide.*

In the next lemma, we give a variant of this lemma in relative setting.

Lemma 4.3.11. *Let* $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{R}$ *be two monomial ideals and* $B \subset \mathcal{J} \setminus \mathcal{I}$ *a set of terms Pommaret autoreduced relative to* $\mathcal{I}$ *such that* $\langle B \rangle + \mathcal{I} = \mathcal{J}$*. Then, the following statements hold:*

(1) *For any term* $x^\mu \in B$*, any Pommaret multiplicative variable relative to* $\mathcal{I}$ *is also Janet multiplicative relative to* $\mathcal{I}$*.*
(2) *If for all terms in* B *the sets of Janet and Pommaret multiplicative variables relative to* $\mathcal{I}$ *coincide, then* $\mathcal{J}$ *is quasi-stable relative to* $\mathcal{I}$*.*
(3) *Let* $\mathcal{J}$ *be quasi-stable relative to* $\mathcal{I}$ *and* B *a Janet basis for* $\mathcal{J}$ *relative to* $\mathcal{I}$*. Assume that for the term* $x^\mu \in B$ *the variable* x_i *is Janet multiplicative relative to* $\mathcal{I}$ *and* $x_i^s x^\mu \notin \mathcal{I}$ *for any exponent* s*. Then,* x_i *is also Pommaret multiplicative relative to* $\mathcal{I}$ *for* x^μ*.*

Proof. (1) Assume that x_i is Pommaret multiplicative relative to $\mathcal{I}$ for x^μ. Then two cases may arise. If $x_i x^\mu \in \mathcal{I}$, then, by definition, it is Janet multiplicative relative to $\mathcal{I}$ as well and we are done. Otherwise, x_i is Pommaret multiplicative for x^μ. It is easy to see that B is Pommaret autoreduced. Then, by [47, Prop. 3.10], it follows that x_i is Janet multiplicative with respect to B and this proves the claim.

(2) Suppose that $\mathcal{J}$ is not quasi-stable relative to $\mathcal{I}$. Then there exists a term $x^\mu \in \mathcal{J} \setminus \mathcal{I}$ and an index $\operatorname{cls}(x^\mu) < i \le n$ such that for each $s \ge 0$ we have $x_i^s x^\mu \notin \mathcal{I}$ and $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \notin \mathcal{J}$. If x_i is Janet multiplicative for x^μ, then by assumption it is also Pommaret multiplicative for x^μ, contradicting $\operatorname{cls}(x^\mu) < i$. Otherwise, there exists a term $x^\nu \in B$ such that x_i is Janet multiplicative for x^ν, $\mu_{i+1} = \nu_{i+1}, \ldots, \mu_n = \nu_n$ and $\mu_i < \nu_i$ where $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$. If $\operatorname{cls}(x^\nu) < i$ then x_i is Pommaret non-multiplicative and in turn it is Janet non-multiplicative which leads to a contradiction. Otherwise, $\operatorname{cls}(x^\nu) = i$ and it follows that $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \in \mathcal{J}$ for some s which leads again to a contradiction,

(3) Arguing by reductio ad absurdum, suppose that x_i is not Pommaret multiplicative relative to $\mathcal{I}$ for x^μ. Since $x_i^s x^\mu \notin \mathcal{I}$ for each s then x_i is Janet but not Pommaret multiplicative for x^μ and it follows that $\operatorname{cls}(x^\mu) < i$. From assumption, there exists s such that $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \in \mathcal{J}$. On the other hand, B is a Janet basis for $\mathcal{J}$ relative to $\mathcal{I}$. Thus, there exists $x^\nu \in B$ such that x^ν divides $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)}$

using Janet division relative to $\mathcal{I}$. Since $x_i^s x^\mu \notin \mathcal{I}$ we conclude that $x_i^s x^\mu / x_{\mathrm{cls}(x^\mu)} \notin \mathcal{I}$ and in turn x^ν divides $x_i^s x^\mu / x_{\mathrm{cls}(x^\mu)}$ using the ordinary Janet division. It yields that $\mu_{i+1} = \nu_{i+1}, \dots, \mu_n = \nu_n$ and $\mu_i < \nu_i$ where $\mu = (\mu_1, \dots, \mu_n)$ and $\nu = (\nu_1, \dots, \nu_n)$. We obtain a contradiction with the fact that x_i is Janet multiplicative for x^μ, proving the claim. □

Remark 4.3.12. *The converse of the second item in Lemma 4.3.11 does not hold in general. For example, in the ring $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$, let $\mathcal{I} = \langle x_2^3, x_3^3 \rangle$ and $\mathcal{J} = \langle x_1 x_3, x_2^3, x_3^3 \rangle$. One easily sees that $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$ and that for $x_1 x_3$ – the only generator of $\mathcal{J}$ not in $\mathcal{I}$ – all variables are Janet multiplicative relative to $\mathcal{I}$, whereas only x_1 is also Pommaret multiplicative relative to $\mathcal{I}$.*

Algorithm 24 uses this lemma to compare the Pommaret and the Janet multiplicative variables relative to $\mathcal{I}$ for a set which is a Pommaret autoreduced Janet basis relative to $\mathcal{I}$.

Algorithm 24: Relative Janet-Pommaret Test

Data: A monomial ideal $\mathcal{I}$ and a term set B which is a Janet basis for the monomial ideal $\mathcal{J}$ relative to the monomial ideal $\mathcal{I}$ and Pommaret autoreduced relative to $\mathcal{I}$.

Result: True if for each $x^\mu \in B$ and each x_i, we have either $x_i^s x^\mu \in \mathcal{I}$ for some s or x_i is Janet and Pommaret multiplicative relative to $\mathcal{I}$ and false otherwise.

begin
  for $x^\mu \in B$ **do**
    for i *from* $\mathrm{cls}(x^\mu) + 1$ *to* n **do**
      if *for each s we have $x_i^s x^\mu \notin \mathcal{I}$ and x_i is Janet multiplicative for x^μ* **then**
        return $(false, x_i, x_{\mathrm{cls}(x^\mu)})$
  return *true*

Algorithm 25 follows a similar strategy as Algorithm 23: with the help of Algorithm 24 it constructs deterministically a linear change of coordinates such that $\mathcal{J}$ is in quasi-stable position relative to $\mathcal{I}$. However, instead of reduced Gröbner bases is uses Janet bases relative to $\mathcal{I}$. Precisely, the function `RelativeJanetBasis`(G, F) computes a Janet basis for $\langle F \rangle$ relative to $\langle G \rangle$ which is Pommaret autoreduced relative to $\langle G \rangle$. While classically a Pommaret autoreduced Janet basis of an ideal in quasi-stable position is automatically also a Pommaret basis, the situation is slightly more complicated in the relative case and we need the following additional construction.

Definition 4.3.13. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{R}$ be two monomial ideals and $B \subset \mathcal{J} \setminus \mathcal{I}$ a set of terms with $\langle B \rangle + \mathcal{I} = \mathcal{J}$. Then* Pommaret completion *of B relative to $\mathcal{I}$, denoted by* $\mathrm{PommComp}(\mathcal{I}, \mathrm{B})$, *is the set of all terms $x_{i_1}^{j_1} \cdots x_{i_k}^{j_k} x^\mu \notin \mathcal{I}$ such that $x^\mu \in B$ and for each ℓ we have $i_\ell > \mathrm{cls}(x^\mu)$ and $x_{i_\ell}^{s_\ell} x^\mu \in \mathcal{I}$ for some s_ℓ.*

Corollary 4.3.14. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{R}$ be two monomial ideals and B a Janet basis for $\mathcal{J}$ relative to $\mathcal{I}$ which is Pommaret autoreduced relative to $\mathcal{I}$. Assume that for each term $x^\mu \in B$ and each variable x_i we have either $x_i^s x^\mu \in \mathcal{I}$ for some exponent s or x_i is Janet multiplicative relative to $\mathcal{I}$, if and only if it is also Pommaret multiplicative relative to $\mathcal{I}$. Then $B \cup \mathrm{PommComp}(\mathcal{I}, \mathrm{B})$ is a finite weak Pommaret basis for $\mathcal{J}$ relative to $\mathcal{I}$.*

Algorithm 25: Relative Pommaret Basis

Data: A homogeneous Gröbner basis G of $\mathcal{I} := \langle G \rangle \trianglelefteq \mathcal{R}$, finite set of homogeneous polynomials $F \subset \mathcal{R}$ with $F \cap \mathcal{I} = \emptyset$, $\mathrm{NF}_G(F) = F$ and $\langle F \rangle + \mathcal{I} = \mathcal{J}$ and the term ordering $\prec$.

Result: A linear change Φ such that $\Phi(\mathcal{J})$ has a finite Pommaret basis relative to $\Phi(\mathcal{I})$ and such a basis.

begin
- $\Phi \longleftarrow$ The identity linear change
- $K \longleftarrow G$
- $H \longleftarrow$ `RelativeJanetBasis`(K, F)
- $A \longleftarrow$ `RelativeJanetPommaretTest`$(\langle \mathrm{lt}(K) \rangle, \mathrm{lt}(H))$
- **while** $A \neq$ true **do**
  - $\phi \longleftarrow (A[3] \mapsto A[3] + A[2])$; $\Phi \longleftarrow \phi \circ \Phi$
  - $\tilde{K} \longleftarrow$ `ReducedGrobnerBasis`$(\phi(K))$
  - $\tilde{H} \longleftarrow$ `RelativeJanetBasis`$(\tilde{K}, \phi(H))$
  - $\tilde{A} \longleftarrow$ `RelativeJanetPommaretTest`$(\langle \mathrm{lt}(\tilde{K}) \rangle, \mathrm{lt}(\tilde{H}))$
  - **while** $A \neq \tilde{A}$ **do**
    - $\phi \longleftarrow (A[3] \mapsto A[3] + A[2])$; $\Phi \longleftarrow \phi \circ \Phi$
    - $\tilde{K} \longleftarrow$ `ReducedGrobnerBasis`$(\phi(\tilde{K}))$
    - $\tilde{H} \longleftarrow$ `RelativeJanetBasis`$(\tilde{K}, \phi(\tilde{H}))$
    - $\tilde{A} \longleftarrow$ `RelativeJanetPommaretTest`$(\langle \mathrm{lt}(\tilde{K}) \rangle, \mathrm{lt}(\tilde{H}))$
  - $K \longleftarrow \tilde{K}$
  - $H \longleftarrow \tilde{H}$
  - $A \longleftarrow \tilde{A}$
- **return** $(\Phi, H \cup \mathrm{PommComp}(\mathcal{I}, \mathrm{B})(\langle K \rangle, H))$

Theorem 4.3.15. *Algorithm 25 is correct and terminates in finitely many steps.*

Proof. Assume that we are given a finite generating set F of $\mathcal{J}$ (that is $\langle F \rangle + \mathcal{I} = \mathcal{J}$) and a Gröbner basis G of ideal $\mathcal{I}$. If the algorithm `RelativeJanetPommaretTest` finds an obstruction $(x_i, x_{\mathrm{cls}(x^\mu)})$ for a term $x^\mu \in \mathrm{lt}(F)$, then we claim that it remains also an obstruction for some monomial in $\mathrm{lt}(F) \cup \mathrm{lt}(G)$. We know that $x_i^s x^\mu \notin \mathrm{lt}(\mathcal{I})$ for each s and x_i is Janet but not Pommaret multiplicative for $x^\mu \in \mathrm{lt}(F)$. Suppose that x_i is not Janet multiplicative for $x^\mu \in \mathrm{lt}(F) \cup \mathrm{lt}(G)$, but Janet multiplicative for some $x^\nu \in \mathrm{lt}(G)$. Two cases may occur. If $\mathrm{cls}(x^\nu) < \mathrm{cls}(x^\mu)$, then x_i is not Pommaret multiplicative for x^ν and therefore $(x_i, x_{\mathrm{cls}(x^\mu)})$ remains an obstruction for $x^\nu \in \mathrm{lt}(F) \cup \mathrm{lt}(G)$. Otherwise, we must have $\mathrm{cls}(x^\nu) = i$ and it follows that $x_i^s x^\mu \in \mathrm{lt}(\mathcal{I})$ for some s and this leads to a contradiction, proving the claim. Thus, based on Corollary 4.3.14 and the correctness and termination of the algorithm similar to Algorithm 25 in the classical setting (see [97, Thm. 4.3.12]), the correctness and termination of Algorithm 25 is guaranteed. Finally, we note that, $\mathrm{PommComp}(\mathcal{I}, \mathrm{B})(\langle K\rangle, H)$ is a finite set. $\square$

Example 4.3.16. *To illustrate the steps of Algorithm 25, let us consider again the ideals given in Example 4.3.9. We know that $G = \{x_1x_2 + x_2^2\}$ is the reduced Gröbner basis for $\mathcal{I}$ and $H = \{x_1x_3\}$ is the Janet basis for $\mathcal{J}$ relative to $\mathcal{I}$ which is Pommaret autoreduced relative to $\mathcal{I}$. Since $x_2^2x_1x_3 \in \langle \mathrm{lt}(G)\rangle$, Algorithm* `RelativeJanetPommaretTest` *returns $(false, x_3, x_1)$. By performing the linear change $\phi := x_1 \mapsto x_1 + x_3$ on $\mathcal{I}$ and $\mathcal{J}$, we get $\tilde{G} = \{x_1x_2 + x_2^2 + x_2x_3\}$ and $\tilde{H} = \{x_1x_3 + x_3^2,\ x_1x_2^2 + x_2^3\}$. One sees that $\{x_1, x_2, x_3\}$ is the set of the Janet multiplicative variables for x_3^2 and x_2^3 relative to $\langle x_2x_3\rangle$. Since $x_3x_2^3 \in \mathrm{lt}(\phi(\mathcal{I}))$, Algorithm* `RelativeJanetPommaretTest` *returns true and in turn $\tilde{H}$ is the weak Pommaret basis for $\phi(\mathcal{J})$ relative to $\phi(\mathcal{I})$.*

Example 4.3.17. *Consider in the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, x_2, x_3]$ the monomial ideals $\mathcal{I} = \langle x_2^3,\ x_3^3\rangle$ and $\mathcal{J} = \langle x_1x_3,\ x_2^3,\ x_3^3\rangle$. Since* `RelativeJanetPommaretTest` *returns true, the set $\{x_1x_3,\ x_1x_2x_3,\ x_1x_2^2x_3,\ x_1x_3^2,\ x_1x_2x_3^2,\ x_1x_2^2x_3^2\}$ is a weak Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$.*

4.4 Relative Involutive-like Divisions and Bases

In this section, we present a generalization of the concepts of relative involutive divisions and involutive-like divisions. For a detailed explanation of relative involutive divisions, see Subsection 4.2.1. For the definition and properties of involutive-like divisions, see Section 3.4.

Definition 4.4.1. *Let $\{0\} \neq \mathcal{I} \trianglelefteq \mathcal{R}$ be a nonzero monomial ideal. An* involutive-like division $L_{\mathcal{I}}$ on $\mathcal{T} \setminus \mathcal{I}$ relative to $\mathcal{I}$ *associates to any finite set $U \subset \mathcal{T} \setminus \mathcal{I}$ of terms and any term $u \in U$ a set of* $L_{\mathcal{I}}$-non-multipliers $\bar{L}_{\mathcal{I}}(u, U)$ *given by the terms contained in an irreducible monomial ideal. The powers generating this irreducible ideal are called the* non-multiplicative powers $\mathrm{NMP}_{L_{\mathcal{I}}}(u, U)$ *of $u \in U$. The set of* $L_{\mathcal{I}}$-multipliers $L_{\mathcal{I}}(u, U)$ *is given by the order ideal $\mathcal{T} \setminus \bar{L}_{\mathcal{I}}(u, U)$. For any term $u \in U$, its* relative involutive-like cone *is defined as $\mathcal{C}_{L_{\mathcal{I}}}(u, U) = u \cdot L_{\mathcal{I}}(u, U) \setminus \mathcal{I}$. For a relative involutive-like division, the relative involutive-like cones must satisfy the following conditions:*

(i) For two terms $v \neq u \in U$ with $\mathcal{C}_{L_{\mathcal{I}}}(u, U) \cap \mathcal{C}_{L_{\mathcal{I}}}(v, U) \neq \emptyset$, we have $u \in \mathcal{C}_{L_{\mathcal{I}}}(v, U)$ or $v \in \mathcal{C}_{L_{\mathcal{I}}}(u, U)$.

(ii) If a term $v \in U$ lies in an involutive cone $\mathcal{C}_{L_{\mathcal{I}}}(u, U)$, then $L_{\mathcal{I}}(v, U) \subset L_{\mathcal{I}}(u, U)$.

It is straightforward to prove that from an involutive-like division L on $\mathcal{T}$, one can derive a relative involutive-like division $L_{\mathcal{I}}$ by using the same rule for the assignment of non-multiplicative powers as for L and merely adapting the cones to make them subsets of $\mathcal{T} \setminus \mathcal{I}$. One can do this, in particular, for the important special case of the Janet-like division J:

Definition 4.4.2. *Let $\mathcal{I} \lhd \mathcal{R}$ be a nonzero monomial ideal and let $U \subset \mathcal{T} \setminus \mathcal{I}$ be a finite set of terms. Let $u \in U$ be a term. Then the non-multiplicative powers of u with respect to U, $\mathcal{I}$ and the relative Janet-like division $J_{\mathcal{I}}$ are defined as follows:*

$$\mathrm{NMP}_{J_{\mathcal{I}}}(u, U) = \mathrm{NMP}_J(u, U) \setminus (\mathcal{I} : u). \tag{4.9}$$

In other words, the relative Janet-like division uses the same rule for the assignment of non-multiplicative powers as the Janet-like division J, but it excludes variable powers that form part of the ideal quotient associated to the term u in question.

If x_a is a variable for which a relative Janet-like non-multiplicative power for u exists, then we write the exponent of this power as

$$p(J_{\mathcal{I}}, u, U, a).$$

Remark 4.4.3. *The relative Janet-like division $J_{\mathcal{I}}$ is an involutive like division relative to $\mathcal{I}$. This fact can be easily proven by using the properties of the Janet-like division J. Also other properties like the continuity of the Janet-like division J are inherited by $J_{\mathcal{I}}$.*

There are a number of different options how one could define the Pommaret-like division $P_{\mathcal{I}}$ relative to a monomial ideal $\mathcal{I}$. One possibility is using the same assignment of non-multiplicative powers that the Pommaret-like division P also employs. However, this is not an optimal choice for the definition. The definition should aim to guarantee that the following properties are fulfilled:

(1) Cones should of course be disjoint if they are not contained in each other.
(2) For $u \in U$, no non-multiplicative powers should be assigned for variables $x_1, , \dots, x_{\mathrm{cls}(u)}$.
(3) Relative Pommaret-like bases for an ideal $\mathcal{A} \supset \mathcal{I}$ should exist if and only if $\mathcal{A}$ is quasi-stable relative to $\mathcal{I}$.
(4) A unique minimal relative Pommaret-like bases should exist for any ideal $\mathcal{A}$ that is quasi-stable relative to $\mathcal{I}$.
(5) The minimal relative Pommaret-like basis should be as small as possible.

The following definition is designed such as to guarantee properties (1)–(5):

Definition 4.4.4. *Let $\{0\} \neq \mathcal{I} \trianglelefteq \mathcal{R}$ be a nonzero monomial ideal. The* Pommaret-like division $P_{\mathcal{I}}$ relative to $\mathcal{I}$ *assigns to each term $u \in \mathcal{T}$ contained in a finite set of terms $U \subset \mathcal{T} \setminus \mathcal{I}$ non-multiplicative powers as follows: For each x_a with $a > \mathrm{cls}(u)$, if $x_a \in \mathrm{NM}_{\mathcal{J}}(u, U)$, then set $p(P_{\mathcal{I}}, u, U, a) = p(J_{\mathcal{I}}, u, U, a)$. If $x_a \in \mathrm{M}_{\mathcal{J}}(u, U)$ and there does not exist any exponent $s \in \mathbb{N}$ with $u \cdot x_a^s \in \mathcal{I}$, set $p(P_{\mathcal{I}}, u, U, a) = 1$. No other variable gets assigned a non-multiplicative power with respect to the relative Pommaret-like division $P_{\mathcal{I}}$. In particular, no variable x_b with $b \leq \mathrm{cls}(u)$ is assigned a relative Pommaret-like non-multiplicative power for the term u.*

Proposition 4.4.5. *The relative Pommaret-like division $P_{\mathcal{I}}$ is a relative involutive-like division.*

Proof. Let $u \neq v \in U$ be two terms contained in the finite subset $U \subset \mathcal{T} \setminus \mathcal{I}$. Let $k = \max\{\operatorname{cls}(u), \operatorname{cls}(v)\}$. If $k = n = \operatorname{cls}(u) = \operatorname{cls}(v)$, the disjointness of the relative Pommaret-like cones is easily seen, as also in the case where $k = n$ but one of $\operatorname{cls}(u), \operatorname{cls}(v)$ is less than n. If $k < n$ and the projections $u|_{\mathbb{K}[x_{k+1},\dots,x_n]}$, $v|_{\mathbb{K}[x_{k+1},\dots,x_n]}$ are equal, then either disjointness or containment of the relative Pommaret-like cones is also easily seen. It remains the case when $k < n$ but the projections on the subring $\mathbb{K}[x_{k+1}, \dots, x_n]$ are not equal. There, note that from any two elements $u' = su$ and $v' = tv$, where s and t are multiplicative terms, we get in the subring that the projections of s and t are Janet-like multipliers of the projections of u and v. Hence the projections of the relative Pommaret-like cones of u and v on the same subring are either contained one in the other or they are disjoint. If they are disjoint, the same also holds true for the full cones in the whole ring $\mathcal{R}$. If they are contained one in the other, then checking the k-degrees of u and v will yield that the full cones are either disjoint or contained. A containment will hold if and only if the term with larger class, without loss of generality v, has a smaller or equal x_k-degree compared to that of the other term and the projection of the cone of v in the subring is a superset of the other cone projection. □

Definition 4.4.6. *Let $\{0\} \neq \mathcal{I} \trianglelefteq \mathcal{R}$ be a nonzero monomial ideal and let $\mathcal{A} \supset \mathcal{I}$ be a further monomial ideal in $\mathcal{R}$. Let $L_{\mathcal{I}}$ be an involutive-like division relative to $\mathcal{I}$. A finite set of terms $H \subset \mathcal{T} \cap (\mathcal{A} \setminus \mathcal{I})$ is called a* weak $L_{\mathcal{I}}$-involutive like basis of $\mathcal{A}$ relative to $\mathcal{I}$ *if every term $t \in \mathcal{T} \cap (\mathcal{A} \setminus \mathcal{I})$ has an $L_{\mathcal{I}}$ involutive-like divisor in the set H. The weak $L_{\mathcal{I}}$-involutive like basis is called* strong $L_{\mathcal{I}}$-involutive like basis of $\mathcal{A}$ relative to $\mathcal{I}$ *if every term $t \in \mathcal{T} \cap (\mathcal{A} \setminus \mathcal{I})$ has a unique $L_{\mathcal{I}}$ involutive-like divisor in the set H.*

Proposition 4.4.7. *Let $\mathcal{I} \subset \mathcal{A} \trianglelefteq \mathcal{R}$ be two monomial ideals such that $\mathcal{A}$ is quasi-stable relative to $\mathcal{I}$. Then there exists a relative Pommaret-like basis of $\mathcal{A}$.*

Proof. By Proposition 4.3.4, we know that there exists a relative Pommaret basis H of $\mathcal{A}$. Since relative Pommaret-like cones are always supersets of relative Pommaret cones, the set H is also a relative Pommaret-like basis of $\mathcal{A}$. (It need not be a minimal relative Pommaret-like basis, though.) □

Proposition 4.4.8. *Let $\mathcal{I} \subset \mathcal{A} \trianglelefteq \mathcal{R}$ be two monomial ideals such that $\mathcal{A}$ is not quasi-stable relative to $\mathcal{I}$. Then there does not exist any relative Pommaret-like basis of $\mathcal{A}$ relative to $\mathcal{I}$.*

Proof. Assume that there exists a Pommaret-like basis $H \subset \mathcal{T} \cap (\mathcal{A} \setminus \mathcal{I})$ of $\mathcal{A}$ relative to $\mathcal{I}$. In particular, H is a generating set of $\mathcal{A}$ relative to $\mathcal{I}$. Since $\mathcal{A}$ is not quasi-stable relative to $\mathcal{I}$, there is a term $1 \neq h \in H$ and an index $j > k =: \operatorname{cls}(h)$ such that for every exponent $s \in \mathbb{N}$ we have $(h/x_k)x_j^s \notin \mathcal{A}$ and $hx_j^s \notin \mathcal{I}$. Consider the Janet class $C := H_{[\deg_{j+1}(h),\dots,\deg_n(h)]}$. Among the terms in C, there is one with maximal x_j-degree. Let this degree be denoted by d.

Now, since H is a relative Pommaret-like basis, the term $h \cdot x_j^{d-\deg_j(h)}$ has a $P_{\mathcal{I}}$-divisor u in H. By the definition of the $P_{\mathcal{I}}$-like division, u must be an element of the Janet class $H_{[d,\deg_{j+1}(h),\ldots,\deg_n(h)]}$. Moreover, it must be a divisor (in the non-involutive sense) of $h \cdot x_j^{d-\deg_j(h)}$, so there is a term $t \in \mathcal{T}$ with $h \cdot x_j^{d-\deg_j(h)} = u \cdot t$. Now, if there were an exponent $e \in \mathbb{N}$ with $u \cdot x_j^e \in \mathcal{I}$, then also $u \cdot t \cdot x_j^e \in \mathcal{I}$ and hence $h \cdot x_j^{e+d-\deg_j(h)} \in \mathcal{I}$, in contradiction to the assumptions made for h. Hence, such an exponent e does not exist. Moreover, by construction, $x_j \in M_{\mathcal{J}}(u, H)$. Additionally, it is not possible that $\mathrm{cls}(u) \geq j$, because otherwise u would be a divisor of $(h/x_k)x_j^{d-\deg_j(h)}$, again in contradiction to the assumptions made for h.

By the statements just shown, and by Definition 4.4.4, the $P_{\mathcal{I}}$-non-multiplicative power of u with respect to the set H is x_j^1. Now, $u \cdot x_j \in \mathcal{A} \setminus \mathcal{I}$, and it cannot have any $P_{\mathcal{I}}$-divisor in the set H, since such a divisor would be an element of a Janet class $H_{[d+1,\deg_j(h),\ldots,\deg_n(j)]}$. But this Janet class is empty by the maximality property of d. All in all, we have shown that there is a term in $\mathcal{A} \setminus \mathcal{I}$ which has no $P_{\mathcal{I}}$-like divisor in the set H. This contradicts the assumption that H is a relative Pommaret-like basis of H, finishing the proof. $\square$

Example 4.4.9. *Consider the ideals $\mathcal{I} = \langle x^6, y^6, z^6 \rangle$ and $\mathcal{A} = \langle \mathcal{I}, xy, yz \rangle$. These ideals are taken from [42, Ex. 4.12]. Here, the set $H = \{xz, yz\}$ is a relative Pommaret-like basis of $\mathcal{A}$.*

To see this, use as a first step the Janet division to see that $\mathrm{M}_{\mathcal{J}}(yz, H) = \{x, y, z\}$ and $\mathrm{M}_{\mathcal{J}}(xz, H) = \{x, z\}$. The Janet-like non-multiplicative powers of xz are $\mathrm{NMP}(xz, H) = \{y\}$. Note that for each term $h \in H$ and for each variable in the ring, by multiplying h with a high enough power of the variable, we get a term in $\mathcal{I}$. Hence, the relative Pommaret-like non-multiplicative powers of the terms in H are $\mathrm{NMP}_{P_{\mathcal{I}}}(yz, H) = \emptyset$, $\mathrm{NMP}_{P_{\mathcal{I}}}(xz, H) = \{y\}$.

Now, it is clear that all non-multiplicative multiples of xz are in the relative Pommaret-like cone of yz and H is a relative Pommaret-like basis as claimed.

Example 4.4.9 can be generalized. First, we need two definitions. The first is taken from [42]. The second goes back essentially to [29]. We adapt both to our conventions on variable orderings.

Definition 4.4.10. *We call a monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ generated by squarefree terms* squarefree Borel, *if for any (necessarily squarefree) term $s \in \mathrm{Min}(\mathcal{I})$ the following holds: For any variable $x_i \in \mathrm{supp}(s)$ and any index j with $i < j \leq n$ such that $x_j \notin \mathrm{supp}(s)$, $(s/x_i) \cdot x_j \in \mathcal{I}$.*

Definition 4.4.11. *We say that an irreducible, non-zero monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is* Clements-Lindström, *if $\mathrm{Min}(\mathcal{I})$ is of the form $\{x_i^{a_i}, x_{i+1}^{a_{i+1}}, \ldots, x_n^{a_n}\}$ with $2 \leq a_n \leq a_{n-1} \leq \cdots \leq a_{i+1} \leq a_i$. We call $\mathcal{R}/\mathcal{I}$ a* Clements-Lindström ring.

Proposition 4.4.12. *Let $\mathcal{I}$ be a zero-dimensional Clements-Lindström ideal and let H be the minimal generating set of a squarefree monomial ideal that is squarefree Borel. Then the Ideal $\mathcal{A} = \langle \mathcal{I}, H \rangle$ is quasi-stable relative to $\mathcal{I}$ and the set H is the minimal relative Pommaret-like basis of $\mathcal{A}$ relative to $\mathcal{I}$.*

Proof. As a zero-dimensional Clements-Lindström ideal, $\mathcal{I} = \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle$ with $a_1 \geq \ldots \geq a_n \geq 2$. The squarefree minimal generating set H of $\mathcal{A}$ is disjoint from $\mathcal{I}$. Hence, it is the minimal generating set of $\mathcal{A}$ relative to $\mathcal{I}$. Exclude in the following the trivial special case $H = \{1\}$. H fulfils the squarefree Borel property. Hence, for any term $1 \neq h \in H$ and any index $j > k = \mathrm{cls}(h)$ such that $x_j \notin \mathrm{supp}(h)$, there is another term $u \in H$ dividing $(h/x_k)x_j$. Since h is a minimal generator, $\mathrm{cls}(u) \leq j$. Applying the squarefree Borel property repeatedly on u and further derived terms of class $\leq j$, dividing out variables with indices $< j$ and multiplying by variables of $\mathrm{supp}(h)$ with indices $> j$, we arrive at a term $v \in H$ dividing $(h/x_k)x_j$ with $\mathrm{cls}(v) \leq j$ and

$$\mathrm{supp}(v) = \{x_j\} \cup \{x_a \in \mathrm{supp}(h) \mid a \geq \mathrm{cls}(v)\}.$$

Both h and v are in the Janet class $H_{[\deg_{j+1}(h),\ldots,\deg_n(h)]}$ and this shows that $x_j \in \mathrm{NM}_{\mathcal{J}}(h, H)$ and x_j is a $P_{\mathcal{I}}$-non-multiplicative power of h with respect to H. On the other hand, each variable $x_a \in \mathrm{supp}(h)$ with $a > \mathrm{cls}(h)$ is in $\mathrm{M}_{\mathcal{J}}(h, H)$, because H is squarefree and $\deg_a(h) = 1$. Additionally, for each such variable x_a, of course there is an exponent $e \in \mathbb{N}$ such that $hx_a^e \in \mathcal{I}$, because $\mathcal{I}$ is zero-dimensional. Hence, for such variables x_a, no $P_{\mathcal{I}}$-non-multiplicative power exists for h.

Applying a local involution argument, we see that H is a relative Pommaret-like basis of $\mathcal{A}$ and this finishes the proof. □

Chapter 5

Resolutions Induced by Relative Pommaret-like Bases

In Subsection 4.2.1 we saw that for ideals in relative quasi-stable position with respect to an ideal $\mathcal{I}$ in quasi-stable position, the relative Pommaret division $\mathcal{P}_{\mathcal{I}}$ is of relative Schreyer type. Hence, it can be applied to compute syzygies and resolutions of ideals in quasi-stable position relative to $\mathcal{I}$. Moreover, in Section 3.5 we established a syzygy theory also for Pommaret-like bases and in Section 4.4 we studied relative involutive-like bases. The present chapter will combine these results and present an analysis of the resolutions induced by Pommaret and Pommaret-like bases.

In Section 5.1 we will start by analysing the resolutions induced by relative Pommaret bases. We focus on obtaining minimal Pommaret bases for the syzygy modules in each homological degree and observe phenomena that distinguish the relative situation from the case of resolutions over $\mathcal{R}$. Pommaret-like bases are generally smaller than their Pommaret counterparts and hence they provide better chances to obtain minimal resolutions. In Section 5.2 we study Pommaret-like induced resolutions over $\mathcal{R}$ and in Section 5.3 we analyse Pommaret-like induced resolutions over Clements-Lindström rings. We obtain a combinatorial formula for the bigraded Betti numbers of the induced resolutions when they are minimal. In Section 5.4, we obtain for some classes of monomial ideals in $\mathcal{R}$ explicit formulas for the differential of the Pommaret-like induced resolution, generalizing for example constructions by Eliahou and Kervaire [37] and by Seiler et al. [97, 4].

5.1 Resolutions via Relative Pommaret Bases

Let $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$, let $\mathcal{I} \trianglelefteq \mathcal{R}$ be an ideal and let $\mathcal{R}/\mathcal{I}$ denote the quotient ring defined by $\mathcal{I}$. Let $\mathcal{J} \supset \mathcal{I}$ be an ideal in $\mathcal{R}$, then the quotient ring $\mathcal{R}/\mathcal{J}$ can be regarded as an $\mathcal{R}/\mathcal{I}$-module. After a suitable linear change of coordinates, we can assume that both ideals, $\mathcal{I}$ and $\mathcal{J}$ are simultaneously in quasi-stable position. Hence, they possess minimal Pommaret bases and the ideal $\mathcal{J}$, in particular, has a minimal Pommaret basis $H \subset \mathcal{R} \setminus \mathcal{I}$

relative to $\mathcal{I}$. We can obtain a (not necessarily minimal) Pommaret basis of the syzygies of $\mathcal{J}$ relative to $\mathcal{I}$ consisting of the A-syzygies of H relative to $\mathcal{I}$ and the non-multiplicative prolongations of the elements of H.

The procedure described above can be interpreted as the first step in the computation of a resolution of $\mathcal{R}/\mathcal{J}$ via free $\mathcal{R}/\mathcal{I}$-modules; however, note that for this first step, *any* basis of the syzygy module of the relative Pommaret basis H suffices. Hence, we can extract the minimal relative Pommaret basis of the syzygies of H; this minimal basis can be extracted by inspection of leading module terms. Once this minimal basis is computed, one can apply the construction again, obtaining a relative Pommaret basis of the second syzygy module, which can again be minimized, and so on. Note that the obtained resolution will be infinite in general.

The relative basis H with which the computation starts and also the relative Pommaret bases obtained during the computation will in general not be minimal generating sets of the modules they generate. This is compensated for by additional combinatorial structure of the bases, and for the analysis of the resolution, we want to show that some of its properties can be read off already from the (relative) Pommaret bases of $\mathcal{J}$ and $\mathcal{I}$ given at the beginning.

Let us look at the following example taken from [73, Ex. 5.2]: $\mathcal{R} = \mathbb{K}[x, y]$, $\mathcal{I} = \langle x^3, y^3\rangle$, $\mathcal{J} = \langle x^2, xy, y^2\rangle$. Note that the monomial ideals $\mathcal{I}$ and $\mathcal{J}$ are quasi-stable. The minimal Pommaret basis of $\mathcal{I}$ is given by $G = \{x^3, x^3y, x^3y^2, y^3\}$ and the minimal Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$ is gven by $H = \{x^2, xy, y^2\}$. We can now apply Proposition 4.2.14 to obtain a Pommaret basis for $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}(H)$—note that G and H are already ordered according to a $\mathcal{P}_{\mathcal{I}}$ ordering. Precisely, the enumerations are $g_1 = x^3, g_2 = x^3y, g_3 = x^3y^2, g_4 = y^3$ and $h_1 = x^2, h_2 = xy, h_3 = y^2$.

Let us recall Definition 4.2.12 of the concept of an involutive division $\mathcal{L}_{\mathcal{I}}$ relative to a monomial ideal $\mathcal{I}$ of *Schreyer type*. Central to it was the definition of the following set, where $x^\mu \in H$ is a given term:

$$B_\mu = \left(\left\{\frac{\mathrm{lcm}(x^\nu, x^\mu)}{x^\mu} \mid x^\nu \in G\right\} \setminus \mathcal{I}\right) \cup \left(\mathrm{NM}_{\mathcal{P}_{\mathcal{I}}}(x^\mu, H)\right).$$

By Proposition 4.2.13, the A-syzygies and the S-syzygies induced by $\mathcal{P}_{\mathcal{I}}$-non-multiplicative prolongations of elements of H form a Pommaret basis of $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}(H)$, and the module leading terms in this basis in the module component of $(\mathcal{R}/\mathcal{I})^{|H|}$ associated to x^μ are exactly the terms in B_μ, and $\bigcup_{x^\mu \in H} B_\mu$ forms a relative Pommaret basis of $\mathbf{lt}(\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}(H))$ with respect to the Schreyer ordering.

We can now compute the first syzygy module of H relative to $\mathcal{I}$, being a subset of the free $\mathcal{R}/\mathcal{I}$-module $(\mathcal{R}/\mathcal{I})^3$ with the canonical basis $\{\mathbf{e}_1^{(1)}, \mathbf{e}_2^{(1)}, \mathbf{e}_3^{(1)}\}$ (the superscript encodes the homological degree.) We underline the leading module terms.

- As A-syzygies, we obtain $\mathbf{A}_1 = \underline{x}\mathbf{e}_1^{(1)}$, $\mathbf{A}_2 = \underline{xy}\mathbf{e}_1^{(1)}$, and $\mathbf{A}_3 = \underline{xy^2}\mathbf{e}_1^{(1)}$ for h_1, $\mathbf{A}_4 = \underline{x^2}\mathbf{e}_2^{(1)}$, $\mathbf{A}_5 = \underline{x^2y}\mathbf{e}_2^{(1)}$, and $\mathbf{A}_6 = \underline{y^2}\mathbf{e}_2^{(1)}$ for h_2, as well as $\mathbf{A}_7 = \underline{y}\mathbf{e}_3^{(1)}$ for h_3.
- As syzygies from non-multiplicative prolongations, we obtain $\mathbf{S}_1 = \underline{y}\mathbf{e}_1^{(1)} - x\mathbf{e}_2^{(1)}$ for h_1 and $\mathbf{S}_2 = \underline{y}\mathbf{e}_2^{(1)} - x\mathbf{e}_3^{(1)}$ for h_2.

We notice immediately that the relative Pommaret basis $\{\mathbf{A}_1,\dots,\mathbf{A}_7,\mathbf{S}_1,\mathbf{S}_2\}$ is not minimal. The leading ideals in the three module components are as follows:

- in the first component, $\{x, xy, xy^2, y\}$ with corresponding syzygies $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{S}_1$;
- in the second component, $\{x^2, x^2y, y^2, y\}$ with corresponding syzygies $\mathbf{A}_4$, $\mathbf{A}_5$, $\mathbf{A}_6$, $\mathbf{S}_2$;
- in the last component, $\{y\}$, with corresponding syzygy $\mathbf{A}_7$.

The syzygies $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_5$, $\mathbf{A}_6$ are redundant, because their module leading terms do not belong to the minimal relative Pommaret bases, which are $\{x, y\}$ in the first component, $\{x^2, y\}$ in the second, and $\{y\}$ in the last component. Leaving out these redundant elements, we obtain the minimal relative Pommaret basis $H^{(1)} = \{\mathbf{A}_1,\mathbf{S}_1,\mathbf{A}_4,\mathbf{S}_2,\mathbf{A}_7\}$ of $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}(H)$. Rename the elements in the given order as $\mathbf{H}^{(1)}_1,\dots,\mathbf{H}^{(1)}_5$. This is indeed a $\mathcal{P}_{\mathcal{I}}$-ordering respecting the involutive structure, as one can check easily. From the formula for the Betti numbers of the $\mathcal{R}/\mathcal{I}$-module $\mathcal{R}/\mathcal{J}$ given in [73, Ex. 5.2], we know that the basis just obtained is still not a minimal generating system. In order to study the nontrivial relations among the elements of the Pommaret basis $H^{(1)}$, we may as well proceed with the next step in our computation of the Pommaret-induced resolution; we expect to find a set of two independent relations between the elements of $H^{(1)}$ which allow to minimise $H^{(1)}$ as a generating set.

However, we first optimize the construction of relative Pommaret bases of syzygy modules. As seen in the example computation just performed, the relative Pommaret bases computed for the syzygy modules of an $\mathcal{R}/\mathcal{I}$-free resolution are generally non-minimal. While it is true that relative divisions of Schreyer type in the sense of Definition 4.2.12 are suitable for the computation of free resolutions, the construction is not optimal. The reason for this is that the set B in Equation (4.7) is not chosen optimally. Indeed, it is in general not autoreduced with respect to classical (non-restricted) division, because, the multipliers of the form $\mathrm{lcm}(x^\nu, x^\mu)/x^\mu$, which are needed for the A-syzygies, may be divisible by non-multiplicative variables. Thus, it is natural to propose the following adapted definition:

Definition 5.1.1. *Let $\mathcal{I}$ and $\mathcal{L}_{\mathcal{I}}$ be as in Definition 4.2.12. Then $\mathcal{L}_{\mathcal{I}}$* induces strong bases for syzygies *if, whenever H is a strong $\mathcal{L}_{\mathcal{I}}$-involutive basis of $\langle H\rangle + \mathcal{I}$ relative to $\mathcal{I}$ and G is a strong $\mathcal{L}$-involutive basis for $\mathcal{I}$, then for all $x^\mu \in \mathrm{lt}(H)$, for the set of* multiplicative A-multipliers $M(x^\mu, \mathrm{lt}(H), \mathrm{lt}(G))$ *defined by*

$$M(x^\mu, \mathrm{lt}(H), \mathrm{lt}(G)) = \left\{ \frac{\mathrm{lcm}(x^\nu, x^\mu)}{x^\mu} \mid x^\nu \in \mathrm{lt}(G) \right\} \setminus (\mathrm{lt}(\mathcal{I}) + \langle \mathrm{NM}_{\mathcal{L}_{\mathcal{I}}}(x^\mu, \mathrm{lt}(H))\rangle) \tag{5.1}$$

the set

$$M(x^\mu, \mathrm{lt}(H), \mathrm{lt}(G)) \cup \mathrm{NM}_{\mathcal{L}_{\mathcal{I}}}(x^\mu, \mathrm{lt}(H))$$

is a strong $\mathcal{L}_{\mathcal{I}}$-involutive basis for the monomial ideal it generates relative to $\mathrm{lt}(\mathcal{I})$.

Note that the set $M(x^\mu, \mathrm{lt}(H), \mathrm{lt}(G)) \cup \mathrm{NM}_{\mathcal{L}_{\mathcal{I}}}(x^\mu, \mathrm{lt}(H))$ from Definition 5.1.1 is a subset of the set B defined in Equation (4.7). Hence, one would expect that any division $\mathcal{L}_{\mathcal{I}}$ which induces strong bases for syzygies is also of Schreyer type. For the relative divisions derived from the Pommaret division, this is the case.

Proposition 5.1.2. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a polynomial ideal in quasi-stable position and $\mathcal{P}$ the Pommaret division on $\mathcal{T}$. Then the relative involutive division $\mathcal{P}_{\mathrm{lt}(\mathcal{I})}$ induced by $\mathcal{P}$ induces strong bases for syzygies.*

Proof. Let G be the strong Pommaret basis of $\mathcal{I}$ and let H be a strong Pommaret basis of the ideal $\langle H\rangle+\mathcal{I}$ relative to $\mathcal{I}$. For each $x^\mu \in \mathrm{lt}(H)$, we have to show that the set $B' := M(x^\mu, \mathrm{lt}(H), \mathrm{lt}(G)) \cup \mathrm{NM}_{\mathcal{P}_{\mathrm{lt}(\mathcal{I})}}(x^\mu)$ is a strong Pommaret basis of the ideal it generates relative to $\mathrm{lt}(\mathcal{I})$. We know that $B' \subseteq B$, where B is defined as in Equation (4.7). Moreover, from the definitions, it is easy to see that $\langle B, \mathrm{lt}(\mathcal{I})\rangle = \langle B', \mathrm{lt}(\mathcal{I})\rangle$. We still have to show that B' is a strong relative Pommaret basis. Note that B is a weak Pommaret basis and each term in $t \in B \setminus B'$ is divisible by a variable x_j with $j > \mathrm{cls}(x^\mu)$, i.e., by a non-multiplicative variable for x^μ. Hence, $t \in \mathcal{C}_{\mathcal{P}}(x_j)$. We can deduce that B' is also a weak Pommaret basis. Also, it is clear that the Pommaret cones $\mathcal{C}_P(x_j)$ and $\mathcal{C}_P(t)$, where $x_j \in \mathrm{NM}_{\mathcal{P}}(x^\mu)$ and $t \in K[M_P(x^\mu)] \cap B'$, have empty intersection (look at the x_j-degrees). Finally, we need to show that all Pommaret cones $\mathcal{C}_{\mathcal{P}}(t)$, $\mathcal{C}_{\mathcal{P}}(s)$, where $s \neq t \in K[\mathrm{M}_{\mathcal{P}}(x^\mu)] \cap B'$, have empty intersection. For this, first note that $\mathrm{cls}(s) = \mathrm{cls}(\overline{s})$ and $\mathrm{cls}(t) = \mathrm{cls}(\overline{t})$, where $\overline{s}, \overline{t} \in \mathrm{lt}(G)$ are the terms inducing the multipliers s, t for x^μ. Hence, a non-empty intersection of the Pommaret cones of s and t would imply a non-empty intersection of the Pommaret cones $\overline{s}$ and $\overline{t}$. This is impossible, because $\overline{s}, \overline{t}$ are elements of the strong Pommaret basis of the ideal $\mathrm{lt}(I)$. □

Proposition 5.1.2 ensures that we get minimal Pommaret bases in each step of the resolution computation. We use Schreyer orderings for these Pommaret bases, which depend on $\mathcal{P}_{\mathcal{I}}$-orderings (i.e., orderings adapted to the Pommaret-involutive structure). There is an easy procedure by which $\mathcal{P}_{\mathcal{I}}$-orderings can be obtained automatically for the next syzygy module. Indeed, for any given generator of the current module, we need to take first the multiplicative A-syzygies in the order that is induced by the ordering on G. Then we take the non-multiplicative variables in ascending order. We do this for each generator sequentially, and we obtain a minimal Pommaret basis, already $\mathcal{P}_{\mathcal{I}}$-ordered, for the next syzygy module.

We compute the second syzygy module, being a subset of the free $\mathcal{R}/\mathcal{I}$-module $(\mathcal{R}/\mathcal{I})^5$ with the canonical basis $\{\mathbf{e}_1^{(2)}, \ldots, \mathbf{e}_5^{(2)}\}$. As always, we underline leading module terms.

- As A-syzygies, we obtain $\mathbf{A}_1 = \underline{x^2}\mathbf{e}_1^{(2)}$ for $H_1^{(1)}$, $\mathbf{A}_2 = \underline{y^2}\mathbf{e}_2^{(2)} + xy\mathbf{e}_4^{(2)} + x^2\mathbf{e}_5^{(2)}$ for $H_2^{(1)}$, $\mathbf{A}_3 = \underline{x}\mathbf{e}_3^{(2)}$ for $H_3^{(1)}$, $\mathbf{A}_4 = \underline{y^2}\mathbf{e}_4^{(2)} + xy\mathbf{e}_5^{(2)}$ for $H_4^{(1)}$, and $\mathbf{A}_5 = \underline{y^2}\mathbf{e}_5^{(2)}$ for $H_5^{(1)}$.

- As syzygies from non-multiplicative prolongations, we obtain $\mathbf{S}_1 = \underline{y}\mathbf{e}_1^{(2)} - x\mathbf{e}_2^{(2)} - \mathbf{e}_3^{(2)}$ for $\mathbf{H}_1^{(1)}$ and $\mathbf{S}_2 = \underline{y}\mathbf{e}_3^{(2)} - x^2\mathbf{e}_4^{(2)}$ for $\mathbf{H}_3^{(1)}$.

Note that $\{\mathbf{A}_1, \ldots, \mathbf{A}_5, \mathbf{S}_1, \mathbf{S}_2\}$ is here already a minimal Pommaret basis of the syzygy module of $H^{(1)}$ relative to $\mathcal{I}$. It is convenient to rename these syzygies as follows, to obtain a $\mathcal{P}$-ordering: $\mathbf{H}_1^{(2)} = \mathbf{A}_1$, $\mathbf{H}_2^{(2)} = \mathbf{S}_1$, $\mathbf{H}_3^{(2)} = \mathbf{A}_2$, $\mathbf{H}_4^{(2)} = \mathbf{A}_3$, $\mathbf{H}_5^{(2)} = \mathbf{S}_2$, $\mathbf{H}_6^{(2)} = \mathbf{A}_4$, and $\mathbf{H}_7^{(2)} = \mathbf{A}_5$. The syzygy $\mathbf{S}_1$, with constant term at $\mathbf{e}_3^{(2)}$, shows that one can eliminate the element $\mathbf{H}_3^{(1)}$ from the generating set $H^{(1)}$ of the syzygy module of H. This elimination procedure can be regarded as the first step needed to minimise the Pommaret-induced resolution in order to obtain a minimal $\mathcal{R}/\mathcal{I}$-free resolution of $\mathcal{R}/\mathcal{J}$.

The data obtained so far, i.e., the minimal generating set H of $\mathcal{J}$ and the Pommaret bases $H^{(1)}$ and $H^{(2)}$ for its first and second syzygy modules relative to $\mathcal{I}$, can be concisely presented in the following three matrices D_0, D_1, D_2, with the properties $D_i D_{i+1} = 0$. Note that i^{th} syzygies are written as columns of D_i.

$$D_0 = \begin{pmatrix} x^2 & xy & y^2 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} x & y & 0 & 0 & 0 \\ 0 & -x & x^2 & y & 0 \\ 0 & 0 & 0 & -x & y \end{pmatrix},$$

$$D_2 = \begin{pmatrix} x^2 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & y^2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & x & y & 0 & 0 \\ 0 & 0 & xy & 0 & -x^2 & y^2 & 0 \\ 0 & 0 & x^2 & 0 & 0 & xy & y^2 \end{pmatrix}.$$

Continuing the syzygy computations, we get the following next matrices:

$$D_3 = \begin{pmatrix} x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x^2 & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & x^2 & y & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & -x & y^2 & 0 & 0 \\ 0 & 0 & 0 & -x & 0 & 0 & x^2 & y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x & y \end{pmatrix},$$

$$D_4 = \begin{pmatrix} x^2 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x & y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^2 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & x & y & 0 & 0 & 0 & 0 \\ 0 & 0 & xy & 0 & 0 & 0 & -x^2 & y^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & x & y & 0 & 0 \\ 0 & 0 & 0 & 0 & xy & 0 & 0 & 0 & -x^2 & y^2 & 0 \\ 0 & 0 & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & xy & y^2 \end{pmatrix}, \ldots$$

In the example computations just performed, we can observe differences in the behavior of the resolutions induced by relative Pommaret bases over $\mathcal{R}/\mathcal{I}$ as opposed to the resolutions induced by Pommaret bases over $\mathcal{R}$:

Remark 5.1.3. *Constants can appear in some homological degree i of the resolution induced by a relative Pommaret basis even if there are no constants in the differential at the previous homological degree $i-1$. The matrix D_4 contains constants even though D_3 doesn't. This behaviour of the induced resolution is new, compared to the Pommaret-induced resolutions for $\mathcal{R}$-modules. Compare [97, Lem. 5.5.1], where it is shown that a Pommaret-induced resolution over $\mathcal{R}$ is minimal if and only if the first differential does not contain any constant terms.*

Example 5.1.4. *Consider the ideals $\mathcal{I} = \langle z^3 \rangle$, $\mathcal{J} = \langle xz, xz^2, xyz, y^2z, yz^2, \mathcal{I} \rangle$ in $\mathcal{R} = \mathbb{K}[x,y,z]$. Since $\mathcal{I}$ is a principal ideal and hence also a complete intersection, one would expect a resolution of any ideal $\mathcal{J}$ by free $\mathcal{R}/\mathcal{I}$-modules to be highly structured. Since the relative Pommaret basis of $\mathcal{J}$ does not coincide with its minimal relative generating set, the Pommaret-induced resolution will not be minimal. But we will see by the following computation that it is 2-periodic. As always, we denote by D_i the matrix in whose columns the elements of the minimal Pommaret basis of the i^{th} syzygy module are encoded.*

$$D_0 = \begin{pmatrix} xz & xz^2 & xyz & y^2z & yz^2 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} y & z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & y & z & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & y & z & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 0 & z & 0 \\ 0 & 0 & -x & 0 & 0 & -x & -y & z \end{pmatrix},$$

$$D_2 = \begin{pmatrix} z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -y & z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & z & -y & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -y & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & z^2 & 0 \\ 0 & 0 & x & 0 & 0 & xz & yz & z^2 \end{pmatrix},$$

$$D_3 = \begin{pmatrix} z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y & z & 0 & 0 & 0 & 0 & 0 & 0 \\ z & 0 & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & y & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z^2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & y & z & 0 & 0 \\ 0 & 0 & 0 & 0 & -x & 0 & z & 0 \\ 0 & 0 & -x & 0 & 0 & -x & -y & z \end{pmatrix},$$

$$D_4 = D_2.$$

Example 5.1.5. *We can change the ideals from Example 5.1.4 slightly to get two stable ideals. We leave out the generators xz and xz^2 of $\mathcal{J}$ and hence we get $\mathcal{I} = \langle z^3 \rangle$,*

$\mathcal{J} = \langle xyz, y^2z, yz^2, \mathcal{I}\rangle$. With the usual notation, we verify by computation that the Pommaret-induced resolution is the minimal $\mathcal{R}/\mathcal{I}$-free resolution of the $\mathcal{R}/\mathcal{I}$-module $\mathcal{J}$:

$$
\begin{aligned}
D_0 &= \begin{pmatrix} xyz & y^2z & yz^2 \end{pmatrix}, \\
D_1 &= \begin{pmatrix} y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix}, \\
D_2 &= \begin{pmatrix} z & 0 & 0 & 0 \\ -y & z^2 & 0 & 0 \\ x & 0 & z^2 & 0 \\ 0 & xz & yz & z^2 \end{pmatrix}, \\
D_3 &= \begin{pmatrix} z^2 & 0 & 0 & 0 \\ y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix}, \\
D_4 &= D_2.
\end{aligned}
$$

Note that the Betti numbers are eventually constant (equal to 4). This is also guaranteed by a result of Eisenbud's which states that 2-periodic infinite $\mathcal{R}/\mathcal{I}$-free resolutions always have eventually constant Betti numbers. It is not known whether there exist periodic resolutions with non-constant Betti numbers [73, Problem 4.16].

5.2 Resolutions via Pommaret-like Bases

In the last section, we have seen that the resolutions induced by relative Pommaret bases over quotient rings are generally not minimal. Pommaret-like bases are in general much more compact than Pommaret bases for the same ideals, and so they can be expected to yield resolutions that contain much less redundant elements. In order to apply relative Pommaret-like bases to resolutions over quotient rings, as a first step it is necessary to understand how the resolutions iduced by Pommaret-like bases over $\mathcal{R}$ are structured. This section presents these basic properties. A special focus is on monomial ideals.

If H is a Pommaret basis of a quasi-stable monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$, then the structure of the induced resolution is known. For results about this, see [97, Sec. 5.4]. The resolution is minimal if and only H is simultaneously the minimal monomial generating set of $\mathcal{I}$ [97, Lem. 5.5.1]. As a first step to a similar result for Pommaret-like bases, some combinatorial characterization of monomial ideals whose minimal generating set is also a Pommaret-like basis may be helpful.

However, it is *not* true in general that, if the minimal generating set G of a monomial ideal $\mathcal{I}$ is simultaneously a Pommaret-like basis of $\mathcal{I}$, the following must hold: "For all $t \in G$, we have $\mathrm{M}_{\mathcal{J}}(t, G) = \{x_1, \dots, x_{\mathrm{cls}(t)}\}$ and for each Janet-like non-multiplicative power $x_j^{p_j}$ of t with respect to the set G, the term $(t/x_{\mathrm{cls}(t)})x_j^{p_j}$ has a Pommaret-like divisor in G." See the next example:

Example 5.2.1. *In the polynomial ring $\mathbb{K}[x, y, z]$, consider the monomial ideal $\mathcal{I}$ with minimal generating set $G = \{xy, y^3, xz, y^2z, z^2\}$. As one can check, G is also a Pommaret-like basis. The generator $t = xy$ has the non-multiplicative powers y^2 and z. While it is true that $(t/x) \cdot y^2 = y^3 \in \mathcal{I}$, we have $(t/x) \cdot z = yz \notin \mathcal{I}$. Only if we increase the exponent of the variable z to 2, i.e., higher than the non-multiplicative power, we reach the term $yz^2 \in \mathcal{I}$.*

There seems to be no obvious "quick" condition to check whether a minimal generating set is also a Pommaret-like basis. However, one may check this by combining the conditions for quasi-stability with the conditions described in [56] for checking whether one has a minimal Janet-like basis.

Remark 5.2.2. *If a Pommaret-like basis H of a monomial ideal $\mathcal{I}$ is given, then ordering the elements ascendingly with respect to the lexicographic ordering with $x_1 \prec \cdots \prec x_n$ gives a P-ordering. (That is, an ordering from which one can derive a Schreyer ordering in the syzygy module which has non-multiplicative powers as leading terms.) One can easily see this by considering the Janet-like tree of H and observing that the Pommaret-like divisor of $h \cdot x_j^{p_j}$, where $h \in H$ and $x_j^{p_j} \in \mathrm{NMP}_P(h, H)$ will always lie to the right of h in the Janet-like tree.*

A Pommaret-like basis H of an ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ in quasi-stable position induces a free resolution of $\mathcal{I}$ over $\mathcal{R}$, and at each homological degree, the corresponding syzygy module is generated by a Pommaret-like basis [56]. There are special classes of ideals for which this induced resolution is in fact the minimal free resolution. One class of ideals for which this is true is the class of *componentwise linear ideals*. We can apply [97, Thm. 5.5.2] to see this, even though that result is concerned with Pommaret bases, because Pommaret bases are a special kind of Pommaret-like bases. Moreover, for *stable* monomial ideals the induced Pommaret-like resolution is also minimal because the Pommaret resolution is [97, Prop. 5.5.6]. The following result shows that the class of ideals for which the Pommaret-like resolution is minimal is larger than the class of ideals for which the Pommaret resolution is minimal:

Proposition 5.2.3. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a quasi-stable monomial ideal, $\mathcal{I} \notin \{\{0\}, \mathcal{R}\}$, generated by the minimal Pommaret-like basis $H \subset \mathcal{I} \cap \mathcal{T}$. Assume that H is simultaneously the minimal monomial generating set of $\mathcal{I}$. Moreover, let H be such that for all $t \in H$, we have $\mathrm{M}_{\mathcal{J}}(t, H) = \{x_1, \ldots, x_{\mathrm{cls}(t)}\}$ and for each Janet-like non-multiplicative power $x_j^{p_j}$ of t with respect to the set H, the term $(t/x_{\mathrm{cls}(t)})x_j^{p_j}$ has a Pommaret-like divisor in H. Then the free resolution of $\mathcal{I}$ over $\mathcal{R}$ induced by the basis H is the minimal free resolution of $\mathcal{I}$ over $\mathcal{R}$.*

Proof. We need to show that no constant terms can appear in the matrices describing the differential of the induced resolution. Write the resolution as

$$\mathbf{F} : \cdots \xrightarrow{d_3} \mathcal{R}^{b_2} \xrightarrow{d_2} \mathcal{R}^{b_1} \xrightarrow{d_1} \mathcal{R}^{|H|} \xrightarrow{d_0} \mathcal{I} \to 0.$$

The matrix D_0 describing d_0 consists of one row containing the elements of H as entries. Hence, no constant terms appear there. As the next step, we show that in the matrix D_1 describing d_1 there are no constant terms. By construction, each column of D_1 contains only two non-zero entries: $x_a^{p_a}$ (a non-multiplicative power of a term $t \in H$) and a cofactor $f \in \mathcal{T}$ such that

$$t \cdot x_a^{p_a} = s \cdot f, \tag{5.2}$$

where $s \in H$ is the unique term such that $t \cdot x_a^{p_a} \in \mathcal{C}_P(s, H)$. Since the set H is by assumption the minimal monomial generating set of $\mathcal{I}$, we have $f \neq 1$. Hence, no column of D_1 contains any constant term and the whole matrix D_1 is, hence, free of constant terms.

The columns of D_1 represent a minimal Pommaret-like basis of the first syzygy module $\operatorname{Syz}(H) \subset \mathcal{R}^{|H|}$ of H. The leading module terms $x^\mu \mathbf{e}_i$ of this syzygy module are exactly of the form $x_a^{p_a}\mathbf{e}_i$ where $x_a^{p_a}$ is a non-multiplicative power of the i^{th} element of H. They are found in the i^{th} row of D_1. There may be other non-zero entries in the said row, but they are cofactors f as given in Equation (5.2). Moreover, in the situation of Equation (5.2), it is clear that $\operatorname{cls}(t) \leq \operatorname{cls}(s)$ and hence, by assumption on H,

$$\operatorname{cls}(f) \leq \operatorname{cls}(s). \tag{5.3}$$

From this it follows that $\operatorname{cls}(f) < \operatorname{cls}(x_b^{p_b})$ for all non-multiplicative powers $x_b^{p_b}$ of s.

The matrix D_2 has as many rows as D_1 has columns. Each column of D_2 contains at least the non-zero entry $x_c^{p_c}$, a non-multiplicative power of a generator of the leading module of $\operatorname{Syz}(H)$. Since this leading module is generated by module terms whose polynomial parts are the non-multiplicative powers of the set H, also the non-multiplicative powers of this leading module will have polynomial parts of the same form. These non-multiplicative powers are obviously not constants. The further non-zero entries of a column of D_2 result from the involutive-like standard representation of the column vector $\mathbf{c} \cdot x_c^{p_c}$ with respect to the set of columns of D_1 (also $\mathbf{c}$ is a column of D_1). We focus on the possible non-zero entries that can be generated by the cancellations which happen in row i. During the involutive-like reduction process, it can happen that an intermediate result has a non-zero entry there, but this entry will be of the form $f \cdot p$, where p is some monomial and f is a term with the properties given in (5.3). In the column of D_2 encoding the involutive-like reduction we are studying at present, a non-zero entry (other than the one already analysed) can be created in row j only if the j^{th} column $\mathbf{c}_j$ of D_1 has as its leading module term $x_b^{p_b}\mathbf{e}_i$, where $x_b^{p_b}$ is as studied in (5.2). The class condition given in (5.3) now guarantees that the non-zero entry generated in the j^{th} row of the column of D_2 will be free of constant terms. What is more, all terms in the support of this entry will have class less or equal to $\operatorname{cls}(f)$. Now, since the indices i and j in the discussion above were "as arbitrary as possible", we have proved that also the matrix D_2 does not contain any constant terms.

The last thing we need to prove is that, also in D_2, we have a condition on the classes of terms analogous to that given in (5.3). If we can show this, then an iteration of the arguments used for the analysis of D_2 can be applied to all successive matrices in the resolution.

To prove this class condition, again consider the j^{th} row of D_2, where a non-zero entry q with $\operatorname{cls}(q) \leq \operatorname{cls}(f)$ is located resulting from a step in an involutive-like reduction which uses the leading module term $x_b^{p_b}\mathbf{e}_i$ of the j^{th} column of D_1. We need to compare this class with the classes of all leading module terms of $\operatorname{Syz}^2(H)$ of the form $u \cdot \mathbf{e}_j$. But these leading module terms arise from non-multiplicative powers of the leading module term $x_b^{p_b}\mathbf{e}_i$ in the leading module of $\operatorname{Syz}(H)$, and hence $u = x_d^{p_d}$ for some index $b < d \leq n$. Using (5.3), it is now clear that $\operatorname{cls}(q) \leq \operatorname{cls}(f) < \operatorname{cls}(x_b^{p_b}) < \operatorname{cls}(x_d^{p_d})$, i.e., the class condition we need is fulfilled. This finishes the proof. $\square$

We continue by giving two examples for minimal free resolutions induced by Pommaret-like bases.

Example 5.2.4. *Let $a, b, c \geq 1$ be any three positive integers and let $\mathcal{I} = \langle x^a, y^b, z^c \rangle$ be an irreducible monomial ideal minimally generated by $H = \{x^a, y^b, z^c\}$, which is easily seen to be also a Pommaret-like basis. Moreover, H satisfies the additional assumptions of Proposition 5.2.3. Hence, it induces a minimal Pommaret-like free resolution of $\mathcal{I}$. The matrices of the differentials are given as follows:*

$$D_0 = \begin{pmatrix} x^a & y^b & z^c \end{pmatrix}, \; D_1 = \begin{pmatrix} y^b & z^c & 0 \\ -x^a & 0 & z^c \\ 0 & -x^a & -y^b \end{pmatrix}, \; D_2 = \begin{pmatrix} z^c \\ -y^b \\ x^a \end{pmatrix}.$$

Example 5.2.5. *In the polynomial ring $\mathbb{K}[w, x, y, z]$ with $w \prec x \prec y \prec z$, consider the monomial ideal $\mathcal{I} = \langle H \rangle$ with*

$$H = \{w^9x^3y^2z^2, x^5y^2z^2, w^7y^4z^2, x^3y^4z^2, y^6z^2, x^3y^2z^4, y^4z^4, z^8\}.$$

(The elements have been ordered lexicographically from lowest to highest.) One can verify that H is simultaneously the minimal generating system of $\mathcal{I}$ and a Pommaret-like basis satisfying the additional assumptions of Proposition 5.2.3. Hence, it induces a minimal Pommaret-like free resolution of $\mathcal{I}$. The matrices of the differentials are given as follows:

$$D_0 = \begin{pmatrix} w^9x^3y^2z^2 & x^5y^2z^2 & w^7y^4z^2 & x^3y^4z^2 & y^6z^2 & x^3y^2z^4 & y^4z^4 & z^8 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} x^2 & y^2 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -w^9 & 0 & 0 & y^2 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x^3 & y^2 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -w^9 & 0 & -x^2 & 0 & -w^7 & 0 & 0 & y^2 & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -w^7 & 0 & -x^3 & 0 & z^2 & 0 & 0 & 0 \\ 0 & 0 & -w^9 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 & 0 & y^2 & z^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -w^7 & 0 & -x^3 & -y^2 & -x^3 & 0 & z^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x^3y^2 & -y^4 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} y^2 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x^2 & 0 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x^2 & -y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ w^9 & 0 & 0 & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & w^9 & 0 & -y^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y^2 & z^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x^3 & 0 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x^3 & -y^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & w^7 & 0 & 0 & z^2 & 0 \\ 0 & 0 & w^9 & x^2 & 0 & w^7 & 0 & -y^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w^7 & x^3 & 0 \\ 0 & 0 & -w^9 & -x^2 & 0 & 0 & 0 & 0 & z^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^3 \end{pmatrix}, \quad D_3 = \begin{pmatrix} z^2 & 0 \\ -y^2 & 0 \\ x^2 & 0 \\ -w^9 & 0 \\ 0 & z^2 \\ 0 & -y^2 \\ 0 & x^3 \\ 0 & -w^7 \\ 0 & 0 \end{pmatrix}.$$

Proposition 5.2.3 does not completely cover the class of quasi-stable monmial ideals whose Pommaret-like bases induce minimal free resolutions. In other words, there exist quasi-stable monomial ideal that do not satisfy the proposition's assumptions but whose Pommaret-like bases nevertheless induce the minimal free resolution:

Example 5.2.6. *Let us continue Example 5.2.1. Set $\mathcal{I} = \langle xy, y^3, xz, y^2z, z^2\rangle$. This Pommaret-like basis induces a minimal free resolution with differential represented by the following matrices:*

$$D_0 = \begin{pmatrix} xy & y^3 & xz & y^2z & z^2 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} y^2 & z & 0 & 0 & 0 & 0 \\ -x & 0 & z & 0 & 0 & 0 \\ 0 & -y & 0 & y^2 & z & 0 \\ 0 & 0 & -y & -x & 0 & z \\ 0 & 0 & 0 & 0 & -x & -y^2 \end{pmatrix}, \quad D_2 = \begin{pmatrix} z & 0 \\ -y^2 & 0 \\ x & 0 \\ -y & z \\ 0 & -y^2 \\ 0 & x \end{pmatrix}.$$

We finish this section with a result that is useful for relating resolutions induced by Pommaret-like bases to other free resolutions.

Proposition 5.2.7. *Let $\mathcal{I}$ with $\{0\} \neq \mathcal{I} \neq \mathcal{R}$ be a polynomial ideal in quasi-stable position and let H be its minimal Pommaret-like basis. Then the free resolution induced by H consists of* reduced *Gröbner bases for all syzygy modules $\mathrm{Syz}^m(H)$, $m \geq 1$. In other words, in each homological degree, the set of columns of the matrix describing the differential is the unique autoreduced Gröbner basis of $\mathrm{Syz}^m(H)$ for the chosen module term order.*

Proof. For the first syzygy module, the statement is easily seen, because the tail terms of the first syzygies arise from an involutive-like reduction computation, while the leading terms stem from the non-mutiplicative powers. This means that the tail terms are not divsible by any of the leading terms, and reducedness of the Gröbner basis for $\mathrm{Syz}(H)$ follows.

Now, let $m > 1$. From the structure of the Pommaret-like resolution, we know that

- The leading module terms of the Pommaret-like basis of $\mathrm{Syz}^{m-1}(H)$ are of the form $x_a^{p_a} \cdot \mathbf{e}_i$, and for fixed i, the leading terms form a sequence $x_b^{p_b} \cdot \mathbf{e}_i, x_{b+1}^{p_{b+1}} \cdot \mathbf{e}_i \ldots, x_n^{p_n}\mathbf{e}_i$; moreover, each leading term belongs to such a sequence.

- For fixed i, the sequence of leading terms considered in the preceding item induces some tail terms with polynomial parts $x_a^{p_a}$ $(b \leq a \leq n-1)$ in the Pommaret-like basis of $\mathrm{Syz}^m(H)$. They arise via non-multiplicative powers of terms of the same form, from the first step of the ensuing involutive-like reduction. Let us denote the free basis of $\mathcal{R}^{b_m} \supseteq \mathrm{Syz}^m(H)$ by $(\mathbf{f}_j)_{1 \leq j \leq b_m}$. Then if we consider such a tail term $x_a^{p_a} \cdot \mathbf{f}_j$, and it comes from a non-multiplicative power $x_c^{p_c}$ with $c > a$, then the leading module terms in the same free component are given by $x_{c+1}^{p_{c+1}} \cdot \mathbf{f}_j, \ldots, x_n^{p_n} \cdot \mathbf{f}_j$, and none of these divides $x_a^{p_a} \cdot \mathbf{f_j}$.

- All other tail terms arise from further steps (not the first) in the involutive-like divisions induced by non-multiplicative powers. Concretely, for some index i, if a step in the division algorithm is performed because $x^\mu \mathbf{e}_i$ is Pommaret-like divisible by $x_c^{p_c}\mathbf{e}_i$, then by the definition of the Pommaret-like division, it is not Pommaret-like divisible by any term $x_a^{p_a}\mathbf{e}_i$ with $a > c$. The division step induces a tail term $x^\mu / x_c^{p_c}\mathbf{f}_j$ in the Pommaret-like basis of $\mathrm{Syz}^m(H)$. But, as seen in the last item, the leading module terms in this free component are exactly $x_{c+1}^{p_{c+1}} \cdot \mathbf{f}_j, \ldots, x_n^{p_n} \cdot \mathbf{f}_j$, whose polynomial parts do not divide $x^\mu \mathbf{f}_j$, let alone $(x^\mu / x_c^{p_c})\mathbf{f}_j$. Hence, these tail terms do not destroy reducedness of the Gröbner basis.

- No tail terms other than those already analysed occur.

We have now shown that no tail term is divisible by any leading term and hence we have shown autoreducedness. All Gröbner basis elements are monic by construction and hence the Gröbner basis is reduced. □

5.3 Resolutions over Clements-Lindström Rings

Since relative Pommaret-like bases are a special kind of relative Gröbner bases, they induce free resolutions via the relative involutive Schreyer Theorem 4.2.13. If we assume that we work in the quotient ring $\mathcal{R}/\mathcal{I}$, where $\mathcal{I}$ is a quasi-stable monomial ideal, and if we complete the relative Pommaret-like basis to a relative Pommaret basis, then the induced resolution will consist of Pommaret bases for the syzygy modules in each homological degree. In this section, we will show that if we restrict to the class of *irreducible* quasi-stable monomial ideals, then we can skip the completion step from Pommaret-like basis to Pommaret basis: The relative

Pommaret-like basis will then induce a free resolution which consists of Pommaret-like bases for each syzygy module. Up to a permutation of coordinates, the class of irreducible quasi-stable monomial ideals is equivalent to the class of Clements-Lindström ideals. We will formulate our results in the most general form possible, but for simplicity one can think of the ring in which computations take place as a Clements-Lindström ring $\mathcal{R}/\langle x_k^{h_k},\dots,x_n^{h_n}\rangle$ with $h_k \geq \cdots \geq h_n \geq 2$.

Proposition 5.3.1. *Let $\mathcal{I}$ be an irreducible quasi-stable monomial ideal $\mathcal{R}/\mathcal{I}$, and let H be a Pommaret-like basis relative to $\mathcal{I}$ of the (polynomial) ideal $\mathcal{J} \supset \mathcal{I}$. If H is ordered according to a P-ordering for its set of leading terms, then a Pommaret-like basis of $\operatorname{Syz}_{\mathcal{R}/\mathcal{I}}(H)$ is given by the S-polynomials of H induced by non-multiplicative multiples of the leading terms and the A-polynomials induced by multiplicatively annihilating leading terms of H modulo $\mathcal{I}$. Only those A-polynomials whose annihilating factor is not identical to a generator of $\mathcal{I}$ contribute non-zero syzygies.*

Iteration of this result implies that a free resolution is induced consisting of relative Pommaret-like bases in each homological degree.

Sketch of Proof. The proof is similar to that of Proposition 4.2.14, where relative Pommaret bases are treated. The proof uses, firstly, the continuity of the Pommaret-like division, as guaranteed by 3.4.14, for the existence of a P-ordering. Also the relative Schreyer Theorem 4.1.11, together with the notions of A-polynomials and S-polynomials discussed in Subsection 4.1.1, is used.

Note that "gaps" can appear in the lists of leading module terms in some syzygy module components. These gaps appear for the variables where one can reach $\mathcal{I}$ by multiplying by a power of that variable. That one can reach $\mathcal{I}$ implies that relative quasi-stability is not destroyed by these gaps. □

As in the non-relative case, we are interested in a description of at least a part of the class of monomial ideals $\mathcal{J} \supset \mathcal{I}$ quasi-stable relative to $\mathcal{I}$ whose relative Pommaret-like bases induce *minimal* free resolutions by the process of Proposition 5.3.1. Recall that an estimate for the classes of tail terms compared to the classes of leading module terms was central to the proof of Proposition 5.2.3. In order to be able to use a similar argument, we need to impose even stricter assumptions on the relative Pommaret-like basis generating $\mathcal{J}$ than we had to impose in Proposition 5.2.3. The reason for this is that in the relative case, the contributions of A-polynomials have an effect which amounts to a "non-increasing" property for the classes of leading module terms in the resolution.

Proposition 5.3.2. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be an irreducible quasi-stable monomial ideal (with $\mathcal{I} \notin \{\{0\},\mathcal{R}\}$) and let $\mathcal{J} \supset \mathcal{I}$ be a larger monomial ideal generated by the minimal Pommaret-like basis $\{1\} \neq H \subset (\mathcal{J} \setminus \mathcal{I}) \cap \mathcal{T}$ relative to $\mathcal{I}$. Assume that H is simultaneously the minimal monomial generating set of $\mathcal{J}$ relative to $\mathcal{I}$. Moreover, let H be such that for each $t \in H$ and $x_a^{p_a} \in \operatorname{NMP}_{P_{\mathcal{I}}}(t,H)$, the unique $P_{\mathcal{I}}$-divisor $s \in H$ of $t \cdot x_a^{p_a}$ is of greater class than t: $\operatorname{cls}(s) > \operatorname{cls}(t)$. Then the free resolution of $\mathcal{J}$ over $\mathcal{R}/\mathcal{I}$ induced by the basis H is the minimal free resolution of $\mathcal{J}$ over R.*

Moreover, for each $m \geq 1$, the set of columns of the matrix D_m describing the differential consists of the unique reduced Gröbner basis of $\mathrm{Syz}^m_{\mathcal{R}/\mathcal{I}}(H)$ *for the chosen module monomial ordering.*

Proof. We need to show that the matrices describing the differential do not contain any constant terms. By assumption, $H \neq \{1\}$ and hence it does not contain any constant. We now analyse the matrices $D_1, D_2, \ldots$ iteratively. For every $h \in H$, the matrix contains as leading module terms the non-multiplicative powers of h as well as, for $k = \mathrm{cls}(h)$, a factor $x_k^{d_k - \deg_k(h)}$ if $x_k^{d_k}$ is a minimal generator of $\mathcal{I}$. The tail terms in D_1 arise by division of terms $h \cdot x_j^{p_j}$, where $x_j^{p_j}$ is a non-multiplicative power, by their unique Pommaret-like divisor u in H:

$$h \cdot x_j^{p_j} = u \cdot s \tag{5.4}$$

Since H is the minimal relative generating set of $\mathcal{A}$, these tail term are not constant. Moreover, by assumption, $\mathrm{cls}(s) = \mathrm{cls}(h)$ and $\mathrm{cls}(u) > \mathrm{cls}(h)$. A tail term s will be found in the row corresponding to the generator $u \in H$, and the leading terms in that row will be of class $\geq \mathrm{cls}(u)$, and so the s has strictly smaller class than the leading terms in the same row. Note that columns of D_1 belonging to annihilating factors do not have any tail term. Summarizing, D_1 does not contain any constant terms and all tail terms have a strictly smaller class than the leading terms in the same row.

It is now straightforward to proceed analogously as in the proof of Proposition 5.2.3, showing by induction on homological degree that no constant terms appear in the resolution, and thus to show its minimality.

Central to this induction proof is the fact that tail terms always have strictly smaller class than leading terms in the same row. The reducedness of the Gröbner bases in each degree is an obvious consequence. □

Remark 5.3.3. *Note that squarefree stable squarefree monomial ideals belong to the class of ideals which are minimally resolved by relative Pommaret-like bases modulo an irreducible quasi-stable monomial ideal (provided that they are relatively quasi-stable with respect to the said irreducible ideal). This is guaranteed by Proposition 5.3.2 in conjunction with Proposition 4.4.12, the condition on the classes being obvious because every (minimal) generator is squarefree.*

Example 5.3.4. *Let us continue Example 4.4.9. Consider the ideals $\mathcal{I} = \langle x^6, y^6, z^6 \rangle$ and $\mathcal{A} = \langle \mathcal{I}, xz, yz \rangle$. The set $H = \{xz, yz\}$ is the minimal generating system of the ideal $\mathcal{A}$ relative to $\mathcal{I}$, and it is simultaneously a relative Pommaret-like basis, as proven in Example 4.4.9. Since $\mathrm{cls}(xz) < \mathrm{cls}(yz)$, the additional conditions of Proposition 5.3.2 are also fulfilled. Hence, H induces an infinite minimal free resolution of $\mathcal{A}$ over $\mathcal{R}/\mathcal{I}$, with the first differential matrices given by:*

$$D_0 = \begin{pmatrix} xz & yz \end{pmatrix}, \quad D_1 = \begin{pmatrix} x^5 & y & z^5 & 0 & 0 \\ 0 & -x & 0 & y^5 & z^5 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} x & y & z^5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x^5 & 0 & y^5 & z^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x^5 & 0 & -y & z & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & y & z^5 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & -y^5 & z \end{pmatrix}.$$

Remark 5.3.5. *The minimal free resolution constructed by Gasharov, Murai, and Peeva [42, Constr. 4.4] for squarefree borel ideals relative to a zero-dimensional Clements-Lindström ring is necessarily isomorphic to the Pommaret-like resolution of the same ideal, since both resolutions are minimal.*

In fact, one can always find an isomorphism that consists only of permutations of bases. One can prove this by assigning leading terms to the syzygies defined in [42, Eqn. 4.10]. This assignment can be done in such a way that the leading terms for each homological degree will coincide with the leading terms in the Pommaret-like resolution. The sets of leading terms being equal, we can conclude that the szygies of [42, Eqn. 4.10] form Gröbner bases in each homological degree; the reducedness can then be shown in a straightforward manner using a basic result on Borel monomial ideals.

The uniqueness of the reduced Gröbner basis then shows that the resolution of [42, Construction 4.4] and the Pommaret-like resolution coincide. This also gives an explicit formula for the differential, depending only on the data contained in the first two matrices D_0 and D_1.

The next example demonstrates that our construction covers many elementary cases:

Example 5.3.6. *Let $a_1, \dots, a_n$ be positive integers, let $i \in \{1, \dots, n\}$, and let $1 \leq b_i < a_i$ be another integer. Then, relative to the irreducible monomial ideal $\mathcal{I} = \langle x_1^{a_1}, \dots, x_n^{a_n} \rangle$, the set $H = \{x_i^{b_i}\}$ is a Pommaret-like basis of $\mathcal{A} = \langle H, \mathcal{I} \rangle$ and the induced resolution over $\mathcal{R}/\mathcal{I}$ is the obvious 2-periodic minimal free resolution with differentials described by the following matrices:*

$$D_0 = \left(x_i^{b_i}\right), \quad D_1 = \left(x_i^{a_i - b_i}\right), \quad D_2 = \left(x_i^{b_i}\right) = D_0.$$

A final, more or less "generic", example, shows the general behavior of the construction:

Example 5.3.7. *Let $\mathcal{I} = \langle y^4, z^5 \rangle$ and $\mathcal{A} = \langle \mathcal{I}, x^2y^3, xy^2z^2, y^3z^2, z^3 \rangle$. Then $H = \{x^2y^3, xy^2z^2, y^3z^2, z^3\}$ is the minimal relative generating set of $\mathcal{A}$, and it is simultaneously a Pommaret-like basis satisfying the additional conditions of Proposition 5.3.2. Hence, it induces a minimal free resolution of $\mathcal{A}$ over $\mathcal{R}/\mathcal{I}$, with the first maps of the differential represented by the following matrices:*

$$D_0 = \begin{pmatrix} x^2y^3 & xy^2z^2 & y^3z^2 & z^3 \end{pmatrix}, \quad D_1 = \begin{pmatrix} y & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & z & 0 & 0 & 0 \\ 0 & -x^2 & -x & 0 & y & z & 0 \\ 0 & 0 & 0 & -xy^2 & 0 & -y^3 & z^2 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} y^3 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y & z^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^3 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y & z^4 & 0 & 0 & 0 & 0 \\ 0 & -x^2 & 0 & xy^2 & 0 & 0 & y^3 & z & 0 & 0 \\ 0 & 0 & x^2z^2 & 0 & x & 0 & 0 & -y & z^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy^2z^2 & 0 & 0 & y^3z^2 & z^3 \end{pmatrix}.$$

5.3.1 Betti Numbers and Poincaré Series

The results of this subsection apply to all (minimal) resolutions induced by Pommaret-like bases over factor rings of the form $\mathcal{R}/\mathcal{I}$, where $\mathcal{I}$ is an irreducible quasi-stable monomial ideal. We understand this to include also the case $\mathcal{I} = \{0\}$, and hence all resolutions induced by Pommaret-like bases over the ordinary polynomial ring $\mathcal{R} = \mathcal{R}/\{0\} = \mathbb{K}[x_1, \ldots, x_n]$.

We will derive formulas for the Betti numbers of these resolutions. The results can also be applied to non-minimal free resolutions induced by Pommaret-like bases, but then one gets only formulas for the ranks of the free modules in these non-minimal resolutions. These ranks can be understood as *pseudo*-Betti numbers of these resolutions. They yield, degree by degree, upper bounds for the true Betti numbers of the resolved ideals.

Let $\mathcal{J} \supseteq \mathcal{I}$ be any homogeneous polynomial ideal in quasi-stable position relative to $\mathcal{I}$ with respect to the degrevlex term ordering. Moreover, assume that the minimal Pommaret-like basis of $\mathcal{J}$ relative to $\mathcal{I}$ is a minimal homogeneous generating set and that the resolution induced by this Pommaret-like basis over the ring $\mathcal{R}/\mathcal{I}$ is minimal. In what follows, we construct a basis for the bigraded free $\mathcal{R}/\mathcal{I}$-module supporting the resolution, only using the Pommaret-like basis of the leading ideal of $\mathcal{J}$ relative to $\mathcal{I}$, and also give a formula for the Poincaré series of the resolution using only these data. Note that the Poincaré series encodes the bigraded Betti numbers of the resolution.

Remark 5.3.8. *In this section, we use for a term $t \in \mathcal{T}$ the notation* $\operatorname{supp}(t) := \{x_i \mid \deg_i(t) > 0\}$ *for the set of variables on which t is supported.*

Definition 5.3.9. *Let $\mathcal{R} \neq \mathcal{I} = \langle x_k^{h_k}, x_{k+1}^{h_{k+1}} \ldots, x_n^{h_n} \rangle$ be an irreducible quasi-stable monomial ideal. Then we write* $\operatorname{cls}(\mathcal{I}) := k$ *and* $\operatorname{supp}(\mathcal{I}) := \{x_k, x_{k+1}, \ldots, x_n\}$.

Consider a Pommaret-like basis H relative to $\mathcal{I}$ that induces a minimal free resolution. The resolution is supported on free $\mathcal{R}/\mathcal{I}$-modules. The first free $\mathcal{R}/\mathcal{I}$-module M_0 has a basis that we enumerate as $\{\mathbf{e}_\alpha \mid h_\alpha \in H\}$. Write $t_\alpha = \operatorname{lt}(h_\alpha)$ for each $h_\alpha \in H$. As always, we order H according to a P-order. The next free module M_1 has a basis whose cardinality equals that of the minimal Pommaret-like basis of $\operatorname{Syz}(H)$ with respect to the Schreyer module term order—note that this is a reduced Gröbner basis. Hence, the free basis of M_1 is in bijection with the elements of this Gröbner basis; in other words, it is in bijection with the leading module terms of the Gröbner basis. These leading module terms are given as follows (cf. Proposition 5.3.1):

- $x_a^{p_a} \cdot \mathbf{e}_\alpha$, where $x_a^{p_a} \in \mathrm{NMP}_{P_\mathcal{I}}(t_\alpha, \mathrm{lt}(H))$,
- $x_i^{h_i - \deg_i(t_\alpha)} \cdot \mathbf{e}_\alpha$, where $x_i \in \mathrm{supp}(t_\alpha) \cap \mathrm{supp}(\mathcal{I})$ and there is no $P_\mathcal{I}$-non-multiplicative power for t_α at x_i. (If $\ell = \mathrm{cls}(t_\alpha) \geq \mathrm{cls}(\mathcal{I})$, then this case will always include $x_\ell^{h_\ell - \deg_\ell(t_\alpha)} \cdot \mathbf{e}_\alpha$.)

Since the two cases are mutually exclusive, and each concerns leading module terms whose polynomial parts are pure variable powers, we can identify each leading module term by its position and the variable involved. Thus, a free basis of M_1 can be enumerated as

$$\{\mathbf{e}_{\alpha,x_i} \mid x_i \geq \mathrm{cls}(t_\alpha) \wedge (x_i \in \mathrm{NM}_{P_\mathcal{I}}(t_\alpha, \mathrm{lt}(H)) \vee x_i \in \mathrm{supp}(t_\alpha) \cap \mathrm{supp}(\mathcal{I}))\}.$$

We keep the condition "$x_i \geq \mathrm{cls}(t_\alpha)$" for clarity, even though it could be omitted, being implicit in the other conditions. At this stage, it is useful to introduce notation for the leading ideals in each module component of M_1, because we can use them to describe, by an iteration, all further leading terms in the resolution. Set

$$\mathcal{J}_\alpha = \langle x_i^{d_i} \mid x_i^{d_i} \cdot \mathbf{e}_\alpha \in \mathrm{lt}(\mathrm{Syz}(H)) \rangle.$$

These ideals are irreducible and we will use the notation $\mathrm{supp}(\mathcal{J}_\alpha)$ for the set of variables appearing in their respective generating sets.

Consider now the leading terms of the Pommaret-like basis of $\mathrm{Syz}^2(H)$, which are in bijection to a free basis of the next module in the resolution, M_2. Each of them is induced by a leading term of the basis of $\mathrm{Syz}(H)$. Such a leading term, $x_i^{d_i} \cdot \mathbf{e}_\alpha$, say, induces exactly the following leading terms in $\mathrm{Syz}^2(H)$:

- $x_j^{d_j} \cdot \mathbf{e}_{\alpha,x_i}$, where $x_j \in \mathrm{supp}(\mathcal{J}_\alpha)$ and $j > i$,
- $x_i^{h_i - d_i} \cdot \mathbf{e}_{\alpha,x_i}$, if $x_i \in \mathrm{supp}(\mathcal{I})$.

Note that the polynomial part of the new leading term will be supported on a variable whose index is not less than that of the polynomial part of the term which induces it. We can now list the free basis of M_2: Leading terms induced as in the first case correspond to basis elements $\mathbf{e}_{\alpha,x_ix_j}$, whereas leading terms induced as in the second case correspond to basis elements $\mathbf{e}_{\alpha,x_i^2}$.

We can iterate this construction. For the rth module in the resolution, M_r, it yields a basis consisting of elements of the form $\mathbf{e}_{\alpha,x^\mu}$, where x^μ is a term of degree r with $\mathrm{cls}(x^\mu) \geq \mathrm{cls}(t_\alpha)$. Moreover, x^μ is supported on $\mathrm{supp}(\mathcal{J}_\alpha)$, and if for each variable $x_i \in \mathrm{supp}(\mathcal{I})$ we substitute 1 into x^μ, we get a squarefree term supported on $\mathrm{supp}(\mathcal{J}_\alpha) \setminus \mathrm{supp}(\mathcal{I})$.

From this description of the free bases, we obtain the following formula for the total Betti numbers of the resolution, where we write $\mathcal{S}_\alpha$ for $\mathrm{supp}(\mathcal{J}_\alpha)$ and $\mathcal{S}$ for $\mathrm{supp}(\mathcal{I})$: For $r = 0$, $\mathrm{rank}(M_0) = |H|$; for $r \geq 1$,

$$\begin{aligned}\mathrm{rank}(M_r) = &\sum_{\substack{h_\alpha \in H \\ \mathcal{S}_\alpha \cap \mathcal{S} \neq \emptyset}} \sum_{j=0}^{\min\{r,\, |\mathcal{S}_\alpha \setminus \mathcal{S}|\}} \binom{|\mathcal{S}_\alpha \setminus \mathcal{S}|}{j} \cdot \binom{|\mathcal{S}_\alpha \cap \mathcal{S}| + r - j - 1}{|\mathcal{S}_\alpha \cap \mathcal{S}| - 1} \\ &+ \sum_{\substack{h_\alpha \in H \\ \mathcal{S}_\alpha \cap \mathcal{S} = \emptyset}} [r \leq |\mathcal{S}_\alpha|] \binom{|\mathcal{S}_\alpha|}{r},\end{aligned} \tag{5.5}$$

where the product of binomial coefficients counts the number of terms x^μ of degree r supported on $\mathcal{J}_\alpha$ with the additional restriction of being squarefree outside $\mathrm{supp}(\mathcal{I})$. Moreover, the term $[r \leq |\,\mathrm{supp}(\mathcal{J}_\alpha)|]$ is defined as having values in $\{0,1\}$, yielding 1 exactly when the statement enclosed in the square brackets is true.

We now turn to the bigraded Betti numbers, which we will compute in the form of a Poincaré series, which is a formal power series in two variables, which we name s and u. The first variable encodes homological degrees and the second encodes degrees as given by the ordinary grading of the polynomial ring S. Recall that each basis element $\mathbf{e}_{\alpha,x^\mu}$ has homological degree $\deg(x^\mu)$. Its polynomial degree is the sum of $\deg(t_\alpha)$ (recall $t_\alpha = \mathrm{lt}(h_\alpha)$) and the degrees of the polynomial parts of all leading module terms involved in the building up of the syzygy $\mathbf{S}_{\alpha,x^\mu} \in \mathrm{Syz}^{\deg(x^\mu)}(H)$. These polynomial parts are pure powers of variables from $\mathrm{supp}(\mathcal{J}_\alpha)$. Moreover, their indices form a non-decreasing sequence. There can be repeated indices in this sequence, and if an index j is repeated, it means that the next syzygy is formed from the annihilation of the current leading term. So if a module term with polynomial part $x_j^{c_j}$ is to annihilate, the next leading term will have polynomial part $x_j^{h_j-c_j}$ (recall that $\mathcal{I}$ is generated by the $x_j^{h_j}$). More repetitions of the same index will cause the involved leading terms to have polynomial parts oscillating between $x_j^{c_j}$ and $x_j^{h_j-c_j}$. This means that the contribution of x_j-terms to the overall polynomial degree of $\mathbf{S}_{\alpha,x^\mu}$ depends, on one hand, on the parity of μ_j, and the remaining part is just $h_j \cdot \lfloor \mu_j/2 \rfloor$. Since $\mathcal{J}_\alpha$ is generated by terms $x_j^{d_j}$, (with $d_j = h_j - \deg_j(t_\alpha)$ if x_j is multiplicative for t_α, and $d_j = p_j$, where $x_j^{p_j}$ is a non-multiplicative power, otherwise), we get the following formula for the Poincaré series of our resolution, where we write $\mathcal{S}_\alpha$ for $\mathrm{supp}(\mathcal{J}_\alpha)$ and $\mathcal{S}$ for $\mathrm{supp}(\mathcal{I})$:

$$\sum_{h_\alpha \in H} \left(u^{\deg(t_\alpha)} \cdot \left(1 + \sum_{B \subseteq \mathcal{S}_\alpha} \binom{|\mathcal{S}_\alpha|}{|B|} s^{|B|} \prod_{x_b \in B} u^{d_b} \prod_{x_a \in \mathcal{S}_\alpha \cap \mathcal{S}} \frac{1}{1 - s^2 u^{h_a}} \right) \right). \tag{5.6}$$

Example 5.3.10. *Let us continue Example 5.3.7. Recall that $\mathcal{I} = \langle y^4, z^5 \rangle$ and $H = \{x^2y^3, xy^2z^2, y^3z^2, z^3\}$ in that example. We will use Equation (5.5) to compute the Betti numbers of the ideal generated by H relative to $\mathcal{I}$ and then compare it with the results of Example 5.3.7.*

We write $h_\alpha = x^2y^3$, $h_\beta = xy^2z^2$, $h_\gamma = y^3z^2$, and $h_\delta = z^3$. An analysis of the Pommaret-like non-mutiplicative powers of these generators shows that $\mathcal{J}_\alpha = \{y, z^2\}$, $\mathcal{J}_\beta = \{y, z\}$, $\mathcal{J}_\gamma = \{y, z\}$, and $\mathcal{J}_\delta = \{z^2\}$. Since $\mathrm{supp}(\mathcal{I}) = \{x, y\}$, we have $\mathrm{supp}(\mathcal{J}_\alpha) = \mathrm{supp}(\mathcal{J}_\alpha) \cap \mathrm{supp}(\mathcal{I})$, and the same equation holds also for the other indices. Thus, Equation (5.5) reduces to:

$$\mathrm{rank}(M_r) = \sum_{h_\alpha \in H} \binom{|\,\mathrm{supp}(\mathcal{J}_\alpha)| + r - 1}{|\,\mathrm{supp}(\mathcal{J}_\alpha)| - 1},$$

and this gives, since we have three generators with $|\,\mathrm{supp}(\mathcal{J}_\bullet)| = 2$ and one *generator with* $|\operatorname{supp}(\mathcal{J}_\bullet)| = 1$, *the formula*

$$\operatorname{rank}(M_r) = 3\binom{1+r}{1} + \binom{r}{0} = 4 + 3r,$$

which is for $r \in \{1, 2, 3\}$ *in perfect agreement with the results of Example 5.3.7, as expected.*

5.4 Explicit Formulas for the Differential

In this section, we will give explicit formulas for the differentials of resolutions of some monomial ideals induced by Pommaret-like bases over the ordinary polynomial ring $\mathcal{R} = K[x_1, \ldots, x_n]$. These formulas will generalize those described in [97] for resolutions induced by Pommaret bases. While in [97, Sec. 5.4], such a formula was found for all quasi-stable ideals and their minimal Pommaret bases, we will here restrict our attention to a smaller class of ideals.

Our first goal is to establish a subclass of quasi-stable ideals whose minimal Pommaret-like basis satisfy conditions analogous to those found in [97, Lemma 5.4.17] for minimal Pommaret bases of arbitrary quasi-stable ideals. For this subclass, we will then have the technical tools needed to give an explicit formula for the differential of the induced resolution.

Definition 5.4.1. *Let* $H = \{h_\alpha \mid \alpha \in A\} \subset \mathcal{T}$ *be the minimal Pommaret-like bases of the quasi-stable ideal* $\mathcal{I} = \langle H \rangle$. A *is a finite index set. For each* $\alpha \in A$, *and for each of its Pommaret-like non-multiplicative powers* $x_a^{p_a} = x_a^{p(P,h_\alpha,H,a)}$, *there exists exactly one generator* $h_\beta \in H$ *with* $x_a^{p_a} \cdot h_\alpha \in \mathcal{C}_P(h_\beta)$. *For such a configuration of terms, we write*

$$\Delta(\alpha, a) = \beta \tag{5.7}$$

for the index of the Pommaret-like divisor, and

$$t_{\alpha,a} = (x_a^{p_a} \cdot h_\alpha)/h_\beta \tag{5.8}$$

for the Pommaret-like multiplicative cofactor involved.

The following result states some elementary properties satisfied by the objects just defined.

Lemma 5.4.2. *Let* $H = \{h_\alpha \mid \alpha \in A\} \subset \mathcal{T}$ *be the minimal Pommaret-like basis of the quasi-stable ideal* $\mathcal{I} = \langle H \rangle$. *The associated function* Δ *and the terms* $t_{\alpha,a}$ *(as given in Definition 5.4.1) satisfy the following properties:*

(i) The inequality $\operatorname{cls}(h_\alpha) \le \operatorname{cls}(h_{\Delta(\alpha,a)}) \le a$ *holds for all non-multiplicative indices* $a > \operatorname{cls}(h_\alpha)$.

(ii) Let $b > a > \operatorname{cls}(h_\alpha)$ *be two non-multiplicative indices.*

- *The variable* x_b *is non-multiplicative for* $h_{\Delta(\alpha,a)}$ *and the non-multiplicative power of* $h_{\Delta(\alpha,a)}$ *at* x_b *equals that of* h_α *at* x_b.

- *If* $\operatorname{cls}(h_{\Delta(\alpha,b)}) \geq a$*, then* $\Delta(\Delta(\alpha,a),b) = \Delta(\alpha,b)$ *and* $x_a^{p_a} \cdot t_{\alpha,b} = t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b}$.

Proof. Property (i) follows from the minimality of the Pommaret-like basis H: $h_{\Delta(\alpha,a)}$ is a divisor of $x_a^{p_a} \cdot h_\alpha$ and thus its class must be at least as high as that of h_α; it cannot be higher than a, because otherwise $h_{\Delta(\alpha,a)}$ would be a strict Pommaret-like divisor of h_α, contradicting minimality.

Property (ii) is split into two items. The first item follows from property (i) and the definition of the Pommaret-like division, because the terms $h_{\Delta(\alpha,a)}$ and h_α must agree in their x_j-degrees for all $j > a$. Now if, to prove the second item, we additionally assume $\operatorname{cls}(h_{\Delta(\alpha,b)}) \geq a$, then since $x_b^{p_b} \cdot h_{\Delta(\alpha,a)}$ and $x_b^{p_b} \cdot h_\alpha$ agree in their x_j-degrees for all indices $j > a$, the same must be true for $h_{\Delta(\Delta(\alpha,a)),b}$ and $h_{\Delta(\alpha,b)}$. We also know that $\deg_a(h_{\Delta(\alpha,b)}) \leq \deg_a(h_\alpha) < \deg_a(x_b^{p_b} \cdot h_{\Delta(\alpha,a)})$. By the class assumption on $h_{\Delta(\alpha,b)}$, we can now conclude that $h_{\Delta(\alpha,b)}$ is the unique Pommaret-like divisor in H of $x_b^{p_b} \cdot h_{\Delta(\alpha,a)}$. Hence, we have shown $\Delta(\Delta(\alpha,a),b) = \Delta(\alpha,b)$. The remaining statement is a consequence of the following chain of equations:

$$\begin{aligned}
x_a^{p_a} \cdot t_{\alpha,b} \cdot h_{\Delta(\alpha,b)} &= x_a^{p_a} \cdot (x_b^{p_b} \cdot h_\alpha) \\
&= x_b^{p_b} \cdot (x_a^{p_a} \cdot h_\alpha) \\
&= x_b^{p_b} \cdot t_{\alpha,a} \cdot h_{\Delta(\alpha,a)} \\
&= t_{\alpha,a} \cdot x_b^{p_b} \cdot h_{\Delta(\alpha,a)} \\
&= t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} \cdot h_{\Delta(\Delta(\alpha,a),b)} \\
&= t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} \cdot h_{\Delta(\alpha,b)}
\end{aligned}$$

□

For arbitrary minimal *Pommaret* bases, the associated Δ functions satisfy a commutativity property of the form

$$\Delta(\Delta(\alpha,a),b) = \Delta(\Delta(\alpha,b),a) \tag{5.9}$$

whenever both of these terms are defined, *i.e.*, when the involved variable indices a, b are always non-multiplicative [97, Lem. 5.4.17]. In general, minimal Pommaret-like bases do not have this property. What is more, for *Pommaret* bases, also the equation $t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} = t_{\alpha,b} \cdot t_{\Delta(\alpha,b),a}$ holds in this situation. In contrast to this, there are minimal Pommaret-like bases for which the commutativity property holds, but not the equation just mentioned. This is caused by differences of degrees of non-multiplicative powers for the same variable.

Example 5.4.3. *Consider the minimal Pommaret-like basis* $H = \{h_\alpha, h_\beta, h_\gamma, h_\delta, h_\epsilon\}$ *with* $h_\alpha = xy$*,* $h_\beta = y^4$*,* $h_\gamma = xz$*,* $h_\delta = y^2 z$*, and* $h_\epsilon = z^3$*. Its associated* Δ *function satisfies the commutativity property of Equation* (5.9)*. For this only one condition needs to be checked:*

$$\Delta(\Delta(\alpha,y),z) = \delta = \Delta(\Delta(\alpha,z),y).$$

However, we have $t_{\alpha,y} = x$, $t_{\Delta(\alpha,y),z} = y^2$, $t_{\alpha,z} = y$, *and* $t_{\Delta(\alpha,z),y} = x$, *so that* $t_{\alpha,y} \cdot t_{\Delta(\alpha,y),z} = xy^2 \neq xy = t_{\alpha,z} \cdot t_{\Delta(\alpha,z),y}$. *This is caused by a difference in the degrees of the non-multiplicative powers at the variable* y *between* h_α *(degree* 3*) and* h_γ *(degree* 2*).*

We now define a subclass of quasi-stable ideals having Δ-functions with properties useful for the analysis of their Pommaret-like resolutions:

Definition 5.4.4. *Let* $H = \{h_\alpha \mid \alpha \in A\} \subset \mathcal{T}$ *be the minimal Pommaret-like basis of the quasi-stable ideal* $\mathcal{I} = \langle H \rangle$. *The ideal* $\mathcal{I}$ *together with the basis* H *is called* Δ-commuting *if the associated function* Δ *and the terms* $t_{\alpha,a}$ *(as given in Definition 5.4.1) satisfy the following properties:*

(i) If $b > a > \operatorname{cls}(h_\alpha)$ *be two non-multiplicative indices and* $\operatorname{cls}(h_{\Delta(\alpha,b)}) < a$, *then the exponent of the non-multiplicative power of* $h_{\Delta(\alpha,b)}$ *at the variable* x_a *equals that of the non-multiplicative power of* h_α *at the variable* x_a.

(ii) We have $\Delta(\Delta(\alpha,a),b) = \Delta(\Delta(\alpha,b),a)$.

(iii) We have $t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} = t_{\alpha,b} \cdot t_{\Delta(\alpha,b),a}$.

For Δ-commuting quasi-stable ideals, we are able to give an explicit formula for the differential of the resolution induced by the minimal Pommaret-like basis. As is usual for such formulas, the summands obey a certain sign rule, and for this we need the following definition:

Definition 5.4.5. *Let* $x_i \in A \subseteq \{x_1, \dots, x_n\}$ *be a variable contained in a subset* A *of variables. Then we write* $\operatorname{sgn}(x_i, A) := |\{x_j \in A \mid j > i\}|$. *Thus,* $\operatorname{sgn}(x_i, A)$ *counts the variables in* A *which have a higher index than* x_i *has.*

Theorem 5.4.6. *Let* $H = \{h_\alpha \mid \alpha \in A\} \subset \mathcal{T}$ *be the minimal Pommaret-like basis of the* Δ*-commuting quasi-stable ideal* $\mathcal{I} = \langle H \rangle$. *We write* $\operatorname{NMP}(h_\alpha, H) = \{x_j^{p_j} \mid j > \operatorname{cls}(h_\alpha)\}$. *The Pommaret-like induced resolution of* $\mathcal{I}$ *is supported on free generators of the form* $\mathbf{e}_{h_\alpha, x^\mu}$, *where the* x^μ *are squarefree terms supported on* $\{x_j \mid j > \operatorname{cls}(h_\alpha)\}$. *The differential* δ *of the resolution is given by* $\delta(\mathbf{e}_{\alpha,1}) = h_\alpha$, *and, for* $\deg(x^\mu) > 0$,

$$\delta(\mathbf{e}_{\alpha,x^\mu}) = \sum_{x_j \in \operatorname{supp}(x^\mu)} (-1)^{\operatorname{sgn}(x_j, \operatorname{supp}(x^\mu))} \cdot \left(x_j^{p_j} \mathbf{e}_{\alpha, x^\mu/x_j} - t_{\alpha,j} \mathbf{e}_{\Delta(\alpha,j), x^\mu/x_j} \right). \tag{5.10}$$

In this formula, we interpret all summands to be zero which involve a non-existent free generator $\mathbf{e}_{\beta,x^\nu}$, *i.e., an expression of this form for which* $\operatorname{supp}(x^\nu) \not\subseteq \{x_j \mid j > \operatorname{cls}(h_\beta)\}$.

Proof. The proof is a straightforward adaptation of the proof of [97, Thm. 5.4.18], replacing non-multiplicative variables by their associated non-multiplicative powers where appropriate. □

Corollary 5.4.7. *Let* $\mathcal{I} = \langle H \rangle$ *be a* Δ*-commuting quasi-stable ideal minimally generated by the set* $H \subset \mathcal{T}$, *for which* H *is also a Pommaret-like basis. Then the Pommaret-like resolution of* $\mathcal{I}$ *induced by* H *is minimal.*

Proof. By minimality of H, we have $t_{\alpha,a} \neq 1$ for the terms defined in Definition 5.4.1. Now, the minimality of the induced resolution is a trivial consequence of the explicit differential formula 5.10, which applies because all assumptions of Theorem 5.4.6 are fulfilled for $\mathcal{I}$ and H. $\square$

Example 5.4.8. *Let us continue Example 5.2.5. We have the minimal Pommaret-like basis*

$$H = \{h_\alpha = w^9x^3y^2z^2, h_\beta = x^5y^2z^2, h_\gamma = w^7y^4z^2, h_\delta = x^3y^4z^2,$$
$$h_\epsilon = y^6z^2, h_\zeta = x^3y^2z^4, h_\eta = y^4z^4, h_\theta = z^8\}.$$

Using Formula (5.10), we obtain the following values of the differential δ of the induced resolution for basis elements of homological degrees 2 and 3:

$$\begin{aligned}
\delta(\mathbf{e}_{\alpha,xy}) &= y^2\mathbf{e}_{\alpha,x} & - x^2\mathbf{e}_{\alpha,y} && +w^9\mathbf{e}_{\beta,y}\\
\delta(\mathbf{e}_{\alpha,xz}) &= z^2\mathbf{e}_{\alpha,x} & - x^2\mathbf{e}_{\alpha,z} && +w^9\mathbf{e}_{\beta,z}\\
\delta(\mathbf{e}_{\alpha,yz}) &= z^2\mathbf{e}_{\alpha,y} & -w^9\mathbf{e}_{\zeta,z} - y^2\mathbf{e}_{\alpha,z} && +x^2\mathbf{e}_{\delta,z}\\
\delta(\mathbf{e}_{\beta,yz}) &= z^2\mathbf{e}_{\beta,y} & -x^2\mathbf{e}_{\zeta,y} - y^2\mathbf{e}_{\beta,z} && +x^2\mathbf{e}_{\delta,z}\\
\delta(\mathbf{e}_{\gamma,xy}) &= y^2\mathbf{e}_{\gamma,x} & - x^3\mathbf{e}_{\gamma,y} && +w^7\mathbf{e}_{\delta,y}\\
\delta(\mathbf{e}_{\gamma,xz}) &= z^2\mathbf{e}_{\gamma,x} & - x^3\mathbf{e}_{\gamma,z} && +w^7\mathbf{e}_{\delta,z}\\
\delta(\mathbf{e}_{\gamma,yz}) &= z^2\mathbf{e}_{\gamma,y} & - y^2\mathbf{e}_{\gamma,z} && +w^7\mathbf{e}_{\epsilon,z}\\
\delta(\mathbf{e}_{\delta,yz}) &= z^2\mathbf{e}_{\delta,y} & - y^2\mathbf{e}_{\delta,z} && +x^3\mathbf{e}_{\epsilon,z}\\
\delta(\mathbf{e}_{\zeta,yz}) &= z^4\mathbf{e}_{\zeta,y} & - y^2\mathbf{e}_{\zeta,z} && +x^3\mathbf{e}_{\eta,z}
\end{aligned}$$

$$\begin{aligned}
\delta(\mathbf{e}_{\alpha,xyz}) &= z^2\mathbf{e}_{\alpha,xy} & -y^2\mathbf{e}_{\alpha,xz} + x^2\mathbf{e}_{\alpha,yz} && -w^9\mathbf{e}_{\beta,yz}\\
\delta(\mathbf{e}_{\gamma,xyz}) &= z^2\mathbf{e}_{\gamma,xy} & -y^2\mathbf{e}_{\gamma,xz} + x^3\mathbf{e}_{\gamma,yz} && -w^7\mathbf{e}_{\delta,yz}
\end{aligned}$$

The P-graph of this Pommaret-like basis is given in Figure 5.1.

Example 5.4.9. *Let us continue Example 5.2.6. Thus, we consider the minimal Pommaret-like basis $H = \{xy, y^3, xz, y^2z, z^2\}$ and we write $h_\alpha = xy$, $h_\beta = y^3$, $h_\gamma = xz$, $h_\delta = y^2z$, and $h_\epsilon = z^2$. Using Formula (5.10), we obtain the following values of the differential δ of the induced resolution for basis elements of homological degree 2:*

$$\begin{aligned}
\delta(\mathbf{e}_{\alpha,yz}) &= z\mathbf{e}_{\alpha,y} & -y\mathbf{e}_{\gamma,y} - y^2\mathbf{e}_{\alpha,z} && +x\mathbf{e}_{\beta,z}\\
\delta(\mathbf{e}_{\gamma,yz}) &= z\mathbf{e}_{\gamma,y} & - y^2\mathbf{e}_{\gamma,z} && +x\mathbf{e}_{\delta,z}
\end{aligned}$$

The P-graph of this Pommaret-like basis is given in Figure 5.2.

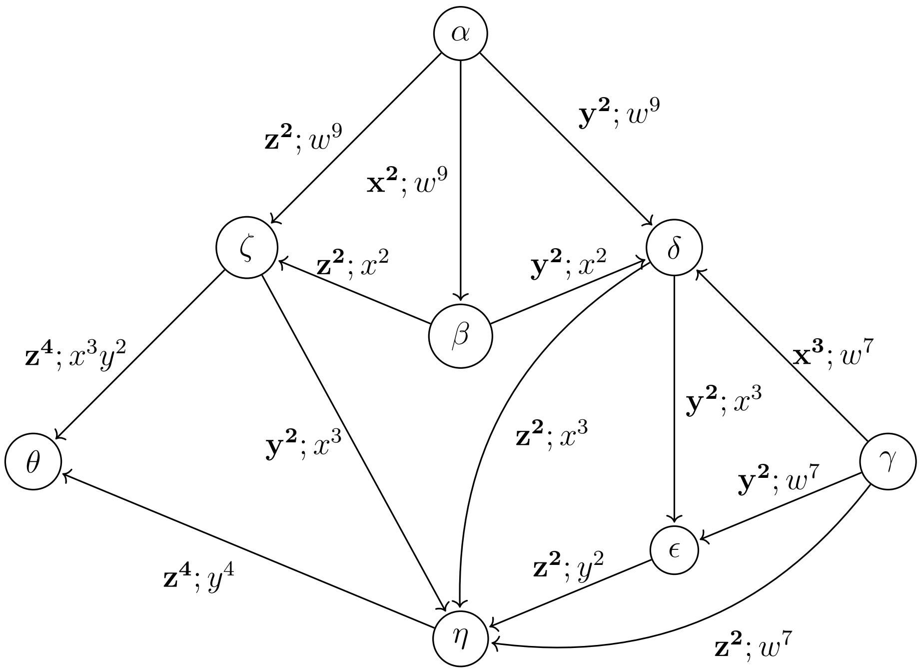

Figure 5.1: P-graph of Pommaret-like basis of Example 5.4.8. Each arrow is labelled with a Pommaret-like non-multiplicative power of the basis element belonging to the source. This non-multiplicative power is printed bold. Moreover, the label contains the associated cofactor, which is Pommaret-like multiplicative for the basis element belonging to the target of the arrow.

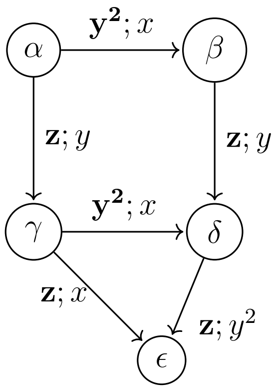

Figure 5.2: P-graph of Pommaret-like basis of Example 5.4.9. See Figure 5.1 for instructions on how to read this graph.

Chapter 6

Relative Marked Bases

Let $R = \mathbb{K}[x_0, \dots, x_n]$ be the polynomial ring over a field $\mathbb{K}$ in $n+1$ variables, with $x_0 < \cdots < x_n$, and I be a homogeneous ideal of R. A first goal of this chapter is to define a special kind of marked basis over a quasi-stable ideal, called *relative marked basis*, that is suitable to work on R/I and to the study of some open subsets of a Hilbert scheme $\mathbf{Hilb}_X^{p(z)}$, where $X = \mathrm{Proj}(R/I) \subset \mathbb{P}^n_{\mathbb{K}}$. This is particularly interesting in the case $I = (x_n^{d_n}, \dots, x_t^{d_t})$, with $d_n \leq \cdots \leq d_t$ and $t > 0$, in other words when R/I is a Clements-Lindström ring of positive Krull-dimension (see [29]), because on these rings an analogon of Macaulay's Theorem characterizing the Hilbert functions of homogeneous ideals in a polynomial ring holds.

In Sections 6.1 and 6.2 we describe the already known main features about marked bases and marked functors in polynomial rings.

We defined and discussed relative Gröbner bases and involutive bases for ideals in quotient rings in Chapter 4. We now look for analogous results in the context of marked bases over quasi-stable ideals given an ideal I. Due to the interest in Clements-Lindström rings, we will focus on the case in which I coincides with a quasi-stable ideal (Sections 6.3, 6.4, 6.5).

A second goal of this chapter is to apply the notion of relative marked basis together with its properties and functorial features to the study of Hilbert schemes over some quotient rings. We obtain two main results.

When the quotient ring is Cohen-Macaulay on a quasi-stable ideal (for instance, Clements-Lindström rings), we describe an open cover of these Hilbert schemes by means of suitable changes of variables (Section 6.6). This result is achieved thanks to a generalization of a method that is described in [6] and which is based on deterministically computable suitable linear changes of variables.

A quotient ring S is called Macaulay-Lex if all possible Hilbert functions in this ring are attained by the class of lex-ideals in S. Macaulay's classical theorem states that $R/\{0\}$ is Macaulay-Lex. Clements and Lindström [29] proved that all rings of the form $S = R/(x_0^{a_0}, x_1^{a_1}, \dots, x_n^{a_n})$, where $2 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, are Macaulay-Lex. Shakin [100] classified all monomial factor rings of $R = \mathbb{K}[x_0, x_1]$ that are Macaulay-Lex. In further work [101], he proved that strongly stable monomial ideals give Macaulay-Lex quotients if and only if they are piecewise lex-segment ideals. Abedelfatah [2] generalized this result to stable monomial ideals and also gave some further classes of examples of monomial ideals that give Macaulay-Lex quotients. For further results on Macaulay-Lex rings and connected ideas, see the article by Mermin and Peeva [77].

When the quotient ring is Macaulay-Lex on a quasi-stable ideal (again, Clements-Lindström rings are an example), the computational techniques of relative marked bases allow the investigation of the lex-point of such Hilbert schemes. Indeed, it is well-known that every non-empty Hilbert scheme over a polynomial ring on a field has a unique point that is defined by a lex-ideal. Moreover this point, which is called the *lex-point*, is smooth (see [87]) and is characterized by the property that its defining saturated lex-ideal has the minimal possible Hilbert function among the points of the same Hilbert scheme. It is also true that every non-empty Hilbert scheme over a Macaulay-Lex ring over a quasi-stable ideal has the lex-point, which moreover has the minimal Hilbert function. However, in the case of Macaulay-Lex rings over quasi-stable ideals this lex-point can be singular, as we show by explicit examples in Section 6.7. The exhibited examples have been computed with the help of CoCoA [1] and Macaulay 2 [54].

6.1 Marked Sets and Bases

In this section, we recall basic facts about marked sets and bases and their associated polynomial reduction procedures. Moreover, we fix notation that we will use throughout this chapter.

Let $\mathbb{K}$ be a field, A be a Noetherian $\mathbb{K}$-algebra with $1_A = 1_{\mathbb{K}}$. We consider the polynomial ring $R = \mathbb{K}[x_0, \dots, x_n]$ with $x_0 < \cdots < x_n$, and $R_A := R \otimes_A A = A[x_0, \dots, x_n]$. A *term* is a power product $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. We denote by $\mathcal{T}$ the set of terms in R. For every $x^\alpha \in \mathcal{T} \setminus \{1\}$, we denote by $\min(x^\alpha)$ the smallest variable dividing x^α. The Pommaret division $\mathcal{P}$ and Pommaret bases are defined as usual, keeping in mind that $\mathrm{M}_{\mathcal{P}}(x^\alpha) = \{x_0, x_1, \dots, \min(x^\alpha)\}$ in the present context. Of course, $\mathrm{NM}_{\mathcal{P}}(x^\alpha) = \{x_0, \dots, x_n\} \setminus \mathrm{M}_{\mathcal{P}}(x^\alpha)$.

If N is a finite subset of R, we denote by $\langle N \rangle_A$ the A-module generated by N, and by (N) the ideal generated by N. We use standard grading on R_A, that is $\deg(x_j) = 1$, for all $j \in \{0, \dots, n\}$, and $\deg(a) = 0$, for all $a \in A$. Hence we have $\deg(x^\alpha) = |\alpha| = \sum \alpha_i$.

We assume that the polynomials, the ideals and A-modules involved in our discussion are homogeneous using the standard grading on R_A. Thanks to this assumption, when for example we write an equality of the kind $I = B_1 \oplus B_2$, where $I \subseteq R_A$ is an ideal and B_1, B_2 are A-modules or ideals, we also mean $I_s = (B_1)_s \oplus (B_2)_s$ for every $s \geq 0$. In such cases we will say that the equality is *graded*.

A monomial ideal $\tilde{J} \subseteq R$ has a unique minimal set of generators made of terms and we call it *the monomial basis* of $\tilde{J}$, denoted by $\mathcal{B}_{\tilde{J}}$. We define $\mathcal{N}(\tilde{J}) \subseteq \mathcal{T}$ as the set of terms in $\mathcal{T}$ not belonging to $\tilde{J}$. For every polynomial $f \in R_A$, $\mathrm{supp}(f)$ is the set of terms appearing in f with a non-zero coefficient. For every polynomial $f \in R_A$, an *x-coefficient* of f is the coefficient in A of a term in $\mathcal{T} \cap \mathrm{supp}(f)$.

Definition 6.1.1. *A* quasi-stable *ideal is a monomial ideal having a Pommaret basis. If $\tilde{J}$ is a quasi-stable ideal, we denote by* $\mathrm{MinPB}(\tilde{J})$ *its Pommaret basis. A monomial ideal $\tilde{J}$ is* stable *if it is quasi-stable and* $\mathrm{MinPB}(\tilde{J}) = \mathcal{B}_{\tilde{J}}$.

Remark 6.1.2. *If $\tilde{J} \subset R$ is a quasi-stable ideal, observe that if $x^\alpha \in \mathrm{MinPB}(\tilde{J})\setminus\mathcal{B}_{\tilde{J}}$, then there is $x^\beta \in \mathcal{B}_{\tilde{J}}$ such that $x^\alpha = x^\beta \cdot x^\delta$ with x^δ made of variables in $\mathrm{NM}_{\mathcal{P}}(x^\alpha)$.*

Definition 6.1.3. *[88] A* marked polynomial *is a polynomial $f \in R_A$ together with a fixed term $x^\alpha \in \mathrm{supp}(f)$ whose coefficient is equal to 1_A. This term is called* head term *of f and denoted by $\mathrm{Ht}(f)$. If $f \in R_A$ is a marked polynomial, we set $\mathrm{M}_{\mathcal{P}}(f) := \mathrm{M}_{\mathcal{P}}(\mathrm{Ht}(f))$ and call it the set of* multiplicative variables of *f.*

Definition 6.1.4. *Let $\tilde{I} \subseteq R_A$ be a quasi-stable ideal generated by terms in R, and let $\mathrm{MinPB}(\tilde{I})$ be its Pommaret basis. A* $\mathrm{MinPB}(\tilde{I})$-marked set *is a finite set $F \subset R_A$ of exactly $|\mathrm{MinPB}(\tilde{I})|$ marked homogeneous polynomials f_α with pairwise distinct head terms $\mathrm{Ht}(f_\alpha) = x^\alpha \in \mathrm{MinPB}(\tilde{I})$ and $\mathrm{supp}(f_\alpha - x^\alpha) \subset \langle \mathcal{N}(\tilde{I})\rangle_A$. A $\mathrm{MinPB}(\tilde{I})$-marked set F is a* $\mathrm{MinPB}(\tilde{I})$-marked basis *of the ideal (F) if the graded decomposition $(R_A) = \langle F\rangle_A \oplus \langle \mathcal{N}(\tilde{I})\rangle_A$ holds.*

Definition 6.1.5. *[27, Defs. 5.3 and 5.6] Let $\tilde{I} \subseteq R_A$ be a quasi-stable ideal generated by terms in R and $F = \{f_\alpha\}_{x^\alpha \in \mathrm{MinPB}(\tilde{I})}$ be a $\mathrm{MinPB}(\tilde{I})$-marked set. We denote by F^* the set of homogeneous polynomials*

$$F^* = \{x^\eta f_\alpha \mid x^\eta x^\alpha \in \mathcal{C}_{\mathcal{P}}(x^\alpha)\} \subseteq (F)$$

that are marked on the terms of $\tilde{I}$ in the natural way: $\mathrm{Ht}(x^\eta f_\alpha) = x^\eta \mathrm{Ht}(f_\alpha)$.

We denote by $\longrightarrow_{F^*}$ the reflexive and transitive closure of the following reduction relation on R_A: f is in relation with f' if $f' = f - \lambda x^\eta f_\alpha$, where $x^\eta f_\alpha \in F^*$ and $\lambda \neq 0_A$ is the coefficient of the term $x^{\eta+\alpha}$ in f.

We will write $f \longrightarrow^+_{F^*} f_0$ if $f \in R_A$, $f \longrightarrow_{F^*} f_0$ and $f_0 \in \langle \mathcal{N}(\tilde{I})\rangle_A$. In this case we say that "f is reduced to f_0 by F^*", and that "f_0 is reduced with respect to F^*".

Theorem 6.1.6. *[27, Lem. 5.8, Thms. 5.9 and 5.13 and Cor. 5.11] Let $\tilde{I} \subseteq R_A$ be a quasi-stable ideal generated by terms in R and F be a $\mathrm{MinPB}(\tilde{I})$-marked set. Then, the reduction relation $\longrightarrow_{F^*}$ is Noetherian and confluent and the following statements are equivalent:*

(i) F is a $\mathrm{MinPB}(\tilde{I})$-marked basis;

(ii) $\langle F^*\rangle_A = (F)$ and this equality is graded;

(iii) $f \longrightarrow^+_{F^*} 0$, for every $f \in (F)$;

(iv) $x_i f_\alpha \longrightarrow^+_{F^*} 0$ for every $f_\alpha \in F$ and for every $x_i \in \mathrm{NM}_{\mathcal{P}}(f_\alpha)$.

Definition 6.1.7. *[27, Def. 3.4] Let $\tilde{I} \subseteq R_A$ be a quasi-stable ideal generated by terms in R and $F = \{f_\alpha\}_{x^\alpha \in \mathrm{MinPB}(\tilde{I})}$ be a $\mathrm{MinPB}(\tilde{I})$-marked set in R_A. For every $H \subset F$, H is called a* substructure *of F, and we have a reduction relation $\longrightarrow_{H^*}$ obviously defined as a subreduction of $\longrightarrow_{F^*}$.*

Lemma 6.1.8. *Let $\tilde{I} \subseteq R_A$ be a quasi-stable ideal generated by terms in R, $F = \{f_\alpha\}_{x^\alpha \in \mathrm{MinPB}(\tilde{I})}$ be a $\mathrm{MinPB}(\tilde{I})$-marked set in R_A and H be a subset of F. The reduction relation $\longrightarrow_{H^*}$ is Noetherian and confluent, where $H^* = \{x^\eta h_\beta \mid x^\eta x^\beta \in \mathcal{C}_\mathcal{P}(x^\beta), h_\beta \in H\} \subseteq F^*$.*

Proof. For what concerns Noetherianity, it is sufficient to observe that $\longrightarrow_{H^*}$ is a subreduction of $\longrightarrow_{F^*}$, which is Noetherian, as recalled in Theorem 6.1.6. Confluency of $\longrightarrow_{H^*}$ is immediate because this reduction is Noetherian and it has disjoint cones (see also [27, Remark 7.2]). □

Remark 6.1.9. *Observe that, in the hypotheses of Lemma 6.1.8, if $p \longrightarrow^+_{H^*} h$, then $\mathrm{supp}(h)$ is included in $\mathcal{T} \setminus \left(\cup_{x^\beta \in \{\mathrm{Ht}(h) | h \in H\}} \mathcal{C}_\mathcal{P}(x^\beta)\right)$.*

6.2 Marked Functors

For a given quasi-stable ideal $\tilde{J} \subset R$, it is possible to parameterize the set of ideals J in R_A having a $\mathrm{MinPB}(\tilde{J})$-marked basis by means of a functor from the category of Noetherian $\mathbb{K}$-Algebras to that of Sets, which turns out to be represented by an affine scheme. We briefly recall the definition of this functor and the construction of this affine scheme.

Let $\tilde{J} \subseteq R$ be a quasi-stable ideal and A a Noetherian $\mathbb{K}$-algebra. The *marked functor* from the category of Noetherian $\mathbb{K}$-algebras to the category of sets

$$\underline{\mathbf{Mf}}_{\tilde{J}} : \underline{\text{Noeth } \mathbb{K}\text{–Alg}} \longrightarrow \underline{\text{Sets}}$$

associates to any Noetherian $\mathbb{K}$-algebra A the set

$$\underline{\mathbf{Mf}}_{\tilde{J}}(A) := \{(G) \subset R_A \mid G \text{ is a } \mathrm{MinPB}(\tilde{J})\text{-marked basis}\}$$

and to any morphism of $\mathbb{K}$-algebras $\sigma : A \to A'$ the map

$$\begin{array}{rccc} \underline{\mathbf{Mf}}_{\tilde{J}}(\sigma) : & \underline{\mathbf{Mf}}_{\tilde{J}}(A) & \longrightarrow & \underline{\mathbf{Mf}}_{\tilde{J}}(A') \\ & (G) & \longmapsto & (\sigma(G)) \ . \end{array}$$

Note that the image $\sigma(G)$ under this map is indeed again a $\mathrm{MinPB}(\tilde{J})$-marked basis, as we are applying the functor $-\otimes_A A'$ to the decomposition $(R_A)_s = (G)_s \oplus \langle \mathcal{N}(\tilde{J})_s \rangle_A$ for every degree s.

Remark 6.2.1. *Generalizing [70, Prop. 2.1] to quasi-stable ideals, we obtain*

$$\{(G) \subset R_A \mid G \textit{ is a } \mathrm{MinPB}(\tilde{J})\textit{-mkd. basis}\} = \{J \subset R_A \textit{ ideal } \mid R_A = J \oplus \langle \mathcal{N}(\tilde{J}) \rangle_A\}.$$

The functor $\underline{\mathbf{Mf}}_{\tilde{J}}$ is represented by the affine scheme that can be explicitly constructed by the following procedure. We consider the $\mathbb{K}$-algebra $\mathbb{K}[C]$, where C denotes the finite set of variables $\{C_{\alpha\eta} \mid x^\alpha \in \mathrm{MinPB}(\tilde{J}), x^\eta \in \mathcal{N}(\tilde{J}), \deg(x^\eta) =$

$\deg(x^\alpha)\}$, and construct the $\mathrm{MinPB}(\tilde{J})$-marked set $\mathscr{G} \subset R_{\mathbb{K}[C]}$ consisting of the following marked polynomials

$$g_\alpha = \left(x^\alpha - \sum_{x^\eta \in \mathcal{N}(\tilde{J})_{|\alpha|}} C_{\alpha\eta} x^\eta \right) \tag{6.1}$$

with $x^\alpha \in \mathrm{MinPB}(\tilde{J})$. According to Definition 6.1.5, we consider

$$\mathscr{G}^* = \{x^\delta g_\alpha | g_\alpha \in \mathscr{G}, x^\delta \in \mathrm{M}_{\mathcal{P}}(g_\alpha)\}.$$

Then, by the Noetherian and confluent reduction procedure given in Definition 6.1.5, for every term $x^\alpha \in \mathrm{MinPB}(\tilde{J})$ and every non-multiplicative variable $x_i \in \mathrm{NM}_{\mathcal{P}}(x^\alpha)$, we compute a polynomial $p_{\alpha,i} \in \langle \mathcal{N}(J)_{|\alpha|+1} \rangle_A$ such that $x_i g_\alpha - p_{\alpha,i} \in \langle \mathscr{G}^* \rangle_A$. We then denote by $\mathscr{U}$ the ideal generated in $\mathbb{K}[C]$ by the x-coefficients of the polynomials $p_{\alpha,i}$.

Theorem 6.2.2. ([26, Rem. 6.3],[6, Thm. 5.1]) *The functor* $\underline{\mathbf{Mf}}_{\tilde{J}}$ *is represented by the scheme* $\mathrm{Spec}(\mathbb{K}[C]/\mathscr{U})$*, which we denote by* $\mathbf{Mf}_{\tilde{J}}$*.*

Let now $\tilde{J} \subset R$ be a *saturated* quasi-stable ideal. Consider $\tilde{J}_{\geq t} := \bigoplus_{s \geq t} \tilde{J}_s$ and the integer $\rho_{\tilde{J}} := \max\{\deg(x^\alpha) \mid x^\alpha \in \mathrm{MinPB}(\tilde{J}) \text{ is divisible by } x_1\}$. When $\mathrm{MinPB}(\tilde{J})$ does not contain any term divisible by x_1 we set $\rho_{\tilde{J}} := 1$.

Theorem 6.2.3. *[14, Prop. 6.13] Let $\tilde{J} \subset R$ be a saturated quasi-stable ideal. For every $t \geq \rho_{\tilde{J}}$, we have an isomorphism $\mathbf{Mf}_{\tilde{J}_{\geq \rho_{\tilde{J}}-1}} \simeq \mathbf{Mf}_{\tilde{J}_{\geq t}}$. Moreover, letting $p(z)$ be the Hilbert polynomial of $R/\tilde{J}$, for every $t \geq \rho_{\tilde{J}} - 1$, $\mathbf{Mf}_{\tilde{J}_{\geq t}}$ is an open subscheme of the Hilbert scheme $\mathrm{Hilb}^n_{p(z)}$ that parameterizes closed subschemes of $\mathbb{P}^n$ having Hilbert polynomial $p(z)$.*

Remark 6.2.4. *It is noteworthy that, if $\tilde{J} \subset R$ is a saturated quasi-stable ideal then, for every Noetherian $\mathbb{K}$-algebra A and integer t, an ideal (G) belonging to $\mathbf{Mf}_{\tilde{J}_{\geq t}}(A) = \{(G) \subset R_A \mid G \text{ is a } \mathrm{MinPB}(\tilde{J}_{\geq t})\text{-marked basis}\}$ is of type $J_{\geq t}$, where J is a saturated ideal (see [14, Cor. 3.7]). More precisely, $J = {J^{sat}}_{\geq t}$. Furthermore, x_0 is generic for J, meaning that $J^{sat} = (J : x_0^\infty)$ [14, Thm. 3.5].*

We now assume that I is a saturated ideal in R. Then we consider the projective closed scheme $X := \mathrm{Proj}(R/I)$ with its Hilbert polynomial $p_X(z)$, and a *smaller* admissible Hilbert polynomial $p(z)$, in the sense that $p(t) \leq p_X(t)$, for $t \gg 0$.

Proposition 6.2.5. *Let I be a saturated ideal in R and $\tilde{J} \subseteq R$ be a saturated quasi-stable ideal having Hilbert polynomial $p(z)$. Let $\mathbf{Hilb}^{p(z)}_X$ be the Hilbert scheme that parameterizes the closed subschemes of $X = \mathrm{Proj}(R/I)$ with Hilbert polynomial $p(z)$. Then, for every $t \geq \rho_{\tilde{J}} - 1$, $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}^{p(z)}_X$ is an open subscheme of $\mathbf{Hilb}^{p(z)}_X$.*

Proof. First, recall that $\mathbf{Hilb}^{p(z)}_X$ is a closed subscheme of $\mathbf{Hilb}^{p(z)}_{\mathbb{P}^n}$ (e.g. [35, Exercise VI-26]). Then, we apply Theorem 6.2.3. ☐

Let $G \subseteq R_A$ be a $\mathrm{MinPB}(\tilde{J}_{\geq t})$-marked basis, for some integer t. We now prove a technical lemma, that allows us to characterize the ideals in $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}_X^{p(z)}$.

Lemma 6.2.6. *Let I be a saturated ideal in R and F a set of polynomials generating I. Let $\tilde{J} \subseteq R$ be a saturated quasi-stable ideal with Hilbert polynomial $p(z)$. If $G \subseteq R_A$ is a $\mathrm{MinPB}(\tilde{J}_{\geq t})$-marked basis for some $t \geq \rho_{\tilde{J}} - 1$, then the following statements are equivalent:*

(i) $(G)^{sat} \supseteq I$;

(ii) $(G) \supseteq I_{\geq t}$;

(iii) $(G) \supseteq \{x_0^{\max\{0,t-\deg(f)\}} f | f \in F\}$.

Proof. Recall that $(G) = {(G)^{sat}}_{\geq t}$ and $(G)^{sat} = ((G) : x_0^{\infty})$, by Remark 6.2.4.

It is immediate that (i) implies (ii) and (ii) implies (iii).

We now prove that item (iii) implies item (i). By hypothesis, for every $f \in F$, we have that either f belongs to $(G) \subseteq (G)^{sat}$, if $\deg(f) \geq t$, or $x_0^{t-\deg(f)} f$ belongs to (G), if $\deg(f) < t$. In this latter case, $f \in ((G) : x_0^{\infty}) = (G)^{sat}$. □

Using Lemma 6.2.6, we now prove that we can parameterize the ideals in $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}_X^{p(z)}$ by adding some polynomial conditions to those generating the ideal $\mathscr{U} \subseteq \mathbb{K}[C]$ which defines $\mathbf{Mf}_{\tilde{J}_{\geq t}}$.

Let $F \subseteq R$ be a set of polynomials generating I. For every $f \in F$, we take an integer d in the following way: if $\deg(f) \geq t$, then $d := 0$, otherwise $d := t - \deg(f)$. By a reduction relation like in Definition 6.1.5, we compute a polynomial $p_f \in \langle \mathcal{N}(\tilde{J}_{\geq t})_d \rangle_A$ such that $x_0^d f - p_f \in \langle \mathscr{G}^* \rangle_A$. We then denote by $\mathscr{V}_F$ the ideal generated in $\mathbb{K}[C]$ by the x-coefficients of the polynomials p_f.

Theorem 6.2.7. *With the notation above, $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}_X^{p(z)}$ is the affine scheme $\mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_F))$.*

Proof. Consider the $\mathrm{MinPB}(\tilde{J}_{\geq t})$-marked set $\mathscr{G} \subset R_{\mathbb{K}[C]}$ made of the polynomials in (6.1). For every $\mathbb{K}$-algebra A, a $\mathrm{MinPB}(\tilde{J})$-marked set in R_A is uniquely and completely given by a $\mathbb{K}$-algebra morphism $\varphi : \mathbb{K}[C] \to A$ defined by $\varphi(C_{\alpha\gamma}) = c_{\alpha\gamma} \in A$, for every $x^{\alpha} \in \mathrm{MinPB}(\tilde{J}_{\geq t}), x^{\gamma} \in \mathcal{N}(\tilde{J}_{\geq t})_{|\alpha|}$. We extend φ to a morphism from $R_{\mathbb{K}[C]}$ to R_A in the obvious way.

It is sufficient to observe that $\varphi(\mathscr{G}) \subset R_A$ is a $\mathrm{MinPB}(\tilde{J}_{\geq t})$-marked basis if and only if the generators of $\mathscr{U}$ vanish at $c_{\alpha\gamma} \in A$. Furthermore, by Lemma 6.2.6, the saturation of the ideal generated by $\varphi(\mathscr{G})$ in R_A contains I if and only if the generators of $\mathscr{V}_F$ vanish at $c_{\alpha\gamma} \in A$.

Hence, $\varphi(\mathscr{G})$ is a $\mathrm{MinPB}(\tilde{J}_{\geq t})$-marked basis in R_A and the saturation of the ideal it generates in R_A contains I if only if $\ker(\varphi) \supseteq \mathscr{U} + \mathscr{V}_F$. In this case, φ factors through $\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_F)$. The induced $\mathbb{K}$-algebra morphism from $\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_F)$ to A defines a scheme morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_F))$. Therefore, the scheme $\mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_F))$ is isomorphic to $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}_X^{p(z)}$. □

Remark 6.2.8. *Observe that the ideal $\mathscr{V}_F \subseteq \mathbb{K}[C]$ depends on the chosen generating set F of I. However, if $F' \subseteq \mathbb{K}[C]$ is another set of polynomials generating I, by Yoneda's Lemma we have that* $\mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_F)) \simeq \mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_{F'}))$.

In the following sections we define the notion of marked basis relative to an ideal I and investigate its features. This will give, for some ideals I, an alternative construction of the affine scheme $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}_X^{p(z)}$.

6.3 Relative Marked Bases

Let $\tilde{J} \supseteq \tilde{I}$ be quasi-stable ideals in R. With an abuse of notation, we keep on writing $\tilde{J}$ (resp. $\tilde{I}$) for $\tilde{J} \cdot R_A$ (resp. $\tilde{I} \cdot R_A$). Observe that the set of terms $\mathrm{MinPB}(\tilde{J}) \setminus \tilde{I}$ coincides with $\mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})$.

Let I be an ideal belonging to $\underline{\mathbf{Mf}}_{\tilde{I}}(\mathbb{K})$. This means that I is generated by a $\mathrm{MinPB}(\tilde{I})$-marked basis, which we denote by $F \subseteq R$. With an abuse of notation, we keep on writing I for $I \cdot R_A$. We also have the graded decomposition $R_A = I \oplus \langle \mathcal{N}(\tilde{I})\rangle_A$.

For every polynomial $p \in R_A$, we denote by $\mathrm{NF}_I(p)$ the normal form of p modulo I, which is the unique polynomial in $\langle \mathcal{N}(\tilde{I})\rangle_A$ such that $p - \mathrm{NF}_{\tilde{I}}(p) \in I$. Furthermore, we denote by $[p]_I$ the equivalence class of p in R_A/I.

We now prove a Lemma that will be useful later.

Lemma 6.3.1. *With the notation above, let $J \subseteq R_A$ be an ideal containing I.*

(i) For every polynomial $p \in J$, $\mathrm{NF}_I(p)$ belongs to $\langle \mathcal{N}(\tilde{I})\rangle_A \cap J$. In particular, $\mathrm{NF}_I(p)$ belongs to $J \setminus I$, unless it is null.

(ii) The graded decomposition $J = I \oplus (\langle \mathcal{N}(\tilde{I})\rangle_A \cap J)$ holds.

Proof.

(i) For every polynomial $p \in J$, it is sufficient to observe that $\mathrm{NF}_I(p)$ belongs to $J \cap \langle \mathcal{N}(\tilde{I})\rangle_A$ because $I \subseteq J$. The second assertion holds because $I \cap \langle \mathcal{N}(\tilde{I})\rangle_A = \{0\}$ by the hypotheses.

(ii) The map $\phi : [p]_I \in R_A/I \mapsto \mathrm{NF}_I(p) \in \langle \mathcal{N}(\tilde{I})\rangle_A$ is an isomorphism of A-modules because we have the graded decomposition $R_A = I \oplus \langle \mathcal{N}(\tilde{I})\rangle_A$, and hence the map
$$[p]_I \in J/I \mapsto \mathrm{NF}_I(p) \in \langle \mathcal{N}(\tilde{I})\rangle_A \cap J$$
is an isomorphism of A-modules too, obtained from ϕ by restriction. Hence, $J \simeq I \oplus (\langle \mathcal{N}(\tilde{I})\rangle_A \cap J)$ and this isomorphism is graded and is an equality thanks to item (i). □

Definition 6.3.2. *Let $\tilde{J} \supseteq \tilde{I}$ be quasi-stable ideals in R_A, I be a homogeneous ideal generated by a $\mathrm{MinPB}(\tilde{I})$-marked basis $F \subset R$, $H \subset R_A$ be a subset of a $\mathrm{MinPB}(\tilde{J})$-marked set, such that the head terms of the marked polynomials in H are the terms in*

$\mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})$, *and let* $J \subseteq R_A$ *be the ideal generated by* $F \cup H$. *We say that* H *is a* $\mathrm{MinPB}(\tilde{J})$-marked basis relative to I *if the following graded decomposition of* R_A *holds:*

$$R_A = I \oplus (\langle \mathcal{N}(\tilde{I})\rangle_A \cap J) \oplus \langle \mathcal{N}(\tilde{J})\rangle_A.$$

Theorem 6.3.3. *Let* $\tilde{J} \supseteq \tilde{I}$ *be quasi-stable ideals in* R_A, I *be a homogeneous ideal generated by a* $\mathrm{MinPB}(\tilde{I})$-*marked basis* $F \subset R$, $H \subset R_A$ *be a subset of a* $\mathrm{MinPB}(\tilde{J})$-*marked set, such that the head terms of the marked polynomials in* H *are the terms in* $\mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})$, *and* $J \subseteq R_A$ *be the ideal generated by* $F \cup H$.

The set H *is a* $\mathrm{MinPB}(\tilde{J})$-*marked basis relative to* I *if and only if* J *is generated by a* $\mathrm{MinPB}(\tilde{J})$-*marked basis containing* H.

Proof. Assume that J is generated by a $\mathrm{MinPB}(\tilde{J})$-marked basis G containing H. Then we have $R_A = J \oplus \langle \mathcal{N}(\tilde{J})\rangle_A$. Since J contains I, by Lemma 6.3.1(ii) we have that $J = I \oplus \langle \mathcal{N}(\tilde{I})\rangle_A \cap J$, and we obtain that H is a $\mathrm{MinPB}(\tilde{J})$-marked basis relative to I.

Assume now that H is a $\mathrm{MinPB}(\tilde{J})$-marked basis relative to I, hence the following decomposition holds:

$$R_A = I \oplus \Big(\langle \mathcal{N}(\tilde{I})\rangle_A \cap J\Big) \oplus \langle \mathcal{N}(\tilde{J})\rangle_A,$$

where we recall that $J = (F \cup H) \subseteq R_A$, with F the $\mathrm{MinPB}(\tilde{I})$-marked basis of I.

Observe again that by Lemma 6.3.1(ii), $J = I \oplus (\langle \mathcal{N}(\tilde{I})\rangle_A \cap J)$. By the above decomposition, we hence obtain $R_A = J \oplus \langle \mathcal{N}(\tilde{J})\rangle_A$, which means that J is generated by a $\mathrm{MinPB}(\tilde{J})$-marked basis G (see Remark 6.2.1).

Since $H \subseteq J$, $h \longrightarrow^{+}_{G^*} 0$ for every $h \in H$. By the hypotheses, for every $h \in H$ there is $g \in G$ with $\mathrm{Ht}(h) = \mathrm{Ht}(g)$. Hence, the polynomial $h - g$ belongs to $J \cap \langle \mathcal{N}(\tilde{J})\rangle_A = \{0\}$ and then $h = g$, by definition of $\mathrm{MinPB}(\tilde{J})$-marked basis. □

Remark 6.3.4. *Thanks to Theorem 6.3.3,* $\mathrm{MinPB}(\tilde{J})$-*marked bases relative to an ideal* I *are in bijective correspondence with the ideals* J *that are generated by a* $\mathrm{MinPB}(\tilde{J})$-*marked basis and contain the ideal* I.

Example 6.3.5. *In the polynomial ring* $\mathbb{K}[x_0, x_1, x_2, x_3]$, *consider the quasi-stable ideal* $\tilde{I} = (x_3^2, x_3x_2^3, x_2^3)$, *the ideal* I *generated by the* $\mathrm{MinPB}(\tilde{I})$-*marked basis* $F = \{x_3^2, x_3x_2^3, x_2^3 - 3x_3x_2^2\}$ *(which is not a Gröbner basis) and the stable ideal* $\tilde{J} = (x_3^2, x_3x_2, x_2^3)$ *(which contains* $\tilde{I}$*). Let* h_1 *be the marked polynomial* $x_2x_3 - x_1x_2$ *with* $\mathrm{Ht}(h_1) = x_2x_3$, *which is the unique term in* $\mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})$. *Applying the results of Section 6.2, we now see that the ideal* J *generated by* $F \cup \{h_1\}$ *is not generated by a* $\mathrm{MinPB}(\tilde{J})$-*marked basis.*

We consider the $\mathrm{MinPB}(\tilde{J})$-*marked set* $\mathscr{G}$ *whose marked polynomials are defined in (6.1) and impose that the marked polynomial of* $\mathscr{G}$ *with head term* x_2x_3 *is exactly* h_1. *We then compute the ideal* $\mathscr{U}$, *obtaining the scheme parameterizing* $\mathrm{MinPB}(\tilde{J})$-*marked bases having* h_1 *among its elements. However, if we impose to these marked bases to contain* I *(computing* $\mathscr{U} + \mathscr{V}_F$ *that we consider in Theorem 6.2.7), we verify that* $\mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_F))$ *is empty, in this case. This means that there are no ideals* *generated by a* $\mathrm{MinPB}(\tilde{J})$*-marked basis containing* h_1 *which also contain the marked set* F.

We now consider the marked polynomial $h_2 = x_2x_3 - 4x_2^2$, *with* $\mathrm{Ht}(h_2) = x_2x_3$. *The set* $H = \{h_2\}$ *is actually a* $\mathrm{MinPB}(\tilde{J})$*-marked basis relative to* I. *Indeed, the ideal* J *generated by* $F \cup H$, *with* F *the* $\mathrm{MinPB}(\tilde{I})$*-marked basis of* I, *is also generated by the* $\mathrm{MinPB}(\tilde{J})$*-marked basis* $G = \{x_3^2, x_3x_2 - 4x_2^2, x_2^3\}$. *Observe that* $h_2 \in G$.

Corollary 6.3.6. *Let* $\tilde{J} \supseteq \tilde{I}$ *be quasi-stable ideals in* R_A, I *be a homogeneous ideal generated by a* $\mathrm{MinPB}(\tilde{I})$*-marked basis* $F \subseteq R$, $H \subset R_A$ *be a subset of a* $\mathrm{MinPB}(\tilde{J})$*-marked set, such that the head terms of the marked polynomials in* H *are the terms in* $\mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})$, *and* $J \subseteq R_A$ *be the ideal generated by* $F \cup H$.

Let $F' := \{f \in F | \mathrm{Ht}(f) \in \mathrm{MinPB}(\tilde{I}) \cap \mathrm{MinPB}(\tilde{J})\}$ *and assume that* $H \cup F'$ *is a* $\mathrm{MinPB}(\tilde{J})$*-marked set.*

Then, H *is a* $\mathrm{MinPB}(\tilde{J})$*-marked basis relative to* I *if and only if* $H \cup F'$ *is the* $\mathrm{MinPB}(\tilde{J})$*-marked basis of* J.

Proof. In the setting of the statement, $f - \mathrm{Ht}(f) \in \langle \mathcal{N}(\tilde{J}) \rangle_A$, for every $f \in F'$, and $\mathrm{MinPB}(\tilde{J}) = (\mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})) \cup (\mathrm{MinPB}(\tilde{I}) \cap \mathrm{MinPB}(\tilde{J}))$. Moreover, $H \cup F'$ is contained in J by construction.

Hence, if $H \cup F'$ is the $\mathrm{MinPB}(\tilde{J})$-marked basis of J, then H is a $\mathrm{MinPB}(\tilde{J})$-marked basis relative to I by Theorem 6.3.3.

Vice versa, if H is a $\mathrm{MinPB}(\tilde{J})$-marked basis relative to I, then J is generated by a $\mathrm{MinPB}(\tilde{J})$-marked basis, that we denote by G, and J also contains the $\mathrm{MinPB}(\tilde{J})$-marked set $H \cup F'$. By Theorem 6.3.3, $H \subseteq G$. Let $f \in F'$, $g \in G$ be two marked polynomials with head term $x^\beta \in \mathrm{MinPB}(\tilde{I}) \cap \mathrm{MinPB}(\tilde{J})$. Then $f - g$ belongs to $J \cap \langle \mathcal{N}(\tilde{J}) \rangle_A$, but the last module is $\{0\}$, because J is generated by the $\mathrm{MinPB}(\tilde{J})$-marked basis G. Hence, we can conclude that $G = H \cup F'$. $\square$

Example 6.3.7. *In the polynomial ring* $\mathbb{K}[x_0, x_1, x_2]$, *we take* $I = \tilde{I}$, *which is generated by the* $\mathrm{MinPB}(\tilde{I})$*-marked basis* $F = \mathrm{MinPB}(\tilde{I}) = \{x_2^3, x_2^2x_1, x_2x_1^2, x_1^3\}$. *Let* $\tilde{J}$ *be the ideal* (x_2^2, x_1^2) *(which contains* $\tilde{I}$*) with* $\mathrm{MinPB}(\tilde{J}) = \{x_2^2, x_2x_1^2, x_1^2\}$. *Consider the set of marked polynomials* $H = \{h_1 = x_2^2 - ax_0x_2, h_2 = x_1^2 - bx_1x_2\}$, $\mathrm{Ht}(h_1) = x_2^2$, $\mathrm{Ht}(h_2) = x_1^2$, $a, b \in \mathbb{K} \setminus \{0\}$. *In this case* $F' = \{x_2x_1^2\}$ *and the marked set* $G = H \cup F'$ *is a* $\mathrm{MinPB}(\tilde{J})$*-marked set. However, we can check by Theorem 6.1.6 (iii) that* G *is not a* $\mathrm{MinPB}(\tilde{J})$*-marked basis, because* $x_2h_2 \longrightarrow^+_{G^*} -abx_0x_1x_2$, *which is non-zero, because* $a \neq 0, b \neq 0$. *By Corollary 6.3.6, we can conclude that* H *is not a* $\mathrm{MinPB}(\tilde{J})$*-marked basis relative to* I.

6.4 Algorithms for Relative Marked Bases

We now investigate constructive methods to check whether a subset of a $\mathrm{MinPB}(\tilde{J})$-marked set, such that the head terms of its marked polynomials are the terms in $\mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})$, is a relative marked basis.

The following theorem is obtained by rephrasing [26, Lem. 5.8, Thms. 5.9, 5.13, Cor. 5.11].

Theorem 6.4.1. *Let $\tilde{J} \supseteq \tilde{I}$ be quasi-stable ideals in R_A, I be a homogeneous ideal generated by a* $\mathrm{MinPB}(\tilde{I})$*-marked basis $F \subseteq R$, $H \subset R_A$ be a subset of a* $\mathrm{MinPB}(\tilde{J})$*-marked set, such that the head terms of the marked polynomials in H are the terms in* $\mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})$*, and $J \subseteq R_A$ be the ideal generated by $F \cup H$.*

Then H is a $\mathrm{MinPB}(\tilde{J})$*-marked basis relative to I if and only if J contains a* $\mathrm{MinPB}(\tilde{J})$*-marked set G containing H such that the following conditions are satisfied:*

(i) $\forall h_\beta \in H$, $\forall x_i > \min(\mathrm{Ht}(h_\beta))$, $x_i h_\beta \longrightarrow^+_{G^*} 0$

(ii) $\forall g_\alpha \in G \setminus H$, $\forall x_i > \min(\mathrm{Ht}(f_\alpha))$, $x_i g_\alpha \longrightarrow^+_{G^*} 0$

(iii) $\forall f \in F \setminus G$, $f \longrightarrow^+_{G^*} 0$.

Proof. Thanks to Theorem 6.3.3, it is enough to observe that items (i) and (ii) are equivalent to the fact that G is a $\mathrm{MinPB}(\tilde{J})$-marked basis, by [26, Lem. 5.8, Thms. 5.9, 5.13, Cor. 5.11], and that item (iii) is equivalent to the fact that I is contained in J. □

Remark 6.4.2. *In condition (iii) of Theorem 6.4.1, we could take any generating set L for I, and impose that for every $f \in L \setminus G$, $f \longrightarrow^+_{G^*} 0$.*

We now focus on the case in which the set of marked polynomials $F' = \{f \in F | \mathrm{Ht}(f) \in \mathrm{MinPB}(\tilde{I}) \cap \mathrm{MinPB}(\tilde{J})\}$ is contained in $\langle \mathcal{N}(\tilde{J})\rangle_A$.

Proposition 6.4.3. *Let $\tilde{J} \supseteq \tilde{I}$ be quasi-stable ideals in R_A, I be a homogeneous ideal generated by a* $\mathrm{MinPB}(\tilde{I})$*-marked basis $F \subseteq R$, $H \subset R_A$ be a subset of a* $\mathrm{MinPB}(\tilde{J})$*-marked set, such that the head terms of the marked polynomials in H are the terms in* $\mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})$*, and $J \subseteq R_A$ be the ideal generated by $F \cup H$.*

Let $F' := \{f \in F | \mathrm{Ht}(f) \in \mathrm{MinPB}(\tilde{I}) \cap \mathrm{MinPB}(\tilde{J})\}$ and assume that $G := H \cup F'$ is a $\mathrm{MinPB}(\tilde{J})$*-marked set. Then, H is a* $\mathrm{MinPB}(\tilde{J})$*-marked basis relative to $\tilde{I}$ if and only if:*

(i) $\forall h_\beta \in H$, $\forall x_i > \min(\mathrm{Ht}(h_\beta))$, $x_i h_\beta \longrightarrow^+_{H^*} r_{\beta,i} \in \langle F'^*\rangle_A$

(ii) $\forall f_\alpha \in F'$, $\forall x_i > \min(\mathrm{Ht}(f_\alpha))$, $x_i f_\alpha \longrightarrow^+_{H^*} r_{\alpha,i} \in \langle F'^*\rangle_A$

(iii) $\forall f \in F \setminus F'$, $f \longrightarrow^+_{H^*} r_f \in \langle F'^*\rangle_A$.

Proof. The reduction relation $\longrightarrow_{H^*}$ is Noetherian and confluent, see Definition 6.1.7 and Lemma 6.1.8. We prove that for every $p \in R_A$ the following statements are equivalent:

(1) $p \longrightarrow^+_{G^*} 0$

(2) $p \longrightarrow^+_{H^*} r$ with $r \in \langle F'^*\rangle$.

Assume that p reduces to 0 by G^*. Since G is equal to $H \cup F'$, this means that we have the following expression:

$$p = \sum_{H^*} c_{\beta\eta} x^\eta h_\beta + \sum_{F'^*} c_{\alpha\gamma} x^\gamma f_\alpha.$$

The above equality gives us that $p \longrightarrow^+_{H^*} \sum_{F'^*} c_{\alpha\gamma} x^\gamma f_\alpha$, and the latter is obviously an element in $\langle F'^* \rangle_A$.

Vice versa, if p reduces to an element $r \in \langle F'^* \rangle_A$ by H^*, then $r \longrightarrow^+_{G^*} 0$, being $F' \subset G$. The statement is now a direct consequence of Theorem 6.4.1. $\square$

Example 6.4.4. *With the same notation as in Proposition 6.4.3, if $H \cup F'$ is not a $\mathrm{MinPB}(\tilde{J})$-marked set, the fact that a polynomial p is reduced to 0 by G^* is not equivalent to the fact that p is reduced to some element in $\langle F'^* \rangle_A$ by H^*. Consider again $\tilde{I}$, $\tilde{J}$, F and H as in Example 6.3.5. In this case $F' = \{x_3^2, x_2^3 - 3x_3x_2^2\}$. Consider the polynomial $p = x_3(x_3x_2 - 4x_2^2)$, which belongs to $J = (F \cup H)$. Hence, $p \longrightarrow^+_{G^*} 0$, where G is the $\mathrm{MinPB}(\tilde{J})$-marked basis of J. However, $p \longrightarrow^+_{H^*} r = x_3^2x_2 - 16x_2^3$ and r does not belong to $\langle F'^* \rangle_A$.*

Remark 6.4.5. *Analogously to Remark 6.4.2, in item (iii) of Proposition 6.4.3 we can replace the set $F \setminus F'$ with $L \setminus F'$, where L is any generating set of I.*

We can further improve Proposition 6.4.3 in the case $I = \tilde{I}$. We define $\mathcal{T}_{\tilde{I},\tilde{J}}$ to be the set $\mathrm{MinPB}(\tilde{I}) \cap \mathrm{MinPB}(\tilde{J})$.

Corollary 6.4.6. *Let $\tilde{J} \supseteq \tilde{I}$ be quasi-stable ideals in R_A, $H \subset R_A$ be a subset of a $\mathrm{MinPB}(\tilde{J})$-marked set, such that the head terms of the marked polynomials in H are the terms in $\mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})$, and $J \subseteq R_A$ be the ideal generated by $\mathrm{MinPB}(\tilde{I}) \cup H$.*

Then H is a $\mathrm{MinPB}(\tilde{J})$-marked basis relative to $\tilde{I}$ if and only if:

(i) $\forall h_\beta \in H$, $\forall x_i > \min(\mathrm{Ht}(h_\beta))$, $x_i h_\beta \longrightarrow^+_{H^*} r_{\beta,i} \in \tilde{I}$

(ii) $\forall x^\alpha \in \mathbb{R}$, $\forall x_i > \min(x^\alpha)$, $x_i x^\alpha \longrightarrow^+_{H^*} r_{\alpha,i} \in \tilde{I}$

(iii) $\forall x^\gamma \in B_{\tilde{I}} \setminus \mathrm{MinPB}(\tilde{J})$, $x^\gamma \longrightarrow^+_{H^*} r_\gamma \in \tilde{I}$.

Proof. As highlighted in the proof of Proposition 6.4.3, for every $p \in R_A$, we have $p \longrightarrow^+_{G^*} 0$ if and only if $p \longrightarrow^+_{H^*} r$ with $r \in \langle \mathcal{T}^*_{\tilde{I},\tilde{J}} \rangle \subseteq \tilde{I}$. It is sufficient to observe that if $p \longrightarrow^+_{H^*} r \in \tilde{I}$, then r belongs to $\langle \mathcal{T}^*_{\tilde{I},\tilde{J}} \rangle \subseteq \tilde{I}$, as pointed out in Remark 6.1.9.

Hence, we can now replace the three items of Theorem 6.4.1 by the three ones in the statement. Indeed, taking into account also Remark 6.4.2, we can consider $\mathcal{B}_{\tilde{I}} \setminus \mathrm{MinPB}(\tilde{J})$ instead of $\mathrm{MinPB}(\tilde{I}) \setminus \mathrm{MinPB}(\tilde{J})$ in item (iii). $\square$

6.5 Relative Marked Functor

Let $\tilde{J} \supseteq \tilde{I}$ be quasi-stable ideals in R and $I \subseteq R_A$ be a homogeneous ideal generated by a $\mathrm{MinPB}(\tilde{I})$-marked basis F.

Definition 6.5.1. *With the notation above we define the functor*

$$\underline{\mathbf{Mf}}_{I,\tilde{J}} : \underline{\textit{Noeth } \mathbb{K}\text{–}\textit{Alg}} \longrightarrow \underline{\textit{Sets}}$$

such that

$$\underline{\mathbf{Mf}}_{I,\tilde{J}}(A) := \{H \subseteq R_A \mid H \subseteq R_A \textit{ is a } \mathrm{MinPB}(\tilde{J})\textit{-marked basis relative to } I\}$$

and, if $\phi : A \to B$ is a morphism of Noetherian $\mathbb{K}$-algebras (with $\phi(1_{\mathbb{K}}) = 1_{\mathbb{K}} \in B$), then the map $\underline{\mathbf{Mf}}_{I,\tilde{J}}(\phi)$ associates to every $H \in \underline{\mathbf{Mf}}_{I,\tilde{J}}(A)$ the MinPB$(\tilde{J})$*-marked basis $H \otimes_A B \in \underline{\mathbf{Mf}}_{I,\tilde{J}}(B) \subset R_B$ relative to I.*

We call $\underline{\mathbf{Mf}}_{I,\tilde{J}}$ the marked functor on $\tilde{J}$ relative to I*, or, when $\tilde{J}$ and I are well-understood, simply the* relative marked functor.

As the reader might expect, $\underline{\mathbf{Mf}}_{I,\tilde{J}}$ is strictly related to $\underline{\mathbf{Mf}}_{\tilde{J}}$ and to $\mathrm{Hilb}^X_{p(z)}$, with $X = \mathrm{Proj}(R/I)$.

Theorem 6.5.2. *With the notation above, the following statements hold:*

(i) The relative marked functor $\underline{\mathbf{Mf}}_{I,\tilde{J}}$ is a closed subfunctor of $\underline{\mathbf{Mf}}_{\tilde{J}}$;

(ii) If the ideals $\tilde{I}$ and $\tilde{J}$ are both saturated, then for every integer $t \geq \rho_{\tilde{J}} - 1$ the relative marked functor $\underline{\mathbf{Mf}}_{I_{\geq t},\tilde{J}_{\geq t}}$ is an open subfunctor of $\underline{\mathbf{Hilb}}^{p(z)}_X$ and it is represented by $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}^{p(z)}_X$.

Proof. For what concerns item (i), thanks to Theorem 6.3.3 there is a bijection between the set $\underline{\mathbf{Mf}}_{I,\tilde{J}}(A)$ and the set

$$\underline{\mathbf{Mf}}_{\tilde{J}}(A) \cap \{J \subseteq R_A : J \text{ is a homogeneous ideal containing } I\}.$$

As we already showed in Section 6.2, the condition "J contains I" can be imposed on the ideals in $\underline{\mathbf{Mf}}_{\tilde{J}}(A)$ by further closed conditions on the polynomials generating the ideal that defines the scheme which represents the functor $\underline{\mathbf{Mf}}_{\tilde{J}}$. So, we obtain item (i).

Thanks to item (i), Remark 6.2.4 and Lemma 6.2.6(ii), for every integer t, $\underline{\mathbf{Mf}}_{I_{\geq t},\tilde{J}_{\geq t}}$ is a closed subfunctor of $\underline{\mathbf{Mf}}_{\tilde{J}_{\geq t}}$. Moreover, if $t \geq \rho_{\tilde{J}} - 1$, for every Noetherian $\mathbb{K}$-algebra A, we have $\underline{\mathbf{Mf}}_{I_{\geq t},\tilde{J}_{\geq t}}(A) = \underline{\mathbf{Mf}}_{\tilde{J}_{\geq t}}(A) \cap \underline{\mathbf{Hilb}}^{p(z)}_X(A)$, and item (ii) holds thanks to Proposition 6.2.5. □

In the particular case $I = \tilde{I}$, we now give a construction of the scheme representing $\underline{\mathbf{Mf}}_{\tilde{I},\tilde{J}}$ which is alternative to the construction that is described in Section 6.2 for $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}^{p(z)}_X$. We use the computational method that arises from Corollary 6.4.6 in order to characterize relative marked bases.

Let C' denote the finite set of variables

$$\{C_{\beta\eta} \mid x^\beta \in \mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I}), x^\eta \in \mathcal{N}(\tilde{J}), \deg(x^\eta) = \deg(x^\beta)\}$$

and consider the $\mathbb{K}$-algebra $\mathbb{K}[C']$. Then, we construct the set $\mathscr{H} \subset R_{\mathbb{K}[C']}$ consisting of the following marked polynomials

$$h_\beta = x^\beta - \sum_{x^\eta \in \mathcal{N}(\tilde{J})_{|\beta|}} C_{\beta\eta} x^\eta \tag{6.2}$$

with $x^\beta \in \mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})$. Moreover, we define

$$\mathscr{H}^* = \{x^\delta h_\beta \mid h_\beta \in \mathscr{H}, x^\delta \in \mathrm{M}_{\mathcal{P}}(h_\beta)\}.$$

We highlight that the set C' can be identified to a subset of the set C given in Section 6.2, and up to this identification we can consider $\mathscr{H}$ as a subset of $\mathscr{G}$.

Then, we explicitly compute the following polynomials in $R_{\mathbb{K}[C']}$ by $\longrightarrow_{\mathscr{H}^*}$:

- $\forall h_\beta \in \mathscr{H}$, $\forall x_i > \min(\mathrm{Ht}(h_\beta))$, let $r_{\beta,i}$ be reduced with respect to $\mathscr{H}^*$ such that $x_i h_\beta \longrightarrow_{\mathscr{H}^*} r_{\beta,i}$;
- $\forall x^\alpha \in \mathbb{R}$, $\forall x_i > \min(x^\alpha)$, let $r_{\alpha,i}$ be reduced with respect to $\mathscr{H}^*$ such that $x_i x^\alpha \longrightarrow_{\mathscr{H}^*} r_{\alpha,i}$;
- $\forall x^\gamma \in \mathcal{B}_{\tilde{I}} \setminus \mathrm{MinPB}(\tilde{J})$, let r_γ be reduced with respect to $\mathscr{H}^*$ such that $f \longrightarrow_{\mathscr{H}^*} r_\gamma$.

For every $h_\beta \in \mathscr{H}$, and for every $x_i > \min(\mathrm{Ht}(h_\beta))$, we collect the coefficients in $\mathbb{K}[C']$ of the terms in $\mathrm{supp}(r_{\beta,i})$ not belonging to $\tilde{I}$, and the same for all the polynomials $r_{\alpha,i}$ and r_γ. Let $\mathscr{R} \subset \mathbb{K}[C']$ be the ideal generated by these coefficients.

Theorem 6.5.3. *The functor* $\underline{\mathbf{Mf}}_{\tilde{I},\tilde{J}}$ *is the functor of points of* $\mathrm{Spec}(\mathbb{K}[C']/\mathscr{R})$*, which we denote by* $\mathbf{Mf}_{\tilde{I},\tilde{J}}$.

Proof. Let $H \subseteq R_A$ be a $\mathcal{P}_{\tilde{J}}$-marked basis relative to $\tilde{I}$ and denote by ϕ_H the evaluation morphism $\phi_H : \mathbb{K}[C'] \to A$ that associates to every variable in C' the corresponding coefficient in the polynomials of H.

It is sufficient to observe that H is a $\mathcal{P}_{\tilde{J}}$-marked basis relative to $\tilde{I}$ if and only if ϕ_H factors through $\mathbb{K}[C']/\mathscr{R}$, or in other words if and only if the following diagram commutes

$$\begin{tikzcd} \mathbb{K}[C'] \arrow[rr, "\phi_H"] \arrow[dr] & & A \\ & \mathbb{K}[C']/\mathscr{R} \arrow[ur] & \end{tikzcd}$$

Equivalently, H is a $\mathcal{P}_{\tilde{J}}$-marked basis relative to $\tilde{I}$ if and only if $\mathscr{R}$ is contained in $\ker(\phi_H)$, which is true thanks to Corollary 6.4.6. □

Remark 6.5.4. *The scheme* $\mathrm{Spec}(\mathbb{K}[C']/\mathscr{R})$ *is computationally more advantageous compared with* $\mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_F))$ *considered in Theorem 6.2.7. Indeed, whenever* $\mathrm{MinPB}(\tilde{I}) \cap \mathrm{MinPB}(\tilde{J}) \neq \emptyset$*, we have* $|C'| < |C|$ *and the reduction* $\longrightarrow_{\mathscr{H}^*}$ *involves the relative marked set* $\mathscr{H}$*, which contains less polynomials than* $\mathscr{G}$*. Actually, in principle we perform less reduction steps using* $\longrightarrow_{\mathscr{H}^*}$*, which is a subreduction of* $\longrightarrow_{\mathscr{G}^*}$.

If the ideals $\tilde{I}$ and $\tilde{J}$ are saturated, we now give a further presentation of the scheme representing $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ for t given. Take the following polynomials in $\mathbb{K}[C']$:

$$\begin{array}{c}\forall x^\gamma \in \mathrm{MinPB}(\tilde{I}) \setminus \mathrm{MinPB}(\tilde{J}),\ \text{take } r_\gamma \text{ reduced with respect to } \mathscr{H}^* \\ \text{such that}\quad x_0^{\max\{0,t-\deg(x^\gamma)\}} x^\gamma \longrightarrow_{\mathscr{H}^*} r_\gamma\end{array} \tag{$\star$}$$

Observe that if t is strictly bigger than the initial degree of $\tilde{I}$, then $|\mathrm{MinPB}(\tilde{I}) \setminus \mathrm{MinPB}(\tilde{J})|$ is strictly smaller than $|\mathrm{MinPB}(\tilde{I}_{\geq t}) \cap \mathrm{MinPB}(\tilde{J}_{\geq t})|$.

Let $\mathscr{R}' \subset \mathbb{K}[C']$ be the ideal generated by the coefficients in $\mathbb{K}[C']$ of the terms not belonging to $\tilde{I}$ of the polynomials $r_{\beta,i}$, $r_{\alpha,i}$ considered for $\mathscr{R}$, and by the coefficients in $\mathbb{K}[C']$ of the terms not belonging to $\tilde{I}$ of the polynomials r_γ in ($\star$).

In Algorithm 26 we collect the instructions to compute the ideal $\mathscr{R}'$.

Theorem 6.5.5. *The functor* $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ *is the functor of points of* $\mathrm{Spec}(\mathbb{K}[C']/\mathscr{R}')$.

Proof. The proof is analogous to that of Theorem 6.5.3 thanks to Lemma 6.2.6. □

Algorithm 26: Algorithm for computing the defining ideal $\mathscr{R}'$ representing the relative marked functor $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$

```
Data: saturated quasi-stable ideals J̃ ⊇ Ĩ and a non-negative integer t
Result: generators of the ideal ℛ' representing Mf_{Ĩ≥t, J̃≥t}
begin
    let ℋ ⊆ K[C'] be the set of the polynomials defined in (6.1) with
      respect to the quasi-stable ideals J̃≥t and Ĩ≥t
    ℛ' := (0)
    for h_β ∈ ℛ' do
        for x_i > min(Ht(h_β)) do
            compute r_{β,i} such that x_i h_β →⁺_{ℋ*} r_{β,i}
            ℛ' := ℛ' + (coefficients in r_{β,i} of the terms not belonging to Ĩ)
    for x^α ∈ MinPB(J̃≥t) ∩ MinPB(Ĩ≥t) do
        for x_i > min(x^α) do
            compute r_{α,i} such that x_i x^α →⁺_{ℋ*} r_{α,i}
            ℛ' := ℛ' + (coefficients in r_{α,i} of the terms not belonging to Ĩ)
    for x^γ ∈ MinPB(Ĩ) \ MinPB(J̃) do
        compute r_γ such that x_0^{max{0,t-deg(x^γ)}} x^γ →⁺_{ℋ*} r_γ
        ℛ' := ℛ' + (coefficients in r_γ of the terms not belonging to Ĩ)
    return ℛ'
```

6.6 Open Covers for Hilbert Schemes over some Quotient Rings

In this section we consider Hilbert schemes defined over Cohen-Macaulay quotient rings $S = R/\tilde{I}$ of positive Krull-dimension, where $\tilde{I}$ is a quasi-stable ideal.

We first recall some notions and set notations when $S := R/M$ is more generally a quotient over any monomial ideal M of R. In this setting, when we consider the image in S of an element f of R we mean its image $[f]$ by the projection $\pi : R \to R/M$.

Following [82], we say that a term of R is M-*free* if its image in the quotient ring S is non-null, i.e. it does not belong to M. A term of S is the image in S of an M-free term of R. If W is any set of terms in S, by abuse of notation we will use the symbol W to also denote the set of terms in $\pi^{-1}(W)$.

An ideal $U \subseteq S$ is *monomial* if is the image in S of a monomial ideal of R. Every monomial ideal U of S has a unique minimal generating set B_U made of terms of S.

We now assume that $M = \tilde{I}$ is quasi-stable in R and hence $S = R/\tilde{I}$ is a quotient over such an ideal. In this setting, a monomial ideal U of S will be said *quasi-stable in* S if the ideal $(B_U \cup \mathcal{B}_{\tilde{I}})$ is quasi-stable in R.

Thus, an ideal in R containing $\tilde{I}$ is quasi-stable if, and only if, its image in S is quasi-stable.

Now, the following definition naturally arises.

Definition 6.6.1. *Let U be a quasi-stable ideal in S and let $\tilde{J} = (B_U \cup \mathcal{B}_{\tilde{I}})$. A finite set F of elements of S is a U-*marked set *if the elements of F are the images of polynomials that are marked over the terms in $\mathrm{MinPB}(\tilde{J}) \setminus \mathrm{MinPB}(\tilde{I})$ and are contained in a $\mathrm{MinPB}(\tilde{J})$-marked set. We say that F is a U-*marked basis *if F is the image of a $\mathrm{MinPB}(\tilde{J})$-marked basis relative to $\tilde{I}$.*

Since we want to consider quotients $S = R/\tilde{I}$ over quasi-stable ideals $\tilde{I}$ that are even Cohen-Macaulay, we now recall a Cohen-Macaulay characterization for quasi-stable ideals. For any quasi-stable ideal, there is an index $0 \leq i \leq n$ such that for each $i \leq j \leq n$, a pure power of the form $x_j^{a_j}$ is contained in the ideal. Moreover, for every index $\ell < i$ there is no pure variable power $x_\ell^{b_\ell}$ in the ideal. This minimal index of a variable appearing in a pure power can be used to characterize Cohen-Macaulay quasi-stable ideals.

Proposition 6.6.2. *[97, Thm. 5.2.9] Let $\tilde{I}$ be a quasi-stable ideal in R and let $\mathrm{MinPB}(\tilde{I})$ be its minimal Pommaret basis. Then $S = R/\tilde{I}$ is Cohen-Macaulay if and only if, for the integer $m = \min\{\min(x^\alpha) \mid x^\alpha \in \mathrm{MinPB}(\tilde{I})\}$, there is a pure variable power $x_m^{a_m}$ contained in $\tilde{I}$.*

Remark 6.6.3. *Note that the above criterion can be applied to a quasi-stable ideal $\tilde{I}$ by looking at the minimal value of $\min(x^\alpha)$ for x^α in the minimal generating set of $\tilde{I}$.*

From now, let $S := R/\tilde{I}$ be a Cohen-Macaulay ring of positive Krull-dimension with $\tilde{I}$ quasi-stable. Under these assumptions, the ideal $\tilde{I}$ is necessarily saturated (as it is evident for example thanks to Proposition 6.6.2) and defines the projective scheme $X = \mathrm{Proj}(S)$. Hence we can consider the Hilbert scheme $\mathbf{Hilb}_X^{p(z)}$, for an admissible Hilbert polynomial $p(z)$.

Let $x_{k+1}, \dots, x_n$ be the variables that divide some minimal generators of $\tilde{I}$. In the following discussion, we will write $\mathcal{T}'$ for the set of all terms in the polynomial ring $\mathbb{K}[x_{k+1}, \dots, x_n]$ and $\mathcal{T}''$ for the set of all terms in the polynomial ring $\mathbb{K}[x_0, \dots, x_k]$.

We need to adapt [6, Proposition 7.2 and Corollary 7.4] to our current setting. We make as a first step the following observation.

Lemma 6.6.4. *The Cohen-Macaulay quotient ring $S = R/\tilde{I}$ is a finitely generated graded free $\mathbb{K}[x_0, \dots, x_k]$-module*

$$S = \bigoplus_{e=0}^{d} \left(\mathbb{K}[x_0, \dots, x_k](-e)\right)^{m_e},$$

where $d = \max\{\deg(t) \mid t \in \mathcal{T}' \setminus \tilde{I}\}$ and, for each $0 \leq e \leq d$, $m_e = |\{t \in \mathcal{T}' \setminus \tilde{I} \mid \deg(t) = e\}|$.

Proof. First, note that $\tilde{I}$ contains pure powers of all variables x_j with $j > k$; hence, the set $\mathcal{T}(e) = \{t \in \mathcal{T}' \setminus \tilde{I} \mid \deg(t) = e\}$ is finite. Since the generators of $\tilde{I}$ are terms in $\mathbb{T}'$, we have for each $t \in \mathcal{T}(e)$ an injection

$$\iota : \mathbb{T}'' \to S, u \mapsto [u \cdot t].$$

Now, we turn to the graded decomposition. The ring S inherits a grading from R: indeed, a term $[t] \in S$ (i.e., a residue class of a term $t \in \mathcal{T}$) has degree $q \geq 0$ if and only if t has the degree q in R. Moreover, it is easy to see that the set of terms of degree $q \geq 0$ in S is disjointly decomposed as follows:

$$S_q \cap \{[u] \mid u \in \mathcal{T}\} = \bigsqcup_{e=0}^{d} \bigsqcup_{t \in \mathcal{T}(e)} t \cdot \left(S_{q-e} \cap \{[u] \mid u \in \mathcal{T}''\}\right). \tag{6.3}$$

It is important to note that all elements in the sets of the right hand side of (6.3) are non-zero; this is guaranteed by the Cohen-Macaulay property of $\tilde{I}$. The claim follows. $\square$

In the present setting, thanks to Lemma 6.6.4, a quasi-stable ideal U of S can be even considered as a $\mathbb{K}[x_0, \dots, x_k]$-submodule of S. We now highlight that the definition of quasi-stable ideal U of S as the image in S of a quasi-stable ideal $\tilde{J}$ in R containing $\tilde{I}$ is equivalent to the definition of quasi-stable submodule of a free module given in [6, Definition 3.2 item (i)].

Proposition 6.6.5. *Let $U \subseteq S$ be a monomial ideal with minimal generating set B_U. Then U is quasi-stable if and only if B_U, interpreted as a monomial subset of the $\mathbb{K}[x_0, \dots, x_k]$-module S, generates a quasi-stable submodule.*

Proof. The proof is by routine verification of quasi-stability conditions for the minimal generators of the ideals and submodules that are considered. □

However, the notions of U-marked set introduced in Definition 6.6.1 and of marked set over a submodule given in [6, Definition 4.3] are different, as we will see in Example 6.6.7.

In the remaining part of this section, we assume that the field $\mathbb{K}$ that we work with is infinite.

First, we apply [6, Corollary 7.4] to a finite subset of a degree component S_q of the $\mathbb{K}[x_0, \dots, x_k]$-module S. We denote by $\mathrm{PGL}(k+1)$ the subset of $\mathrm{PGL}_{\mathbb{K}}(n+1)$ whose elements define invertible change of coordinates of the following kind:

$$x_i \mapsto x_i \quad \text{for } i = k+1, \dots, n, \quad x_j \mapsto \sum_{t=0}^{k} g_{jt} x_t \quad \text{for } j = 0, \dots, k.$$

For any element $g \in \mathrm{PGL}(k+1)$ we denote by $\tilde{g}$ the automorphism induced by g on S.

Proposition 6.6.6. *For a given degree $q \geq 0$, let $F \subset S_q$ be a finite set of elements of S. Then there exists a transformation $g \in \mathrm{PGL}(k+1)$ such that $\tilde{g}(F)$ is a marked set over a quasi-stable monomial submodule of S.*

Proof. We can directly apply [6, Cor. 7.4], because F is a subset of a single degree component of the finitely generated free graded $\mathbb{K}[x_0, \dots, x_k]$-module S by Lemma 6.6.4. □

Example 6.6.7. *Consider $R = \mathbb{K}[x_0, x_1, x_2]$, $\tilde{I} = (x_2^7)$. Then one can transform the set $F = \{x_0 x_1\}$ to a set $\hat{F} = \{x_0 x_1 + x_1^2\}$ by letting $x_0 \mapsto x_0 + x_1$, and this is marked on the term x_1^2 generating a quasi-stable ideal of $U = (x_1^2) \subseteq S = R/\tilde{I}$. However, defining $\tilde{J} = (x_1^2, x_2^7) \subseteq R$, we get the Pommaret basis $\mathrm{MinPB}(\tilde{J}) = \{x_1^2, x_1^2 x_2, \dots, x_1^2 x_2^6, x_2^7\}$. Thus, according to Definition 6.6.1, $\hat{F} \subseteq S$ is not a U-marked set (for such a set, we would need additionally polynomials marked on each of the terms $x_1^2 x_2, \dots, x_1^2 x_2^6$). Nevertheless, $\hat{F}$ is marked on the Pommaret basis of the quasi-stable monomial $\mathbb{K}[x_0, x_1]$-submodule of S generated by $\{x_1^2 \cdot [1]\}$, because this singleton set is also the Pommaret basis of the submodule generated by it (see [6, Definition 3.1]).*

Nevertheless, for high degrees q, the marked set over a quasi-stable submodule obtained by a transformation as in Proposition 6.6.6 is indeed a U-marked set for a quasi-stable ideal $U \subseteq S$:

Proposition 6.6.8. *Let F be as in Proposition 6.6.6, $J := (F, \tilde{I})$ and assume that $q \geq \max\{\mathrm{reg}(\tilde{I}), \mathrm{reg}(J)\}$. Consider $g \in \mathrm{PGL}(k+1)$ as in Proposition 6.6.6. Then the marked set $\tilde{g}(F)$ is not only a set marked over a quasi-stable $\mathbb{K}[x_0, \dots, x_k]$-submodule of S, but also a U-marked set, where $U = (\mathrm{Ht}(\tilde{g}(F))) \subseteq S$. Moreover, the Pommaret basis $\mathrm{MinPB}(\tilde{J})$ of $\tilde{J} = (\mathrm{Ht}(\tilde{g}(F)), \tilde{I})$ is given by the disjoint union of $\mathrm{Ht}(\tilde{g}(F))$ and $\mathrm{MinPB}(\tilde{I})$.*

Proof. Under the made assumptions, and by Proposition 6.6.5, the head terms of $\tilde{g}(F)$ minimally generate a quasi-stable ideal $U \subseteq S$. Since $q \geq \mathrm{reg}(\tilde{I})$, none of the head terms divides any term of $\mathrm{MinPB}(\tilde{I})$. Hence, $(\mathrm{Ht}(\tilde{g}(F)), \mathrm{MinPB}(\tilde{I}))$, being a generating set of maximal degree q, must also be the Pommaret basis of $\tilde{J}$, by the q-regularity of J. This proves the statement about $\mathrm{MinPB}(\tilde{J})$, and it now follows easily that $\tilde{g}(F)$ is a U-marked set. □

Example 6.6.9. *Consider $\tilde{I} = (x_2^7) \subseteq R = \mathbb{K}[x_0, x_1, x_2]$, $S = R/\tilde{I}$, and $F = \{x_0 x_1^6\}$. Note that $F \subset S_7$, and $7 = \mathrm{reg}(\tilde{I})$. While the transformation $x_0 \mapsto x_0 + x_1$ applied to F yields $\hat{F} = \{x_0 x_1^6 + x_1^7\}$, which is marked on $\{x_1^7\}$, and $U = (x_1^7)$ is a quasi-stable ideal in S, $\hat{F}$ is not a U-marked set in the sense of Definition 6.6.1, because the Pommaret basis $\mathrm{MinPB}(\tilde{J})$ of $\tilde{J} = (x_1^7, x_2^7) \subseteq R$ includes also the terms $x_1^7 x_2^a$ for $1 \leq a \leq 6$. Note that the degrevlex leading ideal of $J = (\hat{F}, \tilde{I})$ is exactly $\tilde{J}$; this implies that J (and hence also $(F, \tilde{I})$) is 13-regular (13 being the highest degree of an element of $\mathrm{MinPB}(\tilde{J})$).*

Now consider, as above, $\tilde{I} = (x_2^7)$, but set $F = \{x_1 x_2^6\}$. We have $F \subset S_7$, and $7 = \mathrm{reg}(\tilde{I})$; moreover F is already marked on $\{x_1 x_2^6\}$ which generates a quasi-stable ideal $U = (x_1 x_2^6) \subseteq S$. Since the Pommaret basis $\mathrm{MinPB}(\tilde{J})$ of $\tilde{J} = (F, \tilde{I})$ is exactly $F \cup \{x_2^7\}$, F is a U-marked set in the sense of Definition 6.6.1. Note that the ideal $J = \tilde{J} = (F, \tilde{I}) \subseteq R$ is 7-regular, because it is a quasi-stable monomial ideal whose minimal Pommaret basis has maximal degree 7.

Proposition 6.6.8 guarantees that any finite set of homogeneous elements in S of the same *big enough* degree q can be transformed into a U-marked set, for a suitable quasi-stable ideal U. The following result is a consequence.

Corollary 6.6.10. *For every field extension $\mathbb{L}$ of $\mathbb{K}$, let $J \subseteq R_{\mathbb{L}}$ be a saturated ideal containing $\tilde{I}$ and t be an integer such that $t \geq \max\{\mathrm{reg}(J), \mathrm{reg}(\tilde{I})\}$. Then, there exists $g \in \mathrm{PGL}(k+1)$ such that the ideal $\tilde{g}(J_t) \cdot S$ is generated by the image in S of a $\mathrm{MinPB}(\tilde{J})$-marked basis H relative to $\tilde{I}$ that belongs to $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t}, \tilde{J}_{\geq t}}(\mathbb{L})$, for some saturated quasi-stable ideal $\tilde{J}$ containing $\tilde{I}$.*

Proof. First we observe that $(\tilde{I}_t)$ must be contained in (J_t). Let F be a set of generators of (J_t) made only of polynomials of degree t. We can assume that the minimal monomial generators of $\tilde{I}_{\geq t}$ are contained in F. Recall that we are now assuming that $\mathbb{K}$ is infinite, and hence Zariski dense in any field extension $\mathbb{L}$. Then the thesis follows from Proposition 6.6.8 applied to the image in S of the given set F and from Corollary 6.3.6 and Lemma 6.2.6. Indeed, note that the saturation of $(\mathrm{Ht}(\tilde{g}(F)), \mathrm{MinPB}(\tilde{I}_{\geq t}))$ contains $\tilde{I}$ and its regularity is $\leq t$ by construction. □

Recall that if J belongs to $\underline{\mathbf{Mf}}_{\tilde{J}}(\mathbb{K})$ then $J_{\geq t}$ belongs to $\underline{\mathbf{Mf}}_{\tilde{J}_{\geq t}}(\mathbb{K})$, but the converse is not true (e.g. [14, Example 3.8]). However, by Lemma 6.2.6, if $J_{\geq t}$ belongs to $\underline{\mathbf{Mf}}_{\tilde{J}_{\geq t}}(\mathbb{K})$ and contains $\tilde{I}_{\geq t}$, then J contains $\tilde{I}$.

Given the quasi-stable ideal $\tilde{I}$, let $p(z)$ be any Hilbert polynomial as in Section 6.2, and consider the sets

$$Q_{p(z)} := \{\tilde{J} \text{ saturated quasi-stable} \mid S/\tilde{J} \text{ has Hilbert polynomial } p(z)\},$$

$Q_{p(z),\tilde{I}} := \{\tilde{J}$ saturated quasi-stable $\mid \tilde{J} \supseteq \tilde{I}$ and $S/\tilde{J}$ has Hilbert polynomial $p(z)\}$. The *Gotzmann number* r of the Hilbert polynomial $p(z)$ is the smallest integer such that $r \geq \operatorname{reg}(J)$ for every saturated ideal J defining a scheme lying on $\mathbf{Hilb}^{p(z)}_{\mathbb{P}^n}$.

For every $g \in \mathrm{PGL}(k+1)$, we consider the functor $\underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ that assigns to every $\mathbb{K}$-algebra A the set $\{\tilde{g}^{-1}(F) \subset A[x] | F \in \underline{\mathbf{Mf}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}(A)\}$ and to every $\mathbb{K}$-algebra morphism $\sigma : A \to A'$, the map

$$\begin{aligned} \underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}(\sigma) : \underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}(A) &\to \underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}(A') \\ \tilde{g}^{-1}(F) &\mapsto \tilde{g}^{-1}(\sigma(F)). \end{aligned}$$

The transformation $\tilde{g}^{-1}$ induces a natural isomorphism of functors between $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ and $\underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$, hence $\underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ is an open subfunctor of $\underline{\mathbf{Hilb}}^{p(z)}_X$ for every $g \in \mathrm{PGL}(k+1)$ thanks to Theorem 6.5.2 item (ii). Analogously, for every $g \in \mathrm{PGL}_{\mathbb{K}}(n+1)$, $\underline{\mathbf{Mf}}^{g}_{\tilde{J}_{\geq t}}$ is the open subfunctor of $\underline{\mathbf{Hilb}}^{p(z)}_{\mathbb{P}^n}$ that we obtain from $\underline{\mathbf{Mf}}_{\tilde{J}_{\geq t}}$ by the natural isomorphism induced by g^{-1}.

Theorem 6.6.11. *Let $\tilde{I} \subseteq R$ be a saturated quasi-stable ideal such that $S = R/\tilde{I}$ is a Cohen-Macaulay ring and let $X = \mathrm{Proj}(S)$ be the scheme defined by $\tilde{I}$. Let $p(z)$ be a Hilbert polynomial such that $p(t) \leq p_X(t)$ for $t \gg 0$, r be the Gotzmann number of $p(z)$ and $t := \max\{\operatorname{reg}(\tilde{I}), r\}$. Then, there is the open covering*

$$\underline{\mathbf{Hilb}}^{p(z)}_X = \bigcup_{g \in \mathrm{PGL}(k+1)} \left(\bigcup_{\tilde{J} \in Q_{p(z),\tilde{I}}} \underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}} \right).$$

Proof. We can apply [6, Prop. 10.3] to the Hilbert functor $\underline{\mathbf{Hilb}}^{p(z)}_{\mathbb{P}^n}$, obtaining the following open cover

$$\underline{\mathbf{Hilb}}^{p(z)}_{\mathbb{P}^n} = \bigcup_{g \in \mathrm{PGL}_{\mathbb{K}}(n+1)} \left(\bigcup_{\tilde{J} \in Q_{p(z)}} \underline{\mathbf{Mf}}^{g}_{\tilde{J}_{\geq t}} \right). \tag{6.4}$$

Since $\underline{\mathbf{Hilb}}^{p(z)}_X$ is a closed subfunctor of $\underline{\mathbf{Hilb}}^{p(z)}_{\mathbb{P}^n}$, we have an open cover of $\underline{\mathbf{Hilb}}^{p(z)}_X$ intersecting it with the open subfunctors of (6.4). In order to cover $\underline{\mathbf{Hilb}}^{p(z)}_X$ it is enough to consider $J \in Q_{p(z),\tilde{I}}$, thanks to Theorem 6.3.3.

We now observe that it is even enough to only take $g \in \mathrm{PGL}(k+1)$ thanks to Corollary 6.6.10. We can now conclude taking into account that $\underline{\mathbf{Mf}}^{g}_{\tilde{J}_{\geq t}} \cap \underline{\mathbf{Hilb}}^{p(z)}_X = \underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ by Theorem 6.5.2 item (ii), combined with [35, Exercise VI-11]. □

6.7 Lex-points over some Quasi-stable Macaulay-Lex Quotients

Recall that we are denoting by $R = \mathbb{K}[x_0, \ldots, x_n]$ the polynomial ring over a field $\mathbb{K}$ in $n+1$ variables $x_0 < x_1 < \cdots < x_n$.

Using the notation and terminology introduced in Section 6.6, we now recover the notion of lex-ideal in quotient rings $S := R/M$ where M is a monomial ideal of R.

Definition 6.7.1. *(see [74, 82]) A set W of terms of S is called a* lex-segment *of S if, for all terms $u, v \in S$ of the same degree, if u belongs to W and $v >_{lex} u$ then v belongs to W. A monomial ideal U of S is called a* lex-ideal *if the set of terms in U is a lex-segment of S.*

Example 6.7.2. *The image of a lex-ideal of R in S is a lex-ideal of S. However, there are lex-ideals of S that are not the image of a lex-ideal of R. For example, consider $n = 3$ and $\tilde{I} = (x_3^2, x_2^5)$. Then, the image U in S of the ideal $\tilde{J} = (x_3^2, x_3x_2, x_3x_1^2, x_2^5) \subseteq R$ is a lex-ideal in S, but $\tilde{J}$ is not a lex-ideal in R.*

The quotient ring S is called a *Macaulay-Lex ring* if, for any homogeneous ideal U of S, there exists a lex-ideal of S having the same Hilbert function as U (e.g. [74]). If the monomial ideal M induces a Macaulay-Lex quotient ring, then we say that M is *Macaulay-Lex*.

Example 6.7.3. *Various families of examples of Macaulay-Lex monomial ideals $M \subseteq R$ are known. We list some of them explicitly and point to references in other cases.*

1. *The most well-known class are the* Clements-Lindström ideals *[29]. They are ideals generated by regular sequences, of the form $M = (x_0^{d_0}, x_1^{d_1}, \dots, x_n^{d_n})$ with $2 \leq d_n \leq \cdots \leq d_1 \leq d_0 \leq \infty$. Formal expressions of the form x_i^∞ are interpreted as 0 for this purpose. One may also allow $1 \leq d_n$, but if $d_n = 1$, one may as well work in a quotient of $\mathbb{K}[x_0, \dots, x_{n-1}]$ and drop the generator x_n. Note that all Clements-Lindström ideals are quasi-stable.*

2. *Abedelfatah [2, Theorem 4.5] discovered two families of Macaulay-Lex ideals, whose generating sets show some similarities to the generators of Clements-Lindström ideals. In our conventions, they are given as follows, under the conditions $2 \leq e_n \leq e_{n-1} \leq \cdots \leq e_0 \leq \infty$ and $t_i < e_i$ for all i:*

   - $I = (x_n^{e_n}, x_n^{t_n}x_{n-1}^{e_{n-1}}, \dots, x_n^{t_n}x_0^{e_0})$,
   - $I = (x_n^{e_n}, x_n^{e_n-1}x_{n-1}^{e_{n-1}}, x_n^{e_n-1}x_{n-1}^{t_{n-1}}x_{n-2}^{e_{n-2}}, \dots, x_n^{e_n-1}x_{n-1}^{t_{n-1}} \cdots x_1^{t_1}x_0^{e_0})$.

   One can show that every such ideal is quasi-stable.

3. *Mermin [75] showed that a monomial regular sequence generates a Macaulay-Lex ideal if and only if it is of the form*

$$(x_n^{e_n}, x_{n-1}^{e_{n-1}}, \dots, x_{r+1}^{e_{r+1}}, x_r^{e_r-1}x_i),$$

   where $e_n \leq e_{n-1} \leq \ldots \leq e_r$ and $i \leq r$. Note that such an ideal is quasi-stable if and only if $i = r$, i.e., if it is a Clements-Lindström ideal.

4. *A complete characterization of all Macaulay-Lex monomial ideals in* $\mathbb{K}[x_0, x_1]$ *is known (see e.g. [63]). In particular, there are many quasi-stable Macaulay-Lex ideals in the polynomial ring with two variables.*

5. *Given* n *zero-dimensional Macaulay-Lex monomial ideals* M_i*, each of them in a polynomial ring with two variables, one can construct [64] a zero-dimensional Macaulay-Lex ideal in* R *from* $M_1, \dots, M_n$*. Being zero-dimensional, this ideal is also quasi-stable. Note that the construction in [64] covers also more general cases.*

6. *For each Macaulay-Lex monomial ideal* $M_i \subseteq R_i = \mathbb{K}[x_i, \dots, x_n]$*, where* $i \in \{1, \dots, n\}$*, also the extension ideal* $(M_i)_R \subseteq R$ *is Macaulay-Lex.*

From now we assume that $S = R/M$ is a Macaulay-Lex ring with $M := \tilde{I}$ quasi-stable in R. Recall that a monomial ideal U of S is quasi-stable in S if the ideal $(B_U \cup \mathcal{B}_{\tilde{I}})$ is quasi-stable in R.

Lemma 6.7.4. *With the notation above,*

(i) If W *is a lex-segment in* S *then* $\{x_0, \dots, x_n\} \cdot W$ *is a lex-segment in* S*.*

(ii) A lex-segment ideal U *in* S *is quasi-stable.*

(iii) If U *is a lex-segment ideal of* S*, then* $(B_U \cup \mathcal{B}_{\tilde{I}})^{sat}/\tilde{I}$ *is a lex-segment ideal.*

Proof. For item (i) see [76, Proposition 2.5]. For item (ii), let τ be a term of U with minimal variable x_i and let $x_j > x_i$. Since $x_j \frac{\tau}{x_i} >_{lex} \tau$, we must have that $x_j \frac{\tau}{x_i}$ belongs to U unless it belongs to $\tilde{I}$. Then, we conclude because $\tilde{I}$ is quasi-stable. Item (iii) now follows from item (ii) and from the properties of the lexicographic term order, because thanks to the properties of quasi-stable ideals we obtain the saturation replacing x_0 by 1 in every generator. □

If $\tilde{I}$ is a saturated ideal and $S = R/\tilde{I}$ has positive Krull-dimension, then $\tilde{I}$ defines the projective scheme $X = \mathrm{Proj}(S)$. Hence we can consider the Hilbert scheme $\mathbf{Hilb}_X^{p(z)}$ on the Macaulay-Lex ring S, for an admissible Hilbert polynomial $p(z)$.

Theorem 6.7.5. *Let* $S = R/\tilde{I}$ *be a Macaulay-Lex ring with positive Krull-dimension and* $X = \mathrm{Proj}(S)$*. Then* $\mathbf{Hilb}_X^{p(z)}$ *is non-empty if and only if it contains a (unique) point* Y *defined by a lex-ideal of* S*. Moreover,* Y *has the minimal possible Hilbert function in* $\mathbf{Hilb}_X^{p(z)}$*.*

Proof. Let r be the maximum between the Gotzmann number of $p(z)$ and the regularity of $\tilde{I}$. If $\mathbf{Hilb}_X^{p(z)}$ is non-empty, then there exists at least a lex-ideal U of S such that S/U has Hilbert polynomial $p(z)$, because S is Macaulay-Lex. In particular, letting $\tilde{p}(z)$ be the Hilbert polynomial of $S = R/\tilde{I}$, the set W made of the $\tilde{p}(r) - p(r)$ lex-largest terms of U of degree r is a lex-segment. Thanks to Lemma 6.7.4 item (iii), the ideal $(W \cup \mathcal{B}_{\tilde{I}})^{sat}/\tilde{I}$ is a lex-ideal too and by construction defines the desired point Y in $\mathbf{Hilb}_X^{p(z)}$. Indeed, by definition of saturation we obtain the same point Y starting from any other lex-ideal of S with Hilbert polynomial $p(z)$, so that the last assertion also follows. □

Definition 6.7.6. *Let S be a Macaulay-Lex ring over a saturated quasi-stable ideal $\tilde{I}$ and $X = \mathrm{Proj}(S)$. If $\mathbf{Hilb}_X^{p(z)}$ is non-empty, then its unique point defined by a lex-ideal of S is called the* lex-point *of $\mathbf{Hilb}_X^{p(z)}$.*

Remark 6.7.7. *With the notation above, let r be the maximum between the Gotzmann number of $p(z)$ and the regularity of $\tilde{I}$. If W is the lex-segment of S made of $\tilde{p}(r) - p(r)$ terms of degree r and $|\{x_0, \dots, x_n\} \cdot W \cup \mathcal{B}_{\tilde{I}_{r+1}}| = p(r+1)$, then the ideal generated by W is a lex-ideal in S by Lemma 6.7.4 item (i) and defines the lex-point of $\mathbf{Hilb}_X^{p(z)}$, thanks to Gotzmann's persistence theorem.*

6.7.1 Examples of Singular Lex-Points

As in Example 6.7.3 item (1), we consider a Macaulay-Lex monomial ideal $\tilde{I}$ generated by a regular sequence $x_n^{d_n}, x_{n-1}^{d_{n-1}}, \dots, x_0^{d_0}$, where $1 \leq d_n \leq d_{n-1} \cdots \leq d_0$ are integers or ∞ with $x_i^\infty = 0$. Observe that $\tilde{I}$ is quasi-stable. The ideal $\tilde{I}$ is called Clements-Lindström and the quotient ring $S := R/\tilde{I}$ is a *Clements-Lindström ring* (see [29]). If $d_0 = \infty$, then $\tilde{I}$ is a saturated ideal.

We now exhibit two Hilbert schemes, defined on Clements-Lindström rings, and show that their lex-points are singular. These investigations were inspired by [20, Remark 1.6], where the authors recall that it is unknown whether the lex-point in a Hilbert scheme over a Clements-Lindström ring is smooth or not. More precisely, it is not possible to extend the proof for the smoothness of the lex-point of the Hilbert scheme $\mathbf{Hilb}_{\mathbb{P}^n}^{p(z)}$ given in [87] because the tangent space is not the same (see [43, Section 1.7] and [84, Proposition 2.1]).

Our computations follow the method that arises from Corollary 6.4.6 and use Theorem 6.5.2 item (ii) and Theorem 6.5.3 on which Algorithm 26 is based.

Example 6.7.8. *Consider $\mathbb{K} = \mathbb{Q}$, $R := \mathbb{K}[x_0, \dots, x_3]$, the ring $S = R/\tilde{I}$, with $\tilde{I} = (x_3^2, x_2^5) \subseteq R$, and the lex-ideal $\tilde{J}/\tilde{I} = (x_3^2, x_3x_2, x_3x_1^2, x_2^5)/\tilde{I}$ of S which has been introduced in Example 6.7.2. It defines the lex-point Y of the Hilbert scheme on $X = \mathrm{Proj}(R/\tilde{I})$ with Hilbert polynomial $p(z) = 5z - 3$. We now show that Y is a singular point.*

Following Algorithm 26 and using CoCoA [1], we compute the ideal $\mathscr{R}'$ defining the relative marked functor $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t}, \tilde{J}_{\geq t}}$ for $t = \rho_{\tilde{J}} - 1 = 2$. Note that in this case we have $\tilde{J}_{\geq t} = \tilde{J}$ and $\tilde{I}_{\geq t} = \tilde{I}$. Recall that, thanks to Theorem 6.5.2 item (ii), $\mathscr{R}'$ defines an open subscheme of $\mathbf{Hilb}_X^{p(z)}$ containing Y. Hence, the tangent space to this open subscheme at Y is equal to the tangent space to $\mathbf{Hilb}_X^{p(z)}$ at Y (see also [15, Corollary 1.9]). The Pommaret basis of $\tilde{J}$ is $\mathrm{MinPB}(\tilde{J}) = \{x_3^2, x_3x_2, x_3x_1^2, x_2^5\}$ and that of $\tilde{I}$ is $\mathrm{MinPB}(\tilde{I}) = \{x_3^2, x_3x_2^5, x_2^5\}$. The set $\mathscr{H}$ is made of the following polynomials in the ring $\mathbb{Q}[c_1, \dots, c_{20}][x_0, \dots, x_3]$:

$h_1 = c_1x_0^2 + c_2x_0x_1 + c_3x_1^2 + c_4x_0x_2 + c_5x_1x_2 + c_6x_2^2 + c_7x_0x_3 + c_8x_1x_3 + x_2x_3,$

$h_2 = c_9x_0^3 + c_{10}x_0^2x_1 + c_{11}x_0x_1^2 + c_{12}x_1^3 + c_{13}x_0^2x_2 + c_{14}x_0x_1x_2 + c_{15}x_1^2x_2 + c_{16}x_0x_2^2 + c_{17}x_1x_2^2 + c_{18}x_2^3 + c_{19}x_0^2x_3 + c_{20}x_0x_1x_3 + x_1^2x_3$.
By $\longrightarrow_{\mathscr{H}^*}$ *we reduce the polynomials* x_3h_1, x_3h_2, x_2h_2, $x_3x_2^5$ *and then consider the reduced polynomials modulo* $\tilde{I}$, *obtaining the ideal* $\mathscr{R}' \subseteq \mathbb{Q}[c_1, \dots, c_{20}]$ *that parameterizes all the saturated ideals having a* $\mathrm{MinPB}(\tilde{J})$*-marked basis relative to* $\tilde{I}$ *in* $\mathbf{Hilb}_X^{p(z)}$. *This ring* $\mathbb{Q}[c_1, \dots, c_{20}]/\mathscr{R}'$ *has Krull-dimension* 2. *So the marked scheme defined by* $\mathscr{R}'$ *has dimension* 2. *Moreover, we compute that the tangent space to* $\mathbf{Hilb}_X^{p(z)}$ *at* Y *has dimension* 7. *Hence,* Y *is singular in* $\mathbf{Hilb}_X^{p(z)}$.

Using Macaulay 2 [54] we compute the irreducible components of $\mathscr{R}'$, *obtaining that the associated primes have both dimension* 2 *and are*

$$\mathscr{P}_0 = (c_{18},\, c_{17},\, c_{16},\, c_{15},\, c_{14},\, c_{13},\, c_{12},\, c_{11},\, c_{10},\, c_9,\, c_8,\, c_7,\, c_6,\, c_5,\, c_4,\, c_3,\, c_2,\, c_1)\,,$$

$$\begin{aligned}\mathscr{P}_1 = (&c_{18},\, c_{17},\, c_{16},\, c_{15},\, c_{14},\, c_{13},\, c_{12},\, c_{11},\, c_{10},\, c_9,\, c_6,\, c_5,\, c_4,\, c_3,\, c_2,\, c_1\, c_{20}^2 - 4c_{19},\\ &c_8c_{20} - 2\,c_7,\, 2\,c_8c_{19} - c_7c_{20}).\end{aligned}$$

The defining ideals of both the two irreducible components are not prime and contain the point Y *as a singular point, because the tangent spaces at* Y *to these components have dimensions* 7 *too. Finally, the marked scheme defined by* $\mathscr{R}'$ *in* $\mathbf{Hilb}_X^{p(z)}$ *is neither irreducible nor reduced. Some ancillary material related to this example is available at* *http://wpage.unina.it/cioffifr/AncillaryRelative*.

Example 6.7.9. *Consider* $\mathbb{K} = \mathbb{Q}$, $R := \mathbb{K}[x_0, \dots, x_3]$, *the ring* $S = R/\tilde{I}$, *with* $\tilde{I} = (x_3^2, x_2^5) \subseteq R$. *The image in* S *of the lex-ideal* $\tilde{J} = (x_3, x_2^5)$ *of* R *defines the lex-point* Y *of the Hilbert scheme on* $X = \mathrm{Proj}(R/\tilde{I})$ *with Hilbert polynomial* $p(z) = 5z - 5$. *We now show that* Y *is a singular point.*

Following Algorithm 26 and using CoCoA [1], we compute the ideal $\mathscr{R}'$ *defining the relative marked functor* $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t}, \tilde{J}_{\geq t}}$ *for* $t = \rho_{\tilde{J}} - 1 = 0$. *Note that also in this case we have* $\tilde{J}_{\geq t} = \tilde{J}$ *and* $\tilde{I}_{\geq t} = \tilde{I}$. *Recall that* $\mathscr{R}'$ *defines an open subscheme of* $\mathbf{Hilb}_X^{p(z)}$ *containing* Y, *thanks to Theorem 6.5.2 item (ii). Hence, the tangent space to this open subscheme at* Y *is equal to the tangent space to* $\mathbf{Hilb}_X^{p(z)}$ *at* Y *(see also [15, Corollary 1.9]).*

The Pommaret basis of $\tilde{J}$ *is* $\mathrm{MinPB}(\tilde{J}) = \{x_3, x_2^5\}$ *and that of* $\tilde{I}$ *is* $\mathrm{MinPB}(\tilde{I}) = \{x_3^2, x_3x_2^5, x_2^5\}$. *The set* $\mathscr{H}$ *is made only of the polynomial*

$$h = c_1x_0 + c_2x_1 + c_3x_2 + x_3.$$

By $\longrightarrow_{\mathscr{H}^*}$ *we reduce the terms* $x_3x_2^5$ *and* x_3^2 *and then consider the reduced polynomials modulo* $\tilde{I}$, *obtaining the ideal* $\mathscr{R}'$:

$$\mathscr{R}' = (c_3^2, c_2^2, c_1^2, c_2c_3, c_1c_2, c_1c_3).$$

The affine scheme $\mathrm{Spec}(\mathbb{Q}[c_1, c_2, c_3]/\mathscr{R}')$ *is a zero-dimensional scheme supported over the origin and with Zariski-tangent space of dimension* 3 *at* Y. *The multiplicity of this point is* 4, *being* $1, 3, 0$ *the Hilbert function of* $\mathbb{Q}[C]/\mathscr{R}'$.

Note that the marked scheme $\mathrm{Spec}(\mathbb{Q}[c_1, c_2, c_3]/\mathscr{R}')$ *in* $\mathbf{Hilb}_X^{p(z)}$ *is irreducible, but not reduced.*

More generally, Examples 6.7.8 and 6.7.9 show that there are smooth points in a Hilbert scheme over a projective space that correspond to singular points in a Hilbert scheme over a Clements-Lindström ring. On the other hand, even the contrary can occur.

Example 6.7.10. *If $n = 3$, $\tilde{I} = (x_3)$ and $p(z)$ is constant, the Hilbert scheme $\mathbf{Hilb}_X^{p(z)}$ is an Hilbert scheme over the projective plane, in which every point is smooth. Hence, in this case every singular point in $\mathbf{Hilb}_{\mathbb{P}^3_{\mathbb{K}}}^{p(z)}$ corresponds to a smooth point in $\mathbf{Hilb}_X^{p(z)}$.*

Chapter 7

Conclusion and Outlook

In this thesis, we generalized the concept of involutive bases in two directions: Firstly, involutive-like bases, which inherit many of the properties of involutive bases, with the additional advantage of needing less generators. Secondly, we introduced involutive(-like) bases for ideals in polynomial quotient rings. We applied these generalized types of bases to the computation and analysis of infinite free resolutions. Moreover we generalized the concept of marked bases to relative marked bases for ideals in quotient rings.

In Chapter 3, we saw that recursive and tree based structures underlie the combinatorial and algorithmic properties of involutive bases. These structures are, for the important cases of the Pommaret and Janet divisions, inherited by the more general involutive-like bases. We exploited the tree structures to give alternative efficient algorithms for many tasks in involutive basis theory. We saw that Janet-like and Pommaret-like bases are closely related, as are their involutive counterparts.

In Chapter 4, we generalized the concepts of involutive and involutive-like bases also to ideals in quotient rings. Especially relative Pommaret and Pommaret-like bases turned out to be well-adapted to this generalization; in contrast to the non-relative situation, where both Pommaret and Janet bases induce directly Pommaret and Janet bases for their syzygy modules, we showed that A-polynomials cause additional complications for relative Janet bases, but not for relative Pommaret bases.

In Chapter 5, we analysed the free resolutions induced by relative Pommaret and Pommaret-like bases. We saw that they have good algorithmic properties like Gröbner-reducedness in all higher syzygy modules. Moreover, for special types of quasi-stable monomial ideals, we showed that the induced resolutions are even minimal. For some ideal types we even obtained explicit formulas for the differential which only depend on the data needed to compute the original Pommaret-like bases.

In Chapter 6, we introduced relative marked bases. These new types of bases yielded a framework for computationally analysing Hilbert schemes over some types of quotient rings of positive Krull dimension defined by quasi-stable monomial ideals. We obtained a quasi-stable open cover and information on the lex-point for Hilbert schemes on these classes of rings.

The results obtained in this thesis present several possibilites for further research. This applies to all chapters starting from Chapter 3 up to Chapter 6.

While the Pommaret and Janet divisions are the most well-known types of involutive divisions, there are also other types, and this naturally leads to the question if they can also be generalized to involutive-like counterparts. The recursive test for obstructions to quasi-stability provided us with a new approach to compute linear coordinate transformations to achieve quasi-stable position; it is nevertheless still an interesting open problem to optimize this algorithm, especially when variable permutations are included in the possible transformations.

We have presented an algorithm to transform homogeneous ideals to relative quasi-stable position. Optimizing this algorithm is an open research topic; especially finding a good way to incorporate the recursive checks for obstruction to quasi-stability also to the relative case is a promising approach here.

Many interesting topics are related to the results we got in Chapter 5. While we successfully applied Pommaret-like bases especially in Clements-Lindström rings, a similar analysis of Janet-like bases is desirable. While A-polynomials can pose problems for resolutions induced by relative Janet-bases, in the special situation of Clements-Lindström rings, one may expect that these problems can be managed.

We found several different classes of quasi-stable monomial ideals whose Pommaret-like bases induce minimal free resolutions. It is of interest to investigate possible alternative characterizations of these classes of ideals, resembling stability conditions. Moreover, it remains to be seen whether there are still other classes of quasi-stable ideals whose Pommaret-like bases induce free resolutions that are minimal or have other desirable properties. Another open problem is to generalize the notion of *componentwise quasi-stability* (closely related to the structure of the free resolution) to polynomial ideals in quotient rings.

Another promising direction is the determination of types of quotient rings defined by monomial ideals (other than Clements-Lindström rings) and corresponding classes of relative quasi-stable monomial ideals in these quotient rings which admit, for instance, an explicit formula for the differential of their Pommaret-like induced free resolutions. Once a specific type of quotient ring has been chosen, a starting point can always be the analysis of the homogeneous maximal ideal $\langle x_1, \dots, x_n \rangle$ in this ring.

Once a specific class of quotient rings has been found, especially if they are defined by quasi-stable monomial ideals, one can also use relative marked bases to computationally analyse Hilbert schemes defined over these rings. Already the example of Clements-Lindström rings shows that similar classes of quotient rings are suitable for both kinds of investigations: Properties of free resolutions and of Hilbert schemes.

As a final topic, it is of interest to generalize the concept of *resolving decomposition* [5]. Both involutive bases and marked bases (on quasi-stable monomial modules) are known to induce resolving decompositions [5]. The current framework only allows non-multiplicative variables, not non-multiplicative powers [5]. Nevertheless, a generalization to involutive-like bases appears possible.

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Zusammenfassung

In dieser Arbeit verallgemeinern wir einige Typen von involutiven und markierten Basen für Ideale in Quotientenringen von Polynomringen. Wir wenden diese neuen Basen an, um unendliche freie Auflösungen und Hilbert-Schemata über manchen Arten von Quotientenringen zu untersuchen. Wir benutzen vorrangig Janet- und Pommaretbasen; markierte Basen sind markiert über quasi-stabilen monomialen Untermoduln, also solchen, die eine Pommaretbasis haben.

Die betrachteten involutiven Basen induzieren freie Auflösungen der durch sie erzeugten Ideale. Damit liefern sie auch Abschätzungen für Homologieinvarianten dieser Ideale, zum Beispiel für die Bettizahlen. Im Spezialfall monomialer Pommaretbasen erhält man sogar eine explizite Formel für das Differential der induzierten Auflösung. Aber die induzierte Auflösung ist nicht unbedingt minimal, weil schon die involutiven Basen für sich nicht unbedingt minimale Erzeugendensysteme sind. Des Weiteren war die Anwendung von involutiven und markierten Basen bis jetzt beschränkt auf Ideale in gewöhnlichen Polynomringen.

Beide Problemstellungen werden in dieser Arbeit behandelt, und zwar in vier Teilen. Zunächst definieren wir quasi-involutive Basen; das sind Arten von Gröbnerbasen, die algorithmische und kombinatorische Vorteile involutiver Basen bewahren, aber dafür meist weniger Erzeuger benötigen. Wir zeigen, dass quasi-Janet- und quasi-Pommaretbasen für ihre Syzygienmoduln quasi-involutive Basen jeweils gleicher Art induzieren. Damit induzieren sie, wie involutive Basen auch, freie Auflösungen. Außerdem benutzen wir quasi-involutive Basen, um neue effiziente Algorithmen für die Bestimmung kombinatorischer komplementärer Zerlegungen und Hilbertfunktionen zu beschreiben.

Im zweiten Teil verallgemeinern wir das Konzept involutiver und quasi-involutiver Basen auch auf Ideale in Quotientenringen. Dabei stützen wir uns auf eine ausführliche Untersuchung von Gröbnerbasen und deren Konstruktion in diesen Ringen. Wir legen dar, dass Pommaretbasen in Quotientenringen auf natürliche Art Pommaretbasen ihrer Syzygienmoduln induzieren.

Drittens behandeln wir die Anwendung dieser neuen Typen von Basen auf die Berechnung und Untersuchung ihrer induzierten unendlichen freien Auflösungen. Für den wichtigen Spezialfall der Clements-Lindström-Ringe erhalten wir Formeln für die Bettizahlen der Auflösung. Wir identifizieren einige Klassen quasi-stabiler monomialer Ideale in diesen Quotientenringen, für die die induzierte Auflösung minimal ist. Dadurch verallgemeinern wir einige bekannte Konstruktionen, etwa für stabile Ideale und für quadratfreie Borelsche monomiale Ideale. Für manche Idealtypen finden wir explizite Formeln für das Differential der Auflösung.

Im letzten Teil definieren wir relative markierte Basen in Quotientenringen, die selbst durch Pommaret-markierte Basen definiert sind. Wir finden einen Algorithmus zur Konstruktion relativer markierter Familien wenn der Quotient durch ein monomiales Ideal definiert ist. Schließlich benutzen wir diese Basen, um Hilbert-Schemata über monomialen Quotienten zu untersuchen; wir erhalten eine offene Überdeckung und Informationen über den lexikographischen Punkt.

Abstract

In this thesis, we generalize several types of involutive and marked bases for ideals in quotient rings of commutative polynomial rings. We apply these new types of bases to the analysis of infinite free resolutions and of Hilbert schemes defined over certain types of quotient rings. We are mostly concerned with Pommaret and Janet bases; the marked bases we consider are marked over monomial submodules that are quasi-stable, i.e., that possess finite Pommaret bases.

Involutive bases of the types we consider induce free resolutions of the ideals they generate and hence they yield estimates for homological invariants of these ideals, for example, for their Betti numbers. In the special case of monomial Pommaret bases, one even obtains an explicit formula for the differential of the resolution. However, the induced resolution is not necessarily minimal, because already the involutive bases themselves are in general not minimal generating systems. Moreover, the application of involutive and marked bases was up to now confined to ideals in ordinary polynomial rings.

The thesis addresses both of these problems. Its contributions are split into four parts. In the first part, we introduce involutive-like bases, which are types of Gröbner bases that preserve many of the algorithmic and combinatorial advantages of involutive bases, while needing in general much less generators. We show that Janet-like and Pommaret-like bases induce involutive-like bases of the same types for their syzygy modules, and thus induce free resolutions in the same way that involutive bases do. Moreover, we use involutive-like bases to design new efficient algorithms for the determination of complementary decompositions of monomial ideals and Hilbert functions.

Next, we generalize involutive and involutive-like bases to include also ideals in quotient rings. Our discussion is based on a comprehensive treatment of Gröbner bases for ideals in such rings, together with algorithms for their construction. We establish that Pommaret bases in quotient rings also induce Pommaret bases of their syzygy modules in a natural way.

The third part of contributions treats the application of these new types of bases to the computation and analysis of their induced infinite free resolutions. For the important special case of Clements-Lindström rings, we obtain closed formulas for the Betti numbers of the resolution. We identify several classes of quasi-stable monomial ideals in these quotient rings for which the induced resolution is minimal. Thus, we generalize several well-known resolution constructions, e.g. for stable and for squarefree Borel ideals. We obtain explicit formulas for the differential which apply to some classes of quasi-stable ideals and their Pommaret-like bases.

In the final part, we introduce relative marked bases for ideals in quotient rings defined by ideals generated by Pommaret marked bases. We give an algorithm for the construction of relative marked families in case the quotient ring is defined by a monomial ideal. Lastly, we use these bases to obtain information about the lex-points and about quasi-stable open coverings of the Hilbert schemes defined on some quotient rings, e.g., Clements-Lindström rings.