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Optimal attitude maneuvers in the presence of prohibited directions

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We consider the problem of realigning a space telescope from one observation target to the next in the presence of prohibited directions. More specifically, we want to steer the space telescope – modelled as a gyrostat – from an initial attitude at rest to a final attitude at rest within a fixed time interval. During the motion, the line of sight of the telescope must be kept away from forbidden directions towards bright objects like the sun, moon or earth due to power or thermal requirements. The kinematics of the spacecraft motion are governed by a differential equation on the rotation group SO(3). Treating the angular velocities as control variables, this equation takes the form of a controlled dynamical system. To ensure reorientation maneuvers satisfying these pointing constraints, we introduce a cost functional penalizing proximity of the line of sight of the telescope to any of the forbidden directions. Furthermore, we include penalty terms which provide a smooth motion of the satellite and ensure the execution of a rest-to-rest maneuver. The chosen cost functional is minimized over all possible trajectories of the controlled dynamical system between the prescribed initial and target attitudes, which leads to an optimal control problem on SO(3), which is solved by applying a version of Pontryagin's Maximum Principle tailor-made for optimal control problems on Lie groups. Parametrizing SO(3) in terms of Cardan angles, the solution is formulated as a boundary-value problem on Euclidean space and hence can be solved numerically by conventional methods. The existence of two first integrals is established and exploited to reduce the computational effort. The applicability of this approach is shown in concrete examples.

1 Introduction

To reorient a space telescope from one observation target to the next, typically large-angle attitude changes are required. For thermal and power reasons, but also to prevent straylight from reaching the instruments on board, a number of constraints on the pointing directions allowed for the telescope must be obeyed during such attitude maneuvers. This raises the issue of planning attitude maneuvers, desirably optimal in some sense, compatible with the pointing constraints. This question has been studied for some time (see, e.g., [1], [2], [3], [4]) and has, in recent years, attracted renewed interest (see, e.g., [5], [6], [7], [8], [9], [10]) due both to new demands from space missions and to progress in control-theoretical methods. In this paper we show that the approach used in [2] for the case of a single forbidden direction can be extended to the case of an arbitrary number of forbidden directions.

2 Problem formulation

Let (e_1, e_2, e_3) be a space-fixed coordinate system, which is used as a fixed reference frame, and let $\big(g_1(t), g_2(t), g_3(t)\big)$ be a time-dependent coordinate system rigidly attached to the spacecraft considered, where the components of $g_i(t) \in \mathbb{R}^3$ are taken with respect to the space-fixed system. (By a coordinate system, we always mean a right-handed orthonormal system.) The matrix $g(t) = \big(g_1(t) \mid g_2(t) \mid g_3(t)\big)$ with columns $g_i(t)$, which is an element of the three-dimensional rotation group SO(3), is called the *attitude* of the spacecraft at time t. The *body-referenced angular velocity* of the spacecraft at time t is the unique vector $\omega(t) \in \mathbb{R}^3$ such that $\dot{g}_i(t) = g_i(t) \times \omega(t)$ for all t. This can be rewritten as $\dot{g}(t) = g(t)L\big(\omega(t)\big)$ where, in general, given any vector $u \in \mathbb{R}^3$, we write

$$L(u) \;:=\; \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{1}$$

so that $L(u)\xi = u \times \xi$ for all $\xi \in \mathbb{R}^3$. From the point of view of attitude control it is more natural to express the angular velocities (and the torques governing them) in terms of the body-fixed system, because the location of the momentum wheels or gas jets used for attitude control is known in the body-fixed system, and because the matrix expression of the inertia tensor is time-invariant with respect to the body-fixed system, but not generally with respect to the space-fixed system.

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So the attitude kinematics are given by the equation

$$\dot{g}(t) \;=\; g(t)L\big(\omega(t)\big), \tag{2}$$

which is a differential equation evolving on the rotation group $\mathrm{SO}(3)$. This equation is left-invariant in the sense that if $t \mapsto g(t)$ is a solution of this differential equation, then so is $t \mapsto \gamma\, g(t)$ for any fixed element $\gamma \in \mathrm{SO}(3)$. Loosely speaking, equation (2) is a linear differential equation evolving on the nonlinear space $\mathrm{SO}(3)$. If we treat the angular velocities as control variables, (2) takes the form of a controlled dynamical system. Note that using the angular velocities (rather than the torques) as control variables simplifies the dynamics, allowing us to use the elegant theory of invariant control systems on Lie groups. We now intruduce a time-interval $[0, T]$ in which the desired attitude maneuver has to be carried out, steering the spacecraft from a given initial attitude $g(0) = g_0$ to a specified target attitude $g(T) = g_T$. Furthermore, we introduce unit vectors $d_1, \ldots, d_m \in \mathbb{R}^3$, representing the coordinate expressions of prohibited directions with respect to the space-fixed system, and a unit vector $b = (b_1, b_2, b_3)^T \in \mathbb{R}^3$, representing the coordinate expression of the pointing direction of an on-board telescope with respect to the body-fixed system. The pointing direction of the telescope in the space-fixed system is then $b_1 g_1(t) + b_2 g_2(t) + b_3 g_3(t) = g(t)b$. We assume that the telescope is required to never point towards any of the prohibited directions d_k, so that $d_k \neq g(t)b$ for all times t and all indices $1 \leq k \leq m$, preferably with a safety margin. So in this paper we want to address the following problem: Given an initial attitude $g_0 \in \mathrm{SO}(3)$, a target attitude $g_T \in \mathrm{SO}(3)$, a time interval $[0, T]$, prohibited directions with coordinate representations $d_1, \ldots, d_m \in \mathbb{R}^3$ in the space-fixed system and an on-board telescope with coordinate expression $b \in \mathbb{R}^3$ in the body-fixed system, find a control law $t \mapsto \omega(t)$ such that the solution of the initial value problem

$$\dot{g}(t) \;=\; g(t)L\big(\omega(t)\big), \quad g(0) \;=\; g_0 \tag{3}$$

satisfies $g(T) = g_T$ and $\langle g(t)b, d_k\rangle < c_k$ with $c_k = 1$ for $1 \leq k \leq m$ and all $t \in [0, T]$ (and preferably even $\langle g(t)b, d_k\rangle \leq c_k$ with given constants $c_k < 1$).

3 Solution approach

To invoke the power of optimal control theory, we cast our problem as the question of choosing angular velocities $t \mapsto \omega_i(t)$ satisfying equation (3) and the conditions thereafter while minimizing an integral $\int_0^T \Psi\big(g(t), \omega(t), t\big)\,\mathrm{d}t$ where the integrand is defined by

$$\Psi(g, \omega, t) \;:=\; \sum_{k=1}^{m} \chi_k\big(\langle gb, d_k\rangle\big)\, q(t)\, \big(\omega_1^2 + \omega_2^2 + \omega_3^2\big) \tag{4}$$

with a function $q : (0, T) \to (0, \infty)$ satisfying $q(t) \to \infty$ as $t \to 0$ and $t \to T$ and functions $\chi_k : [-1, c_k) \to (0, \infty)$ where $c_k \leq 1$ and $\chi_k(x) \to \infty$ as $x \to c_k$. The idea behind this choice of cost functional is as follows:

- since $\langle gb, d_k\rangle$ is the cosine of the angle between the telescope direction and the k-th forbidden direction, the factors involving the functions χ_k make close approximations of the space telescope to any of the forbidden directions prohibitively expensive, thereby ensuring abidence by the constraints (with a safety margin which can be controlled by the choice of c_k);

- the factor $q(t)$ makes nonzero values of the angular velocities at the start and the end of the maneuver prohibitively expensive, ensuring the execution of a rest-to-rest maneuver and hence imposing boundary conditions on the angular velocities, which are state variables in a physical sense, but are here formally used as control variables;

- the quadratic term $\omega_1^2 + \omega_2^2 + \omega_3^2$ ensures a "smooth" attitude slew and is easy to handle mathematically.

We are now able to write the problem described above as an optimal control problem:

$$\dot{g}(t) = g(t)L\big(\omega(t)\big), \qquad g(0) = g_0, \qquad g(T) = g_T, \qquad \min\left\{\int_0^T \Psi(g, \omega, t)\,\mathrm{d}t\right\}. \tag{5}$$

This optimal control problem is solved by applying a version of Pontryagin's Maximum Principle tailored to invariant control systems on Lie groups (see [11], [12]).

4 Solution

Pontryagin's Maximum Principle asserts that if $t \mapsto \omega(t)$ is an optimal control steering the system $\dot{g} = gL(\omega)$ between given attitudes over a time interval $[0, T]$ while minimizing $\int_0^T \Psi(g, \omega, t)\,\mathrm{d}t$ then, along the optimal trajectory $t \mapsto \big(g(t), \omega(t), t\big)$, we must have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\Psi}{\partial\omega_i}\right] \;=\; \frac{\partial\Psi}{\partial g}[E_i] \qquad \text{for } 1 \le i \le 3, \tag{6}$$

where

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

Writing $\Omega_i := q\omega_i$, this equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[2\sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big)\Omega_i\right] \;=\; \sum_{k=1}^{m}\chi_k'\big(\langle gb, d_k\rangle\big)\langle gE_ib, d_k\rangle\frac{\Omega_1^2+\Omega_2^2+\Omega_3^2}{q} \tag{8}$$

Taking the derivative on the left-hand side using the product rule, this becomes

$$\begin{aligned} &\dot{\Omega}_i\left[2\sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big)\right] \\ &\quad+\frac{2\Omega_i}{q}\sum_{k=1}^{m}\chi_k'\big(\langle gb, d_k\rangle\big)\big(\Omega_1\langle gE_1b, d_k\rangle + \Omega_2\langle gE_2b, d_k\rangle + \Omega_3\langle gE_3b, d_k\rangle\big) \\ &=\sum_{k=1}^{m}\chi_k'\big(\langle gb, d_k\rangle\big)\langle gE_ib, d_k\rangle\frac{\Omega_1^2+\Omega_2^2+\Omega_3^2}{q}. \end{aligned} \tag{9}$$

We write $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T$ and observe that

$$\begin{bmatrix} \langle gE_1b, d_k\rangle \\ \langle gE_2b, d_k\rangle \\ \langle gE_3b, d_k\rangle \end{bmatrix} = \begin{bmatrix} \langle b, (gE_1)^T d_k\rangle \\ \langle b, (gE_2)^T d_k\rangle \\ \langle b, (gE_3)^T d_k\rangle \end{bmatrix} = \begin{bmatrix} b_2(g^{-1}d_k)_3 - b_3(g^{-1}d_k)_2 \\ b_3(g^{-1}d_k)_1 - b_1(g^{-1}d_k)_3 \\ b_1(g^{-1}d_k)_2 - b_2(g^{-1}d_k)_1 \end{bmatrix} = b \times (g^{-1}d_k), \tag{10}$$

therefore equation (9) becomes

$$\begin{aligned} &\dot{\Omega}\left[2\sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big)\right] + \frac{2\Omega}{q}\sum_{k=1}^{m}\chi_k'\big(\langle gb, d_k\rangle\big)\langle\Omega, b \times (g^{-1}d_k)\rangle \\ &=\sum_{k=1}^{m}\chi_k'\big(\langle gb, d_k\rangle\big)\frac{\Omega_1^2+\Omega_2^2+\Omega_3^2}{q}\,\big(b \times (g^{-1}d_k)\big). \end{aligned} \tag{11}$$

Introducing the vector-valued function

$$\Phi \;:=\; \frac{\sum_{k=1}^{m}\chi_k'\big(\langle gb, d_k\rangle\big)\cdot\big(b \times (g^{-1}d_k)\big)}{2q\sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big)}, \tag{12}$$

equation (11) reads

$$\dot{\Omega} + 2\langle\Phi, \Omega\rangle\Omega \;=\; \Phi\,\|\Omega\|^2. \tag{13}$$

Thus by eliminating the adjoint variables from Pontryagin's Maximum Principle, we arrive at a differential equation which any optimal control $t \mapsto \Omega(t)$ and resulting optimal trajectory $t \mapsto g(t)$ must satisfy. Hence to find a solution to the problem described above we have to solve a boundary-value problem on $\mathrm{SO}(3) \times \mathbb{R}^3$, namely

$$\dot{g} = gL(q^{-1}\Omega), \qquad \dot{\Omega} + 2\langle\Phi, \Omega\rangle\Omega = \Phi\,\|\Omega\|^2, \qquad g(0) = g_0, \qquad g(T) = g_T. \tag{14}$$

In order to shorten the following notations, we introduce the function $F : \mathrm{SO}(3) \to \mathbb{R}$ defined by

$$F(g) \;:=\; \sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big). \tag{15}$$

We now establish the existence of two first integrals.

Theorem 4.1 *Along the trajectories of* (13)*, constants C_1 and C_2 exist such that*

$$C_1 = F\big(g(t)\big)\|\Omega(t)\|^2 \quad \textit{and} \quad C_2 = F\big(g(t)\big)\langle\Omega(t), b\rangle \quad \textit{for all } t. \tag{16}$$

P r o o f. Taking the derivative of the function $t \mapsto F\big(g(t)\big)\|\Omega(t)\|^2$ while taking into account the differential equation (13), we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big)\|\Omega\|^2 &= \sum_{k=1}^{m}\chi_k'\big(\langle gb, d_k\rangle\big)\langle\Omega, b\times g^{-1}d_k\rangle\frac{1}{q}\|\Omega\|^2 + \sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big)\cdot 2\langle\dot\Omega, \Omega\rangle\\ = 2\Big(\langle\Phi,\Omega\rangle\|\Omega\|^2 + \langle\dot\Omega,\Omega\rangle\Big)\sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big) &= 2\big(\langle\Phi,\Omega\rangle\|\Omega\|^2 + \langle\Phi\|\Omega\|^2 - 2\langle\Phi,\Omega\rangle\Omega,\Omega\rangle\big)\sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big)\\ = 2\big(\langle\Phi,\Omega\rangle\|\Omega\|^2 + \langle\Phi,\Omega\rangle\|\Omega\|^2 - 2\langle\Phi,\Omega\rangle\|\Omega\|^2\big)\sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big) &= 0.\end{aligned} \tag{17}$$

Similarly, the derivative of the function $t \mapsto F\big(g(t)\big)\langle\Omega(t), b\rangle$ is seen to be

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big)\langle\Omega, b\rangle &= \sum_{k=1}^{m}\chi_k'\big(\langle gb, d_k\rangle\big)\langle\Omega, b\times g^{-1}d_k\rangle\frac{1}{q}\langle\Omega, b\rangle + \sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big)\langle\dot\Omega, b\rangle\\ = \big(2\langle\Phi,\Omega\rangle\langle\Omega, b\rangle + \langle\Phi\|\Omega\|^2 - 2\langle\Phi,\Omega\rangle\Omega, b\rangle\big)\sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big) &= \langle\Phi, b\rangle\|\Omega\|^2\sum_{k=1}^{m}\chi_k\big(\langle gb, d_k\rangle\big) = 0.\end{aligned} \tag{18}$$

The last equality in the above equation holds because Φ is perpendicular to b by definition. □

The existence of these two first integrals can be exploited as follows. The first integral $C_2 = \langle b, \Omega\rangle \cdot F(g)$ is linear in Ω and hence can be solved for one of the components of Ω, say $\Omega_3 = f(\Omega_1, \Omega_2, g)$. Plugging this into the other first integral $C_1 = \|\Omega\|^2 \cdot F(g)$ yields an equation of the form $\Omega_1^2 + \Omega_2^2 = R(g)^2$. Thus we are able to use polar coordinates to write

$$\begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix} = R(g)\begin{bmatrix}\cos w\\ \sin w\end{bmatrix} \tag{19}$$

with a scalar function w. Taking derivatives and plugging in the differential equation for Ω yields a differential equation for w, which has the form $\dot w(t) = W\big(g(t), w(t), t; C_1, C_2\big)$, and the time derivatives of Ω_1 and Ω_2 can be expressed in terms of w. Thus our original boundary-value problem can be replaced by a new boundary-value problem of the form

$$\dot g = gL\big(V(g, w, t; C_1, C_2)\big), \quad \dot w = W(g, w, t; C_1, C_2), \quad \dot C_1 = 0, \quad \dot C_2 = 0, \quad g(0) = g_0, \quad g(T) = g_T. \tag{20}$$

In this new boundary-value problem, the knowledge of the existence of two first integrals is exploited since two of the six functions sought are constant. The kinematical equation is $\dot g = gL(\omega)$ where

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} = \frac{1}{q}\cdot\begin{bmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{bmatrix} = \frac{1}{q}\cdot\begin{bmatrix}R(g)\cos w\\ R(g)\sin w\\ C_2/F(g)\end{bmatrix}. \tag{21}$$

We now parametrize the elements g of SO(3) by Cardan angles

$$\begin{bmatrix}g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{bmatrix} = \begin{bmatrix}\cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta\\ \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta\\ \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta\end{bmatrix} \tag{22}$$

and find the kinematical equation in terms of Cardan angles by substitution:

$$\begin{aligned}\dot\alpha &= q^{-1}(R(g)\cos w\sec\beta\cos\gamma - R(g)\sin w\sec\beta\sin\gamma)\\ &= q^{-1}R(g)\sec\beta\cos(w+\gamma),\\ \dot\beta &= q^{-1}(R(g)\cos w\sin\gamma + R(g)\sin w\cos\gamma)\\ &= q^{-1}R(g)\sin(w+\gamma),\\ \dot\gamma &= q^{-1}(-R(g)\cos w\tan\beta\cos\gamma + R(g)\sin w\tan\beta\sin\gamma + C_2/F(g))\\ &= q^{-1}\left(C_2/F(g) - R(g)\tan\beta\cos(w+\gamma)\right).\end{aligned} \tag{23}$$

We observe that, as a consequence of equation (19), we have $\tan w = \Omega_2/\Omega_1$. To derive the remaining differential equation for w we take the derivative with respect to t on both sides of this equation and find that

$$\big(1 + \tan(w)^2\big)\dot{w} \;=\; \frac{\dot{\Omega}_2\Omega_1 - \Omega_2\dot{\Omega}_1}{\Omega_1^2}. \tag{24}$$

Without loss of generality we let $b = (0, 0, 1)^T$. Using the equation $1 + \tan(w)^2 = (\Omega_1^2 + \Omega_2^2)/\Omega_1^2$, we find that

$$\begin{aligned}
\dot{w} \;&=\; \frac{\dot{\Omega}_2\Omega_1 - \Omega_2\dot{\Omega}_1}{\Omega_1^2 + \Omega_2^2} \;=\; \frac{\big(\|\Omega\|^2\Phi_2 - 2\langle\Phi,\Omega\rangle\Omega_2\big)\Omega_1 - \big(\|\Omega\|^2\Phi_1 - 2\langle\Phi,\Omega\rangle\Omega_1\big)\Omega_2}{\Omega_1^2 + \Omega_2^2} \;=\; \|\Omega\|^2 \cdot \frac{\Phi_2\Omega_1 - \Phi_1\Omega_2}{\Omega_1^2 + \Omega_2^2} \\
&=\; \frac{C_1}{F(g)} \cdot \frac{\Phi_2 \cdot R(g)\cos(w) - \Phi_1 \cdot R(g)\sin(w)}{R(g)^2} \;=\; \frac{C_1}{F(g)} \cdot \frac{\Phi_2\cos(w) - \Phi_1\sin(w)}{R(g)} \\
&=\; \frac{C_1}{F(g)} \cdot \frac{\langle\Phi, e_2\rangle\cos(w) - \langle\Phi, e_1\rangle\sin(w)}{R(g)} \;=\; \frac{C_1}{R(g)F(g)} \cdot \frac{\sum_{k=1}^m \chi_k'(\langle gb, d_k\rangle) \cdot \langle\cos(w)e_1 + \sin(w)e_2, g^{-1}d_k\rangle}{2q \cdot F(g)} \\
&=\; \frac{C_1}{2qF(g)^2R(g)} \cdot \sum_{k=1}^m \chi_k'(\langle gb, d_k\rangle) \cdot \langle\big(\cos(w), \sin(w), 0\big)^T, g^{-1}d_k\rangle.
\end{aligned} \tag{25}$$

The boundary value problem (20) can now be solved with a straightforward shooting method. To do so, we need the partial derivatives of the state variable with respect to the parameters C_1, C_2, w_0, and these are obtained via the variational equations. At this point we skip the calculation of the variational equations because of space limitations. The following examples are calculated with a classic Runge-Kutta method using the step size $h = 0.04$.

5 Numerical examples

We present two test cases to show the applicability of our algorithm. In both cases, there are four prohibited directions and the telescope direction in the body-fixed system is $b = (0, 0, 1)^T$. The target attitude g_T and the initial attitudes $g_0^{(i)}$ (where upper indices are used to number the test cases) are given by

$$g_0^{(1)} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \text{ where } \beta = 75^\circ, \quad g_0^{(2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad g_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The prohibited directions in the first case are

$$d_1^{(1)} = \frac{1}{\sqrt{11}}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad d_2^{(1)} = \frac{1}{\sqrt{21}}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \quad d_3^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad d_4^{(1)} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

The forbidden directions in the second case are

$$d_1^{(2)} = \frac{1}{\sqrt{19}}\begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}, \quad d_2^{(2)} = \begin{bmatrix} 0.9755 \\ -0.2185 \\ 0.0245 \end{bmatrix}, \quad d_3^{(2)} = \frac{1}{\sqrt{42}}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}, \quad d_4^{(2)} = \begin{bmatrix} 0.2061 \\ 0.5721 \\ 0.7939 \end{bmatrix}.$$

In both cases the data were chosen in such a way that an eigenaxis slew from g_0 to g_T is not admissible, because the telescope points towards one of the forbidden directions during such a slew. An admissible replacement maneuver, avoiding all forbidden directions, is then found with our algorithm, using the choices $T = 10$, $q(t) := 1/\big(t^2(T - t)^2\big)$ and $\chi_k(x) := 1/(1 - x)$ for $1 \le k \le 4$. All maneuvers are visualized, with the forbidden directions marked as red dots on the unit sphere. See Fig. 1 for the first, and Fig. 2 for the second test case.

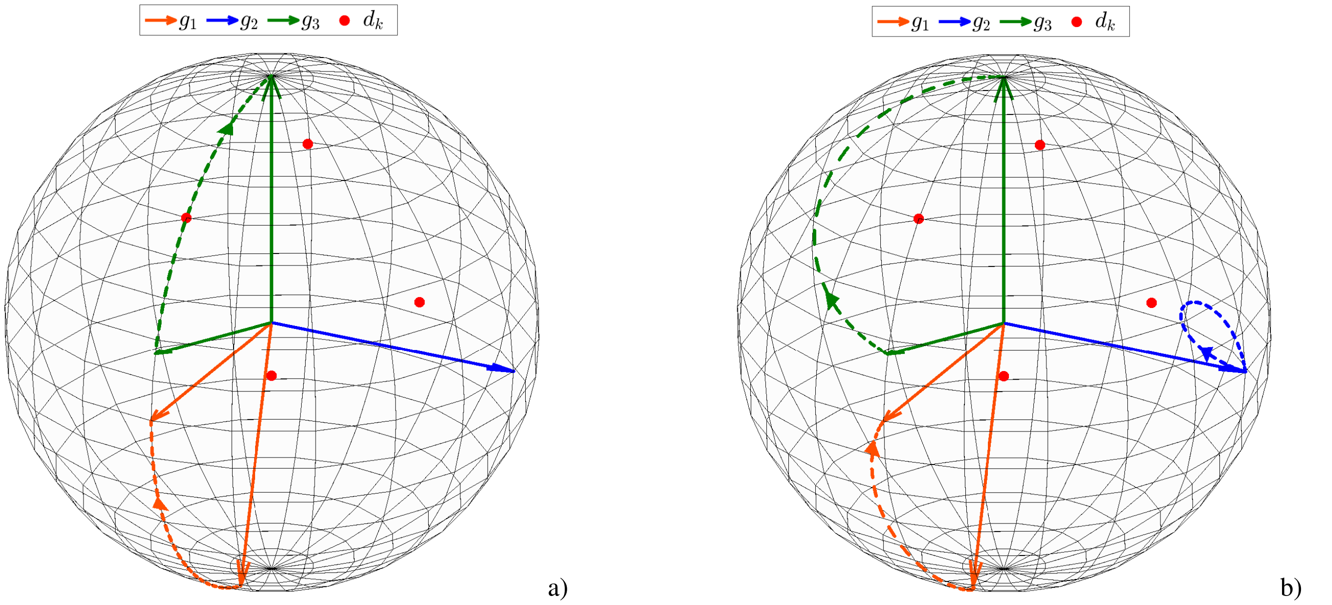

Fig. 1: First test case: Figure **a** shows the eigenaxis slew violating one of the constraints and figure **b** shows the replacement maneuver.

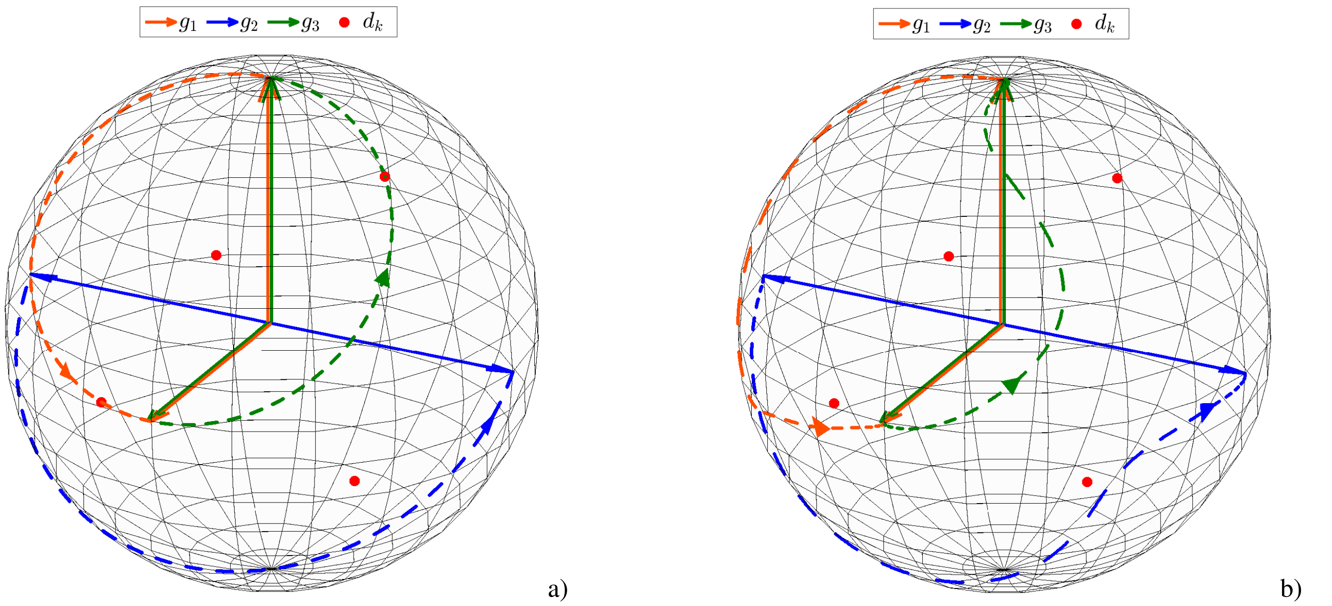

Fig. 2: Second test case: Figure **a** shows the eigenaxis slew violating one of the constraints and figure **b** shows the replacement maneuver.

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