Existence and Asymptotic Behavior of Solutions to the Time-Periodic Navier-Stokes Equations in a Layer Domain with Nonhomogeneous Boundary Data

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Abstract

This dissertation is dedicated to the analysis of the Navier-Stokes equations in a time-periodic framework in the so-called layer domain $\Pi = \mathbb{R}^2 \times (0,1)$, described by:

$$\begin{aligned} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } [0,T]\times\Pi\,, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } [0,T]\times\Pi\,, \\ \mathbf{u}|_{\partial\Pi} &= \mathbf{a} && \text{for all } t\in[0,T]\,, \\ \mathbf{u}|_{t=0} &= \mathbf{u}|_{t=T} && \text{in } \Pi\,. \end{aligned}$$

The velocity field $\mathbf{u}$ and the pressure p are unknowns, while the external force $\mathbf{f}$ is prescribed. Challenges arise due to unboundedness of the layer Π and from introduction of a nonhomogeneous boundary condition $\mathbf{a}$. The investigated topics regarding this system of differential equations are the theory of existence and the theory of asymptotics.

In the existence theory a case distinction with respect to the boundary condition has to be made: For boundary values having zero flux – where flux is the balance of in- and out-flow through the boundary – existence of solutions is proved without restrictions on the (size of the) data. In the case of non-zero flux a statement of existence is achieved for boundary values being small in a certain norm.

The theory of asymptotics is concerned with the behavior of solutions towards spatial infinity. At first, the linear Stokes system is analyzed, continuing the work of Pileckas and Specovius-Neugebauer in [42]. An asymptotic representation for solutions to this problem is derived, which is a generalization of Pileckas and Specovius-Neugebauer's main result. Then, in investigations of the non-linear Navier-Stokes equations, this theorem is employed to prove an asymptotic representation for solutions to the non-linear system as well, where the leading term in fact coincides with that of the Stokes problem.

Zusammenfassung

Diese Dissertation ist der Analyse der Navier-Stokes Gleichungen mit zeit-periodischem Setting in der sogenannten Schicht $\Pi = \mathbb{R}^2 \times (0,1)$ gewidmet, welche beschrieben werden durch:

$$\begin{aligned} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } [0,T] \times \Pi\,, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } [0,T] \times \Pi\,, \\ \mathbf{u}|_{\partial \Pi} &= \mathbf{a} && \text{for all } t \in [0,T]\,, \\ \mathbf{u}|_{t=0} &= \mathbf{u}|_{t=T} && \text{in } \Pi\,. \end{aligned}$$

Das Geschwindigkeitsfeld $\mathbf{u}$ und der Druck p sind Unbekannte, wohingegen die äußere Kraft $\mathbf{f}$ vorgegeben ist. Besondere Herausforderungen stellen die Unbeschränktheit der Schicht Π und die Einführung eines Randwerts $\mathbf{a}$ dar. Die untersuchten Themen hinsichtlich dieses Systems partieller Differentialgleichungen sind die Existenztheorie sowie die Theorie der Asymptotiken.

In der Existenztheorie müssen wir eine Fallunterscheidung bezüglich der Randbedingung vornehmen: Für Randwerte mit Nullfluss – wobei Fluss für die Ein- und Ausflussbilanz durch den Rand des Gebiets steht – wird Existenz von Lösungen ohne Einschränkungen an die (Größe der) Daten gezeigt. Im Falle eines Nichtnullflusses wird eine Existenzaussage unter einer zusätzlichen Kleinheitsbedingung an den Randwert erzielt.

Die Theorie der Asymptotiken befasst sich mit dem Verhalten von Lösungen im räumlich Unendlichen. Zunächst wird die Arbeit von Pileckas and Specovius-Neugebauer in [42] fortgesetzt: das lineare Stokes System wird analysiert. Eine asymptotische Darstellung von Lösungen dieses Problems wird hergeleitet und somit eine Verallgemeinerung des Hauptresultats von Pileckas and Specovius-Neugebauer erreicht. Bei der Untersuchung der nicht-linearen Navier-Stokes Gleichungen wird dieses Theorem dann verwendet um für Lösungen des nicht-linearen Systems ebenfalls eine asymptotische Darstellungen nachzuweisen, wobei der führende Term mit demjenigen des Stokes Problems übereinstimmt.

Contents

1 Introduction

The Navier-Stokes equations are a system of partial differential equations describing the motion of an incompressible, viscous fluid in a region Ω contained in the euclidean space $\mathbb{R}^n$. They are constituted by the balance of momentum

$$\rho\big(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) = \mu\Delta\mathbf{u} - \nabla\pi - \rho\mathbf{f}$$

and the conservation of mass

$$\operatorname{div}\mathbf{u} = 0\,,$$

where the following quantities occur: The time t, x is a point in Ω, ρ denotes the constant density, $\mu > 0$ the shear viscosity coefficient and $\mathbf{f} = \mathbf{f}(t,x)$ is the external force. The Eulerian velocity field $\mathbf{u} = \mathbf{u}(t,x)$ and the pressure $\pi = \pi(t,x)$ are unknown. We set $p := \frac{\pi}{\rho}$ and $\nu := \frac{\mu}{\rho}$, calling p pressure still and ν the kinematic viscosity coefficient. Further, we append a Dirichlet boundary condition in form of the function $\mathbf{a} := \mathbf{a}(t,x)$. Most commonly, a fixed initial state is prescribed, whereas we consider a time-periodic regime, reducing the time scope of interest to a single period $[0,T]$. Finally, we arrive at the system of Navier-Stokes equations investigated in this thesis:

$$\begin{aligned}
\partial_t\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } [0,T]\times\Pi\,,\\
\operatorname{div}\mathbf{u} &= 0 && \text{in } [0,T]\times\Pi\,,\\
\mathbf{u}|_{\partial\Pi} &= \mathbf{a} && \text{for all } t\in[0,T]\,,\\
\mathbf{u}|_{t=0} &= \mathbf{u}|_{t=T} && \text{in } \Pi\,,
\end{aligned}$$

where $\Pi := \mathbb{R}^2 \times (0,1)$ is the layer domain.

Historically, these equations were developed by the French engineer C.L.M.H. Navier in 1822 and clarified by the Irish mathematician G.G. Stokes in 1845 and are nowadays regarded as one standard model for the analysis of hydrodynamic processes. In 1934 Leray [29, 30] proposed a reformulation of the problem, which marks the beginning of the modern theory. The great contributions of Hopf [15] and Ladyzhenskaya [24] led to the fundamental results of global in time existence of weak solutions respectively of local in time existence of strong solutions in $\mathbb{R}^3$, extending Leray's findings. In the 1950s and 1960s, studies of Ladyzhenskaya, Prodi and Serrin ([18], [44], [50], [25]) revealed a gap in regularity of so-called *Leray-Hopf solutions* and, up to this date, the questions of uniqueness and smoothness of these solutions present themselves to be infeasible to answer in a general way.

As foundation and motivation to this work, two specific areas of research in the context of the Navier-Stokes equations are of great importance: The time-periodic framework and domains having a noncompact boundary.

Situated "between" the steady-state and initial-value problem, the time-periodic setting features characteristics of both. Consideration of this framework was first suggested by Serrin [49]. Prodi [43] and Yudovič [61] (independently) expanded the theory by a new approach – without providing a rigorous proof though. Prouse [45, 46] eventually showed its validity, yielding existence of solutions to the time-periodic Navier-Stokes equations in bounded domains. Their technique – referred to as *Prodi-Yudovič method* - is a refinement of Hopf's ideas and couples a Galerkin method with the so-called Poincaré map to derive time-periodicity from a classical initial-value setting. It since has been employed broadly (by Morimoto, Miyakawa, Teramoto, Maremonti, Padula, to name a few – Kyed put together a very comprehensive overview on this in [21]) and takes a pivotal role in the proofs of existence herein.

Examination of domains having multiple exits to infinity started in 1959 with the proposal of *Leray's problem* by Ladyzhenskaya [22, 23], which is the problem of determining a motion in an unbounded "distorted tube". To be more precise, a domain with cylindrical exits to infinity is considered, where, in each bounded cross-section of the outlets, the flux is prescribed. Picked up by Amick [3, 4], research in this direction accelerated in the 1980s with contributions from Ladyzhenskaya & Solonnikov [26, 27], Solonnikov [53, 54], Kapitanskiĭ [16], Nazarov & Pileckas [33, 36] and Kapitanskiĭ & Pileckas [17]. For this work the more recent papers by Nazarov & Pileckas [34, 35], Pileckas [39] and Pileckas & Specovius-Neugebauer [41] served as inspiration. Therein, results on existence and asymptotic behavior of solutions to the stationary Stokes and Navier-Stokes equations were achieved, focusing on layer-like domains, which coincide with the layer $\Pi = \mathbb{R}^2 \times (0, 1)$ outside a fixed ball.

In [5] Beirão da Veiga brought together these two branches, combining Leray's problem in an unbounded cylindrical region with a time-periodic setting. Beirão da Veiga and Galdi & Robertson [10] accomplished a complete analysis of the associated existence theory, including uniqueness of solutions. Hence, an extension to further domains with noncompact boundaries was a logical next step and Pileckas & Specovius-Neugebauer [42] started this by investigating the time-periodic Stokes equations in a layer. They obtained an asymptotic representation of solutions and the corresponding existence theory was treated by the author in [47].

This eventually led to the present thesis, which is structured as follows:

In Chapter 2 we introduce our conventions of notation and collect important inequalities and statements regarding the Stokes operator, the nonlinear term, weighted Sobolev spaces and Bochner spaces.

Chapter 3 is concerned with the existence theory. At first we investigate a "perturbed" Navier-Stokes system to incorporate for nonhomogeneous boundary conditions. The considered perturbation inhabits some practical properties, which can be exploited for certain boundary values. An existence result for bounded subdomains of Π is established and, in a separate theorem, expanded to the whole layer domain by an "invading domains" technique. Then, two types of problems need to be distinguished: Those subject to a zero flux and those with non-zero flux, where the flux F is defined as flow rate through cylindrical cross-sections of the layer. Due to Gauss's Theorem, the flux thereby equals

$$F = -\int_{\partial\Pi} \mathbf{a} \cdot \mathbf{n} \, dS$$

imposing a respective requirement on the boundary datum $\mathbf{a}$. In the case of zero flux we derive an extension operator equipping the resulting extension of a boundary value with the aforementioned sufficient properties of a perturbation. Existence of solutions follows at once through direct application of the theory developed beforehand – without any restrictions on the size of the data. Regarding non-zero flux we construct an extension function of a prescribed boundary value, which carries the present flux. Under a smallness assumption on the boundary data, its properties allow for a proof of existence.

Chapter 4 is dedicated to the spatial asymptotic behavior of solutions. To fit our purposes, a variant of the main theorem in Pileckas & Specovius-Neugebauer's paper [42] – on asymptotics of solutions to the time-periodic Stokes equations – is achieved. Appropriate estimates of the nonlinear term (in weighted Sobolev spaces) enable application of this result, yielding an asymptotic representation of solutions of the time-periodic Navier-Stokes equations.

2 Preliminaries

2.1 Basic notation

This first section is dedicated to the introduction of all basic notions used throughout this thesis starting with the most common ones.

The symbol $\mathbb{N}$ denotes the natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ contains all rational and $\mathbb{R}$ all real numbers.

Let $n \in \mathbb{N}$. A multi-index α is a n-tuple of non-negative integers, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, and we set $|\alpha| := \sum_{i=1}^n \alpha_i$. By δ_{ij} we mean the Kronecker delta.

The n-dimensional Euclidean space $\mathbb{R}^n$ is equipped with the usual norm and scalar product

$$|x| := \sqrt{x_1^2 + \dots + x_n^2} \qquad \text{respectively} \qquad x \cdot x' = \sum_{i=1}^n x_i x_i' \,.$$

Elements of $\mathbb{R}^3$ are frequently interpreted as $x = (y, z)$, where $y \in \mathbb{R}^2$, $z \in \mathbb{R}$, and $r := |y|$. The function $\operatorname{dist}(\cdot, U)$ maps a vector to its distance from a set $U \subset \mathbb{R}^n$:

$$\operatorname{dist}(x_0, U) := \inf_{x \in U} |x - x_0| \,.$$

In the course of Section 3.2 the following geometric objects occur: The open ball with radius $R > 0$ having $(0,0,0)$ or $(0,0,\frac{1}{2})$ as its center is defined by

$$B_R := B_R(0) = \{x \in \mathbb{R}^3 : |x - 0| < R\}$$

respectively

$$\widetilde{B}_R := B_R\big((0,0,\tfrac{1}{2})\big) \,.$$

By

$$C_R := \{x = (y,z) \in \mathbb{R}^2 \times \mathbb{R} : \; |y| < R, \, 0 < z < 1\}$$

we mean the open cylinder with height 1 and radius R. The sets

$$\begin{aligned} S_R &:= \{x = (y,z) \in \mathbb{R}^2 \times \mathbb{R} : \; |y| = R, \, 0 < z < 1\} \,, \\ \Gamma_R &:= \partial C_R \backslash \overline{S_R} = \{x = (y,z) \in \mathbb{R}^2 \times \mathbb{R} : \; |y| < R, \, z = 0 \text{ or } z = 1\} \end{aligned}$$

are its lateral surface and union of top and bottom base, respectively.

By Ω we denote a general domain in $\mathbb{R}^n$, not necessarily bounded. Vector fields

$$\mathbf{u} \colon \Omega \to \mathbb{R}^n \,, \; x \mapsto \mathbf{u}(x) = (u_1(x), \dots, u_n(x))^\top$$

are printed in bold letters, whereas scalar functions

$$u \colon \Omega \to \mathbb{R} \,, \; x \mapsto u(x)$$

are displayed in the standard font. The notation $\mathbf{u}'$ collects the first $n-1$ components of a vector field $\mathbf{u}$: $\mathbf{u}' := (u_1, \dots, u_{n-1})^\top$. Partial derivatives of a function $u\colon \Omega \to \mathbb{R}$ in spatial directions are described by

$$\partial_i u := \frac{\partial}{\partial x_i} u\,, \quad i = 1, \dots, n\,,$$

and

$$\partial^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} u$$

is a partial derivative of order $|\alpha|$. The derivative in time direction of a time-dependent function u is written as $\partial_t u$, alternatively $\frac{d}{dt} u$ or u'. The gradient ∇ is a vector containing all (spatial) partial derivatives, noted as

$$\nabla u := (\partial_1 u, \dots, \partial_n u) \qquad \text{and} \qquad \nabla \mathbf{u} := (\partial_1 \mathbf{u}, \dots, \partial_n \mathbf{u})$$

for scalar respectively vector fields. By ∇^2 we mean the collection of all second order derivatives giving the Hessian for scalar functions and a tensor for vector fields. The divergence operator acts on vector fields as follows:

$$\operatorname{div} \mathbf{u} := \nabla \cdot \mathbf{u} = \sum_{i=1}^{n} \partial_i u_i\,.$$

And the curl of a vector field in $\mathbb{R}^3$ is set to

$$\nabla \times \mathbf{u} := \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}\,.$$

The Laplacian of a function is the divergence of its gradient and is applied componentwise for vector fields, i.e.

$$\Delta u := \operatorname{div}(\nabla u) = \sum_{i=1}^{n} \partial_i^2 u\,, \quad \Delta \mathbf{u} := \begin{pmatrix} \Delta u_1 \\ \vdots \\ \Delta u_n \end{pmatrix}\,.$$

Lastly, concerning differential operators, the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{v}$ is

$$(\mathbf{u} \cdot \nabla)\mathbf{v} = (u_1 \partial_1 + \dots + u_n \partial_n)\mathbf{v} = \begin{pmatrix} \sum_{i=1}^n u_i \partial_i v_1 \\ \vdots \\ \sum_{i=1}^n u_i \partial_i v_n \end{pmatrix}\,.$$

We use the index y in ∂_y^α, ∇_y, Δ_y to indicate derivatives taken with respect to plane directions only.

For $k \in \mathbb{N}_0$, the space $C^k(\Omega)$ consists of all functions $u\colon \Omega \to \mathbb{R}$ having continuous derivatives $\partial^\alpha u$ for all $0 \le |\alpha| \le k$. If $k = 0$ we simply write $C(\Omega) := C^0(\Omega)$. A function is called smooth, if it is infinitely many times continuously differentiable and we define

$$C^\infty(\Omega) := \bigcap_{k \in \mathbb{N}_0} C^k(\Omega)\,.$$

On $C^k(\Omega)$ a norm is given by

$$\|u\|_{C^k(\Omega)} := \sup_{|\alpha|\le k,\, x\in\Omega} |\partial^\alpha u(x)|\,,$$

where, in the case $k=\infty$, we substitute "$|\alpha|\le k$" by "$|\alpha|<\infty$". The support of a function $u\colon \Omega\to\mathbb{R}$ is the closure of the set of arguments giving non-zero values:

$$\operatorname{supp} u := \overline{\{x\in\Omega : u(x)\neq 0\}}\,.$$

Then, for $k\in\mathbb{N}_0$ or $k=\infty$, the space $C_0^k(\Omega)$ is comprised of all functions $u\in C^k(\Omega)$ having a bounded support in Ω. The restriction of a mapping to a subset $\Omega'\subset\Omega$ is denoted by $u|_{\Omega'}$. We define $C^k(\overline{\Omega})$, $k\in\mathbb{N}_0\cup\{\infty\}$, to be the space of functions $u\in C^k(\mathbb{R}^n)$ restricted to $\overline{\Omega}$ and, further,

$$C_0^k(\overline{\Omega}) := \{u\in C^k(\overline{\Omega}) : \operatorname{supp} u\subset\overline{\Omega}\}\,.$$

A function $u\colon\Omega\to\mathbb{R}$ is called Lipschitz continuous, if it fulfills the inequality

$$|u(x)-u(y)|\le C|x-y|$$

for some constant $C>0$ and all $x,y\in\Omega$; the corresponding function space is $C^{0,1}(\Omega)$ with the norm

$$\|u\|_{C^{0,1}(\Omega)} := \|u\|_{C^0(\Omega)} + \sup_{x,y\in\Omega,\, x\neq y}\frac{|u(x)-u(y)|}{|x-y|}\,.$$

We say a continuous function u is absolutely continuous on an interval $I\subset\mathbb{R}$ and write $u\in AC(I)$, provided for each $\varepsilon>0$ there exists a $\delta>0$ such that

$$\sum_{i=1}^m |u(b_i)-u(a_i)|<\varepsilon$$

for every finite number of pairwise disjoint intervals $(a_i,b_i)\subset I$, $i=1,\dots,m$, with $\sum_{i=1}^m b_i-a_i\le\delta$.

Based on this terminology, we describe the smoothness of the boundary $\partial\Omega$ of a domain Ω. For $x^0\in\partial\Omega$ we consider appropriate new coordinates $\xi=(\xi_1,\dots,\xi_n)$ by rotation and translation, such that x^0 lies in the origin, and set $\xi':=(\xi_1,\dots,\xi_{n-1})$. Let $r>0$, $s>0$ and

$$h\colon \xi'\mapsto h(\xi')\,,\quad |\xi'|<r\,,$$

a continuous function. Then, we define

$$U_{r,s,h}(x^0) := \{(\xi',\xi_n)\in\mathbb{R}^n : h(\xi')-s<\xi_n<h(\xi')+s\,,\ |\xi'|<r\}\,.$$

A domain Ω is said to be of class C^k or just called C^k-domain, if for each $x^0\in\partial\Omega$ there exists a local coordinate system and r, s and h as introduced above, with h being a C^k-function, such that

$$U_{r,s,h}(x^0)\cap\partial\Omega = \{(\xi',\xi_n) : \xi_n = h(\xi')\,,\ |\xi'|<r\}$$

and

$$U_{r,s,h}(x^0)\cap\Omega = \{(\xi',\xi_n) : h(\xi')-s<\xi_n<h(\xi')\,,\ |\xi'|<r\}\,.$$

In general, the constants r, s and the function h may depend on x^0. If the constants can be chosen independently of x^0 and additionally

$$\|h_{x^0}\|_{C^k} \leq M\,,$$

for some $M > 0$, for all $x^0 \in \partial\Omega$, we say Ω is a uniform C^k-domain. Naturally, these definitions coincide for bounded domains, due to compactness of the boundary $\partial\Omega$. Likewise, Ω is called a Lipschitz or uniform Lipschitz domain, if the properties above are fulfilled for a Lipschitz function h and

$$\|h_{x^0}\|_{C^{0,1}} \leq M\,,$$

for some constant $M > 0$ and all $x^0 \in \partial\Omega$, in the latter case.

Let $p \in [1, \infty)$. The linear space of all (equivalence classes of) Lebesgue measurable functions $u\colon \Omega \to \mathbb{R}$ satisfying

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p\,dx\right)^{1/p} < \infty$$

is denoted by $L^p(\Omega)$. For $p = \infty$ the condition above is replaced by

$$\|u\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| < \infty\,.$$

Equipped with the norm defined above $L^p(\Omega)$ becomes a Banach space and, in the special case $p = 2$, $L^2(\Omega)$ is a Hilbert space under the scalar product

$$(u, v)_{L^2(\Omega)} := \int_\Omega uv\,dx\,, \quad u, v \in L^2(\Omega)\,.$$

Throughout this work – where no confusion might arise – we abbreviate the L^2-norm and scalar product by $\|\cdot\|$ respectively $(\cdot,\cdot)$. We write $u \in L^p_{loc}(\Omega)$ if for each bounded subdomain $\Omega' \subset \Omega$ with $\overline{\Omega'} \subset \Omega$ it holds $u \in L^p(\Omega')$.

The symbol $W^{k,p}(\Omega)$ with $k \in \mathbb{N}_0$, $1 \leq p \leq \infty$ stands for the Sobolev space consisting of functions $u \in L^p(\Omega)$, which possess all weak derivatives $\partial^\alpha u$ up to order $|\alpha| \leq k$ fulfilling $\partial^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leq k$. Its norm is set to

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|\leq k} \|\partial^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p}$$

for $1 \leq p < \infty$ and to

$$\|u\|_{W^{k,\infty}(\Omega)} := \max_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)}$$

in the case $p = \infty$, rendering $W^{k,p}(\Omega)$ a complete space. With the scalar product

$$\sum_{|\alpha|\leq k} (\partial^\alpha u, \partial^\alpha v)\,, \quad u, v \in W^{k,2}(\Omega)\,,$$

the Sobolev space $W^{k,2}(\Omega)$ becomes a Hilbert space and we alternatively denote $H^k(\Omega) := W^{k,2}(\Omega)$. Additionally, we set

$$((u, v)) := (\nabla u, \nabla v) := \sum_{i=1}^n (\partial_i u, \partial_i v)$$

where $u, v \in W^{1,2}(\Omega)$. For $k=0$, the definition of $W^{0,p}(\Omega)$ coincides with $L^p(\Omega)$, hence we observe $W^{0,p}(\Omega) = L^p(\Omega)$. The subspace $W_0^{k,p}(\Omega)$ of $W^{k,p}(\Omega)$ is defined as closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ and we say $u \in W_{loc}^{k,p}(\Omega)$ if $\partial^\alpha u \in L_{loc}^p(\Omega)$ for all $|\alpha| \le k$.

Let Ω be a Lipschitz domain, $k \ge 1$. Then, for every function u in $W^{k,p}(\Omega)$ a trace v on $\partial\Omega$ exists; we write $u|_{\partial\Omega} = v$. Provided Ω is of class C^k, we denote by $W^{k-1/p,p}(\partial\Omega)$ the space of traces of functions in $W^{k,p}(\Omega)$, while

$$\|v\|_{W^{k-1/p,p}(\partial\Omega)} := \inf\left\{ \|u\|_{W^{k,p}(\Omega)} : u \in W^{k,p}(\Omega) \text{ such that } u|_{\partial\Omega} = v \right\}$$

for $v \in W^{k-1/p,p}(\partial\Omega)$. We say $v \in W_{loc}^{k-1/p,p}(\partial\Omega)$, if $\varphi v \in W^{k-1/p,p}(\partial\Omega)$ for all smooth, compactly supported functions $\varphi\colon \partial\Omega \to \mathbb{R}$. Observe that for $k \ge 2$

$$W_0^{k,p}(\Omega) = \left\{ u \in W^{k,p}(\Omega) : u|_{\partial\Omega} = \tfrac{\partial}{\partial n} u|_{\partial\Omega} = \ldots = \tfrac{\partial^{k-1}}{\partial n^{k-1}} u|_{\partial\Omega} = 0 \right\},$$

if Ω is a C^k-domain and especially

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) : u|_{\partial\Omega} = 0 \right\},$$

if Ω is Lipschitz.[1]

A vector field $\mathbf{u}\colon \Omega \to \mathbb{R}^n$ is contained in the Lebesgue space $L^p(\Omega)^n$ or Sobolev space $W^{k,p}(\Omega)^n$ or trace space $W^{k-1/p,p}(\partial\Omega)^n$ provided each component function lies in the respective space. The corresponding norm is set to

$$\|\mathbf{u}\|_{X^n} := \left(\sum_{i=1}^n \|u_i\|_X^p \right)^{1/p} < \infty,$$

where X is to be substituted by $L^p(\Omega)$, $W^{k,p}(\Omega)$ respectively $W^{k-1/p,p}(\partial\Omega)$. Despite introducing this notation, we do not distinguish between spaces of scalar functions and vector fields and leave out the exponent n. Likewise, the spaces of continuous vector fields – $C^k(\Omega;\mathbb{R}^n)$, $C_0^k(\Omega;\mathbb{R}^n)$, $C^k(\overline{\Omega};\mathbb{R}^n)$, $C_0^k(\overline{\Omega};\mathbb{R}^n)$ – are abbreviated as $C^k(\Omega)$, $C_0^k(\Omega)$, $C^k(\overline{\Omega})$, $C_0^k(\overline{\Omega})$. A norm on these spaces is given by

$$\|\mathbf{u}\|_{C^k(\Omega;\mathbb{R}^n)} := \|\mathbf{u}\|_{C_0^k(\Omega;\mathbb{R}^n)} := \max_{i=1,\ldots,n} \|u_i\|_{C^k(\Omega)}.$$

For $C^k(\overline{\Omega};\mathbb{R}^n)$ and $C_0^k(\overline{\Omega};\mathbb{R}^n)$, naturally, the supremum in the latter norm is taken over $\overline{\Omega}$ instead of Ω.

In the theory of the Navier-Stokes equations special types of function spaces are considered, in particular spaces constituted by divergence-free (also called solenoidal) vector fields. Starting from

$$C_{0,\sigma}^\infty(\Omega) := \{ \boldsymbol{\varphi} \in C_0^\infty(\Omega) : \operatorname{div} \boldsymbol{\varphi} = 0 \},$$

we define $L_\sigma^2(\Omega)$ and $W_{0,\sigma}^{1,2}(\Omega)$ as closure of $C_{0,\sigma}^\infty(\Omega)$ in the norm $\|\cdot\|_{L^2(\Omega)}$ respectively $\|\cdot\|_{W^{1,2}(\Omega)}$. Being closed subspaces of $L^2(\Omega)$, $W^{1,2}(\Omega)$, these are Hilbert spaces under the corresponding scalar products. In particular, for a Lipschitz domain the following identities hold[2]:

$$\begin{aligned} L_\sigma^2(\Omega) &= \{ \mathbf{u} \in L^2(\Omega) : \operatorname{div} \mathbf{u} = 0 \text{ and } \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ weakly} \}, \\ W_{0,\sigma}^{1,2}(\Omega) &= \{ \mathbf{u} \in W_0^{1,2}(\Omega) : \operatorname{div} \mathbf{u} = 0 \} = \{ \mathbf{u} \in W^{1,2}(\Omega) : \operatorname{div} \mathbf{u} = 0,\ \mathbf{u}|_{\partial\Omega} = 0 \}. \end{aligned}$$

[1] cf. [28, Theorems 18.7 + 18.8]

[2] cf. [11, Thm. III.2.3] and [58, Thm. I.1.6]

The Helmholtz-Weyl decomposition splits a vector field $\mathbf{u} \in L^2(\Omega)$ into a solenoidal part and a gradient field, i.e.

$$L^2(\Omega) = L^2_\sigma(\Omega) \oplus G^2(\Omega)\,,$$

where $G^2(\Omega) = \{\nabla p \in L^2(\Omega)\,,\text{ for some } p \in W^{1,2}_{loc}(\Omega)\}$.

For a Banach space X the symbol X' stands for its dual space and $\langle\cdot,\cdot\rangle_{X',X}$ is the corresponding dual pairing.

The letters c, C denote local constants, which may vary from line to line. To highlight dependence on certain variables or terms we append these quantities in brackets. Global constants are displayed as C and have a referencing index.

All norms introduced above may be written without noting the underlying domain, if this is clear from context. For the sake of brevity, we collect a sum of norms of elements $u_1, u_2, \ldots, u_k \in X$, $k \in \mathbb{N}$, from a Banach space X in the following expression:

$$\|u_1, u_2, \ldots, u_k\|_X^2 := \sum_{j=1}^{k} \|u_j\|_X^2\,.$$

Furthermore, the important concepts of weighted function spaces and Bochner spaces are presented separately in Sections 2.5, 2.6.

2.2 Inequalities

For the sake of quick referencing we collect some frequently applied inequalities. A formal division by ∞ in the requirements of some of these lemmas is interpreted as "$\frac{1}{\infty} = 0$".

Lemma 2.1 (Young's inequality)**.** *Let $a, b > 0$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}\,.$$

Lemma 2.2 (Hölder's inequality)**.** *Let $1 \leq p, q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then*

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}\,.$$

Lemma 2.3 (Lyapunov's interpolation inequality)**.** *Let $1 \leq p \leq q \leq \infty$ and $u \in L^p(\Omega) \cap L^q(\Omega)$. For any intermediate exponent $p \leq r \leq q$ there exists $\theta \in [0,1]$ such that*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

and the following estimate holds

$$\|u\|_{L^r(\Omega)} \leq \|u\|^{\theta}_{L^p(\Omega)} \|u\|^{1-\theta}_{L^q(\Omega)}\,.$$

In particular, this implies the inclusion $L^p(\Omega) \cap L^q(\Omega) \subset L^r(\Omega)$.

Lemma 2.4 (Poincaré's inequality)**.** *Let Ω be a domain lying between two parallel hyperplanes of distance less equal $d > 0$. All $u \in W^{1,p}_0(\Omega)$, $1 \leq p < \infty$, fulfill the inequality*

$$\|u\|_{L^p(\Omega)} \leq \frac{d}{p^{1/p}} \|\nabla u\|_{L^p(\Omega)}\,.$$

Remark. The most common use case of Poincaré's inequality in this thesis is for $p = 2$ in the layer domain $\Pi = \mathbb{R}^2 \times (0, 1)$. Therefore, we denote the corresponding constant by $\mathsf{C}_P := \frac{1}{\sqrt{2}}$.

Proof. Cf. [1, Thm. 6.30]. □

Lemma 2.5 (Friedrich's inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\{\mathbf{e}_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2(\Omega)$. For any $\eta > 0$, there exists $N_\eta \in \mathbb{N}$ such that*

$$\|\mathbf{u}\|^2_{L^2(\Omega)} \le \sum_{j=1}^{N_\eta} (\mathbf{u}, \mathbf{e}_j)^2_{L^2(\Omega)} + \eta \|\nabla \mathbf{u}\|^2_{L^2(\Omega)}$$

for all $\mathbf{u} \in W^{1,2}_0(\Omega)$.

Proof. At first, we reformulate the assertion:

For any $\eta > 0$ and $\delta > 0$ there exists $N_{\eta,\delta} \in \mathbb{N}$ such that

$$\|\mathbf{u}\|^2_{L^2(\Omega)} \le (1+\delta) \sum_{j=1}^{N_{\eta,\delta}} (\mathbf{u}, \mathbf{e}_j)^2_{L^2(\Omega)} + \eta \|\nabla \mathbf{u}\|^2_{L^2(\Omega)} . \tag{2.1}$$

If (2.1) is proven, then the original statement follows by the Bessel and Poincaré inequalities:

$$\begin{aligned} \|\mathbf{u}\|^2_{L^2(\Omega)} &\le \sum_{j=1}^{N_{\eta,\delta}} (\mathbf{u}, \mathbf{e}_j)^2_{L^2(\Omega)} + \delta \|\mathbf{u}\|^2_{L^2(\Omega)} + \eta \|\nabla \mathbf{u}\|^2_{L^2(\Omega)} \\ &\le \sum_{j=1}^{N_{\eta,\delta}} (\mathbf{u}, \mathbf{e}_j)^2_{L^2(\Omega)} + (c\delta + \eta) \|\nabla \mathbf{u}\|^2_{L^2(\Omega)} . \end{aligned}$$

Assume (2.1) does not hold, i.e. there exist $\eta_0, \delta_0 > 0$ and a sequence $(\mathbf{u}_k)_{k\in\mathbb{N}} \subset W^{1,2}_0(\Omega)$ satisfying

$$\|\mathbf{u}_k\|^2_{L^2(\Omega)} > (1+\delta_0) \sum_{j=1}^{k} (\mathbf{u}_k, \mathbf{e}_j)^2_{L^2(\Omega)} + \eta_0 \|\nabla \mathbf{u}_k\|^2_{L^2(\Omega)}$$

for all $k \in \mathbb{N}$. Set $\mathbf{v}_k := \|\mathbf{u}_k\|^{-1}_{L^2(\Omega)} \mathbf{u}_k$, then we obtain

$$1 = \|\mathbf{v}_k\|^2_{L^2(\Omega)} > (1+\delta_0) \sum_{j=1}^{k} (\mathbf{v}_k, \mathbf{e}_j)^2_{L^2(\Omega)} + \eta_0 \|\nabla \mathbf{v}_k\|^2_{L^2(\Omega)} . \tag{2.2}$$

Therefore $(\mathbf{v}_k)_k$ is bounded in $W^{1,2}_0(\Omega)$ and we extract a weakly converging subsequence (still denoted $(\mathbf{v}_k)_k$). Let $\mathbf{v} \in W^{1,2}_0(\Omega)$ be the limit. Due to the compact embedding $W^{1,2}_0(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, we find

$$\|\mathbf{v}_k - \mathbf{v}\|_{L^2(\Omega)} \to 0 , \quad k \to \infty .$$

Regarding the right-hand side in (2.2), we infer

$$\begin{aligned}\left|\sum_{j=1}^{k}(\mathbf{v}_k,\mathbf{e}_j)^2_{L^2(\Omega)}-\|\mathbf{v}\|^2_{L^2(\Omega)}\right| &= \left|\sum_{j=1}^{k}(\mathbf{v}_k,\mathbf{e}_j)^2_{L^2(\Omega)}-\sum_{j=1}^{\infty}(\mathbf{v},\mathbf{e}_j)^2_{L^2(\Omega)}\right|\\ &\le \left|\sum_{j=1}^{k}(\mathbf{v}_k-\mathbf{v},\mathbf{e}_j)^2_{L^2(\Omega)}\right|+\left|\sum_{j=k+1}^{\infty}(\mathbf{v},\mathbf{e}_j)^2_{L^2(\Omega)}\right|\\ &\le \|\mathbf{v}_k-\mathbf{v}\|^2_{L^2(\Omega)}+\left|\sum_{j=k+1}^{\infty}(\mathbf{v},\mathbf{e}_j)^2_{L^2(\Omega)}\right|.\end{aligned}$$

Parseval's identity then implies

$$\sum_{j=1}^{k}(\mathbf{v}_k,\mathbf{e}_j)^2_{L^2(\Omega)}\xrightarrow{k\to\infty}\|\mathbf{v}\|^2_{L^2(\Omega)}.$$

From (2.2) we get

$$1=\|\mathbf{v}_k\|^2_{L^2(\Omega)}>(1+\delta_0)\sum_{j=1}^{k}(\mathbf{v}_k,\mathbf{e}_j)^2_{L^2(\Omega)}$$

for all $k\in\mathbb{N}$ and passage to the limit yields

$$1=\|\mathbf{v}\|^2_{L^2(\Omega)}\ge(1+\delta_0)\|\mathbf{v}\|^2_{L^2(\Omega)}=1+\delta_0\,.$$

A contradiction. □

Lemma 2.6 (Gagliardo-Nirenberg's embedding inequality)**.** *Let Ω be a domain in $\mathbb{R}^3$ and suppose $u\in W_0^{1,2}(\Omega)$. Then:*

$$\|u\|_{L^4(\Omega)}\le\sqrt{2}\,\|u\|^{1/4}_{L^2(\Omega)}\|\nabla u\|^{3/4}_{L^2(\Omega)}$$

Proof. Cf. [58, Lemma III.3.5]. □

Lemma 2.7 (Differential Gronwall inequality)**.** *Let $u\in AC([0,T])$ and $\alpha,\beta\in L^1(0,T)$ be scalar functions. Suppose u satisfies the inequality*

$$u'(t)\le\alpha(t)+\beta(t)u(t)$$

for almost all $t\in(0,T)$, then

$$u(t)\le u(0)e^{\int_0^t\beta(\tau)\,d\tau}+\int_0^t\alpha(s)e^{\int_s^t\beta(\tau)\,d\tau}\,ds\qquad\forall t\in[0,T]\,.$$

Proof. We cite the arguments from the author's master thesis [47]. Multiplying the required inequality by $\exp(-\int_0^t\beta(\tau)\,d\tau)$, we find

$$\begin{aligned}&u'(t)e^{-\int_0^t\beta(\tau)\,d\tau}\le\alpha(t)e^{-\int_0^t\beta(\tau)\,d\tau}+\beta(t)u(t)e^{-\int_0^t\beta(\tau)\,d\tau}\\ \Leftrightarrow\ &\frac{d}{dt}\left(u(t)e^{-\int_0^t\beta(\tau)\,d\tau}\right)\le\alpha(t)e^{-\int_0^t\beta(\tau)\,d\tau}\end{aligned}$$

for almost all $t \in (0,T)$. Integration with respect to t yields

$$u(t)e^{-\int_0^t \beta(\tau)\,d\tau} - u(0) \leq \int_0^t \alpha(s)e^{-\int_0^s \beta(\tau)\,d\tau}\,ds$$
$$\Leftrightarrow\ u(t) \leq u(0)e^{\int_0^t \beta(\tau)\,d\tau} + \int_0^t \alpha(s)e^{\int_s^t \beta(\tau)\,d\tau}\,ds\,.$$

□

2.3 The Stokes operator

Let $\Omega \subset \mathbb{R}^3$ be a bounded C^2-domain. The (stationary) Stokes equations are defined by the system

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega\,, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega\,, \\ \mathbf{u}|_{\partial\Omega} &= 0\,, \end{aligned} \tag{S$_\mathrm{s}$}$$

where the velocity field $\mathbf{u}$ and pressure p are unknown. A function $\mathbf{u}$ is called a *weak solution* of (S$_\mathrm{s}$), if $\mathbf{u} \in W^{1,2}_{0,\sigma}(\Omega)$ and

$$\nu \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v}\,dx = \int_\Omega \mathbf{f} \cdot \mathbf{v}\,dx \qquad \text{for all } \ \mathbf{v} \in W^{1,2}_{0,\sigma}(\Omega)\,.$$

For $\mathbf{f} \in L^2(\Omega)$, the Stokes problem is uniquely solvable - in the weak sense (cf. [58, Theorem I.2.1]). Based on this result, we define the Stokes operator $\Lambda\colon D(\Lambda) \to L^2_\sigma(\Omega)$ as the operator, assigning to a solution $\mathbf{u}$ the corresponding external force $\mathbf{f}$, where

$$D(\Lambda) := \{\mathbf{u} \in W^{1,2}_{0,\sigma}(\Omega) : \exists \mathbf{f} \in L^2_\sigma(\Omega) \text{ s.t. } \mathbf{u} \text{ is the weak solution to } \mathbf{f}\}\,.$$

Moreover, we have $D(\Lambda) = L^2_\sigma(\Omega) \cap W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)$ (cf. [52, Theorem III.2.1.1]), due to the restrictions on Ω and Λ is a one-to-one correspondence. We note some convenient properties of the Stokes operator:

Theorem 2.8. *The Stokes operator is self-adjoint. Its inverse Λ^{-1}, regarded as operator from $L^2_\sigma(\Omega)$ to $L^2_\sigma(\Omega)$, is compact.*

Proof. This is a well-known result, which can be found in the classic book of Ladyzhenskaya [24, Theorem 2.6]. Compare also [14, p. 650], [58, sec. I.2.6]. □

Corollary 2.9. *There exists an orthonormal basis of $L^2_\sigma(\Omega)$ consisting of eigenvectors of Λ^{-1}. Further, these eigenvectors are orthogonal with respect to the inner product $((\mathbf{u},\mathbf{v})) = \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v}\,dx$ and form a complete system in $W^{1,2}_{0,\sigma}(\Omega)$ as well.*

Proof. Λ is self-adjoint, so its inverse Λ^{-1} is self-adjoint too. Paired with compactness, this guarantees the existence of an orthonormal basis $\{\boldsymbol{\phi}_k\}_{k\in\mathbb{N}} \subset L^2_\sigma(\Omega)$ of eigenvectors (cf. [8, Thm. 6.11]) to the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ related to Λ, $\lambda_k > 0$ for all $k \in \mathbb{N}$, $\lambda_k \to \infty$. Moreover, $\{\boldsymbol{\phi}_k\}_{k\in\mathbb{N}}$ is contained in $W^{1,2}_{0,\sigma}(\Omega)$ and they fulfill

$$((\boldsymbol{\phi}_k,\mathbf{v})) = \int_\Omega \nabla\boldsymbol{\phi}_k \cdot \nabla\mathbf{v}\,dx = \lambda_k \int_\Omega \boldsymbol{\phi}_k \cdot \mathbf{v}\,dx = \lambda_k(\boldsymbol{\phi}_k,\mathbf{v}) \tag{2.3}$$

for all $\mathbf{v} \in W^{1,2}_{0,\sigma}(\Omega)$, hence $((\boldsymbol{\phi}_k,\boldsymbol{\phi}_l)) = \lambda_k\delta_{kl}$. Let $\mathbf{v} \in W^{1,2}_{0,\sigma}(\Omega)$ with $((\boldsymbol{\phi}_k,\mathbf{v})) = 0$ for all k. In view of (2.3) we find $(\boldsymbol{\phi}_k,\mathbf{v}) = 0$ for all k, implying $\mathbf{v} = 0$. Therefore, $\{\boldsymbol{\phi}_k\}_{k\in\mathbb{N}}$ is a (orthogonal) basis of $W^{1,2}_{0,\sigma}(\Omega)$ as well. □

2.4 The trilinear form b

Let Ω be an arbitrary domain of $\mathbb{R}^3$, having a Lipschitz boundary. For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W^{1,2}(\Omega)$ we define the trilinear form b as

$$\begin{aligned} b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &:= \big((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}\big)_{L^2(\Omega)} \\ &= \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w} \, dx = \sum_{i,j=1}^{3} \int_\Omega u_i (\partial_i v_j) w_j \, dx \end{aligned}$$

and collect some properties, the first being continuity:

Lemma 2.10. *The trilinear form b is continuous on the product space $W^{1,2}(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$.*

Proof. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W^{1,2}(\Omega)$. We analyze $(\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w}$ pointwise:

$$\begin{aligned} |(\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w}| &\le |(\mathbf{u} \cdot \nabla)\mathbf{v}||\mathbf{w}| \\ &= \left(\sum_{i=1}^{3} |\mathbf{u} \cdot \nabla v_i|^2 \right)^{1/2} |\mathbf{w}| \\ &\le \left(\sum_{i=1}^{3} |\mathbf{u}|^2 |\nabla v_i|^2 \right)^{1/2} |\mathbf{w}| \\ &= |\mathbf{u}| \left(\sum_{i=1}^{3} \sum_{j=1}^{3} |\partial_j v_i|^2 \right)^{1/2} |\mathbf{w}| \\ &= |\mathbf{u}||\nabla \mathbf{v}||\mathbf{w}| \,. \end{aligned}$$

Consequently

$$\begin{aligned} |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| &\le \int_\Omega |\mathbf{u}||\nabla \mathbf{v}||\mathbf{w}| \, dx \\ &\le \|\mathbf{u}\|_{L^4} \|\nabla \mathbf{v}\|_{L^2} \|\mathbf{w}\|_{L^4} \\ &\le c \, \|\mathbf{u}\|_{W^{1,2}} \|\mathbf{v}\|_{W^{1,2}} \|\mathbf{w}\|_{W^{1,2}} \,, \end{aligned} \tag{2.4}$$

by Hölder's inequality and the Sobolev embedding theorem (cf. [1, Thm. 4.12]). The constant in the last line depends only on the domain Ω. □

Lemma 2.11. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain and let $\mathbf{u} \in W^{1,2}(\Omega)$ with $\operatorname{div} \mathbf{u} = 0$, $\mathbf{v} \in W^{1,2}(\Omega)$ and $\mathbf{w} \in W_0^{1,2}(\Omega)$. Then it holds*

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \tag{2.5}$$

and further

$$b(\mathbf{u}, \mathbf{w}, \mathbf{w}) = 0 \,.$$

Proof. Let $\mathbf{w} \in C_0^\infty(\Omega)$ at first. We employ an integration by parts formula:

$$\begin{aligned} b(\mathbf{u},\mathbf{v},\mathbf{w}) &= \sum_{i,j=1}^{3} \int_\Omega u_i(\partial_i v_j) w_j \, dx \\ &= \sum_{i,j=1}^{3} \left(-\int_\Omega (\partial_i u_i) v_j w_j \, dx - \int_\Omega u_i v_j (\partial_i w_j) \, dx \right) \\ &= -\sum_{j=1}^{3} \int_\Omega (\operatorname{div} \mathbf{u}) v_j w_j \, dx - \sum_{i,j=1}^{3} \int_\Omega u_i (\partial_i w_j) v_j \, dx \\ &= -b(\mathbf{u},\mathbf{w},\mathbf{v}) \, . \end{aligned}$$

For $\mathbf{w} \in W_0^{1,2}(\Omega)$ there exists a sequence $(\mathbf{w}_n)_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ converging to $\mathbf{w}$ with respect to the $W^{1,2}$-norm. By Lemma 2.10 we deduce

$$b(\mathbf{u},\mathbf{v},\mathbf{w}_n) \to b(\mathbf{u},\mathbf{v},\mathbf{w}) \qquad \text{and} \qquad b(\mathbf{u},\mathbf{w}_n,\mathbf{v}) \to b(\mathbf{u},\mathbf{w},\mathbf{v})$$

as claimed. Now, the second identity is a direct consequence, since

$$b(\mathbf{u},\mathbf{w},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{w}) \, .$$

□

2.5 Weighted function spaces and interpolation

To investigate the spatial asymptotics of solutions to the Navier-Stokes system, we need function spaces describing properly the changing behavior by differentiation in either tangential or normal direction. These reflect the structure of the layer domain $\Pi = \mathbb{R}^2 \times (0,1)$ by inclusion of an anisotropic weight distribution in the norm. First, we start with the definition of the Kondratiev space V_β^l, having isotropic weights. Let $\beta \in \mathbb{R}$, $1 \le p < \infty$ and let l, κ be integers with $l \ge 0$, $0 \le \kappa \le l$. The function spaces $V_\beta^{l,p}(\Pi)$ and $V_\beta^{l,p}(\mathbb{R}^2)$ are derived by closure of $C_0^\infty(\overline{\Pi})$ and $C_0^\infty(\mathbb{R}^2)$ in the norm

$$\|u\|_{V_\beta^{l,p}(\Pi)} := \left(\int_\Pi \sum_{|\gamma|=0}^{l} (1+r^2)^{p(\beta-l+|\gamma|)/2} \, |\partial_x^\gamma u(x)|^p \, dx \right)^{1/p}$$

respectively

$$\|u\|_{V_\beta^{l,p}(\mathbb{R}^2)} := \left(\int_{\mathbb{R}^2} \sum_{|\gamma|=0}^{l} (1+r^2)^{p(\beta-l+|\gamma|)/2} \, |\partial_y^\gamma u(y)|^p \, dy \right)^{1/p} .$$

Most frequently these spaces are utilized with $p = 2$, which is why we abbreviate

$$V_\beta^l(\Pi) := V_\beta^{l,2}(\Pi) \, , \quad V_\beta^l(\mathbb{R}^2) := V_\beta^{l,2}(\mathbb{R}^2) \, .$$

To incorporate anisotropic weights we define $\mathcal{V}^l_{\beta,\kappa}(\Pi)$ as the closure of $C_0^\infty(\overline{\Pi})$ in the norm

$$\|u\|_{\mathcal{V}^l_{\beta,\kappa}(\Pi)} := \left(\int_\Pi \sum_{\alpha+|\gamma|\le l} (1+r^2)^{\beta+|\gamma|-(|\gamma|-\kappa)_+} \, |\partial_z^\alpha \partial_y^\gamma u(x)|^2 \, dx \right)^{1/2} ,$$

where $(t)_+ = \frac{t+|t|}{2}$ is the positive part of $t \in \mathbb{R}$. For vector fields these spaces are understood just as explained in Section 2.1. Observe, in particular, that $\mathcal{V}^0_{\beta,0}(\Pi) = V^0_\beta(\Pi)$ and $V^0_0(\Pi) = L^2(\Pi)$.

These weighted Sobolev spaces are complete and separable and equipped with the scalar products

$$(u,v)_{V^l_\beta(\Pi)} := \sum_{|\gamma|=0}^{l} \int_\Pi (1+r^2)^{\beta-l+|\gamma|} \, \partial_x^\gamma u(x) \cdot \partial_x^\gamma v(x) \, dx ,$$

$$(u,v)_{V^l_\beta(\mathbb{R}^2)} := \sum_{|\gamma|=0}^{l} \int_{\mathbb{R}^2} (1+r^2)^{\beta-l+|\gamma|} \, \partial_y^\gamma u(y) \cdot \partial_y^\gamma v(y) \, dy ,$$

$$(u,v)_{\mathcal{V}^l_{\beta,\kappa}(\Pi)} := \sum_{\alpha+|\gamma|\le l} \int_\Pi (1+r^2)^{\beta+|\gamma|-(|\gamma|-\kappa)_+} \, \partial_z^\alpha \partial_y^\gamma u(x) \cdot \partial_z^\alpha \partial_y^\gamma v(x) \, dx$$

the spaces $V^l_\beta(\Pi)$, $V^l_\beta(\mathbb{R}^2)$ and $\mathcal{V}^l_{\beta,\kappa}(\Pi)$, respectively, become Hilbert spaces.

In the next two lemmas from Pileckas's paper [39] we note down the embedding properties of the latter spaces.

Lemma 2.12. (i) *The embeddings*

$$\mathcal{V}^l_{\beta,\kappa}(\Pi) \hookrightarrow \mathcal{V}^{l-1}_{\beta,\kappa}(\Pi) , \qquad l \ge 1, \ 0 \le \kappa \le l-1 ,$$
$$\mathcal{V}^l_{\beta_1,\kappa}(\Pi) \hookrightarrow \mathcal{V}^l_{\beta,\kappa}(\Pi) , \qquad l \ge 0, \ 0 \le \kappa \le l, \ \beta_1 > \beta ,$$

are continuous.

(ii) *Let* $v \in \mathcal{V}^l_{\beta,\kappa}(\Pi)$, $l \ge 1$, $0 \le \kappa \le l$, $\beta \in \mathbb{R}$. *Then* $\partial_y v \in \mathcal{V}^{l-1}_{\beta+1,\kappa-1}(\Pi)$ *and* $\partial_z v \in \mathcal{V}^{l-1}_{\beta,\kappa}(\Pi)$, *and*

$$\|\partial_y v\|_{\mathcal{V}^{l-1}_{\beta+1,\kappa-1}(\Pi)} + \|\partial_z v\|_{\mathcal{V}^{l-1}_{\beta,\kappa}(\Pi)} \le c \, \|v\|_{\mathcal{V}^l_{\beta,\kappa}(\Pi)} .$$

Proof. These assertions are direct consequences of the definition. □

In the following lemma a further weighted space occurs:

$$C^0_\beta(\overline{\Pi}) := \{ u \in C^0(\overline{\Pi}) : \|u\|_{C^0_\beta(\overline{\Pi})} < \infty \} ,$$
$$\|u\|_{C^0_\beta(\overline{\Pi})} := \sup_{x \in \overline{\Pi}} (1+r^2)^{\beta/2} |u(x)| .$$

The proposition provides an analogue to Sobolev's embedding theorem for these specific weighted spaces.

Lemma 2.13. (i) *Let* $u \in \mathcal{V}^l_{\beta,\kappa}(\Pi)$, $l \geq 2$, $0 \leq \kappa \leq l$, $\beta \in \mathbb{R}$.
If $\alpha + |\gamma| \leq l-2$ *and* $|\gamma| \leq \kappa - 2$, *then* $\partial_z^\alpha \partial_y^\gamma u \in C^0_{\beta+1+|\gamma|}(\overline{\Pi})$ *and*

$$\|\partial_z^\alpha \partial_y^\gamma u\|_{C^0_{\beta+1+|\gamma|}(\overline{\Pi})} \leq c\,\|u\|_{\mathcal{V}^l_{\beta,\kappa}(\Pi)}\,.$$

But if $|\gamma| \geq \kappa - 1$, *then* $\partial_z^\alpha \partial_y^\gamma u \in C^0_{\beta+|\gamma|-(|\gamma|-\kappa)_+}(\overline{\Pi})$ *and*

$$\|\partial_z^\alpha \partial_y^\gamma u\|_{C^0_{\beta+|\gamma|-(|\gamma|-\kappa)_+}(\overline{\Pi})} \leq c\,\|u\|_{\mathcal{V}^l_{\beta,\kappa}(\Pi)}\,.$$

(ii) *Let* $u \in \mathcal{V}^l_{\beta,\kappa}(\Pi)$, $l \geq 1$, $\beta \in \mathbb{R}$. *If* $|\gamma| + \alpha = l-1$ *and* $|\gamma| \leq \kappa - 1$, *then* $\partial_z^\alpha \partial_y^\gamma u \in V^{0,p}_{\beta+|\gamma|+1-2/p}(\Pi)$, $p \in [2,6]$, *and*

$$\|\partial_z^\alpha \partial_y^\gamma u\|_{V^{0,p}_{\beta+|\gamma|+1-2/p}(\Pi)} \leq c\,\|u\|_{\mathcal{V}^l_{\beta,\kappa}(\Pi)}\,. \tag{2.6}$$

But if $|\gamma| \geq \kappa$, *then* $\partial_z^\alpha \partial_y^\gamma u \in V^{0,p}_{\beta+|\gamma|-(|\gamma|-\kappa)_+}(\Pi)$ *and*

$$\|\partial_z^\alpha \partial_y^\gamma u\|_{V^{0,p}_{\beta+|\gamma|-(|\gamma|-\kappa)_+}(\Pi)} \leq c\,\|u\|_{\mathcal{V}^l_{\beta,\kappa}(\Pi)}\,. \tag{2.7}$$

Proof. See [39, Lemma 2.4]. □

In the context of Bochner spaces – specifically Theorem 2.17 – we need a statement on interpolation of weighted Sobolev spaces. Therefore we present the following notation: Let X, Y be Banach spaces, then $(X,Y)_{\theta,p}$ with $0 < \theta < 1$, $1 \leq p \leq \infty$, stands for the interpolation space derived by the K-method (cf. [57, Chapter 22]).

Lemma 2.14. *Suppose* $l \geq 2$, $\beta \in \mathbb{R}$. *Then the following relation holds:*

$$\big(\mathcal{V}^l_{\beta,0}(\Pi),\, \mathcal{V}^{l-2}_{\beta,0}(\Pi)\big)_{1/2,2} = \mathcal{V}^{l-1}_{\beta,0}(\Pi)\,.$$

Proof. Based on the well-know result on classical Sobolev spaces, where

$$\big(W^{l,2}(\Omega),\, W^{l-2,2}(\Omega)\big)_{1/2,2} = W^{l-1,2}(\Omega)\,,$$

we want to derive an analogous identity for weighted spaces. Consider the linear mapping

$$\omega\colon \mathcal{V}^l_{\beta,0}(\Pi) \to W^{l,2}(\Pi)\,,$$
$$u \mapsto (1+r^2)^{\beta/2} u\,.$$

Then ω as well as its inverse ω^{-1} are bounded (for any $l \geq 0$), since $\big|\nabla_y^k (1+|y|^2)^{\beta/2}\big| \leq c\,(1+|y|^2)^{\beta/2}$, $0 \leq k \leq l$. Now, Lemma 22.3 from Tartar's book [57] implies that

$$\omega\colon \big(\mathcal{V}^l_{\beta,0}(\Pi),\, \mathcal{V}^{l-2}_{\beta,0}(\Pi)\big)_{1/2,2} \to \big(W^{l,2}(\Pi),\, W^{l-2,2}(\Pi)\big)_{1/2,2}\,,$$
$$\omega^{-1}\colon \big(W^{l,2}(\Pi),\, W^{l-2,2}(\Pi)\big)_{1/2,2} \to \big(\mathcal{V}^l_{\beta,0}(\Pi),\, \mathcal{V}^{l-2}_{\beta,0}(\Pi)\big)_{1/2,2}$$

are linear and bounded too. Thereby, we obtain the desired identity:

$$\begin{aligned}
\big(\mathcal{V}^l_{\beta,0}(\Pi),\, \mathcal{V}^{l-2}_{\beta,0}(\Pi)\big)_{1/2,2} &= \mathrm{id}\Big(\big(\mathcal{V}^l_{\beta,0}(\Pi),\, \mathcal{V}^{l-2}_{\beta,0}(\Pi)\big)_{1/2,2}\Big)\\
&= \omega^{-1}\circ\omega\Big(\big(\mathcal{V}^l_{\beta,0}(\Pi),\, \mathcal{V}^{l-2}_{\beta,0}(\Pi)\big)_{1/2,2}\Big)\\
&= \omega^{-1}\Big(\big(W^{l,2}(\Pi),\, W^{l-2,2}(\Pi)\big)_{1/2,2}\Big)\\
&= \omega^{-1}\Big(W^{l-1,2}(\Pi)\Big)\\
&= \mathcal{V}^{l-1}_{\beta,0}(\Pi)\,.
\end{aligned}$$

□

2.6 Bochner spaces

Let X be a Banach space. The Bochner space $L^p(0,T;X)$, $1 \le p \le \infty$, consists of all (equivalence classes of) measurable functions $u\colon (0,T) \to X$ having finite norm

$$\|u\|_{L^p(0,T;X)} := \|\|u\|_X\|_{L^p(0,T)}$$

and is a Banach space as well. A function $v \in L^p(0,T;X)$ is called weak derivative of $u \in L^p(0,T;X)$, if

$$\int_0^T u\varphi' \, dt = -\int_0^T v\varphi \, dt$$

for all $\varphi \in C_0^\infty((0,T))$ and we write $\partial_t u = v$. The vector-valued Sobolev space $W^{1,p}(0,T;X)$ is comprised of all functions $u \in L^p(0,T;X)$ which possess a weak derivative $\partial_t u \in L^p(0,T;X)$ and its norm is set to

$$\|u\|_{W^{1,p}(0,T;X)} := \|u\|_{L^p(0,T;X)} + \|\partial_t u\|_{L^p(0,T;X)} .$$

The notion $C^0(I;X)$ stands for all strongly continuous (i.e. with respect to $\|\cdot\|_X$) functions $u\colon I \to X$, where $I \subset \mathbb{R}$ is an interval. If for $u \in C^0(I;X) =: C(I;X)$ the limit

$$\partial_t u(t) := \lim_{\delta \to 0} \frac{1}{\delta}\big(u(t+\delta) - u(t)\big)$$

exists for all $t \in I$ and satisfies $\partial_t u \in C(I;X)$, we write $u \in C^1(I;X)$. Likewise $C^k(I;X)$ denotes the space of all functions $u \in C(I;X)$ whose derivatives $\partial_t u, \partial_t^2 u, \dots, \partial_t^k u$ exist and are continuous. Then, let

$$C^\infty(I;X) := \bigcap_{k=0}^{\infty} C^k(I;X) .$$

And lastly, $C_0^k((0,T);X)$, $0 \le k \le \infty$, is the space of functions $u \in C^k((0,T);X)$ having bounded support in $(0,T)$.

We recall some important statements in the context of Bochner spaces.

Lemma 2.15. *Let S be a bounded, linear operator on a Banach space X into a Banach space Y. Suppose $u \in L^1(0,T;X)$, then $S(u) \in L^1(0,T;Y)$ and*

$$\int_0^T S(u) \, dt = S\Big(\int_0^T u \, dt\Big) .$$

Proof. Cf. Corollary V.5.2 in [60]. □

Theorem 2.16. *Let X be a reflexive Banach space and $1 < p < \infty$. Then $L^p(0,T;X)$ is reflexive and its dual space is isomorphic to $L^{p'}(0,T;X')$, where $\frac{1}{p} + \frac{1}{p'} = 1$, in the sense that for $F \in L^p(0,T;X)'$ there exists exactly one $f \in L^{p'}(0,T;X')$ such that for all $u \in L^p(0,T;X)$:*

$$\langle F, u\rangle_{L^p(0,T;X)', L^p(0,T;X)} = \int_0^T \langle f(t), u(t)\rangle_{X',X} \, dt .$$

Proof. Cf. [20, pp. 124f]. □

Theorem 2.17 (Lions-Magenes Lemma)**.** *Let X, Y be two separable Hilbert spaces, $X \subset Y$ dense, with continuous injection. Suppose $u \in L^2(0,T;X)$ with time-derivative $\partial_t u \in L^2(0,T;Y)$, then*

$$u \in C([0,T];(X,Y)_{1/2,2})$$

with the a-priori estimate

$$\sup_{t\in[0,T]} \|u(t)\|_{(X,Y)_{1/2,2}} \leq c \left(\|u\|^2_{L^2(0,T;X)} + \|\partial_t u\|^2_{L^2(0,T;Y)} \right)^{1/2} .$$

Proof. A general version of this theorem can be found in Lions and Magenes's book [31] (Theorem 1.3.1). The original statement is formulated with the interpolation space $[X,Y]_\theta$, described in [31, Definition 1.2.1]. Due to Theorem 15.1 of [31], $[X,Y]_\theta$ is equal to the interpolation space $(X,Y)_{\theta,2}$, derived by the K-Method. □

Theorem 2.18 (Aubin-Lions Lemma)**.** *Assume $X \subset B \subset Y$ are Banach spaces with compact respectively continuous embeddings, which we denote by*

$$X \hookrightarrow\hookrightarrow B \quad \text{and} \quad B \hookrightarrow Y .$$

Let F be a bounded set in $L^p(0,T;X)$, where $1 \leq p < \infty$, and assume $\partial_t F = \{\partial_t f : f \in F\}$ is bounded in $L^1(0,T;Y)$. Then F is relatively compact in $L^p(0,T;B)$.

Proof. The lemma including a proof is contained in Simon's paper [51], see Corollary 4 therein. □

3 Existence

To incorporate for nonhomogeneous boundary data, in Section 3.1 we firstly start with an investigation of a "perturbed" Navier-Stokes system. Sections 3.2 and 3.3 are dedicated to the zero flux case. An extension operator is derived, such that the existence theory of the perturbed problem is applicable. In Sections 3.4 and 3.5 we consider the Navier-Stokes equations subject to a non-zero flux. Construction of an extension function – in line with the present flux – leads to existence of solutions, under an additional smallness assumption on the boundary data.

3.1 Solving the perturbed Navier-Stokes equations

In this section $\widetilde{\Pi}$ denotes a subdomain of the layer Π (which may coincide with Π itself). The perturbed time-periodic Navier-Stokes system reads as follows:

$$\begin{aligned} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \big((\mathbf{u}+\mathbf{w})\cdot\nabla\big)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w} + \nabla p &= \mathbf{f} && \text{in } [0,T]\times\widetilde{\Pi}\,, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } [0,T]\times\widetilde{\Pi}\,, \\ \mathbf{u}|_{\partial\widetilde{\Pi}} &= 0 && \text{for all } t\in[0,T]\,, \\ \mathbf{u}|_{t=0} &= \mathbf{u}|_{t=T} && \text{in } \widetilde{\Pi}\,. \end{aligned} \tag{NS$_\text{p}$}$$

This system plays an important role in treatment of the Navier-Stokes equations with nonhomogeneous boundary condition. Therefore it is in our interest to investigate its solvability. We need a suitable notion of solution first.

Definition 3.1. *Suppose* $\mathbf{f} \in L^2(0,T; W^{1,2}_{0,\sigma}(\widetilde{\Pi})')$, $\mathbf{w} \in W^{1,2}((0,T)\times\widetilde{\Pi})$ *are time-periodic and* $\operatorname{div}\mathbf{w} = 0$. *A function* $\mathbf{u} \in L^2(0,T; W^{1,2}_{0,\sigma}(\widetilde{\Pi})) \cap L^\infty(0,T; L^2_\sigma(\widetilde{\Pi}))$ *is called a* weak solution *of the perturbed time-periodic Navier-Stokes equations* (NS$_\text{p}$)*, if it satisfies*

$$\begin{aligned} \int_0^T &-(\mathbf{u},\partial_t\boldsymbol{\varphi}) + \nu\,(\nabla\mathbf{u},\nabla\boldsymbol{\varphi}) + \big((\mathbf{u}\cdot\nabla)\mathbf{u},\boldsymbol{\varphi}\big) \\ &\qquad + \big((\mathbf{u}\cdot\nabla)\mathbf{w},\boldsymbol{\varphi}\big) + \big((\mathbf{w}\cdot\nabla)\mathbf{u},\boldsymbol{\varphi}\big)\,dt \\ &= \int_0^T \langle\mathbf{f},\boldsymbol{\varphi}\rangle\,dt \end{aligned} \tag{3.1}$$

for all time-periodic test functions $\boldsymbol{\varphi} \in C^\infty([0,T]; C^\infty_{0,\sigma}(\widetilde{\Pi}))$.

Based on this definition, we directly deduce the following:

Corollary 3.2. *A weak solution* $\mathbf{u}$ *of* (NS$_\text{p}$) *satisfies* $\partial_t\mathbf{u} \in L^1(0,T; W^{1,2}_{0,\sigma}(\widetilde{\Pi})')$ *and* $\mathbf{u} \in C([0,T]; W^{1,2}_{0,\sigma}(\widetilde{\Pi})')$. *Additionally,* $\mathbf{u}$ *is weakly continuous from* $[0,T]$ *to* $L^2_\sigma(\widetilde{\Pi})$.

Proof. We set $\mathbf{g} := \mathbf{f} + \nu\Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{u}$, which lies in the space $L^1(0,T;W^{1,2}_{0,\sigma}(\widetilde{\Pi})')$. To verify this, we evaluate $\mathbf{g}(t)$ in $\mathbf{v} \in W^{1,2}_{0,\sigma}(\widetilde{\Pi})$:

$$\begin{aligned}
&|\langle \mathbf{g},\mathbf{v}\rangle| \\
&\quad \le |\langle \mathbf{f},\mathbf{v}\rangle| + \nu\,|\langle \Delta\mathbf{u},\mathbf{v}\rangle| + |\langle (\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}\rangle| + |\langle (\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}\rangle| + |\langle (\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v}\rangle| \\
&\quad \le \|\mathbf{f}\|_{(W^{1,2}_{0,\sigma})'}\|\mathbf{v}\|_{W^{1,2}} + \nu\,|(\nabla\mathbf{u},\nabla\mathbf{v})| + |b(\mathbf{u},\mathbf{u},\mathbf{v})| + |b(\mathbf{u},\mathbf{w},\mathbf{v})| + |b(\mathbf{w},\mathbf{u},\mathbf{v})| \\
&\quad \overset{(2.4)}{\le} \|\mathbf{f}\|_{(W^{1,2}_{0,\sigma})'}\|\mathbf{v}\|_{W^{1,2}} + \nu\,\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{v}\|_{L^2} + c\,\|\mathbf{u}\|^2_{W^{1,2}}\|\mathbf{v}\|_{W^{1,2}} \\
&\qquad\quad + 2c\,\|\mathbf{u}\|_{W^{1,2}}\|\mathbf{w}\|_{W^{1,2}}\|\mathbf{v}\|_{W^{1,2}} \\
&\quad \le c\left(\|\mathbf{f}\|_{(W^{1,2}_{0,\sigma})'} + \nu\,\|\mathbf{u}\|_{W^{1,2}} + \|\mathbf{u}\|^2_{W^{1,2}} + \|\mathbf{u}\|_{W^{1,2}}\|\mathbf{w}\|_{W^{1,2}}\right)\|\mathbf{v}\|_{W^{1,2}}\,.
\end{aligned}$$

This just means

$$\|\mathbf{g}(t)\|_{(W^{1,2}_{0,\sigma})'} \le c\left(\|\mathbf{f}(t)\|_{(W^{1,2}_{0,\sigma})'} + \|\mathbf{u}(t)\|_{W^{1,2}} + \|\mathbf{u}(t)\|^2_{W^{1,2}} + \|\mathbf{w}(t)\|^2_{W^{1,2}}\right)$$

and integration over t yields the regularity claimed above. Let $\boldsymbol{\phi} \in W^{1,2}_{0,\sigma}(\widetilde{\Pi})$ and $h \in C_0^\infty((0,T))$. Especially, h is time-periodic, so the integral identity (3.1) holds (by approximation of $\boldsymbol{\phi}$):

$$\begin{aligned}
&\int_0^T -(\mathbf{u}(t),\boldsymbol{\phi})h'(t)\,dt \\
&\quad = \int_0^T \Big[\langle \mathbf{f}(t),\boldsymbol{\phi}\rangle - \nu(\nabla\mathbf{u}(t),\nabla\boldsymbol{\phi}) - \big((\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t),\boldsymbol{\phi}\big) \\
&\qquad\quad - \big((\mathbf{u}(t)\cdot\nabla)\mathbf{w}(t),\boldsymbol{\phi}\big) - \big((\mathbf{w}(t)\cdot\nabla)\mathbf{u}(t),\boldsymbol{\phi}\big)\Big]h(t)\,dt \\
&\quad = \int_0^T \big\langle \mathbf{f}(t) + \nu\Delta\mathbf{u}(t) - (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t) - (\mathbf{u}(t)\cdot\nabla)\mathbf{w}(t) - (\mathbf{w}(t)\cdot\nabla)\mathbf{u}(t),\boldsymbol{\phi}\big\rangle\,h(t)\,dt \\
&\quad = \int_0^T \langle \mathbf{g}(t),\boldsymbol{\phi}\rangle\,h(t)\,dt\,.
\end{aligned}$$

This identity can be rewritten as

$$\left\langle \int_0^T \mathbf{u}(t)h'(t)\,dt + \int_0^T \mathbf{g}(t)h(t)\,dt,\ \boldsymbol{\phi}\right\rangle = 0\,.$$

Hence, $\int_0^T \mathbf{u}(t)h'(t)\,dt = -\int_0^T \mathbf{g}(t)h(t)\,dt$ in $W^{1,2}_{0,\sigma}(\widetilde{\Pi})'$, meaning that $\mathbf{g}$ is the weak derivative of $\mathbf{u}$:

$$\partial_t\mathbf{u} = \mathbf{g} \ \in L^1(0,T;W^{1,2}_{0,\sigma}(\widetilde{\Pi})')\,.$$

In particular, $\mathbf{u}$ is absolutely continuous (cf. [2, Chap. III.1.2]), which yields $\mathbf{u} \in C([0,T];W^{1,2}_{0,\sigma}(\widetilde{\Pi})')$. Trivially $\mathbf{u}$ is weakly continuous from $[0,T]$ to $W^{1,2}_{0,\sigma}(\widetilde{\Pi})'$. We obtain weak continuity of $\mathbf{u}$ from $[0,T]$ to $L^2_\sigma(\widetilde{\Pi})$ by [58, Lemma III.1.4], since $\mathbf{u} \in L^\infty(0,T;L^2_\sigma(\widetilde{\Pi}))$. □

As stated above a weak solution is determined for each point in time. Still open to this point though, is the question, whether a solution adopts periodic behavior, which is demanded in $(\mathrm{NS_p})_4$, but not reflected in Definition 3.1 at first glance. In the following remark we derive an answer to this question.

Remark. Let $\mathbf{u}$ be a weak solution with corresponding data $\mathbf{f}$ and $\mathbf{w}$. As seen in Corollary 3.2 the function $\mathbf{u}$ is determined pointwise, assuming values in $L^2_\sigma(\widetilde{\Pi})$. Now, it is in our interest to prove $\mathbf{u}(T) = \mathbf{u}(0)$ in $L^2(\widetilde{\Pi})$.

Take $\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\widetilde{\Pi})$ and $h \in C^\infty([0,T])$ with $h(0) = 1 = h(T)$, furnishing an admissible test function $h\boldsymbol{\varphi}$ for (3.1). Let $\mathbf{g}$ be defined as above. The integration by parts formula from Theorem 30.1 of [6] justifies the following deduction:

$$
\begin{aligned}
&\big(\mathbf{u}(T) - \mathbf{u}(0), \boldsymbol{\varphi}\big)\\
&\quad = \langle \mathbf{u}(T)h(T) - \mathbf{u}(0)h(0), \boldsymbol{\varphi}\rangle_{W^{1,2}_{0,\sigma}(\widetilde{\Pi})', W^{1,2}_{0,\sigma}(\widetilde{\Pi})}\\
&\quad = \left\langle \int_0^T \mathbf{u}'(t)h(t)\,dt + \int_0^T \mathbf{u}(t)h'(t)\,dt, \boldsymbol{\varphi}\right\rangle\\
&\quad = \int_0^T \langle \mathbf{u}'(t), h(t)\boldsymbol{\varphi}\rangle\,dt + \int_0^T \langle \mathbf{u}(t), h'(t)\boldsymbol{\varphi}\rangle\,dt\\
&\quad = \int_0^T \langle \mathbf{g}(t), h(t)\boldsymbol{\varphi}\rangle\,dt + \int_0^T \langle \mathbf{u}(t), h'(t)\boldsymbol{\varphi}\rangle\,dt\\
&\quad = \int_0^T \langle \mathbf{f} + \nu\Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{u}, h\boldsymbol{\varphi}\rangle\;dt + \int_0^T \langle \mathbf{u}, h'\boldsymbol{\varphi}\rangle\,dt\\
&\quad = -\int_0^T -(\mathbf{u}, h'\boldsymbol{\varphi}) + \nu(\nabla\mathbf{u}, h\nabla\boldsymbol{\varphi}) + \big((\mathbf{u}\cdot\nabla)\mathbf{u}, h\boldsymbol{\varphi}\big)\\
&\qquad\quad + \big((\mathbf{u}\cdot\nabla)\mathbf{w}, h\boldsymbol{\varphi}\big) + \big((\mathbf{w}\cdot\nabla)\mathbf{u}, h\boldsymbol{\varphi}\big)\,dt + \int_0^T \langle \mathbf{f}, h\boldsymbol{\varphi}\rangle\,dt\\
&\quad = 0\,.
\end{aligned}
$$

By density of $C^\infty_{0,\sigma}(\widetilde{\Pi})$ in $L^2_\sigma(\widetilde{\Pi})$, the identity above holds for $\boldsymbol{\varphi}$ in this greater function space. With respect to Lemma II.2.5.1 in Sohr's monograph [52] we conclude, that $\mathbf{u}(T) - \mathbf{u}(0)$ must be a gradient field. But at the same time $\mathbf{u}(T) - \mathbf{u}(0) \in L^2_\sigma(\widetilde{\Pi})$. So we find $\mathbf{u}(T) - \mathbf{u}(0) = 0$ in $L^2(\widetilde{\Pi})$ as a consequence of the Helmholtz-Weyl decomposition.

In the following theorem we need to impose some restrictions on the subdomain $\widetilde{\Pi}$.

Theorem 3.3. *Let $\widetilde{\Pi} \subset \Pi$ be a bounded C^2-subdomain. Suppose $\mathbf{f} \in L^2(0,T; W^{1,2}_{0,\sigma}(\widetilde{\Pi})')$ is a volume force and $\mathbf{w} \in W^{1,2}((0,T)\times\widetilde{\Pi})$ a divergence-free perturbation, time-periodic (as in Definition 3.1). Further, let*

$$\left|\big((\mathbf{v}\cdot\nabla)\mathbf{w}(t), \mathbf{v}\big)_{L^2(\widetilde{\Pi})}\right| \le \frac{\nu}{4}\|\nabla\mathbf{v}\|^2_{L^2(\widetilde{\Pi})}\,,$$

for all $\mathbf{v} \in W^{1,2}_{0,\sigma}(\widetilde{\Pi})$, $t \in (0,T)$. Then, there exists at least one solution $\mathbf{u} \in L^2(0,T; W^{1,2}_{0,\sigma}(\widetilde{\Pi}))$ of the integral identity (3.1) *with a-priori estimate*

$$\|\mathbf{u}(s)\|^2_{L^2(\widetilde{\Pi})} + \nu\int_0^s \|\nabla\mathbf{u}\|^2_{L^2(\widetilde{\Pi})}\,d\tau \le c\,\|\mathbf{f}\|^2_{L^2(0,T;W^{1,2}_{0,\sigma}(\widetilde{\Pi})')}$$

for almost all $s \in (0,T)$. Hence, $\mathbf{u}$ is, in fact, a weak solution of the perturbed time-periodic Navier-Stokes equations.

Proof. The proof is divided into five steps. We reduce the problem to finite dimensional function spaces at first and show existence of time-periodic approximating solutions in these spaces. The toughest part is to get sufficient convergence properties for the approximating sequence (or rather a suitable subsequence). In the end, an energy inequality for the weak solution is developed.

Step 1: Galerkin ansatz.
Let $\{\boldsymbol{\phi}_k\}_{k\in\mathbb{N}} \subset L^2_\sigma(\widetilde{\Pi})$ be the complete orthonormal system composed of eigenvectors of the Stokes operator (described in Corollary 2.9). In particular, $\{\boldsymbol{\phi}_k\}_{k\in\mathbb{N}}$ is an orthogonal basis of $W^{1,2}_{0,\sigma}(\widetilde{\Pi})$ as well. To approximate a solution $\mathbf{u}$ of the differential equation we start with the ansatz

$$\mathbf{u}_n(t) = \sum_{j=1}^{n} a_j^n(t)\, \boldsymbol{\phi}_j\,.$$

To derive the coefficients a_j^n, $j = 1, \dots, n$, we demand $\mathbf{u}_n$ to satisfy the equations

$$\begin{aligned}
&\frac{d}{dt}\,(\mathbf{u}_n(t), \boldsymbol{\phi}_k) + \nu\,(\nabla \mathbf{u}_n(t), \nabla \boldsymbol{\phi}_k) + \big((\mathbf{u}_n(t)\cdot\nabla)\mathbf{u}_n(t), \boldsymbol{\phi}_k\big) \\
&\qquad + \big((\mathbf{u}_n(t)\cdot\nabla)\mathbf{w}(t), \boldsymbol{\phi}_k\big) + \big((\mathbf{w}(t)\cdot\nabla)\mathbf{u}_n(t), \boldsymbol{\phi}_k\big) \\
&\quad = \tfrac{d}{dt} a_k^n(t) + \nu c_k a_k^n(t) + \sum_{i,j=1}^{n} \big((\boldsymbol{\phi}_i\cdot\nabla)\boldsymbol{\phi}_j, \boldsymbol{\phi}_k\big) a_i^n(t) a_j^n(t) \\
&\qquad + \sum_{i=1}^{n} \big((\boldsymbol{\phi}_i\cdot\nabla)\mathbf{w}(t), \boldsymbol{\phi}_k\big) a_i^n(t) + \sum_{i=1}^{n} \big((\mathbf{w}(t)\cdot\nabla)\boldsymbol{\phi}_i, \boldsymbol{\phi}_k\big) a_i^n(t) \\
&\quad = \langle \mathbf{f}(t), \boldsymbol{\phi}_k\rangle\,,
\end{aligned} \tag{3.2}$$

for $k = 1, \dots, n$. Defining

$$\begin{aligned}
X(t) &:= (a_1^n(t), \dots, a_n^n(t))^T\,, \\
C(t) &:= \big(\nu c_i \delta_{ij} + \big((\boldsymbol{\phi}_i\cdot\nabla)\mathbf{w}(t), \boldsymbol{\phi}_j\big) + \big((\mathbf{w}(t)\cdot\nabla)\boldsymbol{\phi}_i, \boldsymbol{\phi}_j\big)\big)_{i,j=1,\dots,n}\,, \\
D\colon \mathbb{R}^n &\to \mathbb{R}^n, a \mapsto \left(\sum_{i,j=1}^{n} \big((\boldsymbol{\phi}_i\cdot\nabla)\boldsymbol{\phi}_j, \boldsymbol{\phi}_k\big)\, a_i a_j\right)_{k=1,\dots,n} =: \big(a^T D_k a\big)_{k=1,\dots,n}\,, \\
F(t) &:= \big(\langle \mathbf{f}(t), \boldsymbol{\phi}_k\rangle\big)_{k=1,\dots,n}\,,
\end{aligned}$$

we see that (3.2) is actually a system of ordinary differential equations, having the form

$$X'(t) = -C(t)^T X(t) - D(X(t)) + F(t)\,. \tag{3.3}$$

Due to the linear and quadratic occurrence of X,

$$G(t, X) := -C(t)^T X - D(X) + F(t)$$

satisfies a generalized Lipschitz condition in X on each closed ball in $\mathbb{R}^n$ (depending on t). Further, $G(t, X)$ is measurable with respect to t (for fixed X) and the Lipschitz constant, in the corresponding estimate, is integrable over $[0, T]$. Thus Carathéodory's existence theorem (Theorem I.5.3 in [13] or Theorem 10.XX in [59]) guarantees the existence of a unique solution $X \in AC([0, T^*])$ of (3.3) respectively (3.2) for every prescribed initial value $X_0 = (b_1^n, \dots, b_n^n) \in \mathbb{R}^n$. Therefore we set $\mathbf{u}_{n,0} := \sum_{j=1}^n b_j^n \boldsymbol{\phi}_j$ for an arbitrary but fixed vector $(b_1^n, \dots, b_n^n)$ and define $\mathbf{u}_n$ based on the corresponding solution

$$\mathbf{u}_n(t) := \sum_{j=1}^n a_j^n(t) \boldsymbol{\phi}_j \,.$$

Now, either $T^* = T$, if $\|\mathbf{u}_n(t)\|_{L^2} < \infty$ for all $t \in [0, T]$, or $T^* < T$, in case of a blow up $\limsup_{t \to T^*} \|\mathbf{u}_n(t)\|_{L^2} = \infty$. The latter case can be excluded, due to an a-priori estimate, which we derive in the next step.

Step 2: A-priori estimate.
In order to prove boundedness of the term $\|\mathbf{u}_n(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla \mathbf{u}_n\|_{L^2}^2 \, d\tau$, we multiply (3.2) by $a_k^n(t)$ and sum over $k = 1, \dots, n$:

$$\begin{aligned}
&(\partial_t \mathbf{u}_n, \mathbf{u}_n) + \nu\,(\nabla \mathbf{u}_n, \nabla \mathbf{u}_n) + \big((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, \mathbf{u}_n\big) + \big((\mathbf{u}_n \cdot \nabla)\mathbf{w}, \mathbf{u}_n\big) + \big((\mathbf{w} \cdot \nabla)\mathbf{u}_n, \mathbf{u}_n\big) \\
&\quad = \frac{1}{2}\frac{d}{dt}\|\mathbf{u}_n\|_{L^2}^2 + \nu \|\nabla \mathbf{u}_n\|_{L^2}^2 + \big((\mathbf{u}_n \cdot \nabla)\mathbf{w}, \mathbf{u}_n\big) \\
&\quad = \langle \mathbf{f}, \mathbf{u}_n \rangle \,,
\end{aligned}$$

where

$$\begin{aligned}
\big((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, \mathbf{u}_n\big) &= b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{u}_n) = 0 \,, \\
\big((\mathbf{w} \cdot \nabla)\mathbf{u}_n, \mathbf{u}_n\big) &= b(\mathbf{w}, \mathbf{u}_n, \mathbf{u}_n) = 0 \,,
\end{aligned}$$

due to Lemma 2.11. Hence

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_n\|_{L^2}^2 + \nu \|\nabla \mathbf{u}_n\|_{L^2}^2 = \langle \mathbf{f}, \mathbf{u}_n \rangle - \big((\mathbf{u}_n \cdot \nabla)\mathbf{w}, \mathbf{u}_n\big) \,.$$

To estimate the right hand-side of this equation, we examine the last term and obtain

$$\big|\big((\mathbf{u}_n \cdot \nabla)\mathbf{w}, \mathbf{u}_n\big)\big| \le \frac{\nu}{4} \|\nabla \mathbf{u}_n\|_{L^2}^2 \,,$$

by the assumptions on $\mathbf{w}$. Application of Poincaré's and Young's inequality leads to

$$\begin{aligned}
\langle \mathbf{f}, \mathbf{u}_n \rangle - \big((\mathbf{u}_n \cdot \nabla)\mathbf{w}, \mathbf{u}_n\big) &\le |\langle \mathbf{f}, \mathbf{u}_n \rangle| + \big|\big((\mathbf{u}_n \cdot \nabla)\mathbf{w}, \mathbf{u}_n\big)\big| \\
&\le \frac{\mathsf{C}_P^2 + 1}{\nu} \|\mathbf{f}\|_{(W_{0,\sigma}^{1,2})'}^2 + \frac{\nu}{4}\|\nabla \mathbf{u}_n\|_{L^2}^2 + \frac{\nu}{4}\|\nabla \mathbf{u}_n\|_{L^2}^2 \\
&\le \frac{3}{2\nu} \|\mathbf{f}\|_{(W_{0,\sigma}^{1,2})'}^2 + \frac{\nu}{2}\|\nabla \mathbf{u}_n\|_{L^2}^2 \,,
\end{aligned}$$

taking into account that the Poincaré constant C_P is less equal $\frac{1}{\sqrt{2}}$ in the layer Π. Altogether we conclude

$$\frac{d}{dt}\|\mathbf{u}_n\|_{L^2}^2 + \nu \|\nabla \mathbf{u}_n\|_{L^2}^2 \le \frac{3}{\nu} \|\mathbf{f}\|_{(W_{0,\sigma}^{1,2})'}^2 \tag{3.4}$$

for almost all t. Since $AC([0, T^*]) \subset W^{1,1}(0, T^*)$ and

$$\left| \int_0^{T^*} \tfrac{d}{dt}\|\mathbf{u}_n\|_{L^2}^2 \, dt \right| = \left| \int_0^{T^*} 2 \sum_{j=1}^n a_j^n \tfrac{d}{dt} a_j^n \, dt \right| \le 2\|X\|_{C^0} \sum_{j=1}^n \int_0^{T^*} |\tfrac{d}{dt} a_j^n| \, dt \,,$$

we are able to integrate over t, which yields the energy inequality we were looking for:

$$\|\mathbf{u}_n(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla \mathbf{u}_n\|_{L^2}^2 \, d\tau \le \frac{3}{\nu} \int_0^t \|\mathbf{f}\|_{(W_{0,\sigma}^{1,2})'}^2 \, d\tau + \|\mathbf{u}_n(0)\|_{L^2}^2 \, , \tag{3.5}$$

for all $t \in [0, T^*]$, at first. Extending the region of integration to $[0, T]$ on the right, we find that the resulting expression is finite. Thus, we are in a position to exclude a blow-up of $\mathbf{u}_n$ implying $T^* = T$.

Step 3: Time-periodicity of approximating solutions.
So far we discovered that to each prescribed initial value $\mathbf{u}_{n,0}$ there exists a solution $\mathbf{u}_n$ of (3.2) satisfying the energy inequality (3.5). Our next goal is to find an initial value to (3.2) giving a time-periodic solution, i.e. $\mathbf{u}_n(T) = \mathbf{u}_{n,0} = \mathbf{u}_n(0)$. Rearranging (3.4) yields

$$\frac{d}{dt} \|\mathbf{u}_n\|_{L^2}^2 \le \frac{3}{\nu} \|\mathbf{f}\|_{(W_{0,\sigma}^{1,2})'}^2 - 2\nu \|\mathbf{u}_n\|_{L^2}^2 \, .$$

Then, by Gronwall's inequality (Lemma 2.7), we derive

$$\begin{aligned} \|\mathbf{u}_n(t)\|_{L^2}^2 &\le \|\mathbf{u}_n(0)\|_{L^2}^2 \, e^{-\int_0^t 2\nu \, ds} + \int_0^t \frac{3}{\nu} \|\mathbf{f}(\tau)\|_{(W_{0,\sigma}^{1,2})'}^2 \, e^{-\int_\tau^t 2\nu \, ds} \, d\tau \\ &\le \|\mathbf{u}_{n,0}\|_{L^2}^2 \, e^{-2\nu t} + \frac{3}{\nu} \int_0^t \|\mathbf{f}(\tau)\|_{(W_{0,\sigma}^{1,2})'}^2 \, d\tau \\ &\le \|\mathbf{u}_{n,0}\|_{L^2}^2 \, e^{-2\nu t} + \frac{3}{\nu} \|\mathbf{f}\|_{L^2(0,T;W_{0,\sigma}^{1,2}(\widetilde{\Pi})')}^2 \, . \end{aligned}$$

Choosing R large enough, i.e.

$$R := \left(\frac{3}{(1 - e^{-2\nu T})\nu} \right)^{1/2} \|\mathbf{f}\|_{L^2(0,T;W_{0,\sigma}^{1,2}(\widetilde{\Pi})')} \, ,$$

we obtain $|(a_1^n(T), \dots, a_n^n(T))| = \|\mathbf{u}_n(T)\|_{L^2} < R$, if $|(b_1^n, \dots, b_n^n)| = \|\mathbf{u}_{n,0}\|_{L^2} < R$.

Based on these findings, we define the Poincaré map assigning to each initial condition $\mathbf{u}_{n,0}$ or rather $(b_1^n, \dots, b_n^n)$ the value $\mathbf{u}_n(T)$ respectively $(a_1^n(T), \dots, a_n^n(T))$:

$$\mathcal{P} \colon B_R(0) \to B_R(0) \, , \; (b_1^n, \dots, b_n^n)^\top \mapsto (a_1^n(T), \dots, a_n^n(T))^\top \, .$$

In particular, $\mathcal{P}$ is continuous as a composition of continuous functions: The function, which maps an initial value to its corresponding solution,

$$\mathcal{P}_1 \colon \mathbb{R}^n \to C([0,T]; \mathbb{R}^n) \, ,$$

and evaluation of a continuous function at time T,

$$\mathcal{P}_2 \colon C([0,T]; \mathbb{R}^n) \to \mathbb{R}^n \, , \; \mathbf{h} \mapsto \mathbf{h}(T) \, .$$

Clearly, the latter is continuous and $\mathcal{P}_1$ is continuous due to continuous dependence of solutions to the ordinary differential equations (3.2) on the initial value ([13, Theorem I.5.3]). As pointed out beforehand, $\mathcal{P} = \mathcal{P}_2 \circ \mathcal{P}_1$ maps the ball of radius R into itself and, moreover, is continuous. By Brouwer's fixed-point theorem, $\mathcal{P}$ has a fixed point, meaning

$$\mathbf{u}_n(0) = \mathbf{u}_{n,0} = \sum_{j=1}^n b_j^n \boldsymbol{\phi}_j = \sum_{j=1}^n a_j^n(T) \boldsymbol{\phi}_j = \mathbf{u}_n(T) \, .$$

Henceforth, $\mathbf{u}_n$ denotes such a specific (time-periodic) function belonging to a fixed point $\mathbf{u}_{n,0}$.

We summarize the properties of $\mathbf{u}_n$ found so far. By (3.2), $\mathbf{u}_n$ satisfies

$$\begin{aligned}(\partial_t \mathbf{u}_n, \alpha\boldsymbol{\phi}_k) + \nu\,(\nabla\mathbf{u}_n, \alpha\nabla\boldsymbol{\phi}_k) + \big((\mathbf{u}_n\cdot\nabla)\mathbf{u}_n, \alpha\boldsymbol{\phi}_k\big) \\ + \big((\mathbf{u}_n\cdot\nabla)\mathbf{w}, \alpha\boldsymbol{\phi}_k\big) + \big((\mathbf{w}\cdot\nabla)\mathbf{u}_n, \alpha\boldsymbol{\phi}_k\big) \\ = \langle \mathbf{f}, \alpha\boldsymbol{\phi}_k\rangle\end{aligned} \tag{3.6}$$

pointwise for each $k = 1,\dots,n$ and an arbitrary mapping $\alpha\colon [0,T]\to\mathbb{R}$. Continuing with a smooth and time-periodic function $h\colon [0,T]\to\mathbb{R}$ (in place of α), we find

$$\begin{aligned}\int_0^T -\big(\mathbf{u}_n, h'\boldsymbol{\phi}_k\big) + \nu\,(\nabla\mathbf{u}_n, h\nabla\boldsymbol{\phi}_k) + \big((\mathbf{u}_n\cdot\nabla)\mathbf{u}_n, h\boldsymbol{\phi}_k\big) \\ + \big((\mathbf{u}_n\cdot\nabla)\mathbf{w}, h\boldsymbol{\phi}_k\big) + \big((\mathbf{w}\cdot\nabla)\mathbf{u}_n, h\boldsymbol{\phi}_k\big)\,dt \\ = \int_0^T \langle \mathbf{f}, h\boldsymbol{\phi}_k\rangle\,dt\,,\end{aligned} \tag{3.7}$$

by integration over t.

Further, the energy inequality (3.5) can now be reformulated as

$$\|\mathbf{u}_n(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla\mathbf{u}_n\|_{L^2}^2\,d\tau \le \left(1+\frac{1}{1-e^{-2\nu T}}\right)\frac{3}{\nu}\|\mathbf{f}\|^2_{L^2(0,T;W^{1,2}_{0,\sigma}(\widetilde{\Pi})')}\,. \tag{3.8}$$

Step 4: Passage to the limit.
According to inequality (3.8), the sequence $(\nabla\mathbf{u}_n)_n$ is bounded in $L^2(0,T;L^2_\sigma(\widetilde{\Pi}))$ and then, by Poincaré's inequality, $(\mathbf{u}_n)_n$ is bounded in $L^2(0,T;W^{1,2}_{0,\sigma}(\widetilde{\Pi}))$. Therefore it exists a subsequence – still denoted by the index n – converging weakly to a limit $\mathbf{u}$ in $L^2(0,T;W^{1,2}_{0,\sigma}(\widetilde{\Pi}))$. This gives reason for the following limit passages in (3.7) ($n\to\infty$):

$$\int_0^T \big(\mathbf{u}_n, h'\boldsymbol{\phi}_k\big)\;dt \to \int_0^T \big(\mathbf{u}, h'\boldsymbol{\phi}_k\big)\;dt\,, \tag{3.9}$$

$$\int_0^T (\nabla\mathbf{u}_n, h\nabla\boldsymbol{\phi}_k)\;dt \to \int_0^T (\nabla\mathbf{u}, h\nabla\boldsymbol{\phi}_k)\;dt\,. \tag{3.10}$$

Recall now that b is continuous and trilinear (see Lemma 2.10). Thus,

$$\int_0^T b(\cdot,\mathbf{w},h\boldsymbol{\phi}_k)\,dt \qquad \text{as well as} \qquad \int_0^T b(\mathbf{w},\cdot,h\boldsymbol{\phi}_k)\,dt$$

define linear, continuous functionals on $L^2(0,T;W^{1,2}_{0,\sigma}(\widetilde{\Pi}))$ – due to boundedness of h in $C([0,T])$ – meaning

$$\int_0^T \big((\mathbf{u}_n\cdot\nabla)\mathbf{w}, h\boldsymbol{\phi}_k\big)\,dt \to \int_0^T \big((\mathbf{u}\cdot\nabla)\mathbf{w}, h\boldsymbol{\phi}_k\big)\,dt\,, \tag{3.11}$$

$$\int_0^T \big((\mathbf{w}\cdot\nabla)\mathbf{u}_n, h\boldsymbol{\phi}_k\big)\,dt \to \int_0^T \big((\mathbf{w}\cdot\nabla)\mathbf{u}, h\boldsymbol{\phi}_k\big)\,dt\,. \tag{3.12}$$

Furthermore, $(\mathbf{u}_{n,0})_n$ is bounded in $L^2_\sigma(\widetilde{\Pi})$, since $\|\mathbf{u}_{n,0}\|_{L^2} < R$, and $(\mathbf{u}_n)_n$ is bounded in $L^\infty(0,T;L^2_\sigma(\widetilde{\Pi}))$, by inequality (3.8). Yet another subsequence satisfies $\mathbf{u}_{n,0}\rightharpoonup\mathbf{u}_0$ in $L^2_\sigma(\widetilde{\Pi})$ – thus also $\|\mathbf{u}_0\|_{L^2} < R$ – and $\mathbf{u}_n \overset{*}{\rightharpoonup} \mathbf{u}$ in $L^\infty(0,T;L^2_\sigma(\widetilde{\Pi}))$.

Weak convergence, however, does not suffice to pass to the limit in the convective term $\big((\mathbf{u}_n\cdot\nabla)\mathbf{u}_n, h\boldsymbol{\phi}_k\big)$. To gain better convergence results we wish to utilize the Aubin-Lions Lemma (Theorem 2.18), for which boundedness of the sequence of time-derivatives $(\partial_t\mathbf{u}_n)_n$ in $L^1(0,T;W^{1,2}_{0,\sigma}(\widetilde{\Pi})')$ is needed.

Let $\mathbf{v}\in L^4(0,T;W^{1,2}_{0,\sigma}(\widetilde{\Pi}))$. Then $\mathbf{v}$ has a Fourier series representation $\mathbf{v}(x,t)=\sum_{j=1}^{\infty}\alpha_j(t)\boldsymbol{\phi}_j(x)$, where $\alpha_j(t)=(\mathbf{v}(t),\boldsymbol{\phi}_j)$. We define $\mathbf{v}_n:=\sum_{j=1}^{n}\alpha_j\boldsymbol{\phi}_j$. Taking into account Theorem 2.16, the space $L^{4/3}(0,T;W^{1,2}_{0,\sigma}(\widetilde{\Pi})')$ is isomorphic to $L^4(0,T;W^{1,2}_{0,\sigma}(\widetilde{\Pi}))'$. Hence, it suffices to show that the sequence of functionals $(\partial_t\mathbf{u}_n)_n$ is bounded in the latter:

$$
\begin{aligned}
\Big|\langle\partial_t\mathbf{u}_n,\mathbf{v}\rangle_{L^4(W^{1,2}_{0,\sigma})',L^4(W^{1,2}_{0,\sigma})}\Big| &= \Big|\int_0^T(\partial_t\mathbf{u}_n,\mathbf{v})\,dt\Big| = \Big|\int_0^T(\partial_t\mathbf{u}_n,\mathbf{v}_n)\,dt\Big| \\
&\overset{(3.6)}{=} \Big|\int_0^T\langle\mathbf{f},\mathbf{v}_n\rangle-\nu(\nabla\mathbf{u}_n,\nabla\mathbf{v}_n)-\big((\mathbf{u}_n\cdot\nabla)\mathbf{u}_n,\mathbf{v}_n\big) \\
&\qquad -\big((\mathbf{u}_n\cdot\nabla)\mathbf{w},\mathbf{v}_n\big)-\big((\mathbf{w}\cdot\nabla)\mathbf{u}_n,\mathbf{v}_n\big)\,dt\Big| \\
&\overset{(2.5)}{\le} \int_0^T|\langle\mathbf{f},\mathbf{v}_n\rangle|\,dt+\nu\int_0^T|(\nabla\mathbf{u}_n,\nabla\mathbf{v}_n)|\,dt+\int_0^T\big|\big((\mathbf{u}_n\cdot\nabla)\mathbf{v}_n,\mathbf{u}_n\big)\big|\,dt \\
&\qquad +\int_0^T\big|\big((\mathbf{u}_n\cdot\nabla)\mathbf{v}_n,\mathbf{w}\big)\big|\,dt+\int_0^T\big|\big((\mathbf{w}\cdot\nabla)\mathbf{v}_n,\mathbf{u}_n\big)\big|\,dt \\
&\le \|\mathbf{f}\|_{L^2((W^{1,2}_{0,\sigma})')}\|\mathbf{v}_n\|_{L^2(W^{1,2})}+\nu\|\nabla\mathbf{u}_n\|_{L^2(L^2)}\|\nabla\mathbf{v}_n\|_{L^2(L^2)}+I_1+I_2+I_3 \\
&\le T^{1/4}\|\mathbf{f}\|_{L^2((W^{1,2}_{0,\sigma})')}\|\mathbf{v}_n\|_{L^4(W^{1,2})}+T^{1/4}\nu\|\nabla\mathbf{u}_n\|_{L^2(L^2)}\|\mathbf{v}_n\|_{L^4(W^{1,2})} \\
&\qquad +I_1+I_2+I_3\,.
\end{aligned}
$$

For I_1, I_2, I_3, by the Gagliardo-Nirenberg (Lemma 2.6) and Lyapunov interpolation (Lemma 2.3) inequalities, we find

$$
\begin{aligned}
I_1 &\le \int_0^T\int_{\widetilde{\Pi}}|\mathbf{u}_n|^2|\nabla\mathbf{v}_n|\,dx\,dt \\
&\le \int_0^T\|\mathbf{u}_n\|_{L^4}^2\|\nabla\mathbf{v}_n\|_{L^2}\,dt \\
&\le \sqrt{2}\int_0^T\|\nabla\mathbf{v}_n\|_{L^2}\|\mathbf{u}_n\|_{L^2}^{1/2}\|\nabla\mathbf{u}_n\|_{L^2}^{3/2}\,dt \\
&\le \sqrt{2}\,\|\mathbf{u}_n\|_{L^\infty(L^2)}^{1/2}\|\nabla\mathbf{u}_n\|_{L^2(L^2)}^{3/2}\|\nabla\mathbf{v}_n\|_{L^4(L^2)}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2+I_3 &\le 2\int_0^T\int_{\widetilde{\Pi}}|\mathbf{u}_n||\nabla\mathbf{v}_n||\mathbf{w}|\,dt\,dt \\
&\le 2\int_0^T\|\mathbf{u}_n\|_{L^4}\|\mathbf{w}\|_{L^4}\|\nabla\mathbf{v}_n\|_{L^2}\,dt \\
&\le \int_0^T\left(\|\mathbf{u}_n\|_{L^4}^2+\|\mathbf{w}\|_{L^4}^2\right)\|\nabla\mathbf{v}_n\|_{L^2}\,dt \\
&\le \sqrt{2}\int_0^T\|\mathbf{u}_n\|_{L^2}^{1/2}\|\nabla\mathbf{u}_n\|_{L^2}^{3/2}\|\nabla\mathbf{v}_n\|_{L^2}\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^T \|\mathbf{w}\|_{L^2}^{1/2} \|\mathbf{w}\|_{L^6}^{3/2} \|\nabla \mathbf{v}_n\|_{L^2} \, dt \\
&\leq \sqrt{2}\, \|\mathbf{u}_n\|_{L^\infty(L^2)}^{1/2} \int_0^T \|\nabla \mathbf{u}_n\|_{L^2}^{3/2} \|\nabla \mathbf{v}_n\|_{L^2} \, dt \\
&\quad + c(\widetilde{\Pi})\, \|\mathbf{w}\|_{L^\infty(L^2)}^{1/2} \int_0^T \|\mathbf{w}\|_{W^{1,2}}^{3/2} \|\nabla \mathbf{v}_n\|_{L^2} \, dt \\
&\leq \sqrt{2}\, \|\mathbf{u}_n\|_{L^\infty(L^2)}^{1/2} \|\nabla \mathbf{u}_n\|_{L^2(L^2)}^{3/2} \|\nabla \mathbf{v}_n\|_{L^4(L^2)} \\
&\quad + c(\widetilde{\Pi})\, \|\mathbf{w}\|_{L^\infty(L^2)}^{1/2} \|\mathbf{w}\|_{L^2(W^{1,2})}^{3/2} \|\nabla \mathbf{v}_n\|_{L^4(L^2)} .
\end{aligned}
$$

In view of (3.8) we altogether obtain

$$
\begin{aligned}
&\left| \langle \partial_t \mathbf{u}_n, \mathbf{v} \rangle_{L^4(W^{1,2}_{0,\sigma})', L^4(W^{1,2}_{0,\sigma})} \right| \\
&\quad \leq c \Big(\|\mathbf{f}\|_{L^2((W^{1,2}_{0,\sigma})')} + \nu \|\nabla \mathbf{u}_n\|_{L^2(L^2)} + \|\mathbf{u}_n\|_{L^\infty(L^2)}^{1/2} \|\nabla \mathbf{u}_n\|_{L^2(L^2)}^{3/2} \\
&\qquad + \|\mathbf{w}\|_{L^\infty(L^2)}^{1/2} \|\mathbf{w}\|_{L^2(W^{1,2})}^{3/2} \Big) \|\mathbf{v}_n\|_{L^4(W^{1,2})} \\
&\quad \overset{(3.8)}{\leq} c \Big(\|\mathbf{f}\|_{L^2((W^{1,2}_{0,\sigma})')} + \|\mathbf{f}\|_{L^2((W^{1,2}_{0,\sigma})')}^2 + \|\mathbf{w}\|_{L^\infty(L^2)}^{1/2} \|\mathbf{w}\|_{L^2(W^{1,2})}^{3/2} \Big) \|\mathbf{v}\|_{L^4(W^{1,2})} .
\end{aligned}
$$

The right-hand side is independent of n, due to the estimate $\|\mathbf{v}_n\|_{L^4(W^{1,2})} \leq \|\mathbf{v}\|_{L^4(W^{1,2})}$, as a consequence of Parseval's identity. Hence, $(\partial_t \mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^{4/3}(0,T; W^{1,2}_{0,\sigma}(\widetilde{\Pi})') \subset L^1(0,T; W^{1,2}_{0,\sigma}(\widetilde{\Pi})')$.

The Aubin-Lions Lemma (applied with $X = W^{1,2}_{0,\sigma}$, $B = L^2_\sigma$, $Y = (W^{1,2}_{0,\sigma})'$) gives relative compactness of $(\mathbf{u}_n)_n$ in the space $L^2(0,T; L^2_\sigma(\widetilde{\Pi}))$, implying strong convergence of a subsequence in the corresponding norm. Coupling Lyapunov's interpolation inequality (Lemma 2.3) with a Sobolev embedding we have

$$
\|\mathbf{v}\|_{L^3(\widetilde{\Pi})}^2 \leq \|\mathbf{v}\|_{L^2(\widetilde{\Pi})} \|\mathbf{v}\|_{L^6(\widetilde{\Pi})} \leq c\, \|\mathbf{v}\|_{L^2(\widetilde{\Pi})} \|\mathbf{v}\|_{W^{1,2}(\widetilde{\Pi})} ,
$$

where $\mathbf{v} \in W^{1,2}(\widetilde{\Pi})$. Therefore $(\mathbf{u}_n)_n$ is converging strongly in $L^2(0,T; L^3(\widetilde{\Pi}))$ as well. This allows us to calculate

$$
\begin{aligned}
&\left| \int_0^T \big((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, h\boldsymbol{\phi}_k \big)\, dt - \int_0^T \big((\mathbf{u} \cdot \nabla)\mathbf{u}, h\boldsymbol{\phi}_k \big)\, dt \right| \\
&\quad = \left| \int_0^T b(\mathbf{u}_n, \mathbf{u}_n, h\boldsymbol{\phi}_k) - b(\mathbf{u}_n, \mathbf{u}, h\boldsymbol{\phi}_k) + b(\mathbf{u}_n, \mathbf{u}, h\boldsymbol{\phi}_k) - b(\mathbf{u}, \mathbf{u}, h\boldsymbol{\phi}_k)\, dt \right| \\
&\quad = \left| \int_0^T b(\mathbf{u}_n, \mathbf{u}_n - \mathbf{u}, h\boldsymbol{\phi}_k) + b(\mathbf{u}_n - \mathbf{u}, \mathbf{u}, h\boldsymbol{\phi}_k)\, dt \right| \\
&\quad \overset{(2.5)}{\leq} \int_0^T |b(\mathbf{u}_n, \boldsymbol{\phi}_k, \mathbf{u}_n - \mathbf{u})||h| + |b(\mathbf{u}_n - \mathbf{u}, \boldsymbol{\phi}_k, \mathbf{u})||h|\, dt \\
&\quad \leq \int_0^T \int_{\tilde{\Pi}} |\mathbf{u}_n||\nabla \boldsymbol{\phi}_k||\mathbf{u}_n - \mathbf{u}| + |\mathbf{u}_n - \mathbf{u}||\nabla \boldsymbol{\phi}_k||\mathbf{u}|\, dx\, |h|\, dt \\
&\quad \leq \|h\|_{C^0} \int_0^T \|\mathbf{u}_n\|_{L^6} \|\nabla \boldsymbol{\phi}_k\|_{L^2} \|\mathbf{u}_n - \mathbf{u}\|_{L^3} + \|\mathbf{u}_n - \mathbf{u}\|_{L^3} \|\nabla \boldsymbol{\phi}_k\|_{L^2} \|\mathbf{u}\|_{L^6}\, dt \\
&\quad \leq c\, \|h\|_{C^0} \|\boldsymbol{\phi}_k\|_{W^{1,2}} \big(\|\mathbf{u}_n\|_{L^2(W^{1,2})} + \|\mathbf{u}\|_{L^2(W^{1,2})} \big) \|\mathbf{u}_n - \mathbf{u}\|_{L^2(L^3)} ,
\end{aligned}
\tag{3.13}
$$

which converges to 0 for $n \to \infty$.

Summarizing all convergences of occurring terms in the integral identity leads to the equation

$$\begin{aligned}\int_0^T &- \big(\mathbf{u}, h'\boldsymbol{\phi}_k\big) + \nu\,(\nabla\mathbf{u}, h\nabla\boldsymbol{\phi}_k) + \big((\mathbf{u}\cdot\nabla)\mathbf{u}, h\boldsymbol{\phi}_k\big)\\ &+ \big((\mathbf{u}\cdot\nabla)\mathbf{w}, h\boldsymbol{\phi}_k\big) + \big((\mathbf{w}\cdot\nabla)\mathbf{u}, h\boldsymbol{\phi}_k\big)\,dt\\ &= \int_0^T \langle \mathbf{f}, h\boldsymbol{\phi}_k\rangle\,dt\end{aligned}$$

for all $k \in \mathbb{N}$. Let $\boldsymbol{\varphi} \in C^\infty([0,T]; C^\infty_{0,\sigma}(\widetilde{\Pi}))$ be time-periodic. The test function $\boldsymbol{\varphi}$ can be approximated by a sequence $(h_l\boldsymbol{\psi}_l)_l$, where $h_l \in C^\infty([0,T])$ is time-periodic and $\boldsymbol{\psi}_l \in C^\infty_{0,\sigma}(\widetilde{\Pi})$. Each $\boldsymbol{\psi}_l$ then has a representation as countable linear combination of the base elements $\boldsymbol{\phi}_k$. In conclusion the identity from above is also valid for $\boldsymbol{\varphi}$:

$$\begin{aligned}\int_0^T &- (\mathbf{u}, \partial_t\boldsymbol{\varphi}) + \nu\,(\nabla\mathbf{u}, \nabla\boldsymbol{\varphi}) + \big((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}\big) + \big((\mathbf{u}\cdot\nabla)\mathbf{w}, \boldsymbol{\varphi}\big) + \big((\mathbf{w}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}\big)\,dt\\ &= \int_0^T \langle \mathbf{f}, \boldsymbol{\varphi}\rangle\,dt\,,\end{aligned}$$

which is just (3.1). Hence, $\mathbf{u}$ is a weak solution of the perturbed time-periodic Navier-Stokes equations.

Step 5: Energy inequality.
In this step we wish to show that (3.8) is valid for the limit $\mathbf{u}$ too. To this end, we set $h \in C_0^\infty((0,T))$ to be a non-negative function, multiply (3.8) by h and integrate over $[0,T]$:

$$\begin{aligned}\int_0^T h(t)\int_{\widetilde{\Pi}} |\mathbf{u}_n(t)|^2\,dx\,dt &+ \nu\int_0^T h(t)\int_0^t\int_{\widetilde{\Pi}} |\nabla\mathbf{u}_n(\tau)|^2\,dx\,d\tau\,dt\\ &\le \left(1 + \frac{1}{1-e^{-2\nu T}}\right)\frac{3}{\nu}\|\mathbf{f}\|_{L^2((W^{1,2}_{0,\sigma})')}\int_0^T h(t)\,dt\,.\end{aligned}$$

Recall that $(\mathbf{u}_n)_n$ converges weakly in $L^2(0,T; W^{1,2}_{0,\sigma}(\widetilde{\Pi}))$, implying $h^{1/2}\,\mathbf{u}_n \rightharpoonup h^{1/2}\,\mathbf{u}$ in $L^2(0,T; L^2(\widetilde{\Pi}))$. By weak lower-semicontinuity of the norms, we infer

$$\begin{aligned}\liminf_{n\to\infty}\int_0^T h(t)\int_{\widetilde{\Pi}} |\mathbf{u}_n(t)|^2\,dx\,dt &= \liminf_{n\to\infty}\|h^{1/2}\,\mathbf{u}_n\|^2_{L^2(0,T;L^2(\widetilde{\Pi}))}\\ &\ge \|h^{1/2}\,\mathbf{u}\|^2_{L^2(0,T;L^2(\widetilde{\Pi}))}\\ &= \int_0^T h(t)\int_{\widetilde{\Pi}} |\mathbf{u}(t)|^2\,dx\,dt\end{aligned}$$

and, taking into account Fatou's Lemma,

$$\begin{aligned}\liminf_{n\to\infty}\int_0^T h(t)\int_0^t\int_{\widetilde{\Pi}} |\nabla\mathbf{u}_n|^2\,dx\,d\tau\,dt &\ge \int_0^T h(t)\left(\liminf_{n\to\infty}\int_0^t\int_{\widetilde{\Pi}} |\nabla\mathbf{u}_n|^2\,dx\,d\tau\right)dt\\ &\ge \int_0^T h(t)\int_0^t\int_{\widetilde{\Pi}} |\nabla\mathbf{u}|^2\,dx\,d\tau\,dt\,.\end{aligned}$$

Passing to the limes inferior on both sides, we obtain

$$\int_0^T h \int_{\widetilde{\Pi}} |\mathbf{u}|^2 \, dx \, dt + \nu \int_0^T h \int_0^t \int_{\widetilde{\Pi}} |\nabla \mathbf{u}|^2 \, dx \, d\tau \, dt \\ \leq \left(1 + \frac{1}{1 - e^{-2\nu T}}\right) \frac{3}{\nu} \, \|\mathbf{f}\|_{L^2((W^{1,2}_{0,\sigma})')} \int_0^T h \, dt \, .$$

As an explicit test function we choose the standard mollifier (in time) $h = \eta_\varepsilon$ (cf. [9, §C.4]) centered in $s \in (0,T)$, where $\varepsilon > 0$ such that $[s-\varepsilon, s+\varepsilon] \subset [0,T]$. Sending ε to 0, we conclude

$$\|\mathbf{u}(s)\|^2_{L^2} + \nu \int_0^s \|\nabla \mathbf{u}\|^2_{L^2} \, d\tau \leq \left(1 + \frac{1}{1 - e^{-2\nu T}}\right) \frac{3}{\nu} \, \|\mathbf{f}\|^2_{L^2((W^{1,2}_{0,\sigma})')}$$

for almost all $s \in (0,T)$. □

The next step is to "extend" the previous result to the whole domain Π.

Theorem 3.4. *Let* $\mathbf{f} \in L^2(0,T; W^{1,2}_{0,\sigma}(\Pi)')$ *and* $\mathbf{w} \in W^{1,2}((0,T) \times \Pi)$ *be time-periodic. Assume the perturbation* $\mathbf{w}$ *is solenoidal in* Π *and satisfies the inequality*

$$\left| \big((\mathbf{v} \cdot \nabla)\mathbf{w}(t), \mathbf{v}\big)_{L^2(\Pi)} \right| \leq \frac{\nu}{4} \|\nabla \mathbf{v}\|^2_{L^2(\Pi)} \, ,$$

for all $\mathbf{v} \in W^{1,2}_{0,\sigma}(\Pi)$, $t \in (0,T)$. *Then, there exists a weak solution* $\mathbf{u} \in L^2(0,T; W^{1,2}_{0,\sigma}(\Pi)) \cap L^\infty(0,T; L^2_\sigma(\Pi))$ *of the perturbed time-periodic Navier-Stokes equations fulfilling the a-priori estimate*

$$\|\mathbf{u}(s)\|^2_{L^2(\Pi)} + \nu \int_0^s \|\nabla \mathbf{u}\|^2_{L^2(\Pi)} \, d\tau \leq c \, \|\mathbf{f}\|^2_{L^2(0,T;(W^{1,2}_{0,\sigma}(\Pi))')} \tag{3.14}$$

for almost all $s \in (0,T)$.

Proof. The proof is carried out by an "invading domains" approach. We consider an ascending sequence of C^2-subdomains, $\Pi_1 \subsetneq \Pi_2 \subsetneq \cdots \subsetneq \Pi$, such that $\bigcup_{k \in \mathbb{N}} \Pi_k = \Pi$, and define the operator $\mathcal{E} = \mathcal{E}_k \colon W^{1,2}_{0,\sigma}(\Pi_k) \to W^{1,2}_{0,\sigma}(\Pi)$, which extends a function by zero outside of Π_k. Note that the operator $\mathcal{E}$ is an isometry.

Step 1: Solutions in the subdomains Π_k.
Trivially, we get the following for all $k \in \mathbb{N}$:

$$\mathbf{w} \in L^2(0,T; W^{1,2}(\Pi_k)) \, , \; \partial_t \mathbf{w} \in L^2(0,T; L^2(\Pi_k)) \, , \; \operatorname{div} \mathbf{w} = 0 \;\text{ a.e. in } \Pi_k \, ,$$

as well as

$$\begin{aligned} \left| \big((\mathbf{v} \cdot \nabla)\mathbf{w}(t), \mathbf{v}\big)_{L^2(\Pi_k)} \right| &= \left| \big((\mathcal{E}\mathbf{v} \cdot \nabla)\mathbf{w}(t), \mathcal{E}\mathbf{v}\big)_{L^2(\Pi)} \right| \\ &\leq \frac{\nu}{4} \|\nabla(\mathcal{E}\mathbf{v})\|^2_{L^2(\Pi)} \\ &= \frac{\nu}{4} \|\nabla \mathbf{v}\|^2_{L^2(\Pi_k)} \end{aligned}$$

for all $\mathbf{v} \in W^{1,2}_{0,\sigma}(\Pi_k)$. At last we define $\mathbf{f}_k(t) \colon W^{1,2}_{0,\sigma}(\Pi_k) \to \mathbb{R}$ by

$$\langle \mathbf{f}_k(t), \mathbf{v} \rangle := \langle \mathbf{f}(t), \mathcal{E}(\mathbf{v}) \rangle \, .$$

Hence, $\mathbf{f}_k(t)$ is linear and continuous, since

$$|\langle \mathbf{f}_k(t), \mathbf{v}\rangle| = |\langle \mathbf{f}(t), \mathcal{E}(\mathbf{v})\rangle| \leq \|\mathbf{f}(t)\|_{W^{1,2}_{0,\sigma}(\Pi)'} \|\mathcal{E}(\mathbf{v})\|_{W^{1,2}_{0,\sigma}(\Pi)} = \|\mathbf{f}(t)\|_{W^{1,2}_{0,\sigma}(\Pi)'} \|\mathbf{v}\|_{W^{1,2}_{0,\sigma}(\Pi_k)} .$$

Therefore also $\mathbf{f}_k \in L^2(0,T; W^{1,2}_{0,\sigma}(\Pi_k)')$ and $\|\mathbf{f}_k\|_{L^2(W^{1,2}_{0,\sigma}(\Pi_k)')} \leq \|\mathbf{f}\|_{L^2(W^{1,2}_{0,\sigma}(\Pi)')}$.

Now we are in a position to apply Theorem 3.3. For each subdomain Π_k, we obtain a weak solution $\tilde{\mathbf{u}}_k \in L^2(0,T; W^{1,2}_{0,\sigma}(\Pi_k))$ satisfying the inequality

$$\begin{aligned} \|\tilde{\mathbf{u}}_k(s)\|^2_{L^2(\Pi_k)} + \nu \int_0^s \|\nabla \tilde{\mathbf{u}}_k\|^2_{L^2(\Pi_k)}\, d\tau &\leq c_0\, \|\mathbf{f}_k\|^2_{L^2(W^{1,2}_{0,\sigma}(\Pi_k)')} \\ &\leq c_0\, \|\mathbf{f}\|^2_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} , \end{aligned} \tag{3.15}$$

where $c_0 := \frac{3}{\nu}\left(1 + \frac{1}{1-e^{-2\nu T}}\right)$. Setting $\mathbf{u}_k := \mathcal{E}(\tilde{\mathbf{u}}_k)$, gives us a sequence $(\mathbf{u}_k)_{k\in\mathbb{N}}$ in the space $L^2(0,T; W^{1,2}_{0,\sigma}(\Pi))$. The estimate (3.15) holds for $\mathbf{u}_k$ too, with Π in place of Π_k. By the Poincaré inequality, we find that $(\mathbf{u}_k)_k$ is bounded in $L^2(0,T; W^{1,2}_{0,\sigma}(\Pi))$. Hence, it is possible to extract a subsequence, which is denoted by the same index, converging weakly in $L^2(0,T; W^{1,2}_{0,\sigma}(\Pi))$ and weak-star in $L^\infty(0,T; L^2_\sigma(\Pi))$ to a function $\mathbf{u}$.

Step 2: Developing convergence properties.
Let $\boldsymbol{\varphi}_N \in C^\infty([0,T]; C^\infty_{0,\sigma}(\Pi_N))$ be time-periodic, where $N \in \mathbb{N}$ is arbitrary (but fixed in this step). For $k \geq N$ we deduce – notice the change of domains in the first and last step:

$$\begin{aligned} &\int_0^T -(\mathbf{u}_k, \partial_t \boldsymbol{\varphi}_N)_{L^2(\Pi_N)} + \nu\, (\nabla \mathbf{u}_k, \nabla \boldsymbol{\varphi}_N)_{L^2(\Pi_N)} + \big((\mathbf{u}_k \cdot \nabla)\mathbf{u}_k, \boldsymbol{\varphi}_N\big)_{L^2(\Pi_N)} \\ &\qquad + \big((\mathbf{u}_k \cdot \nabla)\mathbf{w}, \boldsymbol{\varphi}_N\big)_{L^2(\Pi_N)} + \big((\mathbf{w} \cdot \nabla)\mathbf{u}_k, \boldsymbol{\varphi}_N\big)_{L^2(\Pi_N)}\, dt \\ &= \int_0^T -(\mathbf{u}_k, \partial_t(\mathcal{E}\boldsymbol{\varphi}_N))_{L^2(\Pi_k)} + \nu\, (\nabla \mathbf{u}_k, \nabla(\mathcal{E}\boldsymbol{\varphi}_N))_{L^2(\Pi_k)} \\ &\qquad + \big((\mathbf{u}_k \cdot \nabla)\mathbf{u}_k, \mathcal{E}\boldsymbol{\varphi}_N\big)_{L^2(\Pi_k)} + \big((\mathbf{u}_k \cdot \nabla)\mathbf{w}, \mathcal{E}\boldsymbol{\varphi}_N\big)_{L^2(\Pi_k)} \\ &\qquad + \big((\mathbf{w} \cdot \nabla)\mathbf{u}_k, \mathcal{E}\boldsymbol{\varphi}_N\big)_{L^2(\Pi_k)}\, dt \\ &= \int_0^T \langle \mathbf{f}_k, \mathcal{E}\boldsymbol{\varphi}_N \rangle_{W^{1,2}_{0,\sigma}(\Pi_k)', W^{1,2}_{0,\sigma}(\Pi_k)}\, dt \\ &= \int_0^T \langle \mathbf{f}, \mathcal{E}\boldsymbol{\varphi}_N \rangle_{W^{1,2}_{0,\sigma}(\Pi)', W^{1,2}_{0,\sigma}(\Pi)}\, dt , \end{aligned} \tag{3.16}$$

since $\mathbf{u}_k$ is a weak solution with respect to Π_k. Further, from (3.15) we get $(k \geq N)$

$$\begin{aligned} \|\mathbf{u}_k\|^2_{L^\infty(L^2(\Pi_N))} + \nu\, \|\nabla \mathbf{u}_k\|^2_{L^2(W^{1,2}(\Pi_N))} &\leq \|\mathbf{u}_k\|^2_{L^\infty(L^2(\Pi_k))} + \nu\, \|\nabla \mathbf{u}_k\|^2_{L^2(W^{1,2}(\Pi_k))} \\ &\leq c_0\, \|\mathbf{f}\|^2_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} , \end{aligned} \tag{3.17}$$

implying that yet another subsequence $(\mathbf{u}_k^N)_k$ converges weakly in $L^2(0,T; W^{1,2}(\Pi_N))$, where the limit is just $\mathbf{u}|_{\Pi_N}$.

Our goal is to show that a subsequence of $(\mathbf{u}_k^N)_k$ is strongly convergent in $L^2(0,T; L^3(\Pi_N))$. To this end we want to apply, as before, the Aubin-Lions Lemma (Theorem 2.18) and need to verify that $(\partial_t \mathbf{u}_k^N)_k$ is bounded in $L^{4/3}(0,T; W^{1,2}_{0,\sigma}(\Pi_N)')$, which is isomorphic to

$L^4(0,T;W^{1,2}_{0,\sigma}(\Pi_N))'$. Let $\hat{\boldsymbol{\varphi}}_N \in C_0^\infty((0,T);C^\infty_{0,\sigma}(\Pi_N))$. We examine $\partial_t \mathbf{u}_k^N$ in the dual space, as a linear functional:

$$\begin{aligned}
&\left|\left\langle \partial_t \mathbf{u}_k^N, \hat{\boldsymbol{\varphi}}_N \right\rangle_{L^4(W^{1,2}_{0,\sigma}(\Pi_N))', L^4(W^{1,2}_{0,\sigma}(\Pi_N))}\right| \\
&\quad = \left| - \left\langle \mathbf{u}_k^N, \partial_t \hat{\boldsymbol{\varphi}}_N \right\rangle_{L^4(W^{1,2}_{0,\sigma}(\Pi_N))', L^4(W^{1,2}_{0,\sigma}(\Pi_N))}\right| \\
&\quad = \left| - \int_0^T \left(\mathbf{u}_k^N, \partial_t \hat{\boldsymbol{\varphi}}_N\right)_{L^2(\Pi_N)} \, dt \right| \\
&\quad \overset{(3.16)}{=} \Big| \int_0^T \langle \mathbf{f}, \mathcal{E}\hat{\boldsymbol{\varphi}}_N \rangle_{W^{1,2}_{0,\sigma}(\Pi)', W^{1,2}_{0,\sigma}(\Pi)} - \nu \left(\nabla \mathbf{u}_k^N, \nabla \hat{\boldsymbol{\varphi}}_N\right)_{L^2(\Pi_N)} \\
&\qquad - \left((\mathbf{u}_k^N \cdot \nabla)\mathbf{u}_k^N, \hat{\boldsymbol{\varphi}}_N\right)_{L^2(\Pi_N)} - \left((\mathbf{u}_k^N \cdot \nabla)\mathbf{w}, \hat{\boldsymbol{\varphi}}_N\right)_{L^2(\Pi_N)} \\
&\qquad - \left((\mathbf{w} \cdot \nabla)\mathbf{u}_k^N, \hat{\boldsymbol{\varphi}}_N\right)_{L^2(\Pi_N)} dt \Big| \\
&\quad \overset{(2.5)}{\leq} \|\mathbf{f}\|_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} \|\hat{\boldsymbol{\varphi}}_N\|_{L^2(W^{1,2}(\Pi_N))} + \nu \|\nabla \mathbf{u}_k^N\|_{L^2(L^2(\Pi_N))} \|\nabla \hat{\boldsymbol{\varphi}}_N\|_{L^2(L^2(\Pi_N))} \\
&\qquad + \int_0^T \left| \left((\mathbf{u}_k^N \cdot \nabla)\hat{\boldsymbol{\varphi}}_N, \mathbf{u}_k^N\right)_{L^2(\Pi_N)} \right| dt + \int_0^T \left| \left((\mathbf{u}_k^N \cdot \nabla)\hat{\boldsymbol{\varphi}}_N, \mathbf{w}\right)_{L^2(\Pi_N)} \right| dt \\
&\qquad + \int_0^T \left| \left((\mathbf{w} \cdot \nabla)\hat{\boldsymbol{\varphi}}_N, \mathbf{u}_k^N\right)_{L^2(\Pi_N)} \right| dt \,.
\end{aligned}$$

From here on the arguments follow the exact same lines as in the proof of Theorem 3.3 (step 4). Observe that, for the application of the Gagliardo-Nirenberg inequality, we must consider $\mathbf{u}_k^N$ in Π_k (to guarantee zero trace), meaning that in the treatment of I_1, I_2 and I_3 the term $\|\mathbf{u}_k^N\|_{L^4(\Pi_N)}$ needs to be estimated to $\|\mathbf{u}_k^N\|_{L^4(\Pi_k)}$ directly. To obtain an estimate independent of the subdomain Π_N, we further have to pay attention to the Sobolev embedding of $\mathbf{w}$. Namely $\|\mathbf{w}\|_{L^6(\Pi_N)} \leq \|\mathbf{w}\|_{L^6(\Pi)} \leq c(\Pi)\|\mathbf{w}\|_{W^{1,2}(\Pi)}$, in that specific order. Eventually this yields (by density of the test functions):

$$\begin{aligned}
&\|\partial_t \mathbf{u}_k^N\|_{L^4(W^{1,2}_{0,\sigma}(\Pi_N))'} \\
&\quad \leq c(\Pi) \Big(\|\mathbf{f}\|_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} + \nu \|\nabla \mathbf{u}_k^N\|_{L^2(L^2(\Pi_N))} \\
&\qquad + \|\mathbf{u}_k^N\|^{1/2}_{L^\infty(L^2(\Pi_k))} \|\nabla \mathbf{u}_k^N\|^{3/2}_{L^2(L^2(\Pi_k))} + \|\mathbf{w}\|^{1/2}_{L^\infty(L^2(\Pi_N))} \|\mathbf{w}\|^{3/2}_{L^2(W^{1,2}(\Pi))} \Big) \\
&\quad \overset{(3.17)}{\leq} c(\Pi) \Big(\|\mathbf{f}\|_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} + \|\mathbf{f}\|^2_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} + \|\mathbf{w}\|^{1/2}_{L^\infty(L^2(\Pi))} \|\mathbf{w}\|^{3/2}_{L^2(W^{1,2}(\Pi))} \Big) .
\end{aligned}$$

Now, due to the Lemma of Aubin-Lions, there is a subsequence, converging strongly in $L^2(0,T;L^3(\Pi_N))$. Therefore, all terms in (3.16) converge to the desired limits,

$$\begin{aligned}
\int_0^T \left(\mathbf{u}_k^N, \boldsymbol{\varphi}_N\right)_{L^2(\Pi_N)} dt &\to \int_0^T \left(\mathbf{u}, \boldsymbol{\varphi}_N\right)_{L^2(\Pi_N)} dt \,, \\
\int_0^T \left(\nabla \mathbf{u}_k^N, \boldsymbol{\varphi}_N\right)_{L^2(\Pi_N)} dt &\to \int_0^T \left(\nabla \mathbf{u}, \boldsymbol{\varphi}_N\right)_{L^2(\Pi_N)} dt \,, \\
\int_0^T \left((\mathbf{u}_k^N \cdot \nabla)\mathbf{w}, \boldsymbol{\varphi}_N\right)_{L^2(\Pi_N)} dt &\to \int_0^T \left((\mathbf{u} \cdot \nabla)\mathbf{w}, \boldsymbol{\varphi}_N\right)_{L^2(\Pi_N)} dt \,, \\
\int_0^T \left((\mathbf{w} \cdot \nabla)\mathbf{u}_k^N, \boldsymbol{\varphi}_N\right)_{L^2(\Pi_N)} dt &\to \int_0^T \left((\mathbf{w} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}_N\right)_{L^2(\Pi_N)} dt \,,
\end{aligned}$$

$$\int_0^T \big((\mathbf{u}_k^N \cdot \nabla)\mathbf{u}_k^N, \boldsymbol{\varphi}_N\big)_{L^2(\Pi_N)}\, dt \to \int_0^T \big((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}_N\big)_{L^2(\Pi_N)}\, dt\,,$$

for $\boldsymbol{\varphi}_N$ from above, which can be justified as in step 4 of the proof of Theorem 3.3 (see (3.9),(3.10),(3.11),(3.12), (3.13)).

Step 3: Passage to the limit for an appropriate subsequence.
Starting with $N = 1$, we iteratively find, for each $N \in \mathbb{N}$, a subsequence with the convergence properties shown above, allowing us to select a further subsequence $(\mathbf{u}_n)_n$ consisting of the diagonal elements $\mathbf{u}_n^n$.

Let $\boldsymbol{\varphi} \in C^\infty([0,T]; C_{0,\sigma}^\infty(\Pi))$ be a time-periodic test function (on the whole domain Π). In particular, $\boldsymbol{\varphi}$ is compactly supported in $[0,T] \times \Pi$, meaning that there exists an integer $N \in \mathbb{N}$ such that $\operatorname{supp} \boldsymbol{\varphi} \subset [0,T] \times \Pi_N$. Exploiting now all established convergence properties, we conclude

$$\begin{aligned}
&\int_0^T - (\mathbf{u}, \partial_t \boldsymbol{\varphi})_{L^2(\Pi)} + \nu\, (\nabla \mathbf{u}, \nabla \boldsymbol{\varphi})_{L^2(\Pi)} + \big((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}\big)_{L^2(\Pi)} \\
&\qquad + \big((\mathbf{u} \cdot \nabla)\mathbf{w}, \boldsymbol{\varphi}\big)_{L^2(\Pi)} + \big((\mathbf{w} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}\big)_{L^2(\Pi)}\, dt \\
&= \int_0^T - (\mathbf{u}, \partial_t \boldsymbol{\varphi})_{L^2(\Pi_N)} + \nu\, (\nabla \mathbf{u}, \nabla \boldsymbol{\varphi})_{L^2(\Pi_N)} + \big((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}\big)_{L^2(\Pi_N)} \\
&\qquad + \big((\mathbf{u} \cdot \nabla)\mathbf{w}, \boldsymbol{\varphi}\big)_{L^2(\Pi_N)} + \big((\mathbf{w} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}\big)_{L^2(\Pi_N)}\, dt \\
&= \lim_{\substack{n\to\infty,\\ n \geq N}} \int_0^T - (\mathbf{u}_n, \partial_t \boldsymbol{\varphi})_{L^2(\Pi_N)} + \nu\, (\nabla \mathbf{u}_n, \nabla \boldsymbol{\varphi})_{L^2(\Pi_N)} + \big((\mathbf{u}_n \cdot \nabla)\mathbf{u}_n, \boldsymbol{\varphi}\big)_{L^2(\Pi_N)} \\
&\qquad + \big((\mathbf{u}_n \cdot \nabla)\mathbf{w}, \boldsymbol{\varphi}\big)_{L^2(\Pi_N)} + \big((\mathbf{w} \cdot \nabla)\mathbf{u}_n, \boldsymbol{\varphi}\big)_{L^2(\Pi_N)}\, dt \\
&\overset{(3.16)}{=} \int_0^T \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{W_{0,\sigma}^{1,2}(\Pi)', W_{0,\sigma}^{1,2}(\Pi)}\, dt\,.
\end{aligned}$$

Thereby, $\mathbf{u}$ satisfies the integral identity (3.1) regarding the whole layer domain Π. Also $\mathbf{u} \in L^2(0,T; W_{0,\sigma}^{1,2}(\Pi)) \cap L^\infty(0,T; L_\sigma^2(\Pi))$, meaning that $\mathbf{u}$ in fact is a weak solution.

Step 4: Energy inequality.
At last, we need to verify the energy inequality for $\mathbf{u}$. As noted before, estimate (3.15) is valid for $\mathbf{u}_n$, with Π instead of Π_n on the left-hand side. With this inequality at hand we are able to reason in the same way as before, in step 5 of the preceding proof, to get

$$\|\mathbf{u}(s)\|_{L^2(\Pi)}^2 + \nu \int_0^s \|\nabla \mathbf{u}\|_{L^2(\Pi)}^2\, d\tau \leq c_0\, \|\mathbf{f}\|_{L^2(0,T;(W_{0,\sigma}^{1,2}(\Pi))')}^2\,,$$

for almost all $s \in (0,T)$. This completes the proof. □

3.2 A solenoidal extension operator

First we prove a statement, which is essential for our derivation of a solenoidal extension operator in Lemma 3.6.

Lemma 3.5. *Let Ω be a bounded domain and suppose $\mathbf{v} \in W_{0,\sigma}^{1,2}(\Omega)$. Then, there exists a function $\mathbf{w} \in W^{2,2}(\Omega)$ such that*

$$\mathbf{v} = \nabla \times \mathbf{w}$$

almost everywhere. Furthermore, the following estimate holds:

$$\|\mathbf{w}\|_{W^{2,2}(\Omega')} \leq c\,\|\mathbf{v}\|_{W^{1,2}(\Omega)}$$

for any subdomain Ω' *with* $\overline{\Omega'} \subset \Omega$. *The constant* c *in the inequality depends only on* Ω' *and* Ω.

Proof. By $\Phi(x) := -\frac{1}{4\pi}\frac{1}{|x|}$ we denote the fundamental solution of Laplace's equation. Consider $\boldsymbol{\varphi} \in C^\infty_{0,\sigma}(\Omega)$ at first. The convolution

$$\begin{aligned}\mathbf{W}(x) := \Phi * \boldsymbol{\varphi}(x) &= \int_\Omega \Phi(x-y)\boldsymbol{\varphi}(y)\,dy\\ &= \int_{\mathbb{R}^3} \Phi(x-y)\boldsymbol{\varphi}(y)\,dy\\ &= \int_{\mathbb{R}^3} \Phi(y)\boldsymbol{\varphi}(x-y)\,dy\end{aligned}$$

defines a classical solution to Poisson's equation $-\Delta\mathbf{W} = \boldsymbol{\varphi}$ in Ω. Notice that $\boldsymbol{\varphi}$ was implicitly extended to $\mathbb{R}^3$. We also find that

$$\operatorname{div}\mathbf{W}(x) = \int_{\mathbb{R}^3} \Phi(y)\,\operatorname{div}\boldsymbol{\varphi}(x-y)\,dy = 0$$

for all $x \in \Omega$. Set $\mathbf{w} := \nabla \times \mathbf{W}$. Then, we obtain

$$\nabla \times \mathbf{w}(x) = \nabla \times \big(\nabla \times \mathbf{W}\big)(x) = \nabla(\operatorname{div}\mathbf{W})(x) - \Delta\mathbf{W}(x) = \boldsymbol{\varphi}(x)\,, \qquad x \in \Omega\,.$$

The stated inequality follows from potential theory. Since Φ and $\partial_i\Phi$ are weakly singular and $\partial_i\partial_j\Phi$ is a singular integral kernel, the Theorems of Sobolev and Calderón-Zygmund (cf. [11, Theorems II.11.2, II.11.3, III.11.4] or [24, pp. 21f]) contain the desired estimate: Let Ω' be an arbitrary subdomain of Ω, $\overline{\Omega'} \subset \Omega$, then

$$\|\mathbf{w}\|_{W^{2,2}(\Omega')} = \|\nabla \times \mathbf{W}\|_{W^{2,2}(\Omega')} \leq c\,\|\mathbf{W}\|_{W^{3,2}(\Omega')} \leq c(\Omega',\Omega)\,\|\boldsymbol{\varphi}\|_{W^{1,2}(\Omega)}\,.$$

Due to density, we find a sequence $(\boldsymbol{\varphi}_n)_n \subset C^\infty_{0,\sigma}(\Omega)$ converging to $\mathbf{v}$ in $W^{1,2}_{0,\sigma}(\Omega)$. Passage to the limit leads to

$$\|\mathbf{w}\|_{W^{2,2}(\Omega')} \leq c(\Omega',\Omega)\,\|\mathbf{v}\|_{W^{1,2}(\Omega)}$$

and

$$\nabla \times \mathbf{w} = \mathbf{v}\,.$$

□

Our next lemma was inspired by Lemma IX.4.2 of Galdi's monograph [11], constructing a - in some sense - controllable extension of a function from the trace space $W^{1/2,2}(\partial\Pi)$ having zero flux.

Lemma 3.6. *Let* $\Pi \subset \mathbb{R}^3$ *be the layer domain and* $\mathbf{a} \in W^{1/2,2}(\partial\Pi)$ *with bounded support* $\operatorname{supp} \mathbf{a} \subset \Gamma_{R_0}$*,* $R_0 > 0$*, and* $\int_{\partial\Pi} \mathbf{a} \cdot \mathbf{n}\, dS = 0$*. Let* $\eta > 0$*, then there exists an extension* $\mathbf{v} = \mathbf{v}(\eta) \in W^{1,2}(\Pi)$ *of* $\mathbf{a}$ *fulfilling*

$$\operatorname{div} \mathbf{v} = 0\,, \quad \|\mathbf{v}\|_{W^{1,2}(\Pi)} \le c\,\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)}\,, \quad \operatorname{supp} \mathbf{v} \subset \overline{C_{R_0+\lambda}}\,,$$

where c *depends on* R_0 *only and* $\lambda > \frac{\sqrt{17}}{2}$*. In particular, for all* $\mathbf{u} \in W^{1,2}_{0,\sigma}(\Pi)$*,*

$$\left(\int_\Pi |\mathbf{u}|^2 |\mathbf{v}|^2\, dx\right)^{1/2} \le \eta\,\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)}\,\|\nabla \mathbf{u}\|_{L^2(\Pi)}\,,$$

implying

$$\left|\int_\Pi (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}\, dx\right| \le \eta\,\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)}\|\nabla \mathbf{u}\|^2_{L^2(\Pi)}\,.$$

The extension operator

$$E_{\eta,R_0}\colon D(E_{\eta,R_0}) \to W^{1,2}(\Pi)\,,$$

for $D(E_{\eta,R_0}) := \left\{\mathbf{a} \in W^{1/2,2}(\partial\Pi) : \operatorname{supp}\mathbf{a} \subset \overline{\Gamma}_{R_0}\,,\ \int_{\partial\Pi} \mathbf{a}\cdot\mathbf{n}\, dS = 0\right\}$*, is linear and continuous.*

Remark. The domain $D(E_{\eta,R_0})$ of E_{η,R_0} is a closed subspace of $W^{1/2,2}(\partial\Pi)$, thus it is a Banach space as well.

Proof. At first we make use of an extension operator from Nečas's book [38, Lemma 2.5.6] coupled with a suitable partition of unity over $\partial\Pi$, to obtain the linear and continuous operator

$$E_0\colon W^{1/2,2}(\partial\Pi) \to W^{1,2}(\Pi)\,.$$

Setting $\mathbf{v}_0 := E_0(\mathbf{a})$, we find

$$\|\mathbf{v}_0\|_{W^{1,2}(\Pi)} \le c_0\,\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)}$$

and the support of $\mathbf{v}_0$ lies in $C_{R_0+2} \cup \Gamma_{R_0+2}$. Note that the constant c_0 in the inequality depends on Γ_{R_0} only.

The next step is to find a solenoidal vector field having the same trace as $\mathbf{v}_0$. To this end we employ the linear, continuous operator resulting from Bogovskiĭ's formula (see [7, Lemma 1] or [11, Lemma III.3.1]) to the scalar function $\operatorname{div}\mathbf{v}_0$. The cylinder C_{R_0+2} is a star-like, Lipschitz domain and $\operatorname{div}\mathbf{v}_0 \in L^2(C_{R_0+2})$. The requirement $\int_{C_{R_0+2}} \operatorname{div}\mathbf{v}_0\, dx = 0$ still needs to be checked. This, however, is a direct consequence of the assumptions on $\mathbf{a}$, due to Gauss's Theorem. Therefore, the Lemma yields a function $\tilde{\mathbf{v}}_1 \in W^{1,2}_0(C_{R_0+2})$ satisfying

$$\operatorname{div}\tilde{\mathbf{v}}_1 = \operatorname{div}\mathbf{v}_0$$

and, further,

$$\|\tilde{\mathbf{v}}_1\|_{W^{1,2}(C_{R_0+2})} \le \tilde{c}_1\,\|\operatorname{div}\mathbf{v}_0\|_{L^2(C_{R_0+2})} \le \tilde{c}_1\,\|\mathbf{v}_0\|_{W^{1,2}(C_{R_0+2})}\,,$$

with $\tilde{c}_1 = \tilde{c}_1(C_{R_0+2})$. The solenoidal vector field we were looking for is given by $\mathbf{v}_1 := \mathbf{v}_0 - \tilde{\mathbf{v}}_1$ and, hence,

$$\|\mathbf{v}_1\|_{W^{1,2}(C_{R_0+2})} \leq \|\mathbf{v}_0\|_{W^{1,2}(C_{R_0+2})} + \|\tilde{\mathbf{v}}_1\|_{W^{1,2}(C_{R_0+2})} \leq \underbrace{(1+\tilde{c}_1)}_{=:c_1} \|\mathbf{v}_0\|_{W^{1,2}(C_{R_0+2})} .$$

Let $R_0 + \frac{\sqrt{17}}{2} \leq R_1 < R_0 + \lambda$. Since

$$\begin{aligned}(R_0 + \tfrac{\sqrt{17}}{2})^2 &= \big((R_0+2) - 2 + \tfrac{\sqrt{17}}{2}\big)^2 \\ &= (R_0+2)^2 + 2\tfrac{\sqrt{17}-4}{2}(R_0+2) + (\tfrac{\sqrt{17}-4}{2})^2 \\ &> (R_0+2)^2 + 4\tfrac{\sqrt{17}-4}{2} + (\tfrac{\sqrt{17}-4}{2})^2 \\ &= (R_0+2)^2 + \tfrac{1}{4} \\ &= \max\{|x-(0,0,\tfrac{1}{2})|^2 : x \in \overline{C_{R_0+2}}\} ,\end{aligned}$$

we have $\overline{C_{R_0+2}} \subset \widetilde{B}_{R_1} = B_{R_1}\big((0,0,\frac{1}{2})\big)$. Next, we apply Corollary III.3.1 of [11] to extend the solenoidal function $\mathbf{v}_1$, receiving a function $\mathbf{v}_2 \in W_0^{1,2}(\widetilde{B}_{R_1})$, $\mathbf{v}_2 = \mathbf{v}_1$ in C_{R_0+2}, with

$$\operatorname{div} \mathbf{v}_2 = 0 \ \text{ in } \widetilde{B}_{R_1} , \quad \|\mathbf{v}_2\|_{W^{1,2}(\widetilde{B}_{R_1})} \leq c_2 \|\mathbf{v}_1\|_{W^{1,2}(C_{R_0+2})} ,$$

$c_2 = c_2(C_{R_0+2}, \widetilde{B}_{R_1})$.

Redefine $\mathbf{v}_2$ as the extension by 0 from $\widetilde{B}_{R_1}$ to $\widetilde{B}_{R_2}$, where $R_1 < R_2 < R_0 + \lambda$ and choose $R_1 < R_2' < R_2$. Then, $\mathbf{v}_2 \in W_{0,\sigma}^{1,2}(\widetilde{B}_{R_2})$ and by Lemma 3.5 we get $\mathbf{w} \in W^{2,2}(\widetilde{B}_{R_2})$ satisfying $\mathbf{v}_2 = \nabla \times \mathbf{w}$ in $\widetilde{B}_{R_2}$ and

$$\|\mathbf{w}\|_{W^{2,2}(\Omega')} \leq c_3 \|\mathbf{v}_2\|_{W^{1,2}(\widetilde{B}_{R_2})} = c_3 \|\mathbf{v}_2\|_{W^{1,2}(\widetilde{B}_{R_1})}$$

for any domain Ω' with $\overline{\Omega'} \subset \widetilde{B}_{R_2}$, $c_3 = c_3(\Omega', \widetilde{B}_{R_2})$. Let $\mathbf{w} := 0$ outside of $\widetilde{B}_{R_2}$, such that $\mathbf{w} \colon \mathbb{R}^3 \to \mathbb{R}^3$.

To get hold of the integral estimate we need a cut-off function limiting the extension to a strip near the boundary of Π. Therefore, we define $\psi_\varepsilon \colon \Pi \to [0,1]$, $\varepsilon > 0$, as the cut-off function from Lemma III.6.2 of [11], being a type of regularized distance with respect to the boundary $\partial\Pi$. In particular,

$$\psi_\varepsilon(x) = 1 \ \text{ for } \ \delta(x) < \tfrac{1}{2\kappa_1} e^{-2/\varepsilon} , \quad \psi_\varepsilon(x) = 0 \ \text{ for } \ \delta(x) \geq 2e^{-1/\varepsilon}$$

and $|\nabla\psi_\varepsilon(x)| \leq \frac{\kappa_2 \varepsilon}{\delta(x)}$, where $\delta(x) = \operatorname{dist}(x, \partial\Pi)$. Furthermore, $\psi_\varepsilon \in C^\infty(\overline{\Pi})$. Let ϕ be another smooth cut-off function fulfilling $0 \leq \phi \leq 1$,

$$\phi(x) = 1 \ \text{ for } \ x \in \widetilde{B}_{R_1} , \quad \phi(x) = 0 \ \text{ for } \ x \in \mathbb{R}^3 \backslash \widetilde{B}_{R_2'} .$$

Since $\phi \in C_0^\infty(\mathbb{R}^3)$, the gradient $\nabla\phi$ is bounded, so $|\nabla\phi| \leq M$ for some constant $M = M(\widetilde{B}_{R_2'}) > 0$.

Now we are in a position to the define the actual extension function and verify its stated properties. Set

$$\mathbf{v}_\varepsilon \colon \Pi \to \mathbb{R}^3 , \quad \mathbf{v}_\varepsilon(x) := \nabla \times \big(\psi_\varepsilon \, \phi \, \mathbf{w}\big)(x) .$$

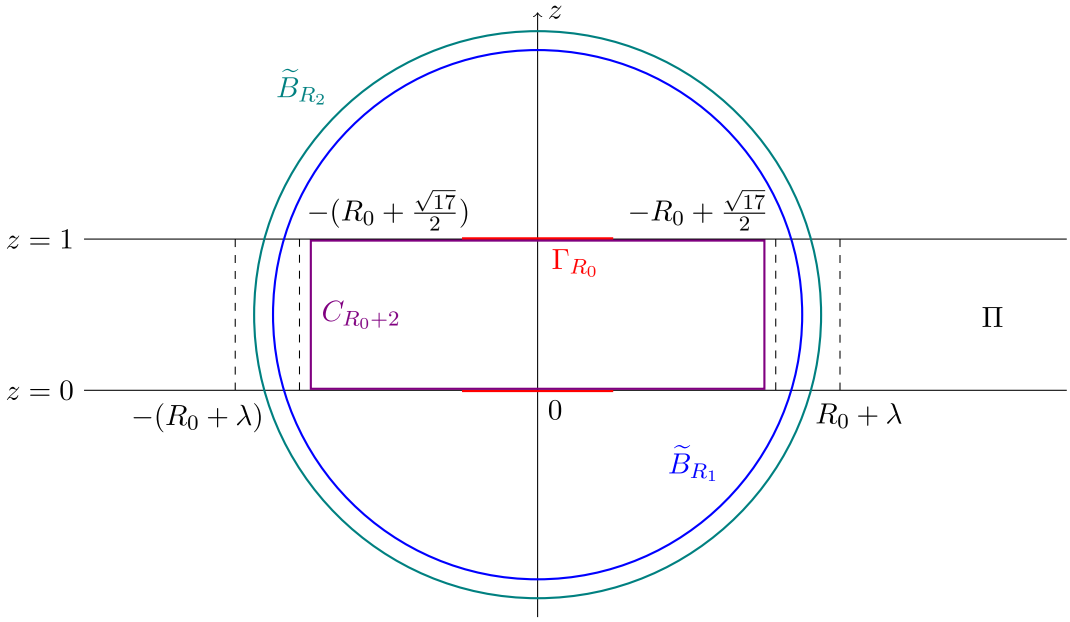

Figure 3.1: Stepwise expansion of domains in a 2D cross-section.

By definition we obtain directly:

$$\mathbf{v}_\varepsilon \in W^{1,2}(\Pi)\,, \quad \operatorname{div} \mathbf{v}_\varepsilon = 0\,, \quad \mathbf{v}_\varepsilon|_{\partial\Pi} = \mathbf{a}$$

and

$$\operatorname{supp} \mathbf{v}_\varepsilon \subset \overline{\Pi} \cap \widetilde{B}_{R_2'} \subset \overline{C_{R_2}} \subset \overline{C_{R_0+\lambda}}\,.$$

Let $\mathbf{u} \in W^{1,2}_{0,\sigma}(\Pi)$. We notice

$$\begin{aligned}
\left| \int_\Pi (\mathbf{u} \cdot \nabla)\, \mathbf{v}_\varepsilon \cdot \mathbf{u}\, dx \right| &= \left| \int_{C_{R_2}} (\mathbf{u} \cdot \nabla)\, \mathbf{v}_\varepsilon \cdot \mathbf{u}\, dx \right| \\
&\overset{(2.5)}{=} \left| \int_{C_{R_2}} (\mathbf{u} \cdot \nabla)\, \mathbf{u} \cdot \mathbf{v}_\varepsilon\, dx \right| \\
&= \left| \int_\Pi (\mathbf{u} \cdot \nabla)\, \mathbf{u} \cdot \mathbf{v}_\varepsilon\, dx \right| \\
&\le \|\nabla \mathbf{u}\|_{L^2(\Pi)} \left(\int_\Pi |\mathbf{u}|^2 |\mathbf{v}_\varepsilon|^2\, dx \right)^{1/2}
\end{aligned}$$

and, hence, it suffices to investigate an appropriate estimate for $\int_\Pi |\mathbf{u}|^2 |\mathbf{v}_\varepsilon|^2\, dx$. Let

$x \in \Pi$, and $\Pi_\varepsilon := \Pi_{\varepsilon,R_2'} := \{x \in \Pi : \delta(x) < 2e^{-1/\varepsilon}\} \cap \widetilde{B}_{R_2'}$. At first we consider

$$\begin{aligned}
|\mathbf{v}_\varepsilon(x)| &= |\nabla \times (\psi_\varepsilon \phi \mathbf{w})| \\
&\le |\nabla(\psi_\varepsilon\phi) \times \mathbf{w}| + |\psi_\varepsilon\phi\,(\nabla \times \mathbf{w})| \\
&\le |\nabla(\psi_\varepsilon\phi)||\mathbf{w}| + |\psi_\varepsilon||\phi||\nabla\mathbf{w}| \\
&\le |\nabla\psi_\varepsilon||\phi||\mathbf{w}| + |\psi_\varepsilon||\nabla\phi||\mathbf{w}| + |\psi_\varepsilon||\phi||\nabla\mathbf{w}| \\
&\le \begin{cases} \frac{\varepsilon\kappa_2}{\delta(x)}|\mathbf{w}(x)| + M|\mathbf{w}(x)| + |\nabla\mathbf{w}(x)|\,, & x \in \Pi_\varepsilon\,, \\ 0\,, & x \in \Pi\backslash\Pi_\varepsilon\,. \end{cases}
\end{aligned}$$

Thereby, we infer

$$\begin{aligned}
&\left(\int_\Pi |\mathbf{u}|^2|\mathbf{v}_\varepsilon|^2\,dx\right)^{1/2} \\
&= \left(\int_{\Pi_\varepsilon} |\mathbf{u}|^2|\mathbf{v}_\varepsilon|^2\,dx\right)^{1/2} \\
&\le \left(\int_{\Pi_\varepsilon} \left(\frac{\varepsilon\kappa_2}{\delta}|\mathbf{w}| + M|\mathbf{w}| + |\nabla\mathbf{w}|\right)^2 |\mathbf{u}|^2\,dx\right)^{1/2} \\
&\le c\left(\int_{\Pi_\varepsilon} \left(\frac{\varepsilon\kappa_2}{\delta}\right)^2 |\mathbf{w}|^2|\mathbf{u}|^2\,dx\right)^{1/2} + \tilde{c}\left(\int_{\Pi_\varepsilon} \left(|\mathbf{w}|^2 + |\nabla\mathbf{w}|^2\right)|\mathbf{u}|^2\,dx\right)^{1/2} \\
&\le c\,\kappa_2\varepsilon\left(\int_{\Pi_\varepsilon} \|\mathbf{w}\|^2_{W^{2,2}(\Pi_\varepsilon)}|\mathbf{u}\delta^{-1}|^2\,dx\right)^{1/2} \\
&\qquad + \tilde{c}\left(\|\mathbf{w}\|^2_{L^3(\Pi_\varepsilon)} + \|\nabla\mathbf{w}\|^2_{L^3(\Pi_\varepsilon)}\right)^{1/2}\|\mathbf{u}\|_{L^6(\Pi_\varepsilon)} \\
&\le c\,\kappa_2\varepsilon\,\|\mathbf{w}\|_{W^{2,2}(\widetilde{B}_{R_2'})}\,\|\mathbf{u}\delta^{-1}\|_{L^2(\Pi)} + \tilde{c}\,\|\nabla\mathbf{w}\|_{L^3(\Pi_\varepsilon)}\,\|\mathbf{u}\|_{W^{1,2}(\Pi)} \\
&\le c\,\kappa_2\varepsilon\,\|\mathbf{a}\|_{W^{1/2,2}(\Gamma_{R_0})}\,\|\nabla\mathbf{u}\|_{L^2(\Pi)} + \tilde{c}\,|\Pi_\varepsilon|^{1/12}\,\|\nabla\mathbf{w}\|_{L^4(\Pi_\varepsilon)}\,\|\mathbf{u}\|_{W^{1,2}(\Pi)} \\
&\le c\,\kappa_2\varepsilon\,\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)}\,\|\nabla\mathbf{u}\|_{L^2(\Pi)} \\
&\qquad + \tilde{c}\left(2\cdot\pi {R_2}^2\cdot 2e^{-1/\varepsilon}\right)^{1/12}\,\|\mathbf{w}\|_{W^{2,2}(\widetilde{B}_{R_2'})}\,\|\nabla\mathbf{u}\|_{L^2(\Pi)} \\
&\le C\,\varepsilon\,\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)}\,\|\nabla\mathbf{u}\|_{L^2(\Pi)}\,,
\end{aligned}$$

due to the Sobolev embeddings $W^{2,2}(\Pi_\varepsilon) \hookrightarrow C^0(\Pi_\varepsilon)$ and $W^{2,2}(\Pi_\varepsilon) \hookrightarrow W^{1,4}(\Pi_\varepsilon)$, and the elementary estimate $-\frac{1}{12\varepsilon} \le \ln(12\varepsilon)$. Moreover,

$$\begin{aligned}
&\|\mathbf{u}\delta^{-1}\|^2_{L^2(\Pi)} \\
&= \sum_{i=1}^3 \left(\int_{\Pi_+} |u_i|^2\,\delta^{-2}\,dx + \int_{\Pi_-} |u_i|^2\,\delta^{-2}\,dx\right) \\
&= \sum_{i=1}^3 \left(\int_{\mathbb{R}^2}\int_{\frac12}^1 |u_i|^2\,(1-z)^{-2}\,dz\,d(y_1,y_2) + \int_{\mathbb{R}^2}\int_0^{\frac12} |u_i|^2\,z^{-2}\,dz\,d(y_1,y_2)\right) \\
&\le 2\sum_{i=1}^3 \left(\int_{\mathbb{R}^2}\int_{\frac12}^1 |\partial_z u_i|^2\,dz\,d(y_1,y_2) + \int_{\mathbb{R}^2}\int_0^{\frac12} |\partial_z u_i|^2\,dz\,d(y_1,y_2)\right) \\
&\le 2\,\|\nabla\mathbf{u}\|^2_{L^2(\Pi)}
\end{aligned}$$

is a consequence of Hardy's inequality

$$\int_0^\infty |f(s)|^2 s^{-2}\, ds \le 2 \int_0^\infty |f'(s)|^2\, ds\,, \quad f \in C_0^\infty(\mathbb{R}_+)\,,$$

since $\mathbf{u} \in W_0^{1,2}(\Pi)$. The subdomains Π_+ and Π_- above denote the upper respectively the lower half of Π. The constant C depends only on $\widetilde{B}_{R_2}$, $\widetilde{B}_{R_2'}$, $\widetilde{B}_{R_1}$, C_{R_0+2}, Γ_{R_0} and R_2, R_2', R_1 were successively derived from R_0. So, we essentially find

$$C = C(\widetilde{B}_{R_2}, \widetilde{B}_{R_2'}, \widetilde{B}_{R_1}, C_{R_0+2}, \Gamma_{R_0}) = C(R_0)\,.$$

Fixing $\varepsilon_0 \le \frac{\eta}{C}$, we altogether conclude:

$$\begin{aligned}
\left| \int_\Pi (\mathbf{u}\cdot\nabla)\,\mathbf{v}_{\varepsilon_0}\cdot\mathbf{u}\,dx \right| &\le \|\nabla\mathbf{u}\|_{L^2(\Pi)} \left(\int_\Pi |\mathbf{u}|^2 |\mathbf{v}_{\varepsilon_0}|^2\,dx \right)^{1/2} \\
&\le \|\nabla\mathbf{u}\|_{L^2(\Pi)}\; C\frac{\eta}{C} \|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)} \|\nabla\mathbf{u}\|_{L^2(\Pi)} \\
&= \eta\, \|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)} \|\nabla\mathbf{u}\|^2_{L^2(\Pi)}\,.
\end{aligned}$$

To justify continuity of the operator E_{η,R_0}, we prove that $\|\mathbf{v}_{\varepsilon_0}\|_{W^{1,2}(\Pi)}$ can be estimated by $\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)}$:

$$\begin{aligned}
\|\mathbf{v}_{\varepsilon_0}\|_{W^{1,2}(\Pi)} &= \|\mathbf{v}_{\varepsilon_0}\|_{W^{1,2}(\Pi_{\varepsilon_0})} \\
&= \|\nabla\times(\psi_{\varepsilon_0}\phi\mathbf{w})\,\|_{W^{1,2}(\Pi_{\varepsilon_0})} \\
&\le \|\nabla(\psi_{\varepsilon_0}\phi)\times\mathbf{w}\|_{W^{1,2}(\Pi_{\varepsilon_0})} + \|\psi_{\varepsilon_0}\phi\,(\nabla\times\mathbf{w})\|_{W^{1,2}(\Pi_{\varepsilon_0})} \\
&\le c\,\|\psi_{\varepsilon_0}\phi\|_{C^2(\Pi_\varepsilon)} \Big(\|\mathbf{w}\|_{W^{1,2}(\Pi_{\varepsilon_0})} + \|\nabla\mathbf{w}\|_{W^{1,2}(\Pi_{\varepsilon_0})} \Big) \\
&\le c\,\|\mathbf{w}\|_{W^{2,2}(\widetilde{B}_{R_2'})} \\
&\le c\,c_3\,c_2\,c_1\,c_0\,\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)}\,.
\end{aligned}$$

Thus, $\mathbf{v} := \mathbf{v}_{\varepsilon_0}$ is an extension with the asserted properties. □

3.3 Existence in the case of zero flux

With the preparatory results of Sections 3.1 and 3.2 we are now able to investigate the time-periodic Navier-Stokes equations with inhomogeneous boundary condition:

$$\begin{aligned}
\partial_t\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } [0,T]\times\Pi\,, \\
\operatorname{div}\mathbf{u} &= 0 && \text{in } [0,T]\times\Pi\,, \\
\mathbf{u}|_{\Gamma_{R_0}} &= \mathbf{a} && \text{for all } t\in[0,T]\,, \\
\mathbf{u}|_{\partial\Pi\setminus\Gamma_{R_0}} &= 0 && \text{for all } t\in[0,T]\,, \\
\mathbf{u}|_{t=0} &= \mathbf{u}|_{t=T} && \text{in } \Pi\,.
\end{aligned} \qquad (\mathrm{NS_{bc}})$$

For the definition of a corresponding weak solution, we need an appropriate extension of the boundary value $\mathbf{a}$, which is achieved with the help of Lemma 3.6.

Let $\mathbf{a}$ be a T-periodic function in $W^{1,2}(0,T;W^{1/2,2}(\partial\Pi))$. We suppose that $\mathbf{a}$ has bounded support $\operatorname{supp}\mathbf{a}(t)\subset\Gamma_{R_0}$ for all $t\in[0,T]$, $R_0>0$, and satisfies the zero flux condition:

$$F(t)=-\int_{\partial\Pi}\mathbf{a}(t)\cdot\mathbf{n}\,dS=0 \quad \text{f.a.a. } t\,, \tag{3.18}$$

with $\mathbf{n}$ the unit outer normal vector to $\partial\Pi$. Due to a Sobolev-type embedding (see Theorem 2.17, or [48, Lemma 7.1] for a direct version), we find $\mathbf{a}\in C([0,T];W^{1/2,2}(\partial\Pi))$. Hence,

$$\alpha:=\|\mathbf{a}\|_{C([0,T];W^{1/2,2}(\partial\Pi))}=\sup_{t\in[0,T]}\|\mathbf{a}(t)\|_{W^{1/2,2}(\partial\Pi)}<\infty\,.$$

Let E_{η,R_0} be the extension operator of Lemma 3.6 with η set to $\frac{\nu}{4\alpha}$. We define $\mathbf{A}_0(t):=E_{\eta,R_0}(\mathbf{a}(t))$ for all $t\in[0,T]$, implying $\|\mathbf{A}_0(t)\|_{W^{1,2}(\Pi)}\le c\,\|\mathbf{a}(t)\|_{W^{1/2,2}(\partial\Pi)}$. Thereby, $\mathbf{A}_0$ has the following properties:

$$\mathbf{A}_0\in L^2(0,T;W^{1,2}(\Pi))\,,\quad \operatorname{div}\mathbf{A}_0(t)=0 \ \text{ in } \Pi\,,\quad \operatorname{supp}\mathbf{A}_0\subset\overline{C_{R_0+\lambda}}\,,$$

for $\lambda>\frac{\sqrt{17}}{2}$, and

$$\left|\int_\Pi(\mathbf{v}\cdot\nabla)\mathbf{A}_0(t)\cdot\mathbf{v}\,dx\right|\le\frac{\nu}{4\alpha}\|\mathbf{a}(t)\|_{W^{1/2,2}(\partial\Pi)}\|\nabla\mathbf{v}\|^2_{L^2(\Pi)}\le\frac{\nu}{4}\|\nabla\mathbf{v}\|^2_{L^2(\Pi)}\,,$$

for all $\mathbf{v}\in W^{1,2}_{0,\sigma}(\Pi)$, $t\in[0,T]$. Obviously $\mathbf{A}_0$ is T-periodic too. Regularity of the time-derivative $\partial_t\mathbf{A}_0$ follows due to Lemma 2.15, exploiting linearity and continuity of E_{η,R_0}:

$$\begin{aligned}
-\int_0^T E_{\eta,R_0}(\mathbf{a})\varphi'\,dt &= -\int_0^T E_{\eta,R_0}(\mathbf{a}\varphi')\,dt\\
&= E_{\eta,R_0}\Big(-\int_0^T\mathbf{a}\varphi'\,dt\Big)\\
&= E_{\eta,R_0}\Big(\int_0^T(\partial_t\mathbf{a})\varphi\,dt\Big)\\
&= \int_0^T E_{\eta,R_0}\big((\partial_t\mathbf{a})\varphi\big)\,dt\\
&= \int_0^T E_{\eta,R_0}(\partial_t\mathbf{a})\varphi\,dt
\end{aligned}$$

for all $\varphi\in C_0^\infty((0,T))$, therefore $\partial_t\mathbf{A}_0=\partial_tE_{\eta,R_0}(\mathbf{a})=E_{\eta,R_0}(\partial_t\mathbf{a})$. We conclude $\partial_t\mathbf{A}_0\in L^2(0,T;W^{1,2}(\Pi))$ as well as $\|\partial_t\mathbf{A}_0(t)\|_{W^{1,2}(\Pi)}\le c\,\|\partial_t\mathbf{a}(t)\|_{W^{1/2,2}(\partial\Pi)}$.

With this explicit extension at hand, we note:

Definition 3.7. *Let* $\mathbf{f}\in L^2(0,T;W^{1,2}_{0,\sigma}(\Pi)')$ *be* T*-periodic and* $\mathbf{A}_0$ *the extension of* $\mathbf{a}$ *derived above. We call a function* $\mathbf{u}\in L^2(0,T;W^{1,2}(\Pi))$ a weak solution *of the inhomogeneous Navier-Stokes equations* $(\mathrm{NS_{bc}})$ *with zero flux, if*

$$\mathbf{u}-\mathbf{A}_0\in L^2(0,T;W^{1,2}_{0,\sigma}(\Pi))$$

and if the integral identity

$$\int_0^T -(\mathbf{u}, \partial_t \boldsymbol{\varphi}) + \nu\,(\nabla\mathbf{u}, \nabla\boldsymbol{\varphi}) + \big((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}\big)\,dt = \int_0^T \langle \mathbf{f}, \boldsymbol{\varphi}\rangle\,dt$$

is satisfied for all time-periodic test functions $\boldsymbol{\varphi} \in C^\infty([0,T]; C^\infty_{0,\sigma}(\Pi))$.

Now, this notion enables us to tackle the question of existence.

Theorem 3.8. *Let* $\mathbf{f} \in L^2(0,T; W^{1,2}_{0,\sigma}(\Pi)')$ *and* $\mathbf{a} \in W^{1,2}(0,T; W^{1/2,2}(\partial\Pi))$ *be* T*-periodic,* $\mathbf{A}_0 \in W^{1,2}(0,T; W^{1,2}(\Pi))$ *the extension of* $\mathbf{a}$*. Then, there exists a weak solution* $\mathbf{u}$ *of the time-periodic Navier-Stokes equations* ($\mathrm{NS}_{\mathrm{bc}}$) *satisfying an a-priori estimate:*

$$\begin{aligned}
&\|\mathbf{u}\|^2_{L^\infty(L^2(\Pi))} + \nu\|\nabla\mathbf{u}\|^2_{L^2(L^2(\Pi))} \\
&\quad \le c\left(\|\mathbf{f}\|^2_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} + \|\mathbf{a}\|^2_{W^{1,2}(W^{1/2,2}(\partial\Pi))} + \|\mathbf{a}\|^4_{W^{1,2}(W^{1/2,2}(\partial\Pi))}\right).
\end{aligned}$$

Proof. We reformulate the Navier-Stokes equations by demanding $\mathbf{u} = \mathbf{u}^* + \mathbf{A}_0$, which leads to the perturbed system

$$\begin{aligned}
\partial_t\mathbf{u}^* - \nu\Delta\mathbf{u}^* + \big((\mathbf{u}^* + \mathbf{A}_0)\cdot\nabla\big)\mathbf{u}^* + (\mathbf{u}^*\cdot\nabla)\mathbf{A}_0 + \nabla p &= \mathbf{f}^* && \text{in } [0,T]\times\Pi\,,\\
\operatorname{div}\mathbf{u}^* &= 0 && \text{in } [0,T]\times\Pi\,,\\
\mathbf{u}^*|_{\partial\Pi} &= 0 && \text{for all } t\in[0,T]\,,\\
\mathbf{u}^*|_{t=0} &= \mathbf{u}^*|_{t=T} && \text{in } \Pi\,,
\end{aligned}$$

with $\mathbf{f}^* = \mathbf{f} - \partial_t\mathbf{A}_0 + \nu\Delta\mathbf{A}_0 - (\mathbf{A}_0\cdot\nabla)\mathbf{A}_0$. We examine the functionals $\Delta\mathbf{A}_0(t)\colon W^{1,2}_{0,\sigma}(\Pi) \to \mathbb{R}$ and $(\mathbf{A}_0\cdot\nabla)\mathbf{A}_0(t)\colon W^{1,2}_{0,\sigma}(\Pi) \to \mathbb{R}$, where

$$\begin{aligned}
\langle \Delta\mathbf{A}_0, \mathbf{v}\rangle &:= -\int_\Pi \nabla\mathbf{A}_0\cdot\nabla\mathbf{v}\,dx\,,\\
\langle (\mathbf{A}_0\cdot\nabla)\mathbf{A}_0, \mathbf{v}\rangle &:= \big((\mathbf{A}_0\cdot\nabla)\mathbf{A}_0, \mathbf{v}\big) = \int_\Pi (\mathbf{A}_0\cdot\nabla)\mathbf{A}_0\cdot\mathbf{v}\,dx\,.
\end{aligned}$$

Since

$$\begin{aligned}
\|\Delta\mathbf{A}_0(t)\|_{W^{1,2}_{0,\sigma}(\Pi)'} &= \sup_{\|\mathbf{v}\|_{W^{1,2}}=1} |\langle \Delta\mathbf{A}_0(t), \mathbf{v}\rangle| \\
&\le \sup_{\|\mathbf{v}\|_{W^{1,2}}=1} \|\nabla\mathbf{A}_0(t)\|_{L^2(\Pi)}\|\nabla\mathbf{v}\|_{L^2(\Pi)} \\
&\le \|\mathbf{A}_0(t)\|_{W^{1,2}(\Pi)} \\
&\le c\,\|\mathbf{a}(t)\|_{W^{1/2,2}(\partial\Pi)}\,,
\end{aligned}$$

it follows

$$\|\Delta\mathbf{A}_0\|_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} \le c\,\|\mathbf{a}\|_{L^2(W^{1/2,2}(\partial\Pi))}\,.$$

Further, we calculate

$$\begin{aligned}
\|(\mathbf{A}_0\cdot\nabla)\mathbf{A}_0(t)\|_{W^{1,2}_{0,\sigma}(\Pi)'} &= \sup_{\|\mathbf{v}\|_{W^{1,2}}=1} |\langle(\mathbf{A}_0\cdot\nabla)\mathbf{A}_0(t),\mathbf{v}\rangle| \\
&\overset{(2.5)}{=} \sup_{\|\mathbf{v}\|_{W^{1,2}}=1} \left|\int_\Pi (\mathbf{A}_0(t)\cdot\nabla)\mathbf{v}\cdot\mathbf{A}_0(t)\,dx\right| \\
&\le \sup_{\|\mathbf{v}\|_{W^{1,2}}=1} \int_\Pi |\mathbf{A}_0(t)||\nabla\mathbf{v}||\mathbf{A}_0(t)|\,dx \\
&\le \sup_{\|\mathbf{v}\|_{W^{1,2}}=1} \|\mathbf{A}_0(t)\|^2_{L^4(\Pi)}\|\nabla\mathbf{v}\|_{L^2(\Pi)} \\
&\le c\,\|\mathbf{A}_0(t)\|^2_{W^{1,2}(\Pi)} \\
&\le c\,\|\mathbf{a}(t)\|^2_{W^{1/2,2}(\partial\Pi)}\,,
\end{aligned}$$

giving us the estimate

$$\|(\mathbf{A}_0\cdot\nabla)\mathbf{A}_0\|_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} \le c\,\|\mathbf{a}\|^2_{L^4(W^{1/2,2}(\partial\Pi))} \le c\,\|\mathbf{a}\|^2_{W^{1,2}(W^{1/2,2}(\partial\Pi))}\,.$$

Therefore $\mathbf{f}^* \in L^2(0,T;W^{1,2}_{0,\sigma}(\Pi)')$. Checking closely, we find that all assumptions of Theorem 3.4 are satisfied and receive a weak solution

$$\mathbf{u}^* \in L^2(0,T;W^{1,2}_{0,\sigma}(\Pi))\cap L^\infty(0,T;L^2_\sigma(\Pi))$$

of the perturbed Navier-Stokes equations. Then $\mathbf{u}=\mathbf{u}^*+\mathbf{A}_0$ is a weak solution of the inhomogeneous problem, as described in Definition 3.7.

With the weak solution $\mathbf{u}^*$ of the perturbed system, we also obtain inequality (3.14). Thereby, we conclude

$$\begin{aligned}
&\|\mathbf{u}^*\|^2_{L^\infty(L^2(\Pi))} + \nu\|\nabla\mathbf{u}^*\|^2_{L^2(L^2(\Pi))} \\
&\quad\le c_0\,\|\mathbf{f}^*\|^2_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} \\
&\quad\le c_0\Big(\|\mathbf{f}\|_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} + \|\partial_t\mathbf{A}_0\|_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} + \nu\,\|\Delta\mathbf{A}_0\|_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} \\
&\qquad\quad + \|(\mathbf{A}_0\cdot\nabla)\mathbf{A}_0\|_{L^2(W^{1,2}_{0,\sigma}(\Pi)')}\Big)^2 \\
&\quad\le c\Big(\|\mathbf{f}\|_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} + \|\partial_t\mathbf{A}_0\|_{L^2(W^{1,2}(\Pi))} + \nu\,\|\mathbf{a}\|_{L^2(W^{1/2,2}(\partial\Pi))} \\
&\qquad\quad + \|\mathbf{a}\|^2_{W^{1,2}(W^{1/2,2}(\partial\Pi))}\Big)^2 \\
&\quad\le c\left(\|\mathbf{f}\|^2_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} + \|\mathbf{a}\|^2_{W^{1,2}(W^{1/2,2}(\partial\Pi))} + \|\mathbf{a}\|^4_{W^{1,2}(W^{1/2,2}(\partial\Pi))}\right),
\end{aligned}$$

leading to the a-priori estimate

$$\begin{aligned}
&\|\mathbf{u}\|^2_{L^\infty(L^2(\Pi))} + \nu\|\nabla\mathbf{u}\|^2_{L^2(L^2(\Pi))} \\
&\quad\le 2\left(\|\mathbf{u}^*\|^2_{L^\infty(L^2(\Pi))} + \nu\|\nabla\mathbf{u}^*\|^2_{L^2(L^2(\Pi))}\right) + 2\,\|\mathbf{A}_0\|^2_{L^\infty(L^2(\Pi))} + 2\nu\,\|\nabla\mathbf{A}_0\|^2_{L^2(L^2(\Pi))} \\
&\quad\le c\left(\|\mathbf{f}\|^2_{L^2(W^{1,2}_{0,\sigma}(\Pi)')} + \|\mathbf{a}\|^2_{W^{1,2}(W^{1/2,2}(\partial\Pi))} + \|\mathbf{a}\|^4_{W^{1,2}(W^{1/2,2}(\partial\Pi))}\right).
\end{aligned}$$

□

3.4 A flux-carrying extension

Our next goal is to find an appropriate flux driver function to compensate a non-zero balance of flow through the surface $\partial\Pi$. The flux driver $\mathbf{A}_F$ we construct is an extension of the boundary value $\mathbf{a} \in L^2(0,T;W^{3/2,2}(\partial\Pi)) \cap W^{1,2}(0,T;W^{1/2,2}(\partial\Pi))$ with support in Γ_{R_0} and consists of three components.

The first one being the function $\mathbf{D}$, which describes the asymptotic behavior we assume a solution to have and thereby in particular transports the flux

$$F(t) = -\int_{\partial\Pi} \mathbf{a}(t)\cdot\mathbf{n}\,dS\,.$$

In [42] Pileckas and Specovius-Neugebauer develop an asymptotic expansion of a solution to the time-periodic Stokes system in Π. Therein, the flux-carrying term is identified to be

$$\mathbf{D}(t,x) := \left(w^{(0,-)}(t,z)\,\nabla_y P^{(0,-)}(y), 0\right)^T\,,$$

where $P^{(0,-)}$ is the harmonic function $P^{(0,-)}(y) = -\frac{1}{2\pi}\ln r$, hence $\nabla_y P^{(0,-)}(y) = -\frac{1}{2\pi}\frac{y}{r^2}$, and $w^{(0,-)}$ is part of the unique solution $\left(w^{(0,-)}, s^{(0,-)}\right)$ of the inverse heat equation

$$\begin{aligned}
\partial_t w - \partial_z^2 w &= s && \text{in } [0,T]\times(0,1)\,,\\
w|_{z=0} = w|_{z=1} &= 0 && \text{in } [0,T]\,,\\
w|_{t=0} &= w|_{t=T} && \text{in } (0,1)\,,\\
\int_0^1 w\,dz &= F && \text{in } [0,T]\,,
\end{aligned}$$

for prescribed F. This problem is thoroughly investigated by Galdi and Robertson in [10]. Especially, the following a-priori estimate holds:

$$\begin{aligned}
&\|w^{(0,-)}\|^2_{L^\infty(W^{1,2}(0,1))} + \int_0^T \|\partial_t w^{(0,-)}\|^2_{L^2(0,1)} + \|\partial_z^2 w^{(0,-)}\|^2_{L^2(0,1)} + |s^{(0,-)}|^2\,dt\\
&\quad\le c\int_0^T |F|^2 + |F'|^2\,dt\,.
\end{aligned} \tag{3.19}$$

We collect some properties of $\mathbf{D}$. At first, as desired, it carries the flux, meaning f.a.a. t

$$\begin{aligned}
\int_{S_R} \mathbf{D}(t)\cdot\mathbf{n}\,dS &= \int_{S_R} \left(w^{(0,-)}(t,z)\nabla_y P^{(0,-)}(y),0\right)^T \cdot \left(\frac{y_1}{R},\frac{y_2}{R},0\right)^T dx\\
&= -\frac{1}{2\pi}\int_{S_R} w^{(0,-)}(t,z)\frac{y_1^2+y_2^2}{R^3}\,dx\\
&= -\frac{1}{2\pi}\int_0^1 w^{(0,-)}(t,z)\,dz\int_0^{2\pi}\frac{1}{R}R\,d\theta\\
&= -F(t)\,.
\end{aligned}$$

Furthermore, $\mathbf{D}|_{\partial\Pi} = 0$ in $[0,T]$ and

$$\begin{aligned}
\partial_t\mathbf{D} - \Delta\mathbf{D} &= (\partial_t w^{(0,-)} - \partial_z^2 w^{(0,-)})\left(\nabla_y P^{(0,-)},0\right)^T - w^{(0,-)}\left(\Delta_y\nabla_y P^{(0,-)},0\right)^T\\
&= \left(s^{(0,-)}\nabla_y P^{(0,-)},0\right)^T\\
&= s^{(0,-)}\nabla_x P^{(0,-)}\,,
\end{aligned} \tag{3.20}$$

as well as

$$\begin{aligned}\operatorname{div}\mathbf{D} &= \partial_{y_1}\left(w^{(0,-)}\partial_{y_1}P^{(0,-)}\right) + \partial_{y_2}\left(w^{(0,-)}\partial_{y_2}P^{(0,-)}\right)\\ &= w^{(0,-)}\Delta_y P^{(0,-)}\\ &= 0\end{aligned}$$

in $[0,T]\times\Pi$.

The next component is an extension of $\mathbf{a}$ from $\partial\Pi$ to Π. Let E_0 be the linear, continuous extension operator $W^{3/2,2}(\partial\Pi)\to W^{2,2}(\Pi)$ based on Lemma 2.5.6 of Nečas's book [38] combined with a partition of unity corresponding to squares covering $\Gamma_{R_0}\supset\operatorname{supp}(\mathbf{a})$. We have the pointwise estimate

$$\|E_0(\mathbf{a}(t))\|_{W^{2,2}(\Pi)} \le c(\Gamma_{R_0})\|\mathbf{a}(t)\|_{W^{3/2,2}(\partial\Pi)} \tag{3.21}$$

for almost all t and the support of $E_0(\mathbf{a})$ is contained in $\overline{C_{R_0+2}}$. The explicit construction of the extension allows for an additional estimate:

$$\|E_0(\mathbf{a}(t))\|_{W^{1,2}(\Pi)} \le c(\Gamma_{R_0})\|\mathbf{a}(t)\|_{W^{1/2,2}(\partial\Pi)}\,. \tag{3.22}$$

Due to linearity and continuity, we further conclude $\partial_t E_0(\mathbf{a}) = E_0(\partial_t\mathbf{a})$ (cf. Section 3.3 for an analogous derivation of the time derivative of $\mathbf{A}_0$). Hence,

$$\|\partial_t E_0(\mathbf{a}(t))\|_{W^{1,2}(\Pi)} \le c(\Gamma_{R_0})\|\partial_t\mathbf{a}(t)\|_{W^{1/2,2}(\partial\Pi)}\,. \tag{3.23}$$

Altogether this furnishes: $E_0(\mathbf{a})\in L^2(0,T;W^{2,2}(\Pi))\cap W^{1,2}(0,T;W^{1,2}(\Pi))$.

The purpose of the third component is to establish a divergence-free extension. Consider the differential equation

$$\begin{aligned}\operatorname{div}(\mathbf{g}) &= \operatorname{div}(E_0(\mathbf{a})) - \nabla\chi\cdot\mathbf{D} \qquad \text{in } C_{R_2}\,,\\ \mathbf{g}|_{\partial C_{R_2}} &= 0\,,\end{aligned} \tag{3.24}$$

for almost all t, where χ is a smooth truncation function in Π depending on y only:

$$\chi\equiv 0 \ \text{ for } |y|\le 1 \quad\text{and}\quad \chi\equiv 0 \ \text{ for } |y|\ge R_1\,,$$

with $R_1>1$ large enough to allow $|\nabla\chi|\le 1$ and $R_2>\max(R_0+2,R_1)$. The domain C_{R_2} is star-like and Lipschitz. Based on Bogovskiĭ's formula a continuous, linear solution operator to (3.24) is defined, mapping $W_0^{1,2}(C_{R_2})\cap L_0^2(C_{R_2})$ to $W_0^{2,2}(C_{R_2})$ as well as $L_0^2(C_{R_2})\to W_0^{1,2}(C_{R_2})$ (cf. Lemma III.3.1 and Remark III.3.2 in [11]), where $L_0^2(C_{R_2}) := \{f\in L^2(C_{R_2}) : \int_{C_{R_2}} f = 0\}$. Since

$$\begin{aligned}\int_{C_{R_2}} \operatorname{div}(E_0(\mathbf{a})) - \nabla\chi\cdot\mathbf{D}\,dx &= \int_{C_{R_2}} \operatorname{div}\left(E_0(\mathbf{a}) - \chi\mathbf{D}\right)dx\\ &= \int_{\partial\Pi} \mathbf{a}\cdot\mathbf{n}\,dS - \int_{S_{R_2}} \mathbf{D}\cdot\mathbf{n}\,dS\\ &= -F + F\\ &= 0\,,\end{aligned}$$

we obtain a solution $\mathbf{g}(t) \in W_0^{2,2}(C_{R_2})$ for almost all t. Continuity implies

$$\begin{aligned} \|\mathbf{g}\|_{W^{2,2}(C_{R_2})} &\le c\,\|\operatorname{div}(E_0(\mathbf{a})) - \nabla\chi \cdot \mathbf{D}\|_{W^{1,2}(C_{R_2})} \\ &\le c\Big(\|E_0(\mathbf{a})\|_{W^{2,2}(\Pi)} + \|\mathbf{D}\|_{W^{1,2}(C_{R_1}\setminus C_1)}\Big) \\ &\overset{(3.21)}{\le} c\Big(\|\mathbf{a}\|_{W^{3/2,2}(\partial\Pi)} + \|\mathbf{D}\|_{W^{1,2}(C_{R_1}\setminus C_1)}\Big) \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \|\mathbf{g}\|_{W^{1,2}(C_{R_2})} &\le c\,\|\operatorname{div}(E_0(\mathbf{a})) - \nabla\chi \cdot \mathbf{D}\|_{L^2(C_{R_2})} \\ &\le c\Big(\|E_0(\mathbf{a})\|_{W^{1,2}(\Pi)} + \|\mathbf{D}\|_{L^2(C_{R_1}\setminus C_1)}\Big) \\ &\overset{(3.22)}{\le} c\Big(\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)} + \|\mathbf{D}\|_{L^2(C_{R_1}\setminus C_1)}\Big) \end{aligned} \tag{3.26}$$

for almost all t. Further, linearity and continuity of the corresponding operator yield

$$\begin{aligned} \|\partial_t\mathbf{g}\|_{W^{1,2}(C_{R_2})} &\le c\,\|\partial_t\big(\operatorname{div}(E_0(\mathbf{a})) - \nabla\chi \cdot \mathbf{D}\big)\|_{L^2(C_{R_2})} \\ &\le c\Big(\|\partial_t E_0(\mathbf{a})\|_{W^{1,2}(\Pi)} + \|\partial_t\mathbf{D}\|_{L^2(C_{R_1}\setminus C_1)}\Big) \\ &\overset{(3.23)}{\le} c\Big(\|\partial_t\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)} + \|\partial_t\mathbf{D}\|_{L^2(C_{R_1}\setminus C_1)}\Big) \qquad \text{f.a.a. } t\,. \end{aligned}$$

Altogether we find $\mathbf{g} \in L^2(0,T;W_0^{2,2}(C_{R_2})) \cap W^{1,2}(0,T;W_0^{1,2}(C_{R_2}))$ and extend $\mathbf{g}$ to the whole domain Π (without changing notation) by setting $\mathbf{g} \equiv 0$ in $\Pi\setminus C_{R_2}$. In conclusion,

$$\mathbf{g} \in L^2(0,T;W^{2,2}(\Pi)) \cap W^{1,2}(0,T;W^{1,2}(\Pi))\,,$$

the support is bounded, $\operatorname{supp}\mathbf{g} \subset \overline{C_{R_2}}$, and zero trace on the boundary $\partial\Pi$ is preserved.

Now, we are in a position to define the actual flux-driving extension

$$\mathbf{A}_F := E_0(\mathbf{a}) - \chi\mathbf{D} - \mathbf{g}\,,$$

where χ is the truncating function as before. In the following lemma we summarize properties of $\mathbf{A}_F$.

Lemma 3.9. *The function $\mathbf{A}_F$ is a solenoidal extension of $\mathbf{a} \in L^2(0,T;W^{3/2,2}(\partial\Pi)) \cap W^{1,2}(0,T;W^{1/2,2}(\partial\Pi))$ with $\operatorname{supp}(\mathbf{a}) \subset \Gamma_{R_0}$, carrying the flux:*

$$\int_{S_R} \mathbf{A}_F \cdot \mathbf{n}\, dS = F$$

for large $R > 0$. Additionally,

$$\mathbf{A}_F \in L^2(0,T;W_{loc}^{2,2}(\Pi)) \cap W^{1,2}(0,T;L_{loc}^2(\Pi))$$

and, in particular, $(\mathbf{A}_F \cdot \nabla)\mathbf{A}_F \in L^2(0,T;L^2(\Pi))$. For $\mathbf{v} \in W_{0,\sigma}^{1,2}(\Pi)$ there holds the estimate

$$\left|\int_\Pi (\mathbf{v}\cdot\nabla)\mathbf{A}_F \cdot \mathbf{v}\, dx\right| \le \mathsf{C}_{\mathbf{A}_F}\, \kappa\, \|\nabla\mathbf{v}\|_{L^2(\Pi)}^2 \quad \text{f.a.a. } t\,, \tag{3.27}$$

with $\kappa := \|\mathbf{a}\|_{W^{1,2}(0,T;W^{1/2,2}(\partial\Pi))} + (\sqrt{\ln(R_1)} + 1)\|F\|_{W^{1,2}(0,T)}$ and $\mathsf{C}_{\mathbf{A}_F}$ depending on R_0, R_1, R_2 only.

Proof. The properties $\mathbf{A}_F|_{\partial\Pi} = \mathbf{a}$ and $\operatorname{div} \mathbf{A}_F = 0$ are clear by construction. Let $R > R_2$. Then, since supports of $E_0(\mathbf{a})$ and $\mathbf{g}$ are contained in C_{R_2}, we find

$$\int_{S_R} \mathbf{A}_F \cdot \mathbf{n} \, dS = \int_{S_R} -\mathbf{D} \cdot \mathbf{n} \, dS = F \, .$$

Concerning the regularity, we recall

$$E_0(\mathbf{a}),\, \mathbf{g} \in L^2(0,T;W^{2,2}(\Pi)) \cap W^{1,2}(0,T;W^{1,2}(\Pi)) \, ,$$

hence investigation of $\chi\mathbf{D}$ is sufficient for this part of the assertion. Since, by definition, $\mathbf{D} = (w^{(0,-)}\nabla_y P^{(0,-)}, 0)^T$ with

$$w^{(0,-)} \in L^2(0,T;W^{2,2}(0,1)) \cap W^{1,2}(0,T;L^2(0,1)) \cap L^\infty(0,T;W^{1,2}(0,1)) \, ,$$

we can focus on $\nabla_y P^{(0,-)}(y) = -\frac{1}{2\pi}\frac{y}{r^2}$. The gradient of $P^{(0,-)}$ asymptotically behaves like $\frac{1}{r}$ and the second and third derivatives decline like $\frac{1}{r^2}$ respectively $\frac{1}{r^3}$ for $|y| \to \infty$. The singularity in $y = 0$ gets cut out by χ, leading to the conclusion that $\chi\mathbf{D}$, $\nabla\chi\mathbf{D}$, $\nabla^2\chi\mathbf{D}$ and $\partial_t\chi\mathbf{D}$ are locally squaresummable (in space) for almost all t. Therefore,

$$\chi\mathbf{D} \in L^2(0,T;W^{2,2}_{loc}(\Pi)) \cap W^{1,2}(0,T;L^2_{loc}(\Pi))$$

and notice, in particular: $\chi\mathbf{D} \in L^\infty(0,T;W^{1,2}_{loc}(\Pi))$.

Next, we analyze $(\mathbf{A}_F \cdot \nabla)\mathbf{A}_F$. Denote $\mathbf{V} := E_0(\mathbf{a}) - \mathbf{g}$ and observe

$$(\mathbf{A}_F \cdot \nabla)\mathbf{A}_F = (\mathbf{V} \cdot \nabla)\mathbf{V} - (\mathbf{V} \cdot \nabla)\chi\mathbf{D} - (\chi\mathbf{D} \cdot \nabla)\mathbf{V} + (\chi\mathbf{D} \cdot \nabla)\chi\mathbf{D} \, . \tag{3.28}$$

Due to bounded support of $\mathbf{V}$, the first three terms in (3.28) are $L^2(0,T;L^2(\Pi))$-functions. The last term requires a separate treatment. We deduce the estimate

$$\begin{aligned}
\int_0^T \int_{C_{R_1}\setminus C_1} |(\chi\mathbf{D} \cdot \nabla)\chi\mathbf{D}|^2 \, dx \, dt &\le \int_0^T \int_{C_{R_1}\setminus C_1} |\chi\mathbf{D}|^2 |\nabla(\chi\mathbf{D})|^2 \, dx \, dt \\
&\le \int_0^T \|\chi\mathbf{D}\|^2_{L^4(C_{R_1}\setminus C_1)} \|\nabla(\chi\mathbf{D})\|^2_{L^4(C_{R_1}\setminus C_1)} \, dt \\
&\le c \int_0^T \|\chi\mathbf{D}\|^2_{W^{1,2}(C_{R_1}\setminus C_1)} \|\chi\mathbf{D}\|^2_{W^{2,2}(C_{R_1}\setminus C_1)} \, dt \\
&\le c \, \|\chi\mathbf{D}\|^2_{L^\infty(W^{1,2}(C_{R_1}\setminus C_1))} \|\chi\mathbf{D}\|^2_{L^2(W^{2,2}(C_{R_1}\setminus C_1))}
\end{aligned}$$

and pointwise there holds the following

$$|(\mathbf{D} \cdot \nabla)\mathbf{D}| \le |\mathbf{D}||\nabla_y \mathbf{D}| = |w^{(0,-)}|^2 |\nabla_y P^{(0,-)}||\nabla_y^2 P^{(0,-)}| = c \, |w^{(0,-)}|^2 \frac{1}{r^3} \, .$$

These two combined yield

$$\begin{aligned}
&\int_0^T \int_\Pi |(\chi\mathbf{D} \cdot \nabla)\chi\mathbf{D}|^2 \, dx \, dt \\
&= \int_0^T \int_{C_{R_1}\setminus C_1} |(\chi\mathbf{D} \cdot \nabla)\chi\mathbf{D}|^2 \, dx \, dt + \int_0^T \int_{\Pi\setminus C_{R_1}} |(\mathbf{D} \cdot \nabla)\mathbf{D}|^2 \, dx \, dt \\
&\le c \, \|\chi\mathbf{D}\|^2_{L^\infty(W^{1,2}(C_{R_1}\setminus C_1))} \|\chi\mathbf{D}\|^2_{L^2(W^{2,2}(C_{R_1}\setminus C_1))}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^T \int_{\Pi\setminus C_{R_1}} c\,|w^{(0,-)}|^4 \frac{1}{r^6}\,dx\,dt \\
&\le c\,\|\chi\mathbf{D}\|^2_{L^\infty(W^{1,2}(C_{R_1}\setminus C_1))}\|\chi\mathbf{D}\|^2_{L^2(W^{2,2}(C_{R_1}\setminus C_1))} \\
&\quad + c\int_0^T \int_0^1 |w^{(0,-)}|^4\,dz\,dt \int_0^{2\pi}\int_{R_1}^\infty \frac{1}{r^6} r\,dr\,d\theta \\
&= c\,\|\chi\mathbf{D}\|^2_{L^\infty(W^{1,2}(C_{R_1}\setminus C_1))}\|\chi\mathbf{D}\|^2_{L^2(W^{2,2}(C_{R_1}\setminus C_1))} + c\frac{T}{{R_1}^4}\|w^{(0,-)}\|^4_{L^\infty(W^{1,2}(0,1))} \\
&< \infty\,,
\end{aligned}
$$

meaning $(\chi\mathbf{D}\cdot\nabla)\chi\mathbf{D} \in L^2(0,T;L^2(\Pi))$.

To verify the last assertion, we need the following estimates: For almost all t,

$$
\begin{aligned}
\|\mathbf{D}(t)\|^2_{L^2(C_{R_1}\setminus C_1)} &= \int_0^1 |w^{(0,-)}(t)|^2\,dz \int_{1\le|y|\le R_1} |\nabla_y P^{(0,-)}|^2\,dy \\
&= \frac{1}{4\pi^2}\,\|w^{(0,-)}(t)\|^2_{L^2(0,1)} \int_0^{2\pi}\int_1^{R_1} \frac{1}{r^2} r\,dr\,d\theta \\
&= \frac{1}{2\pi}\ln(R_1)\,\|w^{(0,-)}(t)\|^2_{L^2(0,1)}
\end{aligned}
\tag{3.29}
$$

and, pointwise for $x\in\Pi$,

$$
\begin{aligned}
&|\nabla(\chi\mathbf{D})| \\
&\quad \le |(\nabla\chi)\otimes\mathbf{D}| + |\chi\nabla\mathbf{D}| \\
&\quad \le \begin{cases} |w^{(0,-)}||\nabla_y P^{(0,-)}| + |w^{(0,-)}||\nabla_y^2 P^{(0,-)}| + |\partial_z w^{(0,-)}||\nabla_y P^{(0,-)}|\,, & |y|>1\,, \\ 0\,, & |y|\le 1 \end{cases} \\
&\quad \le 2\left(|w^{(0,-)}| + |\partial_z w^{(0,-)}|\right) \sup_{|y|\ge 1}\left(|\nabla_y P^{(0,-)}| + |\nabla_y^2 P^{(0,-)}|\right) \\
&\quad = \frac{1+\sqrt{2}}{\pi}\left(|w^{(0,-)}| + |\partial_z w^{(0,-)}|\right).
\end{aligned}
\tag{3.30}
$$

Now, we are able to prove inequality (3.27). Let $\mathbf{v}\in W^{1,2}_{0,\sigma}(\Pi)$. Using a Gagliardo-Nirenberg-type inequality (Lemma 2.6), we deduce for almost all t

$$
\begin{aligned}
&\left|\int_\Pi (\mathbf{v}\cdot\nabla)\mathbf{A}_F\cdot\mathbf{v}\,dx\right| \\
&\quad \le \int_\Pi |\mathbf{v}|^2|\nabla\mathbf{A}_F|\,dx \\
&\quad \le \int_\Pi |\mathbf{v}|^2\left(|\nabla E_0(\mathbf{a})| + |\nabla\mathbf{g}| + |\nabla(\chi\mathbf{D})|\right)dx \\
&\quad \overset{(3.30)}{\le} \left(\int_\Pi |\mathbf{v}|^4\,dx\right)^{1/2}\left(\left(\int_\Pi |\nabla E_0(\mathbf{a})|^2\,dx\right)^{1/2} + \left(\int_\Pi |\nabla\mathbf{g}|^2\,dx\right)^{1/2}\right) \\
&\qquad + \frac{1+\sqrt{2}}{\pi}\int_{\mathbb{R}^2}\int_0^1 |\mathbf{v}|^2\left(|w^{(0,-)}| + |\partial_z w^{(0,-)}|\right)dz\,dy \\
&\quad \overset{(3.22),(3.26)}{\le} c\,\|\nabla\mathbf{v}\|^2_{L^2(\Pi)}\left(\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)} + \left(\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)} + \|\mathbf{D}\|_{L^2(C_{R_1}\setminus C_1)}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1+\sqrt{2}}{\pi} \int_{\mathbb{R}^2} \left(\int_0^1 |\mathbf{v}|^4 \, dz \right)^{1/2} \left(2 \int_0^1 |w^{(0,-)}|^2 + |\partial_z w^{(0,-)}|^2 \, dz \right)^{1/2} dy \\
&\overset{(3.29)}{\leq} c \left(\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)} + \sqrt{\ln(R_1)} \, \|w^{(0,-)}\|_{L^2(0,1)} \right) \|\nabla \mathbf{v}\|^2_{L^2(\Pi)} \\
&\quad + \tilde{c} \, \|w^{(0,-)}\|_{W^{1,2}(0,1)} \int_{\mathbb{R}^2} \|\mathbf{v}\|^2_{W^{1,2}(0,1)} \, dy \\
&\leq \mathsf{C}_{\mathbf{A}_F} \left(\|\mathbf{a}\|_{W^{1/2,2}(\partial\Pi)} + \big(\sqrt{\ln(R_1)} + 1\big) \|w^{(0,-)}\|_{W^{1,2}(0,1)} \right) \|\nabla \mathbf{v}\|^2_{L^2(\Pi)} \, .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left| \int_{\Pi} (\mathbf{v} \cdot \nabla) \mathbf{A}_F \cdot \mathbf{v} \, dx \right| \\
&\overset{(3.19)}{\leq} \mathsf{C}_{\mathbf{A}_F} \left(\|\mathbf{a}\|_{W^{1,2}(W^{1/2,2}(\partial\Pi))} + \big(\sqrt{\ln(R_1)} + 1\big) \|F\|_{W^{1,2}(0,T)} \right) \|\nabla \mathbf{v}\|^2_{L^2(\Pi)} \, .
\end{aligned}
$$

□

At last, we wish to investigate the expression $\partial_t \mathbf{A}_F - \nu \Delta \mathbf{A}_F$.

Remark. Inserting the definition of $\mathbf{A}_F$ we obtain

$$
\partial_t \mathbf{A}_F - \nu \Delta \mathbf{A}_F = \partial_t E_0(\mathbf{a}) - \nu \Delta E_0(\mathbf{a}) - \partial_t \mathbf{g} + \nu \Delta \mathbf{g} - \partial_t (\chi \mathbf{D}) + \nu \Delta (\chi \mathbf{D}) \, .
$$

We break down further the terms including $\mathbf{D}$ - recall (3.20):

$$
\begin{aligned}
&\partial_t (\chi \mathbf{D}) - \nu \Delta (\chi \mathbf{D}) \\
&\quad = \chi \partial_t \mathbf{D} - \nu \left((\Delta \chi) \mathbf{D} + 2 \nabla \chi \cdot \nabla \mathbf{D} + \chi \Delta \mathbf{D} \right) \\
&\quad = \chi \left(\partial_t \mathbf{D} - \nu \Delta \mathbf{D} \right) - 2 \nabla \chi \cdot \nabla \mathbf{D} - \nu (\Delta \chi) \mathbf{D} \\
&\quad = \chi s^{(0,-)} \nabla_x P^{(0,-)} - 2 \nabla \chi \cdot \nabla \mathbf{D} - \nu (\Delta \chi) \mathbf{D} \\
&\quad = s^{(0,-)} \nabla_x \big(\chi P^{(0,-)} \big) - s^{(0,-)} (\nabla_x \chi) P^{(0,-)} - 2 \nabla \chi \cdot \nabla \mathbf{D} - \nu (\Delta \chi) \mathbf{D} \, ,
\end{aligned}
$$

where $\nabla \chi \cdot \nabla \mathbf{D} = w^{(0,-)} \big(\nabla_y \chi \cdot \nabla_y (\partial_{y_1} P^{(0,-)}) \, , \ \nabla_y \chi \cdot \nabla_y (\partial_{y_2} P^{(0,-)}) \, , \ 0 \big)^T$.

Note, in particular, that all terms except $\nabla_x (s^{(0,-)} \chi P^{(0,-)})$ are in $L^2(0,T; L^2(\Pi))$, which is the reason we split these up in the following section.

3.5 Existence in the case of non-zero flux

With the preparatory results of the preceding section we are now in a position to study the non-zero flux problem, which differs from the situation in Section 3.3 only in the "detail" that the balance of flow through the boundary is not necessarily zero. The physical setting presents itself in the absence of requirement (3.18), namely

$$
F(t) = - \int_{\partial\Pi} \mathbf{a}(t) \cdot \mathbf{n} \, dS = 0 \, ,
$$

where $\mathbf{a}$ is a prescribed boundary condition.

Once more, we need to formulate clearly the definition of a weak solution, to investigate existence in the following.

Definition 3.10. *Let* $\mathbf{f} \in L^2(0,T;L^2(\Pi))$ *be* T*-periodic and* $\mathbf{A}_F$ *the extension of the* T*-periodic boundary value* $\mathbf{a} \in L^2(0,T;W^{3/2,2}(\partial\Pi)) \cap W^{1,2}(0,T;W^{1/2,2}(\partial\Pi))$*,* $\operatorname{supp}(\mathbf{a}) \subset \Gamma_{R_0}$*, derived in Section 3.4. We call a function* $\mathbf{u} \in L^2(0,T;W^{1,2}_{loc}(\Pi))$ *a* weak solution *of the inhomogeneous Navier-Stokes equations* (NS$_{\text{bc}}$) *with non-zero flux, if it suffices the integral identity*

$$\int_0^T -(\mathbf{u},\partial_t\boldsymbol{\varphi}) + \nu\,(\nabla\mathbf{u},\nabla\boldsymbol{\varphi}) + \big((\mathbf{u}\cdot\nabla)\mathbf{u},\boldsymbol{\varphi}\big)\,dt = \int_0^T \big(\mathbf{f},\boldsymbol{\varphi}\big)\,dt$$

for all time-periodic test functions $\boldsymbol{\varphi} \in C^\infty([0,T];C^\infty_{0,\sigma}(\Pi))$*. Further, we demand* $\mathbf{u}-\mathbf{A}_F$ *to be an element of* $L^2(0,T;W^{1,2}_{0,\sigma}(\Pi))$*.*

Theorem 3.11. *Let* $\mathbf{f}$*,* $\mathbf{a}$ *be as in the definition above. Suppose the following condition on the data is fulfilled:*

$$\kappa = \|\mathbf{a}\|_{W^{1,2}(0,T;W^{1/2,2}(\partial\Pi))} + \left(\sqrt{\ln(R_1)+1}\right)\|F\|_{W^{1,2}(0,T)} < \frac{\nu}{\mathsf{C}_{\mathbf{A}_F}}\,.$$

Then, there exists a weak solution $\mathbf{u}$ *of the inhomogeneous Navier-Stokes equations with non-zero flux.*

Proof. To show existence of a weak solution $\mathbf{u}$ we reformulate the problem (NS$_{\text{bc}}$) by setting $\mathbf{u} = \mathbf{u}^* + \mathbf{A}_F$. The system we obtain is

$$\begin{aligned}
\partial_t\mathbf{u}^* - \nu\Delta\mathbf{u}^* + \big((\mathbf{u}^*+\mathbf{A}_F)\cdot\nabla\big)\mathbf{u}^* + (\mathbf{u}^*\cdot\nabla)\mathbf{A}_F + \nabla p^* &= \mathbf{f}^* && \text{in } [0,T]\times\Pi\,,\\
\operatorname{div}\mathbf{u}^* &= 0 && \text{in } [0,T]\times\Pi\,,\\
\mathbf{u}^*|_{\partial\Pi} &= 0 && \text{for all } t\in[0,T]\,,\\
\mathbf{u}^*|_{t=0} &= \mathbf{u}^*|_{t=T} && \text{in } \Pi\,,
\end{aligned} \tag{3.31}$$

where $p^* = p - s^{(0,-)}\chi P^{(0,-)}$ and

$$\begin{aligned}
\mathbf{f}^* = \mathbf{f} &- \partial_t E_0(\mathbf{a}) + \nu\Delta E_0(\mathbf{a}) + \partial_t\mathbf{g} - \nu\Delta\mathbf{g}\\
&- s^{(0,-)}(\nabla\chi)P^{(0,-)} - 2\nu\nabla\chi\cdot\nabla\mathbf{D} - \nu(\Delta\chi)\mathbf{D} - (\mathbf{A}_F\cdot\nabla)\mathbf{A}_F\,.
\end{aligned}$$

As described in the previous section, especially $\mathbf{f}^* \in L^2(0,T;L^2(\Pi))$.

The proof is executed in four steps. We start by breaking down the problem to finite dimensional spaces.

Step 1: Galerkin ansatz.
Let $\{\boldsymbol{\phi}_k,\ k\in\mathbb{N}\} \subset C^\infty_{0,\sigma}(\Pi)$ be a complete orthonormal system of $L^2_\sigma(\Pi)$ and define $H_n \subset L^2_\sigma(\Pi)$ as the space spanned by the first n basis vectors. Hence, we are looking for approximating solutions $\mathbf{u}^*_n(t,x) = \sum_{k=1}^n a^n_k(t)\boldsymbol{\phi}_j(x)$. In H_n problem (3.31) translates to a system of ordinary differential equations:

$$\begin{aligned}
&\frac{d}{dt}a^n_k(t) + \nu\sum_{i=1}^n \big(\nabla\boldsymbol{\phi}_i,\nabla\boldsymbol{\phi}_k\big)a^n_i(t) + \sum_{i,j=1}^n \big((\boldsymbol{\phi}_i\cdot\nabla)\boldsymbol{\phi}_j,\boldsymbol{\phi}_k\big)a^n_i(t)a^n_j(t)\\
&\quad + \sum_{i=1}^n \big((\boldsymbol{\phi}_i\cdot\nabla)\mathbf{A}_F(t),\boldsymbol{\phi}_k\big)a^n_i(t) + \sum_{i=1}^n \big((\mathbf{A}_F(t)\cdot\nabla)\boldsymbol{\phi}_i,\boldsymbol{\phi}_k\big)a^n_i(t)\\
&= \big(\mathbf{f}^*(t),\boldsymbol{\phi}_k\big)\,,\quad 1\le k\le n\,.
\end{aligned} \tag{3.32}$$

Per Carathéodory's theory – for each prescribed initial condition $(b_1^n, \dots, b_n^n) \in \mathbb{R}^n$ – there exists a corresponding (unique) solution $(a_1^n, \dots, a_n^n) \in AC([0,T^*])$, where $T^* < T$ in case of a blow-up (cf. [13, Theorem I.5.3], [59, Theorem 10.XX] or the analogous derivation in the proof of Theorem 3.3). In step 2 we establish an energy inequality, excluding a blow-up, hence $T^* = T$.

Step 2: A-priori estimate.
Multiplying (3.32) with a_k^n and summing $k = 1, \dots, n$ gives

$$\begin{aligned}\big(\partial_t \mathbf{u}_n^*, \mathbf{u}_n^*\big) + \nu\big(\nabla \mathbf{u}_n^*, \nabla \mathbf{u}_n^*\big) + \big((\mathbf{u}_n^* \cdot \nabla)\mathbf{u}_n^*, \mathbf{u}_n^*\big) \\ + \big((\mathbf{u}_n^* \cdot \nabla)\mathbf{A}_F, \mathbf{u}_n^*\big) + \big((\mathbf{A}_F \cdot \nabla)\mathbf{u}_n^*, \mathbf{u}_n^*\big) \\ = \big(\mathbf{f}^*, \mathbf{u}_n^*\big)\,.\end{aligned}$$

There is a bounded Lipschitz domain Ω with $\operatorname{supp} \mathbf{u}_n^* \subset \Omega \subset \Pi$. Thus, due to Lemma 2.11, the third term on the left-hand side vanishes and, since $\mathbf{A}_F \in W^{1,2}(\Omega)$ and $\operatorname{div} \mathbf{A}_F = 0$, so does the last:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_n^*\|_{L^2}^2 + \nu\|\nabla \mathbf{u}_n^*\|_{L^2}^2 = \big(\mathbf{f}^*, \mathbf{u}_n^*\big) - \big((\mathbf{u}_n^* \cdot \nabla)\mathbf{A}_F, \mathbf{u}_n^*\big)\,.$$

Applying Poincaré's and Young's inequalities we obtain

$$\big|\big(\mathbf{f}^*, \mathbf{u}_n^*\big)\big| \le \|\mathbf{f}^*\|_{L^2}\|\mathbf{u}_n^*\|_{L^2} \le \mathsf{C}_P\,\|\mathbf{f}^*\|_{L^2}\|\nabla \mathbf{u}_n^*\|_{L^2} \le c(\varepsilon,\nu)\,\|\mathbf{f}^*\|_{L^2}^2 + \varepsilon\nu\,\|\nabla \mathbf{u}_n^*\|_{L^2}^2$$

and from Lemma 3.9 we know

$$\big|\big((\mathbf{u}_n^* \cdot \nabla)\mathbf{A}_F, \mathbf{u}_n^*\big)\big| \le \mathsf{C}_{\mathbf{A}_F}\kappa\,\|\nabla \mathbf{u}_n^*\|_{L^2}^2$$

for almost all t. Together this yields

$$\frac{d}{dt}\|\mathbf{u}_n^*\|_{L^2}^2 + 2\nu\left(1 - \frac{1}{\nu}\mathsf{C}_{\mathbf{A}_F}\kappa - \varepsilon\right)\|\nabla \mathbf{u}_n^*\|_{L^2}^2 \le 2c(\varepsilon,\nu)\|\mathbf{f}^*\|_{L^2}^2\,. \tag{3.33}$$

Since $AC([0,T^*]) \subset W^{1,1}(0,T^*)$ and $\frac{d}{dt}\|\mathbf{u}_n\|_{L^2}^2 = 2\sum_{j=1}^n a_j^n \frac{d}{dt}a_j^n$, we find $\frac{d}{dt}\|\mathbf{u}_n^*\|_{L^2}^2 \in L^1(0,T^*)$ and deduce through integration

$$\begin{aligned}\|\mathbf{u}_n^*(t)\|_{L^2}^2 + 2\nu\left(1 - \frac{1}{\nu}\mathsf{C}_{\mathbf{A}_F}\kappa - \varepsilon\right)\int_0^t \|\nabla \mathbf{u}_n^*(s)\|_{L^2}^2\,ds \\ \le c_\varepsilon \int_0^t \|\mathbf{f}^*(s)\|_{L^2}^2\,ds + \|\mathbf{u}_n^*(0)\|_{L^2}^2\,,\end{aligned}$$

for all $t \in [0,T^*]$. The right-hand side is finite for $t = T$, thereby a blow-up of $\mathbf{u}_n^*$ is not possible and $T^* = T$ as preannounced. Note, $\frac{\mathsf{C}_{\mathbf{A}_F}}{\nu}\kappa < 1$, which enables $1 - \frac{1}{\nu}\mathsf{C}_{\mathbf{A}_F}\kappa - \varepsilon > 0$ for sufficiently small ε.

Step 3: Time-periodicity of approximating solutions.
We rearrange (3.33) with the help of Poincaré's inequality:

$$\frac{d}{dt}\|\mathbf{u}_n^*\|_{L^2}^2 \le c_\varepsilon\,\|\mathbf{f}^*\|_{L^2}^2 - 4\nu\left(1 - \frac{1}{\nu}\mathsf{C}_{\mathbf{A}_F}\kappa - \varepsilon\right)\|\mathbf{u}_n^*\|_{L^2}^2\,.$$

By Gronwall's inequality (Lemma 2.7) we then find for all t:

$$\|\mathbf{u}_n^*(t)\|_{L^2}^2 \leq \|\mathbf{u}_n^*(0)\|_{L^2}^2 \, e^{-4\nu\left(1-\frac{1}{\nu}\mathsf{C}_{\mathbf{A}_F}\kappa-\varepsilon\right)t} + c_\varepsilon \int_0^t \|\mathbf{f}^*(s)\|_{L^2}^2 \, ds \,,$$

which, in particular, holds for $t = T$. Motivated by this estimate, we define the operator – called Poincaré map – assigning to $\mathbf{u}_{n,0}^* \sim (b_1^n, \dots, b_n^n) \in \mathbb{R}^n$ its corresponding solution evaluated in $t = T$: $\mathbf{u}_{n,0}^* \mapsto \mathbf{u}_n^*(T)$ respectively $b \mapsto (a_1^n(T), \dots, a_n^n(T)) \in \mathbb{R}^n$. Choosing

$$R := \left(\frac{c_\varepsilon}{1 - e^{-4\nu\left(1-\nu^{-1}\mathsf{C}_{\mathbf{A}_F}\kappa-\varepsilon\right)T}} \right)^{1/2} \|\mathbf{f}^*\|_{L^2(0,T;L^2(\Pi))} \,,$$

this operator maps the ball $B_R(0) \subset \mathbb{R}^n$ into itself. Furthermore, the Poincaré map is continuous as composition of two continuous functions (cf. step 3 in the proof of Theorem 3.3), guaranteeing the existence of a fixed point by the theorem of Brouwer. In the following $\mathbf{u}_n^*$ denotes the solution emerging from this fixed point and therefore $\mathbf{u}_n^*(0) = \mathbf{u}_n^*(T)$. Moreover, $\mathbf{u}_n^*$ is a solution of (3.32), hence, it fulfills the identity

$$\begin{aligned}\big(\partial_t \mathbf{u}_n^*, \alpha\boldsymbol{\phi}_k\big) + \nu\big(\nabla \mathbf{u}_n^*, \alpha\nabla\boldsymbol{\phi}_k\big) + \big((\mathbf{u}_n^* \cdot \nabla)\mathbf{u}_n^*, \alpha\boldsymbol{\phi}_k\big) \\ + \big((\mathbf{u}_n^* \cdot \nabla)\mathbf{A}_F, \alpha\boldsymbol{\phi}_k\big) + \big((\mathbf{A}_F \cdot \nabla)\mathbf{u}_n^*, \alpha\boldsymbol{\phi}_k\big) \\ = \big(\mathbf{f}^*, \alpha\boldsymbol{\phi}_k\big) \,,\end{aligned} \tag{3.34}$$

for all $k = 1, \dots, n$, where $\alpha\colon [0,T] \to \mathbb{R}$ is an arbitrary mapping. Given a smooth and time-periodic function $h\colon [0,T] \to \mathbb{R}$, we conclude

$$\begin{aligned}\int_0^T -\big(\mathbf{u}_n^*, h'\boldsymbol{\phi}_k\big) + \nu\big(\nabla \mathbf{u}_n^*, h\nabla\boldsymbol{\phi}_k\big) + \big((\mathbf{u}_n^* \cdot \nabla)\mathbf{u}_n^*, h\boldsymbol{\phi}_k\big) \\ + \big((\mathbf{u}_n^* \cdot \nabla)\mathbf{A}_F, h\boldsymbol{\phi}_k\big) + \big((\mathbf{A}_F \cdot \nabla)\mathbf{u}_n^*, h\boldsymbol{\phi}_k\big) \, dt \\ = \int_0^T \big(\mathbf{f}^*, h\boldsymbol{\phi}_k\big) \, dt \,.\end{aligned} \tag{3.35}$$

Another immediate consequence is the energy inequality

$$\|\mathbf{u}_n^*(t)\|_{L^2}^2 + 2\nu \left(1 - \frac{1}{\nu}\mathsf{C}_{\mathbf{A}_F}\kappa - \varepsilon\right) \int_0^t \|\nabla \mathbf{u}_n^*(s)\|_{L^2}^2 \, ds \leq c \, \|\mathbf{f}^*\|_{L^2(0,T;L^2(\Pi))}^2 \,, \tag{3.36}$$

for all $t \in [0,T]$, since $\|\mathbf{u}_n^*(0)\|_{L^2}^2 \leq R^2 = \left(\frac{c_\varepsilon}{1-e^{-4\nu\left(1-\nu^{-1}\mathsf{C}_{\mathbf{A}_F}\kappa-\varepsilon\right)T}} \right) \|\mathbf{f}^*\|_{L^2(0,T;L^2(\Pi))}^2$.

Step 4: Passage to the limit.
For each $n \in \mathbb{N}$ we found an approximating solution. Considering now the arising sequence $(\mathbf{u}_n^*)_n$ we want to examine convergence properties to allow for a proper passage to the limit. Firstly, coupling (3.36) with Poincaré's inequality grants boundedness of $(\mathbf{u}_n^*)_n$ in $L^2(0,T;W_{0,\sigma}^{1,2}(\Pi))$. Thus, there is a subsequence - still denoted by $(\mathbf{u}_n^*)_n$ - converging weakly in $L^2(0,T;W_{0,\sigma}^{1,2}(\Pi))$ to a limit $\mathbf{u}^*$. This convergence directly justifies the following limit processes, $n \to \infty$:

$$\int_0^T \big(\mathbf{u}_n^*, h'\boldsymbol{\phi}_k\big) \, dt \longrightarrow \int_0^T \big(\mathbf{u}^*, h'\boldsymbol{\phi}_k\big) \, dt \,,$$
$$\int_0^T \big(\nabla\mathbf{u}_n^*, h\nabla\boldsymbol{\phi}_k\big) \, dt \longrightarrow \int_0^T \big(\nabla\mathbf{u}^*, h\nabla\boldsymbol{\phi}_k\big) \, dt \,,$$

and

$$\int_0^T \big((\mathbf{u}_n^* \cdot \nabla)\mathbf{A}_F, h\boldsymbol{\phi}_k\big)\,dt = \int_0^T b(\mathbf{u}_n^*, \mathbf{A}_F, h\boldsymbol{\phi}_k)\,dt$$
$$\longrightarrow \int_0^T b(\mathbf{u}^*, \mathbf{A}_F, h\boldsymbol{\phi}_k)\,dt = \int_0^T \big((\mathbf{u}^* \cdot \nabla)\mathbf{A}_F, h\boldsymbol{\phi}_k\big)\,dt\,,$$
$$\int_0^T \big((\mathbf{A}_F \cdot \nabla)\mathbf{u}_n^*, h\boldsymbol{\phi}_k\big)\,dt = \int_0^T b(\mathbf{A}_F, \mathbf{u}_n^*, h\boldsymbol{\phi}_k)\,dt$$
$$\longrightarrow \int_0^T b(\mathbf{A}_F, \mathbf{u}^*, h\boldsymbol{\phi}_k)\,dt = \int_0^T \big((\mathbf{A}_F \cdot \nabla)\mathbf{u}^*, h\boldsymbol{\phi}_k\big)\,dt\,,$$

where we exploit $\mathbf{A}_F$'s local integrability in combination with the compact support of $\boldsymbol{\phi}_k$. Also, recall that b is a continuous trilinear form (cf. Lemma 2.10).

The remaining convective term needs more careful treatment. Our goal is to prove strong convergence of $\mathbf{u}_n^*$ in $L^2(0,T;L^2(\Omega))$ for each bounded subdomain $\Omega \subset \Pi$. To this end we first show $\mathbf{u}_n^*(t)$ is weakly converging in $L^2(\Pi)$ for almost all t. Investigating on the sequences $\big((\mathbf{u}_n^*(\cdot), \boldsymbol{\phi}_k)\big)_{n\in\mathbb{N}} \subset C([0,T])$, $k \in \mathbb{N}$, we find each is bounded due to inequality (3.36). Further, for $t, s \in [0,T]$, we establish the estimate

$$\begin{aligned}
&\big|\big(\mathbf{u}_n^*(t), \boldsymbol{\phi}_k\big) - \big(\mathbf{u}_n^*(s), \boldsymbol{\phi}_k\big)\big| \\
&\quad = \left|\int_s^t \frac{d}{d\tau}\big(\mathbf{u}_n^*(\tau), \boldsymbol{\phi}_k\big)\,d\tau\right| \\
&\quad \overset{(3.34)}{=} \Bigg|\int_s^t -\nu\big(\nabla\mathbf{u}_n^*, \nabla\boldsymbol{\phi}_k\big) - \big((\mathbf{u}_n^* \cdot \nabla)\mathbf{u}_n^*, \boldsymbol{\phi}_k\big) \\
&\qquad\quad - \big((\mathbf{u}_n^* \cdot \nabla)\boldsymbol{\phi}_k, \mathbf{A}_F\big) - \big((\mathbf{A}_F \cdot \nabla)\mathbf{u}_n^*, \boldsymbol{\phi}_k\big) + \big(\mathbf{f}^*, \boldsymbol{\phi}_k\big)\,d\tau\Bigg| \\
&\quad \le \int_s^t \nu\,\|\nabla\mathbf{u}_n^*\|_{L^2}\|\nabla\boldsymbol{\phi}_k\|_{L^2} + \int_\Pi |\mathbf{u}_n^*||\nabla\mathbf{u}_n^*||\boldsymbol{\phi}_k|\,dx \\
&\qquad\quad + \int_\Pi |\mathbf{u}_n^*||\nabla\boldsymbol{\phi}_k||\mathbf{A}_F|\,dx + \int_\Pi |\mathbf{A}_F||\nabla\mathbf{u}_n^*||\boldsymbol{\phi}_k|\,dx + \|\mathbf{f}^*\|_{L^2}\|\boldsymbol{\phi}_k\|_{L^2}\,d\tau \\
&\quad \le \int_s^t \nu\,\|\nabla\mathbf{u}_n^*\|_{L^2}\|\nabla\boldsymbol{\phi}_k\|_{L^2} + \|\mathbf{u}_n^*\|_{L^2}\|\nabla\mathbf{u}_n^*\|_{L^2}\|\boldsymbol{\phi}_k\|_{C^0} \\
&\qquad\quad + \mathsf{C}_P\,\|\nabla\mathbf{u}_n^*\|_{L^2}\|\nabla\boldsymbol{\phi}_k\|_{C^0}\|\mathbf{A}_F\|_{L^2(\operatorname{supp}\boldsymbol{\phi}_k)} \\
&\qquad\quad + \|\mathbf{A}_F\|_{L^2(\operatorname{supp}\boldsymbol{\phi}_k)}\|\nabla\mathbf{u}_n^*\|_{L^2}\|\boldsymbol{\phi}_k\|_{C^0} \\
&\qquad\quad + \|\mathbf{f}^*\|_{L^2}\|\boldsymbol{\phi}_k\|_{L^2}\,d\tau \\
&\quad \le \nu\,\|\nabla\boldsymbol{\phi}_k\|_{L^2}\left(\int_s^t 1\,d\tau\right)^{1/2}\|\nabla\mathbf{u}_n^*\|_{L^2(L^2)} \\
&\qquad\quad + \|\mathbf{u}_n^*\|_{L^\infty(L^2)}\|\boldsymbol{\phi}_k\|_{C^0}\left(\int_s^t 1\,d\tau\right)^{1/2}\|\nabla\mathbf{u}_n^*\|_{L^2(L^2)} \\
&\qquad\quad + \mathsf{C}_P\,\|\mathbf{A}_F\|_{L^\infty(L^2(\operatorname{supp}\boldsymbol{\phi}_k))}\|\nabla\boldsymbol{\phi}_k\|_{C^0}\left(\int_s^t 1\,d\tau\right)^{1/2}\|\nabla\mathbf{u}_n^*\|_{L^2(L^2)} \\
&\qquad\quad + \|\mathbf{A}_F\|_{L^\infty(L^2(\operatorname{supp}\boldsymbol{\phi}_k))}\|\boldsymbol{\phi}_k\|_{C^0}\left(\int_s^t 1\,d\tau\right)^{1/2}\|\nabla\mathbf{u}_n^*\|_{L^2(L^2)}
\end{aligned}$$

$$
\begin{aligned}
&+ \|\boldsymbol{\phi}_k\|_{L^2} \left(\int_s^t 1\, d\tau \right)^{1/2} \|\mathbf{f}^*\|_{L^2(L^2)} \\
\overset{(3.36)}{\leq}\ & c \Big(\nu \|\nabla \boldsymbol{\phi}_k\|_{L^2} \|\mathbf{f}^*\|_{L^2(L^2)} + \|\boldsymbol{\phi}_k\|_{C^0} \|\mathbf{f}^*\|^2_{L^2(L^2)} \\
&\quad + \mathsf{C}_P \|\mathbf{A}_F\|_{L^\infty(L^2(\operatorname{supp} \boldsymbol{\phi}_k))} \|\nabla \boldsymbol{\phi}_k\|_{C^0} \|\mathbf{f}^*\|_{L^2(L^2)} \\
&\quad + \|\mathbf{A}_F\|_{L^\infty(L^2(\operatorname{supp} \boldsymbol{\phi}_k))} \|\boldsymbol{\phi}_k\|_{C^0} \|\mathbf{f}^*\|_{L^2(L^2)} \\
&\quad + \|\boldsymbol{\phi}_k\|_{L^2} \|\mathbf{f}^*\|_{L^2(L^2)} \Big) (t-s)^{1/2} ,
\end{aligned}
$$

implying uniform equicontinuity of $\big((\mathbf{u}_n^*, \boldsymbol{\phi}_k)\big)_{n\in\mathbb{N}}$ for each fixed $k \in \mathbb{N}$. Therefore, Arzelà-Ascoli's Theorem enables extraction of a subsequence (iteratively for every k) converging uniformly to a continuous limit function. Selecting a further subsequence diagonally from this array of sequences, we obtain a sequence – still denoted by index n – which satisfies the aforementioned convergence for all $k \in \mathbb{N}$:

$$
\big((\mathbf{u}_n^*, \boldsymbol{\phi}_k)\big) \xrightarrow{C([0,T])} G_k .
$$

Due to (3.36), for fixed $t \in [0,T]$ there is a subsequence $(\mathbf{u}_{n'}^*(t))_{n'}$ converging weakly in $L^2_\sigma(\Pi)$ to a limit $\hat{\mathbf{u}}(t)$ and we find

$$
(\hat{\mathbf{u}}(t), \boldsymbol{\phi}_k) = \lim_{n'\to\infty} (\mathbf{u}_{n'}^*(t), \boldsymbol{\phi}_k) = G_k(t) .
$$

Since $\{\boldsymbol{\phi}_k\}_k$ is a basis of $L^2_\sigma(\Pi)$, we derive through this identity that $\hat{\mathbf{u}}(t)$ is the weak limit of $\mathbf{u}_n^*(t)$ for all t. Note that $\mathbf{u}_n^*(t)$ is in fact weakly convergent in $L^2(\Pi)$, because of the Helmholtz-Weyl decomposition. Now, let $\mathbf{v} \in L^2(0,T; L^2_\sigma(\Pi))$. We observe

$$
\lim_{n\to\infty} \int_0^T \big(\mathbf{u}_n^*(t), \mathbf{v}(t)\big)\, dt = \int_0^T \lim_{n\to\infty} \big(\mathbf{u}_n^*(t), \mathbf{v}(t)\big)\, dt = \int_0^T \big(\hat{\mathbf{u}}(t), \mathbf{v}(t)\big)\, dt ,
$$

by the dominated convergence theorem, where the integrable upper bound is provided by (3.36). Therefore, $\mathbf{u}_n^* \rightharpoonup \hat{\mathbf{u}}$ in $L^2(0,T; L^2_\sigma(\Pi))$. On the other hand, we already know $\mathbf{u}_n^* \rightharpoonup \mathbf{u}^*$ in $L^2(0,T; L^2_\sigma(\Pi))$, implying $\hat{\mathbf{u}} = \mathbf{u}^*$. Furthermore, notice time-periodicity of $\mathbf{u}^*$ is preserved thanks to pointwise weak convergence:

$$
\mathbf{u}^*(0) = \operatorname*{w-lim}_{n\to\infty} \mathbf{u}_n^*(0) = \operatorname*{w-lim}_{n\to\infty} \mathbf{u}_n^*(T) = \mathbf{u}^*(T) .
$$

Now, we apply Friedrich's inequality (Lemma 2.5): Let $\Omega \subset \Pi$ be a bounded subdomain and $\eta > 0$, then

$$
\|\mathbf{u}_n^*(t) - \mathbf{u}^*(t)\|^2_{L^2(\Omega)} \leq \sum_{j=1}^{N_\eta} \big(\mathbf{u}_n^*(t) - \mathbf{u}^*(t), \mathbf{e}_j\big)^2_{L^2(\Omega)} + \eta \|\nabla \mathbf{u}_n^*(t) - \nabla \mathbf{u}^*(t)\|^2_{L^2(\Omega)}
$$

for almost all $t \in [0,T]$. Integration over t gives

$$
\int_0^T \|\mathbf{u}_n^* - \mathbf{u}^*\|^2_{L^2(\Omega)}\, dt \leq \sum_{j=1}^{N_\eta} \int_0^T \big(\mathbf{u}_n^* - \mathbf{u}^*, \mathbf{e}_j\big)^2_{L^2(\Omega)}\, dt + \eta \int_0^T \|\nabla \mathbf{u}_n^* - \nabla \mathbf{u}^*\|^2_{L^2(\Omega)}\, dt .
$$

Weak convergence of $(\mathbf{u}_n^*)_n$ in $L^2(0,T;W^{1,2}_{0,\sigma}(\Pi))$ guarantees boundedness of the last term. The inner product $\big(\mathbf{u}_n^* - \mathbf{u}^*, \mathbf{e}_j\big)_{L^2(\Omega)}$ converges to zero pointwise. By the dominated convergence theorem (combined with (3.36) again), we infer

$$\int_0^T \big(\mathbf{u}_n^* - \mathbf{u}^*, \mathbf{e}_j\big)^2_{L^2(\Omega)}\, dt \to 0\,.$$

Altogether, appropriately choosing η leads to an arbitrary small right-hand side, when $n \to \infty$, i.e.

$$\mathbf{u}_n^* \to \mathbf{u}^* \qquad \text{in } L^2(0,T;L^2(\Omega)) \text{ for each bounded } \Omega \subset \Pi\,.$$

Finally, we are in a position to justify convergence of the convective term. Due to compact support of the base functions $\boldsymbol{\phi}_k$, there are bounded Lipschitz domains $\Omega_k \supset \operatorname{supp} \boldsymbol{\phi}_k$. We conclude

$$\begin{aligned}
&\left|\int_0^T \big((\mathbf{u}_n^*\cdot\nabla)\mathbf{u}_n^*, h\boldsymbol{\phi}_k\big) - \big((\mathbf{u}^*\cdot\nabla)\mathbf{u}^*, h\boldsymbol{\phi}_k\big)\, dt\right| \\
&\overset{(2.5)}{=} \left|\int_0^T h\left[\big((\mathbf{u}_n^*\cdot\nabla)\boldsymbol{\phi}_k, \mathbf{u}_n^*\big) - \big((\mathbf{u}^*\cdot\nabla)\boldsymbol{\phi}_k, \mathbf{u}^*\big)\right]\, dt\right| \\
&= \left|\int_0^T h\left[\big(((\mathbf{u}_n^* - \mathbf{u}^*)\cdot\nabla)\boldsymbol{\phi}_k, \mathbf{u}_n^*\big) + \big((\mathbf{u}^*\cdot\nabla)\boldsymbol{\phi}_k, \mathbf{u}_n^* - \mathbf{u}^*\big)\right]\, dt\right| \\
&\le \int_0^T |h|\Big[\int_{\Omega_k} |\mathbf{u}_n^* - \mathbf{u}^*||\nabla\boldsymbol{\phi}_k||\mathbf{u}_n^*|\, dx + \int_{\Omega_k} |\mathbf{u}^*||\nabla\boldsymbol{\phi}_k||\mathbf{u}_n^* - \mathbf{u}^*|\, dx\Big]\, dt \\
&\le \|h\|_{C([0,T])}\|\boldsymbol{\phi}_k\|_{C^1(\Pi)}\Big[\int_0^T \|\mathbf{u}_n^* - \mathbf{u}^*\|_{L^2(\Omega_k)}\|\mathbf{u}_n^*\|_{L^2(\Pi)}\, dt \\
&\qquad + \int_0^T \|\mathbf{u}^*\|_{L^2(\Pi)}\|\mathbf{u}_n^* - \mathbf{u}^*\|_{L^2(\Omega_k)}\, dt\Big] \\
&\le \|h\|_{C([0,T])}\|\boldsymbol{\phi}_k\|_{C^1(\Pi)}\|\mathbf{u}_n^* - \mathbf{u}^*\|_{L^2(L^2(\Omega_k))}\left[\|\mathbf{u}_n^*\|_{L^2(L^2(\Pi))} + \|\mathbf{u}^*\|_{L^2(L^2(\Pi))}\right]
\end{aligned}$$

This last expression tends to zero, since weak convergence of $(\mathbf{u}_n^*)_n$ implies boundedness of the term in brackets.

We pass to the limit in (3.35) and recall that $\{h\boldsymbol{\phi}_k :\ k \in \mathbb{N},\, h \in C^\infty([0,T])\ T\text{-periodic}\}$ is dense in the space of time-periodic test functions $C^\infty([0,T];C^\infty_{0,\sigma}(\Pi))$. Thus, for such a test function $\boldsymbol{\varphi}$, we deduce

$$\begin{aligned}
\int_0^T &-\big(\mathbf{u}^*, \partial_t\boldsymbol{\varphi}\big) + \nu\big(\nabla\mathbf{u}^*, \nabla\boldsymbol{\varphi}\big) + \big((\mathbf{u}^*\cdot\nabla)\mathbf{u}^*, \boldsymbol{\varphi}\big) \\
&\qquad + \big((\mathbf{u}^*\cdot\nabla)\mathbf{A}_F, \boldsymbol{\varphi}\big) + \big((\mathbf{A}_F\cdot\nabla)\mathbf{u}^*, \boldsymbol{\varphi}\big)\, dt \\
&= \int_0^T \big(\mathbf{f}^*, \boldsymbol{\varphi}\big)\, dt\,.
\end{aligned}$$

Reformulating this identity in terms of $\mathbf{u}$ yields

$$\int_0^T -\left(\mathbf{u}, \partial_t\boldsymbol{\varphi}\right) + \nu\left(\nabla\mathbf{u}, \nabla\boldsymbol{\varphi}\right) + \big((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}\big)\, dt = \int_0^T \big(\mathbf{f}, \boldsymbol{\varphi}\big)\, dt\,.$$

And since $\mathbf{u} - \mathbf{A}_F = \mathbf{u}^* \in L^2(0,T;W^{1,2}_{0,\sigma}(\Pi))$, the function $\mathbf{u}$ indeed is a weak solution of the inhomogeneous Navier-Stokes equations with non-zero flux. □

4 Asymptotics

The aim of this chapter is to identify an asymptotic representation of solutions of the time-periodic Navier-Stokes equations in the layer Π, containing the flux driver as leading term. At first, in Section 4.1, we achieve a variant of Pileckas and Specovius-Neugebauer's main theorem of [42]. Their theorem states the asymptotic decomposition of solutions to the time-periodic Stokes problem, which is crucial for the investigation of spatial behavior of solutions to the Navier-Stokes system. In Section 4.2 we prove estimates for the nonlinear term in weighted Sobolev spaces. This enables application of the linear theory to derive an asymptotic expansion of solutions of the Navier-Stokes equations.

Note that we set $T = 2\pi$ and $\nu = 1$ in this chapter in alignment with the setting in Pileckas and Specovius-Neugebauer's paper [42].

4.1 Asymptotics in the linear case

Before analyzing the spatial behavior of solutions of the full Navier-Stokes system we treat the linear Stokes cases first, which is given by

$$\begin{aligned} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } [0,2\pi] \times \Pi\,, \\ \operatorname{div} \mathbf{u} &= g && \text{in } [0,2\pi] \times \Pi\,, \\ \mathbf{u}|_{\partial\Pi} &= \mathbf{a} && \text{for all } t \in [0,2\pi]\,, \\ \mathbf{u}|_{t=0} &= \mathbf{u}|_{t=2\pi} && \text{in } \Pi\,. \end{aligned} \tag{S}$$

In [42] an asymptotic expansion for distributional solutions $(\mathbf{u}, p)$ was implemented and we develop a variant of their main theorem ([42, Theorem 2.2]) in this section.

Firstly, we cite Lemma 3.4 from [42], which proves to be an essential tool:

Lemma 4.1. *Let $\mathbf{f} \in L^2(V^0_{\beta+1}(\Pi))$, $g, \nabla g, \partial_t g \in L^2(V^0_{\beta+2}(\Pi))$ be time-periodic, $\beta \in \mathbb{R}$, and $\mathbf{a} = 0$. Suppose $(\mathbf{u}, p) \in L^2(V^0_\beta(\Pi))$ is a distributional solution of the Stokes equations* (S)*, $\partial_t \mathbf{u} \in L^2(L^2_{loc}(\overline{\Pi}))$. Then $\mathbf{u}, \nabla\mathbf{u}, \nabla^2\mathbf{u}, \partial_t\mathbf{u}, \nabla p \in L^2(V^0_{\beta+1}(\Pi))$ and the following estimates hold true:*

$$\begin{aligned} \|\mathbf{u}, \nabla\mathbf{u}\|_{L^2(V^0_{\beta+1}(\Pi))} &\le c\left(\|\mathbf{f}\|_{L^2(V^0_{\beta+1}(\Pi))} + \|g\|_{L^2(V^0_{\beta+2}(\Pi))} + \|\mathbf{u}, p\|_{L^2(V^0_\beta(\Pi))}\right)\,, \\ \|\partial_t\mathbf{u}\|_{L^2(V^0_{\beta+1}(\Pi))} &\le c\left(\|\mathbf{f}\|_{L^2(V^0_{\beta+1}(\Pi))} + \|g, \partial_t g\|_{L^2(V^0_{\beta+2}(\Pi))} + \|\mathbf{u}, p\|_{L^2(V^0_\beta(\Pi))}\right)\,, \\ \|\nabla p, \nabla^2\mathbf{u}\|_{L^2(V^0_{\beta+1}(\Pi))} &\le c\left(\|\mathbf{f}\|_{L^2(V^0_{\beta+1}(\Pi))} + \|g, \nabla g, \partial_t g\|_{L^2(V^0_{\beta+2}(\Pi))} + \|\mathbf{u}, p\|_{L^2(V^0_\beta(\Pi))}\right)\,. \end{aligned}$$

To develop an asymptotic representation the plane harmonics $P^{(k,\pm)}\colon \mathbb{R}^2\backslash\{0\} \to \mathbb{R}$, with

$$P^{(k,+)}(y) = \frac{1}{\sqrt{2\pi|k|}} r^k \cos(k\phi)\,, \quad P^{(k,-)}(y) = \frac{1}{\sqrt{2\pi|k|}} r^k \sin(k\phi)$$

for $k \in \mathbb{Z}\backslash\{0\}$ and

$$P^{(0,+)}(y) = 1\,, \quad P^{(0,-)}(y) = -\frac{1}{\sqrt{2\pi}}\ln(r)\,,$$

(written in polar coordinates (r,ϕ)) play a crucial role, making its appearance in the following lemma already:

Lemma 4.2. *Let $\mathcal{F} \in V^2_{-\gamma_1}(\mathbb{R}^2)' \subset V^2_{-\gamma}(\mathbb{R}^2)'$, $\gamma, \gamma_1 \in \mathbb{R}\backslash\mathbb{Z}$, $\gamma_1 > \gamma$. Suppose $\psi \in V^0_\gamma(\mathbb{R}^2)$ is a solution of Poisson's equation*

$$-\Delta\psi = \mathcal{F}\,,$$

with $-\Delta\colon V^0_\gamma(\mathbb{R}^2) \to V^2_\gamma(\mathbb{R}^2)'$, $\psi \mapsto (\psi, -\Delta\cdot)$. Then, ψ admits the representation

$$\psi(y) = \chi(r)\Bigg(\sum_{\substack{-\gamma_1-1<k<-\gamma-1,\\ k\in\mathbb{Z}}}\sum_{\pm} c^{(k,\pm)}P^{(k,\pm)}(y)\Bigg) + \psi^{(1)}(y)\,,$$

where χ is a smooth cut-off function, $\chi(r) = 1$ for $r \geq 2$, $\chi(r) = 0$ for $r \leq 1$ and $c^{(k,\pm)}$ are constants. Additionally, there holds $\psi^{(1)} \in V^0_{\gamma_1}(\mathbb{R}^2)$ and

$$\|\psi^{(1)}\|_{V^0_{\gamma_1}(\mathbb{R}^2)} + \sum_{-\gamma_1-1<k<-\gamma-1}\left(|c^{(k,+)}| + |c^{(k,-)}|\right) \leq c\left(\|\mathcal{F}\|_{V^2_{-\gamma_1}(\mathbb{R}^2)'} + \|\psi\|_{V^0_\gamma(\mathbb{R}^2)}\right).$$

This lemma is a derivation of a far more general statement on elliptic equations in corner domains from Nazarov and Plamenevskiĭ's book [37] (see Theorems 3.5.7 and 4.2.4 therein).

A key ingredient to obtain an asymptotic representation of solutions to the Stokes equations is the investigation of the mean pressure $\overline{p}$ in the plane $\mathbb{R}^2$ and its behavior as $|y| \to \infty$. We define

$$p = \overline{p} + p_\perp\,, \quad \text{with } \overline{p}(t,y) := \int_0^1 p(t,y,z)\,dz\,.$$

Theorem 4.3. *Let $(\mathbf{u}, p) \in L^2(0,2\pi; V^0_\gamma(\Pi)) \times L^2(0,2\pi; V^0_\gamma(\Pi))$ be a distributional solution pair of the Stokes equations* (S) *with time-periodic data $\mathbf{f} \in L^2(0,2\pi; V^0_{\gamma_1+1}(\Pi))$, $g, \nabla g, \partial_t g \in L^2(0,2\pi; V^0_{\gamma_1+2}(\Pi))$, where $\gamma_1, \gamma \in \mathbb{R}\backslash\mathbb{Z}$, $\gamma \leq \gamma_1 \leq \gamma+1$, and $\mathbf{a} = 0$. Further, suppose $\partial_t\mathbf{u} \in L^2(0,2\pi; L^2_{loc}(\overline{\Pi}))$.*

(i) *The mean pressure $\overline{p}$ admits the representation*

$$\overline{p}(t,y) = \chi(r)\Bigg(\sum_{\substack{-\gamma_1-1<k<-\gamma-1,\\ k\in\mathbb{Z}}} -s^{(k,+)}(t)P^{(k,+)}(y) - s^{(k,-)}(t)P^{(k,-)}(y)\Bigg) + \overline{p}^{(1)}(t,y)\,,$$

with $\overline{p}^{(1)} \in L^2(0,2\pi; V^0_{\gamma_1}(\mathbb{R}^2))$ and χ as before (cf. Lemma 4.2). There holds the a-priori estimate

$$\begin{aligned}\|\overline{p}^{(1)}\|_{L^2(V^0_{\gamma_1}(\mathbb{R}^2))} + \sum_{\substack{-\gamma_1-1<k<-\gamma-1,\\ k\in\mathbb{Z}}}\Big(\|s^{(k,+)}\|_{L^2(0,2\pi)} + \|s^{(k,-)}\|_{L^2(0,2\pi)}\Big)\\ \leq c\left(\|\mathbf{f}\|_{L^2(V^0_{\gamma_1+1}(\Pi))} + \|g, \nabla g, \partial_t g\|_{L^2(V^0_{\gamma_1+2}(\Pi))} + \|\mathbf{u}, p\|_{L^2(V^0_\gamma(\Pi))}\right).\end{aligned}$$

(ii) *If the interval* (γ,γ_1) *does not contain any integers,* $(\gamma,\gamma_1)\cap\mathbb{Z}=\emptyset$*, then* $p\in L^2(0,2\pi;V^0_{\gamma_1}(\Pi))$ *and* $\mathbf{u},\nabla\mathbf{u},\nabla^2\mathbf{u},\partial_t\mathbf{u},\nabla p\in L^2(0,2\pi;V^0_{\gamma_1+1}(\Pi))$ *with*

$$\begin{aligned}&\|p\|_{L^2(V^0_{\gamma_1}(\Pi))}+\|\mathbf{u},\nabla\mathbf{u},\nabla^2\mathbf{u},\partial_t\mathbf{u},\nabla p\|_{L^2(V^0_{\gamma_1+1}(\Pi))}\\&\quad\le c\Big(\|\mathbf{f}\|_{L^2(V^0_{\gamma_1+1}(\Pi))}+\|g,\nabla g,\partial_t g\|_{L^2(V^0_{\gamma_1+2}(\Pi))}+\|\mathbf{u},p\|_{L^2(V^0_{\gamma}(\Pi))}\Big)\,.\end{aligned}$$

Proof. *ad*(i): At first we use Lemma 4.1 inserting $\mathbf{u},p\in L^2(V^0_\gamma(\Pi))$, $\mathbf{f}\in L^2(V^0_{\gamma+1}(\Pi))$, $g,\nabla g,\partial_t g\in L^2(V^0_{\gamma+2}(\Pi))$ and Lemma 3.5 of [42] with $\mathbf{u},p\in L^2(V^0_{\gamma_1-1}(\Pi))$, $\mathbf{f}\in L^2(V^0_{\gamma_1+1}(\Pi))$, $g,\partial_z g,\partial_t g\in L^2(V^0_{\gamma_1+1}(\Pi))$, $\nabla_y g\in L^2(V^0_{\gamma_1+2}(\Pi))$ to improve the weight exponent of $\mathbf{u},p$. We find

$$\begin{aligned}&\|\mathbf{u},\nabla\mathbf{u},\nabla^2\mathbf{u},\partial_t\mathbf{u},\nabla p\|_{L^2(V^0_{\gamma+1}(\Pi))}\\&\quad\le c\Big(\|\mathbf{f}\|_{L^2(V^0_{\gamma+1}(\Pi))}+\|g,\nabla g,\partial_t g\|_{L^2(V^0_{\gamma+2}(\Pi))}+\|\mathbf{u},p\|_{L^2(V^0_{\gamma}(\Pi))}\Big)\\&\quad\le c\Big(\|\mathbf{f}\|_{L^2(V^0_{\gamma_1+1}(\Pi))}+\|g,\nabla g,\partial_t g\|_{L^2(V^0_{\gamma_1+2}(\Pi))}+\|\mathbf{u},p\|_{L^2(V^0_{\gamma}(\Pi))}\Big)\,,\end{aligned}\tag{4.1}$$

$$\begin{aligned}&\|\nabla_y\mathbf{u},\nabla\nabla_y\mathbf{u}\|_{L^2(V^0_{\gamma_1+1}(\Pi))}\\&\quad\le c\Big(\|\mathbf{f}\|_{L^2(V^0_{\gamma_1+1})}+\|g,\partial_z g,\partial_t g\|_{L^2(V^0_{\gamma_1+1})}+\|\nabla_y g\|_{L^2(V^0_{\gamma_1+2})}+\|\mathbf{u},p\|_{L^2(V^0_{\gamma_1-1})}\Big)\\&\quad\le c\Big(\|\mathbf{f}\|_{L^2(V^0_{\gamma_1+1}(\Pi))}+\|g,\nabla g,\partial_t g\|_{L^2(V^0_{\gamma_1+2}(\Pi))}+\|\mathbf{u},p\|_{L^2(V^0_{\gamma}(\Pi))}\Big)\,.\end{aligned}\tag{4.2}$$

In order to apply Lemma 4.2 we write down the Fourier series of $\overline{p}$ with respect to the orthogonal basis $\{\frac12\}\cup\{\cos(nt)\,,\ n\in\mathbb{N}\}\cup\{\sin(nt)\,,\ n\in\mathbb{N}\}$ of $L^2(0,2\pi;\mathcal{H})$ – for any Hilbert space $\mathcal{H}$:

$$\overline{p}(t,y)=\frac{\overline{p}_{c0}(y)}{2}+\sum_{n=1}^{\infty}\overline{p}_{cn}(y)\cos(nt)+\overline{p}_{sn}(y)\sin(nt)\,,$$

$$\|\overline{p}\|^2_{L^2(0,2\pi;V^0_\gamma(\mathbb{R}^2))}=\frac{\pi}{2}\|\overline{p}_{c0}\|^2_{V^0_\gamma(\mathbb{R}^2)}+\pi\sum_{n=1}^{\infty}\Big(\|\overline{p}_{cn}\|^2_{V^0_\gamma(\mathbb{R}^2)}+\|\overline{p}_{sn}\|^2_{V^0_\gamma(\mathbb{R}^2)}\Big)\,.$$

Furthermore, we need auxiliary functions solving the inverse problem

$$\begin{aligned}\partial_t h(t,z)+\partial_z^2 h(t,z)&=s(t) &&\text{in }[0,2\pi]\times(0,1)\,,\\ h(t,0)=h(t,1)&=0 &&\text{in }[0,2\pi]\,,\\ h(0,z)&=h(2\pi,z) &&\text{in }(0,1)\,,\\ \int_0^1 h(t,z)\,dz&=\phi(t) &&\text{in }[0,2\pi]\,,\end{aligned}\tag{4.3}$$

where h,s are unknown and ϕ is a given time-periodic function. This problem is analyzed by Pileckas, Specovius-Neugebauer in [42, Section 4] and by Galdi, Robertson in [10]. We denote by (h_{c0},s_{c0}), (h_{cn},s_{cn}), (h_{sn},s_{sn}) the solutions regarding the data

$$\phi_{c0}(t):=\tfrac12\,,\ \ \phi_{cn}(t):=\cos(nt)\ \ \text{and}\ \ \phi_{sn}(t):=\sin(nt)\,,$$

for $n\in\mathbb{N}$. These functions satisfy the following estimates (see (4.10),(4.11) in [42]):

$$\left\|\int_0^{2\pi}\int_0^1 h_n(t,z)\,w(t,\cdot,z)\,dz\,dt\right\|^2_{V^0_\beta(\mathbb{R}^2)}\le c\left(\|w_{cn}\|^2_{V^0_\beta(\Pi)}+\|w_{sn}\|^2_{V^0_\beta(\Pi)}\right),\tag{4.4}$$

$$\left\| \int_0^{2\pi} \int_0^1 s_n(t)\, w(t,\cdot,z)\, dz\, dt \right\|^2_{V^0_\beta(\mathbb{R}^2)} \le c\, n^2 \Big(\|w_{cn}\|^2_{V^0_\beta(\Pi)} + \|w_{sn}\|^2_{V^0_\beta(\Pi)} \Big)\,, \tag{4.5}$$

$$\left\| \int_0^{2\pi} \int_0^1 s_{c0}\, w(t,\cdot,z)\, dz\, dt \right\|^2_{V^0_\beta(\mathbb{R}^2)} \le c\, \|w_{c0}\|^2_{V^0_\beta(\Pi)}\,, \tag{4.6}$$

with $\beta \in \mathbb{R}$, $w \in L^2(0,2\pi;V^0_\beta(\Pi))$ and its Fourier coefficients w_{c0}, w_{cn}, w_{sn}. The indices in (4.4), (4.5) are abbreviated, such that h_n, may be substituted by either h_{cn}, h_{sn} or h_{c0} and s_n by either s_{cn} or s_{sn}. Then, we set

$$\begin{aligned} H_{c0}(z) &:= \int_0^z h_{c0}(\xi)\, d\xi - \phi_{c0} z\,, \\ H_{cn}(t,z) &:= \int_0^z h_{cn}(t,\xi)\, d\xi - \phi_{cn}(t) z\,, \quad n \in \mathbb{N}\,, \\ H_{sn}(t,z) &:= \int_0^z h_{sn}(t,\xi)\, d\xi - \phi_{sn}(t) z\,, \quad n \in \mathbb{N}\,. \end{aligned}$$

For H_{c0}, H_{cn}, H_{sn} we have at hand the decomposition $H_n(t,z) = H_n^+(z)\cos(nt) + H_n^-(z)\sin(nt)$ and $\|H_n\|_{L^2(0,2\pi;L^2(0,1))} \le \|h_n\|_{L^2(0,2\pi;L^2(0,1))} + \sqrt{\frac{\pi}{3}} \le c$ independent of n (cf. [42, (4.7)]). Therefore,

$$\int_0^1 |H_n^+|^2 + |H_n^-|^2\, dz = \tfrac{1}{\pi} \|H_n\|^2_{L^2(0,2\pi;L^2(0,1))} \le c\,. \tag{4.7}$$

Now, the proof is executed for $\overline{p}_{cn}$; verifying the assertion that $\overline{p}_{cn}$ is a solution of Poisson's equation

$$-\Delta \overline{p}_{cn} = \mathcal{F}$$

for some $\mathcal{F} \in V^2_{-\gamma_1}(\mathbb{R}^2)'$, as in Lemma 4.2. The argumentation is completely analog for $\overline{p}_{sn}$ and with minor adjustments for $\overline{p}_{c0}$ only. The following identity holds[1]

$$\begin{aligned} \sqrt{\pi}\, \overline{p}_{cn} &= \int_0^{2\pi} \phi_{cn}\, \overline{p}\, dt \\ &= \int_0^{2\pi} \int_0^1 \phi_{cn}\, p\, dz\, dt \\ &= \int_0^{2\pi} \int_0^1 (-\partial_z H_{cn} + h_{cn})\, p\, dz\, dt \\ &= \int_0^{2\pi} \int_0^1 H_{cn}\, \partial_z p\, dz\, dt + \int_0^{2\pi} \int_0^1 h_{cn}\, p\, dz\, dt\,. \end{aligned}$$

Recall the notation $\mathbf{u}' = (u_1, u_2)$, $\mathbf{f}' = (f_1, f_2)$. Assuming $\varphi \in C_0^\infty(\mathbb{R}^2)$, we infer

$$\begin{aligned} &\langle -\Delta \overline{p}_{cn}, \varphi \rangle \\ &\quad = \int_{\mathbb{R}^2} \overline{p}_{cn}\, (-\Delta_y \varphi)\, dy \\ &\quad = \int_{\mathbb{R}^2} \tfrac{1}{\sqrt{\pi}} \Big(\int_0^{2\pi} \int_0^1 H_{cn}\, \partial_z p\; dz\, dt + \int_0^{2\pi} \int_0^1 h_{cn}\, p\; dz\, dt \Big) (-\Delta_y \varphi)\, dy \end{aligned}$$

[1]Insert $\sqrt{\frac{\pi}{2}}$ as factor here and in the following, in the case of $\overline{p}_{c0}$.

$$
\begin{aligned}
&=: \int_{\mathbb{R}^2} \mathcal{F}_0\, \Delta_y \varphi \; dy + \tfrac{1}{\sqrt{\pi}} \int_0^{2\pi} \int_0^1 h_{cn} \int_{\mathbb{R}^2} \nabla_y p \cdot \nabla_y \varphi \; dy\, dz\, dt \\
&= (\mathcal{F}_0, \Delta_y \varphi) + \tfrac{1}{\sqrt{\pi}} \int_{\mathbb{R}^2} \int_0^{2\pi} \int_0^1 (\mathbf{f}' - \partial_t \mathbf{u}' + \Delta \mathbf{u}') h_{cn} \; dz\, dt \cdot \nabla_y \varphi \, dy \\
&= (\mathcal{F}_0, \Delta_y \varphi) + \tfrac{1}{\sqrt{\pi}} \int_{\mathbb{R}^2} \int_0^{2\pi} \int_0^1 h_{cn}\, \mathbf{f}' \, dz\, dt \cdot \nabla_y \varphi \, dy \\
&\quad + \tfrac{1}{\sqrt{\pi}} \int_{\mathbb{R}^2} \int_0^{2\pi} \int_0^1 h_{cn}\, \Delta_y \mathbf{u}' \, dz\, dt \cdot \nabla_y \varphi \, dy \\
&\quad + \tfrac{1}{\sqrt{\pi}} \int_{\mathbb{R}^2} \int_0^{2\pi} \int_0^1 (\partial_t h_{cn} + \partial_z^2 h_{cn}) \mathbf{u}' \, dz\, dt \cdot \nabla_y \varphi \, dy \\
&=: \big((\mathcal{F}_0, \Delta_y \varphi) + (\mathcal{F}_1, \nabla_y \varphi) + (\mathcal{F}_2, \nabla_y \varphi) \big) \\
&\quad + \tfrac{1}{\sqrt{\pi}} \int_0^{2\pi} \int_0^1 \int_{\mathbb{R}^2} s_{cn}(t)\, \mathbf{u}' \cdot \nabla_y \varphi \; dy\, dz\, dt \,.
\end{aligned}
$$

We elaborate on the last term:

$$
\begin{aligned}
&\int_0^{2\pi} \int_0^1 \int_{\mathbb{R}^2} s_{cn}\, \mathbf{u}' \cdot \nabla_y \varphi \; dy\, dz\, dt \\
&\quad = - \int_0^{2\pi} s_{cn} \int_0^1 \int_{\mathbb{R}^2} \varphi \; \mathrm{div}_y\, \mathbf{u}' \; dy\, dz\, dt \\
&\quad = - \int_{\mathbb{R}^2} \int_0^{2\pi} s_{cn} \left(\int_0^1 g\, dz - \int_0^1 \partial_z u_3 \, dz \right) \varphi \, dt\, dy \\
&\quad = - \int_{\mathbb{R}^2} \int_0^{2\pi} \int_0^1 s_{cn}\, g\, dz\, dt \; \varphi \, dy \\
&\quad =: (\sqrt{\pi} \mathcal{F}_3, \varphi) \,.
\end{aligned}
$$

In summary, we have $\langle \mathcal{F}, \varphi \rangle = (\mathcal{F}_0, \Delta_y \varphi) + (\mathcal{F}_1, \nabla_y \varphi) + (\mathcal{F}_2, \nabla_y \varphi) + (\mathcal{F}_3, \varphi)$, where

$$
\begin{aligned}
\mathcal{F}_0(y) &= - \tfrac{1}{\sqrt{\pi}} \int_0^{2\pi} \int_0^1 H_{cn}\, \partial_z p \; dz\, dt \,, \\
\mathcal{F}_1(y) &= \tfrac{1}{\sqrt{\pi}} \int_0^{2\pi} \int_0^1 h_{cn}\, \mathbf{f}' \; dz\, dt \,, \\
\mathcal{F}_2(y) &= \tfrac{1}{\sqrt{\pi}} \int_0^{2\pi} \int_0^1 h_{cn}\, \Delta_y \mathbf{u}' \; dz\, dt \,, \\
\mathcal{F}_3(y) &= - \tfrac{1}{\sqrt{\pi}} \int_0^{2\pi} \int_0^1 s_{cn}\, g \, dz\, dt \,,
\end{aligned}
$$

as derived above. We note the following estimates

$$
\begin{aligned}
\|\mathcal{F}_1\|^2_{V^0_{\gamma_1+1}(\mathbb{R}^2)} &= \left\| \tfrac{1}{\sqrt{\pi}} \int_0^{2\pi} \int_0^1 h_{cn}\, \mathbf{f}' \; dz\, dt \right\|^2_{V^0_{\gamma_1+1}(\mathbb{R}^2)} \\
&\overset{(4.4)}{\leq} c \left(\|\mathbf{f}'_{cn}\|^2_{V^0_{\gamma_1+1}(\Pi)} + \|\mathbf{f}'_{sn}\|^2_{V^0_{\gamma_1+1}(\Pi)} \right) ,
\end{aligned}
$$

$$\begin{aligned}\|\mathcal{F}_2\|^2_{V^0_{\gamma_1+1}(\mathbb{R}^2)} &= \left\| \tfrac{1}{\sqrt{\pi}} \int_0^{2\pi}\int_0^1 h_{cn}\,\Delta_y \mathbf{u}'\;dz\,dt \right\|^2_{V^0_{\gamma_1+1}(\mathbb{R}^2)} \\ &\overset{(4.4)}{\leq} c\left(\|(\Delta_y \mathbf{u}')_{cn}\|^2_{V^0_{\gamma_1+1}(\Pi)} + \|(\Delta_y \mathbf{u}')_{sn}\|^2_{V^0_{\gamma_1+1}(\Pi)}\right) \overset{(4.2)}{<} \infty\,,\end{aligned}$$

$$\begin{aligned}\|\mathcal{F}_3\|^2_{V^0_{\gamma_1+2}(\mathbb{R}^2)} &= \left\| \tfrac{1}{\sqrt{\pi}} \int_0^{2\pi}\int_0^1 s_{cn}\,g\;dz\,\mathrm{d}t \right\|^2_{V^0_{\gamma_1+2}(\mathbb{R}^2)} \\ &\overset{(4.5)}{\leq} c\,n^2\left(\|g_{cn}\|^2_{V^0_{\gamma_1+2}(\Pi)} + \|g_{sn}\|^2_{V^0_{\gamma_1+2}(\Pi)}\right)\end{aligned}$$

respectively

$$\|\mathcal{F}_3\|^2_{V^0_{\gamma_1+2}(\mathbb{R}^2)} = \left\| \sqrt{\tfrac{2}{\pi}} \int_0^{2\pi}\int_0^1 s_{c0}\,g\;dz\,\mathrm{d}t \right\|^2_{V^0_{\gamma_1+2}(\mathbb{R}^2)} \overset{(4.6)}{\leq} c\,\|g_{c0}\|^2_{V^0_{\gamma_1+2}(\Pi)}\,.$$

Regarding $\mathcal{F}_0$ we find

$$\begin{aligned}&|\mathcal{F}_0(y)|^2 \\ &\quad= \left| \tfrac{1}{\sqrt{\pi}} \int_0^{2\pi}\int_0^1 H_{cn}\,\partial_z p\;dz\,dt\right|^2 \\ &\quad= \left| \tfrac{1}{\sqrt{\pi}} \int_0^{1}\int_0^{2\pi} \big(H^+_{cn}\cos(nt) + H^-_{cn}\sin(nt)\big)\,\partial_z\Big(\tfrac{p_{c0}}{2} + \sum_{n=1}^{\infty} p_{cn}\cos(nt) + p_{sn}\sin(nt)\Big)dt\,dz\right|^2 \\ &\quad\leq 2\pi\left(\left|\int_0^1 H^+_{cn}\,\partial_z p_{cn}\;dz\right|^2 + \left|\int_0^1 H^-_{cn}\,\partial_z p_{sn}\;dz\right|^2\right) \\ &\quad\leq 2\pi\left(\int_0^1 |H^+_{cn}|^2\,dz \int_0^1 |\partial_z p_{cn}|^2\,dz + \int_0^1 |H^-_{cn}|^2\,dz \int_0^1 |\partial_z p_{sn}|^2\,dz\right) \\ &\quad\overset{(4.7)}{\leq} c\left(\int_0^1 |\partial_z p_{cn}|^2\,dz + \int_0^1 |\partial_z p_{sn}|^2\,dz\right)\,,\end{aligned}$$

resulting in

$$\|\mathcal{F}_0\|^2_{V^0_{\gamma_1}(\mathbb{R}^2)} \leq c\left(\|\partial_z p_{cn}\|^2_{V^0_{\gamma_1}(\Pi)} + \|\partial_z p_{sn}\|^2_{V^0_{\gamma_1}(\Pi)}\right)\,.$$

For $v \in V^2_{-\gamma_1}(\mathbb{R}^2)$ we then obtain

$$\begin{aligned}&\left|\langle \mathcal{F}, v\rangle_{(V^2_{-\gamma_1})', V^2_{-\gamma_1}}\right| \\ &\quad= |(\mathcal{F}_0, \Delta_y v) + (\mathcal{F}_1, \nabla_y v) + (\mathcal{F}_2, \nabla_y v) + (\mathcal{F}_3, v)| \\ &\quad\leq \|\mathcal{F}_0\|_{V^0_{\gamma_1}}\|\Delta_y v\|_{V^0_{-\gamma_1}} + \|\mathcal{F}_1\|_{V^0_{\gamma_1+1}}\|\nabla_y v\|_{V^0_{-\gamma_1-1}} \\ &\qquad\quad + \|\mathcal{F}_2\|_{V^0_{\gamma_1+1}}\|\nabla_y v\|_{V^0_{-\gamma_1-1}} + \|\mathcal{F}_3\|_{V^0_{\gamma_1+2}}\|v\|_{V^0_{-\gamma_1-2}} \\ &\quad\leq \Big(\|\mathcal{F}_0\|_{V^0_{\gamma_1}} + \|\mathcal{F}_1\|_{V^0_{\gamma_1+1}} + \|\mathcal{F}_2\|_{V^0_{\gamma_1+1}} + \|\mathcal{F}_3\|_{V^0_{\gamma_1+2}}\Big)\|v\|_{V^2_{-\gamma_1}}\,,\end{aligned}$$

hence $\mathcal{F} \in V^2_{-\gamma_1}(\mathbb{R}^2)'$. As claimed the functions $\overline{p}_{c0}$, $\overline{p}_{cn}$ and $\overline{p}_{sn}$ are solutions of Poisson's equation with respective right-hand side $\mathcal{F}$ and by Lemma 4.2

$$\overline{p}_{cn} = \chi\Bigg(\sum_{\substack{-\gamma_1-1<k<-\gamma-1,\\ k\in\mathbb{Z}}} c_{cn}^{(k,+)} P^{(k,+)} + c_{cn}^{(k,-)} P^{(k,-)} \Bigg) + \overline{p}^{(1)}_{cn}\,, \quad n \in \mathbb{N}_0\,,$$
$$\overline{p}_{sn} = \chi\Bigg(\sum_{\substack{-\gamma_1-1<k<-\gamma-1,\\ k\in\mathbb{Z}}} c_{sn}^{(k,+)} P^{(k,+)} + c_{sn}^{(k,-)} P^{(k,-)} \Bigg) + \overline{p}^{(1)}_{sn}\,, \quad n \in \mathbb{N}\,,$$

with

$$\begin{aligned} &\|\overline{p}^{(1)}_{cn}\|_{V^0_{\gamma_1}(\mathbb{R}^2)} + \sum_{\substack{-\gamma_1-1<k<-\gamma-1,\\ k\in\mathbb{Z}}} \big(|c_{cn}^{(k,+)}| + |c_{cn}^{(k,-)}|\big) \\ &\le c\left(\|\mathcal{F}\|_{(V^2_{-\gamma_1})'} + \|\overline{p}_{cn}\|_{V^0_\gamma(\mathbb{R}^2)}\right) \\ &\le c\Big(\|\partial_z p_{cn}, \partial_z p_{sn}\|_{V^0_{\gamma_1}(\Pi)} + \|\mathbf{f}'_{cn}, \mathbf{f}'_{sn}\|_{V^0_{\gamma_1+1}(\Pi)} + \|(\Delta_y \mathbf{u}')_{cn}, (\Delta_y \mathbf{u}')_{sn}\|_{V^0_{\gamma_1+1}(\Pi)} \\ &\quad + n\,\|g_{cn}, g_{sn}\|_{V^0_{\gamma_1+2}(\Pi)} + \|\overline{p}_{cn}\|_{V^0_\gamma(\mathbb{R}^2)}\Big) \end{aligned}$$

and an analogous estimate for $\overline{p}^{(1)}_{sn}$, as well as

$$\begin{aligned} &\|\overline{p}^{(1)}_{c0}\|_{V^0_{\gamma_1}(\mathbb{R}^2)} + \sum_{\substack{-\gamma_1-1<k<-\gamma-1,\\ k\in\mathbb{Z}}} \big(|c_{c0}^{(k,+)}| + |c_{c0}^{(k,-)}|\big) \\ &\le c\Big(\|\partial_z p_{c0}\|_{V^0_{\gamma_1}(\Pi)} + \|\mathbf{f}'_{c0}\|_{V^0_{\gamma_1+1}(\Pi)} + \|(\Delta_y \mathbf{u}')_{c0}\|_{V^0_{\gamma_1+1}(\Pi)} \\ &\quad + \|g_{c0}\|_{V^0_{\gamma_1+2}(\Pi)} + \|\overline{p}_{c0}\|_{V^0_\gamma(\mathbb{R}^2)}\Big)\,. \end{aligned}$$

We set

$$-s^{(k,\pm)}(t) := \frac{c_{c0}^{(k,\pm)}}{2} + \sum_{n=1}^{\infty} c_{cn}^{(k,\pm)} \cos(nt) + c_{sn}^{(k,\pm)} \sin(nt)\,,$$
$$\overline{p}^{(1)}(t,y) := \frac{\overline{p}^{(1)}_{c0}(y)}{2} + \sum_{n=1}^{\infty} \overline{p}^{(1)}_{cn}(y) \cos(nt) + \overline{p}^{(1)}_{sn}(y) \sin(nt)\,,$$

which finally furnishes

$$\overline{p}(t,y) = \chi(r)\Bigg(\sum_{\substack{-\gamma_1-1<k<-\gamma-1,\\ k\in\mathbb{Z}}} -s^{(k,+)}(t) P^{(k,+)}(y) - s^{(k,-)}(t) P^{(k,-)}(y) \Bigg) + \overline{p}^{(1)}(t,y)$$

and

$$
\begin{aligned}
&\|\overline{p}^{(1)}\|_{L^2(V^0_{\gamma_1}(\mathbb{R}^2))} + \sum_{\substack{-\gamma_1-1<k<-\gamma-1,\\ k\in\mathbb{Z}}} \left(\|s^{(k,+)}\|_{L^2(0,2\pi)} + \|s^{(k,-)}\|_{L^2(0,2\pi)}\right) \\
&= \left[\frac{\pi}{2}\|\overline{p}^{(1)}_{c0}\|^2_{V^0_{\gamma_1}(\mathbb{R}^2)} + \pi\sum_{n=1}^{\infty}\|\overline{p}^{(1)}_{cn}\|^2_{V^0_{\gamma_1}(\mathbb{R}^2)} + \|\overline{p}^{(1)}_{sn}\|^2_{V^0_{\gamma_1}(\mathbb{R}^2)}\right]^{1/2} \\
&\quad + \sum_{\substack{-\gamma_1-1<k<-\gamma-1,\\ k\in\mathbb{Z}}} \left(\left[\frac{\pi}{2}|c^{(k,+)}_{c0}|^2 + \pi\sum_{n=1}^{\infty}|c^{(k,+)}_{cn}|^2 + |c^{(k,+)}_{sn}|^2\right]^{1/2}\right. \\
&\qquad \left. + \left[\frac{\pi}{2}|c^{(k,-)}_{c0}|^2 + \pi\sum_{n=1}^{\infty}|c^{(k,-)}_{cn}|^2 + |c^{(k,-)}_{sn}|^2\right]^{1/2}\right) \\
&\le c\left[\left(\|\partial_z p_{c0}\|^2_{V^0_{\gamma_1}(\Pi)} + \|\mathbf{f}'_{c0}\|^2_{V^0_{\gamma_1+1}(\Pi)} + \|(\Delta_y\mathbf{u}')_{c0}\|^2_{V^0_{\gamma_1+1}(\Pi)}\right.\right. \\
&\qquad \left. + \|g_{c0}\|^2_{V^0_{\gamma_1+2}(\Pi)} + \|\overline{p}_{c0}\|^2_{V^0_{\gamma}(\mathbb{R}^2)}\right) \\
&\quad + \sum_{n=1}^{\infty}\left(\|\partial_z p_{cn},\partial_z p_{sn}\|^2_{V^0_{\gamma_1}(\Pi)} + \|\mathbf{f}'_{cn},\mathbf{f}'_{sn}\|^2_{V^0_{\gamma_1+1}(\Pi)} + \|(\Delta_y\mathbf{u}')_{cn},(\Delta_y\mathbf{u}')_{sn}\|^2_{V^0_{\gamma_1+1}(\Pi)}\right. \\
&\qquad \left.\left. + n^2\,\|g_{cn},g_{sn}\|^2_{V^0_{\gamma_1+2}(\Pi)} + \|\overline{p}_{cn},\overline{p}_{sn}\|^2_{V^0_{\gamma}(\mathbb{R}^2)}\right)\right]^{1/2} \\
&= c\left[\|\partial_z p\|^2_{L^2(V^0_{\gamma_1}(\Pi))} + \|\mathbf{f}'\|^2_{L^2(V^0_{\gamma_1+1}(\Pi))} + \|\Delta_y\mathbf{u}'\|^2_{L^2(V^0_{\gamma_1+1}(\Pi))}\right. \\
&\quad \left. + \|g_{c0}\|^2_{V^0_{\gamma_1+2}(\Pi)} + \|\partial_t g\|^2_{L^2(V^0_{\gamma_1+2}(\Pi))} + \|\overline{p}\|^2_{L^2(V^0_{\gamma}(\mathbb{R}^2))}\right]^{1/2} \\
&\overset{(4.1),(4.2)}{\le} c\left(\|\mathbf{f}\|_{L^2(V^0_{\gamma_1+1}(\Pi))} + \|g,\nabla g,\partial_t g\|_{L^2(V^0_{\gamma_1+2}(\Pi))} + \|\mathbf{u},p\|_{L^2(V^0_{\gamma}(\Pi))}\right).
\end{aligned}
$$

ad(ii): Let $(\gamma,\gamma_1)\cap\mathbb{Z}=\emptyset$, which implies $\{k\in\mathbb{Z} : -\gamma_1-1<k<-\gamma-1\}=\emptyset$ for the summation index. Hence, the representation of (i) reduces to

$$\overline{p} = \overline{p}^{(1)} \ \in L^2(V^0_{\gamma_1}(\mathbb{R}^2)).$$

Obviously $\int_0^1 p_\perp\,dz = \int_0^1 p - \overline{p}\,dz = 0$, so Poincaré's inequality yields

$$\int_0^1 |p_\perp|^2\,dz \le c\int_0^1 |\partial_z p_\perp|^2\,dz = c\int_0^1 |\partial_z p|^2\,dz\,,$$

leading to

$$
\begin{aligned}
&\int_0^{2\pi}\int_{\mathbb{R}^2}(1+r^2)^{\gamma+1}\int_0^1 |p_\perp|^2\,dz\,dy\,dt \\
&\qquad \le \int_0^{2\pi}\int_{\mathbb{R}^2}(1+r^2)^{\gamma+1}\,c\int_0^1 |\nabla p|^2\,dz\,dy\,dt\,,
\end{aligned}
$$

i.e. $\|p_\perp\|_{L^2(V^0_{\gamma+1}(\Pi))} \le c\,\|\nabla p\|_{L^2(V^0_{\gamma+1}(\Pi))}$. Since $V^0_{\gamma+1}(\Pi)\subset V^0_{\gamma_1}(\Pi)$, we obtain

$$p = \overline{p} + p_\perp \ \in L^2(V^0_{\gamma_1}(\Pi))\,,$$

and

$$\begin{aligned}
&\|p\|_{L^2(V^0_{\gamma_1}(\Pi))}\\
&\quad\le \|\overline{p}^{(1)}\|_{L^2(V^0_{\gamma_1}(\mathbb{R}^2))} + \|p_\perp\|_{L^2(V^0_{\gamma+1}(\Pi))}\\
&\quad\overset{(i),(4.1)}{\le} c\left(\|\mathbf{f}\|_{L^2(V^0_{\gamma_1+1}(\Pi))} + \|g,\nabla g,\partial_t g\|_{L^2(V^0_{\gamma_1+2}(\Pi))} + \|\mathbf{u},p\|_{L^2(V^0_{\gamma}(\Pi))}\right).
\end{aligned}$$

To conclude this proof we apply Lemma 4.1 once more with $\mathbf{u} \in L^2(V^0_{\gamma+1}(\Pi)) \subset L^2(V^0_{\gamma_1}(\Pi))$, $p \in L^2(V^0_{\gamma_1}(\Pi))$, $\mathbf{f} \in L^2(V^0_{\gamma_1+1}(\Pi))$, $g, \nabla g, \partial_t g \in L^2(V^0_{\gamma_1+2}(\Pi))$. This furnishes

$$\begin{aligned}
&\|\mathbf{u},\nabla\mathbf{u},\nabla^2\mathbf{u},\partial_t\mathbf{u},\nabla p\|_{L^2(V^0_{\gamma_1+1}(\Pi))}\\
&\quad\le c\left(\|\mathbf{f}\|_{L^2(V^0_{\gamma_1+1}(\Pi))} + \|g,\nabla g,\partial_t g\|_{L^2(V^0_{\gamma_1+2}(\Pi))} + \|\mathbf{u},p\|_{L^2(V^0_{\gamma_1}(\Pi))}\right)\\
&\quad\le c\left(\|\mathbf{f}\|_{L^2(V^0_{\gamma_1+1}(\Pi))} + \|g,\nabla g,\partial_t g\|_{L^2(V^0_{\gamma_1+2}(\Pi))} + \|\mathbf{u},p\|_{L^2(V^0_{\gamma}(\Pi))}\right).
\end{aligned}$$

□

The final part concerning the linear case is a variant of Theorem 2.2 from Pileckas and Specovius-Neugebauer's paper [42], which presents an asymptotic expansion for solutions of the time-periodic Stokes equations.

Theorem 4.4. *Let $m \in \mathbb{N}_0$, $k_1 \in \mathbb{Z}$ and $\beta \in (-k_1-2, -k_1-1)$. For the data suppose that*

$$\mathbf{f} \in L^2(0,2\pi; V^0_{\beta+2+m}(\Pi)), \quad g,\nabla g,\partial_t g \in L^2(0,2\pi; V^0_{\beta+3+m}(\Pi)), \quad \mathbf{a} = 0$$

and $\mathbf{f}, g$ time-periodic. Let $(\mathbf{u},p) \in L^2(0,2\pi;V^0_\beta(\Pi))\times L^2(0,2\pi;V^0_\beta(\Pi))$ be a distributional solution pair of the Stokes equations (S)*. Additionally, assume $\partial_t\mathbf{u} \in L^2(0,2\pi;L^2_{loc}(\overline{\Pi}))$. Then, $\mathbf{u}$ and p admit the following asymptotic representation:*

$$\begin{aligned}
\mathbf{u}(t,x) &= \sum_{k=k_1}^{k_1+m} \chi(r)\left(w^{(k,+)}(t,z)\nabla_x P^{(k,+)}(y) + w^{(k,-)}(t,z)\nabla_x P^{(k,-)}(y)\right) + \tilde{\mathbf{u}}(t,x),\\
p(t,x) &= \sum_{k=k_1}^{k_1+m} \chi(r)\left(-s^{(k,+)}(t)P^{(k,+)}(y) - s^{(k,-)}(t)P^{(k,-)}(y)\right) + \tilde{p}(t,x).
\end{aligned}$$

Moreover, there holds an a-priori estimate

$$\begin{aligned}
&\sum_{k=k_1}^{k_1+m}\left(\|s^{(k,\pm)}\|_{L^2(0,2\pi)} + \|w^{(k,\pm)}\|_{L^\infty(H^1(0,1))} + \|w^{(k,\pm)}\|_{L^2(H^2(0,1))} + \|\partial_t w^{(k,\pm)}\|_{L^2(L^2(0,1))}\right)\\
&\qquad + \|\tilde{p}\|_{L^2(V^0_{\beta+1+m}(\Pi))} + \|\tilde{\mathbf{u}},\nabla\tilde{\mathbf{u}},\nabla^2\tilde{\mathbf{u}},\partial_t\tilde{\mathbf{u}},\nabla\tilde{p}\|_{L^2(V^0_{\beta+2+m}(\Pi))}\\
&\quad\le c\left(\|\mathbf{f}\|_{L^2(V^0_{\beta+2+m}(\Pi))} + \|g,\nabla g,\partial_t g\|_{L^2(V^0_{\beta+3+m}(\Pi))} + \|\mathbf{u},p\|_{L^2(V^0_\beta(\Pi))}\right).
\end{aligned}$$

Remark. The main difference to the original version lies in differentiability of the data. In [42] $\mathbf{f}$ has to be differentiable twice and for g derivatives up to third order need to exist (leading, naturally, also to statements on higher derivatives of $\tilde{\mathbf{u}}$ and $\tilde{p}$). Further, assumptions on the asymptotic behavior of g are stricter here, but in exchange we achieve better results concerning the weight exponents of $\nabla\tilde{p}$ and $\tilde{\mathbf{u}}, \nabla\tilde{\mathbf{u}}, \nabla^2\tilde{\mathbf{u}}$. Compare inequality (2.6) in [42], in particular.

Proof. The proof mostly resembles the one from [42, Theorem 2.2]; cf. Section 6 in [42]. The crucial difference is application of Theorem 4.3(i) instead of [42, Corollary 5.3].

Set $k = k_1$, $\beta \in (-k_1 - 2, -k_1 - 1)$, for the first iteration. Using Lemma 4.1 we derive

$$\begin{aligned} &\|\mathbf{u}, \nabla \mathbf{u}, \nabla^2 \mathbf{u}, \partial_t \mathbf{u}, \nabla p\|_{L^2(V^0_{\beta+1})} \\ &\quad \le c \left(\|\mathbf{f}\|_{L^2(V^0_{\beta+1})} + \|g, \nabla g, \partial_t g\|_{L^2(V^0_{\beta+2})} + \|\mathbf{u}, p\|_{L^2(V^0_{\beta})} \right) . \end{aligned}$$

Therefore, it holds

$$\|p_\perp\|_{L^2(V^0_{\beta+1})} \le c \, \|\nabla p\|_{L^2(V^0_{\beta+1})} ,$$

by Poincaré's inequality (just as in the proof of Theorem 4.3(ii)). Further, Theorem 4.3(i) yields

$$\begin{aligned} &\overline{p}(t,y) = \chi(r) \left(-s^{(k_1,+)}(t) P^{(k_1,+)}(y) - s^{(k_1,-)}(t) P^{(k_1,-)}(y) \right) + \overline{p}^{(1)}(t,y) , \\ &\|\overline{p}^{(1)}\|_{L^2(V^0_{\beta+1}(\mathbb{R}^2))} + \left(\|s^{(k_1,+)}\|_{L^2(0,2\pi)} + \|s^{(k_1,-)}\|_{L^2(0,2\pi)} \right) \\ &\quad \le c \left(\|\mathbf{f}\|_{L^2(V^0_{\beta+2}(\Pi))} + \|g, \nabla g, \partial_t g\|_{L^2(V^0_{\beta+3}(\Pi))} + \|\mathbf{u}, p\|_{L^2(V^0_{\beta}(\Pi))} \right) . \end{aligned} \tag{4.8}$$

Now, let $w^{(k_1,\pm)}$ be the unique solution to the time-periodic Dirichlet boundary value problem of the one-dimensional heat equation with right-hand side $s^{(k_1,\pm)}$ and boundary condition 0, if $k_1 \neq 0$ and for $(w^{(0,+)}, s^{(0,+)})$. For $(w^{(0,-)}, s^{(0,-)})$, consider the inverse problem

$$\begin{aligned} \partial_t w(t,z) - \partial_z^2 w(t,z) &= s(t) && \text{in } [0,2\pi] \times (0,1) , \\ w(t,0) = w(t,1) &= 0 && \text{in } [0,2\pi] , \\ w(0,z) &= w(2\pi, z) && \text{in } (0,1) , \\ \int_0^1 w(t,z) \, dz &= \phi(t) && \text{in } [0,2\pi] , \end{aligned}$$

with prescribed $\phi(t) = F(t) = \int_\Pi g(t,x) \, dx$. This system is equivalent to (4.3), thus we take $(w^{(0,-)}, s^{(0,-)})$ to be the corresponding solution.

Based on these, we construct the first asymptotic term

$$\begin{aligned} \boldsymbol{\mathcal{U}}^{(k_1)}(t,x) &:= w^{(k_1,+)}(t,z) \nabla_x P^{(k_1,+)}(y) + w^{(k_1,-)}(t,z) \nabla_x P^{(k_1,-)}(y) , \\ \mathcal{P}^{(k_1)}(t,x) &:= -s^{(k_1,+)}(t) P^{(k_1,+)}(y) - s^{(k_1,-)}(t) P^{(k_1,-)}(y) \end{aligned}$$

and therefore, define

$$\begin{aligned} \mathbf{u}^{(1)} &:= \mathbf{u} - \chi \boldsymbol{\mathcal{U}}^{(k_1)} , \\ p^{(1)} &:= p - \chi \mathcal{P}^{(k_1)} = p_\perp + \overline{p}^{(1)} . \end{aligned}$$

Outside of $y = 0$ the functions $P^{(\tilde{k},\pm)}$, $\tilde{k} \in \mathbb{Z}$, are smooth and regarding spatial decay we have

$$\chi P^{(\tilde{k},\pm)} \in V^3_{-\tilde{k}+2-\delta}(\mathbb{R}^2) \quad \text{for all } \delta > 0 .$$

Being a solution to the heat equation, $w^{(k_1,\pm)}$, $k_1 \neq 0$ and $w^{(0,+)}$, fulfill the inequality

$$\begin{aligned}
&\|w^{(k_1,\pm)}\|_{L^\infty(H^1(0,1))} + \|w^{(k_1,\pm)}\|_{L^2(H^2(0,1))} + \|\partial_t w^{(k_1,\pm)}\|_{L^2(L^2(0,1))} \\
&\quad \leq c\,\|s^{(k_1,\pm)}\|_{L^2(0,2\pi)} \\
&\quad \overset{(4.8)}{\leq} \left(\|\mathbf{f}\|_{L^2(V^0_{\beta+2}(\Pi))} + \|g, \nabla g, \partial_t g\|_{L^2(V^0_{\beta+3}(\Pi))} + \|\mathbf{u}, p\|_{L^2(V^0_\beta(\Pi))}\right).
\end{aligned}$$

In case $k_1 = 0$, we require a similar estimate for $(w^{(0,-)}, s^{(0,-)})$, shown in [42, Lemma 4.2] respectively [10, Theorem 2.2]:

$$\begin{aligned}
&\|w^{(0,-)}\|_{L^2(H^1(0,1))} + \|w^{(0,-)}\|_{L^2(H^2(0,1))} + \|\partial_t w^{(0,-)}\|_{L^2(L^2(0,1))} + \|s^{(0,-)}\|_{L^2(0,2\pi)} \\
&\quad \leq c\,\|F\|_{H^1(0,2\pi)} \\
&\quad \leq c\,\|g, \partial_t g\|_{L^2(L^2(\Pi))} \\
&\quad \leq c\,\|g, \partial_t g\|_{L^2(V^0_{\beta+2}(\Pi))},
\end{aligned}$$

where the last step is due to the embedding $V^0_{\beta+2}(\Pi) \hookrightarrow L^2(\Pi)$, since $\beta \in (-2,-1)$ for $k_1 = 0$. Hence, we conclude

$$\begin{aligned}
&\|\chi \boldsymbol{\mathcal{U}}^{(k_1)}\|_{L^2(\mathcal{V}^2_{\beta+1,2}(\Pi))} + \|\chi \partial_t \boldsymbol{\mathcal{U}}^{(k_1)}\|_{L^2(V^0_{\beta+1}(\Pi))} \\
&\quad \leq c\left(\|w^{(k_1,+)}\|_{L^2(H^1(0,1))} + \|\partial_t w^{(k_1,+)}\|_{L^2(L^2(0,1))}\right)\|\chi \nabla_y P^{(k_1,+)}\|_{V^2_{\beta+3}(\mathbb{R}^2)} \\
&\qquad + c\left(\|w^{(k_1,-)}\|_{L^2(H^1(0,1))} + \|\partial_t w^{(k_1,-)}\|_{L^2(L^2(0,1))}\right)\|\chi \nabla_y P^{(k_1,-)}\|_{V^2_{\beta+3}(\mathbb{R}^2)} \\
&\quad \leq c\left(\|\mathbf{f}\|_{L^2(V^0_{\beta+2}(\Pi))} + \|g, \nabla g, \partial_t g\|_{L^2(V^0_{\beta+3}(\Pi))} + \|\mathbf{u}, p\|_{L^2(V^0_\beta(\Pi))}\right),
\end{aligned}$$

and, in particular, $\mathbf{u}^{(1)}, \partial_t \mathbf{u}^{(1)} \in L^2(V^0_{\beta+1}(\Pi))$.

The pair $(\boldsymbol{\mathcal{U}}^{(k_1)}, \mathcal{P}^{(k_1)})$ is a solution of the homogeneous, time-periodic Stokes problem outside of $y = 0$, which stems from the following: The functions $w^{(k_1,\pm)}$ solve the (inverse) heat equation in $[0,2\pi] \times (0,1)$ with right-hand side $s^{(k_1,\pm)}$, suitable periodicity and zero boundary value and $P^{(k_1,\pm)}$ are harmonic in $\mathbb{R}^2 \backslash \{0\}$. Therefore, $(\mathbf{u}^{(1)}, p^{(1)})$ is a distributional solution of the following Stokes system:

$$\begin{aligned}
\partial_t \mathbf{u}^{(1)} - \Delta \mathbf{u}^{(1)} + \nabla p^{(1)} &= \mathbf{f}^{(1)} && \text{in } [0,2\pi] \times \Pi, \\
\operatorname{div} \mathbf{u}^{(1)} &= g^{(1)} && \text{in } [0,2\pi] \times \Pi, \\
\mathbf{u}^{(1)} &= 0 && \text{on } [0,2\pi] \times \partial\Pi, \\
\mathbf{u}^{(1)}|_{t=0} &= \mathbf{u}^{(1)}|_{t=2\pi} && \text{in } \Pi,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{f}^{(1)} &:= \mathbf{f} + 2\nabla\chi \cdot \nabla \boldsymbol{\mathcal{U}}^{(k_1)} + (\Delta\chi)\boldsymbol{\mathcal{U}}^{(k_1)} - (\nabla\chi)\mathcal{P}^{(k_1)}, \\
g^{(1)} &:= g - \nabla\chi \cdot \boldsymbol{\mathcal{U}}^{(k_1)}.
\end{aligned}$$

All additionally occurring terms have compact support, thus the asymptotic behavior of $\mathbf{f}^{(1)}$ and $g^{(1)}$ equals that of $\mathbf{f}$ respectively g: $\mathbf{f}^{(1)} \in L^2(V^0_{\beta+2+m}) \subset L^2(V^0_{\beta+2})$, $g^{(1)}, \nabla g^{(1)}, \partial_t g^{(1)} \in L^2(V^0_{\beta+3+m}) \subset L^2(V^0_{\beta+3})$. Since $p^{(1)} = p_\perp + \overline{p}^{(1)} \in L^2(V^0_{\beta+1})$ and

$\mathbf{u}^{(1)} \in L^2(V^0_{\beta+1})$, we infer with Lemma 4.1:

$$\begin{aligned}
&\|\mathbf{u}^{(1)}, \nabla\mathbf{u}^{(1)}, \nabla^2\mathbf{u}^{(1)}, \partial_t\mathbf{u}^{(1)}, \nabla p^{(1)}\|_{L^2(V^0_{\beta+2})} \\
&\quad\le c\left(\|\mathbf{f}^{(1)}\|_{L^2(V^0_{\beta+2})} + \|g^{(1)}, \nabla g^{(1)}, \partial_t g^{(1)}\|_{L^2(V^0_{\beta+3})} + \|\mathbf{u}^{(1)}, p^{(1)}\|_{L^2(V^0_{\beta+1})}\right) \\
&\quad\le c\Big(\|\mathbf{f}\|_{L^2(V^0_{\beta+2})} \\
&\qquad + c_{k_1}\|w^{(k_1,+)}, w^{(k_1,-)}\|_{L^2(H^1(0,1))} + c_{k_1}\|s^{(k_1,+)}, s^{(k_1,-)}\|_{L^2(0,2\pi)} \\
&\qquad + \|g, \nabla g, \partial_t g\|_{L^2(V^0_{\beta+3})} \\
&\qquad + c_{k_1}\left(\|w^{(k_1,+)}, w^{(k_1,-)}\|_{L^2(H^1(0,1))} + \|\partial_t w^{(k_1,+)}, \partial_t w^{(k_1,-)}\|_{L^2(L^2(0,1))}\right) \\
&\qquad + \|\mathbf{u}, \chi\boldsymbol{\mathcal{U}}^{(k_1)}, p_\perp, \overline{p}^{(1)}\|_{L^2(V^0_{\beta+1})}\Big) \\
&\quad\le c\left(\|\mathbf{f}\|_{L^2(V^0_{\beta+2})} + \|g, \nabla g, \partial_t g\|_{L^2(V^0_{\beta+3})} + \|\mathbf{u}, p\|_{L^2(V^0_{\beta})}\right),
\end{aligned}$$

where c_{k_1} is composed of integrals of $P^{(k_1,\pm)}$, $\nabla P^{(k_1,\pm)}$ and $\nabla^2 P^{(k_1,\pm)}$ over the support of $\nabla\chi$.

In case $m = 0$, we just need to relabel $\tilde{\mathbf{u}} := \mathbf{u}^{(1)}$, $\tilde{p} := p^{(1)}$. For $m \ge 1$, we repeat the argumentation above with $k = k_1 + 1$ for the pair $(\mathbf{u}^{(1)}, p^{(1)}) \in L^2(V^0_{\beta+1}(\Pi)) \times L^2(V^0_{\beta+1}(\Pi))$, $\beta + 1 \in (-k-2, -k-1)$, to derive the next term in the asymptotic representation and a new remainder $(\mathbf{u}^{(2)}, p^{(2)})$.
This iteration is continued until $k = k_1 + m$. Then, we finally set $\tilde{\mathbf{u}} := \mathbf{u}^{(m+1)}, \tilde{p} := p^{(m+1)}$ to close the proof. □

4.2 Spatial behavior of solutions in the nonlinear case

Now, we focus on the full Navier-Stokes system:

$$\begin{aligned}
\partial_t\mathbf{u} - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } [0,2\pi]\times\Pi, \\
\operatorname{div}\mathbf{u} &= g && \text{in } [0,2\pi]\times\Pi, \\
\mathbf{u}|_{\partial\Pi} &= \mathbf{a} && \text{for all } t\in[0,2\pi], \\
\mathbf{u}|_{t=0} &= \mathbf{u}|_{t=2\pi} && \text{in } \Pi.
\end{aligned} \tag{NS}$$

Treatment of the Navier-Stokes equations requires a thorough investigation of the nonlinear part. We start with a preparatory lemma on estimates of a product uv in weighted spaces, enabling us to deduce respective properties regarding $(\mathbf{u}\cdot\nabla)\mathbf{v}$.

Lemma 4.5. *Let $\beta_1, \beta_2 \in \mathbb{R}$.*

(a) *Suppose $u \in \mathcal{V}^1_{\beta_1,1}(\Pi)$, $v \in \mathcal{V}^1_{\beta_2,1}(\Pi)$. Then $w := uv \in V^0_{\beta_1+\beta_2+1}(\Pi)$ and*

$$\|w\|_{V^0_{\beta_1+\beta_2+1}(\Pi)} \le c\,\|u\|_{\mathcal{V}^1_{\beta_1,1}(\Pi)}\|v\|_{\mathcal{V}^1_{\beta_2,1}(\Pi)}\,. \tag{4.9}$$

(b) *Suppose $u \in \mathcal{V}^1_{\beta_1,1}(\Pi)$, $v \in \mathcal{V}^1_{\beta_2,0}(\Pi)$. Then $w \in V^0_{\beta_1+\beta_2+\frac{2}{3}}(\Pi)$ and*

$$\|w\|_{V^0_{\beta_1+\beta_2+\frac{2}{3}}(\Pi)} \le c\,\|u\|_{\mathcal{V}^1_{\beta_1,1}(\Pi)}\|v\|_{\mathcal{V}^1_{\beta_2,0}(\Pi)}\,. \tag{4.10}$$

(c) *Suppose* $u \in \mathcal{V}^1_{\beta_1,0}(\Pi)$, $v \in \mathcal{V}^1_{\beta_2,0}(\Pi)$. *Then* $w \in V^0_{\beta_1+\beta_2}(\Pi)$ *and*

$$\|w\|_{V^0_{\beta_1+\beta_2}(\Pi)} \leq c\,\|u\|_{\mathcal{V}^1_{\beta_1,0}(\Pi)}\|v\|_{\mathcal{V}^1_{\beta_2,0}(\Pi)}\,. \tag{4.11}$$

Proof. The proof relies on a statement about embedding properties of Kondratiev spaces from [39] (Lemma 2.4 therein), which we recall in Lemma 2.13.

ad(a):

$$\begin{aligned}
&\int_\Pi (1+r^2)^{\beta_1+\beta_2+1}|u|^2|v|^2\,dx \\
&\quad\leq \left(\int_\Pi (1+r^2)^{2(\beta_1+\frac{1}{2})}|u|^4\,dx\right)^{1/2}\left(\int_\Pi (1+r^2)^{2(\beta_2+\frac{1}{2})}|v|^4\,dx\right)^{1/2} \\
&\quad= \|u\|^2_{V^{0,4}_{\beta_1+\frac{1}{2}}}\|v\|^2_{V^{0,4}_{\beta_2+\frac{1}{2}}} \\
&\quad\overset{(2.6)}{\leq} c\,\|u\|^2_{\mathcal{V}^1_{\beta_1,1}}\|v\|^2_{\mathcal{V}^1_{\beta_2,1}}\,.
\end{aligned}$$

ad(b):

$$\begin{aligned}
&\int_\Pi (1+r^2)^{\beta_1+\beta_2+\frac{2}{3}}|u|^2|v|^2\,dx \\
&\quad\leq \left(\int_\Pi (1+r^2)^{3(\beta_1+\frac{2}{3})}|u|^6\,dx\right)^{1/3}\left(\int_\Pi (1+r^2)^{\frac{3}{2}\beta_2}|v|^3\,dx\right)^{2/3} \\
&\quad= \|u\|^2_{V^{0,6}_{\beta_1+\frac{2}{3}}}\|v\|^2_{V^{0,3}_{\beta_2}} \\
&\quad\overset{(2.6),(2.7)}{\leq} c\,\|u\|^2_{\mathcal{V}^1_{\beta_1,1}}\|v\|^2_{\mathcal{V}^1_{\beta_2,0}}\,.
\end{aligned}$$

ad(c):

$$\begin{aligned}
&\int_\Pi (1+r^2)^{\beta_1+\beta_2}|u|^2|v|^2\,dx \\
&\quad\leq \left(\int_\Pi (1+r^2)^{2\beta_1}|u|^4\,dx\right)^{1/2}\left(\int_\Pi (1+r^2)^{2\beta_2}|v|^4\,dx\right)^{1/2} \\
&\quad= \|u\|^2_{V^{0,4}_{\beta_1}}\|v\|^2_{V^{0,4}_{\beta_2}} \\
&\quad\overset{(2.7)}{\leq} c\,\|u\|^2_{\mathcal{V}^1_{\beta_1,0}}\|v\|^2_{\mathcal{V}^1_{\beta_2,0}}\,.
\end{aligned}$$

□

The next lemma gathers a couple of estimates for the nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{v}$.

Lemma 4.6. *Let* $\beta_1, \beta_2 \in \mathbb{R}$.

(a) *Suppose* $\mathbf{u} \in L^\infty(0, 2\pi; \mathcal{V}^1_{\beta_1,1}(\Pi))$, $\mathbf{v} \in L^2(0, 2\pi; \mathcal{V}^2_{\beta_2,1}(\Pi))$. *Then*

$$(\mathbf{u}\cdot\nabla)\mathbf{v} \in L^2(0, 2\pi; V^0_{\beta_1+\beta_2+1}(\Pi))$$

and

$$\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|_{L^2(V^0_{\beta_1+\beta_2+1}(\Pi))} \leq c\,\|\mathbf{u}\|_{L^\infty(\mathcal{V}^1_{\beta_1,1}(\Pi))}\|\mathbf{v}\|_{L^2(\mathcal{V}^2_{\beta_2,1}(\Pi))}\,.$$

(b) (i) *Suppose* $\mathbf{u} \in L^\infty(0,2\pi;\mathcal{V}^1_{\beta_1,1}(\Pi))$, $\mathbf{v} \in L^2(0,2\pi;\mathcal{V}^2_{\beta_2,0}(\Pi))$. *Then*

$$(\mathbf{u}\cdot\nabla)\mathbf{v} \in L^2(0,2\pi;V^0_{\beta_1+\beta_2+\frac{2}{3}}(\Pi))$$

and

$$\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|_{L^2(V^0_{\beta_1+\beta_2+\frac{2}{3}}(\Pi))} \le c\,\|\mathbf{u}\|_{L^\infty(\mathcal{V}^1_{\beta_1,1}(\Pi))}\|\mathbf{v}\|_{L^2(\mathcal{V}^2_{\beta_2,0}(\Pi))}\,.$$

(ii) *Suppose* $\mathbf{u} \in L^\infty(0,2\pi;\mathcal{V}^1_{\beta_1,0}(\Pi))$, $\mathbf{v} \in L^2(0,2\pi;\mathcal{V}^2_{\beta_2,1}(\Pi))$. *Then*

$$(\mathbf{u}\cdot\nabla)\mathbf{v} \in L^2(0,2\pi;V^0_{\beta_1+\beta_2+\frac{2}{3}}(\Pi))$$

and

$$\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|_{L^2(V^0_{\beta_1+\beta_2+\frac{2}{3}}(\Pi))} \le c\,\|\mathbf{u}\|_{L^\infty(\mathcal{V}^1_{\beta_1,0}(\Pi))}\|\mathbf{v}\|_{L^2(\mathcal{V}^2_{\beta_2,1}(\Pi))}\,.$$

(c) *Suppose* $\mathbf{u} \in L^\infty(0,2\pi;\mathcal{V}^1_{\beta_1,0}(\Pi))$, $\mathbf{v} \in L^2(0,2\pi;\mathcal{V}^2_{\beta_2,0}(\Pi))$. *Then*

$$(\mathbf{u}\cdot\nabla)\mathbf{v} \in L^2(0,2\pi;V^0_{\beta_1+\beta_2}(\Pi))$$

and

$$\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|_{L^2(V^0_{\beta_1+\beta_2}(\Pi))} \le c\,\|\mathbf{u}\|_{L^\infty(\mathcal{V}^1_{\beta_1,0}(\Pi))}\|\mathbf{v}\|_{L^2(\mathcal{V}^2_{\beta_2,0}(\Pi))}\,.$$

(d) *Suppose* $\mathbf{u} = (\mathbf{u}', u_3) \in L^\infty(0,2\pi;\mathcal{V}^1_{\beta_1,1}(\Pi)) \times L^\infty(0,2\pi;\mathcal{V}^1_{\beta_1+1,0}(\Pi))$, $\mathbf{v} = (\mathbf{v}', v_3) \in L^2(0,2\pi;\mathcal{V}^2_{\beta_2,1}(\Pi)) \times L^2(0,2\pi;\mathcal{V}^2_{\beta_2+1,0}(\Pi))$. *Then*

$$(\mathbf{u}\cdot\nabla)\mathbf{v} \in L^2(0,2\pi;V^0_{\beta_1+\beta_2+\frac{5}{3}}(\Pi))$$

and

$$\begin{aligned}
&\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|_{L^2(V^0_{\beta_1+\beta_2+\frac{5}{3}}(\Pi))} \\
&\quad\le c\left(\|\mathbf{u}'\|^2_{L^\infty(\mathcal{V}^1_{\beta_1,1}(\Pi))} + \|u_3\|^2_{L^\infty(\mathcal{V}^1_{\beta_1+1,0}(\Pi))}\right)^{1/2} \\
&\qquad\cdot\left(\|\mathbf{v}'\|^2_{L^2(\mathcal{V}^2_{\beta_2,1}(\Pi))} + \|v_3\|^2_{L^2(\mathcal{V}^2_{\beta_2+1,0}(\Pi))}\right)^{1/2} \\
&\quad= c\,\|(\mathbf{u}',u_3)\|_{L^\infty(\mathcal{V}^1_{\beta_1,1}(\Pi))\times L^\infty(\mathcal{V}^1_{\beta_1+1,0}(\Pi))}\|(\mathbf{v}',v_3)\|_{L^2(\mathcal{V}^2_{\beta_2,1}(\Pi))\times L^2(\mathcal{V}^2_{\beta_2+1,0}(\Pi))}\,.
\end{aligned}$$

Proof. The proof is based mainly on Lemma 4.5.

ad(a):

$$\begin{aligned}
\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|^2_{L^2(V^0_{\beta_1+\beta_2+1})} &\le \||\mathbf{u}||\nabla\mathbf{v}|\|^2_{L^2(V^0_{\beta_1+\beta_2+1})} \\
&\overset{(4.9)}{\le} \int_0^{2\pi} c\,\|\mathbf{u}\|^2_{\mathcal{V}^1_{\beta_1,1}}\|\nabla\mathbf{v}\|^2_{\mathcal{V}^1_{\beta_2,1}}\,dt \\
&\le c\,\|\mathbf{u}\|^2_{L^\infty(\mathcal{V}_{\beta_1,1})}\|\mathbf{v}\|^2_{L^2(\mathcal{V}^2_{\beta_2,1})}\,.
\end{aligned}$$

ad(b)(i):

$$\begin{aligned}\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|^2_{L^2(V^0_{\beta_1+\beta_2+\frac{2}{3}})} &\le \||\mathbf{u}||\nabla\mathbf{v}|\|^2_{L^2(V^0_{\beta_1+\beta_2+\frac{2}{3}})}\\ &\overset{(4.10)}{\le} \int_0^{2\pi} c\,\|\mathbf{u}\|^2_{\mathcal{V}^1_{\beta_1,1}}\|\nabla\mathbf{v}\|^2_{\mathcal{V}^1_{\beta_1,0}}\,dt\\ &\le c\,\|\mathbf{u}\|^2_{L^\infty(\mathcal{V}^1_{\beta_1,1})}\|\mathbf{v}\|^2_{L^2(\mathcal{V}^2_{\beta_2,0})}\,.\end{aligned}$$

ad(b)(ii):

$$\begin{aligned}\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|^2_{L^2(V^0_{\beta_1+\beta_2+\frac{2}{3}})} &\le \||\mathbf{u}||\nabla\mathbf{v}|\|^2_{L^2(V^0_{\beta_1+\beta_2+\frac{2}{3}})}\\ &\overset{(4.10)}{\le} \int_0^{2\pi} c\,\|\mathbf{u}\|^2_{\mathcal{V}^1_{\beta_1,0}}\|\nabla\mathbf{v}\|^2_{\mathcal{V}^1_{\beta_2,1}}\,dt\\ &\le c\,\|\mathbf{u}\|^2_{L^\infty(\mathcal{V}^1_{\beta_1,0})}\|\mathbf{v}\|^2_{L^2(\mathcal{V}^2_{\beta_2,1})}\,.\end{aligned}$$

ad(c):

$$\begin{aligned}\|(\mathbf{u}\cdot\nabla)\mathbf{v}\|^2_{L^2(V^0_{\beta_1+\beta_2})} &\le \||\mathbf{u}||\nabla\mathbf{v}|\|^2_{L^2(V^0_{\beta_1+\beta_2})}\\ &\overset{(4.11)}{\le} \int_0^{2\pi} c\,\|\mathbf{u}\|^2_{\mathcal{V}^1_{\beta_1,0}}\|\nabla\mathbf{v}\|^2_{\mathcal{V}^1_{\beta_2,0}}\,dt\\ &\le c\,\|\mathbf{u}\|^2_{L^\infty(\mathcal{V}^1_{\beta_1,0})}\|\mathbf{v}\|^2_{L^2(\mathcal{V}^2_{\beta_2,0})}\,.\end{aligned}$$

ad(d): Estimation in this case is a little more involved. We start by decomposing $(\mathbf{u}\cdot\nabla)\mathbf{v}$ into

$$(\mathbf{u}\cdot\nabla)\mathbf{v} = \begin{pmatrix}(\mathbf{u}'\cdot\nabla_y)\mathbf{v}' + u_3\partial_z\mathbf{v}'\\ (\mathbf{u}'\cdot\nabla_y)v_3 + u_3\partial_z v_3\end{pmatrix}\,.$$

For each individual term in this representation we directly obtain by Lemma 4.5:

$$\begin{aligned}\|(\mathbf{u}'\cdot\nabla_y)\mathbf{v}'\|^2_{L^2(V^0_{\beta_1+\beta_2+\frac{5}{3}})} &\le \||\mathbf{u}'||\nabla_y\mathbf{v}'|\|^2_{L^2(V^0_{\beta_1+\beta_2+\frac{5}{3}})}\\ &\overset{(4.10)}{\le} \int_0^{2\pi} c\,\|\mathbf{u}'\|^2_{\mathcal{V}^1_{\beta_1,1}}\|\nabla_y\mathbf{v}'\|^2_{\mathcal{V}^1_{\beta_2+1,0}}\,dt\\ &\le c\,\|\mathbf{u}'\|^2_{L^\infty(\mathcal{V}^1_{\beta_1,1})}\|\mathbf{v}'\|^2_{L^2(\mathcal{V}^2_{\beta_2,1})}\,,\end{aligned}$$

$$\begin{aligned}
\|u_3\partial_z\mathbf{v}'\|^2_{L^2(V^0_{\beta_1+\beta_2+\frac{5}{3}})} &\le \||u_3||\partial_z\mathbf{v}'|\|^2_{L^2(V^0_{\beta_1+\beta_2+\frac{5}{3}})}\\
&\overset{(4.10)}{\le} \int_0^{2\pi} c\,\|u_3\|^2_{\mathcal{V}^1_{\beta_1+1,0}}\|\partial_z\mathbf{v}'\|^2_{\mathcal{V}^1_{\beta_2,1}}\,dt\\
&\le c\,\|u_3\|^2_{L^\infty(\mathcal{V}^1_{\beta_1+1,0})}\|\mathbf{v}'\|^2_{L^2(\mathcal{V}^2_{\beta_2,1})}\,,
\end{aligned}$$

$$\begin{aligned}
\|(\mathbf{u}'\cdot\nabla_y)v_3\|^2_{L^2(V^0_{\beta_1+\beta_2+\frac{5}{3}})} &\le \||\mathbf{u}'||\nabla_y v_3|\|^2_{L^2(V^0_{\beta_1+\beta_2+\frac{5}{3}})}\\
&\overset{(4.10)}{\le} \int_0^{2\pi} c\,\|\mathbf{u}'\|^2_{\mathcal{V}^1_{\beta_1,1}}\|\nabla_y v_3\|^2_{\mathcal{V}^1_{\beta_2+1,0}}\,dt\\
&\le c\,\|\mathbf{u}'\|^2_{L^\infty(\mathcal{V}^1_{\beta_1,1})}\|v_3\|^2_{L^2(\mathcal{V}^2_{\beta_2+1,0})}\,,
\end{aligned}$$

$$\begin{aligned}
\|u_3\partial_z v_3\|^2_{L^2(V^0_{\beta_1+\beta_2+\frac{5}{3}})} &\le \|u_3\partial_z v_3\|^2_{L^2(V^0_{\beta_1+\beta_2+2})}\\
&\le \||u_3||\partial_z v_3|\|^2_{L^2(V^0_{\beta_1+\beta_2+2})}\\
&\overset{(4.11)}{\le} \int_0^{2\pi} c\,\|u_3\|^2_{\mathcal{V}^1_{\beta_1+1,0}}\|\partial_z v_3\|^2_{\mathcal{V}^1_{\beta_2+1,0}}\,dt\\
&\le c\,\|u_3\|^2_{L^\infty(\mathcal{V}^1_{\beta_1+1,0})}\|v_3\|^2_{L^2(\mathcal{V}^2_{\beta_2+1,0})}\,.
\end{aligned}$$

Summarizing these inequalities we derive the asserted estimate. □

With these results we establish the main theorem of this chapter.

Theorem 4.7. *Let* $(\mathbf{u},p)\in L^2(H^2_{loc}(\overline{\Pi}))\times L^2(L^2_{loc}(\overline{\Pi}))$ *be a distributional solution of the time-periodic Navier-Stokes equations* (NS)*. For the outer force and boundary condition, suppose*

$$\begin{aligned}
&\mathbf{f}\in L^2(0,2\pi;V^0_{\beta+2}(\Pi))\,,\ g=0\,,\\
&\mathbf{a}\in L^2(0,2\pi;W^{3/2,2}_{loc}(\partial\Pi))\,,\ \partial_t\mathbf{a}\in L^2(0,2\pi;W^{1/2,2}_{loc}(\partial\Pi))\,,
\end{aligned}$$

all time-periodic, $\beta\in(-2,-1)$*. Let there exist an extension* $\mathbf{A}$ *of* $\mathbf{a}$ *with*

$$\begin{aligned}
\mathbf{A}'&\in L^2(0,2\pi;\mathcal{V}^2_{\beta+2,1}(\Pi))\,, & A_3&\in L^2(0,2\pi;\mathcal{V}^2_{\beta+3,0}(\Pi))\,,\\
\partial_t\mathbf{A}'&\in L^2(0,2\pi;\mathcal{V}^1_{\beta+2,1}(\Pi))\,, & \partial_t A_3&\in L^2(0,2\pi;\mathcal{V}^1_{\beta+3,0}(\Pi))\,.
\end{aligned}$$

Additionally, let the solution satisfy $\partial_t\mathbf{u}\in L^2(0,2\pi;L^2_{loc}(\overline{\Pi}))$*,*

$$\begin{aligned}
\mathbf{u}'&\in L^2(0,2\pi;\mathcal{V}^2_{\gamma+1,1}(\Pi))\cap L^\infty(0,2\pi;\mathcal{V}^1_{\gamma+1,1}(\Pi))\,,\\
u_3&\in L^2(0,2\pi;\mathcal{V}^2_{\gamma+2,0}(\Pi))\cap L^\infty(0,2\pi;\mathcal{V}^1_{\gamma+2,0}(\Pi))\,,
\end{aligned}\tag{4.12}$$

and $p\in L^2(0,2\pi;V^0_\gamma(\Pi))$*, where* $\gamma\in(-\frac{11}{6},-1)$*. Then,* $(\mathbf{u},p)$ *admits the asymptotic representation*

$$\begin{aligned}
\mathbf{u}(t,x)&=\chi(r)w^{(0,-)}(t,z)\nabla_x P^{(0,-)}(y)+\tilde{\mathbf{u}}(t,x)\,,\\
p(t,x)&=\chi(r)\left(-s^{(0,+)}(t)-s^{(0,-)}(t)P^{(0,-)}(y)\right)+\tilde{p}(t,x)\,,
\end{aligned}$$

with $\tilde{p} \in L^2(0, 2\pi; V^0_{\beta+1}(\Pi))$ *and*

$$\tilde{\mathbf{u}}, \nabla\tilde{\mathbf{u}}, \nabla^2\tilde{\mathbf{u}}, \partial_t\tilde{\mathbf{u}}, \nabla\tilde{p} \in L^2(0, 2\pi; V^0_{\beta+2}(\Pi)) .$$

Further, the following a-priori estimate holds

$$\begin{aligned}
&\|s^{(0,\pm)}\|_{L^2(0,2\pi)} + \|w^{(0,-)}\|_{L^\infty(H^1(0,1))} + \|w^{(0,-)}\|_{L^2(H^2(0,1))} + \|\partial_t w^{(0,-)}\|_{L^2(L^2(0,1))} \\
&\quad + \|\tilde{p}\|_{L^2(V^0_{\beta+1})} + \|\tilde{\mathbf{u}}, \nabla\tilde{\mathbf{u}}, \nabla^2\tilde{\mathbf{u}}, \partial_t\tilde{\mathbf{u}}, \nabla\tilde{p}\|_{L^2(V^0_{\beta+2})} \\
&\le c\Big[1 + \|\mathbf{f}\|_{L^2(V^0_{\beta+2})} + \|(\mathbf{A}', A_3)\|_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})} + \|\big(\partial_t\mathbf{A}', \partial_t A_3\big)\|_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})} \\
&\qquad + \|(\mathbf{u}', u_3)\|^2_{L^\infty(\mathcal{V}^1_{\gamma+1,1})\times L^\infty(\mathcal{V}^1_{\gamma+2,0})} + \|(\mathbf{u}', u_3)\|^2_{L^2(\mathcal{V}^2_{\gamma+1,1})\times L^2(\mathcal{V}^2_{\gamma+2,0})} + \|\mathbf{u}, p\|_{L^2(V^0_\gamma)}\Big]^{2^k} ,
\end{aligned}$$

where $k \in \mathbb{N}_0$ *is determined by the inequality* $2^{k+1}\big(\gamma + \frac{11}{6}\big) \ge \beta + 2$.

Remark. This theorem can easily be extended to the case of $g \neq 0$ – provided $g, \nabla g, \partial_t g \in L^2(0, 2\pi; V^0_{\beta+3}(\Pi))$ time-periodic – by setting $g^* := g - \operatorname{div}\mathbf{A}$ in the following proof. Then, corresponding norms need to be added to the right-hand side of the a-priori estimate.

Proof. Define $\mathbf{u}^* := \mathbf{u} - \mathbf{A}$, which thereby is a solution of

$$\begin{aligned}
\partial_t\mathbf{u}^* - \Delta\mathbf{u}^* + \nabla p &= \mathbf{f}^* && \text{in } [0, 2\pi] \times \Pi\,, \\
\operatorname{div}\mathbf{u}^* &= g^* && \text{in } [0, 2\pi] \times \Pi\,, \\
\mathbf{u}^*|_{\partial\Pi} &= 0 && \text{for all } t \in [0, 2\pi]\,, \\
\mathbf{u}^*|_{t=0} &= \mathbf{u}^*|_{t=2\pi} && \text{in } \Pi\,,
\end{aligned}$$

having $\mathbf{f}^* := \mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \partial_t\mathbf{A} + \Delta\mathbf{A}$ and $g^* := -\operatorname{div}\mathbf{A}$. Observe, that the assumptions on $\mathbf{A}$ directly yield $\partial_t\mathbf{A} \in L^2(V^0_{\beta+2})$, $\Delta\mathbf{A} \in L^2(V^0_{\beta+2})$ and $g^*, \nabla g^*, \partial_t g^* \in L^2(V^0_{\beta+3})$.

Now, we infer by Lemma 4.6(d) that $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L^2(V^0_{2\gamma+\frac{11}{3}})$ with

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^2(V^0_{2\gamma+\frac{11}{3}})} \le c\,\|(\mathbf{u}', u_3)\|_{L^\infty(\mathcal{V}^1_{\gamma+1,1})\times L^\infty(\mathcal{V}^1_{\gamma+2,0})}\|(\mathbf{u}', u_3)\|_{L^2(\mathcal{V}^2_{\gamma+1,1})\times L^2(\mathcal{V}^2_{\gamma+2,0})} ,$$

motivating the following case distinction. Set $\gamma_1 + 2 := 2\gamma + \frac{11}{3}$ (implying $\gamma_1 > -2$).

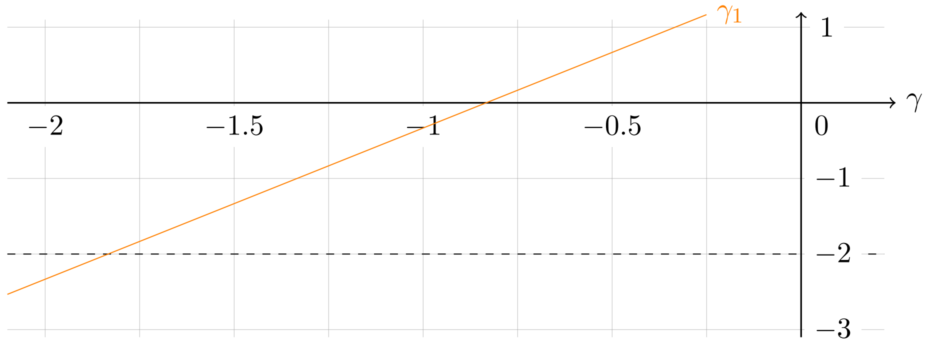

Figure 4.1: γ_1 in dependence of γ.

Case 1: $\gamma_1 + 2 \ge \beta + 2$.
This case implies $V^0_{\gamma_1+2}(\Pi) \subset V^0_{\beta+2}(\Pi)$, hence $\mathbf{f}^* \in L^2(V^0_{\beta+2})$. Moreover it is $\mathbf{u}^*, p \in$

$L^2(V^0_\gamma)$. If $\gamma \geq \beta$, we directly have $V^0_\gamma(\Pi) \subset V^0_\beta(\Pi)$. If $\beta > \gamma$ instead, we need to apply Theorem 4.3(ii) to find: $\mathbf{u}^*, p \in L^2(V^0_\beta)$.

All assumptions of Theorem 4.4 are satisfied, which gives the asymptotic representation

$$\begin{aligned}\mathbf{u}^*(t,x) &= \chi(r) w^{(0,-)}(t,z) \nabla_x P^{(0,-)}(y) + \tilde{\mathbf{u}}^*(t,x)\,,\\ p(t,x) &= \chi(r)\left(-s^{(0,+)}(t) - s^{(0,-)}(t) P^{(0,-)}(y)\right) + \tilde{p}(t,x)\end{aligned}$$

along with the following inequality:

$$\begin{aligned}&\|s^{(0,\pm)}\|_{L^2(0,2\pi)} + \|w^{(0,-)}\|_{L^\infty(H^1(0,1))} + \|w^{(0,-)}\|_{L^2(H^2(0,1))} + \|\partial_t w^{(0,-)}\|_{L^2(L^2(0,1))}\\ &\quad + \|\tilde{p}\|_{L^2(V^0_{\beta+1})} + \|\tilde{\mathbf{u}}^*, \nabla\tilde{\mathbf{u}}^*, \nabla^2\tilde{\mathbf{u}}^*, \partial_t\tilde{\mathbf{u}}^*, \nabla\tilde{p}\|_{L^2(V^0_{\beta+2})}\\ &\leq c\left[\|\mathbf{f}^*\|_{L^2(V^0_{\beta+2})} + \|g^*, \nabla g^*, \partial_t g^*\|_{L^2(V^0_{\beta+3})} + \|\mathbf{u}^*, p\|_{L^2(V^0_\beta)}\right]\\ &\leq c\left[\|\mathbf{f}^*\|_{L^2(V^0_{\beta+2})} + \|g^*, \nabla g^*, \partial_t g^*\|_{L^2(V^0_{\beta+3})} + \|\mathbf{u}^*, p\|_{L^2(V^0_\gamma)}\right]\\ &\leq c\Big[\|\mathbf{f}, (\mathbf{u}\cdot\nabla)\mathbf{u}, \partial_t\mathbf{A}, \Delta\mathbf{A}\|_{L^2(V^0_{\beta+2})} + \|\operatorname{div}\mathbf{A}, \nabla(\operatorname{div}\mathbf{A}), \partial_t(\operatorname{div}\mathbf{A})\|_{L^2(V^0_{\beta+3})}\\ &\qquad + \|\mathbf{u}^*, p\|_{L^2(V^0_\gamma)}\Big]\\ &\leq c\Big[\|\mathbf{f}\|_{L^2(V^0_{\beta+2})} + \|(\mathbf{A}', A_3)\|_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})} + \|\big(\partial_t\mathbf{A}', \partial_t A_3\big)\|_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})}\\ &\qquad + \|(\mathbf{u}', u_3)\|_{L^\infty(\mathcal{V}^1_{\gamma+1,1})\times L^\infty(\mathcal{V}^1_{\gamma+2,0})}\|(\mathbf{u}', u_3)\|_{L^2(\mathcal{V}^2_{\gamma+1,1})\times L^2(\mathcal{V}^2_{\gamma+2,0})} + \|\mathbf{u}^*, p\|_{L^2(V^0_\gamma)}\Big]\\ &\leq c\Big[\|\mathbf{f}\|_{L^2(V^0_{\beta+2})} + \|(\mathbf{A}', A_3)\|_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})} + \|\big(\partial_t\mathbf{A}', \partial_t A_3\big)\|_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})}\\ &\qquad + \|(\mathbf{u}', u_3)\|^2_{L^\infty(\mathcal{V}^1_{\gamma+1,1})\times L^\infty(\mathcal{V}^1_{\gamma+2,0})} + \|(\mathbf{u}', u_3)\|^2_{L^2(\mathcal{V}^2_{\gamma+1,1})\times L^2(\mathcal{V}^2_{\gamma+2,0})} + \|\mathbf{u}, p\|_{L^2(V^0_\gamma)}\Big]\,.\end{aligned}$$

Setting $\tilde{\mathbf{u}} := \tilde{\mathbf{u}}^* + \mathbf{A}$, we finally arrive at

$$\mathbf{u}(t,x) = \chi(r) w^{(0,-)}(t,z) \nabla_x P^{(0,-)}(y) + \tilde{\mathbf{u}}(t,x)$$

and the asserted a-priori estimate lies at hand.

Case 2: $\gamma_1 + 2 < \beta + 2$.
In this case $V^0_{\beta+2}(\Pi) \subset V^0_{\gamma_1+2}(\Pi)$, therefore $\mathbf{f}^* \in L^2(V^0_{\gamma_1+2})$ and further, $g^*, \nabla g^*, \partial_t g^* \in L^2(V^0_{\gamma_1+3})$. Again, it is $\mathbf{u}^*, p \in L^2(V^0_\gamma)$. If $\gamma \geq \gamma_1$, we directly have $V^0_\gamma(\Pi) \subset V^0_{\gamma_1}(\Pi)$. If $\gamma_1 > \gamma$ instead, we need to apply Theorem 4.3(ii) to find: $\mathbf{u}^*, p \in L^2(V^0_{\gamma_1})$. Note that $(\gamma, \gamma_1) \cap \mathbb{Z} = \emptyset$ is granted by $\gamma_1 < \beta$.

Due to Theorem 4.4 we get, just as in Case 1:

$$\begin{aligned}\mathbf{u}^*(t,x) &= \chi(r) w^{(0,-)}(t,z) \nabla_x P^{(0,-)}(y) + \tilde{\mathbf{u}}^*(t,x)\\ &=: \chi(r)\boldsymbol{\mathcal{U}}^{(0,-)}(t,x) + \tilde{\mathbf{u}}^*(t,x)\,,\end{aligned}$$

$$\begin{aligned}p(t,x) &= \chi(r)\left(-s^{(0,+)}(t) - s^{(0,-)}(t) P^{(0,-)}(y)\right) + \tilde{p}(t,x)\\ &=: \chi(r)\left(\mathcal{P}^{(0,+)}(t,y) + \mathcal{P}^{(0,-)}(t,y)\right) + \tilde{p}(t,x)\end{aligned}$$

and

$$\begin{aligned}
&\|s^{(0,\pm)}\|_{L^2(0,2\pi)} + \|w^{(0,-)}\|_{L^\infty(H^1(0,1))} + \|w^{(0,-)}\|_{L^2(H^2(0,1))} + \|\partial_t w^{(0,-)}\|_{L^2(L^2(0,1))} \\
&\quad + \|\tilde{p}\|_{L^2(V^0_{\gamma_1+1})} + \|\tilde{\mathbf{u}}^*, \nabla\tilde{\mathbf{u}}^*, \nabla^2\tilde{\mathbf{u}}^*, \partial_t\tilde{\mathbf{u}}^*, \nabla\tilde{p}\|_{L^2(V^0_{\gamma_1+2})} \qquad (4.13)\\
&\le c\left[\|\mathbf{f}^*\|_{L^2(V^0_{\gamma_1+2})} + \|g^*, \nabla g^*, \partial_t g^*\|_{L^2(V^0_{\gamma_1+3})} + \|\mathbf{u}^*, p\|_{L^2(V^0_{\gamma_1})}\right].
\end{aligned}$$

Thus, our goal now is to improve the weight exponent of the remainder $(\tilde{\mathbf{u}}^*, \tilde{p})$.

In particular, the inequality above contains the information that $\tilde{\mathbf{u}}^* \in L^2(\mathcal{V}^2_{\gamma_1+2,0})$ and $\partial_t\tilde{\mathbf{u}}^* \in L^2(V^0_{\gamma_1+2})$. Lions-Magenes's Lemma (Theorem 2.17) coupled with Lemma 2.14, on interpolation of weighted spaces, implies

$$\begin{aligned}
\tilde{\mathbf{u}}^* &\in C([0,2\pi]; \big(\mathcal{V}^2_{\gamma_1+2,0}(\Pi), V^0_{\gamma_1+2}(\Pi)\big)_{1/2,2}) \\
&= C([0,2\pi]; \mathcal{V}^1_{\gamma_1+2,0}(\Pi)) \\
&\subset L^\infty(0,2\pi; \mathcal{V}^1_{\gamma_1+2,0}(\Pi))
\end{aligned}$$

with

$$\|\tilde{\mathbf{u}}^*\|_{L^\infty(\mathcal{V}^1_{\gamma_1+2,0})} \le c\left(\|\tilde{\mathbf{u}}^*\|^2_{L^2(\mathcal{V}^2_{\gamma_1+2,0})} + \|\partial_t\tilde{\mathbf{u}}^*\|^2_{L^2(V^0_{\gamma_1+2})}\right)^{1/2}.$$

Therefore, Lemma 4.6(c) yields

$$\|(\tilde{\mathbf{u}}^*\cdot\nabla)\tilde{\mathbf{u}}^*\|_{L^2(V^0_{2\gamma_1+4})} \le c\,\|\tilde{\mathbf{u}}^*\|_{L^\infty(\mathcal{V}^1_{\gamma_1+2,0})}\|\tilde{\mathbf{u}}^*\|_{L^2(\mathcal{V}^2_{\gamma_1+2,0})}.$$

Observe, in particular, that $2\gamma_1 + 4 > \gamma_1 + 2$, since $\gamma_1 > -2$.

The remainder $(\tilde{\mathbf{u}}^*, \tilde{p})$ is a solution of the following Stokes system:

$$\begin{aligned}
\partial_t\tilde{\mathbf{u}}^* - \Delta\tilde{\mathbf{u}}^* + \nabla\tilde{p} &= \tilde{\mathbf{f}}^* && \text{in } [0,2\pi]\times\Pi\,,\\
\operatorname{div}\tilde{\mathbf{u}}^* &= \tilde{g}^* && \text{in } [0,2\pi]\times\Pi\,,\\
\tilde{\mathbf{u}}^*|_{\partial\Pi} &= 0 && \text{for all } t\in[0,2\pi]\,,\\
\mathbf{u}|_{t=0} &= \mathbf{u}|_{t=2\pi} && \text{in } \Pi\,,
\end{aligned}$$

where $\tilde{\mathbf{f}}^* := \mathbf{f}^* - \partial_t\chi\boldsymbol{\mathcal{U}}^{(0,-)} + \Delta\big(\chi\boldsymbol{\mathcal{U}}^{(0,-)}\big) - \nabla\big(\chi\mathcal{P}^{(0,+)} + \chi\mathcal{P}^{(0,-)}\big)$, $\tilde{g}^* := g^* - \operatorname{div}\big(\chi\boldsymbol{\mathcal{U}}^{(0,-)}\big)$. Since $\operatorname{div}(\chi\boldsymbol{\mathcal{U}}^{(0,-)}) = \nabla\chi\cdot\boldsymbol{\mathcal{U}}^{(0,-)} + \chi\operatorname{div}\boldsymbol{\mathcal{U}}^{(0,-)} = \nabla\chi\cdot\boldsymbol{\mathcal{U}}^{(0,-)}$ has bounded support, we obviously have: $\tilde{g}^*, \nabla\tilde{g}^*, \partial_t\tilde{g}^* \in L^2(V^0_{\gamma_1+3})$. We need to break down $\tilde{\mathbf{f}}^*$. On one hand:

$$\begin{aligned}
&-\partial_t\chi\boldsymbol{\mathcal{U}}^{(0,-)} + \Delta(\chi\boldsymbol{\mathcal{U}}^{(0,-)}) - \nabla\left(\chi\mathcal{P}^{(0,+)} + \chi\mathcal{P}^{(0,-)}\right)\\
&\quad = (\Delta\chi)\boldsymbol{\mathcal{U}}^{(0,-)} + 2\nabla\chi\cdot\nabla\boldsymbol{\mathcal{U}}^{(0,-)} - (\nabla\chi)\mathcal{P}^{(0,+)} - (\nabla\chi)\mathcal{P}^{(0,-)}\\
&\qquad - \chi\underbrace{\left(\partial_t\boldsymbol{\mathcal{U}}^{(0,-)} - \Delta\boldsymbol{\mathcal{U}}^{(0,-)} + \nabla\mathcal{P}^{(0,-)}\right)}_{=0}.
\end{aligned}$$

And on the other:

$$\begin{aligned}
\mathbf{f}^* &= \mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \partial_t\mathbf{A} + \Delta\mathbf{A}\\
&= \mathbf{f} - \big((\mathbf{u}^*\cdot\nabla)\mathbf{u}^* + (\mathbf{A}\cdot\nabla)\mathbf{u}^* + (\mathbf{u}^*\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{A}\big) - \partial_t\mathbf{A} + \Delta\mathbf{A}\\
&= \mathbf{f} - (\tilde{\mathbf{u}}^*\cdot\nabla)\tilde{\mathbf{u}}^* - (\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\tilde{\mathbf{u}}^* - (\tilde{\mathbf{u}}^*\cdot\nabla)\chi\boldsymbol{\mathcal{U}}^{(0,-)} - (\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\chi\boldsymbol{\mathcal{U}}^{(0,-)}\\
&\quad - (\mathbf{A}\cdot\nabla)\chi\boldsymbol{\mathcal{U}}^{(0,-)} - (\mathbf{A}\cdot\nabla)\tilde{\mathbf{u}}^* - (\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\mathbf{A} - (\tilde{\mathbf{u}}^*\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{A}\\
&\quad - \partial_t\mathbf{A} + \Delta\mathbf{A}\,.
\end{aligned}$$

Having a compact support, we find

$$(\Delta\chi)\boldsymbol{\mathcal{U}}^{(0,-)},\ \nabla\chi\cdot\nabla\boldsymbol{\mathcal{U}}^{(0,-)},\ (\nabla\chi)\mathcal{P}^{(0,+)},\ (\nabla\chi)\mathcal{P}^{(0,-)} \in L^2(V^0_\varepsilon(\Pi))$$

for all $\varepsilon\in\mathbb{R}$ (especially $\varepsilon\in(0,1)$). The statement

$$(\tilde{\mathbf{u}}^*\cdot\nabla)\tilde{\mathbf{u}}^* \in L^2(V^0_{2\gamma_1+4}(\Pi))$$

is derived above and for the remaining terms we conclude with Lemma 4.6:

$$\begin{aligned}
&\chi\boldsymbol{\mathcal{U}}^{(0,-)}\in L^\infty(\mathcal{V}^1_{-\delta,1}),\ \tilde{\mathbf{u}}^*\in L^2(\mathcal{V}^2_{\gamma_1+2,0}) &&\overset{4.6(b)(i)}{\Longrightarrow} && (\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\tilde{\mathbf{u}}^*\in L^2(V^0_{-\delta+\gamma_1+\frac{8}{3}}),\\
&\tilde{\mathbf{u}}^*\in L^\infty(\mathcal{V}^1_{\gamma_1+2,0}),\ \chi\boldsymbol{\mathcal{U}}^{(0,-)}\in L^2(\mathcal{V}^2_{-\delta,1}) &&\overset{4.6(b)(ii)}{\Longrightarrow} && (\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\tilde{\mathbf{u}}^*\in L^2(V^0_{-\delta+\gamma_1+\frac{8}{3}}),\\
&\chi\boldsymbol{\mathcal{U}}^{(0,-)}\in L^\infty(\mathcal{V}^1_{-\delta,1}),\ \chi\boldsymbol{\mathcal{U}}^{(0,-)}\in L^2(\mathcal{V}^2_{-\delta,1}) &&\overset{4.6(a)}{\Longrightarrow} && (\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\chi\boldsymbol{\mathcal{U}}^{(0,-)}\in L^2(V^0_{-2\delta+1}),\\
&\mathbf{A}\in L^\infty(\mathcal{V}^1_{\beta+2,1}),\ \chi\boldsymbol{\mathcal{U}}^{(0,-)}\in L^2(\mathcal{V}^2_{-\delta,1}) &&\overset{4.6(a)}{\Longrightarrow} && (\mathbf{A}\cdot\nabla)\chi\boldsymbol{\mathcal{U}}^{(0,-)}\in L^2(V^0_{-\delta+\beta+3}),\\
&\chi\boldsymbol{\mathcal{U}}^{(0,-)}\in L^\infty(\mathcal{V}^1_{-\delta,1}),\ \mathbf{A}\in L^2(\mathcal{V}^2_{\beta+2,1}) &&\overset{4.6(a)}{\Longrightarrow} && (\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\mathbf{A}\in L^2(V^0_{-\delta+\beta+3}),\\
&\mathbf{A}\in L^\infty(\mathcal{V}^1_{\beta+2,1}),\ \tilde{\mathbf{u}}^*\in L^2(\mathcal{V}^2_{\gamma_1+2,0}) &&\overset{4.6(b)(i)}{\Longrightarrow} && (\mathbf{A}\cdot\nabla)\tilde{\mathbf{u}}^*\in L^2(V^0_{\beta+\gamma_1+\frac{14}{3}}),\\
&\tilde{\mathbf{u}}^*\in L^\infty(\mathcal{V}^1_{\gamma_1+2,0}),\ \mathbf{A}\in L^2(\mathcal{V}^2_{\beta+2,1}) &&\overset{4.6(b)(ii)}{\Longrightarrow} && (\tilde{\mathbf{u}}^*\cdot\nabla)\mathbf{A}\in L^2(V^0_{\beta+\gamma_1+\frac{14}{3}}),\\
&\left.\begin{array}{l}(\mathbf{A}',A_3)\in L^\infty(\mathcal{V}^1_{\beta+2,1})\times L^\infty(\mathcal{V}^2_{\beta+3,0}),\\ (\mathbf{A}',A_3)\in L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})\end{array}\right\} &&\overset{4.6(d)}{\Longrightarrow} && (\mathbf{A}\cdot\nabla)\mathbf{A}\in L^2(V^0_{2\beta+\frac{17}{3}}),
\end{aligned}$$

for all $\delta>0$. Notice $-\delta+\beta+3\geq\beta+2$ (for δ sufficiently small), $\beta+\gamma_1+\frac{14}{3}\geq 2\gamma_1+4$ and $2\beta+\frac{17}{3}\geq\beta+2$ resulting in corresponding embeddings.

With these inclusions at hand, we distinguish further:

(i): $2\gamma_1+4\geq\beta+2$.
If $2\gamma_1+4\geq\beta+2$, then also $-\delta+\gamma_1+\frac{8}{3}\geq\beta+2$ (for δ sufficiently small). This is due to the fact that $\beta<-1$, hence $\frac{\beta}{2}>\beta+\frac{1}{2}$ leading to $\gamma_1+\frac{8}{3}\geq\frac{\beta}{2}-1+\frac{8}{3}>\beta+\frac{1}{2}+\frac{5}{3}>\beta+2$. Checking now all summands of $\tilde{\mathbf{f}}^*$, we find $\tilde{\mathbf{f}}^*\in L^2(V^0_{\beta+2})$ and since $\tilde{g}^*,\nabla\tilde{g}^*,\partial_t\tilde{g}^*\in L^2(V^0_{\beta+3})$, we are able to apply Theorem 4.3(ii) with $\tilde{\mathbf{u}}^*,\tilde{p}\in L^2(V^0_{\gamma_1+1})$. Note that $(\gamma_1+1,\beta+1)\cap\mathbb{Z}=\emptyset$, due to $-2<\gamma_1<\beta<-1$. Thereby, we establish

$$\tilde{p}\in L^2(V^0_{\beta+1}(\Pi)),\quad \tilde{\mathbf{u}}^*,\nabla\tilde{\mathbf{u}}^*,\nabla^2\tilde{\mathbf{u}}^*,\partial_t\tilde{\mathbf{u}}^*,\nabla\tilde{p}\in L^2(V^0_{\beta+2}(\Pi))$$

and the following a-priori estimate:

$$\begin{aligned}
&\|\tilde{p}\|_{L^2(V^0_{\beta+1})}+\|\tilde{\mathbf{u}}^*,\nabla\tilde{\mathbf{u}}^*,\nabla^2\tilde{\mathbf{u}}^*,\partial_t\tilde{\mathbf{u}}^*,\nabla\tilde{p}\|_{L^2(V^0_{\beta+2})}\\
&\leq c\left[\|\tilde{\mathbf{f}}^*\|_{L^2(V^0_{\beta+2})}+\|\tilde{g}^*,\nabla\tilde{g}^*,\partial_t\tilde{g}^*\|_{L^2(V^0_{\beta+3})}+\|\tilde{\mathbf{u}}^*,\tilde{p}\|_{L^2(V^0_{\gamma_1+1})}\right]\\
&\leq c\Big[\big\|\mathbf{f},(\tilde{\mathbf{u}}^*\cdot\nabla)\tilde{\mathbf{u}}^*,(\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\tilde{\mathbf{u}}^*,(\tilde{\mathbf{u}}^*\cdot\nabla)\chi\boldsymbol{\mathcal{U}}^{(0,-)},(\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\chi\boldsymbol{\mathcal{U}}^{(0,-)},\\
&\qquad(\mathbf{A}\cdot\nabla)\chi\boldsymbol{\mathcal{U}}^{(0,-)},(\mathbf{A}\cdot\nabla)\tilde{\mathbf{u}}^*,(\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\mathbf{A},(\tilde{\mathbf{u}}^*\cdot\nabla)\mathbf{A},(\mathbf{A}\cdot\nabla)\mathbf{A},\partial_t\mathbf{A},\Delta\mathbf{A},\\
&\qquad(\Delta\chi)\boldsymbol{\mathcal{U}}^{(0,-)},\nabla\chi\cdot\nabla\boldsymbol{\mathcal{U}}^{(0,-)},(\nabla\chi)\mathcal{P}^{(0,+)},(\nabla\chi)\mathcal{P}^{(0,-)}\big\|_{L^2(V^0_{\beta+2})}\\
&\quad+\big\|\operatorname{div}\mathbf{A},\nabla(\operatorname{div}\mathbf{A}),\partial_t(\operatorname{div}\mathbf{A}),\nabla\chi\cdot\boldsymbol{\mathcal{U}}^{(0,-)},\nabla(\nabla\chi\cdot\boldsymbol{\mathcal{U}}^{(0,-)}),\nabla\chi\cdot\partial_t\boldsymbol{\mathcal{U}}^{(0,-)}\big\|_{L^2(V^0_{\beta+3})}
\end{aligned}$$

$$
\begin{aligned}
&+ \|\tilde{\mathbf{u}}^*, \tilde{p}\|_{L^2(V^0_{\gamma_1+1})}\Big] \\
&\le c\Big[\|\mathbf{f}\|_{L^2(V^0_{\beta+2})} + \|(\tilde{\mathbf{u}}^*\cdot\nabla)\tilde{\mathbf{u}}^*\|_{L^2(V^0_{2\gamma_1+4})} + \|(\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\tilde{\mathbf{u}}^*, (\tilde{\mathbf{u}}^*\cdot\nabla)\chi\boldsymbol{\mathcal{U}}^{(0,-)}\|_{L^2(V^0_{-\delta+\gamma_1+\frac{8}{3}})} \\
&+ \|(\mathbf{A}\cdot\nabla)\chi\boldsymbol{\mathcal{U}}^{(0,-)}, (\chi\boldsymbol{\mathcal{U}}^{(0,-)}\cdot\nabla)\mathbf{A}\|_{L^2(V^0_{-\delta+\beta+3})} + \|(\mathbf{A}\cdot\nabla)\tilde{\mathbf{u}}^*, (\tilde{\mathbf{u}}^*\cdot\nabla)\mathbf{A}\|_{L^2(V^0_{\beta+\gamma_1+\frac{14}{3}})} \\
&+ \|(\mathbf{A}\cdot\nabla)\mathbf{A}\|_{L^2(V^0_{2\beta+\frac{17}{3}})} + \|\partial_t\mathbf{A}, \Delta\mathbf{A}\|_{L^2(V^0_{\beta+2})} \\
&+ \|(\Delta\chi)\boldsymbol{\mathcal{U}}^{(0,-)}, \nabla\chi\cdot\nabla\boldsymbol{\mathcal{U}}^{(0,-)}, (\nabla\chi)\mathcal{P}^{(0,+)}, (\nabla\chi)\mathcal{P}^{(0,-)}\|_{L^2(V^0_{\beta+2})} \\
&+ \big\|\operatorname{div}\mathbf{A}, \nabla(\operatorname{div}\mathbf{A}), \partial_t(\operatorname{div}\mathbf{A}), \nabla\chi\cdot\boldsymbol{\mathcal{U}}^{(0,-)}, \nabla(\nabla\chi\cdot\boldsymbol{\mathcal{U}}^{(0,-)}), \nabla\chi\cdot\partial_t\boldsymbol{\mathcal{U}}^{(0,-)}\big\|_{L^2(V^0_{\beta+3})} \\
&+ \|\tilde{\mathbf{u}}^*, \tilde{p}\|_{L^2(V^0_{\gamma_1+1})}\Big] \\
&\le c\Big[\|\mathbf{f}\|_{L^2(V^0_{\beta+2})} + \|\tilde{\mathbf{u}}^*\|_{L^\infty(\mathcal{V}^1_{\gamma_1+2,0})}\|\tilde{\mathbf{u}}^*\|_{L^2(\mathcal{V}^2_{\gamma_1+2,0})} + \|\chi\boldsymbol{\mathcal{U}}^{(0,-)}\|_{L^\infty(\mathcal{V}^1_{-\delta,1})}\|\tilde{\mathbf{u}}^*\|_{L^2(\mathcal{V}^2_{\gamma_1+2,0})} \\
&+ \|\tilde{\mathbf{u}}^*\|_{L^\infty(\mathcal{V}^1_{\gamma_1+2,0})}\|\chi\boldsymbol{\mathcal{U}}^{(0,-)}\|_{L^2(\mathcal{V}^2_{-\delta,1})} + \|\chi\boldsymbol{\mathcal{U}}^{(0,-)}\|_{L^\infty(\mathcal{V}^1_{-\delta,1})}\|\chi\boldsymbol{\mathcal{U}}^{(0,-)}\|_{L^2(\mathcal{V}^2_{-\delta,1})} \\
&+ \|\mathbf{A}\|_{L^\infty(\mathcal{V}^1_{\beta+2,1})}\|\chi\boldsymbol{\mathcal{U}}^{(0,-)}\|_{L^2(\mathcal{V}^2_{-\delta,1})} + \|\chi\boldsymbol{\mathcal{U}}^{(0,-)}\|_{L^\infty(\mathcal{V}^1_{-\delta,1})}\|\mathbf{A}\|_{L^2(\mathcal{V}^2_{\beta+2,1})} \\
&+ \|\mathbf{A}\|_{L^\infty(\mathcal{V}^1_{\beta+2,1})}\|\tilde{\mathbf{u}}^*\|_{L^2(\mathcal{V}^2_{\gamma_1+2,0})} + \|\tilde{\mathbf{u}}^*\|_{L^\infty(\mathcal{V}^1_{\gamma_1+2,0})}\|\mathbf{A}\|_{L^2(\mathcal{V}^2_{\beta+2,1})} \\
&+ \|(\mathbf{A}', A_3)\|_{L^\infty(\mathcal{V}^1_{\beta+2,1})\times L^\infty(\mathcal{V}^1_{\beta+3,0})}\|(\mathbf{A}', A_3)\|_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})} + \|\partial_t\mathbf{A}, \Delta\mathbf{A}\|_{L^2(V^0_{\beta+2})} \\
&+ \Big(\|w^{(0,-)}\|_{L^2(H^1(0,1))} + \|s^{(0,\pm)}\|_{L^2(0,2\pi)}\Big)\|P^{(0,\pm)}, \nabla_y P^{(0,-)}, \nabla_y^2 P^{(0,-)}\|_{L^2(\operatorname{supp}\nabla\chi)} \\
&+ \|(\mathbf{A}', A_3)\|_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})} + \|(\partial_t\mathbf{A}', \partial_t A_3)\|_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})} \\
&+ \Big(\|w^{(0,-)}\|_{L^2(H^1(0,1))} + \|\partial_t w^{(0,-)}\|_{L^2(L^2(0,1))}\Big)\|\nabla_y P^{(0,-)}, \nabla_y^2 P^{(0,-)}\|_{L^2(\operatorname{supp}\nabla\chi)} \\
&+ \|\tilde{\mathbf{u}}^*, \tilde{p}\|_{L^2(V^0_{\gamma_1+1})}\Big] \\
&\le c\Big[\|\mathbf{f}\|_{L^2(V^0_{\beta+2})} + \|\tilde{\mathbf{u}}^*\|^2_{L^\infty(\mathcal{V}^1_{\gamma_1+2,0})} + \|\tilde{\mathbf{u}}^*\|^2_{L^2(\mathcal{V}^2_{\gamma_1+2,0})} \\
&+ \|w^{(0,-)}\|^2_{L^\infty(H^1(0,1))}\|\chi\nabla_y P^{(0,-)}\|^2_{V^1_{-\delta+1}(\mathbb{R}^2)} + \|w^{(0,-)}\|^2_{L^2(H^2(0,1))}\|\chi\nabla_y P^{(0,-)}\|^2_{V^2_{-\delta+2}(\mathbb{R}^2)} \\
&+ \|(\mathbf{A}', A_3)\|^2_{L^\infty(\mathcal{V}^1_{\beta+2,1})\times L^\infty(\mathcal{V}^1_{\beta+3,0})} + \|(\mathbf{A}', A_3)\|^2_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})} \\
&+ \|(\mathbf{A}', A_3)\|_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})} + \|(\partial_t\mathbf{A}', \partial_t A_3)\|_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})} \\
&+ \|w^{(0,-)}\|_{L^2(H^1(0,1))} + \|\partial_t w^{(0,-)}\|_{L^2(L^2(0,1))} + \|s^{(0,\pm)}\|_{L^2(0,2\pi)} \\
&+ \|\tilde{\mathbf{u}}^*, \tilde{p}\|_{L^2(V^0_{\gamma_1+1})}\Big] \\
&\le c\Big[\|\mathbf{f}\|_{L^2(V^0_{\beta+2})} + \|(\mathbf{A}', A_3)\|^2_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})} + \|(\partial_t\mathbf{A}', \partial_t A_3)\|^2_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})} \\
&+ \|(\mathbf{A}', A_3)\|_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})} + \|(\partial_t\mathbf{A}', \partial_t A_3)\|_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})} \\
&+ \|w^{(0,-)}\|^2_{L^2(H^2(0,1))} + \|w^{(0,-)}\|^2_{L^\infty(H^1(0,1))} \\
&+ \|w^{(0,-)}\|_{L^2(H^2(0,1))} + \|\partial_t w^{(0,-)}\|_{L^2(L^2(0,1))} + \|s^{(0,\pm)}\|_{L^2(0,2\pi)} \\
&+ \|\tilde{\mathbf{u}}^*\|^2_{L^2(\mathcal{V}^2_{\gamma_1+2,0})} + \|\partial_t\tilde{\mathbf{u}}^*\|^2_{L^2(V^0_{\gamma_1+2})} + \|\tilde{\mathbf{u}}^*, \tilde{p}\|_{L^2(V^0_{\gamma_1+1})}\Big]
\end{aligned}
\tag{4.14}
$$

$$
\overset{(4.13)}{\le} c\Big[\|\mathbf{f}\|_{L^2(V^0_{\beta+2})} + \|(\mathbf{A}', A_3)\|^2_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})} + \|(\partial_t\mathbf{A}', \partial_t A_3)\|^2_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})}
$$

$$
\begin{aligned}
&+\|(\mathbf{A}',A_3)\|_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})}+\|(\partial_t\mathbf{A}',\partial_t A_3)\|_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})}\\
&+\Big(\|\mathbf{f}^*\|_{L^2(V^0_{\gamma_1+2})}+\|g^*,\nabla g^*,\partial_t g^*\|_{L^2(V^0_{\gamma_1+3})}+\|\mathbf{u}^*,p\|_{L^2(V^0_{\gamma_1})}\Big)^2\\
&+\Big(\|\mathbf{f}^*\|_{L^2(V^0_{\gamma_1+2})}+\|g^*,\nabla g^*,\partial_t g^*\|_{L^2(V^0_{\gamma_1+3})}+\|\mathbf{u}^*,p\|_{L^2(V^0_{\gamma_1})}\Big)\\
&+\Big(\|\mathbf{f}^*\|_{L^2(V^0_{\gamma_1+2})}+\|g^*,\nabla g^*,\partial_t g^*\|_{L^2(V^0_{\gamma_1+3})}+\|\mathbf{u}^*,p\|_{L^2(V^0_{\gamma_1})}\Big)^2\\
&+\Big(\|\mathbf{f}^*\|_{L^2(V^0_{\gamma_1+2})}+\|g^*,\nabla g^*,\partial_t g^*\|_{L^2(V^0_{\gamma_1+3})}+\|\mathbf{u}^*,p\|_{L^2(V^0_{\gamma_1})}\Big)\Big]\\
\le c\Big[&\|\mathbf{f}\|_{L^2(V^0_{\beta+2})}+\|(\mathbf{A}',A_3)\|^2_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})}+\|(\partial_t\mathbf{A}',\partial_t A_3)\|^2_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})}\\
&+\|(\mathbf{A}',A_3)\|_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})}+\|(\partial_t\mathbf{A}',\partial_t A_3)\|_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})}\\
&+\Big(\|\mathbf{f},(\mathbf{u}\cdot\nabla)\mathbf{u},\partial_t\mathbf{A},\Delta\mathbf{A}\|_{L^2(V^0_{\gamma_1+2})}+\|\operatorname{div}\mathbf{A},\nabla(\operatorname{div}\mathbf{A}),\partial_t(\operatorname{div}\mathbf{A})\|_{L^2(V^0_{\gamma_1+3})}+\|\mathbf{u}^*,p\|_{L^2(V^0_{\gamma})}\Big)^2\\
&+\Big(\|\mathbf{f},(\mathbf{u}\cdot\nabla)\mathbf{u},\partial_t\mathbf{A},\Delta\mathbf{A}\|_{L^2(V^0_{\gamma_1+2})}+\|\operatorname{div}\mathbf{A},\nabla(\operatorname{div}\mathbf{A}),\partial_t(\operatorname{div}\mathbf{A})\|_{L^2(V^0_{\gamma_1+3})}+\|\mathbf{u}^*,p\|_{L^2(V^0_{\gamma})}\Big)\Big]\\
\le c\Big[&1+\|\mathbf{f}\|_{L^2(V^0_{\beta+2})}+\|(\mathbf{A}',A_3)\|_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})}+\|(\partial_t\mathbf{A}',\partial_t A_3)\|_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})}\\
&+\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^2(V^0_{2\gamma+\frac{11}{3}})}+\|\mathbf{u},p\|_{L^2(V^0_{\gamma})}\Big]^2\\
\le c\Big[&1+\|\mathbf{f}\|_{L^2(V^0_{\beta+2})}+\|(\mathbf{A}',A_3)\|_{L^2(\mathcal{V}^2_{\beta+2,1})\times L^2(\mathcal{V}^2_{\beta+3,0})}+\|(\partial_t\mathbf{A}',\partial_t A_3)\|_{L^2(\mathcal{V}^1_{\beta+2,1})\times L^2(\mathcal{V}^1_{\beta+3,0})}\\
&+\|(\mathbf{u}',u_3)\|^2_{L^\infty(\mathcal{V}^1_{\gamma+1,1})\times L^\infty(\mathcal{V}^1_{\gamma+2,0})}+\|(\mathbf{u}',u_3)\|^2_{L^2(\mathcal{V}^2_{\gamma+1,1})\times L^2(\mathcal{V}^2_{\gamma+2,0})}+\|\mathbf{u},p\|_{L^2(V^0_{\gamma})}\Big]^2.
\end{aligned}
$$

Finally, setting $\tilde{\mathbf{u}} := \tilde{\mathbf{u}}^* + \mathbf{A}$ we obtain the claimed representation and estimate.

(ii): $2\gamma_1 + 4 < \beta + 2$.
Otherwise, if $2\gamma_1 + 4 < \beta + 2$, we have $\tilde{\mathbf{f}}^* \in L^2(V^0_{2\gamma_1+4})$, since $2\gamma_1 + 4 < \beta + 2 < 1 \Leftrightarrow \gamma_1 < -\frac{3}{3}$ implies $2\gamma_1 + 4 < -\delta + \gamma_1 + \frac{8}{3}$. Notice, further, that $(\gamma_1 + 1, 2\gamma_1 + 3) \cap \mathbb{Z} = \emptyset$, due to $-2 < \gamma_1 < 2\gamma_1 + 2 < \beta < -1$. By Theorem 4.3(ii), with $\tilde{\mathbf{f}}^* \in L^2(V^0_{2\gamma_1+4})$, $\tilde{g}^*, \nabla\tilde{g}^*, \partial_t\tilde{g}^* \in L^2(V^0_{2\gamma_1+5})$ and $\tilde{\mathbf{u}}^*, \tilde{p} \in L^2(V^0_{\gamma_1+1})$, we infer

$$\tilde{p} \in L^2(V^0_{2\gamma_1+3}(\Pi)), \quad \tilde{\mathbf{u}}^*, \nabla\tilde{\mathbf{u}}^*, \nabla^2\tilde{\mathbf{u}}^*, \partial_t\tilde{\mathbf{u}}^*, \nabla\tilde{p} \in L^2(V^0_{2\gamma_1+4}(\Pi)).$$

Define $\gamma_2 + 2 := 2\gamma_1 + 4$, whereby $\gamma_2 > \gamma_1$. We iteratively repeat the argumentation above starting with an improved embedding $\tilde{\mathbf{u}}^* \in L^\infty(\mathcal{V}^1_{\gamma_2+2,0}(\Pi))$ and, consequently, $(\tilde{\mathbf{u}}^* \cdot \nabla)\tilde{\mathbf{u}}^* \in L^2(V^0_{2\gamma_2+4}(\Pi))$. The distinction (i) / (ii) then becomes: "$2\gamma_2 + 4 \geq \beta + 2$ or $2\gamma_2 + 4 < \beta + 2$".

If $2\gamma_2 + 4 \geq \beta + 2$ we conclude the proof as before, with (i) inserting γ_2. Thus, the final inequality is carried out for γ_2, where

$$\|\tilde{\mathbf{u}}^*\|_{L^2(\mathcal{V}^2_{\gamma_2+2,0})}, \|\partial_t\tilde{\mathbf{u}}^*\|_{L^2(V^0_{\gamma_2+2})}, \|\tilde{\mathbf{u}}^*, \tilde{p}\|_{L^2(V^0_{\gamma_2+1})}$$

in line (4.14) need to be estimated by

$$\|\tilde{\mathbf{f}}^*\|_{L^2(V^0_{2\gamma_1+4})} + \|\tilde{g}^*, \nabla\tilde{g}^*, \partial_t\tilde{g}^*\|_{L^2(V^0_{2\gamma_1+5})} + \|\tilde{\mathbf{u}}^*, \tilde{p}\|_{L^2(V^0_{\gamma_1+1})}$$

due to Theorem 4.3(ii) (instead of using (4.13)). Then, these terms get treated exactly as in (i) for the first iteration – applying appropriate embeddings of weighted Sobolev spaces first (in particular, $V^0_{-\delta+\gamma_1+\frac{8}{3}}(\Pi) \hookrightarrow V^0_{2\gamma_1+4}(\Pi)$ and $V^0_{\beta+\gamma_1+\frac{14}{3}}(\Pi) \hookrightarrow V^0_{2\gamma_1+4}(\Pi)$).

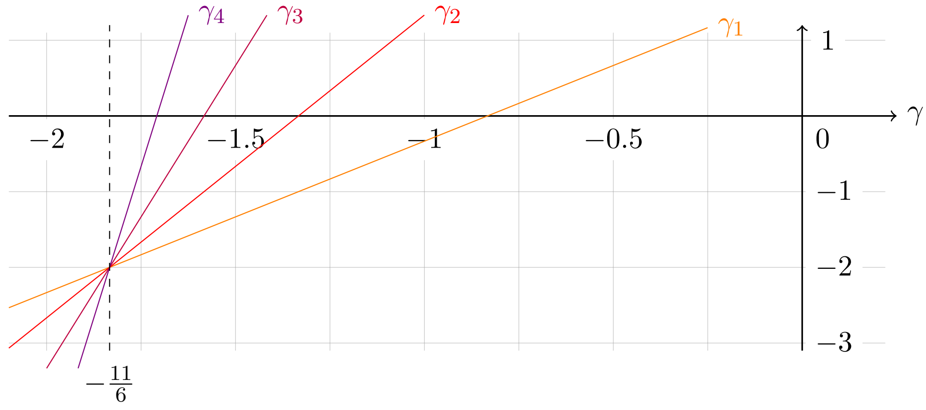

Figure 4.2: Iterated weight indices in dependence of the initial index γ.

In the latter case we continue the iteration up to the point where

$$2\gamma_k + 4 = 2^k(\gamma_1 + 2) = 2^{k+1}\big(\gamma + \tfrac{11}{6}\big) \geq \beta + 2$$

for some $k \in \mathbb{N}$. Then, the proof is closed by (i). □

By a slight adjustment in its proof, the statement of the preceding theorem can be preserved, whilst altering assumptions on $\mathbf{u}$, to furnish an overall greater picture:

Corollary 4.8. *Let* $\mathbf{f}$, g, $\mathbf{a}$, p *and* $\mathbf{u}$ *satisfy the assumptions of Theorem 4.7.*

(i) *Suppose* $\mathbf{u} \in L^2(0, 2\pi; \mathcal{V}^2_{\gamma+2,0}(\Pi)) \cap L^\infty(0, 2\pi; \mathcal{V}^1_{\gamma+2,0}(\Pi))$ *(in place of* (4.12)*), then Theorem 4.7 is valid for* $\gamma \in (-2, -1)$ *and the iteration index* k *results from the inequality* $2^{k+1}(\gamma + 2) \geq \beta + 2$.

(ii) *Suppose* $\mathbf{u} \in L^2(0, 2\pi; \mathcal{V}^2_{\gamma+1,1}(\Pi)) \cap L^\infty(0, 2\pi; \mathcal{V}^1_{\gamma+1,1}(\Pi))$ *(in place of* (4.12)*), then Theorem 4.7 is valid for* $\gamma \in (-\frac{3}{2}, -1)$ *and the iteration index* k *results from the inequality* $2^{k+1}(\gamma + \frac{3}{2}) \geq \beta + 2$.

Proof. The proof is carried out exactly like the one of Theorem 4.7, with the only difference being the usage of another estimate for the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$.

ad(i): In the first lines of the proof of Theorem 4.7 we use Lemma 4.6(c) (instead of Lemma 4.6(d)), due to the new assumptions on $\mathbf{u}$. This yields

$$(\mathbf{u} \cdot \nabla)\mathbf{u} \in L^2(0, 2\pi; V^0_{2\gamma+4}(\Pi))$$

and, thereby, we set $\gamma_1 + 2 := 2\gamma + 4$. To continue with the identical scheme as before, we need $\gamma_1 > -2$, which is satisfied for $\gamma > -2$. This eventually results in the iteration

$$2\gamma_k + 4 = 2^k(\gamma_1 + 2) = 2^{k+1}(\gamma + 2)\,.$$

Hence, the iteration terminates, since there exists an index $k \in \mathbb{N}_0$ such that $2^{k+1}(\gamma+2) \geq \beta + 2$ for all $\gamma \in (-2, -1)$.

ad(ii): Following the same idea as in (i), we start with an application of Lemma 4.6(a) (instead of Lemma 4.6(d)), taking into account the altered assumptions on $\mathbf{u}$. We obtain

$$(\mathbf{u} \cdot \nabla)\mathbf{u} \in L^2(0, 2\pi; V^0_{2\gamma+3}(\Pi))$$

and set $\gamma_1 + 2 := 2\gamma + 3$. In order to proceed as in the proof of Theorem 4.7 we verify $\gamma_1 > -2$: Since $\gamma \in (-\frac{3}{2}, -1)$, $\gamma_1 = 2\gamma + 1 > -2$. This leads to the iteration

$$2\gamma_k + 4 = 2^k(\gamma_1 + 2) = 2^{k+1}(\gamma + \tfrac{3}{2})$$

and, for each $\gamma \in (-\frac{3}{2}, -1)$, there exists an index $k \in \mathbb{N}_0$ with $2^{k+1}(\gamma + \frac{3}{2}) \geq \beta + 2$. $\square$

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