Stability of Preconditioned Finite Volume Schemes at Low Mach Numbers

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Abstract

Abstract. In [4], Guillard and Viozat propose a finite volume method for the simulation of inviscid steady as well as unsteady flows at low Mach numbers, based on a preconditioning technique. The scheme satisfies the results of a single scale asymptotic analysis in a discrete sense and comprises the advantage that this can be derived by a slight modification of the dissipation term within the numerical flux function. Unfortunately, it can be observed by numerical experiments that the preconditioned approach combined with an explicit time integration scheme turns out to be unstable if the time step Δt does not satisfy the requirement to be $\mathcal{O}(M^2)$ as the Mach number M tends to zero, whereas the corresponding standard method remains stable up to $\Delta t = \mathcal{O}(M)$, $M \to 0$, which results from the well-known CFL-condition.

We present a comprehensive mathematical substantiation of this numerical phenomenon by means of a von Neumann stability analysis, which reveals that in contrast to the standard approach, the dissipation matrix of the preconditioned numerical flux function possesses an eigenvalue growing like M^{-2} as M tends to zero, thus causing the diminishment of the stability region of the explicit scheme. Thereby, we present statements for both the standard preconditioner used by Guillard and Viozat [4] and the more general one due to Turkel [21]. The theoretical results are after wards confirmed by numerical experiments.

Keywords: Preconditioning, Low Mach number flow, Asymptotic analysis, Finite volume method, Conservation laws

1 Introduction

The efficient simulation of low Mach number flows is a subject of ongoing discussion in the CFD community. While the flow is expected to be incompressible, in a lot of applications the Mach number or the compressibility properties vary strongly in time or space. This is for example the case in nozzle flow, chemically reacting flows or laminar combustion. A specific example is a fire in a road tunnel, where the strong heat sources make parts of the flow compressible although the Mach number M remains small, i.e. $M \approx 10^{-3}$. It is

well known that purely compressible flow solvers which were developed for transonic flow produce wrong results at low Mach numbers. On the other hand, standard incompressible flow solvers cannot deal with strong temperature or strong density gradients. This sets a demand for codes that can deal with flows at all Mach numbers.

As a lot of expertise and development time was put into currently used codes, the desire to expand an existing code as opposed to writing a completely new one is very natural. Consequently, there are two main approaches to the design of numerical methods for the above mentioned flows: use either the compressible or the incompressible Euler or Navier-Stokes equations as the basic model and improve upon the existing methods. Both approaches are pursued and widely used. One important idea in this context was the artificial compressibility method by Chorin that inspired the preconditioner of Turkel [21] for the compressible equations. These methods incorporate a preconditioning of the time derivative of the PDE, thus allowing faster convergence to steady state but sacrificing time accuracy. Along these lines, other preconditioners were proposed [23, 1]. The crucial idea is, that as the Mach number tends to zero, the original system develops a large disparity in wave speeds, as some of the eigenvalues grow to infinity while others stay $\mathcal{O}(1)$. The preconditioner changes all the wave speeds to $\mathcal{O}(1)$, thus greatly improving the condition number of the system.

For incompressible flows, there are two main techniques that are used to expand the validity of the scheme into the compressible regime. One class of schemes is based on the marker and cell method, MAC for short, by Harlow and Welch [5], which is a finite difference method on a staggered grid. The method is quite fast, but it is very difficult to use the staggered location of the variables in the context of unstructured grids. Recently, Wenneker, Segal and Wesseling proposed a method that faces this difficulty [26]. On the other hand, Patankar and Spalding [16] published their SIMPLE scheme in 1972. Based on an approximation of the pressure, a velocity field is computed using the momentum equations. Then, an elliptic pressure correction equation is solved to improve the approximation of the pressure. These steps are then iterated until convergence is achieved. The approach is not limited to incompressible flows: see [2]. Both SIMPLE and MAC scheme as well as their improved descendants have in common that they work on the velocity field and the pressure distribution. By contrast, codes for compressible flow are usually based on the conserved variables density, momentum and energy. Thus, in the context of methods for all Mach numbers it is useful not to speak of incompressible and compressible solvers, but of pressure based and density based schemes.

In this paper, we will concentrate on the case of density based flow solvers. Here, three main techniques to obtain time accuracy can be distinguished. First of all, there is the technique to use the above mentioned preconditioning methods for steady state flows in a pseudo time stepping scheme [20, 22, 25]. Furthermore, there is the flux correction approach, where an approximation to the Euler or Navier-Stokes equations is solved and then corrected via elliptic correction equations [6, 10, 18]. Finally, there is the flux preconditioning approach, where only the dissipation within the numerical flux function of the flow solver is changed by low Mach number preconditioning [4, 14]. This has several advantages. An important one is that the implementation is quite simple. The only part of the flow solver that needs to be changed is the flux function. The other one is that compared to the flux correction approach, the computational effort per time step is less expensive.

However, numerical experiments indicate that the stability region of an explicit preconditioned method deteriorates as the Mach number tends to zero. In order to overcome this severe disadvantage implicit methods are usually employed. Although this behavior of explicit schemes is often reported, a comprehensive analysis of this phenomenon is still missing. The paper is devoted to close this gap and to present a detailed stability analysis of a class of preconditioned Lax-Friedrich type schemes. Besides the method proposed by Guillard and Viozat [4] a more general preconditioning matrix due to Turkel [21] is investigated. Thereby, it is proved that the whole class of this approaches suffers from similar stability problems and thus, an implicit time stepping scheme should be preferred for this kind of method in the regime of low Mach numbers. The outline of the article is as follows: First, we briefly introduce the Euler equations of gas dynamics. In order to get a deeper insight into the behavior of the corresponding physical quantities, we summarize the main results of an asymptotic analysis in the low Mach number limit. Thereafter, a finite volume approximation of the governing equations using a Lax-Friedrichs type scheme is presented whereby we curtly discuss the asymptotic behavior of the preconditioned as well as the unpreconditioned approach as the Mach number tends to zero. The core of the paper is a L_2-stability analysis of a class of preconditioned methods, where we will prove the general unfavorable behavior of the explicit scheme. The theoretical results are after wards confirmed by numerical experiments.

2 The governing equations

The Euler equations consist of the conservation laws of mass, momentum and energy, closed by an equation of state. Given an open domain $D \subset \mathbb{R}^d$, the dimensional form can be expressed as

$$\partial_{\hat{t}} \hat{\mathbf{u}} + \sum_{j=1}^{d} \partial_{\hat{x}_j} \hat{\mathbf{f}}_j(\hat{\mathbf{u}}) = \mathbf{0} \quad \text{in } D \times \mathbb{R}^+,$$

where $\hat{\mathbf{u}} = (\hat{\rho}, \hat{m}_1, ..., \hat{m}_d, \hat{\rho}\hat{E})^T$ represents the vector of conserved variables. The flux functions $\hat{\mathbf{f}}_j$ are given by

$$\hat{\mathbf{f}}_j(\hat{\mathbf{u}}) = \begin{pmatrix} \hat{m}_j \\ \hat{m}_j \hat{v}_1 + \delta_{1j}\hat{p} \\ \vdots \\ \hat{m}_j \hat{v}_d + \delta_{dj}\hat{p} \\ \hat{H}\hat{m}_j \end{pmatrix}, \ j = 1, ..., d,$$

with δ_{ij} denoting the Kronecker symbol. The dimensional quantities $\hat{\rho}$, $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2)^T$, $\hat{\mathbf{m}} = (\hat{m}_1, \hat{m}_2)^T$, $\hat{E}$ and $\hat{H} = \hat{E} + \frac{\hat{p}}{\hat{\rho}}$ describe the density, velocity, momentum per unit volume, total energy per unit mass and total enthalpy per unit mass, respectively. The pressure is defined by the equation of state for a perfect gas $\hat{p} = (\gamma - 1)\hat{\rho}(\hat{E} - \frac{1}{2}|\hat{\mathbf{v}}|^2)$, where γ denotes the ratio of specific heats, taken as 1.4 for air.

Numerical schemes are usually based on a non-dimensional form of the Euler equations. Such a form is obtained by introducing reference values. Usually, these are the reference

values for the length $\hat{x}_{ref}$, the density $\hat{\rho}_{ref}$ and the velocity $\hat{v}_{ref}$. Reference values for other quantities are derived from these by functional relationships, for example $\hat{t}_{ref} = \hat{x}_{ref}/\hat{v}_{ref}$ and $\hat{p}_{ref} = \hat{\rho}_{ref}\hat{v}_{ref}^2$. This well-known approach leads to an identical non-dimensional system, compared to the dimensional form. Unfortunately, for low Mach numbers, the pressure reference has no longer a valid physical meaning and the non-dimensional quantity p is not $\mathcal{O}(1)$.

Thus, following [9, 8], we employ an independent pressure reference $\hat{p}_{ref}$. We introduce the expression

$$\hat{\varphi} = \widetilde{\varphi} \cdot \hat{\varphi}_{ref}$$

for each physical quantity $\hat{\varphi}$ into the governing equations and hence the resulting non-dimensional formulation of the Euler equations as well as the equation of state read

$$\begin{aligned} \partial_t \widetilde{\rho} + & \quad \nabla_{\mathbf{x}} \cdot \widetilde{\mathbf{m}} & &= 0, \\ \partial_t \widetilde{\mathbf{m}} + & \quad \nabla_{\mathbf{x}} \cdot (\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{v}}) + \tfrac{1}{M^2} \nabla_{\mathbf{x}} \widetilde{p} & &= \mathbf{0}, \\ \partial_t (\widetilde{\rho} \widetilde{E}) + & \quad \nabla_{\mathbf{x}} \cdot (\widetilde{H} \widetilde{\mathbf{m}}) & &= 0 \end{aligned} \tag{1}$$

and

$$\widetilde{p} = (\gamma - 1)\, \widetilde{\rho} \left(\widetilde{E} - \frac{M^2}{2} |\widetilde{\mathbf{v}}|^2 \right). \tag{2}$$

The non-dimensional characteristic number M appearing within (1) and (2) is called the Mach number which is defined as

$$M = \frac{\hat{v}_{ref}}{\hat{c}_{ref}}, \tag{3}$$

where $\hat{c}_{ref} = \sqrt{\hat{p}_{ref}/\hat{\rho}_{ref}}$ denotes the reference value for the speed of sound $\hat{c}$. Note that the non-dimensional solution depends on the reference values and therefore we take into account this fact by writing $\widetilde{\varphi}(\mathbf{x}, t; M)$. Based on the characteristic number M we are now able to investigate the behavior of the physical quantities in the low Mach number limit by means of a formal asymptotic analysis.

Throughout this paragraph we assume that the non-dimensional formulation of the governing equations (1) consists exclusively of physical quantities $\widetilde{\varphi}$ satisfying $\widetilde{\varphi} = \mathcal{O}(1)$ if the Mach number tends to zero.

We expand every physical variable in a single time scale, single space scale asymptotic expansion

$$\widetilde{\varphi}(\mathbf{x}, t; M) = \sum_{m=0}^{j} M^m \widetilde{\varphi}^{(m)}(\mathbf{x}, t) + o(M^j),\ M \to 0,\ j = 0, 1, 2 \tag{4}$$

and introduce it into system (1). We will only briefly present the results of this technique. A comprehensive survey can be found in [12, 9].

Theorem 1 *Let* $\widetilde{\mathbf{u}}$ *be a solution of system* (1)*. Then the leading order pressure satisfies*

$$\widetilde{p}(\mathbf{x},t;M) = \widetilde{p}^{(0)}(t) + M\widetilde{p}^{(1)}(t) + M^2\widetilde{p}^{(2)}(\mathbf{x},t) + o\left(M^2\right), \quad M \to 0.$$

Therefore, the pressure varies in space with $\mathcal{O}(M^2)$ if the Mach number is sufficiently small. This is an important result that the discrete scheme has to take into account.

Theorem 2 *Let* $\widetilde{\mathbf{u}}$ *be a solution of system* (1)*. Thus there holds*

$$\frac{1}{\gamma\widetilde{p}^{(0)}}\frac{d}{dt}\widetilde{p}^{(0)} = -\frac{1}{|\Omega|}\int_{\partial\Omega}\widetilde{\mathbf{v}}^{(0)}\cdot\mathbf{n}\,ds, \quad M \to 0.$$

Theorem 2 implies that the temporal change of the leading order pressure is only due to compression from the boundary or expansion of the gas itself.

Theorem 3 *Let* $\widetilde{\mathbf{v}}$ *be a velocity vector corresponding to a solution of* (1)*. Then it satisfies the divergence constraint*

$$\nabla_{\mathbf{x}}\cdot\widetilde{\mathbf{v}} = \frac{1}{|\Omega|}\int_{\partial\Omega}\widetilde{\mathbf{v}}\cdot\mathbf{n}\,ds + o(1), \quad M \to 0.$$

Note, that for a vanishing right hand side, we obtain the divergence constraint on the velocity field known from incompressible flow.

3 Preconditioned Finite Volume Method

In order to rewrite the Euler equations (1) in a convenient manner for standard compressible flow solvers we define the auxiliary variables

$$p = \frac{\widetilde{p}}{M^2}, \quad c = \frac{\widetilde{c}}{M}, \quad E = \frac{\widetilde{E}}{M^2} \text{ and } H = \frac{\widetilde{H}}{M^2}. \tag{5}$$

Thus, system (1) becomes

$$\partial_t\mathbf{u} + \sum_{\ell=1}^{2}\partial_{x_\ell}\mathbf{f}_\ell(\mathbf{u}) = \mathbf{0} \text{ in } D\times\mathbb{R}^+, \tag{6}$$

where $\mathbf{u} = (\rho, \rho v_1, \rho v_2, \rho E)^T$, $\mathbf{f}_\ell(\mathbf{u}) = (\rho v_\ell,\ \rho v_\ell v_1 + \delta_1^\ell p,\ \rho v_\ell v_2 + \delta_2^\ell p,\ \rho H v_\ell)^T$ and $p = (\gamma-1)\rho\left(E - \frac{|\mathbf{v}|^2}{2}\right)$.

It is well known that smooth solutions of the Euler equations exist in general only for short times and thus one usually introduces the concept of weak solutions. By means of integrating the system (6) over a control volume $\Omega \subset D$ and applying Gauss' integral theorem one obtains

$$\frac{d}{dt}\int_{\mathbf{\Omega}} \mathbf{u}\, d\mathbf{x} + \sum_{\ell=1}^{2} \int_{\partial\mathbf{\Omega}} \mathbf{f}_\ell(\mathbf{u}) n_\ell \, ds = \mathbf{0}, \tag{7}$$

where $\mathbf{n} = (n_1, n_2)^T$ represents the outer unit normal vector on $\partial\Omega$. A mapping $\mathbf{u}$ is called a weak solution of the governing equations (6) if $\mathbf{u}$ satisfies the integral form of the Euler equations (7) on every bounded set $\Omega \subset D$ which allows to utilize Gauss' integral theorem.

Introducing the cell average $\mathbf{u}_i(t) = \frac{1}{|\Omega_i|}\int_{\Omega_i} \mathbf{u}(\mathbf{x}, t)\, d\mathbf{x}$, the integral form with respect to a box Ω_i can be written as

$$\frac{d}{dt}\mathbf{u}_i(t) = -\frac{1}{|\Omega_i|} \sum_{e_{ij}\subset\partial\Omega_i} \int_{e_{ij}} \sum_{\ell=1}^{2} \mathbf{f}_\ell(\mathbf{u}(\mathbf{x}, t)) n_\ell \, d\mathbf{s},$$

where e_{ij} denotes the edge between the adjacent control volumes Ω_i and Ω_j. In order to evaluate the integral we employ a numerical flux function of Lax-Friedrichs type

$$H(\mathbf{u}_i, \mathbf{u}_j, \mathbf{n}) = \frac{1}{2}\left(\sum_{\ell=1}^{2} \left(\mathbf{f}_\ell(\mathbf{u}_i) + \mathbf{f}_\ell(\mathbf{u}_j)\right) n_\ell - \mathbf{D}(\mathbf{u}_i, \mathbf{u}_j, \mathbf{n})(\mathbf{u}_j - \mathbf{u}_i)\right). \tag{8}$$

Numerical methods of this type differ only in the dissipation term $\mathbf{D}$. With respect to the governing equations (6) Hu and Shu [7] defined $\mathbf{D} \in \mathbb{R}$ to be the largest absolute eigenvalue of the corresponding Jacobi matrix

$$\mathbf{F}(\mathbf{u}, \mathbf{n}) = \sum_{\ell=1}^{2} \frac{\partial \mathbf{f}_\ell}{\partial \mathbf{u}_\ell}(\mathbf{u}) n_\ell.$$

Similar to the formulation of Shu and Osher [19] we prefer a matrix-valued term $\mathbf{D}$ which was proposed by Friedrich [3]. It is proved in [13, 14] that the use of this standard numerical flux function leads to an unphysical pressure distribution which contradicts the statement of theorem 1. In particular, variations of the first order pressure field are generated on the space scale $\mathbf{x}$. To extend the validity of the numerical method we utilize a preconditioning technique originally proposed by Guillard and Viozat [4] for the Roe scheme and later on derived in the context of the Lax-Friedrichs method in [14]. Therefore, the dissipation matrix is defined as

$$\mathbf{D}(\mathbf{u}_i, \mathbf{u}_j, \mathbf{n}) \;=\; \mathbf{P}^{-1}\left(\frac{\mathbf{u}_j + \mathbf{u}_i}{2}\right) \mathbf{R}\left(\frac{\mathbf{u}_j + \mathbf{u}_i}{2}, \mathbf{n}\right) |\mathbf{\Lambda}|(\mathbf{u}_i, \mathbf{u}_j, \mathbf{n}) \mathbf{R}^{-1}\left(\frac{\mathbf{u}_j + \mathbf{u}_i}{2}, \mathbf{n}\right).$$

Herein, $\mathbf{R}(\mathbf{u}, \mathbf{n})$ represents the matrix of the right eigenvectors of the corresponding preconditioned Jacobian

$$\mathbf{F}^P(\mathbf{u}, \mathbf{n}) = \mathbf{P}(\mathbf{u})\mathbf{F}(\mathbf{u}, \mathbf{n})$$

and $|\mathbf{\Lambda}|(\mathbf{u}_i, \mathbf{u}_j, \mathbf{n})$ denotes the diagonal matrix defined by

$$|\mathbf{\Lambda}|(\mathbf{u}_i, \mathbf{u}_j, \mathbf{n}) = \operatorname{diag}\left\{ \max_{\mathbf{u}\in\left\{\mathbf{u}_i, \mathbf{u}_j, \frac{\mathbf{u}_i+\mathbf{u}_j}{2}\right\}} |\lambda_1(\mathbf{u}, \mathbf{n})|, \ldots, \max_{\mathbf{u}\in\left\{\mathbf{u}_i, \mathbf{u}_j, \frac{\mathbf{u}_i+\mathbf{u}_j}{2}\right\}} |\lambda_4(\mathbf{u}, \mathbf{n})| \right\},$$

where $\lambda_i(\mathbf{u},\mathbf{n})$, $i=1,\dots,4$ are chosen to be the eigenvalues of the matrix $\mathbf{F}^P(\mathbf{u},\mathbf{n})$. The properties of the derived method strongly depend on the preconditioning matrix used. Obviously, arranging $\mathbf{P}(\mathbf{u},\mathbf{n})$ to be the identity yields the standard Lax-Friedrichs scheme [3]. In order to overcome the failure of the standard Lax-Friedrichs scheme with respect to the pressure distribution in the low Mach number regime it is quite natural to multiply the pressure by a factor associated with the Mach number. Therefore, we extract the pressure by consideration of the so-called entropy variables $\mathbf{w}=(p,v_1,v_2,s)^T$, whereby s denotes the entropy determined as $s=\ln\frac{p}{\rho^\gamma}$. Following Turkel [21] we introduce

$$\mathbf{P}^{-1}(\mathbf{u}) = \left(\mathbf{U}\mathbf{Q}^{-1}\mathbf{W}\right)(\mathbf{u}), \tag{9}$$

where $\mathbf{U}=\frac{\partial \mathbf{u}}{\partial \mathbf{w}}$, $\mathbf{W}=\frac{\partial \mathbf{w}}{\partial \mathbf{u}}$ and

$$\mathbf{Q}^{-1} = \begin{pmatrix} \beta^2 & 0 & 0 & 0 \\ -\frac{\alpha v_1}{\rho c^2} & 1 & 0 & 0 \\ -\frac{\alpha v_2}{\rho c^2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ with } \beta=\mathcal{O}_S(M),\ M\to 0 \tag{10}$$

and $\alpha\in\mathbb{C}$. To ensure that the matrix is always nonsingular we additionally require that $\beta\neq 0$ for all $M>0$. Simple but time-consuming calculations give

$$\begin{aligned}\mathbf{P}^{-1}(\mathbf{u}) \;=\; & \mathbf{I}+\frac{\gamma-1}{c^2}\left\{(\beta^2-1)\begin{pmatrix} \frac{|v|^2}{2} & -v_1 & -v_2 & 1 \\ \frac{|v|^2 v_1}{2} & -v_1^2 & -v_1v_2 & v_1 \\ \frac{|v|^2 v_2}{2} & -v_1v_2 & -v_2^2 & v_2 \\ \frac{|v|^2 H}{2} & -Hv_1 & -Hv_2 & H \end{pmatrix}\right. \\ & \left. -\alpha\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{|v|^2 v_1}{2} & -v_1^2 & -v_1v_2 & v_1 \\ \frac{|v|^2 v_2}{2} & -v_1v_2 & -v_2^2 & v_2 \\ |v|^2 & -v_1|v|^2 & -v_2|v|^2 & |v|^2 \end{pmatrix}\right\}\end{aligned}$$

and

$$\lambda_{1,2}(\mathbf{u},\mathbf{n}) = v_{\mathbf{n}} := \mathbf{v}\cdot\mathbf{n},$$

$$\lambda_{3,4}(\mathbf{u},\mathbf{n}) = \frac{1}{2}\left[(1-\alpha+\beta^2)v_{\mathbf{n}} \pm \sqrt{(1-\alpha+\beta^2)^2 v_{\mathbf{n}}^2 - 4v_{\mathbf{n}}^2\beta^2 + 4\beta^2 c^2}\right].$$

Note that the choice $\alpha=0$ yields the preconditioning matrix proposed by Guillard and Viozat [4].

Interpreting the cell average $\mathbf{u}_i(t)$ as a piecewise constant function on Ω_i and using a simple explicit time marching procedure leads to the first order scheme

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{|\Omega_i|}\sum_{e_{ij}\subset\partial\Omega_i} |e_{ij}| H(\mathbf{u}_i^n,\mathbf{u}_j^n,\mathbf{n}_{ij}) \tag{11}$$

with $\mathbf{u}_i^n=\mathbf{u}_i(t^n)$, $t^{n+1}=t^n+\Delta t$ and $\mathbf{n}_{ij}$ represents the unit outer normal vector on $e_{ij}\subset\partial\Omega_i$. It was recently proven in [13] that utilizing the preconditioned Lax-Friedrichs flux (8) for the choice $\beta=\mathcal{O}(M)$, $M\to 0$, and $\alpha=0$ within the finite volume method associated with (11) yields a pressure distribution satisfying the statement of theorem 1 in a discrete sense. Furthermore, a discrete divergence constraint corresponding to theorem 3 is shown for this scheme in [15] in the absence of compression and expansion over the boundary of the computational domain.

4 Stability Analysis

Due to the characteristic propagation speeds associated with the governing equations, i.e.

$$v_{\mathbf{n}} = \mathcal{O}(1) \;\text{ and }\; v_{\mathbf{n}} \pm c = \mathcal{O}\left(\frac{1}{M}\right), \;\; M \to 0,$$

we can directly conclude that the step size of an explicit time integration scheme decreases at least linearly as the Mach number tends to zero, i.e. we expect

$$\Delta t = \mathcal{O}\left(M\right), \;\; M \to 0$$

in order to fulfill the requirement that the numerical domain of dependence always covers the physical one. However, for the specific scheme used a different condition on the time step could be valid. To obtain a deeper insight into the behavior of the method we perform a von Neumann stability analysis.

Parallel flows show that without stability in the one-dimensional case, we cannot expect stability in higher dimensions. Therefore, the one-dimensional Euler equations are of interest also in our applications, but tremendously easier to analyze. Thus, for the sake of simplicity we restrict ourselves to the consideration of the spatially one-dimensional case. The relevance for multi-dimensional flow fields is obvious and after wards confirmed by numerical experiments.

It is well known that the standard Lax-Friedrichs scheme combined with the explicit Euler time integration is stable in the sense of von Neumann if the CFL-condition is satisfied. In order to perform a von Neumann stability analysis we have to linearize the governing equations as well as the preconditioning matrix. Thus, we consider the linearized one-dimensional Euler equations in the form

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x(\mathbf{u}) = \mathbf{0} \tag{12}$$

where $\mathbf{A} = \mathbf{A}(\bar{\mathbf{u}}) = \frac{\partial f}{\partial u}(\bar{\mathbf{u}})$ with $\bar{\mathbf{u}} = \mathbf{u}_i$ for an arbitrary but fixed vector of conserved variables corresponding to an inner control volume Ω_i. Quantities derived from $\bar{\mathbf{u}}$ will also be denoted with a bar. Note that in one spatial dimension, the conserved variables are ρ, m and ρE, while the entropy variables are p, v and s. Due to the linearization it is appropriate to define the preconditioner also in a global manner by using $\bar{\mathbf{u}}$ instead of the average value of $\mathbf{u}_i$ and $\mathbf{u}_j$ as suggested in (9). Consequently, the preconditioned Lax-Friedrichs scheme for the linearized Euler equations (12) reads

$$H(\mathbf{u}_i, \mathbf{u}_j, n) = \frac{1}{2}\left(\mathbf{A}(\bar{\mathbf{u}})(\mathbf{u}_j + \mathbf{u}_i)n - \mathbf{D}(\bar{\mathbf{u}}, \bar{\mathbf{u}}, n)(\mathbf{u}_j - \mathbf{u}_i)\right). \tag{13}$$

with

$$\mathbf{D}(\bar{\mathbf{u}}, \bar{\mathbf{u}}, n) = \mathbf{P}^{-1}\left(\bar{\mathbf{u}}\right)\mathbf{R}\left(\bar{\mathbf{u}}, n\right)|\mathbf{\Lambda}|(\bar{\mathbf{u}}, \bar{\mathbf{u}}, n)\mathbf{R}^{-1}\left(\bar{\mathbf{u}}, n\right). \tag{14}$$

The one-dimensional preconditioning matrix is obtained from 10 by omitting the third row and column. For the investigation of appropriate stability requirements for the preconditioned approach it is necessary to analyze the spectrum of the dissipation matrix $\mathbf{D}(\bar{\mathbf{u}}, \bar{\mathbf{u}}, n)$. At first, we focus on the case of a vanishing parameter α. Thereafter, we will extend the statement to the general case.

Lemma 4 *Let μ_i, $i = 1, 2, 3$ represent the eigenvalues of the linearized dissipation matrix* $\mathbf{D}$ *corresponding to* (14) *of the preconditioned Lax-Friedrichs scheme* (8) *for equation (12) in one spatial dimension with $\beta = \mathcal{O}_S(M)$, $M \to 0$, $\beta \neq 0$, and $\alpha = 0$. Then the appropriately renumbered eigenvalues of* $\mathbf{D}$ *have the properties*

$$\mu_1, \mu_2 = \mathcal{O}(1), \quad M \to 0$$

and

$$\mu_3 = \mathcal{O}_S\left(\frac{1}{M^2}\right), \quad M \to 0.$$

Proof:
Analogously to the two-dimensional case the eigenvalues of the one-dimensional preconditioned Jacobian $\mathbf{F}^P(\bar{\mathbf{u}}, n)$ read

$$\lambda_1 = \bar{v}_n \text{ and } \lambda_{2,3} = \frac{1}{2}\left[(1+\beta^2)\bar{v}_n \pm \sqrt{(1-\beta^2)^2\bar{v}^2 + (2\beta\bar{c})^2}\right].$$

In the case that M is sufficiently small one gets $\bar{c}^2 > \bar{v}^2$ by which it follows that

$$(1-\beta^2)^2\bar{v}^2 + 4\beta^2\bar{c}^2 > (1+\beta^2)^2\bar{v}^2. \tag{15}$$

This inequality yields $\lambda_2 > 0$ and $\lambda_3 < 0$, and consequently

$$|\lambda_2| - |\lambda_3| = \lambda_2 + \lambda_3 = (1+\beta^2)\bar{v}_n. \tag{16}$$

For the investigation of the spectrum of $\mathbf{D}$ it is advantageous to consider the matrix with respect to the entropy variables. Hence, we write the dissipation matrix in the form

$$\mathbf{D} = \mathbf{U}\mathbf{Q}^{-1}\mathbf{S}|\mathbf{\Lambda}|\mathbf{S}^{-1}\mathbf{W}$$

with $\mathbf{S} = \mathbf{W}\mathbf{R}$. Since $\mathbf{W} = \mathbf{U}^{-1}$ the matrices $\mathbf{D}$ and

$$\bar{\mathbf{D}} = \mathbf{Q}^{-1}\mathbf{S}|\mathbf{\Lambda}|\mathbf{S}^{-1} \tag{17}$$

are related by means of a similarity transformation which maintains the eigenvalues. Utilizing the abbreviations

$$\xi_1 = \lambda_2 - \lambda_1\beta^2, \quad \xi_2 = \lambda_3 - \lambda_1\beta^2, \quad \text{and} \quad \xi_3 = \frac{\lambda_3 - \lambda_2}{2}$$

in combination with equation (16) one can write (see Appendix A)

$$\bar{\mathbf{D}} = \begin{pmatrix} \frac{|\lambda_2|\xi_2 - |\lambda_3|\xi_1}{2\xi_3\beta^2} & -\frac{\bar{c}^2\rho}{2\xi_3}(1+\beta^2)\bar{v}_n & 0 \\ \frac{\xi_1\xi_2}{2\xi_3\beta^2\bar{c}^2\rho}(1+\beta^2)\bar{v}_n & -\frac{1}{2\xi_3}(\xi_1|\lambda_2| + \xi_2|\lambda_3|) & 0 \\ 0 & 0 & |\lambda_1| \end{pmatrix}$$

for sufficiently small M. By means of straightforward calculations we obtain the eigenvalues

$$\mu_1 = |\lambda_1| = |\bar{v}_n| = \mathcal{O}(1)$$

and

$$\mu_{2,3} = -\frac{1}{2\xi_3}\left((1+\beta^2)\bar{c}^2 \mp \sqrt{(1-\beta^2)^2(\bar{c}^4 - \bar{v}^2\bar{c}^2 + \bar{v}^4) + (2\beta\bar{v}\bar{c})^2}\right).$$

Due to the fact that the speed of sound $\widetilde{c}$ is always positive independent of the Mach number we get $\bar{c} = \mathcal{O}_S(M^{-1})$, $M \to 0$ and as a result

$$\begin{aligned} \xi_3 &= -\frac{1}{2}\sqrt{(1-\beta^2)^2\bar{v}^2 + (2\beta\bar{c})^2} = \underbrace{-\frac{1}{2}\sqrt{\bar{v}^2 + (2\beta\bar{c})^2}}_{=\mathcal{O}(1)} + \mathcal{O}(M^2) \\ &= \mathcal{O}(1), \quad M \to 0 \end{aligned} \tag{18}$$

and $\xi_3 > 0$ for all M. Similar to (15) we obtain

$$(1-\beta^2)^2(\bar{c}^4 - \bar{v}^2\bar{c}^2 + \bar{v}^4) + (2\beta\bar{v}\bar{c})^2 > (1-\beta^2)^2(\bar{c}^4 + \bar{v}^4) + (1+\beta^2)^2\bar{v}^2\bar{c}^2 \geq 0$$

for a sufficient small Mach number and hence the eigenvalues $\mu_{2,3}$ are real in the low Mach number regime.

Using (18) in combination with

$$(1+\beta^2)^2\bar{c}^2 = \mathcal{O}_S(M^{-2}), \quad M \to 0$$

and

$$\sqrt{(1-\beta^2)^2(\bar{c}^4 - \bar{v}^2\bar{c}^2 + \bar{v}^4) + (2\beta\bar{v}\bar{c})^2} = \mathcal{O}_S(M^{-4}), \quad M \to 0$$

directly yields

$$\begin{aligned} \mu_3 &= -\underbrace{\frac{1}{2\xi_3}}_{=\mathcal{O}_S(1)} \left(\underbrace{(1+\beta^2)\bar{c}^2}_{=\mathcal{O}_S(M^{-2})} + \underbrace{\sqrt{(1-\beta^2)^2(\bar{c}^4 - \bar{v}^2\bar{c}^2 + \bar{v}^4) + (2\beta\bar{v}\bar{c})^2}}_{=\mathcal{O}_S(M^{-4})} \right) \\ &= \mathcal{O}_S(M^{-2}), \quad M \to 0. \end{aligned}$$

Now we can deduce the asymptotic behavior of the remaining eigenvalue μ_2 via

$$\mu_2 = \frac{\mu_2\mu_3}{\mu_3} = \frac{4\bar{c}^4\beta^2 + \bar{v}^2\bar{c}^2 + \mathcal{O}(M^{-1})}{2\xi_3\mu_3} = \mathcal{O}_S(1), \quad M \to 0.$$

□

Let us now consider the preconditioned Lax-Friedrichs scheme (13) on an equidistant grid with fixed mesh size $\Delta x > 0$. Using the explicit Euler time integration and taking into account $n = \pm 1$ gives

$$\begin{aligned} \mathbf{u}_i^{m+1} &= \mathbf{u}_i^m - \frac{\Delta t}{\Delta x} \sum_{j \in \{i-1, i+1\}} H(\mathbf{u}_i^m, \mathbf{u}_j^m, n) \\ &= \mathbf{u}_i^m - \frac{\Delta t}{\Delta x} \left(H(\mathbf{u}_i^m, \mathbf{u}_{i-1}^m, -1) + H(\mathbf{u}_i^m, \mathbf{u}_{i+1}^m, 1) \right) \\ &= \mathbf{u}_i^m - \frac{\Delta t}{2\Delta x} \Big(\mathbf{A}(\bar{\mathbf{u}}) \left(\mathbf{u}_{i-1}^m - \mathbf{u}_{i+1}^m \right) \\ &\qquad + \mathbf{D}(\bar{\mathbf{u}}, \bar{\mathbf{u}}, 1) \left(\mathbf{u}_{i+1}^m - 2\mathbf{u}_i^m + \mathbf{u}_{i-1}^m \right) \Big). \end{aligned} \tag{19}$$

Starting from this formulation we are now able to prove the following stability condition in the case of a vanishing parameter α.

Theorem 5 *A necessary condition to ensure stability of the linearized preconditioned Lax-Friedrichs scheme with $\beta = \mathcal{O}_S(M)$, $M \to 0$, $\beta \neq 0$ and $\alpha = 0$ is*

$$\Delta t = \mathcal{O}(M^2), \quad M \to 0.$$

Proof:
The investigation of the L_2-stability, known as the von Neumann stability, is based on a Fourier analysis. Therefore, we replace $\mathbf{u}_i^m$ by the corresponding Fourier expansion

$$\mathbf{u}_i^m = \sum_{j=-\infty}^{\infty} \mathbf{h}_j^m e^{ij\Delta x I},$$

where I represents the imaginary unit. Introducing $\phi = j\Delta x$ we obtain the evolution of the j^{th} mode in the form

$$\begin{aligned} \mathbf{h}_j^{m+1} e^{i\phi I} &= \mathbf{h}_j^m e^{i\phi I} + \frac{\Delta t}{2\Delta x}\Bigg(\mathbf{A}(\bar{\mathbf{u}})\mathbf{h}_j^m \left(e^{(i-1)\phi I} - e^{(i+1)\phi I}\right) \\ &\qquad + \mathbf{D}(\bar{\mathbf{u}}, \bar{\mathbf{u}}, 1)\mathbf{h}_j^m \left(e^{(i+1)\phi I} - 2e^{i\phi I} + e^{(i-1)\phi I}\right)\Bigg). \end{aligned}$$

Division by $e^{i\phi I}$ yields

$$\mathbf{h}_j^{m+1} = \mathbf{H}(j, \Delta x, \Delta t)\mathbf{h}_j^m$$

with the amplification matrix

$$\begin{aligned} \mathbf{H}(j, \Delta x, \Delta t) &= \mathbf{I} + \frac{\Delta t}{2\Delta x}\left(\mathbf{A}(\bar{\mathbf{u}})\left(e^{-\phi I} - e^{\phi I}\right) + \mathbf{D}(\bar{\mathbf{u}}, \bar{\mathbf{u}}, 1)\left(e^{\phi I} - 2 + e^{-\phi I}\right)\right) \\ &= \mathbf{I} - \frac{\Delta t}{\Delta x}\mathbf{D}(\bar{\mathbf{u}}, \bar{\mathbf{u}}, 1) + \frac{\Delta t}{\Delta x}\mathbf{D}(\bar{\mathbf{u}}, \bar{\mathbf{u}}, 1)\cos\phi - I\frac{\Delta t}{\Delta x}\mathbf{A}(\bar{\mathbf{u}})\sin\phi. \end{aligned}$$

The scheme is stable in the sense of von Neumann if the spectral radius of the amplification matrix is less than one for all ϕ. The eigenvalues of the matrix $\mathbf{A}(\bar{\mathbf{u}})$ are known to be

$$\nu_1 = v = \mathcal{O}(1), \quad M \to 0$$

and

$$\nu_{2,3} = v \pm c = \mathcal{O}(M^{-1}), \quad M \to 0.$$

Consequently, for fixed $\Delta x > 0$ Lemma 4 yields

$$\rho(\mathbf{H}) = \max_{i=1,2,3}\left|1 - (1 - \cos\phi)\frac{\Delta t}{\Delta x}\mu_i - I\sin\phi\frac{\Delta t}{\Delta x}\nu_i\right| = \mathcal{O}\left(\frac{\Delta t}{M^2}\right), \quad M \to 0.$$

This property proves the requirement $\Delta t = \mathcal{O}(M^2)$, $M \to 0$, which completes the proof. □

Let us now generalize the statement of Lemma 4 to the case of an arbitrary parameter α. It is worth mentioning that the parameter α can of course depend on the physical quantities.

Lemma 6 *If $\beta = \mathcal{O}_S(M)$, $M \to 0$, $\beta \neq 0$, then there exists an eigenvalue μ of the of the linearized dissipation matrix* $\mathbf{D}$ *corresponding to* (14) *of the linearized preconditioned Lax-Friedrichs scheme* (8) *such that*

$$\mu = \mathcal{O}_S\left(\frac{1}{M^2}\right), \quad M \to 0$$

independent of α.

Proof:
As we can learn from the elaborated description of the eigenvalues μ_i, $i = 1, 2, 3$ of the dissipation matrix $\mathbf{D}$ in Appendix A the crucial eigenvalues can be written as

$$\mu_{2,3} = a \pm \frac{1}{2}\sqrt{b} \tag{20}$$

with

$$a = \underbrace{-\beta^2 c^2}_{\mathcal{O}(1)} - \underbrace{c^2}_{\mathcal{O}_S\left(\frac{1}{M^2}\right)} + \alpha \underbrace{\left(\frac{3}{2}v^2 - \frac{\beta^4 v^2}{2}\right)}_{\mathcal{O}(1)} + \alpha^2 \underbrace{\left(\frac{\beta^2 v^2}{2} - \frac{v^2}{2}\right)}_{\mathcal{O}(1)}. \tag{21}$$

In order to ensure that neither μ_2 nor μ_3 is any longer $\mathcal{O}\left(\frac{1}{M^2}\right)$, the parameter α has to be chosen such that the leading order terms cancel out. Due to the fact that the parameter α always appears in form of a product αv within the preconditioner (10) it is obvious that one cannot improve the stability properties in the case of a vanishing velocity field. This is also documented by equation (21) which shows that $a = \mathcal{O}_S\left(\frac{1}{M^2}\right)$, $M \to 0$ independent of α, if $v = 0$.

If $v \neq 0$ we deduce from (21) that we have to define

$$\alpha = \sqrt{-\frac{2c^2}{v^2}} \in \mathbb{C} \tag{22}$$

in order to remove the leading order term. However, introducing (22) into the second addend of (20) yields

$$b = 4c^4 + v^4\alpha^4 + o\left(\frac{1}{M^4}\right) = 8c^4 + o\left(\frac{1}{M^4}\right) = \mathcal{O}_S\left(\frac{1}{M^4}\right), \quad M \to 0.$$

This implies $\sqrt{b} = \mathcal{O}_S\left(\frac{1}{M^2}\right)$, $M \to 0$ and consequently there exists always an eigenvalue

$$\mu = \mathcal{O}_S\left(\frac{1}{M^2}\right), \quad M \to 0.$$

□

We are now able to employ the statement of the above mentioned lemma to prove the following proposition exactly in the way we perform the evidence of theorem 5.

Theorem 7 *A necessary condition to ensure stability of the linearized preconditioned Lax-Friedrichs scheme with $\beta = \mathcal{O}_S(M)$, $M \to 0$, $\beta \neq 0$ and arbitrary α is*

$$\Delta t = \mathcal{O}(M^2), \quad M \to 0.$$

This theoretical result is only valid for the linearized scheme and therefore only a necessary stability condition. Later on we will confirm the significance of the results of the one-dimensional linear theory by numerical tests in the context of the two-dimensional Euler equations. Furthermore, we'd like to point out that the proof works analogously for other schemes, for example the Roe-scheme [17] or the AUSMDV scheme [24] and thus these schemes have the same stability properties.

5 Numerical Experiments

We used a standard finite volume scheme with a first order spatial discretization. The specific scheme employed is described in [11]. Our first test problem is a NACA0012 profile at zero angle of attack with varying inflow Mach numbers. The time step then tends to zero quadratically with M in the case of a preconditioned Lax-Friedrichs flux. If we don't employ preconditioning, the time step still goes to zero, but only linearly in M. These experiments thus confirm the theorem for the nonlinear scheme. The results can be seen in the first tabular. The CFL number are accurate up to the leading digit: 0.1 thus means that a CFL number of 0.2 leads to instabilities.

	Preconditioned		Unpreconditioned	
M	CFL number	Δt	CFL number	Δt
0.1	0.1	7*10E-7	0.9	6*10E-5
0.01	0.01	10E-8	0.9	7*10E-6
0.001	0.001	10E-10	0.9	7*10E-7
0.0001	0.0001	10E-12	0.9	7*10E-8
0.00001	0.00001	10E-14	0.9	7*10E-9

The second test problem is a circular bump in a channel, again with varying inflow Mach numbers. The asymptotic behavior predicted by the theorem can be clearly seen. For the case $M = 0.1$, it can be seen that the preconditioned scheme is as stable as the unpreconditioned one. This is not a contradiction to the theoretical results, where the Mach number is assumed to be small enough. The difference in Δt compared to the first problem is due to a different grid with different cell sizes.

	Preconditioned		Unpreconditioned	
M	CFL number	Δt	CFL number	Δt
0.1	0.9	5*10E-4	0.9	5*10E-4
0.01	0.01	6*10E-7	0.9	5*10E-5
0.001	0.001	6*10E-9	0.9	5*10E-6
0.0001	0.0001	6*10E-11	0.9	5*10E-7
0.00001	0.00001	6*10E-13	0.9	5*10E-8

If we use the implicit Euler scheme instead of the explicit Euler for the time integration, we observe no bound on the CFL number. Not surprisingly, the von Neumann stability

analysis reveals no bound on the time step. Only for $M = 10^{-5}$, a restriction of the CFL number was observed.

6 Conclusions

We presented a theoretical proof that the preconditioned method, combined with an explicit scheme suffers from unphysical and much too restrictive stability conditions. This is due to the behavior of the eigenvalues of the preconditioned dissipation. The result was confirmed by numerical experiments. However, for an implicit method, there is no stability bound on the time step. Therefore, we strongly recommend using implicit methods in this context.

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A Appendix

In one spatial dimension, the transformation matrices between conserved and entropic variables are given by:

$$\mathbf{U} = \frac{\partial \mathbf{u}}{\partial \mathbf{w}} = \begin{pmatrix} \frac{1}{c^2} & 0 & -\frac{\rho}{\gamma} \\ \frac{v}{c^2} & \rho & -\frac{\rho v}{\gamma} \\ \frac{H}{c^2} & \rho v & -\frac{\rho |v|^2}{2\gamma} \end{pmatrix} \quad \text{and} \tag{23}$$

$$\mathbf{W} = \frac{\partial \mathbf{w}}{\partial \mathbf{u}} = \begin{pmatrix} \frac{1}{2}(\gamma-1)|v|^2 & -(\gamma-1)v & \gamma-1 \\ -\frac{v}{\rho} & \frac{1}{\rho} & 0 \\ \frac{(\gamma-1)|v|^2}{2p} - \frac{\gamma}{\rho} & -\frac{(\gamma-1)v}{p} & \frac{\gamma-1}{p} \end{pmatrix}. \tag{24}$$

The preconditioner is given by

$$\mathbf{Q}^{-1} = \begin{pmatrix} \beta^2 & 0 & 0 \\ -\frac{\alpha v}{\rho c^2} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The matrices of eigenvectors that diagonalize the preconditioned Jacobian in entropy formulation are:

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & \frac{\xi_1}{\rho\beta^2 c^2} & \frac{\xi_2}{\rho\beta^2 c^2} \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \mathbf{S}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{\xi_2}{2\xi_3} & -\frac{\rho\beta^2 c^2}{2\xi_3} & 0 \\ -\frac{\xi_1}{2\xi_3} & \frac{\rho\beta^2 c^2}{2\xi_3} & 0 \end{pmatrix}. \tag{25}$$

Thus, we can compute the preconditioned dissipation matrix (17)

$$\bar{\mathbf{D}} = \mathbf{Q}^{-1}\mathbf{S}|\mathbf{\Lambda}|\mathbf{S}^{-1} = (\bar{d}_{ij})_{i,j=1,2,3} \tag{26}$$

with $\bar{d}_{3,1} = \bar{d}_{3,2} = \bar{d}_{2,3} = \bar{d}_{1,3} = 0$, $\bar{d}_{3,3} = |\lambda_1|$,
$\bar{d}_{1,1} = \frac{|\lambda_2|\xi_2 - |\lambda_3|\xi_1}{2\beta^2\xi_3}$, $\bar{d}_{1,2} = \frac{c^2\rho}{2\xi_3}(|\lambda_3| - |\lambda_2|)$,
$\bar{d}_{2,1} = \frac{\alpha v}{\rho c^2 2\xi_3}(|\lambda_2|\xi_2 - |\lambda_3|\xi_1) + \frac{\xi_1\xi_2}{2\beta^2c^2\rho\xi_3}(|\lambda_2| - |\lambda_3|)$ and
$\bar{d}_{2,2} = \frac{\beta^2\alpha v}{2\xi_3}(|\lambda_3| - |\lambda_2|) + \frac{|\lambda_3|\xi_2 - |\lambda_2|\xi_1}{2\xi_3}$.
This matrix has the eigenvalues $\mu_1 = v_n$ and $\mu_{2,3} = \frac{a \pm \sqrt{b}}{4\xi_3}$ with

$$a = -\beta^4v^2\alpha + \beta^2v^2\alpha^2 - \frac{c^2}{2} + 3\alpha v^2 - v^2\alpha^2 - \beta^2c^2 \text{ and}$$

$$\begin{aligned} b \;=\; & \beta^8v^4\alpha^2 - 6\beta^4v^4\alpha^2 + 4v^4 - 4\beta^4c^2v^2 + 4\beta^4c^2v^2\alpha + 4\beta^4c^4 - 4\beta^2c^2\alpha v^2 - 8\beta^2c^4 \\ & +4c^4 - 8v^4\alpha + 13v^4\alpha^2 + 4v^4\beta^4 - 6\alpha^3v^4 + v^4\alpha^4 - 4c^2\alpha v^2 - 8v^4\beta^2\alpha + 6\alpha^3v^4\beta^2 \\ & -2v^4\alpha^4\beta^2 + 2v^4\alpha^3\beta^4 + 24v^2\beta^2c^2 - 8\beta^2v^4 - 4v^2c^2 + 4\beta^6v^2\alpha c^2 - 4\beta^4v^2\alpha^2c^2 \\ & -2\beta^6v^4\alpha^3 + \beta^4v^4\alpha^4. \end{aligned}$$

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