AN ALGEBRAIC PROOF OF IITAKA'S CONJECTURE $C_{2,1}$

by

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Abstract

We give a proof of Iitaka's conjecture $C_{2,1}$ using only elementary methods from algebraic geometry.

1 Introduction

We consider the following situation. Let X be a smooth projective surface and let B be a smooth projective curve, defined over an algebraically closed field k of characteristic zero. Let $f : X \to B$ be a surjective morphism and let $\omega_{X/B}$ denote the relatively canonical sheaf of differentials. Let us assume that the generic fibre is smooth of genus g and let us denote by δ the number of singular points in the fibres. We write Λ_n for the determinant of $f_*\omega^n_{X/B}$ and λ_n for the degree of Λ_n. Finally, let us assume that f is a relatively minimal model, which means that there are no exceptional curves among the fibres.

In this situation, Iitaka's conjecture $C_{2,1}$ is well-known:

Theorem 1.1 *If, in the situation above, X_b denotes a general fibre, we have the subadditivity*

$$\kappa(X) \geq \kappa(B) + \kappa(X_b)$$

of the Kodaira dimensions.

We immediately notice that, in order to give a proof for this, we may assume both B and X_b to have genus greater than zero.

The result 1.1 contributes to the Enriques-Kodaira classification of surfaces, as it is presented in [1], VI., for example. A proof for 1.1 is given in [1], III. 18.4, and as it is worked out there, Iitaka's conjecture $C_{2,1}$ basically follows from:

Theorem 1.2 *Keeping the assumptions made above and assuming in addition that f is non-isotrivial, we have $\lambda_1 > 0$.*

There have been several kinds of proof for 1.2, but most of them used analytic methods. In [1], III.17, for example, a proof is given by considering the period map. In case this map is not constant, 1.2 follows from constructing a section of Λ_n, which locally arises from modular forms. For the constant case, the image of the period map is exactly the period point of all the smooth fibres. After excluding the existence of singular fibres, 1.2 then follows from Torelli's theorem. Other proofs used analytic methods in order to achieve the weak positivity of $f_*\omega_{X/B}$ (see 2.2) like Fujita ([3]), Kawamata ([5]) or Viehweg ([8]). In case algebraic methods were used, those were developed in a more general set up in order to apply for higher dimensions, too.

But in the special case of a familiy of curves over a curve, one can give both an elementary and a purely algebraic proof of 1.2 based on positivity methods. The aim of this paper is to present this proof, which exclusively uses methods from algebraic geometry and hence yields an algebraic proof for 1.1.

2 Positivity

First we want to recall the following positivity notations and give some properties, which we shall need below. We refer to [2], 5., for example. A curve is always assumed to be irreducible.

Definition and Lemma 2.1 Let $\mathcal{L}$ be an invertible sheaf on a smooth projective variety Y. We call $\mathcal{L}$ *big* if its Kodaira dimension is maximal or, equivalently, some power of $\mathcal{L}$ contains an ample subsheaf. We call $\mathcal{L}$ *nef (numerically free)* if for every curve C on X one has $\deg \mathcal{L}_{|C} \geq 0$ or, equivalently, if for every $\nu \geq 0$ and for every ample invertible sheaf $\mathcal{H}$ one has that $\mathcal{L}^{\nu} \otimes \mathcal{H}$ is ample. (Sometimes the notion *numerically effective* is used as well.) This property implies that the self intersection number of $\mathcal{L}$ is not negative and that it is positive if and only if $\mathcal{L}$ is big. Moreover, a locally free sheaf $\mathcal{G}$ on Y is called *weakly positive* if every quotient sheaf of $\mathcal{G}$ has non-negative degree, even after a finite covering.

The following theorem is well known:

Theorem 2.2 *The direct image sheaf $f_*\omega_{X/B}$ is weakly positive.*

There have been proofs by Fujita ([3]) and Kawamata ([5]), using analytic methods, and there have been several generalizations. By means of Kollár's vanishing theorem one obtains a proof ([6], see also [10], Theorem 2.41), which is based on methods from algebraic geometry.

Corollary 2.3 *The relatively canonical sheaf $\omega_{X/B}$ is nef and hence for the self intersection number we have $c_1(\omega_{X/B})^2 \geq 0$.*

Proof: Let C be an irreducible curve on X. Since f is assumed to be a relatively minimal model, we have $c_1(\omega_{X/B}).C \geq 0$ in case C is a fibre. If not, then the natural map $f^*f_*\omega_{X/B} \to \omega_{X/B}$ restricted to C is surjective, and

$$\deg \omega_{X/B\,|C} = c_1(\omega_{X/B}).C \geq 0,$$

since $f_*\omega_{X/B}$ is weakly positive. Thus $\omega_{X/B}$ is nef and by 2.1 we obtain that $c_1(\omega_{X/B})^2 \geq 0$. □

We should remark here that, in case $g > 1$, one even obtains strict positivity for $c_1(\omega_{X/B})^2$. We will, however, not need this in the sequel.

3 Reductions

In order to show 1.2 we may, without loss of generality, make some reductions, which we shall deduce in the following.

First of all, we may assume f to be a semi-stable model, which means that all the fibres are reduced normal crossing divisors. For if this is not the case, then for some finite covering

$\tau : B' \to B$, for the fibre product $X' = X \times_B B'$ with projections $f' : X' \to B'$ and $\tau' : X' \to X$ and for a desingularization $d : X'' \to X'$ the induced map $f'' : X'' \to B'$ will be semi-stable. Considering the trace map $d_*\omega_{X''} \to \omega_{X'}$, which induces an injective map $d_*\omega_{X''/B'} \to \omega_{X'/B'} = {\tau'}^* \omega_{X/B}$, and applying flat base change we obtain an injective map $f''_*\omega_{X''/B'} = f'_*d_*\omega_{X''/B'} \to f'_*{\tau'}^* \omega_{X/B} = \tau^* f_*\omega_{X/B}$. Comparing degrees, we notice that if 1.2 holds for the semi-stable model f'', it will hold for f as well.

Our next reduction is based on the following lemma which is due to Mumford ([7], 5.10) and follows from an easy calculation of the Riemann Roch formulae on X and on B.

Lemma 3.1 *For every $n \in \mathbb{N}$ we have*

$$\lambda_n = \binom{n}{2} \cdot \Big(12\lambda_1 - \delta\Big) + \lambda_1 = \binom{n}{2} \cdot c_1(\,\omega_{X/B}\,)^2 + \lambda_1$$

in case $g > 1$ and $12\lambda_n = n\delta = 12n\lambda_1$ in case $g = 1$.

In case $g > 1$ the above formula tells us that $12\lambda_1 \geq \delta$ (since by 2.3 we have $c_1(\,\omega_{X/B}\,)^2 \geq 0$) and, moreover, $\lambda_1 = 0$ implies $\lambda_n \leq 0$ for all $n \in \mathbb{N}$. In case $g = 1$ we have $12\lambda_1 = \delta$ and $\lambda_n = n \cdot \lambda_1$. Hence in both cases, in order to show $\lambda_1 > 0$, we may assume from now on that all the fibres are smooth and, moreover, it suffices to show $\lambda_n > 0$ for some $n \in \mathbb{N}$.

More reductions can be made considering the genus g of the fibres. We can immediately exclude the case $g = 1$, for if all the fibres are smooth elliptic curves, then associating to each b the j-invariant of the corresponding fibre gives a morphism $j : B \to \mathbb{A}^1$, which has to be constant and thereby forces f to be isotrivial.

Moreover, we may assume that all the fibres are smooth non-hyperelliptic curves of genus $g \geq 2$, as we shall prove now. To this end, we have to distinguish two cases. First, let us assume that all the fibres are smooth and hyperelliptic curves. Then f again turns out to be isotrivial. To see this, let us consider the projective bundle $\pi : \mathbb{P}(\,f_*\omega_{X/B}\,) \to B$ associated to $f_*\omega_{X/B}$ and let us denote by $\mathbb{P}$ the image of the morphism $\varphi : X \to \mathbb{P}(\,f_*\omega_{X/B}\,)$, which corresponds to the surjective map $f^*\, f_*\omega_{X/B} \to \omega_{X/B}$. Since all the fibres are assumed to be hyperelliptic and thus φ is given fibrewise by double coverings of $\mathbb{P}^1$, $\mathbb{P}$ turns out to be a ruled suface. Denoting by Δ the discriminant, the branched covering trick ([1], I.18.2) implies that there exists an étale covering $\gamma : B' \to B$ such that the pull back of Δ to $\mathbb{P}'$ has $2g + 2$ disjoint components, where $\mathbb{P}'$ denotes the fibre product. These components correspond to $2g + 2$ disjoint sections of the bundle $\pi' : \mathbb{P}' \to B'$, forcing it to be trivial, since $g \geq 1$. Hence the corresponding morphism $f' : X' \to B'$ is isotrivial, and by property of the fibre product, so is f.

For the second case we need the following

Lemma 3.2 *For $n \in \mathbb{N}$ sufficiently large, the multiplication map*

$$\mu_n : S^n(\,f_*\omega_{X/B}\,) \longrightarrow f_*\omega^n_{X/B}$$

is surjective outside hyperelliptic fibres.

Proof: Let X_b be a non-hyperelliptic fibre of f, thus ω_{X_b} is very ample and we consider the embedding $X_b \to \mathbb{P}^{g-1}_k$ given by the global sections of ω_{X_b}. We may identify those sections with $H^0(\mathbb{P}^{g-1}_k, \mathcal{O}_{\mathbb{P}^{g-1}_k}(1))$ which implies an isomorphism

$$S^n(H^0(X_b, \omega_{X_b})) \to H^0(\mathbb{P}^{g-1}_k, \mathcal{O}_{\mathbb{P}^{g-1}_k}(n))$$

for every $n \in \mathbb{N}$. Considering the long exact cohomology sequence corresponding to the ideal sheaf of X_b and twisting by n, Serre's Vanishing Theorem implies a surjective map

$$H^0(\mathbb{P}_k^{g-1}, \mathcal{O}_{\mathbb{P}_k^{g-1}}(n)) \to H^0(X_b, \omega_{X_b}^n),$$

if we choose n to be sufficiently large. Thus we obtain a surjective map

$$S^n(H^0(X_b, \omega_{X_b})) \to H^0(X_b, \omega_{X_b}^n),$$

and by base change we are done. □

Now in case some of the fibres are hyperelliptic, we consider the factorization of μ_n over its image sheaf. Since symmetric products of weakly positive sheaves are again weakly positive (see for example [10], 2.20), $S^n(f_*\omega_{X/B})$ is weakly positive by 2.2, and we obtain by definition of weak positivity that the image sheaf has non-negative degree, hence $\lambda_n > 0$ and by 3.1 we are done.

Putting all this together, it remains, in order to prove 1.2, to show the following

Theorem 3.3 *Keeping the assumptions from 1.2 and assuming in addition that f is semi-stable and all the fibres are smooth and non-hyperelliptic curves, we have $\lambda_n > 0$ for some $n \in \mathbb{N}$.*

4 The proof of 3.3

In this section we prove 3.3. Since we know that $f_*\omega_{X/B}$ is weakly positive, the idea is to consider a certain tensor sheaf of $f_*\omega_{X/B}$ and find a quotient, which is a proper subsheaf of some power of Λ_n. This is achieved by the method of the universal basis as it was used by Viehweg in [9]. We shall briefly recall this method, referring to the starting point of the proof of [10], 4.33 and 4.34, respectively.

Let $\mathcal{E}$ be a locally free sheaf on B of rank m and let $\pi : \mathbb{P} \to B$ denote the projective bundle associated to $\bigoplus^m \mathcal{E}^\vee$. From the construction of the projective bundle (see for example [4], II.7.12) we obtain a surjective map

$$\pi^* \bigoplus^m \mathcal{E}^\vee \longrightarrow \mathcal{O}_{\mathbb{P}}(1)$$

and by dualizing we obtain an injective map

$$\mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \pi^* \bigoplus^m \mathcal{E},$$

sending a local section l of $\mathcal{O}_{\mathbb{P}}(-1)$ to an m-tuplet $(s_1, \ldots, s_m)$. This induces a map

$$s : \bigoplus^m \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \pi^*\mathcal{E},$$

injective, too, sending $(f_1 l, \ldots, f_m l)$ to the sum of the $f_i s_i$. We call s the universal basis associated to $\mathcal{E}$ and we observe that, fibrewise, the locus where s is an isomorphism is isomorphic to $\mathbb{P}GL(m, k)$. We call s the universal basis associated to $\mathcal{E}$.

We apply this method to $\mathcal{E} = f_*\omega_{X/B}$. Taking for $n \in \mathbb{N}$ the symmetric product of the corresponding universal basis s and composing with the multiplication map, we obtain a map

$$S^n\Big(\bigoplus^m \mathcal{O}_{\mathbb{P}}(-1)\Big) \to S^n\Big(\pi^* f_*\omega_{X/B}\Big) = \pi^* S^n(f_*\omega_{X/B}) \to \pi^* f_*\omega^n_{X/B},$$

which by 3.2 is surjective outside the zero divisor D of $\det s$, provided we have chosen n to be sufficiently large. We denote the image sheaf of this morphism by $\mathcal{B}$.

Next we consider a blowing up $\tau : \mathbb{P}' \to \mathbb{P}$ with center in D such that $\mathcal{B}' = \tau^*\mathcal{B}$ modulo torsion is locally free. For $\nu \in \mathbb{Z}$ let us write $\mathcal{O}_{\mathbb{P}'}(\nu)$ for the sheaf $\tau^*\mathcal{O}_{\mathbb{P}}(\nu)$ and let $\pi' : \mathbb{P}' \to B$ denote the composed map. We obtain a surjection

$$S^n\Big(\bigoplus^m \mathcal{O}_{\mathbb{P}'}(-1)\Big) \longrightarrow \mathcal{B}'.$$

Let us now consider the Grassmannian manifold G parametrizing the r-dimensional quotients of the linear space $S^n(k^m)$. For $r = \operatorname{rk} f_*\omega^n_{X/B}$, let us moreover denote by V the linear space $\bigwedge^r S^n(k^m)$. By [4], II.7.12, the induced quotient map

$$V \otimes \mathcal{O}_{\mathbb{P}'} = \bigwedge^r S^n\Big(\bigoplus^m \mathcal{O}_{\mathbb{P}'}\Big) \to \bigwedge^r \Big(\mathcal{B}' \otimes \mathcal{O}_{\mathbb{P}'}(n)\Big) = \det \mathcal{B}' \otimes \mathcal{O}_{\mathbb{P}'}(nr)$$

corresponds to a morphism

$$\Phi' : \mathbb{P}' \longrightarrow G \xrightarrow{\subset} \mathbb{P}(V).$$

Whereas everything described above holds for a more general situation, too (see [10], 4.), the proof of the following lemma essentially uses the fact that we are in the case of a family of curves over a curve.

Lemma 4.1 *The morphism Φ' constructed above is generically finite.*

PROOF: We start with the following observation. A point $p \in \mathbb{P}' - \tau^*D$ corresponds to a fibre X_b of the family f together with a basis of $H^0(X_b, \omega_{X_b})$, up to multiplication with a constant. In particular it gives an isomorphism between $\mathbb{P}^{g-1}_k$ and $\mathbb{P}(H^0(X_b, \omega_{X_b}))$ and, since X_b is assumed to be smooth and non-hyperelliptic, this induces an embedding of X_b in $\mathbb{P}^{g-1}_k$.

The point $\Phi'(p) \in G$ is then given by the quotient $\mu_n : S^n(H^0(X_b, \omega_{X_b})) \to H^0(X_b, \omega^n_{X_b})$, where μ_n is the multiplication map on the fibre. So for n sufficiently large the kernel of μ_n determines both X_b and its embedding in projective space.

Now the curve X_b has a finite automorphism group, hence the subgroup of $\mathbb{P}GL(m, k)$ which leaves $X_b \subset \mathbb{P}^{g-1}_k$ invariant has to be finite. So if C is a curve in $\mathbb{P}' - \tau^*D$, mapping to a point in G, we obtain that C can not be contained in a fibre of $\pi' : \mathbb{P}' \to B$. Hence C has to map surjectively to B. But on the other hand, as we have seen, for every point in C the kernel of the associated multiplication map is the same. So in this case $X \times_B C$ turns out to be trivial which contradicts the non-isotriviality of the family f. Hence we obtain that there are no curves contained in the fibres of $\Phi'|_{\mathbb{P}' - \tau^*D}$. □

As an immediate consequence, $\Phi'^*\mathcal{O}_{\mathbb{P}(V)}(1) = \det \mathcal{B}' \otimes \mathcal{O}_{\mathbb{P}'}(nr)$ is ample, and hence the inclusion

$$\Phi'^*\mathcal{O}_{\mathbb{P}(V)}(1) = \det \mathcal{B}' \otimes \mathcal{O}_{\mathbb{P}'}(nr) \to \mathcal{L} := \pi'^*\Lambda_n \otimes \mathcal{O}_{\mathbb{P}'}(nr)$$

implies that $\mathcal{L}$ is big on $\mathbb{P}'$.

For a general fibre F of π' and for every $\nu \in \mathbb{N}$ we have the long exact cohomology sequence

$$0 \to H^0(\mathbb{P}', \mathcal{L}^\nu(-F)) \to H^0(\mathbb{P}', \mathcal{L}^\nu) \to H^0(F, \mathcal{L}^\nu_{|F}) \to \dots$$

Since $\mathcal{L}$ is big, for ν sufficiently large $\mathcal{L}^\nu(-F)$ is going to have a non-trivial section

$$\mathcal{O}_{\mathbb{P}'} \longrightarrow \mathcal{L}^\nu(-F).$$

For a point P in general position the projection formula implies

$$\mathcal{O}_B \longrightarrow \Lambda_n^\nu(-P) \otimes \pi'_* \mathcal{O}_{\mathbb{P}'}(\nu n r) = \Lambda_n^\nu(-P) \otimes S^{\nu n r} \bigoplus^m f_* \omega_{X/B}{}^\vee,$$

where the last equality holds by definition of the projective bundle. Dualizing again, we obtain a non-trivial map

$$S^{\nu n r} \bigoplus^m f_* \omega_{X/B} \longrightarrow \Lambda_n^\nu(-P).$$

Since $f_* \omega_{X/B}$ is weakly positive by 2.2, we obtain by definition $\deg \Lambda_n^\nu(-P) \geq 0$, hence $\lambda_n > 0$, which completes the proof of 3.3.

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