Solution properties of the de Branges differential recurrence equation

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Dedicated to our teacher and colleague Heinrich Begehr on the occasion of his 65th birthday

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Abstract

In his 1984 proof of the Bieberbach and Milin conjectures de Branges used a positivity result of special functions which follows from an identity about Jacobi polynomial sums that was published by Askey and Gasper in 1976.

The de Branges functions $\tau_k^n(t)$ are defined as the solutions of a system of differential recurrence equations with suitably given initial values. The essential fact used in the proof of the Bieberbach and Milin conjectures is the statement $\dot{\tau}_k^n(t) \leqq 0$.

In 1991 Weinstein presented another proof of the Bieberbach and Milin conjectures, also using a special function system $\Lambda_k^n(t)$ which (by Todorov and Wilf) was realized to be directly connected with de Branges', $\dot{\tau}_k^n(t) = -k\Lambda_k^n(t)$, and the positivity results in both proofs $\dot{\tau}_k^n(t) \leqq 0$ are essentially the same.

In this paper we study differential recurrence equations equivalent to de Branges' original ones and show that many solutions of these differential recurrence equations don't change sign so that the above inequality is not as surprising as expected.

Furthermore, we present a multiparameterized hypergeometric family of solutions of the de Branges differential recurrence equations showing that solutions are not rare at all.

1 Introduction

Let S denote the family of analytic and univalent functions $f(z) = z + a_2 z^2 + \ldots$ of the unit disk $\mathbb{D}$. S is compact with respect to the topology of locally uniform convergence so that $k_n := \max\limits_{f \in S} |a_n(f)|$

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exists. In 1916 Bieberbach [4] proved that $k_2 = 2$, with equality if and only if f is a rotation of the *Koebe function*

$$K(z) := \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=1}^{\infty} nz^n \ , \tag{1}$$

and in a footnote he mentioned "Vielleicht ist überhaupt $k_n = n$.". This statement is known as the *Bieberbach conjecture.*

In 1923 Löwner [13] proved the Bieberbach conjecture for $n = 3$. His method was to embed a univalent function $f(z)$ into a *Löwner chain*, i.e. a family $\{f(z,t) \mid t \geqq 0\}$ of univalent functions of the form

$$f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n, \quad (z \in \mathbb{D}, t \geqq 0, a_n(t) \in \mathbb{C}\ (n \geqq 2))$$

which start with f

$$f(z,0) = f(z) \ ,$$

and for which the relation

$$\operatorname{Re} p(z,t) = \operatorname{Re}\left(\frac{\dot{f}(z,t)}{zf'(z,t)}\right) > 0 \qquad (z \in \mathbb{D}) \tag{2}$$

is satisfied. Here $'$ and $\dot{}$ denote the partial derivatives with respect to z and t, respectively. Equation (2) is referred to as the *Löwner differential equation*, and geometrically it states that the image domains of f_t expand as t increases.

The history of the Bieberbach conjecture showed that it was easier to obtain results about the *logarithmic coefficients* of a univalent function f, i.e. the coefficients d_n of the expansion

$$\varphi(z) = \ln \frac{f(z)}{z} =: \sum_{n=1}^{\infty} d_n z^n$$

rather than for the coefficients a_n of f itself. So Lebedev and Milin [12] in the mid sixties developed methods to exponentiate such information. They proved that if for $f \in S$ the *Milin conjecture*

$$\sum_{k=1}^{n} (n+1-k)\left(k|d_k|^2 - \frac{4}{k}\right) \leqq 0$$

on its logarithmic coefficients is satisfied for some $n \in \mathbb{N}$, then the Bieberbach conjecture for the index $n+1$ follows.

In 1984 de Branges [5] verified the Milin, and therefore the Bieberbach conjecture and in 1991, Weinstein [16] gave a different proof. Both proofs use the positivity of special function systems, and independently Todorov [15] and Wilf [17] showed that (the t-derivatives of the) de Branges functions and Weinstein's functions essentially are the same (see also [8]),

$$\dot{\tau}_k^n(t) = -k\Lambda_k^n(t) \ ,$$

$\tau_k^n(t)$ denoting the de Branges functions and $\Lambda_k^n(t)$ denoting the Weinstein functions, respectively. Whereas de Branges applied an identity of Askey and Gasper [2] to his function system, Weinstein applied an addition theorem for Legendre polynomials [11] to his function system to deduce the positivity result needed.

2 The de Branges and Weinstein functions

In [5] de Branges showed that the Milin conjecture is valid if for all $n \geq 1$ the *de Branges functions* $\tau_k^n : \mathbb{R}_{\geqq 0} \to \mathbb{R}$ $(k = 1, \ldots, n)$ defined by the system of differential recurrence equations

$$\tau_{k+1}^n(t) - \tau_k^n(t) \;=\; \frac{\dot{\tau}_k^n(t)}{k} + \frac{\dot{\tau}_{k+1}^n(t)}{k+1} \qquad (k = 1, \ldots, n) \tag{3}$$

$$\tau_{n+1}^n \;\equiv\; 0 \tag{4}$$

with the initial values

$$\tau_k^n(0) = n + 1 - k \tag{5}$$

have the properties

$$\lim_{t \to \infty} \tau_k^n(t) = 0 \;, \tag{6}$$

and

$$\dot{\tau}_k^n(t) \leq 0 \qquad (t \in \mathbb{R}_{\geqq 0}) \;. \tag{7}$$

The relation (6) is easily checked using standard methods for ordinary differential equations, whereas (7) is a deep result.

L. de Branges gave the explicit representation

$$\tau_k^n(t) = e^{-kt} \binom{n+k+1}{2k+1} {}_4F_3\left(\left. \begin{array}{c} k+1/2, n+k+2, k, k-n \\ k+1, 2k+1, k+3/2 \end{array} \right| e^{-t} \right) \tag{8}$$

([5], [6], [14]), with which the proof of the de Branges theorem was completed as soon as de Branges realized that (7) was a theorem previously proved by Askey and Gasper [2].

Note that the function

$${}_pF_q\left(\left. \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right| x \right) := \sum_{k=0}^{\infty} A_k \, x^k = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the *Pochhammer symbol*, is called the *generalized hypergeometric series*. Its coefficient term ratio

$$\frac{A_{k+1} \, x^{k+1}}{A_k \, x^k} = \frac{(k+a_1)\cdots(k+a_p)}{(k+b_1)\cdots(k+b_q)} \frac{x}{(k+1)}$$

is a general rational function, in factorized form. More informations about generalized hypergeometric functions can be found in [3] or [7].

In [16] Weinstein used the Löwner chain

$$w(z,t) := K^{-1}\Big(e^{-t}K(z)\Big) \quad (z \in \mathbb{D}, t \geq 0)$$

of bounded univalent functions in the unit disk $\mathbb{D}$ which is defined in terms of the Koebe function (1), and showed the validity of Milin's conjecture if for all $n \geq 1$ the *Weinstein functions* $\Lambda_k^n : \mathbb{R}_{\geqq 0} \to \mathbb{R}$ $(k = 1, \ldots, n)$ defined by

$$\frac{e^t w(z,t)^{k+1}}{1 - w^2(z,t)} =: \sum_{n=k}^{\infty} \Lambda_k^n(t) z^{n+1} \tag{9}$$

satisfy the relations

$$\Lambda_k^n(t) \geq 0 \qquad (t \in \mathbb{R}_{\geqq 0}, \quad k, n \in \mathbb{N}) \ . \tag{10}$$

Weinstein did not identify the functions $\Lambda_k^n(t)$, but by applying the addition theorem for Legendre polynomials [11] to his function system he deduced (10) without an explicit representation.

Independently, both Todorov [15] and Wilf [17] proved—using the explicit representation (8) of the de Branges functions—that

$$\dot{\tau}_k^n(t) = -k\Lambda_k^n(t) \ , \tag{11}$$

i.e. the (t-derivatives of the) de Branges functions and the Weinstein functions essentially are the same, and the main inequalities (7) and (10) are identical. In [8] another proof of (11) was given that does not use the explicit representation of the de Branges functions. Note furthermore that in [9] we deduced the result (10) using a version of the addition theorem for the Gegenbauer polynomials whose simple proof is contained in the same article.

In this article we study differential recurrence equations equivalent to de Branges' original ones and show that many solutions of these differential recurrence equations don't change sign so that the inequalities (7) and (10), as well as $\tau_k^n \geqq 0$ are not as surprising as expected.

Furthermore, we present a multiparameterized hypergeometric family of solutions of the de Branges differential recurrence equations showing that solutions are not rare at all.

3 Nonnegative solutions of the de Branges differential recurrence equation

In this section, we will deal with the following functions

$$\delta_{kn}(y) = \delta_{kn}(e^{-t}) := \frac{1}{k}\, e^{kt}\, \tau_k^n(t) \tag{12}$$

and

$$\widetilde{\delta}_{kn}(y) = \widetilde{\delta}_{kn}(e^{-t}) := e^{kt}\, \Lambda_k^n(t) \tag{13}$$

instead of $\tau_k^n(t)$ and $\Lambda_k^n(t)$, where we use the variable $y = e^{-t} \in [0,1]$ instead of $t \geqq 0$. Note that our interest to consider $\delta_{kn}(y)$ comes from the fact that this function is decreasing in $[0,1]$ (see [10], Theorem 7(a)).

As boundary values we have

$$\delta_{kn}(0) = \frac{1}{k}\binom{n+k+1}{2k+1} \quad \text{and} \quad \delta_{kn}(1) = \frac{1}{k}(n-k+1)$$

and

$$\widetilde{\delta}_{kn}(0) = \binom{n+k+1}{2k+1} \quad \text{and} \quad \widetilde{\delta}_{kn}(1) = \begin{cases} 1 & \text{if } n-k \text{ is even} \\ 0 & \text{otherwise} \end{cases},$$

respectively. Of course the relation $\tau_k^n(t) \geqq 0$ $(t \geqq 0)$ is equivalent to $\delta_{kn}(y) \geqq 0$ $(y \in [0,1])$ and the relation $\Lambda_k^n(t) \geqq 0$ $(t \geqq 0)$ is equivalent to $\widetilde{\delta}_{kn}(y) \geqq 0$ $(y \in [0,1])$.

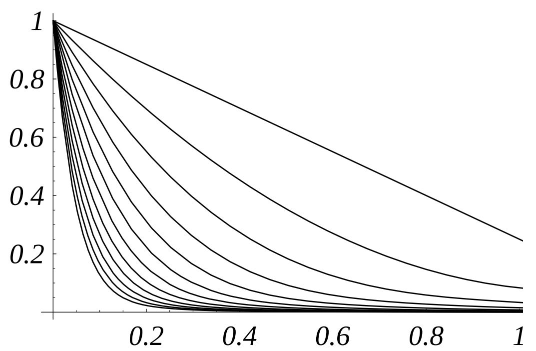

Figure 1: The decreasing functions $\frac{\delta_{kn}(y)}{\delta_{kn}(0)}$ for $k=3$ and $n=4,\ldots,15$

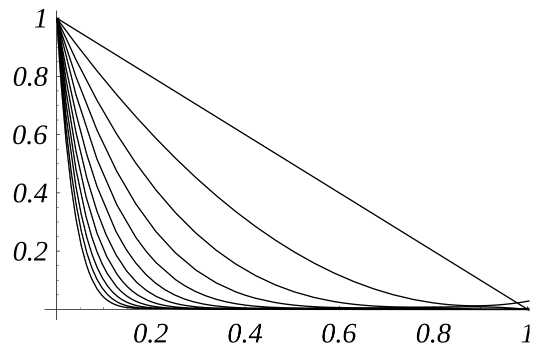

Figure 2: The functions $\frac{\widetilde{\delta}_{kn}(y)}{\widetilde{\delta}_{kn}(0)}$ for $k=3$ and $n=4,\ldots,15$

Note that although at first sight (Figure 2) it seems that $\widetilde{\delta}_{kn}(y)$ are also decreasing functions, this is not the case in a left neighborhood of $y=1$. Using mathematical induction and (14) one can show that

$$\widetilde{\delta}'_{n-2q,n}(1) = 2q(n-q+1) > 0 \quad \text{and} \quad \widetilde{\delta}'_{n-2q+1,n}(1) = -2q(n-q+1) < 0 \quad \text{for } q = 0,1,\ldots\ ,$$

see Figure 4:

The de Branges differential recurrence equation (3) can now be restated in terms of the new functions and give the differential recurrence equations

$$-\delta'_{k-1,n}(y) = 2k\,\delta_{kn}(y) + y\,\delta'_{kn}(y) \tag{14}$$

and

$$-\widetilde{\delta}'_{k-1,n}(y) = 2k\,\widetilde{\delta}_{kn}(y) + y\,\widetilde{\delta}'_{kn}(y)$$

for $\delta_{kn}(y)$ and $\widetilde{\delta}_{kn}(y)$, respectively, hence we see that both $\delta_{kn}(y)$ as well as $\widetilde{\delta}_{kn}(y)$ are solutions of (14).

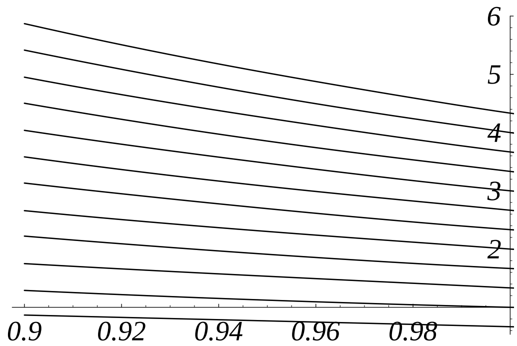

Figure 3: The decreasing functions $\delta_{kn}(y)$ near $y = 1$ for $k = 3$ and $n = 4, \ldots, 15$

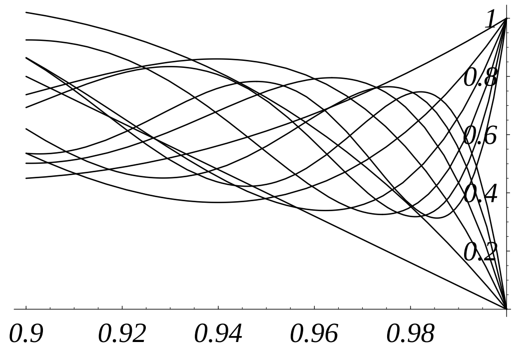

Figure 4: The functions $\widetilde{\delta}_{kn}(y)$ near $y = 1$ for $k = 3$ and $n = 4, \ldots, 15$

Rewriting the Todorov-Wilf identity (11) in terms of the functions δ_{kn} and $\widetilde{\delta}_{kn}$, we get

$$\widetilde{\delta}_{kn}(y) = k\,\delta_{kn}(y) + y\,\delta'_{kn}(y)\ . \tag{15}$$

Combining (14) and (15), we get

$$-\delta'_{k-1,n}(y) = \widetilde{\delta}_{kn}(y) + k\,\delta_{kn}(y)\ . \tag{16}$$

Since both $\widetilde{\delta}_{kn}(y)$ and $\delta_{kn}(y)$ are nonnegative in $[0, 1]$ and every $k = 1, \ldots, n$, this immediately yields that $\delta_{k,n}$ decreases in $[0, 1]$ for $k = 1, \ldots, n-1$. This statement was first given in [10].

We would like to notice that the solutions of recurrence equation (14) must be polynomials of degree $\leqq n - k$ as soon as we start with constant initial values $\delta_{nn}(y) = \delta_{nn}(0)$. This is easily seen by induction. We will now show that the degree of these polynomials is exactly $n - k$. By the above observation, we can set

$$\delta_{kn}(y) = \sum_{j=0}^{n-k} a^n_{jk}\,y^j \quad (a^n_{0k} = \delta_{kn}(0))\ .$$

Substituting this into (14), one gets

$$-\sum_{j=0}^{n-k} (j+1) a^n_{j+1,k-1}\,y^j = 2k\,a^n_{0k} + \sum_{j=1}^{n-k} (2k+j)\,a^n_{jk}\,y^j\ .$$

Comparing the coefficients, we therefore get

$$-(j+1)a^n_{j+1,k-1} = (2k+j)\,a^n_{jk} \quad (j = 0, 1, \ldots, n-k)\ ,$$

and in particular for $j = n - k$

$$-(n-k+1)a^n_{n-k+1,k-1} = (k+n)\,a^n_{n-k,k} \neq 0\ .$$

Note that this finishes our proof that the degree of these polynomials is always exactly $n-k$. From the above recurrence we can furthermore deduce that, given the initial value a_{0n}^n, the highest coefficient of $\delta_{kn}(y)$ is given by the hypergeometric term

$$a_{n-k,k}^n = (-1)^{n-k} \binom{2n}{n+k} a_{0n}^n \; .$$

Since both solution families of (14) that we know are nonnegative in $[0,1]$, the question arises to study the sign of a general solution of (14).

For this purpose, we integrate (14) from 0 to y and get

$$\delta_{k-1,n}(0) - \delta_{k-1,n}(y) = 2k \int_0^y \delta_{kn}(z)dz + \int_0^y z\,\delta'_{kn}(z)dz \; .$$

Partial integration gives

$$\delta_{k-1,n}(0) = \delta_{k-1,n}(y) + y\,\delta_{kn}(y) + (2k-1) \int_0^y \delta_{kn}(z)dz \; . \tag{17}$$

From (17), we immediately see that $\delta_{k-1,n}(y_0) = 0$ is equivalent to

$$\delta_{k-1,n}(0) = y_0\,\delta_{kn}(y_0) + (2k-1) \int_0^{y_0} \delta_{kn}(z)dz \; .$$

With the notation

$$R_{kn} := \left\{ y\,\delta_{kn}(y) + (2k-1) \int_0^y \delta_{kn}(z)dz \;\middle|\; 0 \leqq y \leqq 1 \right\}$$

we therefore learn that if $\delta_{k-1,n}(0) \notin R_{kn}$ then the function $\delta_{k-1,n}(y)$ does not have a zero in the interval $[0,1]$ and vice versa.

If we set $M_{kn} := \max R_{kn}$, then obviously $M_{kn} \geqq 0$ since $0 \in R_{kn}$ (set $y = 0$). Now if one chooses $\delta_{k-1,n}(0) > M_{kn}$, then the function $\delta_{k-1,n}(y)$ must be positive throughout the interval $[0,1]$. Hence if we initialize the recurrence equation (14) with a positive continuously differentiable function $\delta_{nn}(y)$, and choose $\delta_{k-1,n}(0) > M_{kn}$ for every $k = 1, \ldots, n-1$, then the resulting function system $\delta_{kn}(y)$ is positive for all $y \in [0,1]$ and all $k = 1, \ldots, n$.

Hence, as a consequence, positive solution families of the de Branges differential recurrence equation are not rare at all.

Since both $\delta_{kn}(y)$ and $\widetilde{\delta}_{kn}(y)$ are solutions of (14) and since $\widetilde{\delta}_{kn}(y)$ is rather oscillating, see Figure 4, it is clear that the positivity of $\delta_{kn}(y)$ cannot imply the positivity of $\delta'_{kn}(y)$. It is interesting that nevertheless a similar statement is true, where an integration is involved, however.

Multiplying (14) by y^{2k-1} and replacing y by z yields

$$-z^{2k-1}\,\delta'_{k-1,n}(z) = \left(z^{2k}\,\delta_{kn}(z)\right)' \ .$$

Integration from 0 to y gives

$$-\int\limits_0^y z^{2k-1}\,\delta'_{k-1,n}(z)\,dz = y^{2k}\,\delta_{kn}(y) \ .$$

Hence, if $\delta_{kn}(y)$ is positive on $[0,1]$, then $\int_0^y z^{2k-1}\,\delta'_{k-1,n}(z)\,dz$ is negative on $(0,1]$, independent of the choice of the value of $\delta_{k-1,n}(0)$.

4 Hypergeometric solution families of the de Branges differential recurrence equation

The hypergeometric representation (8) leads to the explicit repesentations

$$\delta_{kn}(y) = \frac{1}{k}\binom{n+k+1}{2k+1}\,{}_4F_3\left(\left.\begin{array}{c} k-n, k+1/2, n+k+2, k \\ 2k+1, k+1, k+3/2 \end{array}\right| y\right)$$

of $\delta_{kn}(y)$ and

$$\widetilde{\delta}_{kn}(y) = \binom{n+k+1}{2k+1}\,{}_3F_2\left(\left.\begin{array}{c} k-n, k+1/2, n+k+2 \\ 2k+1, k+3/2 \end{array}\right| y\right)$$

of $\widetilde{\delta}_{kn}(y)$. In this section we will show that many more structurally similar hypergeometric functions are solutions of the recurrence equation (14).

The above two representations suggest to consider the general hypergeometric functions

$$\delta_{kn}(y) = \delta_{kn}(0)\,{}_pF_q\left(\left.\begin{array}{c} k-n, \alpha_2^k, \ldots, \alpha_p^k \\ 2k+1, \beta_2^k, \ldots, \beta_q^k \end{array}\right| y\right) \ .$$

By the hypergeometric derivative rule (see e.g. [7], p. 27, Exercise 2.4), we get

$$2k\,{}_pF_q\left(\left.\begin{array}{c} k-n, \alpha_2^k, \ldots, \alpha_p^k \\ 2k+1, \beta_2^k, \ldots, \beta_q^k \end{array}\right| y\right) + y\,\frac{d}{dy}{}_pF_q\left(\left.\begin{array}{c} k-n, \alpha_2^k, \ldots, \alpha_p^k \\ 2k+1, \beta_2^k, \ldots, \beta_q^k \end{array}\right| y\right) = 2k\,{}_pF_q\left(\left.\begin{array}{c} k-n, \alpha_2^k, \ldots, \alpha_p^k \\ 2k, \beta_2^k, \ldots, \beta_q^k \end{array}\right| y\right) ,$$

and differentiating term-wise one has

$$\frac{d}{dy}{}_pF_q\left(\left.\begin{array}{c} k-1-n, \alpha_2^{k-1}, \ldots, \alpha_p^{k-1} \\ 2k-1, \beta_2^{k-1}, \ldots, \beta_q^{k-1} \end{array}\right| y\right) =$$

$$\frac{(k-1-n)\,\alpha_2^{k-1}\cdots\alpha_p^{k-1}}{(2k-1)\,\beta_2^{k-1}\cdots\beta_q^{k-1}}\cdot{}_pF_q\left(\left.\begin{array}{c}k-n,\alpha_2^{k-1}+1,\ldots,\alpha_p^{k-1}+1\\2k,\beta_2^{k-1}+1,\ldots,\beta_q^{k-1}+1\end{array}\right|y\right).$$

Substituting the above equations in (14) and dividing by $\delta_{k-1,n}(0)$, we arrive at

$$\frac{(n+1-k)\,\alpha_2^{k-1}\cdots\alpha_p^{k-1}}{(2k-1)\,\beta_2^{k-1}\cdots\beta_q^{k-1}}\cdot{}_pF_q\left(\left.\begin{array}{c}k-n,\alpha_2^{k-1}+1,\ldots,\alpha_p^{k-1}+1\\2k,\beta_2^{k-1}+1,\ldots,\beta_q^{k-1}+1\end{array}\right|y\right)$$

$$=2k\,\frac{\delta_{kn}(0)}{\delta_{k-1,n}(0)}\,{}_pF_q\left(\left.\begin{array}{c}k-n,\alpha_2^{k},\ldots,\alpha_p^{k}\\2k,\beta_2^{k},\ldots,\beta_q^{k}\end{array}\right|y\right).$$

For $y=0$ this yields

$$\frac{(n+1-k)\,\alpha_2^{k-1}\cdots\alpha_p^{k-1}}{(2k-1)\,\beta_2^{k-1}\cdots\beta_q^{k-1}}=2k\,\frac{\delta_{kn}(0)}{\delta_{k-1,n}(0)}\,, \tag{18}$$

and therefore

$${}_pF_q\left(\left.\begin{array}{c}k-n,\alpha_2^{k-1}+1,\ldots,\alpha_p^{k-1}+1\\2k,\beta_2^{k-1}+1,\ldots,\beta_q^{k-1}+1\end{array}\right|y\right)={}_pF_q\left(\left.\begin{array}{c}k-n,\alpha_2^{k},\ldots,\alpha_p^{k}\\2k,\beta_2^{k},\ldots,\beta_q^{k}\end{array}\right|y\right).$$

This equation is an identity if we set

$$\alpha_m^{k-1}+1=\alpha_m^k\quad\text{for }\ m=2,\ldots,p\quad\text{and}\quad\beta_r^{k-1}+1=\beta_r^k\quad\text{for }\ r=2,\ldots,q\,.$$

Solving these simple recurrences for α_m^k and β_r^k, we therefore get

$$\delta_{kn}(y)=\delta_{kn}(0)\,{}_pF_q\left(\left.\begin{array}{c}k-n,k+c_2,\ldots,k+c_p\\2k+1,k+d_2,\ldots,k+d_q\end{array}\right|y\right)$$

with constants c_m $(m=2,\ldots,p)$ and d_r $(r=2,\ldots,q)$.

With the aid of (18), we finally compute the initial values

$$\delta_{kn}(0)=\delta_{nn}(0)\,\frac{(2n)!}{(2k)!\,(n-k)!}\,\frac{\prod_{r=2}^{q}(k+d_r)_{n-k}}{\prod_{m=2}^{p}(k+c_m)_{n-k}}$$

by induction. Therefore we have computed the following solution $\delta_{kn}(y)$ $(k=1,\ldots,n)$ of (14)

$$\delta_{kn}(y)=\delta_{nn}(0)\,\frac{(2n)!}{(2k)!\,(n-k)!}\,\frac{\prod_{r=2}^{q}(k+d_r)_{n-k}}{\prod_{m=2}^{p}(k+c_m)_{n-k}}\,{}_pF_q\left(\left.\begin{array}{c}k-n,k+c_2,\ldots,k+c_p\\2k+1,k+d_2,\ldots,k+d_q\end{array}\right|y\right)$$

showing that hypergeometric solutions of the de Branges differential recurrence equation are not rare at all.

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