SPACELIKE MAXIMAL SURFACES IN 3D LORENTZ-MINKOWSKI SPACE

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ABSTRACT. We investigate spacelike maximal surfaces in 3-dimensional Lorentz-Minkowski space, give an Enneper-Weierstrass representation of such surfaces and classify those with a Lorentzian or Euclidean rotation symmetry.

1. INTRODUCTION

Let $\mathbb{R}^3_1$ denote $\mathbb{R}^3$ endowed with the product $< x, y >:= x_1y_1 + x_2y_2 - x_3y_3$. Furthermore let M denote a smooth oriented surface and $X : M \to \mathbb{R}^3_1$ a spacelike immersion, i.e. (M, g) a Riemannian 2-manifold. Here $g := X^* <,>$ is the induced metric on M. Consider an isotherm local chart $z = u + iv$ at $p \in M$:

$$\begin{aligned} < X_z, X_z > &= < X_{\bar z}, X_{\bar z} >= 0, \\ < X_z, X_{\bar z} > &=: \frac{1}{2}e^{\rho}, \end{aligned}$$

and a unit normal field (Gauß map) η on M:

$$\begin{aligned} < \eta, X_z > &= < \eta, X_{\bar z} >= 0, \\ < \eta, \eta > &= -1. \end{aligned}$$

Then the induced metric reads in local coordinates as $ds^2 = e^{\rho}|dz|^2$ and the second normalform is given by $II = Re(\Phi dz^2 + He^{\rho}dzd\bar z)$. Here $H = \frac{1}{2}trace(d\eta)$ denotes the mean curvature and $\Phi := 2 < \eta, X_{zz} >$. Suppose that the X is maximal, i.e. $H \equiv 0$. Then the integrability conditions

$$\begin{aligned} X_{z\bar z z} &= X_{zz\bar z}, \\ \eta_{z\bar z} &= \eta_{\bar z z}, \end{aligned}$$

which are the Gauß- and Codazzi-Mainardi equations read as

$$\begin{aligned} 2\rho_{z\bar z} &= -|\Phi|e^{-\rho} \\ \Phi_{\bar z} &= 0. \end{aligned}$$

Note that the Hopf differential Φdz^2 is invariant under change of coordinates and by the second equation holomorphic. See also [4] and [5].

2. ENNEPER-WEIERSTRASS REPRESENTATION

In this section we present and prove Enneper-Weierstrass representation formulas and derive some subsequences. Let $\Psi := X_z$, thus $Re(\Psi) = \frac{1}{2}X_u$ and $Im(\Psi) = -\frac{1}{2}X_v$. Then we have

$$\Psi_{\bar z} = 0 \tag{2.1}$$

$$\Psi_1^2 + \Psi_2^2 - \Psi_3^2 = 0 \tag{2.2}$$

$$|\Psi_1|^2 + |\Psi_2|^2 - |\Psi_3|^2 > 0. \tag{2.3}$$

A change of the coordinates $z - - > w$ at $p \in M$ results in

$$\Psi = \tilde\Psi \frac{dw}{dz},$$

where $\tilde\Psi = X_w$. Hence the complex vector-valued differential form $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i := \Psi_i dz$, $i = 1, 2, 3$, is invariant under change of coordinates, hence globally defined on M. Suppose now that α has no real periods, i.e. for each closed path c in M we have $\int_c \alpha = 0$ or equivalently $Re \int_r^s \Psi(q)dq$ is for fixed endpoints r and s independent of the integration path. Then $X(z) = Re \int_0^z \Psi(q)dq + Const$. A solution $\Psi : M \to \mathbb{C}$ of the equations

(1)-(3) can be explicitly given in terms of a holomorphic function $f : M \to \mathbb{C}$ and a meromorphic function $g : M \to \mathbb{C}$ as

$$\Psi := \begin{bmatrix} \frac{1}{2}f(1+g^2) \\ \frac{1}{2}if(1-g^2) \\ fg \end{bmatrix}, \tag{2.4}$$

provided

(i) $\Psi_i dz$ has no real periods,
(ii) f has a pole of order m at a zero of order $2m$ of g and
(iii) $|g|^2 - 1$ has no zeros.

Next we show that each solution of the equations (1)-(3) has such a representation. If $\Psi_1 = i\Psi_2$, then (2.2) yields $\Psi_3 = 0$ and the surface is flat. Let $\Psi_1 \neq i\Psi_2$. Then the functions

$$f := \Psi_1 - i\Psi_2,$$
$$g := \frac{\Psi_3}{\Psi_1 - i\Psi_2}.$$

clearly satisfy the desired conditions. Hence we have the following

Theorem 2.1. *Let $f : M \to \mathbb{C}$ be holomorphic and $g : M \to \mathbb{C}$ meromorphic satisfying (i)-(iii). Furthermore let Ψ be defined as (2.4). Then $X(z) := Re \int_r^z \Psi(q)dq$ defines a spacelike immersion of M into $\mathbb{R}^3_1$ with constant mean curvature $H \equiv 0$. Conversely every regular spacelike surface in $\mathbb{R}^3_1$ with $H \equiv 0$ has a parametrization in terms of f and g as given above.*

We denote that by a straightforward calculation the Gaussian curvature reads as

$$k = \frac{16|g'|^2}{|f|^2(1-|g|^2)^4},$$

particularly either $k \equiv 0$ or it has only isolated zeros. Furthermore the Gauß map is given by

$$\eta = \frac{1}{|g|^2 - 1}\begin{bmatrix} 2Re(g) \\ 2Im(g) \\ -(1+|g|^2) \end{bmatrix}.$$

Example 2.2. Let M be the open unit disk and $f(z) :\equiv 1$, $g(z) = z$ for $z = u + iv \in M$. A direct computation yields

$$\Psi(z) = \begin{bmatrix} cosh\, z \\ isinh\, z \\ 1 \end{bmatrix}.$$

By the previous theorem

$$X(u,v) := Re \int_0^z \Psi(q)dq = \begin{bmatrix} \frac{1}{2}u + \frac{1}{6}u^3 - \frac{1}{2}uv^2 \\ -\frac{1}{2}v + \frac{1}{2}u^2v - \frac{1}{6}v^3 \\ \frac{1}{2}(u^2 - v^2) \end{bmatrix}, \; u, v \in \mathbb{R}, u > 0$$

defines a spacelike maximal immersion of M into $\mathbb{R}^3_1$ with the image shown in figure 1. The induced Riemannian metric is $ds^2 = sinh^2 u|dz|^2$. The Gauß curvature reads as $k = \frac{1}{sinh^4 u}$.

Example 2.3. Let M be as above and $f(z) = \frac{4}{1-z^4}$ and $g(z) = z$ for $z \in M$. Then a straightforward computation yields

$$\Psi(z) = \begin{bmatrix} \frac{2}{1-z^2} \\ \frac{2i}{1+z^2} \\ \frac{4z}{1-z^4} \end{bmatrix}.$$

By the previous theorem

$$X(u,v) := Re \int_0^z \Psi(q)dq = \begin{bmatrix} ln|\frac{z+1}{z-1}| \\ ln|\frac{z-i}{z+i}| \\ ln|\frac{1+z^2}{1-z^2}| \end{bmatrix}$$

defines a spacelike maximal immersion of M into $\mathbb{R}^3_1$ with the image (graph of the function $(u,v) \mapsto ln(\frac{cosh\, u}{cosh\, v})$) shown in figure 2.

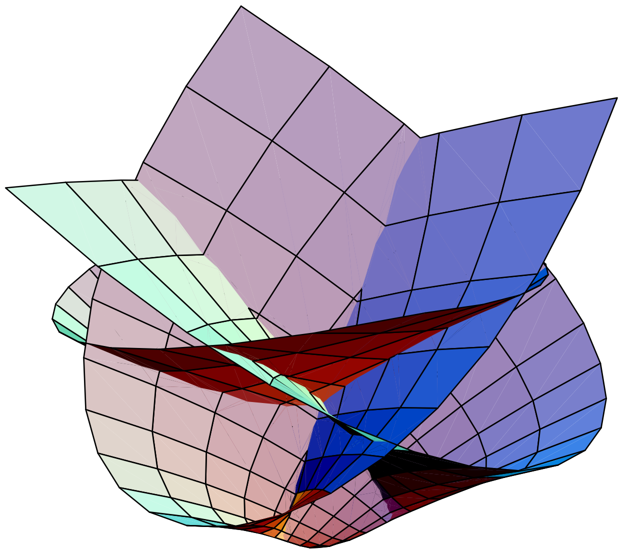

Figure 1

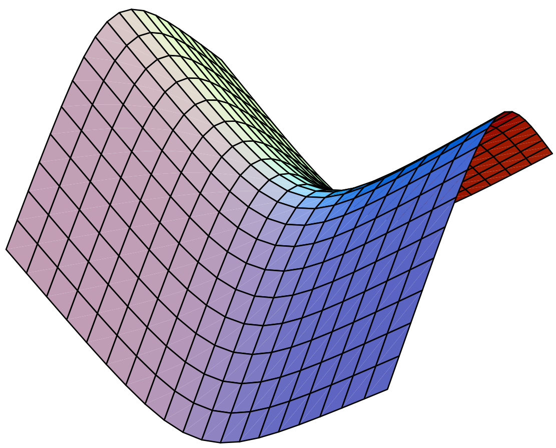

Figure 2

3. Rotation Symmetric Surfaces

The Boost- or Lorentian rotations

$$A_\varphi := \begin{bmatrix} cosh\,\varphi & 0 & \sinh\,\varphi \\ 0 & 1 & 0 \\ sinh\,\varphi & 0 & \cosh\,\varphi \end{bmatrix} \text{ and } B_\theta := \begin{bmatrix} 1 & 0 & 0 \\ 0 & cosh\,\theta & sinh\,\theta \\ 0 & sinh\,\theta & \cosh\,\theta \end{bmatrix}$$

with $\varphi, \theta \in \mathbb{R}$ are Lorentzian isometries of $\mathbb{R}^3_1$. Note that Euclidean rotations around the x_3-axis are Lorentzian isometries of $\mathbb{R}^3_1$ as well. In the following uniqueness means uniqueness up to Lorentzian isometris of $\mathbb{R}^3_1$. The next theorem shows that the only spacelike maximal surfaces of revolution in $\mathbb{R}^3_1$ are those given in the examples 2.1 and 2.2.

Theorem 3.1. *Let S be a non-flat spacelike maximal surface in $\mathbb{R}^3_1$ with Lorentzian rotation symmetry.*
(i) If the rotation-axis is timelike, then S is the surface of revolution of the graph of the function sinh on an intervall not including zero, around the rotation-axis.
(ii) If the rotation-axis is spacelike, then S is the surface of revolution of the graph of the function sin on an intervall not including zero, around the rotation-axis.

Proof. We prove (i). Similar arguments apply to (ii). The rotation-axis can be written as $\mathbb{R}w$ with

$$w = (sinhv, sinhucoshv, coshucoshv)$$

for some $u, v \in \mathbb{R}$. Hence applying the Lorentzian isometry $A_{-v}B_{-u}$ we may and do suppose that $w = (0, 0, 1)$. Now consider the parametrization

$$X(u, v) = (\alpha(v) cos\, u, \alpha(v) sin\, u, \beta(v)),\ u \in \mathbb{R}, v \in I,$$

with some intervall I and smooth functions $\alpha, \beta : I \to \mathbb{R}$. Then we have

$$\begin{aligned} < X_u(u, v), X_u(u, v) > &= \alpha^2(v), \\ < X_v(u, v), X_v(u, v) > &= \dot\alpha^2(v) - \dot\beta^2(v), \\ < X_u, X_v > &\equiv 0. \end{aligned}$$

Hence S is spacelike if and only if

$$\dot\alpha^2(v) - \dot\beta^2(v) > 0.$$

We may and do assume that the curve $I \ni t \mapsto (\alpha(t), \beta(t))$ is parametrised by its arc-lenght, i.e. denoting differentiation with respect to the arc-lenght parameter by a dot,

$$\dot\alpha^2(v) - \dot\beta^2(v) = 1.$$

A direct computation yields

$$2H\alpha\dot\alpha = -\dot\beta + \alpha\dot\alpha\ddot\alpha.$$

From the last two equations we conclude that $H \equiv 0$ if and only if

$$\alpha\dot\beta = C$$

with some constant $C \in \mathbb{R}$. Clearly S is flat if and only if β is constant. Hence assume $\dot\beta \neq 0$. With $Y := \alpha \circ \beta^{-1}$ we arrive at

$$\begin{aligned} (\dot Y \circ \beta)^2 &= \dot\alpha^2/\dot\beta^2 \\ &= 1 + (Y \circ \beta)^2/C^2. \end{aligned}$$

Therefore $Y(v) = sinh(mv + n)$, with some constants $m, n \in \mathbb{R}$, $m \neq 0$ and $-n/m \notin I$. Subsequently

$$X(u, v) = (sinh(mv + n)\cos u, sinh(mv + n)\, sin\, u, mv + C_1),$$

with a constant $C_1 \in \mathbb{R}$. □

The following figure shows two conjugate surfaces with Lorentzian as well as Euclidean rotation symmetry. We denote that the left one has a lightlike singularity and the right one is a Helicoid.

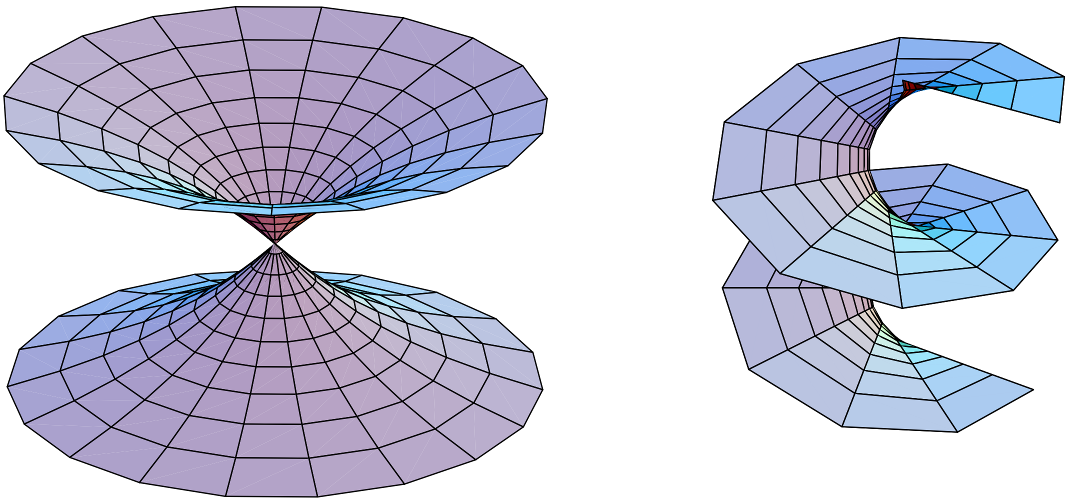

FIGURE 3

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