Approximate Approximations and a Boundary Point Method for the Linearized Stokes System

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Abstract The method of approximate approximations, introduced by Maz'ya [1], can also be used for the numerical solution of boundary integral equations. In this case, the matrix of the resulting algebraic system to compute an approximate source density depends only on the position of a finite number of boundary points and on the direction of the normal vector in these points (Boundary Point Method). We investigate this approach for the Stokes problem in the whole space and for the Stokes boundary value problem in a bounded convex domain $G \subset \mathbb{R}^2$, where the second part consists of three steps: In a first step the unknown potential density is replaced by a linear combination of exponentially decreasing basis functions concentrated near the boundary points. In a second step, integration over the boundary ∂G is replaced by integration over the tangents at the boundary points such that even analytical expressions for the potential approximations can be obtained. In a third step, finally, the linear algebraic system is solved to determine an approximate density function and the resulting solution of the Stokes boundary value problem. Even not convergent the method leads to an efficient approximation of the form $O(h^2) + \varepsilon$, where ε can be chosen arbitrarily small.

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1 Introduction

We consider the linearized Stokes system concerning the slow motion of a viscous incompressible fluid flow in a bounded two- or three-dimensional domain G with a compact smooth boundary ∂G. With methods of hydrodynamical potential theory, this system of differential equations can be reduced to a so-called volume potential , i.e. a convolution integral with the given external force density, and to a uniquely solvable system of boundary integral equations for the unknown source density of the chosen boundary layer potentials. We present an approach for the numerical solution of this boundary value problem based on exponentially decreasing ansatz functions satisfying an approximate partition of the unity, only. This method is called approximate approximations and has been introduced by Maz'ya [1]. Applying it to the numerical solution of the boundary integral equations, the matrix components of the resulting algebraic system for an approximate source density depend only on the position of a finite number of boundary points and on the direction of the normal vector in these points (Boundary Point Method). Our approach is carried out for the Stokes problem in the whole space and for the Stokes boundary value problem in a bounded convex domain $G \subset \mathbb{R}^2$, where the second part covers three steps: In a first step the unknown potential density is replaced by a linear combination of exponentially decreasing basis functions concentrated near the boundary points. In a second step, integration over the boundary ∂G is replaced by integration over the tangents at the boundary points such that even analytical expressions for the potential approximations can be obtained. In a third step, finally, the linear algebraic system is solved to determine an approximate density function and the resulting solution of the Stokes boundary value problem. Even not convergent the method leads to an efficient approximation of the form $O(h^2) + \varepsilon$, where ε can be chosen less than machine precision.

2 Approximation on the real line

It is well known that, in contrast to splines, the function $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\Phi(\mu) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(\mu-k)^2}{2}\right) \tag{1}$$

generates an approximate partition of the unity, only. Anyhow we shall use it for the approximation of a given function $f : \mathbb{R} \to \mathbb{R}$. For this purpose we choose $h > 0$ and define

$$f_h(x) := \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{1}{2}\left(\frac{x-hk}{h}\right)^2\right) f(hk). \tag{2}$$

Since we are not using an exact partition of the unity, we cannot expect convergence of the resulting sequence as $h \to 0$. Let us study the error

$$\varepsilon_h(x) := f_h(x) - f(x)$$

for $h \to 0$. To do so we need the space $C_b^2(\mathbb{R})$ of functions having bounded continuous second order derivatives in $\mathbb{R}$.

Proposition 1 *Let $f \in C_b^2(\mathbb{R})$, $h > 0$, and f_h defined by (2). Then the error $\varepsilon_h(x)$ satisfies in $x \in \mathbb{R}$ the following estimate:*

$$|\varepsilon_h(x)| \le \frac{h^2}{2}\|f''\|_\infty\left(\left|\Phi\left(\tfrac{x}{h}\right)\right| + \left|\Phi''\left(\tfrac{x}{h}\right)\right|\right) + h\,|f'(x)|\left|\Phi'\left(\tfrac{x}{h}\right)\right| + |f(x)|\left|\Phi\left(\tfrac{x}{h}\right) - 1\right|.$$

Here Φ is the function defined by (1) and $\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|$ the norm in $L^\infty(\mathbb{R})$.

The estimate of Proposition 1 shows that we are using an approximation essentially of second order, since in reality only the term

$$\frac{h^2}{2}\|f''\|_\infty \left|\Phi\left(\tfrac{x}{h}\right)\right|$$

has to be taken into account, all other factors are neglectably small. Therefore the expression approximate approximation seems to be reasonable (compare [3]).

The method carries over immediately to the n–dimensional case, where a given function $f : \mathbb{R}^n \to \mathbb{R}$ can be approximated by

$$f_h(x) := \frac{1}{\sqrt{(2\pi)^n}} \sum_{k\in\mathbb{Z}^n} \exp\left(-\frac{1}{2}\left|\frac{x-hk}{h}\right|^2\right) f(hk). \tag{3}$$

All the above statements hold true in this case, too.

3 The Stokes system in $\mathbb{R}^n$ $(n = 2, 3)$

Now we will use the above method to solve numerically the Stokes equations

$$\left.\begin{array}{c} -\triangle u + \nabla p = f \\ \operatorname{div} u = 0 \end{array}\right\} \text{ in } \mathbb{R}^n \qquad (n = 2, 3). \tag{4}$$

Here $u := (u_1, ..., u_n)$ is the unknown velocity field and $\nabla p := (\partial_1 p, ..., \partial_n p)$ the unknown pressure gradient of a viscous incompressible fluid flow, and the external force density $f := (f_1, ... f_n)$ is given.

In the following we approximate the velocity field u, only. The corresponding volume potential part

$$u(x) = Vf(x) := \int_{\mathbb{R}^n} E(x-y) \cdot f(y)\, dy \qquad (n = 2, 3), \tag{5}$$

which solves (4), is defined using the fundamental solution $E(x) = E_{ij}(x)$ $(i, j = 1 \ldots n)$ of the Stokes operator in $\mathbb{R}^n$, i.e.

$$E_{ij}(x) = \frac{1}{2\omega_n} \left\{ \frac{x_i x_j}{|x|^n} + \delta_{ij} \left\{ \begin{array}{ll} \ln \frac{1}{|x|}, & n = 2 \\ |x|^{-1}, & n = 3 \end{array} \right\} \right\}. \tag{6}$$

Here ω_n denotes the surface of the $(n-1)$- dimensional unit ball.

To approximate the volume potential Vf we replace each component f_j $(j = 1, \ldots, n)$ of the given function f by the approximation (3), i.e. by

$$f_j^h(y) := \frac{1}{\sqrt{(2\pi)^n}} \sum_{m \in \mathbb{Z}^n} \exp\left(-\frac{1}{2} \left| \frac{y - hm}{h} \right|^2 \right) f_j(hm).$$

This leads to an approximate solution u of (4) in the form

$$u_h(x) := Vf_h = \int_{\mathbb{R}^n} E(x-y) \cdot f_h(y)\, dy = \sum_{m \in \mathbb{Z}^n} A^{m,h}(x) \cdot f(hm) \tag{7}$$

with $A^{m,h} = A_{ij}^{m,h}$, $(i, j = 1 \ldots n)$ given by

$$A_{ij}^{m,h}(x) := \left\{ \begin{array}{ll} \frac{1}{8\pi^2} \int\limits_{\mathbb{R}^2} \left(\delta_{ij} \ln \frac{1}{|x-y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2} \right) \exp\left(-\frac{1}{2} \left| \frac{y}{h} - m \right|^2 \right) dy \quad , & n = 2, \\ \frac{1}{8\pi\sqrt{(2\pi)^3}} \int\limits_{\mathbb{R}^3} \left(\delta_{ij} \frac{1}{|x-y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3} \right) \exp\left(-\frac{1}{2} \left| \frac{y}{h} - m \right|^2 \right) dy, & n = 3. \end{array} \right.$$

The matrix $A^{m,h}(x)$ can be determined analytically:

Proposition 2 *Let $\xi := x/h - m$. Then for $n = 2$ and $i, j = 1, 2$ and $|\xi| \neq 0$ we have*

$$\begin{aligned} A_{ij}^{m,h}(x) &= \frac{1}{8\pi} h^2 \delta_{ij} \left[-\ln(2h^2) + C - \operatorname{exint}\left(\tfrac{1}{2}|\xi|^2 \right) + \frac{1 - \exp\left(-\frac{1}{2}|\xi|^2 \right)}{\frac{1}{2}|\xi|^2} \right] \\ &+ \frac{1}{8\pi} h^2 \left\{ \frac{\xi_i \xi_j}{\frac{1}{2}|\xi|^2} - \frac{\xi_i \xi_j}{\frac{1}{4}|\xi|^4} \left[1 - \exp\left(-\tfrac{1}{2}|\xi|^2 \right) \right] \right\}, \end{aligned}$$

and for $|\xi| = 0$ *we have*

$$A_{ij}^{m,h}(x) = \frac{1}{8\pi} h^2 \delta_{ij} \left(C - \ln\left(2h^2\right) + 1 \right),$$

where C *is Euler's constant.*

Proposition 3 *Let* $\xi := x/h - m$. *Then for* $n = 3$ *and* $i, j = 1, \ldots 3$ *and* $|\xi| \neq 0$ *we have*

$$\begin{aligned} A_{ij}^{m,h}(x) &= \frac{1}{2\sqrt{(2\pi)^3}} \delta_{ij} h^2 \frac{1}{|\xi|^2} \left\{ -\exp\left(-\frac{|\xi|^2}{2}\right) + \left(|\xi| + \frac{1}{|\xi|}\right) \int_0^{|\xi|} \exp\left(-\frac{t^2}{2}\right) dt \right\} \\ &+ \frac{1}{2\sqrt{(2\pi)^3}} h^2 \frac{\xi_i \xi_j}{|\xi|^4} \left\{ 3\exp\left(-\frac{|\xi|^2}{2}\right) + \left(|\xi| - \frac{3}{|\xi|}\right) \int_0^{|\xi|} \exp\left(-\frac{t^2}{2}\right) dt \right\}, \end{aligned}$$

and for $|\xi| = 0$ *we have*

$$A_{ij}^{m,h}(x) = \frac{2}{3\sqrt{(2\pi)^3}} \delta_{ij} h^2.$$

4 Hydrodynamical potential theory

We consider the interior Dirichlet problem for the spatially homogeneous Stokes equations:

$$-\Delta \mathbf{u} + \nabla p = \mathbf{0} \quad \text{in} \quad G, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad G, \qquad \mathbf{u} = \mathbf{b} \quad \text{on} \quad \partial G. \tag{8}$$

Here $\mathbf{u} := (u_1, u_2)^T$ denotes the velocity field and p some pressure function of a viscous incompressible fluid flow in G, where $G \subset \mathbb{R}^2$ is a bounded simply connected convex domain with a compact boundary $\partial G \in C^2$. We assume that the boundary value $\mathbf{b} \in C(\partial G)$ satisfies the compatibility condition

$$\int_{\partial G} \mathbf{b} \cdot \mathbf{n} \, \mathrm{do} = 0, \tag{9}$$

where $\mathbf{n}$ denotes the outward unit normal on ∂G. It is well known that the interior Dirichlet problem for the Stokes equations has a solution $\mathbf{u}$, p, where $\mathbf{u}$ is unique and p is unique up to an additive constant, only. This solution can be represented by hydrodynamical double layer potentials. In particular, for the velocity field $\mathbf{u}$ this potential has the form

$$\mathbf{u}(x) = D\mathbf{\Phi}(x) := \int_{\partial G} D(\mathbf{x}, \mathbf{y}) \, \mathbf{\Phi}(y) \, \mathrm{do}_{\mathbf{y}}, \ \mathbf{x} \in G, \tag{10}$$

where $D(\mathbf{x}, \mathbf{y})$ is the weakly singular 2×2 double layer tensor, defined by

$$D_{ij}(\mathbf{x}, \mathbf{y}) = -\frac{1}{\pi} \frac{(x_i - y_i)(x_j - y_j)(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^4}, \quad i, j = 1, 2, \tag{11}$$

and $\mathbf{\Phi} := (\Phi_1, \Phi_2)^T$ is some source density. Since these potentials already satisfy the differential equations, it remains to determine the source density $\mathbf{\Phi}$ in such a way that also the boundary condition is satisfied. Due to the jump relations of the hydrodynamical double layer potential $D\mathbf{\Phi}$, this leads to a Fredholm boundary integral equations system of second kind:

$$\frac{1}{2}\,\mathbf{\Phi}(\mathbf{x}) + D\mathbf{\Phi}(\mathbf{x}) = \mathbf{b}(\mathbf{x}), \quad \mathbf{x} \in \partial G. \tag{12}$$

It is known that there exists a solution $\mathbf{\Phi}$ of this system, but it is not unique. In the frame of our approximation procedure, however, it is desirable to work with uniquely solvable systems. Therefore, instead of (12), we use the boundary integral equations system

$$\frac{1}{2}\,\mathbf{\Phi}(\mathbf{x}) + D\mathbf{\Phi}(\mathbf{x}) - \mathbf{n}(\mathbf{x}) \int_{\partial G} \mathbf{\Phi}(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})\, do_{\mathbf{y}} = \mathbf{b}(\mathbf{x}), \quad \mathbf{x} \in \partial G. \tag{13}$$

It can be shown [7] that this system is uniquely solvable, and that its unique solution also solves the system (12), provided that the boundary value $\mathbf{b} \in C(\partial G)$ satisfies the compatibility condition (9).

5 The boundary point method

Starting point for our approximation procedure is the double layer representation (10) of the solution $\mathbf{u}$ of the Stokes equations (8). In the following, we establish three steps to obtain an approximate analytic representation of $\mathbf{u}$. To do so, let $\{\mathbf{x}_k \in \partial G \mid k = 1, \ldots, N\}$ be a set of somehow uniformly distributed boundary points.

Step 1: For every $k = 1, \ldots, N$ we define a radial, exponentially decreasing basis function $g_k : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g_k(\mathbf{x}) := \frac{1}{\sqrt{\pi d}} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_k|^2}{d\,h^2}\right), \qquad \mathbf{x} \in \mathbb{R}^2, \tag{14}$$

where h is the mean distance between adjacent boundary points and d is some positive parameter. Now we replace the source density $\mathbf{\Phi}$ by a linear combination

$$\mathbf{\Phi}_h(\mathbf{x}) := \sum_{k=1}^{N} g_k(\mathbf{x})\; \mathbf{c}_k = \frac{1}{\sqrt{\pi d}} \sum_{k=1}^{N} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_k|^2}{d\,h^2}\right)\; \mathbf{c}_k, \qquad \mathbf{c}_k \in \mathbb{R}^2, \quad \mathbf{x} \in \partial G, \tag{15}$$

of these basis functions and approximate the solution $\mathbf{u} = D\mathbf{\Phi}$ in $\mathbf{x} \in G$ by

$$\begin{aligned}
\mathbf{u}_h(\mathbf{x}) := D\mathbf{\Phi}_h(\mathbf{x}) &= \sum_{k=1}^{N} Dg_k(\mathbf{x})\; \mathbf{c}_k = \sum_{k=1}^{N} \int_{\partial G} g_k(\mathbf{y})\, D(\mathbf{x}, \mathbf{y})\, do_{\mathbf{y}}\; \mathbf{c}_k \\
&= \frac{1}{\sqrt{\pi d}} \sum_{k=1}^{N} \int_{\partial G} \exp\left(-\frac{|\mathbf{y} - \mathbf{x}_k|^2}{d\,h^2}\right) D(\mathbf{x}, \mathbf{y})\, do_{\mathbf{y}}\; \mathbf{c}_k, \qquad \mathbf{x} \in G.
\end{aligned}$$

Step 2: Since the basis function $g_k(\mathbf{x})$ is exponentially decreasing, for $k = 1, \ldots, N$ we substitute the double layer potential matrix $Dg_k(\mathbf{x})$ by the double layer potential matrix $D_k g_k(\mathbf{x})$, defined by

$$D_k g_k(\mathbf{x}) := \int_{T_k} g_k(\mathbf{y})\, D(\mathbf{x}, \mathbf{y})\, do_{\mathbf{y}} = \frac{1}{\sqrt{\pi d}} \int_{T_k} \exp\left(-\frac{|\mathbf{y} - \mathbf{x}_k|^2}{d\,h^2}\right) D(\mathbf{x}, \mathbf{y})\, do_{\mathbf{y}},$$

where T_k denotes the tangent to ∂G in $\mathbf{x}_k \in \partial G$. The corresponding approximation of the solution $\mathbf{u}$ is denoted by

$$\begin{aligned}\mathbf{u}_{\mathrm{T}}(\mathbf{x}) &= \sum_{k=1}^{N} \int_{T_k} g_k(\mathbf{y})\, D(\mathbf{x},\mathbf{y})\, \mathrm{do}_{\mathbf{y}}\, \mathbf{c}_k \\ &= \frac{1}{\sqrt{\pi d}} \sum_{k=1}^{N} \int_{T_k} \exp\left(-\frac{|\mathbf{y}-\mathbf{x}_k|^2}{d\,h^2}\right) D(\mathbf{x},\mathbf{y})\, \mathrm{do}_{\mathbf{y}}\, \mathbf{c}_k, \qquad \mathbf{x} \in G.\end{aligned}$$

Step 3: To determine the unknown coefficients $\mathbf{c}_k$ from the ansatz (15) we consider the boundary integral equations system (13) and replace the source density $\boldsymbol{\Phi}$ by $\boldsymbol{\Phi}_h$, obtaining in $\mathbf{x}_l \in \partial G$ $(l = 1, \ldots, N)$ by an elementary calculation

$$\sum_{k=1}^{N} \left(\frac{1}{2}\, g_k(\mathbf{x}_l) + \int_{\partial G} g_k(\mathbf{y}) \left[D(\mathbf{x}_l,\mathbf{y}) - N(\mathbf{x}_l,\mathbf{y}) \right] \mathrm{do}_{\mathbf{y}} \right) \mathbf{c}_k = \mathbf{b}(\mathbf{x}_l), \quad \mathbf{x}_l \in \partial G,$$

where the 2×2 matrix $N(\mathbf{x},\mathbf{y})$ is defined by

$$N_{ij}(\mathbf{x},\mathbf{y}) := n_i(\mathbf{x})\, n_j(\mathbf{y}), \quad i,j = 1,2.$$

Finally, replacing integration over ∂G by integration over the tangents, we get the following linear algebraic system to determine the unknown vectorial coefficients $\mathbf{c}_k$:

$$\sum_{k=1}^{N} \left(\frac{1}{2}\, g_k(\mathbf{x}_l) + \int_{T_k} g_k(\mathbf{y}) \left[D(\mathbf{x}_l,\mathbf{y}) - N(\mathbf{x}_l,\mathbf{y}) \right] \mathrm{do}_{\mathbf{y}} \right) \mathbf{c}_k = \mathbf{b}(\mathbf{x}_l), \quad \mathbf{x}_l \in \partial G.$$

6 Representation of the double layer potential

Due to the simplicity of the integration domain, we are able to develop exact analytical expressions for the above mentioned double layer potentials:

Proposition 4 *Let $G \subset \mathbb{R}^2$ be a bounded convex domain with boundary $\partial G \in C^2$, and let $\mathbf{x} \in \overline{G} := G \cup \partial G$. Let $\mathbf{x}_k \in \partial G$ be a boundary point and T_k the tangent to ∂G in $\mathbf{x}_k$. Then for $i,j = 1,2$ the double layer potential*

$$\begin{aligned} D_{ij}^k(\mathbf{x}) &:= \int_{T_k} g_k(\mathbf{y}) D_{ij}(\mathbf{x},\mathbf{y})\, \mathrm{do}_{\mathbf{y}} \\ &= -\frac{1}{\pi\sqrt{\pi d}} \int_{T_k} \exp\left(-\frac{|\mathbf{y}-\mathbf{x}_k|^2}{d\,h^2}\right) \frac{(x_i - y_i)\,(x_j - y_j)\,(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^4}\, \mathrm{do}_{\mathbf{y}} \end{aligned}$$

with the radial basis function g_k defined by (14) is given by

$$\begin{aligned} D_{ij}^k(\mathbf{x}) &:= \frac{\xi}{\sqrt{\pi d}} \left(\frac{1}{\sqrt{\pi}} \mathrm{Re}\big[(v(i)\, v(j)\big] - \mathrm{Im}\big[(\eta + i|\xi|)\, w(\eta + i|\xi|)\, v(i)\, v(j)\big] \right) \\ &- \delta_{ij}\, \frac{\mathrm{sign}(\xi)}{2\sqrt{\pi d}}\, \mathrm{Re}\big[w(\eta + i|\xi|)\big], \end{aligned}$$

where δ_{ij} *is the Kronecker symbol and we set*

$$\xi := \frac{(\mathbf{x}-\mathbf{x}_k)\cdot\mathbf{n}(\mathbf{x}_k)}{\sqrt{d}\,h}, \qquad \eta := \frac{(\mathbf{x}-\mathbf{x}_k)\cdot\mathbf{n}^{\perp}(\mathbf{x}_k)}{\sqrt{d}\,h}, \qquad \mathbf{n}^{\perp}(\mathbf{x}_k) := \begin{pmatrix} -n_2(\mathbf{x}_k) \\ n_1(\mathbf{x}_k) \end{pmatrix},$$

$$v(1) := -n_2(\mathbf{x}_k) - i\,n_1(\mathbf{x}_k), \qquad v(2) := i\,v(1), \qquad w(z) := \exp(-z^2)\,\mathrm{erfc}(-iz).$$

7 A whole space simulation

For $4 \le \beta \in \mathbb{N}$ we define the test function $u = (u_1, u_2)$ setting

$$\left.\begin{array}{rcl} u_1(x_1,x_2) &=& 4x_2\left(\frac{16}{3}\right)^{2\beta-1}\left(\frac{1}{4}-x_1^2\right)^{\beta}\left(\frac{1}{4}-x_2^2\right)^{\beta-1} \\ u_2(x_1,x_2) &=& -4x_1\left(\frac{16}{3}\right)^{2\beta-1}\left(\frac{1}{4}-x_2^2\right)^{\beta}\left(\frac{1}{4}-x_1^2\right)^{\beta-1} \end{array}\right\} \text{ in } Q \tag{16}$$

and $u = 0$ in $\mathbb{R}^2\backslash Q$, where Q is the unit square centered at zero. An easy calculation shows $\operatorname{div} u = 0$ in $\mathbb{R}^2$. Moreover, setting

$$p(x_1,x_2) := 16^{\beta-1}\left(\frac{1}{4}-x_1^2\right)^{\beta-1}\left(\frac{1}{4}-x_2^2\right)^{\beta-1} \text{ in } Q \tag{17}$$

and $p = 0$ in $\mathbb{R}^2\backslash Q$, we obtain the function $f := -\triangle u + \nabla p$ in $\mathbb{R}^2$ as potential density. The error $\varepsilon_h := \max|u_i(x) - u_h^i(x)|$ (the results are identical for $i = 1, 2$) for different values of the smoothness parameter β is shown in Table 1.

h	$\beta = 4$	$\beta = 5$	$\beta = 6$	$\beta = 7$
0,1	1,08505e-00	2,32411e-00	4,22133e-00	7,34847e-00
0,05	3,47030e-01	7,68388e-01	1,44394e-00	2,60742e-00
0,025	9,25330e-02	2,07515e-01	3,95278e-01	7,23272e-01
0,0125	2,35129e-02	5,29152e-02	1,01176e-01	1,85807e-01
0,00625	5,90226e-03	1,32948e-02	2,54448e-02	4,67728e-02
0,003125	1,47707e-03	3,32784e-03	6,37069e-03	1,17134e-02
0,0015625	3,69362e-04	8,32221e-04	1,59327e-03	2,92961e-03

Table 1. Maximal error expansion

The corresponding order

$$\alpha_h := \log_2 \frac{\varepsilon_{2h}}{\varepsilon_h}$$

of convergence is presented in Table 2 and confirms an approximate approximation of second order.

h	$\beta = 4$	$\beta = 5$	$\beta = 6$	$\beta = 7$
0,05	1,64463	1,59677	1,54769	1,49482
0,025	1,90702	1,88862	1,86907	1,85001
0,0125	1,97651	1,97146	1,96600	1,96073
0,00625	1,99411	1,99282	1,99142	1,99007
0,003125	1,99853	1,99820	1,99785	1,99751
0,0015625	1,99963	1,99955	1,99946	1,99938

Table 2. Order of convergence

8 A simulation for the boundary value problem

We investigate a test flow in the 2-d unit circle G with prescribed boundary value $\mathbf{b}(x_1, x_2) := \frac{1}{2}(x_2, x_1)$ on ∂G. In this case, the unique solution φ of the corresponding integral equations (13) is $\varphi(x_1, x_2) := (x_2, x_1)$ We calculate the components φ_1, φ_2 using N boundary points. The absolute error $\varepsilon_N = (\varepsilon_N^1, \varepsilon_N^2)$ and the rate of convergence $\alpha_N^i := \log_2 \frac{\varepsilon_{N/2}^i}{\varepsilon_N^i} \quad (i = 1, 2)$ for both components are shown in Table 1. The rate shows convergence of second order.

N	ε_N^1	α_N^1	ε_N^2	α_N^2
7	3,82609e-01	-	3,86710e-01	-
15	3,09887e-01	0,30413	3,16234e-01	0,29026
31	9,49303e-02	1,70680	9,72965e-02	1,70053
62	2,55624e-02	1,89284	2,62163e-02	1,89192
125	6,42047e-03	1,99328	6,58536e-02	1,99313
251	1,60075e-03	2,00393	1,64189e-03	2,00390

Table 3. Maximal error expansion

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