The Navier-Stokes Equations with Time Delay

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Abstract

In the present paper we use a time delay $\varepsilon > 0$ for an energy conserving approximation of the nonlinear term of the non-stationary Navier-Stokes equations. We prove that the corresponding initial value problem (N_ε) in smoothly bounded domains $G \subseteq \mathbb{R}^3$ is well-posed. Passing to the limit $\varepsilon \to 0$ we show that the sequence of stabilized solutions has an accumulation point such that it solves the Navier-Stokes problem (N_0) in a weak sense (Hopf).

1 Introduction

Let $T > 0$ be given and $G \subseteq \mathbb{R}^3$ be a bounded domain with a smooth compact boundary ∂G. In G we consider a non-stationary viscous incompressible fluid flow and assume that it can be described by the Navier-Stokes equations

$$\begin{aligned} \partial_t v - \nu \Delta v + \nabla p + (v \cdot \nabla) v &= F\,, \\ \nabla \cdot v &= 0\,, \qquad (N_0) \\ v_{|\partial G} = 0\,, \qquad v_{|t=0} &= v_0\,. \end{aligned}$$

These equations represent a system of nonlinear partial differential equations concerning four unknown functions: the velocity vector $v = (v_1(t,x), v_2(t,x), v_3(t,x))$ and the (scalar) kinematic pressure function $p = p(t,x)$ of the fluid at time $t \in (0,T)$ in position $x \in G$. The constant $\nu > 0$ (kinematic viscosity), the external force density F, and the initial velocity v_0 are given data. In (N_0) $\partial_t v$ means the partial derivative with respect to the time t, Δ is the Laplace operator in $\mathbb{R}^3$, and $\nabla = (\partial_1, \partial_2, \partial_3)$ the

gradient, where $\partial_j = \frac{\partial}{\partial x_j}$ denotes the partial derivative with respect to x_j $(j = 1, 2, 3)$. From the physical point of view, the nonlinear convective term $(v \cdot \nabla)v$ is a result of the total derivative of the velocity field. Here the operator $(v \cdot \nabla)$ has to be applied to each component v_j of v. In the fourth equation $\nabla \cdot v = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3$ defines the divergence of v, which vanishes due to the incompressibility of the fluid. Finally, the no-slip boundary condition $v_{|\partial G} = 0$ expresses that the fluid adheres to the boundary ∂G.

Let us assume that smooth data are given without any smallness assumptions. Then the problem to construct a solution $v,\ \nabla p$ of (N_0), which is uniquely determined and exists globally in time, has not been solved in the 3-d case considered here (see for example [3], [4]). Consequently, there is no globally stable approximation scheme for (N_0) up to now.

In the present paper we use a smoothing procedure for the Navier-Stokes equations based on a time delay in the nonlinear term: Setting $\varepsilon = T/N > 0$ $(N \in \mathbb{N})$ and

$$v_\varepsilon(\,t\,) = v(t - \varepsilon)$$

we replace (N_0) by

$$\begin{aligned} \partial_t v - \nu\Delta v + \nabla p + (v_\varepsilon \cdot \nabla)v &= F\,, \\ \nabla \cdot v &= 0\,, \\ v_{|\partial G} = 0\,, \qquad v_{|t\le 0} &= v_0\,. \end{aligned} \tag{N_ε}$$

Here the initial value v_0 from (N_0) has to be extended to the time interval $[-\varepsilon, 0]$ in a suitable way. We show that these equations have strongly $H^2(G)$ - continuous global unique solutions (see the next section for the notations):

Theorem 1: *Let $f \in H^1(0, T, H(G))$, $\varepsilon > 0$, and $v_0 \in C([-\varepsilon, 0],\ H^2(G) \cap V(G))$ with $\partial_t v_0 \in C([-\varepsilon, 0],\ H(G)) \cap L^2(G)(-\varepsilon,\ 0,\ V(G))$ be given. Then there is a unique solution $v, \nabla p$ of the equations (N_ε) such that*

$$v \in C([0,T],\ H^2(G) \cap V(G)), \qquad \partial_t v \in C([0,T],\ H(G)) \cap L^2(0,T,\ V(G)),$$

$$\nabla p \in C([0,T],\ L^2(G)).$$

Moreover, v satisfies for all $t \in [0, T]$ the energy equation

$$||v(t)||^2 + 2\nu \int_0^t ||\nabla v(s)||^2 ds = ||v_0||^2 + 2\int_0^t (F(s), v(s))ds \tag{1.1}$$

and the estimates

$$\int_0^T ||\nabla \partial_t v(t)||^2 dt \le K, \quad ||v(t)||_2 \le K, \quad ||\partial_t v(t)|| \le K \quad (t \in [0,T]) \tag{1.2}$$

with some constant $K = K(G, T, \varepsilon, \nu, F, v_0)$.

We investigate the behavior of the solution of (N_ε) if the delay ε tends to zero. Up to now, without any additional smallness assumptions on the data, the only solutions of the Navier-Stokes system (N_0), for which global existence (but not uniqueness) for all time $t \in [0,T]$ has been proved, are weak solutions. Let us recall the definition of a weak solution of the Navier-Stokes equations (N_0) (compare also [5], p. 220, [6], p. 173, [7], p. 72, [9], p. 280):

Definition: *Let $v_0 \in H(G)$ and $f \in L^2(0,T,H(G))$ be given. Then we call a function*

$$v \in L^2(0,T,V(G)) \cap L^\infty(0,T,H(G))$$

a weak solution of the Navier-Stokes equations (N_0), if $v : [0,T] \to H(G)$ is weakly continuous, if $||v(t) - v_0|| \to 0$ as $t \to 0$ (strong $H(G)$ – continuity at $t = 0$), and if for all test function $\Phi \in C_0^\infty((0,T) \times G)$ with $\Phi(t) \in C_{0,\sigma}^\infty(G)$ the following identity holds:

$$\int_0^T \{-(v(t), \partial_t \Phi(t)) + \nu(\nabla v(t), \nabla \phi(t)) - ((v(t) \cdot \nabla)\Phi(t), v(t)) - (F(t), \Phi(t))\} dt \; = \; 0 \, .$$

We show that weak solutions can be constructed by the solution of (N_ε), if the delay parameter $\varepsilon = T/N$ tends to zero (T remains fixed):

Theorem 2: *Let $T > 0$ be fixed. For $\varepsilon^N = T/N > 0$ $(N \in \mathbb{N})$ let v^N denote the solution constructed in Theorem 1. Then the sequence $(v^N)_{N \in \mathbb{N}}$ has an accumulation point v, which solves the Navier-Stokes equations (N_0) weakly and satisfies for all $t \in [0,T]$ the energy inequality*

$$||v(t)||^2 + 2\nu \int_0^t ||\nabla v(s)||^2 ds \;\; \leq \;\; ||v_0||^2 + 2 \int_0^t (F(s), v(s)) ds \, . \tag{1.3}$$

2 Notations and Auxiliaries

Throughout this paper, $G \subseteq \mathbb{R}^3$ is a bounded domain having a compact boundary ∂G of class C^2. In the following, all functions are real valued. As usual, $C_0^\infty(G)$ denotes the spaces of smooth functions defined in G with compact support, and $L^p(G)$ is the Lebesgue space with the norm $||f||_{0,p}$ $(1 \leq p \leq \infty$; if $p = \infty$ we use $||f||_\infty$ instead of $||f||_{0,\infty})$. The space $L^2(G)$ is a Hilbert space with scalar product and norm defined by

$$(f,g) \; = \; \int_G f(x) \, g(x) dx \, , \qquad ||f|| \; = \; ||f||_{0,2} \; = \; (f,f)^{\frac{1}{2}} \, ,$$

respectively.

The Sobolev space $H^m(G)$ ($m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$) is the space of functions f such that $\partial^\alpha f \in L^2(G)$ for all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq m$. Its norm is denoted by

$$||f||_m \;=\; ||f||_{H^m(G)} \;=\; \left(\sum_{|\alpha| \leq m} ||\partial^\alpha f||^2 \right)^{\frac{1}{2}}, \qquad \partial^\alpha \;=\; \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3},$$

where $\partial_k = \frac{\partial}{\partial x_k}$ ($k = 1, 2, 3$) is the distributional derivative.

The completion of $C_0^\infty(G)$ with respect to the norm $||\cdot||_m$ is denoted by $H_0^m(G)$. In particular, we have $H_0^0(G) = H^0(G) = L^2(G)$. For the corresponding spaces of vector functions $u = (u_1, u_2, u_3)$ and their norms we use the same symbols as in the scalar case, hence, for example,

$$(u, v) \;=\; \sum_{k=1}^{3} (u_k, v_k), \qquad ||u|| \;=\; (u, u)^{\frac{1}{2}} \;=\; \int_G |u(x)|^2 dx^{\frac{1}{2}},$$

where $|u(x)| = (u_1(x)^2 + u_2(x)^2 + u_3(x)^2)^{\frac{1}{2}}$ is the Euclidian norm of $u(x) \in \mathbb{R}^3$.

We shall often use the Poincare inequality

$$||u||^2 \;\leq\; c\, ||\nabla u||^2. \tag{2.4}$$

Here the constant $\lambda_1 = c^{-1}$ is the smallest eigenvalue of the Laplace operator $-\Delta$ in G with vanishing boundary condition, and it depends on the size of the domain G ([6], p. 11). We will also use the estimates (see [1], p. 97)

$$||u||_\infty \;\leq\; c_1\, ||u||_2, \qquad ||v||_{0,p} \;=\; c_2\, ||\nabla v|| \quad (1 \leq p \leq 6), \tag{2.5}$$

valid for functions $u \in H^2(G)$ and $v \in H_0^1(G)$, respectively. The completion of the set of vector functions

$$C_{0,\sigma}^\infty(G) \;=\; \{u \in C_0^\infty(G) \mid \text{div } u = 0\}$$

with respect to the norm $||\cdot||$ and $||\cdot||_1$ are basic spaces for the treatment of the Stokes and the Navier-Stokes equations. We denote theses spaces by $H(G)$ and $V(G)$, respectively. In $H_0^1(G)$ and $V(G)$ we also use

$$(\nabla u, \nabla v) \;=\; \sum_{k,j=1}^{3} (\partial_k u_j, \partial_k v_j), \qquad ||\nabla u|| \;=\; (\nabla u, \nabla u)^{\frac{1}{2}}$$

as scalar product and norm ([9], p. 5). Let $P : L^2(G) \longrightarrow H(G)$ denote Weyl's orthogonal projection ([9], p. 15) such that

$$L^2(G) \;=\; H(G) \oplus \{v \in L^2(G) \mid v = \nabla p \text{ for some } p \in H^1(G)\}.$$

Then the stationary linear Stokes system

$$-\Delta u + \nabla p \;=\; f \;\text{ in }\; G\,, \qquad \nabla \cdot u \;=\; 0 \;\text{ in }\; G\,, \qquad u_{|\partial G} \;=\; 0\,, \tag{2.6}$$

can be written as

$$Au \;=\; Pf \;\text{ in }\; G, \tag{2.7}$$

where $A = -P\Delta$ denotes the Stokes operator with domain of definition $D(A) = H^2(G) \cap V(G)$ (see [3], p. 270). It is well known (compare [2]) that for $f = Pf \in H(G)$ this system has a unique solution $u \in H^2(G) \cap V(G)$ satisfying the estimate

$$||u_2|| \;\leq\; c\,||P\Delta u|| \;=\; c\,||Au|| \;=\; c\,||f||\,. \tag{2.8}$$

Finally, we use the B-valued spaces $C^m(J,\ B)$ and $H^m(a,\ b,\ B)$, $m \in \mathbb{N}_0$. Here $J \subseteq \mathbb{R}$ is a compact interval, $a,\ b \in \mathbb{R}$ $(a < b)$, and B is any of the spaces above. For $C^0(,)$ we simply write $C(,)$, and we sometimes use $H,\ V(G),\ H^m,\ \ldots$ instead of $H(G),\ V(G),\ H^m(G),\ \ldots$, if the domain of definition is clear from the context.

All constants appearing in this note are generic, i. e. its value may be different in different estimates. Throughout the paper we deal with two kinds of constants. Constants, which may depend on the domain G (on its size or on the regularity of its boundary ∂G) and on the viscosity ν, but not on the data v_0 and F, are always denoted by $C, C_1, C_2, \ldots$ Constants, which, in addition, may also depend on the data v_0 and F, on T, and on the delay ε, are always denoted by $K, K_1, K_2, \ldots$

For the derivation of energy estimates we need the following version of Gronwall's Lemma:

Lemma 1: *Let $\alpha,\ \beta,\ \gamma \in \mathbb{R}$ $(\beta \geq 0,\ \gamma > 0)$ and let $\Phi,\ f,\ g \in C([0,T])$ be real functions $(f \geq 0,\ g \geq 0$ in $[0,T])$ satisfying for all $t \in [0,T]$ the inequality*

$$\Phi(t) + \int_0^t f(s)ds \;\leq\; \alpha + \beta t + \int_0^t g(s)ds + \int_0^t \Phi(s)ds\,. \tag{2.9}$$

Then for all $t \in [0,T]$

$$\phi(t) + \int_0^t f(s)ds \;=\; \left(\alpha + \beta t + \int_0^t g(s)ds\right) e^{\gamma t}\,. \tag{2.10}$$

Proof: Let

$$G(t) \;=\; \left(\alpha + \beta t + \int_0^t g(s)ds\right), \quad \delta > 0\,,$$

and define $\Psi(t) = (\delta + G(t))e^{\gamma t}$. Then $\Psi'(t) = \gamma\Psi(t) + G'(t)e^{\gamma t}$, which implies

$$\Psi(t) \;=\; \alpha + \delta + \gamma \int_0^t \Psi(s)ds + \beta\gamma^{-1}(e^{\gamma t} - 1) + \int_0^t g(s)e^{\gamma s}ds\,.$$

We show that $\Phi(t) < \Psi(t)$ for all $t \in [0,T]$: Obviously, $\Phi(0) \le \alpha < \alpha + \delta = \Psi(0)$. Now let $t^* > 0$ be the first position with $\Phi(t^*) = \Psi(t^*)$. Then $\Phi(t) \le \Psi(t)$ for all $t \in [0,t^*]$, hence

$$\begin{aligned}\Phi(t^*) &\le \alpha + \beta t^* + \int_0^{t^*} g(s)ds + \gamma \int_0^{t^*} \Phi(s)ds \\ &< \alpha + \beta + \gamma^{-1}(e^{\gamma t^*} - 1) + \int_0^{t^*} g(s)e^{\gamma^s} ds + \gamma \int_0^{t^*} \Psi(s)ds \;=\; \Psi(t^*)\,.\end{aligned}$$

Thus $\Phi < \Psi$ for every $\delta > 0$, which implies

$$\Phi(t) \le \left(\alpha + \beta t + \int_0^t g(s)ds\right) e^{\gamma t}$$

for all $t \in [0,T]$, and the last integral in (2.9) can be estimated by

$$\begin{aligned}&\le \int_0^t (\alpha + \beta s + \int_0^s g(r)dr)\gamma^{e^{\gamma^s}} ds \\ &= \left(\alpha + \beta t + \int_0^t g(s)ds\right) e^{\gamma^t} - \alpha - \beta\gamma^{-1}(e^{\gamma^t} - 1) - \int_0^t g(s)e^{\gamma^s} ds \\ &\le \left(\alpha + \beta t + \int_0^t g(s)ds\right) e^{\gamma^t} - \alpha - \beta t - \int_0^t g(s)ds \\ &= \left(\alpha + \beta t + \int_0^t g(s)ds\right) (e^{\gamma^t} - 1)\,.\end{aligned}$$

Addition of the other terms of the right hand side of (2.9) proves (2.10).

3 Proof of the Theorems

As a first result we prove that the stabilized equations (N_ε) for $\varepsilon > 0$ are well-posed:

Proof of Theorem 1: Let us first assume $t \in [0,\varepsilon]$. Here the system (N_ε) is linear, and a spectral Galerkin method based on the eigenfunctions of the Stokes operator as in [4] shows the existence and uniqueness of the solution $v^1, \nabla p^1$ such

that $v^1 \in C([0,\varepsilon], H^2(G) \cap V(G))$ with $\partial_t v^1 \in C([0,\varepsilon], H(G)) \cap L^2(0,\varepsilon, V(G))$ and $\nabla p^1 \in C([0,\varepsilon], L^2(G))$. Due to the orthogonality relation (see for example [9], p. 163)

$$((u \cdot \nabla)w, w) \;=\; 0 \qquad (u \in V(G), w \in H_0^1(G))\,,$$

v^1 satisfies for $t \in [0,\varepsilon]$ the energy equation (1.1). Thus from the given initial function v_0, defined for $t \in [-\varepsilon, 0]$, all regularity properties carry over to the solution $v = v^1$, defined for $t \in [0,\varepsilon]$.

Now using v^1 as new initial function, we consider the system (N_ε) for $t \in [\varepsilon, 2\varepsilon]$ only. Again, we obtain a unique solution $v^2, \nabla p^2$ having the same regularity properties on $[\varepsilon, 2\varepsilon]$ as above $v^1, \nabla p^1$ on $[0,\varepsilon]$. Because v^1 and v^2 are strongly $H^2(G)$-continuous on $[0,\varepsilon]$ and $[\varepsilon, 2\varepsilon]$, respectively, and because $v^1(\varepsilon) = v^2(\varepsilon)$, we have $\partial_t v^2(\varepsilon) = \partial_t v^1(\varepsilon)$, and the resulting solution v (defined by $v = v^1$ on $[0,\varepsilon]$, $v = v^2$ on $[\varepsilon, 2\varepsilon]$) has on $[0, 2\varepsilon]$ the asserted continuity properties, too (it should be carefully noted that the jump of the time derivative at $t = 0$, which is due to the initial construction, leads to a jump of the second order time derivative at $\varepsilon, 2\varepsilon, \ldots$, only). Repeating this procedure N times on all subintervals of length ε, the global existence, uniqueness, and conservation of energy of the solution $v, \nabla p$ is proved.

Concerning the estimates, it is sufficient to show the existence of some constant $K_1 = K_1\,(G, \varepsilon, \nu, F, v_0)$ such that (1.2) holds for the case $T = \varepsilon$. Because then, by the same procedure, we find some constant $K_2 = K_2\,(G, \varepsilon, \nu, F, v_0)$ such that (1.2) holds for the case $T = 2\varepsilon$, too. Thus, repeating this technique a finite number ($= N$) of times (ε remains fixed), the asserted estimates (1.2) are proved.

To do so, we start with the energy equation (1.1). Here we find

$$||v(t)||^2 + 2\nu \int_0^t ||\nabla v(s)||^2 ds \;\leq\; ||v_0||^2 + \int_0^t ||F(s)||^2 ds + \int_0^t ||v(s)||^2 ds\,,$$

and Lemma 1 leads to

$$\begin{aligned} ||v(t)||^2 + 2\nu \int_0^t ||\nabla v(s)||^2 ds \;&\leq\; \left(||v_0||^2 + \int_0^t ||F(s)||^2 ds \right) e^t \\ &\leq\; (||v_0||^2 + \varepsilon K_1) e^{\varepsilon}\,, \end{aligned} \tag{3.1}$$

where here K_1 depends on F only (note that (3.1) holds even for all $t \in [0,T]$ without any additional considerations). Hence, in particular, $||v(t)|| \leq K$ for all $t \in [0,\varepsilon]$.

Now multiplying the equations (N_ε) scalar in $L^2(G)$ with $-2P\Delta v(t)$, we obtain after integration with respect to t

$$||\nabla v(t)||^2 + 2\nu \int_0^t ||P\Delta v(s)||^2 ds \;\leq\; ||\nabla v_0||^2 + \int_0^t 2(||F(s)|| + ||(v_\varepsilon(s) \cdot \nabla) v(s)||) ||P\Delta v(s)|| ds\,,$$

where the integrand on the right hand side can be estimated as follows:

$$\begin{aligned} & 2(||F(s)|| + ||(v_\varepsilon(s)\cdot\nabla)v(s)||)||P\Delta v(s)|| \\ \leq\ & \nu^{-1}(||F(s)|| + ||(v_\varepsilon(s)\cdot\nabla)v(s)||)^2||P\Delta v(s)||^2 \\ \leq\ & 2\nu^{-1}(||F(s)||^2 + ||(v_\varepsilon(s)\cdot\nabla)v(s)||^2) + \nu||P\Delta v(s)||^2 \\ \leq\ & 2\nu^{-1}(||F(s)||^2 + ||v_\varepsilon(s)||_\infty^2||\nabla v(s)||^2) + \nu||P\Delta v(s)||^2\,. \end{aligned}$$

Thus Lemma 1 yields

$$||\nabla v(t)||^2 + \nu\int_0^t ||P\Delta v(s)||^2 ds \ \leq\ (||\nabla v_0||^2 + K_2\varepsilon)e^{K_2\varepsilon}$$

with some constant $K_2 = K_2(\nu, F, v_0)$. Hence, in particular, $||v(t)||_1 \leq K$ for all $t \in [0,\varepsilon]$.

Next, differentiating the equations (N_ε) with respect to t, multiplying scalar in $L^2(G)$ by $2\partial_t v$, and integrating with respect to t, implies

$$\begin{aligned} & ||\partial_t v(t)||^2 + 2\nu\int_0^t ||\nabla\partial_t v(s)||^2 ds \\ \leq\ & ||\partial_t v(0)||^2 + \int_0^t 2(||\partial_t F(s)||\ ||\partial_t v(s)|| - ((\partial_t v_\varepsilon(s)\cdot\nabla)v(s), \partial_t v(s)))ds\,, \end{aligned}$$

because $((v_\varepsilon(t)\cdot\nabla)\partial_t v(t), \partial_t v(t)) = 0$. The last summand can be estimated using the generalized Hölder inequality $||uv|| \leq ||u||_{0,3}||v||_{0,6}$, and the imbeddings (2.5) as follows:

$$\begin{aligned} -2((\partial_t v_\varepsilon(s)\cdot\nabla)v(s), \partial_t v(s)) &= +2((\partial_t v_\varepsilon(s)\cdot\nabla)\partial_t v(s), v(s)) \\ &\leq 2||\partial_t v_\varepsilon(s)||_{0,6}\ ||\nabla\partial_t v(s)||\ ||v(s)||_{0,3} \\ &\leq 2(c_1||\nabla\partial_t v_\varepsilon(s)||\ ||\nabla v(s)||)(||\nabla\partial_t v(s)||) \\ &\leq c_2||\nabla\partial_t v_\varepsilon(s)||^2||v(s)||^2 + \nu||\nabla\partial_t v(s)||^2 \\ &\leq K_3||\nabla\partial_t v_\varepsilon(s)||^2 + \nu||\nabla\partial_t v(s)||^2\,, \end{aligned}$$

with some constant $K_3 = K_3\,(G, \varepsilon, \nu, F, v_0)$. This implies

$$\begin{aligned} & ||\partial_t v(t)||^2 + \nu\int_0^t ||\nabla\partial_t v(s)||^2 ds \\ \leq\ & ||\partial_t v(0)||^2 + \int_0^t ||\partial_t F(s)||^2 ds + \int_0^t ||\partial_t v(s)||^2 ds + K_3\int_0^t ||\nabla\partial_t v_\varepsilon(s)||^2 ds \\ \leq\ & \int_0^t ||\partial_t v(s)||^2 ds + ||\nu P\Delta v_0||^2 + K_4(G, \varepsilon, \nu, F, v_0)\,, \end{aligned}$$

because $\partial_t F \in L^2(0,T,H(G))$, $\partial_t v_0 \in L^2(-\varepsilon,0,V(G))$, and because

$$||\partial_t v(0)||^2 \;\leq\; ||\nu P\Delta v_0||^2 + K_5(G,\varepsilon,\nu,F,v_0)\,.$$

Hence, in particular,

$$\int_0^\varepsilon ||\partial_t \nabla v(t)||^2 dt \;\leq\; K$$

and $||\partial_t v(t)||^2 \leq K$ for all $t \in [0,\varepsilon]$. Finally, because

$$\nu P\Delta v(t) \;=\; \partial_t v(t) + P((v_\varepsilon(t)\cdot\nabla)v(t)) - F(t)\,,$$

by means of Cattabriga's estimate [2], we find

$$||v(t)||_2 \;\leq\; C(||\partial_t v(t)|| + ||v_\varepsilon(t)||_\infty ||\nabla v(t)|| + ||F(t)||)\,,$$

hence $||v(t)||_2 \leq K$ for all $t \in [0,\varepsilon]$. This proves (1.2) and thus the theorem.

Next we show that the time delay ε in the nonlinear term can be removed again in a certain sense. In this case the resulting sequence of stabilized solutions has an accumulation point v such that v is a weak solution of the Navier-Stokes equations (N_0):

Proof of Theorem 2: Let $T \geq 0$ be given. For every $N = 1,2,\ldots$ let v^N denote the solution of the stabilized equations (N_ε), $\varepsilon = T/N > 0$, according to Theorem 1. By (3.1) and Lemma 1, for all $t \in [0,T]$ we find

$$||v^N(t)||^2 + 2\nu \int_0^t ||\nabla v^N(s)||^2 ds \;\leq\; \left(||v_0||^2 + \int_0^t ||F(s)||^2 ds\right) e^t \;\leq\; K \qquad (3.2)$$

with some constant $K = K(F,v_0)$ independent of N and $t \in [0,T]$. Because $H(G)$, as a Hilbert space, is separable, it has a complete orthonormal system $E = \{e_1, e_2, \ldots\}$ in $C^\infty_{0,\sigma}$, where we use that $C^\infty_{0,\sigma}$ is dense in $H(G)$. For $N, i \in \mathbb{N}$ consider now on $[0,T]$ the function

$$t \longrightarrow g^{i,N}(t) \;=\; (v^N(t), e_i)\,.$$

From (3.2) we find $|g^{i,N}(t)| \leq K_i$ for all $N \in \mathbb{N}$ and $t \in [0,T]$ with some constant $K_i = K_i(e_i,F,v_0)$, i. e. for every $i \in \mathbb{N}$ the function sequence $(g^{i,N})_N$ is uniformly bounded on $[0,T]$. Moreover, for every $i \in \mathbb{N}$ this sequence is also equi-continuous on $[0,T]$. This follows from the fact that the functions $g^{i,N}(N \in \mathbb{N})$ satisfy on $[0,T]$ a Lipschitz condition having the same Lipschitz constant K_i for all $N \in \mathbb{N}$:

$$\begin{aligned}(\partial_t v^N(t), e_i) \;&=\; \nu(v^N(t), P\Delta e_i) + ((v^N_\varepsilon(t)\cdot\nabla)e_i, v^N(t)) + (F(t), e_i)\\ &\leq\; \nu||v^N(t)||\;||P\Delta e_i|| + ||v^N_\varepsilon(t)||\;||\nabla e_i||_\infty ||v^N(t)|| + ||F(t)||\;||e_i|| \qquad (3.3)\\ &\leq\; K_i\,.\end{aligned}$$

Here the Lipschitz constant $K_i = K_i\ (\nu, e_i, F, v_0)$ is independent of N and $t \in [0,T]$ because of (3.2). Now applying the Arzela-Ascoli Theorem we obtain a subsequence

of $(g^{i,N})_N$ being uniformly convergent on $[0,T]$. Moreover, by a suitable diagonal procedure we can find a subsequence of $(v^N)_N$ – which we again denote by $(v^N)_N$ – such that the resulting subsequences $(g^{i,N})$ are uniformly convergent on $[0,T]$ für all $i \in \mathbb{N}$, i. e. there exists a weakly continuous function $v : [0,T] \to H(G)$ such that $v^N(t) \to v(t)$ weakly for all $t \in [0,T]$. Due to (3.2) we even may assume that $v \in L^2(0,T,V(G)) \cap L^\infty(O,T,H(G))$, and that $v^N \to v$ weakly in $L^2(0,T,V(G))$ and strongly in $L^2(0,T,H(G))$.

Next we show that v satisfies the weak Navier-Stokes equations according to Definition 1.3, where we can restrict ourselves to functions $(t,x) \to \Phi(t,x) = s(t)e_i(x)$ with some scalar function $s \in C_0^\infty((0,T))$. Because the stabilized solution v^N certainly fulfills the integral identity

$$\int_0^T \{-s'(v^N,e_i) + \nu s(\nabla v^N, \nabla e_i) - s((v_\varepsilon^N \cdot \nabla)e_i, v^N) - s(F,e_i)\}dt \;=\; 0\,,$$

and because

$$\int_0^T (v^N,e_i)dt \longrightarrow \int_0^T (v,e_i)dt\,, \qquad \int_0^T (\nabla v^N, \nabla e_i)dt \longrightarrow \int_0^T (\nabla v, \nabla e_i)dt$$

as $N \to \infty$ due to the weak convergence in $L^2(0,T,V(G))$, it remains to prove

$$\int_0^T \{((v\cdot\nabla)e_i, v) - ((v_\varepsilon^N \cdot \nabla)e_i, v^N)\}dt \longrightarrow 0 \qquad (N \to \infty)\,. \tag{3.4}$$

To do so, we use the following decomposition of the integral above:

$$\int_0^T \{(((v-v^N)\cdot\nabla)e_i, v) + ((v^N\cdot\nabla)e_i, (v-v^N)) + ((v^N - v_\varepsilon^N\cdot\nabla)e_i, v^N)\}dt \;=\; A_N + B_N + C_N\,.$$

The integrals A_N, B_N, C_N can be estimated as follows:

$$A_N \le \int_0^T ||v-v^N||\;||\nabla e_i||_\infty ||v||dt \le \left(\int_0^T ||v-v^N||^2 dt\right)^{\frac{1}{2}} \left(\int_0^T ||\nabla e||_\infty^2 ||v||^2 dt\right)^{\frac{1}{2}} \longrightarrow 0\,,$$

$$B_N \le \int_0^T ||v-v^n||\;||\nabla e_i||_\infty ||v^N||dt \le \left(\int_0^T ||v-v^N||^2 dt\right)^{\frac{1}{2}} \left(\int_0^T ||\nabla e||_\infty^2 ||v^N||^2 dt\right)^{\frac{1}{2}} \longrightarrow 0\,.$$

Here we used the strong convergence of $(v^N)_N$ in $L^2(0,T,H(G))$ and the estimate (3.2). Finally, again by (3.2),

$$C_N \le \int_0^T ||v^N - v_\varepsilon^N||\;||\nabla e_i||_\infty ||v^N||dt \le K_i \left(\int_0^\varepsilon ||v^N - v_\varepsilon^N||dt + \int_\varepsilon^T ||v^N - v_\varepsilon^N||dt\right)$$

with some constant $K_i = K_i(e_i, F, v_0)$. For the first integral we have

$$\int_0^\varepsilon ||v^N - v_\varepsilon^N|| dt \le \int_0^\varepsilon (||v^N|| + ||v_\varepsilon^N||) dt \le K\varepsilon = KT/N \longrightarrow 0 \qquad (N \to \infty)\,.$$

The second integral can be treated with Friedrichs' inequality (see [8], p. 147): Let $\delta > 0$ be given. Then there is a number $M_\delta \in \mathbb{N}$ such that

$$\begin{aligned} ||v^N - v_\varepsilon^N|| &= ||v^N(t) - v^N(t-\varepsilon)|| \\ &\le \sum_{m=1}^{M_\delta} |(v^N(t) - v^N(t-\varepsilon), e_m rangle| + \delta\{||\nabla v^N(t)|| + ||\nabla v^N(t-\varepsilon)||\}\,. \end{aligned}$$

Using (3.3) we find

$$|(v^N(t) - v^N(t-\varepsilon), e_m)| \;\le\; K_m\varepsilon \;=\; K_m T/N\,,$$

hence the finite sum above can be estimated by

$$\le \left(T\sum_{m=1}^{M_\delta} K_m\right)/N \longrightarrow 0 \qquad (N \to \infty)\,.$$

Finally, the second integral remains bounded independent of ε and N:

$$\int_\varepsilon^T \{||\nabla v^N(t)|| + ||\nabla v^N(t-\varepsilon)||\} dt \;\le\; 2\int_0^T ||\nabla v^N(t)|| dt \;\le\; 2\left(T\int_0^T ||\nabla v^N(t)||^2 dt\right)^{\frac{1}{2}} \;\le\; K\,.$$

This proves that v satisfies the weak Navier-Stokes equations. The validity of the energy inequality follows from the energy equation (1.1) for the stabilized solution by means of (see [8], p. 148)

$$\begin{aligned} ||v(t)||^2 + 2\nu\int_0^t ||\nabla v(s)||^2 ds &\le \liminf_{n\to\infty}\left(||v^N(t)||^2 + 2\nu\int_0^t ||\nabla v^N(s)||^2 ds\right) \\ &\le ||v_0||^2 + 2\int_0^t (F(s), v(s)) ds\,, \end{aligned} \tag{3.5}$$

for all $t \in [0,T]$. In particular, $\lim ||v(t)||^2 \le ||v_0||^2$ as $t \to 0^+$, which, together with the weak continuity of $v : [0,T] \to H(G)$, implies the strong continuity at initial time $t = 0$ as asserted: $||v(t) - v_0|| \to 0$ as $t \to 0^*$. This proves the theorem.

Remark: As already mentioned in the proof of Theorem 1, there is a jump of the time derivative at $t_0 = 0$, i. e. $\partial_t v \in C([0,T], H(G))$ and $\partial_t v_0 \in C([-\varepsilon, 0], H(G))$, but not necessarily $\partial_t v(t_0) = \partial_t v_0(t_0)$ (in a sense of a one-sided derivative).

This jump, which leads to a jump of the second order time derivatives at the points $\varepsilon, 2\varepsilon, \ldots$, can be avoided choosing a special initial function v_0 as follows: Because

$$\partial_t v(0) \;=\; \nu P\Delta v(0) - P((v(-\varepsilon)\cdot\nabla)v(0)) + F(0)\,,$$

we can determine $\alpha = \partial_t v(0)$, if $\beta = v(0)$, $\gamma = v(-\varepsilon)$, and $F(0)$ are given. Now, for example, choose v_0 to be a second order polynomial with $v_0(-\varepsilon) = \gamma, v_0(0) = \beta$ and $v_0'(0) = \alpha$, i. e. $t \to v_0(t) = (\gamma + \varepsilon\alpha - \beta)(t/\varepsilon)^2 + \alpha t + \beta$.

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