Construction of recurrent fractal interpolation surfaces(RFISs) on rectangular grids

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Abstract

A recurrent iterated function system (RIFS) is a genaralization of an IFS and provides nonself-affine fractal sets which are closer to natural objects. In general, it's attractor is not a continuous surface in R^3. A recurrent fractal interpolation surface (RFIS) is an attractor of RIFS which is a graph of bivariate continuous interpolation function. We introduce a general method of generating recurrent interpolation surface which are attractors of RIFSs about any data set on a grid.

Keywords: Recurrent Iterated function system(RIFS); Fractal interpolation function(FIF); Box-counting dimension

1 Introduction

Fractal interpolation surfaces (FISs) are graphs of bivariate fractal interpolation functions (FIFs), which have been used in approximation theory, computer graphics, image compression, metallurgy, physics, geography, geology and so on. Barnsley (1986, [3]) introduced the idea of a FIF as a function whose graph is an attractor of IFS, which by many scientists (Massopust, Elton, Hardin, Geronimo, Zhao, Malysz, Bouboulis etc.) has widely been studied and applied.

Massopust (1990, [16]) presents the construction of self-affine FISs on triangular data sets, where the interpolation points on the boundary data are coplanar, which by Geronimo and Hardin (1993, [13]) was generalized to allow more general boundary data and by Zhao (1996, [18]) more general vertical scaling factor.

Dalla (2002, [10]) introduced the construction of a FIS by an IFS in the case where the interpolation points on the boundary data sets on a grid are collinear and Malysz (2006, [15]) generalized this method to allow more general data set on the grid, where the free contrativity factor is constant. The FISs generated by the above construction are all self-affine. By Metzler and Yun (2008, [17]), the construction of FISs with a free vertical contractivity factor function on grids was presented which generalizes the results in [15].

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There are no objects in nature which have an exact structure of a self-affine set, which is nothing but special model of a fractal for modelling natural objects. That requires more flexible constructions of FISs, for which a recurrent iterated function system (RIFS) is a good way. Bouboulis etc. (2006, [6], 2007, [8]) present more general constructions of nonself-affine FISs on grids by RIFS and apply to image compression, which has a lack because the free vertical constractivity factor is still there constant.

In this paper, we introduce a more flexible construction of recurrent fractal interpolation surfaces which are attractors of RIFSs with a free vertical contractivity factor function on grids and consider the construction in $\mathbf{R}^N$.

2 Recurrent fractal interpolation surfaces (RFISs) on rectangular grid

In general, a recurrent iterated function system (RIFS) is defined as a pair of a collection $w_1, \ldots, w_N$ of Lipschitz mappings in complete metric space (i.e. an IFS) and a row irreducible stochastic matrix $P = (p_{ij})_{N\times N}$ satisfying $\sum_{j=1}^{N} p_{ij} = 1$ for all $i \in \{1, \ldots, N\}$, and for any $i,\ j \in \{1, \ldots, N\}$, there exist $i_1, \ldots, i_k$ such that $p_{ii_1} \cdot p_{i_1 i_2} \cdot \ldots \cdot p_{i_k j} > 0$ (Barnsley, Elton and Hardin[4]). The attractor A of a RIFS is computed as follows: By a stochastic matrix P, for an initial $k_0 \in \{1, \ldots, N\}$ a sequence $\{k_i\}_0^\infty$ is given, which obeys $p_{ab} = \Pr\{k_{i+1} = b | k_i = a\}$. With this, we get a sequence of transformations $\{w_{k_i}\}_0^\infty$ that generates an orbit $\{q_i\}_0^\infty$ for an initial $q_0 \in \mathbf{R}^3$, that is, k_{i+1} is chosen with the probability that $k_{i+1} = j$ equals to $p_{k_i j}$ and then we have $q_{i+1} = w_{k_{i+1}}(q_i)$. In the matrix (p_{kl}), p_{kl} shows the possibility of applying the transformation w_l to the point in state k, so that the system transits to state l. The attractor A is defined as a limit set of these orbits, which consists of the points whose every neighbourhood contains infinitely many q_i for almost all orbits. By Barnsley, Elton and Hardin ([4]) the existance, uniqueness and characteristics of this limit set A was proved. Usually, a RIFS is constructed with a given data set in a complete metric space and it's attractor generally is not a graph of a continuous interpolaion function about the data set. The attractor A which is a graph of a bivariate continuous fractal interpolation function of a given data set in $\mathbf{R}^3$ is called a recurrent fractal interpolation surface (RFIS).

We present a construction of RIFS, whose attractor is a graph of fractal interpolation function of a given data set on a grid. To do this, an IFS and an irreducible row stochastic matrix P should be defined.

2.1 Local IFSs

Because an attractor of RIFS depends on the IFS and only a connection matrix $C = (c_{kl})_{N\times N}$ defined by the stochastic matrix P as follows:

$$c_{kl} = \begin{cases} 1 & : p_{lk} > 0 \\ 0 & : p_{lk} = 0 \end{cases}, \qquad (1)$$

we consider the IFS and the connection matrix C.

We construct a local IFS $\left\{\mathrm{R}^3; w_{ij} = (L_{ij}, F_{ij}),\ i = 1, \ldots, m,\ j = 1, \ldots, n\right\}$ over the grid, whose general definition is as follows ([2])(An idea of such IFS was introduced by Jacquin ([14])).

Definition 1 *Let $(\mathbf{X}, d)$ be a compact metric space. Let R be a nonempty subset of $\mathbf{X}$. Let $w : R \to \mathbf{X}$ and let s be a real number with $0 \le s < 1$. If*

$$d(w(x),\ w(y)) \le s \cdot d(x, y) \quad for\ all\ x,\ y\ in\ R,$$

then w is called a local contraction mapping *on* $(\mathbf{X}, d)$*. The number s is a contractivity factor for w.*

Definition 2 *Let $(\mathbf{X}, d)$ be a compact metric space, and let $w_i : R_i \to \mathbf{X}$ be a local contraction mapping on $(\mathbf{X}, d)$, with contractivity factor s_i, for $i = 1, 2, \ldots, N$, where N is a finite positive integer. Then*

$$\{w_i : R_i \to \mathbf{X} \; : \; i = 1, 2, \ldots, N\}$$

is called a local iterated function system (local IFS) *(or* partitioned IFS ([*12*])*). The number $s = \max\{s_i \; : \; i = 1, 2, \ldots, N\}$ is called the contractivity factor of the local IFS.*

Let the data set on the rectangular grid be

$$\mathrm{S} = \left\{(x_i, y_j, z_{ij}) \in \mathbf{R}^3; \; i = 0, 1, \ldots, m, \; j = 0, 1, \ldots, n\right\},$$

such that $x_0 < x_1 < \ldots < x_m, \; y_0 < y_1 < \ldots < y_n$. Let denote

$$\begin{aligned}
&\mathrm{N}_{mn} = \{1, \ldots, m\} \times \{1, \ldots, n\}, \; \mathrm{I}_x = [x_0, x_m], \; \mathrm{I}_y = [y_0, y_n],\\
&\mathrm{I}_{x_i} = [x_{i-1}, x_i], \; \mathrm{I}_{y_j} = [y_{j-1}, y_j], \; \mathrm{E} = \mathrm{I}_x \times \mathrm{I}_y\\
&\mathrm{E}_{ij} = \mathrm{I}_{x_i} \times \mathrm{I}_{y_j} \text{ (which we call the } \textit{region}\text{)}, \quad \text{for } (i,j) \in \mathrm{N}_{mn},\\
&\mathrm{P}_{x_\alpha} = \{(x_\alpha, y_l, z_{\alpha l}) \in \mathrm{S}; \; l = 0, 1, \ldots, n\}, \quad \text{for } \alpha \in \{0, \ldots, m\},\\
&\mathrm{P}_{y_\beta} = \{(x_k, y_\beta, z_{k\beta}) \in \mathrm{S}; \; k = 0, 1, \ldots, m\}, \quad \text{for } \beta \in \{0, \ldots, n\}.
\end{aligned}$$

Let choose an interval $\tilde{\mathrm{I}}_{x_i}$ to be $\left[x'_{i_{i-1}}, x'_{i_i}\right] = \bigcup_{k=1}^{m_i} \mathrm{I}_{x_k}$ for $i \in \{1, \ldots, m\}$ (then, there exist $x_{k_1}, \; x_{k_2} \in [0, m]$ such that $x'_{i_{i-1}} = x_{k_1}$ and $x'_{i_i} = x_{k_2}$, which we denote by $\gamma_x\left(x'_{i_{i-1}}\right)$ and $\gamma_x\left(x'_{i_i}\right)$ respectively) such that

(i) For any $i \in \{1, \ldots, m\}, \; x_i - x_{i-1} < x'_{i_i} - x'_{i_{i-1}}$

(ii) For any $i \in \{2, \ldots, m-1\}$, there exist $x'_{i-1_\alpha} \in \{x'_{i-1_{i-2}}, x'_{i-1_{i-1}}\}$, $x'_{i+1_\beta} \in \{x'_{i+1_i}, x'_{i+1_{i+1}}\}$ such that $x'_{i-1_\alpha} = x'_{i_{i-1}}$ and $x'_{i_i} = x'_{i+1_\beta}$, which we denote these points - the endpoints of connecting the intervals $\tilde{\mathrm{I}}_{x_i}$ - by $\tilde{x}_{i-1,i}$ and $\tilde{x}_{i,i+1}$ respectively.

An interval $\tilde{\mathrm{I}}_{y_j}$ has the same form for $j \in \{1, \ldots, n\}$. We denote $\tilde{\mathrm{E}}_{ij} = \tilde{\mathrm{I}}_{x_i} \times \tilde{\mathrm{I}}_{y_j}$ and call $\tilde{\mathrm{E}}_{ij}$ the *domain*.

We define the domain contraction transformations $L_{ij} : \tilde{\mathrm{E}}_{ij} \to \mathrm{E}_{ij}$, for $(i,j) \in \mathrm{N}_{mn}$ by

$$L_{ij}(x, y) = \left(L_{x_i}(x), L_{y_j}(y)\right),$$

where $L_{x_i} : \tilde{\mathrm{I}}_{x_i} \to \mathrm{I}_{x_i}, \; L_{y_j} : \tilde{\mathrm{I}}_{y_j} \to \mathrm{I}_{y_j}$ are contractive homeomorphisms with contractivity factors $a_{x_i}, \; a_{y_j}$ obeying

(i) For any $i \in \{1, \ldots, m\}, \; j \in \{1, \ldots, n\}$,

$$L_{x_i} : \left\{x'_{i_{i-1}}, x'_{i_i}\right\} \to \{x_{i-1}, x_i\}, \qquad L_{y_j} : \left\{y'_{j_{j-1}}, y'_{j_j}\right\} \to \{y_{j-1}, y_j\},$$

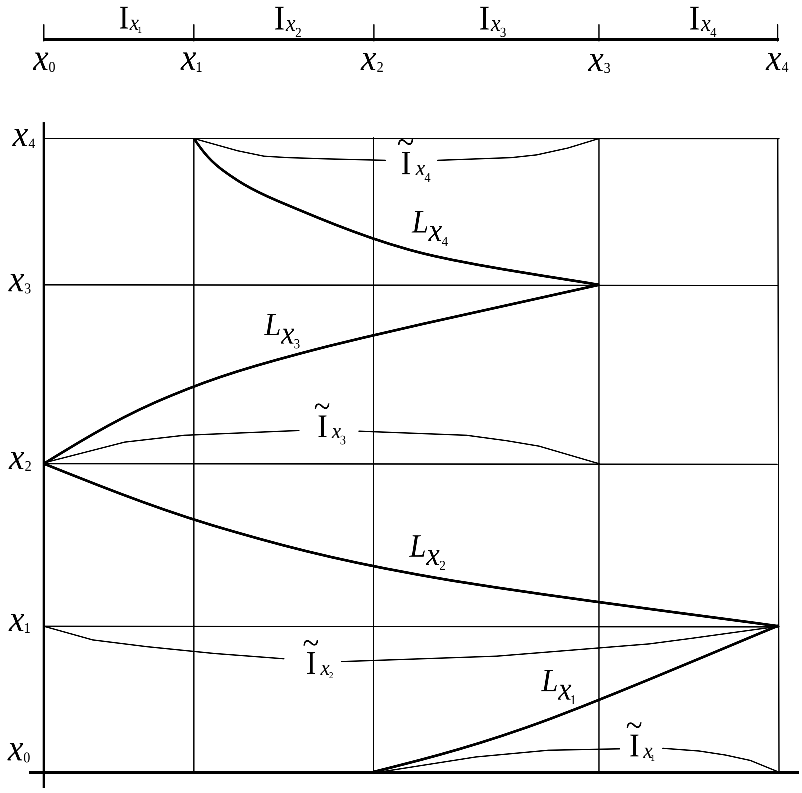

Figure 1: Shape of L_{x_i}

(ii) For any $i \in \{1, \ldots, m-1\}$, $j \in \{1, \ldots, n-1\}$, there exist $\tilde{x}_{i,i+1}$, $\tilde{y}_{j,j+1}$ such that

$$L_{x_{i+1}}(\tilde{x}_{i,i+1}) = L_{x_i}(\tilde{x}_{i,i+1}) = x_i,\ L_{y_{j+1}}(\tilde{y}_{j,j+1}) = L_{y_j}(\tilde{y}_{j,j+1}) = y_j. \tag{2}$$

(See Figure 1). Denote $a_{ij} = \mathrm{Max}\left\{a_{x_i}, a_{y_j}\right\}$, for $(i, j) \in \mathrm{N}_{mn}$. Then, the a_{ij} are contractivity factors of the transformations L_{ij}.

Let $F_{ij} : \tilde{\mathrm{E}}_{ij} \times \mathbf{R} \to \mathbf{R}$, for $(i, j) \in \mathrm{N}_{mn}$ be defined by

$$F_{ij}(x, y, z) = d(L_{ij}(x, y))(z - g(x, y)) + h(L_{ij}(x, y)), \tag{3}$$

where $d(x, y)$ is a vertical continuous contraction such that $|d(x, y)| < 1$ on E , $h(x, y)$ and $g(x, y)$ are continuous Lipschitz mappings on E with the Lipschitz constants L_h, L_g satisfying

$$\begin{aligned} &g\left(x'_{i_\alpha}, y'_{j_\beta}\right) = z_{\tilde{\gamma}(i_\alpha, j_\beta)}, && \text{for } (\alpha, \beta) \in \{i-1, i\} \times \{j-1, j\}, \\ &h(x_i, y_j) = z_{ij}, && \text{for } (i, j) \in \mathrm{N}_{mn}, \end{aligned}$$

where $\tilde{\gamma}(i_\alpha, j_\beta) = \tilde{\gamma}\left(\gamma_x\left(x'_{i_\alpha}\right), \gamma_y\left(y'_{j_\beta}\right)\right) = \tilde{\gamma}(x_k, y_l) = (k, l)$, that is, g goes through 4 end-points of $\tilde{\mathrm{E}}_{ij}$, h all data points of E. Then, the F_{ij} satisfy 'join up' conditions, for $\alpha \in \{i-1, i\}$, $\beta \in \{j-1, j\}$,

$$F_{ij}\left(x'_{i_\alpha}, y'_{j_\beta}, z_{\tilde{\gamma}(i_\alpha, j_\beta)}\right) = z_{\sigma\left(L_{ij}\left(x'_{i_\alpha}, y'_{j_\beta}\right)\right)},$$

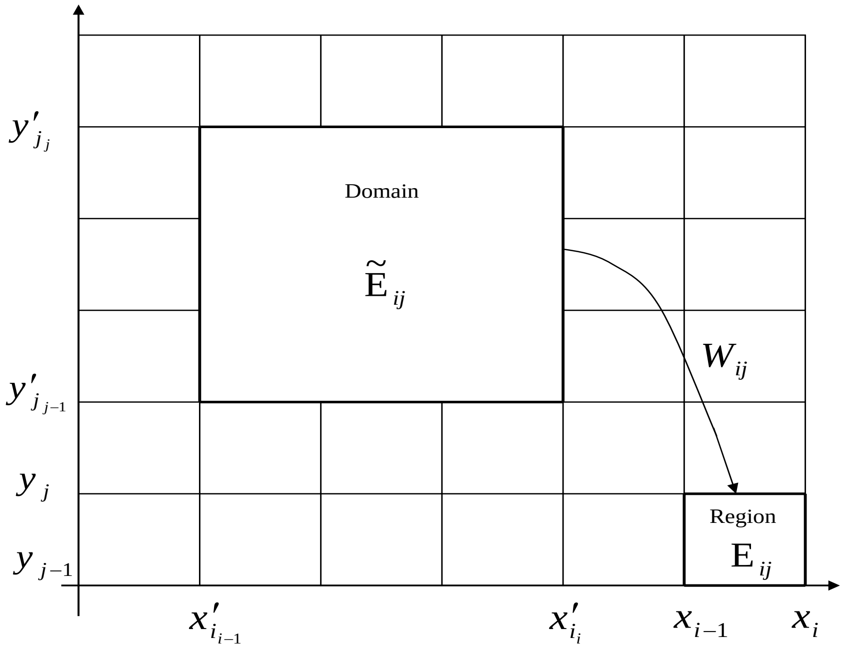

Figure 2: Mapping the domain to the region

where $\sigma\left(L_{ij}\left(x'_{i_\alpha}, y'_{j_\beta}\right)\right) = \sigma\left(x_a, y_b\right) = (a, b) \in \{i-1, i\} \times \{j-1, j\}$. By (2), (3) we have on the common borders $\{x_i\} \times [y_{j-1}, y_j]$ for $(i, j) \in \{1, \ldots, m-1\} \times \{1, \ldots, n\}$ ($[x_{i-1}, x_i] \times \{y_j\}$ for $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n-1\}$)

$$F_{i+1\ j}\left(\tilde{x}_{i,i+1}, y, z\right) = F_{ij}\left(\tilde{x}_{i,i+1}, y, z\right)$$
$$\left(F_{i\ j+1}\left(x, \tilde{y}_{j,j+1}, z\right) = F_{ij}\left(x, \tilde{y}_{j,j+1}, z\right)\right),$$

where $\tilde{x}_{i,i+1},\ x_i,\ \tilde{y}_{j,j+1},\ y_j$ obey (2).

Hence, for $(i, j) \in \mathrm{N}_{mn}$ the transformations w_{ij} coincide on common borders.

Fourthermore, there exists some metric ρ that is equivalent to the Euclidean metric on $\mathbf{R}^3$ such that the w_{ij} are contractions for all $(i, j) \in \mathrm{N}_{mn}$ with respect to ρ, which is given on $\mathbf{R}^3$ for $(x, y, z), (x', y', z') \in \mathbf{R}^3$ by

$$\rho\left((x, y, z), (x', y', z')\right) = |x - x'| + |y - y'| + \theta\,|z - z'|,$$

where

$$\theta = \frac{1 - \mathrm{Max}\left\{a_{x_i},\ a_{y_j};\ i = 1, \ldots, m,\ j = 1, \ldots, n\right\}}{2\mathrm{Max}\left\{d_{max}\mathrm{L}_g + \mathrm{L}_h a_{x_i},\ d_{max}\mathrm{L}_g + \mathrm{L}_h a_{y_j};\ i = 1, \ldots, m,\ j = 1, \ldots, n\right\}}$$

and $d_{max} = \mathrm{Max}_{\mathrm{E}}\,|d(x, y)|\,,\ d_{min} = \mathrm{Min}_{\mathrm{E}}\,|d(x, y)|\,.$ A contractivity of w_{ij} is $\mathrm{Max}\,\{a, d_{max}\}$, where

$$a = \frac{1 + \mathrm{Max}\left\{a_{x_i},\ a_{y_j};\ i = 1, \ldots, m,\ j = 1, \ldots, n\right\}}{2} < 1.$$

Remark 1. To be simple, let the endpoints of interval $[x_0,\ x_m]$ be 0 and 1. If $\tilde{\mathrm{E}}_{ij} = \mathrm{E}$ for any $(i, j) \in \mathrm{N}_{mn}$, then the above L_{x_i} can take two ways of corresponding endpoints of

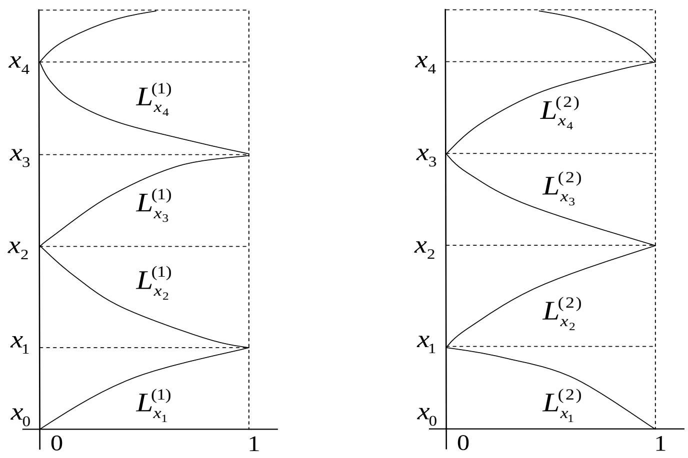

Figure 3: Shapes of $L^{(1)}_{x_i}, \quad L^{(2)}_{x_i}$

intervals as follows:

$$L^{(1)}_{x_i}(\alpha) = \begin{cases} x_{i-1+\alpha} & i : \text{odd} \\ x_{i-1+(1-\alpha)} & i : \text{even} \end{cases}$$

or

$$L^{(2)}_{x_i}(\alpha) = \begin{cases} x_{i-1+(1-\alpha)} & i : \text{odd} \\ x_{i-1+\alpha} & i : \text{even} \end{cases}$$

for $\alpha \in \{0, 1\}$. $L^{(1)}_{y_j}, \ L^{(2)}_{y_j}$ are the same forms. Thus, for L_{ij} there are 4 cases(See Figure 3):

$$\left(L^{(1)}_{x_i}, \ L^{(1)}_{y_j}\right), \ \left(L^{(1)}_{x_i}, \ L^{(2)}_{y_j}\right), \ \left(L^{(2)}_{x_i}, \ L^{(1)}_{y_j}\right), \ \left(L^{(2)}_{x_i}, \ L^{(2)}_{y_j}\right).$$

In the paper[15], L_{ij} has the form $\left(L^{(1)}_{x_i}, \ L^{(1)}_{y_j}\right)$, where $L^{(1)}_{x_i}, \ L^{(1)}_{y_j}$ are linear mappings.

Remark 2. $d\left(L_{ij}(x, y)\right)$ is the vertical contraction factor function on the region E_{ij}. In (3), $d\left(L_{ij}(x, y)\right)$ can be replaced by $d\left(L^u_{x_i}(x), \ L^v_{y_j}(y)\right)$, where

$$L^u_{x_i}(x) = \begin{cases} L^{\theta_{i_1}}_{x_{i_1}} \circ \cdots \circ L^{\theta_{i_u}}_{x_{i_u}}(x) & u \in \mathbf{Z}^+ \\ x & u = 0 \end{cases} \tag{4}$$

and $\theta_{i_k} \in \{-1, \ 1\}, \ i_k \in \{1, \ldots, m\}, \ k = 1, \ldots, u$. $L^v_{y_j}(y)$ is of the same form. This can improve more flexibility of the IFS, but normally the transformations w_{ij}, can not coincide on common borders, excepting the case where in (3) d is given by $d\left(L_{x_i}(x), \ L_{y_j}(y)\right)$ and $d(x, y)$ from (4).

2.2 Row stochastic matrix P

We enumerate the set $\mathrm{N_{nm}} = \{1, \ldots, m\} \times \{1, \ldots, n\}$ by a injective mapping $\tau : \{1, \ldots, m\} \times \{1, \ldots, n\} \to \{1, \ldots, m \cdot n\}$ and denote $\mathrm{M} = \tau\left(\mathrm{N_{mn}}\right)$. For simplicity, we denote $(i, j) = \tau^{-1}(k)$ by k and $m \cdot n$ by N.

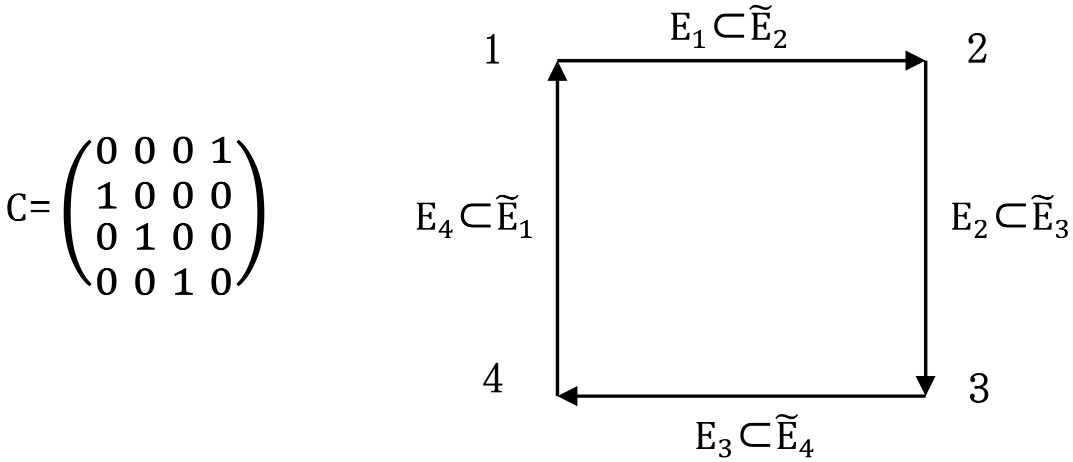

Figure 4: Connection matrix and it's directed graph

On the basis of the above construction of local IFS, we define the connection matrix $C = (c_{kl})_{N\times N}$ as follows:

$$c_{kl} = \begin{cases} 1 & : \mathrm{E}_l \subseteq \tilde{\mathrm{E}}_k \\ 0 & : otherwise \end{cases}$$

The connection matrix C shows that the transformation w_k can follow the transformation w_l iff $c_{kl} = 1$. Because this matrix C hase to be irreducible, in the above construction of the IFS , the relation between the domain $\tilde{\mathrm{E}}_i$ and the region E_j for $i, j \in \{1, \ldots, N\}$ should be taken so that this condition is satisfied. For example, for any $i \in \{2, \ldots, N\}$, $\mathrm{E}_{i-1} \subset \tilde{\mathrm{E}}_i$ (or $\mathrm{E}_i \subset \tilde{\mathrm{E}}_{i-1}$) and $\mathrm{E}_N \subset \tilde{\mathrm{E}}_1$ (or $\mathrm{E}_1 \subset \tilde{\mathrm{E}}_N$)(See Figure 4).

Then we have a row irreducible stochastic matrix $P = (p_{kl})_{N\times N}$ by (1).

A pair consisting of the above local IFS and the row irreducible stochastic matrix P is the RIFS. The existence, uniqueness and characteristic of the attractor A of this RIFS has been proved by Barnsley Elton and Hardin ([4]).

2.3 Attractor of RIFS

In this section, we review some of the work of Barnsley, Elton and Hardin [4] on the attractor A of RIFS. Let (K_i, d_i) be compact metric spaces for $i \in \{1, \ldots, N\}$, (H_i, h_i) denote the associated metric spaces of nonempty compact subsets which use the Hausdorff metrics and $\mathrm{H}^N = \mathrm{H}_1 \times \cdots \times \mathrm{H}_N$. The transformation $W : \mathrm{H}^N \to \mathrm{H}^N$ is defined by

$$W = \begin{pmatrix} c_{11}w_1 & c_{12}w_1 & \ldots & c_{1N}w_1 \\ c_{21}w_2 & c_{22}w_2 & \ldots & c_{2N}w_2 \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1}w_N & c_{N2}w_N & \ldots & c_{NN}w_N \end{pmatrix},$$

for $\mathrm{B} = (\mathrm{B}_1, \ldots, \mathrm{B}_N) \in \mathrm{H}^N$,

$$\begin{aligned} W(\mathrm{B}) &= \begin{pmatrix} c_{11}w_1(\mathrm{B}_1) \bigcup c_{12}w_1(\mathrm{B}_2) \bigcup \ldots \bigcup c_{1N}w_1(\mathrm{B}_N) \\ c_{21}w_2(\mathrm{B}_1) \bigcup c_{22}w_2(\mathrm{B}_2) \bigcup \ldots \bigcup c_{2N}w_2(\mathrm{B}_N) \\ \vdots \\ c_{N1}w_N(\mathrm{B}_1) \bigcup c_{N2}w_N(\mathrm{B}_2) \bigcup \ldots \bigcup c_{NN}w_N(\mathrm{B}_N) \end{pmatrix} \\ &= \begin{pmatrix} \bigcup_{j\in\Lambda(1)} w_1(\mathrm{B}_j) \\ \bigcup_{j\in\Lambda(2)} w_2(\mathrm{B}_j) \\ \vdots \\ \bigcup_{j\in\Lambda(N)} w_N(\mathrm{B}_j) \end{pmatrix}, \end{aligned}$$

where $\Lambda(i) = \{j : \ c_{ij} = 1\}$ for $i \in \{1, \ldots, N\}$. For example, in the case where $P = \begin{pmatrix} 0.3 & 0.7 \\ 0 & 1 \end{pmatrix}$,

$$C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, W = \begin{pmatrix} w_1 & 0 \\ w_2 & w_2 \end{pmatrix},$$

$$W(\mathrm{B}) = \begin{pmatrix} w_1(\mathrm{B}_1) \\ w_2(\mathrm{B}_1) \bigcup w_2(\mathrm{B}_2) \end{pmatrix}.$$

Then, an unique invariant set A of this transformation W exists , which is called an attractor of the RIFS :

$$W(\mathrm{A}) = \mathrm{A}$$

i.e. there exist the unique nonempty compact sets $\mathrm{A}_1, \ldots, \mathrm{A}_N$ such that

$$\mathrm{A}_i = \bigcup_{k\in\Lambda(i)} w_i(\mathrm{A}_k),$$

$$\mathrm{A} = \bigcup_{i=1}^{N} \mathrm{A}_i = \bigcup_{k\in\Lambda(i)} w_i(\mathrm{A}_k). \tag{5}$$

2.4 Recurrent fractal interpolation surfaces

The following theorem shows that this attractor A is a recurrent fractal interpolation surface.

Theorem 1 *Let the set* A *be the attractor of the above RIFS of data set* S*. Then, there exists a continuous interpolation function of data set* S *whose graph is the attractor* A*.*

Proof Let denote

$$\begin{aligned} &\mathbf{C}(\mathrm{E}) = \left\{\varphi \in \mathbf{C}^0(\mathrm{E}) : \ \varphi(x_i, y_j) = z_{ij}, \ i = 0, \ldots, m, \ j = 0, 1 \ldots, n\right\}, \\ &\mathbf{F}(\mathrm{E}) = \{f | f : \mathrm{E} \to \mathbf{R}\}. \end{aligned}$$

Defining an operator $T : \mathbf{C}(\mathrm{E}) \to \mathbf{F}(\mathrm{E})$ by

$$(T\varphi)(x, y) = F_{ij}\left(L_{x_i}^{-1}(x), \ L_{y_j}^{-1}(y), \ \varphi\left(L_{x_i}^{-1}(x), L_{y_j}^{-1}(y)\right)\right), \quad \text{for } (x, y) \in \mathrm{E}_{ij},$$

it can be easily proved that the operator T has the following properties:

Table 1: The interpolation points of RIFS.

	x				
y	0	50	100	150	200
0	45	30	80	60	50
50	65	80	70	90	40
100	80	70	100	70	30
150	70	80	60	90	60
200	50	48	70	50	40

Table 2: The interpolation points of the free vertical contractivity function $d(x, y)$.

	x								
y	0	25	50	75	100	125	150	175	200
0	0.75	0.90	0.60	0.60	0.74	0.65	0.80	0.70	0.80
25	0.65	0.80	0.70	0.90	0.84	0.75	0.60	0.60	0.70
50	0.80	0.70	0.90	0.70	0.30	0.85	0.80	0.50	0.70
75	0.70	0.80	0.60	0.90	0.60	0.65	0.90	0.80	0.60
100	0.65	0.48	0.70	0.50	0.40	0.50	0.80	0.70	0.80
125	0.45	0.30	0.80	0.60	0.50	0.65	0.70	0.60	0.70
150	0.65	0.80	0.70	0.90	0.84	0.85	0.60	0.70	0.60
175	0.80	0.70	0.90	0.70	0.30	0.75	0.70	0.60	0.70
200	0.70	0.80	0.60	0.90	0.60	0.65	0.60	0.90	0.60

(i) $T\varphi \in \mathbf{C}(\mathrm{E})$

(ii) T is contractive in the sup-norm $\|\cdot\|_\infty$ with contractivity factor d_{max}.

Thus, according to the fixed point theorem in the complete metric space $\mathbf{C}(\mathrm{E})$, the operator T has a unique fixed point $f \in \mathbf{C}(\mathrm{E})$ with

$$f(L_i(x, y)) = F_i(x, y, f(x, y)), \qquad \text{for } (x, y) \in \tilde{\mathrm{E}}_i,\ i \in \{1, \dots, N\}.$$

This means that

$$Gr(f) = \bigcup_{i=1}^{N} \bigcup_{k \in \Lambda(i)} w_i\left(Gr\left(f\,|_{\mathrm{E}_k}\right)\right). \tag{6}$$

by (5), (6), we have $Gr(f) = \mathrm{A}$. □

Example. Figure 5 shows the RFIS of the data set in Table 1 with the vertical contractivity factor function $d(x, y)$ which is given by a polynomial interpolation of a data set in Table 2. The transformation $L_{x_i},\ L_{y_j}$ are linear one, the function $g,\ h$ are defined by the polynomial interpolation. Here $d(x, y),\ g,\ h$ can be defined by the fractal interpolation.

3 Box-counting dimension of RFIS

In general, it is not easy to calculate a fractal dimension of the attractor of a RIFS. In the case where the attractor is a RFIS, we estimate the box-counting dimension of a RFIS, which is denoted by $\dim_{\mathrm{B}}(\mathrm{A})$.

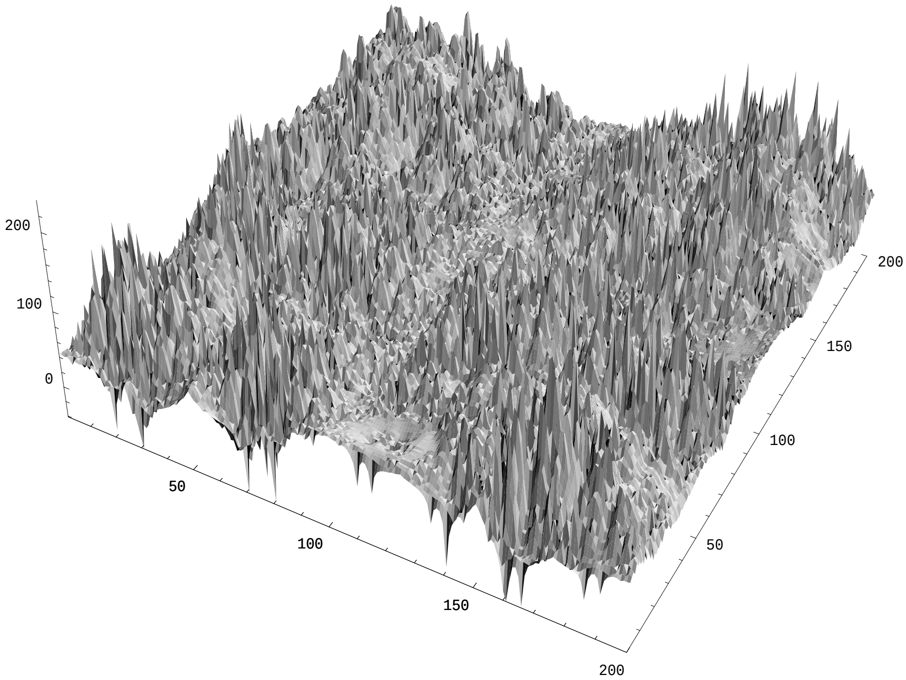

Figure 5: RFIS of data set in Table 1 with the vertical contractivity factor function $d(x,y)$ which is a polynomial interpolation function of a data set in Table 2

Theorem 2 *In th section 2, let the transformation L_i be similitude with scaling a for $i \in \{1,\ldots,N\}$, $d(x,y) = d_0$ and* A *be the RFIS. If there exist $k \in \{1,\ldots,N\}$ and $\alpha \in \{0,\ldots,m\}$ (or $\beta \in \{0,\ldots,n\}$) such that the points of $\mathrm{P}_{x_\alpha} \cap \tilde{\mathrm{E}}_k$ (or $\mathrm{P}_{y_\beta} \cap \tilde{\mathrm{E}}_k$) are noncolinear, and*

$$\rho\left(\mathrm{diag}\left(|d_0|\,a\right)C\right) > 1,$$

then $\dim_{\mathrm{B}}(\mathrm{A})$ *is the unique* D *such that*

$$\rho\left(\mathrm{diag}\left(|d_0|\,a^{\mathrm{D}-1}\right)C\right) = 1,$$

otherwise $\dim_{\mathrm{B}}(\mathrm{A}) = 2$. *Here $\rho(B)$ denotes the spectral radius of the matrix B.*

Proof The proof is similar to Theorem 4.2 in [4], THEOREM 4.1 in [13]. □

4 Construction in $\mathbf{R}^N$

We consider the construction of RIFSs with the data set on grid in $\mathbf{R}^N$. The idea is the same as that in $\mathbf{R}^3$. Therefore, we present the results. Let the data set be denoted by

$$\mathrm{S} = \left\{(x_{1,i_1}, x_{2,i_2}, \ldots, x_{M,i_M}, z_{i_1,i_2,\ldots,i_M}) \in \mathbf{R}^{M+1};\ i_k = 0,1,\ldots,m_k,\ m_k \in \mathbf{N},\ k = 1,\ldots,M\right\},$$

where $x_{k,0} < x_{k,1} < \ldots < x_{k,m_k}$ for $k \in \{1,\ldots,M\}$, $M,\ m_k \in \mathbf{N}$, and denote $\mathrm{P}_{k,i_k} = \left\{(x_{1,i_1}, \ldots, x_{k,i_k}, \ldots, x_{M,i_M}, z_{i_1,\ldots,i_k,\ldots,i_M}) \in \mathrm{P};\ i_l = 0,\ldots,m_l,\ l = 1,\ldots,k-1,k+1,\ldots,M\right\}$, for $k \in \{1, \ldots, M\}$, $i_k \in \{0, \ldots, m_k\}$. We denote

$$\begin{aligned}
&\mathrm{I}_k = [x_{k,0}, x_{k,m_k}] , \quad \mathrm{I}_{k,l} = [x_{k,l-1}, x_{k,l}] , \\
&\mathrm{E}_{\mathrm{i}} = \mathrm{I}_{1,i_1} \times \ldots \times \mathrm{I}_{M,i_M} \text{ (which we call the } \textit{region}) , \\
&\Omega = \{(1, i_1) , \ldots , (M, i_M) ; \ i_k = 1, \ldots , m_k, \ k = 1, \ldots , M\} ,
\end{aligned} \tag{7}$$

where $\mathrm{i} \in \Omega$, $i_k \in \{1, \ldots, m_k\}$, $k \in \{1, \ldots, M\}$. Then $\mathrm{E} = \mathrm{I}_1 \times \ldots \times \mathrm{I}_M = \bigcup_{\mathrm{i}} \mathrm{E}_{\mathrm{i}}$, $\mathrm{I}_k = \bigcup_{l=1}^{m_k} \mathrm{I}_{k,l}$. Let choose an interval $\tilde{\mathrm{I}}_{k,l}$ to be $\left[x'_{k,l_{l-1}}, x'_{k,l_l}\right] = \cup_j \mathrm{I}_{k,j}$ for $k \in \{1, \ldots, M\}$ (then there exist x_{k,l_1}, $x_{k,l_2} \in [0, \ m_k]$ such that $x'_{k,l_{l-1}} = x_{k,l_1}$ and $x'_{k,l_l} = x_{k,l_2}$, which we denote by $\gamma_k \left(x'_{k,l_{l-1}}\right)$ and $\gamma_k \left(x'_{k,l_l}\right)$ respectively) such that

(i) For any $l \in \{1, \ldots, m_k\}$, $x_{k,l} - x_{k,l-1} < x'_{k,l_l} - x'_{k,l_{l-1}}$

(ii) For any $l \in \{2, \ldots, m_k - 1\}$, there exist $x'_{k,l-1_\alpha} \in \left\{x'_{k,l-1_{l-2}}, \ x'_{k,l-1_{l-1}}\right\}$, $x'_{k,l+1_\beta} \in \left\{x'_{k,l+1_l}, \ x'_{k,l+1_{l+1}}\right\}$ such that $x'_{k,l-1_\alpha} = x'_{k,l_{l-1}}$ and $x'_{k,l_l} = x'_{k,l+1_\beta}$ i.e. the endpoints of connecting the intervals $\tilde{\mathrm{I}}_{k,l}$, which we denote by $\tilde{x}_{k,l-1,l}$ and $\tilde{x}_{k,l,l+1}$ respectively, generally $\tilde{x}_{k,j,j+1}$ for $j \in \{1, \ldots, m_k - 1\}$ (which connects the interval $\tilde{\mathrm{I}}_{k,j}$ and the interval $\tilde{\mathrm{I}}_{k,j+1}$). We denote $\tilde{\mathrm{E}}_{\mathrm{i}} = \tilde{\mathrm{I}}_{1,i_1} \times \ldots \times \tilde{\mathrm{I}}_{M,i_M}$ for $\mathrm{i} \in \Omega$, which we call the *domain*. Then $\tilde{\mathrm{E}}_{\mathrm{i}}$ contains some of regions E_{i} for $\mathrm{i} \in \Omega$.

We construct a local IFS$\left\{\mathbf{R}^{M+1}; \ W_{\mathrm{i}} = (L_{\mathrm{i}}, F_{\mathrm{i}}) ; \mathrm{i} \in \Omega\right\}$. The contraction transformations from the domain to the region $L_{\mathrm{i}} : \tilde{\mathrm{E}}_{\mathrm{i}} \to \mathrm{E}_{\mathrm{i}}$ with contractivity factors a_{i} are defined by

$$L_{\mathrm{i}} (x_1, \ldots, x_M) = (L_{1,i_1} (x_1) , \ldots , L_{M,i_M} (x_M)) ,$$

where $L_{k,i_k} : \tilde{\mathrm{I}}_{k,i_k} \to \mathrm{I}_{k,i_k}$, for $i_k \in \{1, \ldots, m_k\}$, $k \in \{1, \ldots, M\}$ are contractive homeomorphisms with the contractivity factors a_{k,i_k} satisfying

(i) $L_{k,i_k} : \left\{x'_{k,i_{k_{i_k-1}}}, x'_{k,i_{k_{i_k}}}\right\} \to \{x_{k,i_k-1}, x_{k,i_k}\}$, $i_k \in \{1, \ldots, m_k\}$,

(ii) For any $x_{k,i_k} \in \{x_{k,1}, \ldots, x_{k,m_k-1}\}$, there exist $\tilde{x}_{k,i_{k_{i_k+1}}}$ such that

$$L_{k,i_k+1} \left(\tilde{x}_{k,i_{k_{i_k+1}}}\right) = L_{k,i_k} \left(\tilde{x}_{k,i_{k_{i_k+1}}}\right) = x_{k,i_k}, \tag{8}$$

and $a_{\mathrm{i}} = \mathrm{Max} \{a_{1,i_1}, \ldots, a_{M,i_M}\}$.

We define the vertical contraction functions $F_{\mathrm{i}} : \tilde{\mathrm{E}}_{\mathrm{i}} \times \mathbf{R} \to \mathbf{R}$ by

$$F_{\mathrm{i}} (\mathbf{x}, z) = d (L_{\mathrm{i}} (\mathbf{x})) (z - g (\mathbf{x})) + h (L_{\mathrm{i}} (\mathbf{x})) , \qquad \text{for } (\mathbf{x}, z) \in \tilde{\mathrm{E}}_{\mathrm{i}} \times \mathbf{R}, \tag{9}$$

where $d (\mathbf{x})$ obeys $|d (\mathbf{x})| < 1$ and g, h are continuous Lipschitz mappings on E with Lipschitz constants $\mathrm{L}_h, \mathrm{L}_g$ satisfying, for $(\alpha_1, \ldots, \alpha_M) \in \{i_1 - 1, i_1\} \times \ldots \times \{i_M - 1, i_M\}$, $(i_1, \ldots, i_M) \in \{0, \ldots, m_1\} \times \ldots \times \{0, \ldots, m_M\}$,

$$\begin{aligned}
&g \left(x'_{1,i_{1_{\alpha_1}}}, \ldots, x'_{M,i_{M_{\alpha_M}}}\right) = z_{\tilde{\gamma}\left(i_{1_{\alpha_1}}, \ldots, i_{M_{\alpha_M}}\right)}, \\
&h (x_{1,i_1}, \ldots, x_{M,i_M}) = z_{i_1, \ldots, i_M} ,
\end{aligned}$$

where $\tilde{\gamma}\left(i_{1\alpha_1},\ldots,i_{M\alpha_M}\right)=\tilde{\gamma}\left(\gamma_1\left(x'_{1,i_{1\alpha_1}}\right),\ldots,\gamma_M\left(x'_{M,i_{M\alpha_M}}\right)\right)=(x_{1,\beta_1},\ldots, x_{M,\beta_M})=(\beta_1,\ldots,\beta_M)$.

Then, the F_{i} satisfy 'join-up' conditions

$$
\begin{aligned}
&F_{\mathrm{i}}\left(x'_{1,i_{1\alpha_1}},\ldots,x'_{M,i_{M\alpha_M}},z_{\tilde{\gamma}\left(i_{1\alpha_1},\ldots,i_{M\alpha_M}\right)}\right)=z_{\sigma\left(L_{\mathrm{i}}\left(x'_{1,i_{1\alpha_1}},\ldots,x'_{M,i_{M\alpha_M}}\right)\right)},\\
&\text{for } (\alpha_1,\ldots,\alpha_M)\in\{i_1-1,i_1\}\times\ldots\times\{i_M-1,i_M\},
\end{aligned}
$$

where $\sigma\left(L_{\mathrm{i}}\left(x'_{1,i_{1\alpha_1}},\ldots,x'_{M,i_{M\alpha_M}}\right)\right)=(x_{1,j_1},\ldots,x_{M,j_M})=(j_1,\ldots,j_M)\in\{i_1-1,i_1\}\times\ldots\times\{i_M-1,i_M\}$.

Consequently, W_{i} are contractive transformations for all $\mathrm{i}\in\Omega$ with respect to some metric which is equivalent to Euclidean metric on $\mathbf{R}^{M+1}$.

Let $m_1\cdot\ldots\cdot m_M$ be denoted by m_{1-M} and enumerate the set Ω by a injective mapping $\tau:\Omega\to\{1,\ldots,m_{1-M}\}$. We defined a connection matrix $C=(c_{kl})_{m_{1-M}}$ by

$$
c_{kl}=\begin{cases}1 & :\mathrm{E}_{\tau^{-1}(l)}\subseteq\tilde{\mathrm{E}}_{\tau^{-1}(k)}\\ 0 & :otherwise\end{cases}.
$$

In the construction of local IFS, the relation between E_{i} and $\tilde{\mathrm{E}}_{\mathrm{i}}$ so that the connection matrix defined above should be irreducible. Then we get a RIFS, whose attractor A is a graph of continuous fractal interpolation function of the data set S.

The following theorem gives the Box-counting dimension of the attractor A.

Theorem 3 *Let $L_{\tau^{-1}(i)}$ be similitude with scaling a for $i\in\{1,\ldots,m_{1-M}\}$, $d(x,y)=d_0$ and* A *be the RIFS. If there exist $k\in\{1,\ldots,m_{1-M}\}$ and $i_\alpha\in\{0,\ldots,m_\alpha\}$ such that the points of $\mathrm{P}_{\alpha,i_\alpha}\cap\tilde{\mathrm{E}}_{\tau^{-1}(k)}$ are not in M-1 dimension space and*

$$
\rho\left(\mathrm{diag}\left(|d_0|\,a\right)C\right)>1,
$$

then $\dim_{\mathrm{B}}(\mathrm{A})$ *is the unique* D *such that*

$$
\rho\left(\mathrm{diag}\left(|d_0|\,a^{\mathrm{D}-1}\right)C\right)=1,
$$

otherwise $\dim_{\mathrm{B}}(\mathrm{A})=M$.

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