An introduction to ordinary differential equations by Computer Algebra–systems

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Abstract: We report on an elementary course in ordinary differential equations (odes) for students in engineering sciences. The course is also intended to become a self–study package for odes and is is based on several interactive computer lessons using REDUCE and MATHEMATICA. The aim of the course is not to do Computer Algebra (CA) by example or to use it for doing classroom examples. The aim is to teach and to learn mathematics by using CA–systems.

Keywords: computer algebra, education, ordinary differential equations

1 Remarks on the use of computers in teaching mathematics as a service subject

For several years now, computers have proliferated into many areas in society, including the educational system, and they are also influencing mathematics teaching (compare, e.g., the international survey given by Fey, 1989). Computers may be used as a means for performing numerical and algebraic calculations, or for drawing graphs and visualizing situations, and as an aid for creating new teaching methods. By the use of computers, **new possibilities** have become available for making mathematical contents accessible to learners, for promoting the intended aims, or for relieving mathematics learning and teaching of some tedious activities. This holds true also and especially for mathematics as a service subject for science, economy or technology (compare, e.g., the survey by Blum/Niss 1991).

However, it should be remarked that computers may also entail many kinds of **problems and risks**. For instance, students may try to replace necessary intellectual efforts by mere button pressing. And, although computers ought to contribute towards treating more real world examples and devoting more time to modelling and applications (which, of course, is particularly important in teaching mathematics as a service subject), computer simulations may replace handling real situation, or computer graphics may serve as substitutes for real objects, so teaching and learning may become even more remote from real life than before. Sometimes, the chances of computers are considerably overestimated. For example, when treating differential equations, computers may only support the mathematical solution process, whereas conceptual problems or difficulties in translating between the real world and mathematics (mathematizing and interpreting) remain nearly unaffected.

In the last few years, powerful so–called **Computer Algebra Systems** such as Derive, Macsyma, Maple, Mathematica or Reduce have considerably increased the possibilities of using computers in mathematics teaching (see, e.g., the review by Leadbetter/Thomas 1990 or the collection of articles in Karian 1992, including the literature mentioned there). Calculating limits, derivatives, integrals or matrix product is easily performed by these systems. Thus, the necessity of re-thinking contents and methods of mathematics teaching has become even much more urgent. Today many people argue for eliminating **schemes and algorithms** such as curve sketching in differential calculus or formally solving differential equations from mathematics teaching since computers are much more effective than human beings. However, this seems to be very short-sighted. For, schemes and algorithms will be relevant for mathematics curricula also in the future, among others since they are still indispensable for exercises or for providing students with experiences of success (also in examinations) since a lot of students get to understanding only by way of performing algorithms, and since the effect of computers can only be appreciated after having experienced the strain of carefully performing calculations. Nevertheless, computers (and in particular CA–systems

*Permanent address: Institute of Theoretical and Applied Mechanics, Novosibirsk 630090, Russia force us to reflect upon the real meaning of schemes and algorithms. We propose to continue treating the essential mathematical topics -including corresponding algorithms- and building up basic conceptions, and afterwards to use computers to perform the algorithms when these are needed in another context. An example dealt with in this paper: When the concept of integral has been elaborated and students have calculated integrals conventionally in integral calculus, a computer is used for calculating integrals when these are needed as a tool for solving differential equations.

2 Description of an odes course and the role of CA–systems

Many papers dealing with the question of how to do mathematics by CA–systems proceed in the following way: a problem from science or engineering is considered and mathematically modelled. Then the mathematical model is prepared in such a way that tools from a certain CA–system can be used for its solution. For example the motion of a forced pendulum may be studied by Mathematica (see Abell/Braselton 1992). After the physical and mathematical modelling is done, the Mathematica procedure DSOLVE is presented and it is shown how DSOLVE can be used for solving the resulting ode. Then the great graphical facilities of Mathematica are used to show the behavior of the trajectories and to demonstrate the influence of initial conditions and parameters upon them.

Such teaching modules are very well suited for visualizing solutions and giving a feeling for the behaviour of the solution space of an ode, (see also several contributions in Zimmermann/Cunningham 1991). However, the solution algorithms do not appear, in contrast the solver acts as a black box. For us this is not appropriate when we teach a first course in odes. Instead, we want students to know some important algorithms (compare the arguments given in section 1), in addition to understanding the basic ideas. Nevertheless, when we teach mathematical algorithms in the field of odes, CA–systems may be very useful in the following sense. First, the computer may do routine calculations or problems from elementary calculus like differentiation or integration. Second, the computer can lead students through the solution procedure. We realized this by designing interactive computer lessons that will be presented in sections 3–6 of this paper.

At Kassel–University we have a basic course in mathematics for students in engineering sciences which is divided into four parts. In part I and II an introduction to calculus and linear algebra is given. In part III and IV odes and numerical mathematics are treated. It is our aim to design CA packages for teaching those materials as well as for self–studying . To begin with we considered odes. The odes course usually covers the following subjects:

1. Odes of first order
    (a) Directional field
    (b) Successive approximation
    (c) Linear equations
    (d) Separable equations
    (e) Exact equations
    (f) Some special equations
2. Linear odes of n–th order
    (a) Characteristic equation, fundamental systems
    (b) Inhomogeneous equations, particular solutions
3. Linear systems of ode's
    (a) 2x2 Systems
        i. Method of elimination
        ii. Eigenvalues, Eigenvectors
    (b) nxn Systems, Jordan normalform
    (c) Inhomogeneous systems

In the following we shall present a few examples such as linear equations of the first order, separable and exact equations as well as 2x2 systems by the method of elimination. In each case we shall briefly recollect the mathematical background and the present a symbolic procedure which step by step guides the student along the solution algorithm.

3 Linear equations

A linear equation of the first order has the form

$$y' + f(x)y = g(x) \tag{1}$$

with continuous functions f and g.
The solution of the homogeneous equation becomes

$$y_h = C\exp(-\int f(x)dx)\,. \tag{2}$$

Next we look for a particular solution, yielding

$$y_p = \int g(x)\exp\left(\int f(x)dx\right)dx \cdot \exp(-\int f(x)dx)dx\,. \tag{3}$$

The general solution then becomes

$$y = y_h + y_p \tag{4}$$

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REDUCE PROCEDURE

     % INPUT OF THE EQUATION

     write "INPUT OF f(x):";
     f:=xread(f);
     write "INPUT OF g(x):";
     g:=xread(g);

     % SOLUTION OF THE HOMOGENEOUS
     % EQUATION

     h: = c * exp(-int(f,x));
     write "---> y=",h;

     % GENERAL SOLUTION OF THE
     % INHOMOGENEOUS EQUATION

     z:=(int(g*exp**(int(f,x)),x)+C)*
         exp**(-int(f,x));
     write "--->  y = ",z;

     % INPUT OF THE INITIAL VALUES

     write "INPUT OF x0:";
     xa:=xread(xa);
     write "INPUT OF y(x0):";
     ya:=xread(ya);

     % COMPUTATION OF THE CONST C

     C:=sub(x=xa,ya*exp(int(f,x))-
        int(g*exp(int(f,x)),x));
     end;
```

4 Separable equations

Ode's of the type

$$y' = f(x)g(y) \tag{1}$$

are solved by separation of variables

$$\frac{1}{g(y)}y' = f(x), \qquad (g(y) \neq 0). \tag{2}$$

Note that any solution of $g(y) = 0$ provides a stationary solution. Starting from (1.2) we obtain the implicit equation for solutions

$$\int \frac{dy}{g(y)} = \int f(x)dx. \tag{3}$$

It is also easily possible to check by CA–systems whether a given equation

$$y' = h(x, y) \tag{4}$$

is separable through

$$\frac{\partial}{\partial y}\left(\frac{h_x}{h}\right) = 0, \quad \text{or} \quad \frac{\partial}{\partial x}\left(\frac{h_y}{h}\right) = 0. \tag{5}$$

```
REDUCE PROCEDURE

   INPUT OF THE EQUATION

   write "INPUT OF f(x):";
   p:=xread(p);
   write "INPUT OF g(y):";
   q:=xread(q);

   %
   % SOLUTION PROCEDURE
   %
   if q=0 then
   begin
     write "STATIONARY SOLUTION y = c";
     d:=c-y;
     ye:=c;
   end
   else
   begin

     a:=int(1/q,y);
     b:=int(p,x);
     d:=b-a+C;
     liste:=(solve(d,y));
     ye:=rhs(first(liste));

     INPUT INITIAL CONDITION

     y0:=xread(y0);
     x0:=xread(x0);

     yc:=sub(x=x0,y=y0,d);
     ce:=rhs(first(solve(yc,c)));
     erg:=sub(c=ce,ye);

     if length liste > 1 then
       begin
       erg1:=sub(c=ce,yf);
       write"y = ",erg1;
       end;
     end;
```

5 Exact equations

An ode of the type

$$P(x, y)dx + Q(x, y)dy = 0 \tag{1}$$

is said to be exact if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{2}$$

Solutions are obtained through

$$\int_{x_0}^{x} P(t,y)dt + \int_{y_0}^{y} Q(x_0,t)dt = 0\,. \qquad (3)$$

If the equation is not exact we may look for a multiplier M such that multiplication by M yields an exact equation. Assuming that the quantity

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \qquad (4)$$

does not depend upon y we may look for a multiplier which depends only upon x. From the exactness condition we obtain the multiplier as

$$M(x,y) = \exp\left(\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx\right) \qquad (5)$$

Similar considerations hold in the case of multipliers depending upon y only.

```
REDUCE PROCEDURE

% INPUT OF THE EQUATION

  write "INPUT OF P(x,y):";
  write;
  p:=xread(p);
  write "INPUT OF Q(x,y):";
  write;
  q:=xread(q);

%SOLUTION IN THE EXACT CASE

  a:=df(p,y,1);
  b:=df(q,x,1);
  if a=b then
   begin
    a:=int(p,x);
    b:=int(q,y);
    b:=sub(x=x0,b);
    a:=a-sub(x=x0,a);
    b:=b-sub(y=y0,b);
    f:=a+b;
    wk:=1;
    write "SOLUTION OF THE EXACT EQUATION;
    write "";
    write "F(x,y)=",f;
    return f;
    end
   else

%THE NONEXACT CASE

 begin
  write "THE EQUATION IS NOT EXACT";
  write "TRY TO FIND A MULTIPLIER SUCH THAT";
  write "M*P(x,y)*dx+M*Q(x,y)*dy=0";
  write "  BECOMES EXACT";

     d:=(a-b);
     if df(d/q,y,1)=0 then
       begin
        a:=int(d/q,x);
        m:=exp(a);
        write "";
        write "MULTIPLIER   M(x):=",m;
        w:=1;
        end
      else
        if df(d/p,x,1)=0 then
         begin
         a:=int(d/p,y);
         m:=exp(-1*a);
         write "";
         write "MULTIPLIER M(y):=",m;
         w:=1;
         end
       else
         begin
          write "";
          write "NO MULTIPLIER FOUND";
          w:=0;
          wk:=0;
           end;
;end;
```

Examples:

```
INPUT OF P(x,y):

        -x        2  2
P(x,y)=e  *(2*x-x -y )

INPUT OF Q(x,y):
        -x
Q(x,y)=e  *2*y

THE EQUATION IS EXACT
AND THE SOLUTION BECOMES

        - x  2    - x  2
F(x,y)=e   *x  + e   *y  -
         -x0    2    -x0    2
      - e    *x0  - e    *y0



INPUT OF P(x,y):

            2  2
P(x,y)=2*x-x -y

INPUT OF Q(x,y):

Q(x,y)=2*y

THE EQUATION IS NOT EXACT

TRY TO FIND A MULTIPLIER SUCH THAT
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```
M*P(x,y)*dx+M*Q(x,y)*dy=0

BECOMES EXACT

                          -x
MULTIPLIER M(x):=e


THE SOLUTION BECOMES

        - x  2      - x  2
F(x,y)=e     *x  + e     *y  -
     -x0      2    -x0      2
   -e      *x0  - e      *y0
```

6 2x2–systems by elimination

We consider 2x2 linear systems with constant coefficients

$$y' = Ay\,, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\,. \tag{1}$$

We want to solve this system by taking recourse to linear equations of second order and by the way give an introduction to methods from linear algebra needed for a systematic treatment of nxn–systems.
We discuss first the special case $a_{12} = 0$. In that case we obtain

$$\begin{aligned} y_1 &= c_1 e^{a_{11}x}\,, & (2)\\ y_2' &= a_{22}y_2 + a_{21}c_1 e^{a_{11}x}\,, & (3) \end{aligned}$$

yielding

$$y = c_1 \begin{pmatrix} 1 \\ \frac{a_{21}}{a_{11}-a_{22}} \end{pmatrix} e^{a_{11}x} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{a_{22}x}\,. \tag{4}$$

if $a_{11} \neq a_{22}$. The vectors appearing in the result are solutions of

$$(A - \lambda E)y = 0\,, \tag{5}$$

with $\lambda = a_{11}\,, \lambda = a_{22}$ respectively.
In the case $a_{11} = a_{22}, a_{21} \neq 0$ we obtain

$$y = \left(c_1 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_{21} \end{pmatrix} x\right) + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) e^{a_{11}x}\,,$$

where eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and a vector $\begin{pmatrix} 0 \\ a_{21} \end{pmatrix}$ satisfying

$$(A - a_{11}E)y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \tag{6}$$

appear.
Now let us discuss the case $a_{12} \neq 0$. Differentiating the first equation of the system and using th second one we are led to

$$y_1'' - \mathrm{tr}(A)y_1' + \det(A)y_1 = 0\,, \tag{7}$$

By solution algorithms for second order equations we can write down the solution of this equation and then discuss again the solution of the system from the point of view of linear algebra.

```
MATHEMATICA PROCEDURE

 y'  =  A y  BY ELIMINATION

 INPUT OF A

 Print[""];
 a11=Input["INPUT OF       a11 = "];
 Print[""];
 a12=Input["INPUT OF       a12 = "];
 Print[""];
 a21=Input["INPUT OF       a21 = "];
 Print[""];
 a22=Input["INPUT OF       a22 = "];
 Print[""];

MATRIX A

a={{a11,a12},{a21,a22}};

SYSTEM

y1'=a11*y1+a12*y2;
y2'=a21*y1+a22*y2;


THE CASE a12 == 0

 If [a12 == 0,
   y1 = c1*Exp[a11*x];
   y2'= a22*y2 + c1*a22*Exp[a11*x];
 If [a11 == a22,
   y2= c1*a21*x*Exp[a11*x]+c2*Exp[a11*x]
    ]
 THE CASE a12 <> 0

 If [a12 != 0,
   y1''=a11*y1'+a12*y2';
   Print["y1'' = a11*y1' + a12*y2'"];
   Print["y1'' = a11*y1' +"];
   Print["+ a12*(a21*y1 + a22*y2) "];
   Print["y1'' = a11*y1' + "];
   Print[" + a22*(y1' - a11*y1) + a12*a21*y1"];
   Print["y1'' - (a11 + a22)*y1' +"];
   Print["+ (a11*a22 - a12*a21)*y1 = 0"];
    ]
```

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