On Oseen Resolvent Estimates: A Negative Result

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We consider the resolvent problem for the scalar Oseen equation in the whole space $\mathbb{R}^3$. We show that for small values of the resolvent parameter it is impossible to obtain an L^2-estimate analogous to the one which is valid for the Stokes resolvent, even if the resolvent parameter has positive real part.

Key words. Oseen equations, resolvent estimate [1]

[1]**AMS subject classifications.** 35Q30, 65N30, 76D05

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a domain, either bounded or exterior (i. e. a domain with compact complement) or a half-space or the whole space $\mathbb{R}^3$. Set $p > 1$ and $0 < \vartheta < \pi$. Then the velocity part u of the solution (u, q) of the Stokes resolvent problem

$$-\Delta u + \lambda u + \nabla q = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \tag{1}$$

with Dirichlet boundary condition

$$u = 0 \text{ on } \partial\Omega \qquad (\Omega \neq \mathbb{R}^3) \tag{2}$$

satisfies the estimate

$$\|u\|_p \ \leq \ \frac{C}{|\lambda|}\|f\|_p, \ \ f \in L^p(\Omega)^3, \ \ 0 \neq \lambda \in \mathbb{C} \ \text{ with } \ |\arg\lambda| \leq \vartheta \tag{3}$$

with a constant C depending only on Ω, p and ϑ. This so-called resolvent estimate is a crucial auxiliary result when a semigroup approach is applied to the non-stationary Navier-Stokes system. The estimate (3) was proved by Giga [9] in the case that Ω is a bounded or an exterior domain and $|\lambda|$ is large, by McCracken [11] if $\Omega = \mathbb{R}^3$ or if Ω is a half-space in $\mathbb{R}^3$, by Borchers, Sohr [1] if $\Omega \subset \mathbb{R}^3 (n \geq 3)$ is an exterior domain and $|\lambda|$ is small, and by Borchers, Varnhorn [2] if $\Omega \subset \mathbb{R}^2$ is an exterior domain and $|\lambda|$ is small. Deuring [3], [4], [5] gave another proof of (3) if Ω is an exterior domain, as did Solonnikov [13] under the additional assumption that $|\lambda|$ is large.

With inequality (3) in mind, we want to consider a different situation here. In fact, the constant motion of a rigid body in an incompressible viscous fluid is usually modeled by the Navier-Stokes system with an Oseen term. Thus, in order to apply a semigroup approach to that system, equation (1) has to be replaced by the Oseen resolvent problem, which reads as follows:

$$-\Delta u + \tau \partial_1 u + \lambda u + \nabla q = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega. \tag{4}$$

Here $\tau > 0$ is the Reynolds number. Again boundary condition (2) has to be imposed if $\Omega \neq \mathbb{R}^3$. According to [10, Theorem 4.4], an estimate as in (3) holds for the velocity part u of a solution to the Oseen system (4), (2) if $\operatorname{Re} \lambda \geq 0$, $|\lambda|$ is large and $\Omega \subset \mathbb{R}^3$ is an exterior domain, with C depending on Ω, p, τ and a lower bound for $|\lambda|$. In [7, Theorem 4.4] this result is generalized to the case of an exterior domain in $\mathbb{R}^n$ with $n \geq 3$. It is further shown in [7] that for a given $\vartheta \in (\frac{\pi}{2}, \pi)$, a sufficiently large value $R_0 > 0$ may be chosen such that (3) holds for $\lambda \in \mathbb{C}$ with $|\lambda| \geq R_0$ and $|\arg \lambda| \leq \vartheta$ ([7, Lemma 4.5]). The results from [10] and [7] are exploited in [12] and [8], respectively, in order to solve the time-dependent Navier-Stokes system with Oseen term, and to prove decay results for solutions of this system.

The preceding observations give rise to the question whether in the Oseen case an estimate as in (3) holds for small values of $|\lambda|$. Of course, it must be taken into account that the spectrum of the Oseen operator touches the imaginary axis from the left, see [10, p. 7] for more details. Therefore it cannot be expected that in the Oseen case inequality (3) is valid for $\lambda \in \mathbb{C}$ with $|\lambda|$ small, $\operatorname{Im} \lambda < 0$ and $|\arg \lambda| \leq \vartheta$, for some $\vartheta \in (\frac{\pi}{2}, \pi)$. However, one might hope that (3) holds for $0 \neq \lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq 0$ and $|\lambda|$ small. We are aware of only one result in this direction: According to [6, Theorem 10, (3.15)], the solution u of the scalar

Oseen equation with resolvent term

$$-\Delta u + \tau\partial_1 u + \lambda u = f \quad \text{in } \mathbb{R}^3 \tag{5}$$

satisfies the estimate

$$\|u\|_p \le \frac{C}{|\lambda|^2}\|f\|_p,\ f \in L^p(\mathbb{R}^3),\ 1 \le p \le 2,\ 0 \ne \lambda \in \mathbb{C},\ |\lambda| \le \left(\tfrac{\tau}{2}\right)^2,\ \operatorname{Re}\lambda \ge 0. \tag{6}$$

In view of (3), it may be asked whether it is possible to improve inequality (6) by replacing the factor $|\lambda|^{-2}$ by $|\lambda|^{-1}$ (note that small values of $|\lambda|$ are considered in (6)). It is the purpose of the present paper to show that such a improved version of (6) does not hold. More precisely, we show that, given any $0 < \alpha < \frac{1}{2}$, there is no constant $C > 0$ with

$$\|u\|_2 \le C|\lambda|^{-3/2+\alpha}\|f\|_2 \quad \text{for } f \in L^2(\mathbb{R}^3),\ \ \lambda = r^2 + ir \ \text{ with } r \in (0,1) \cap \left(0,\ \frac{\tau^2}{8}\right).$$

Here $u \in H^2(\mathbb{R}^3)$ is the solution of (5) for given f and λ, see Theorem 2 below, where a slightly more general result is presented. This leaves open the question for the exponents $\gamma \in [\frac{3}{2}, 2)$ such that the inequality $\|u\|_p \le \frac{C}{|\lambda|^\gamma}\|f\|_p$ is satisfied, at least in the case $p = 2$, if $|\lambda|$ is small and $\operatorname{Re}\lambda \ge 0$. Nevertheless, and this is the point we want to make with the present paper, our result is sufficient to indicate that even for λ in the right complex half-plane, the Stokes resolvent estimate (3) does not carry over to the Oseen case if $|\lambda|$ is small.

2 Notations and known results

For $R > 0$, let B_R denote the open ball in $\mathbb{R}^3$ with radius R and center in the origin. If $A \subset \mathbb{R}^3$, by χ_A we denote the characteristic function of A. For $g \in L^1(\mathbb{R}^3)$ let $\hat{g}$ denote the Fourier transform of g, defined by:

$$\hat{g}(\xi) := \frac{1}{\sqrt{(2\pi)^3}} \int\limits_{\mathbb{R}^3} g(y)\, e^{-i\,\xi\,\cdot\,y}\,\mathrm{d}y, \quad \xi \in \mathbb{R}^3.$$

The inverse Fourier transform of g is given by

$$\check{g}(\xi) := \frac{1}{\sqrt{(2\pi)^3}} \int\limits_{\mathbb{R}^3} g(y)\, e^{i\,\xi\,\cdot\,y}\,\mathrm{d}y, \quad \xi \in \mathbb{R}^3.$$

The Fourier transform $\hat{g}$ of a function $g \in L^2(\mathbb{R}^3)$ is to be defined in the usual way and in accordance with the preceding choice of the Fourier transform of functions in $L^1(\mathbb{R}^3)$. If g, h are measurable functions with $\int_{\mathbb{R}^3} |g(x-y)|\,|h(y)|\,dy < \infty$ for $x \in \mathbb{R}^3$, we denote by $g * h$ the convolution of g and h, defined by,

$$(g * h)(x) := \int\limits_{\mathbb{R}^3} g(x-y)\,h(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^3.$$

Let $\tau > 0$. Then we define the fundamental solution $E^{(\lambda)}$ of the scalar Oseen resolvent equation (5) by

$$E^{(\lambda)}(z) := \frac{1}{4\pi|z|}\, e^{-\sqrt{\lambda+\frac{\tau^2}{4}}|z|+\frac{\tau z_1}{2}}, \quad 0 \neq z \in \mathbb{R}^3,\ \lambda \in \mathbb{C}.$$

Using this fundamental solution, we may solve (5) in the following sense:

Theorem 1. *Let $0 \neq \lambda \in \mathbb{C}$ with $|\lambda| \leq (\frac{\tau}{2})^2$, $Re\,\lambda \geq 0$. Then $E^{(\lambda)} \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ such that, in particular,*

$$\int\limits_{\mathbb{R}^3} |E^{(\lambda)}(x-y)|\,|f(y)|\,dy < \infty \quad \text{for } f \in L^2(\mathbb{R}^3), \;\; x \in \mathbb{R}^3.$$

Moreover, for $f \in L^2(\mathbb{R}^3)$, we have $E^{(\lambda)} * f \in H^2(\mathbb{R}^3)$, and $u := E^{(\lambda)} * f$ solves (5). In addition,

$$\hat{E}^{(\lambda)}(\xi) = \frac{1}{\sqrt{(2\pi)^3}\,(\lambda + |\xi|^2 + i\tau\xi_1)}, \quad \xi \in \mathbb{R}^3. \tag{7}$$

Of course, equation (5) admits a unique solution for any $0 \neq \lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$ if $f \in L^2(\mathbb{R}^3)$. But we restrict our existence result to the case $|\lambda| \leq (\frac{\tau}{2})^2$ since this is sufficient for our purposes, and because an existence result under this assumption may easily be deduced from the results in [6]. This latter point becomes clear by the

Proof of Theorem 1: By [6, Theorem 9] there are constants $C_1(\lambda)$, $C_2 > 0$ such that

$$|E^{(\lambda)}(z)| \leq C_1(\lambda) \left(\frac{\chi_{(0,1)}(|z|)}{|z|} + \frac{\chi_{[1,\infty)}(|z|)}{e^{C_2|\lambda|^2|z|}} \right), \quad 0 \neq z \in \mathbb{R}^3.$$

This implies $E^{(\lambda)} \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. According to [6, (3.15)] there is some constant $C(\lambda) > 0$ with

$$\|E^{(\lambda)} * f\|_2 \leq C(\lambda)\|f\|_2, \quad f \in L^2(\mathbb{R}^3). \tag{8}$$

Moreover, using [6, Theorem 13] we find $E^{(\lambda)} * f \in H^2(\mathbb{R}^3)$ and

$$\|\partial_l \partial_m (E^{(\lambda)} * f)\|_2 \;\le\; C(\lambda)\|f\|_2, \quad f \in L^2(\mathbb{R}^3),\; 1 \le l, m \le 3. \tag{9}$$

The constant $C(\lambda) > 0$ is independent of f. In addition, the relations [6, (3.21)] and [6, (3.17)] yield

$$\|\partial_l (E^{(\lambda)} * f)\|_5 \;\le\; C(\lambda)\|f\|_2, \tag{10}$$

again with a constant $C(\lambda) > 0$ independent of f. According to [6, Corollary 1] the function $u := E^{(\lambda)} * f$ belongs to $C^\infty(\mathbb{R}^3)$ and satisfies (5) if $f \in C_0^\infty(\mathbb{R}^3)$. Now it follows from (8) – (10) that equation (5) is valid also for $f \in L^2(\mathbb{R}^3)$. The uniqueness result stated in the above theorem holds according to [6, Theorem 9].

Equation (7) is stated in [10, p. 19]. For the convenience of the reader, we indicate a proof here. To this end, let $0 \neq \xi \in \mathbb{R}^3$. The integral over $\mathbb{R}^3$ appearing in the definition of the Fourier transform may be written as an integral with respect to $(r, \eta) \in (0, \infty) \times \partial B_1$. This transformation gives rise to a factor r^2, which is removed by a partial integration. Then, integrating with respect to r we obtain

$$\hat{E}^{(\lambda)}(\xi) = \frac{1}{\sqrt{2^7 \pi^5}} \int\limits_{\partial B_1} \frac{1}{\big(\sqrt{\lambda + \kappa^2} + \eta \cdot (i\xi - \kappa e_1)\big)^2} \, \mathrm{d}o_\eta, \tag{11}$$

with $e_1 := (1, 0, 0)$ and $\kappa := \frac{\tau}{2}$. Next we choose an orthonormal matrix $A \in \mathbb{R}^{3\times3}$ such that $\big(A(-\kappa e_1)\big)_1 = (A\xi)_2$ and $(A\xi)_1 = -\big(A(-\kappa e_1)\big)_2$. In other words, the vectors $-\kappa e_1$ and ξ are simultaneously rotated in such a way that their projection onto the x_1 - x_2 - plane verifies the preceding relations. Set $a := A(-\kappa e_1)$ and

$b := A\xi$ such that $a_1 = b_2$ and $-a_2 = b_1$. Then

$$\int\limits_{\partial B_1} \frac{1}{\big(\sqrt{\lambda+\kappa^2} + \eta\cdot(i\xi - \kappa e_1)\big)^2}\,\mathrm{d}o_\eta = \int\limits_{\partial B_1} \frac{1}{\big(\sqrt{\lambda+\kappa^2} + i\eta\cdot b + \eta\cdot a\big)^2}\,\mathrm{d}o_\eta$$
$$= \int\limits_{-\pi/2}^{\pi/2} \cos\vartheta \int\limits_{0}^{2\pi} \frac{1}{\big(\sqrt{\lambda+\kappa^2} + (\sin\vartheta)a_3 + i(\sin\vartheta)b_3 + (\cos\vartheta)e^{i\phi}(b_2 + ib_1)\big)^2}\,\mathrm{d}\phi\,\mathrm{d}\vartheta.$$

By applying Cauchy's formula to the integral with respect to ϕ, we may conclude

$$\int\limits_{\partial B_1} \frac{1}{\big(\sqrt{\lambda+\kappa^2} + \eta\cdot(i\xi - \kappa e_1)\big)^2}\,\mathrm{d}o_\eta \tag{12}$$
$$= 2\pi \int\limits_{-\pi/2}^{\pi/2} \frac{\cos\vartheta}{(\sqrt{\lambda+\kappa^2} + (\sin\vartheta)a_3 + i(\sin\vartheta)b_3)^2}\,\mathrm{d}\vartheta.$$

Finally, integrating with respect to ϑ we obtain from (11) and (12)

$$\begin{aligned}\hat{E}^{(\lambda)}(\xi) &= \frac{1}{\sqrt{2^5\pi^3}\,(a_3+ib_3)} \left(-\frac{1}{\sqrt{\lambda+\kappa^2} + a_3 + ib_3} + \frac{1}{\sqrt{\lambda+\kappa^2} - a_3 - ib_3} \right) \\ &= \frac{1}{\sqrt{(2\pi)^3}\ \left(\lambda+\kappa^2 - (a_3+ib_3)^2\right)}.\end{aligned}$$

Since $a_3 b_3 = ab = -\kappa\xi_1$ and $-a_3^2 + b_3^2 = -|a|^2 + |b|^2 = -\kappa^2 + |\xi|^2$, equation (7) follows. $\square$

3 Main theorem

In this section, we prove the main result of this article, stated in the ensuing theorem.

Theorem 2. *Let* $0 \leq \alpha < \frac{1}{2}$*. For* $n \in \mathbb{N}$ *let* $0 \leq r_n \leq \frac{1}{n^2}$ *and set* $\lambda_n := r_n + \frac{i}{n}$*. Then there is no constant* $C > 0$ *such that*

$$\|u\|_2 \ \leq \ C|\lambda_n|^{-3/2+\alpha}\|f\|_2$$

for $f \in L^2(\mathbb{R}^3)$ *and* $u \in H^2(\mathbb{R}^3)$ *with* $-\Delta u + \tau u + \lambda_n u = f$ *in* $\mathbb{R}^3$*.*

Proof of Theorem 2: For $n \in \mathbb{N}$ we set

$$I_n := \left(-\frac{1}{\tau n}, -\frac{1}{2\tau n}\right) \times \left(0, \frac{1}{n}\right)^2, \quad g_n(\xi) := \sqrt{2\tau\, n^3}\, \chi_{I_n}(\xi) \quad \text{for } \ \xi \in \mathbb{R}^3.$$

Obviously $g_n \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, so we may define $f_n := \check{g}_n$, and we obtain $f_n \in L^2(\mathbb{R}^3)$, $\|f_n\|_2 = \|g_n\|_2$ $(n \in \mathbb{N})$. The latter relation and the definition of g_n imply

$$\|f_n\|_2 = \sqrt{2\tau\, n^3\, |I_n|} = 1 \quad (n \in \mathbb{N}). \tag{13}$$

Now choose $n \in \mathbb{N}$ with $\frac{1}{n} \leq \frac{\tau^2}{8}$ such that $|\lambda_n| \leq (\frac{\tau}{2})^2$. We define $u_n := E^{(\lambda_n)} * f_n$. Then we know by Theorem 1 that $u_n \in H^2(\mathbb{R}^3)$ with

$$-\Delta u_n + \tau \partial_1 u_n + \lambda_n u_n = f_n, \tag{14}$$

and that u_n is the only function in $H^2(\mathbb{R}^3)$ satisfying (14). Actually, it is the only function in the larger class mentioned in Theorem 1. On the other hand,

$$\|u_n\|_2^2 = \|\hat{u}_n\|_2^2 = (2\pi)^3 \|\hat{E}^{(\lambda_n)} \hat{f}_n\|_2^2 = (2\pi)^3 \int\limits_{I_n} |\hat{E}^{(\lambda_n)}(\xi)|^2 2\tau n^3 \; \mathrm{d}\xi$$

$$= 2\tau n^3 \int\limits_{I_n} \frac{1}{\left|\frac{i}{n} + r_n + |\xi|^2 + i\tau\xi_1\right|^2} \,\mathrm{d}\xi$$

$$= 2\tau n^3 \int\limits_{I_n} \frac{1}{(|\xi|^2 + r_n)^2 + (\frac{1}{n} + \tau\xi_1)^2} \,\mathrm{d}\xi$$

$$= 2\tau n^3 \int\limits_{1/(2\tau n)}^{1/(\tau n)} \int\limits_{(0,1/n)^2} \frac{1}{\left(|(\xi_2,\xi_3)|^2 + \xi_1^2 + r_n\right)^2 + (\frac{1}{n} - \tau\xi_1)^2} \,\mathrm{d}(\xi_2,\xi_3)\,\mathrm{d}\xi_1.$$

But $r_n \leq \frac{1}{n^2}$ and $|(\xi_2,\xi_3)| \leq \frac{\sqrt{2}}{n}$ for $(\xi_2,\xi_3) \in (0,\, \frac{1}{n})^2$, so we may conclude

$$\|u_n\|_2^2 \geq 2\tau n^3 \int\limits_{1/(2\tau n)}^{1/(\tau n)} \int\limits_{(0,1/n)^2} \frac{1}{(\frac{2}{n^2} + \xi_1^2 + \frac{1}{n^2})^2 + (\frac{1}{n} - \tau\xi_1)^2} \,\mathrm{d}(\xi_2,\xi_3)\,\mathrm{d}\xi_1$$

$$= 2\tau n \int\limits_{1/(2\tau n)}^{1/(\tau n)} \frac{1}{(\frac{3}{n^2} + \xi_1^2)^2 + (\frac{1}{n} - \tau\xi_1)^2} \,\mathrm{d}\xi_1$$

$$\geq 2\tau n \int\limits_{1/(2\tau n)}^{1/(\tau n)} \frac{1}{\frac{18}{n^4} + 2\xi_1^4 + (\frac{1}{n} - \tau\xi_1)^2} \,\mathrm{d}\xi_1,$$

where we used the relation $(a+b)^2 \le 2(a^2+b^2)$ for $a,b \ge 0$ in the last inequality. Next we perform the change of variable $\xi_1 = \frac{r}{\tau n}$ to obtain

$$\begin{aligned}
\|u_n\|_2^2 &\ge 2\int_{1/2}^{1} \frac{1}{\frac{18}{n^4} + \frac{2r^4}{\tau^4 n^4} + (\frac{1}{n} - \frac{r}{n})^2}\,\mathrm{d}r \\
&= 2n^2 \int_{1/2}^{1} \frac{1}{\frac{18}{n^2} + \frac{2r^4}{\tau^4 n^2} + (1-r)^2}\,\mathrm{d}r \\
&\ge 2n^2 \int_{1/2}^{1} \frac{1}{\frac{18+\frac{2}{\tau^4}}{n^2} + (1-r)^2}\,\mathrm{d}r \\
&\ge 2n^2 \int_{1/2}^{1} \frac{1}{\left(\frac{\sqrt{18+\frac{2}{\tau^4}}}{n} + 1 - r\right)^2}\,\mathrm{d}r,
\end{aligned}$$

where the last inequality holds because $a^2 + b^2 \le (a+b)^2$ for $a,b \ge 0$. By integrating with respect to r we now obtain

$$\begin{aligned}
\|u_n\|_2^2 &\ge 2n^2 \left(\frac{n}{\sqrt{18+\frac{2}{\tau^4}}} - \frac{1}{\frac{\sqrt{18+\frac{2}{\tau^4}}}{n} + \frac{1}{2}} \right) \\
&\ge 2n^2 \left(\frac{n}{\sqrt{18+\frac{2}{\tau^4}}} - 2 \right).
\end{aligned}$$

Thus for $n \in \mathbb{N}$ with $n \ge \max\{\frac{8}{\tau^2},\ 4\sqrt{18+\frac{2}{\tau^4}}\}$ we get

$$\frac{n}{\sqrt{18+\frac{2}{\tau^4}}} - 2 \ge \frac{n}{2\sqrt{18+\frac{2}{\tau^4}}}, \quad \text{hence} \quad \|u_n\|_2^2 \ge \frac{n^3}{\sqrt{18+\frac{2}{\tau^4}}}.$$

It follows with (13) that $\|u_n\|_2 \geq \frac{\sqrt{n^3}}{\sqrt[4]{18+\frac{2}{\tau^4}}}\|f_n\|_2$ for $n \in \mathbb{N}$ with $n \geq \max\{\frac{8}{\tau^2},\ 4\sqrt{18+\frac{2}{\tau^4}}\}$. Since $|\lambda_n| \geq \frac{1}{n}$ we obtain

$$\|u_n\|_2 \geq n^\alpha |\lambda_n|^{-3/2+\alpha} \frac{1}{\sqrt[4]{18+\frac{2}{\tau^4}}} \|f_n\|_2$$

for n as above. Therefore there can be no constant $C > 0$ such that

$$\|u_n\|_2 \leq C|\lambda_n|^{-3/2+\alpha}\|f_n\|_2$$

for $n \in \mathbb{N}$. This implies the theorem. $\square$

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