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# Concepts of Value in Linear Economic Models

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In the last twenty years some mathematical economists have tried to formalize the Marxian system and to establish a link between the value and the price system by means of a so-called "Fundamental Theorem" (Okishio, 1963; Morishima, 1973; Wolfstetter, 1973). Although we consider the Marxian distinction between labor and labor-power as a fruitful starting point for the investigation of authority relationships within the firm, we argue that the above mentioned formalizations do not give additional insights into the nature of the production process. Instead they relate value theory either to the distribution of products, and not the real conditions of production, or they reduce it to a comparison between a stationary economy and a growing system. Since those models are not sociologically specified, we are forced to conclude that exploitation exists whenever workers do not get the whole net product or, alternatively, if we have a non-stationary economy. It is highly doubtful if this comparison gives a useful measure of exploitation since according to Marx (1875) socialism should not be conceived as a stationary state. Hence, we conclude that value analysis should be displaced by a direct investigation of the production process.

Quantity system

(Ia) $x = Ax + y$ $\quad A = (a_{ij}) = (\frac{x_{ij}}{x_j})$ technology matrix, semi-positive and indecomposable

(Ib) $L = a_o'x$ $\quad$ L = number of workers (working units)

$a_o = (a_{oj}) \geq 0$ vector of direct labor requirements per unit of j

x = gross output $\quad$ y = net output (= final demand)

If domA < 1, then $(I-A)^{-1} = I + A + A^2 \ldots + A^n + \ldots$ exists and is strictly positive. Hence, we solve (Ia) to get

(3) $x = (I-A)^{-1}y \gg 0$, whenever $y \geq 0$

(4) $L(y) = L(x(y)) = a_o'(I-A)^{-1}y > 0$

Viability condition

(5) $y - cL \geq 0$ $\quad c \geq 0$ vector of means of subsistence per worker

Rate of exploitation $e = \dfrac{\text{surplus labor}}{\text{necessary labor}} = \dfrac{a_o'(I-A)^{-1}(y-cL)}{a_o'(I-A)^{-1}cL}$

Lemma 1: $e > 0$ if $(y-cL) \geq 0$, i.e. workers are exploited if and only if they do not get the whole net product.

Price system

(IIa) $p'(r) = wa_o' + (1+r)p'A$ $\quad w$ = (money) wage rate
$p$ = price vector
$r$ = interest rate (= profit rate)

(IIb) $w = p'c$ (subsistence wage hypothesis)

If $0 \leqq r < \dfrac{1-\text{dom}A}{\text{dom}A}$, then according to the Frobenius-Perron theorem we can solve (IIa) to get

(11) $p'^* = p'(r^*) = wa_o'(I - (1+r^*)A)^{-1} \gg 0$

as the unique price vector.

Basic accounting identity: From (Ia), (IIa) and (Ib)

(12) $p'x = p'Ax + p'y = wa_o'x + (1+r)p'Ax = p'cL + (1+r)p'Ax$, or

(13a) $p'(y-cL) = rp'Ax$

Value system

(III) $z' = a_o' + z'A \gg 0$ $\quad z$ = vector of (actual) values

(14) $z' = a_o'(I-A)^{-1}$

Second definition of the rate of exploitation:

$$e = \frac{\text{surplus value}}{\text{value of labor-power}} = \frac{\text{value of surplus product}}{\text{value of means of subsistence}}$$

$$= \frac{z'(y-cL)}{z'cL} = \frac{a_o'(I-A)^{-1}(y-cL)}{a_o'(I-A)^{-1}cL}, \quad \text{as before.}$$

Transformation problem

Combining eq. (11) and (14) we get for any given $r^*$ the unique mapping from values into prices

(17) $w^{-1}p'(r^*) = z'(I-A)\left(I - (1+r^*)A\right)^{-1}$

Unsatisfactory solution since

(a) transformation depends on the interest rate, i.e. we have in general not a unique mapping, but a correspondence;

(b) prices were already determined without the value system;

(c) if there is more than one technique $(A, a_o)$, then the underlying technological conditions for the definition of values are dependent from the choice of technique on the basis of prices.

First solution: Rational values (Samuelson- v. Weizsäcker, 1971)

If there is steady state growth at the rate $g > 0$, then it is necessary to incrase the input vector available at the beginning of the period of production at the rate $g$ in order to maintain a constant supply of means of subsistence per worker. If $v_{t-1}$ is the vector of inputs produced in $t-1$ that are available for the production of gross output $x_t$, then we have

(18) $v_t = (1+g)v_{t-1} = (1+g)Ax_t$

(19) $x_t - v_{t-1} = x_t - (1+g)Ax_t = c_tL_t = cL_t$

Defining a new matrix $B = (1+g)A$, restricting $g$ to the interval $0 \leq g < (1-\text{dom}A)/\text{dom}A$, we find that $\text{dom}B < 1$ and hence all mathematical properties of $A$ (semipositivity, indecomposability, productivity) are maintained. Hence we can define a revised value system and a revised quantity system as follows:

(23) $\bar{\bar{z}}' = a_o'(I-B)^{-1}$ (24a) $\bar{\bar{x}} = (I-B)^{-1}cL$ (25) $L = a_o'\bar{\bar{x}}$

The rational value vector $\bar{\bar{z}}$ has the mathematical properties of the (actual) value vector. The nonsubstitution theorem (independence of the optimal technique from the composition of final demand) holds. Yet, $\bar{\bar{z}}$ can be identified as a price vector, if $r = g$:

(26) $\bar{\bar{z}}' = a'\left(I - (1+g)A\right)^{-1} = w^{-1}p'(r=g)$.

Second solution: The Fundamental Theorem

The Samuelson-v. Weizsäcker revision of value theory has been rejected because it does not analyze the social relations of production, but it gives a theory of optimal price planning. Another qualitative link between the price system and the value system has been established by the Fundamental Theorem:

If the assumptions of the price system, the quantity system and the value system are fulfilled, a positive rate of exploitation is a necessary and sufficient condition for the profit rate to be positive.

Proof: In contrast to earlier proofs (Okishio, 1963; Morishima, 1973; Wolfstetter, 1973), our proof makes explicit the crucial role of a semipositive surplus product over the means of subsistence.

If $e > 0$, then from (16) and (4) we obtain $z'(y-cL) = (1-z'c)z'y > 0$ and because of $z'y > 0$ also $1 - z'c > 0$. From the subsistence wage hypothesis we get $w^{-1}p'(r)c = 1$ and hence $w^{-1}p'(r)c - z'c = \left(w^{-1}p'(r) - z'\right)c > 0$ (A). Now, $w^{-1}p(r)$ is a strictly isotonic vector function of $r$ and $z = w^{-1}p(r=0)$; hence $r > 0$ in order to allow for $c \geq 0$ in (A).

On the other hand, if $(1-\text{dom}A)/\text{dom}A > r > 0$, then $rp'Ax = p'(y-cL) > 0$. By the viability condition (5), then $y - cL \geq 0$. But evaluating this by $z >> 0$ we obtain $z'(y-cL) > 0$ and accordingly $e > 0$.

Comment: By means of our basic accounting identity and the viability condition of the system, we relate both surplus labor and profits to the existence of a surplus product over the means of subsistence. If we change the time unit to a working day of length 1 (uniform in all industries) and denote the daily means of subsistence per worker by $V^*$, we have a third definition of the rate of exploitation

(28) $$\frac{1 - z'V^*}{z'V^*} = e,$$

and workers are exploited whenever $z'V^* < 1$. The working time necessary for the means of subsistence is then given as a percentage of the whole working day. The value vector $z$ plays a double role: 1) it evaluates the means of subsistence per day and worker, and 2) it measures the necessary working time. Hence we have in terms of values, by definition, an identity between 'value of labor-power' and 'value of wage goods'.

In the price system, no such identity holds. But as Marx assumes that only capitalists save and invest, and workers spend their whole wage in buying the means of subsistence (equation (IIb))we have, by assumption, the corresponding equality in the price system. The crucial role of the subsistence wage assumption is also emphasized by the following

Buy-Back Theorem (Ellerman, 1976): If workers can save a fraction $\varepsilon > 0$ of their wages (instead of buying the means of subsistence), they can buy back the whole product from the capitalists.

## Third solution: The Fundamental Theorem with optimal values

In the frame of our simple model we are free to replace actual values with techniques selected on the basis of profit maximization by optimal values, determined by minimization of labor requirements to produce the means of subsistence. Then we have the problem to determine the technology $(A^*, a_o^*)$ which minimizes $a_o'x$ subject to the constraint $x \geqq (I-A)^{-1}cL$, $a_o'x \geqq L$.

By the nonsubstitution theorem (Lancaster, 1968, ch. 6.7), the choice of technique is independent of the composition of $cL$. In this situation, values and prices coincide, and $r = e = 0$. But the fundamental theorem can be restated in terms of optimal values by evaluating the surplus product at optimal values $z^* >> 0$ belonging to the labor-minimizing technology $(A^*, a_o^*)$.

In general, with intrinsic joint production and different techniques, there is only one way of maintaining the fundamental theorem, namely by comparing the labor actually expended (determined in the actual quantity system) and the minimum amount of labor necessary to produce the means of subsistence. Again, the fundamental theorem in the single commodity case is easily established by noting that actual surplus labor cannot exceed the difference between labor actually performed and the minimum amount of necessary labor. If we consider this as the final formulation of the fundamental theorem, as Wolfstetter (1976, p. 7n) does, then the following generalization by Morishima (1974) to the joint production case can be applied.

(34) Minimize $a_o'x$ S.T. $(B-A)x \geqq cL, \quad x \geqq 0$

with the dual problem

(35) Maximize $z'cL$ S.T. $z'(B-A) \leqq a_o', \quad z \geqq 0$

A = input matrix, B = output matrix,
x = operation vector = activity vector
Column i of A (B) gives the input (output) coefficients of process i; row j of A (B) gives the input (output) coefficients of good j; A and B are nonnegative and nonzero.

By the duality theorem of linear programming we get

(36) $a_o'x^o = z^{o\prime}cL$ for any $x^o \in X^o$ and $z^o \in Z^o$

($X^o$ = set of optimal solutions of (34), $Z^o$ = set of optimal solutions of (35))

If $x^a$ is the actual operation vector, we have the rate of exploitation as

(37) $e = (a_o'x^a - a_o'x^o)/a_o'x^o$

$e = 0$ <u>iff</u> $x^a \in X^o$, and $e > 0$ <u>iff</u> $x^a$ is feasible, but not in $X^o$

<u>Lemma 1'</u>: Workers are exploited if they perform more labor than necessary for the production of their means of subsistence, or if they do not receive all net products with positive optimal values.

From the constraints of (34) and (35) it is easily shown that any $x^o \in X^o$ and $z^o \in Z^o$ is not consistent with $g > 0$ and $r > 0$.

Hence we have $e = g = r = 0$.

Growth condition of the system:

(39) $Bx - (1+g)Ax \geqq (1+g)cL$ $g$ = smallest growth rate of the processes $i = 1,...,r$

Profitability condition of the system:

(45) $P'B \leqq (1+R)(P'A + a_o')$ $R$ = maximal profit rate of the processes

If $x^a$ is feasible, but $x^a \notin X^o$, then we can associate it with the von Neumann model

(50) Maximize $G = (1+g)$ S.T. $[B - G(A + CA_o)]\, x \geqq 0,\ x \geq 0,\ G \geqq 0$

and for any feasible $P \equiv w^{-1}p \notin Z^o$ we have the dual problem

(51) Minimize $R'' = (1+r)$ S.T. $P'(B - R''(A + CA_o) \leqq 0,\ P \geq 0,\ R'' \geqq 0$

where $CA_o = \begin{pmatrix} ca_o & 0 & 0 \\ 0 & \ddots\ 0 & \\ 0 & ca_o & 0 \end{pmatrix}$ and $P'c = 1$ because of (1) (subsistence wage hypothesis)

<u>Von Neumann equilibrium</u>: There exist optimal solutions $G^*$ and $R''^*$ for problems (50) and (51) such that $G^* \geq R''^*$ and hence $g^* \geqq r^*$. Moreover, if the system is indecomposable, then $g^* = r^*$.

Since any $x^a \notin X^o$ is associated with $p(r^*) \notin Z^o$, $e > 0$ (i.e., $x^a \notin X^o$) is necessary and sufficient for a positive rate of growth and a positive rate of interest. This is Morishima's generalization of the Fundamental Theorem.

<u>Remark</u>: The comparison between a stationary state with $x^a \in X^o$ and $p \in Z^o$ and a growing economy is in general necessary, since with joint production (a) values are not unique (but there is still a unique "value" of necessary labor according to (34), (35)), (b) values are not necessarily nonnegative, (c) the equality between necessary labor to produce $V^*$ and the value of $V^*$ has to be replaced by the weak inequality necessary labor $\leqq$ value of $V^*$.

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