On Connection, Linearization and Duplication Coefficients of Classical Orthogonal Polynomials

By

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Contents

Abstract

In this work, we have mainly achieved the following:

1. we provide a review of the main methods used for the computation of the connection and linearization coefficients between orthogonal polynomials of a continuous variable, moreover using a new approach, the duplication problem of these polynomial families is solved;

2. we review the main methods used for the computation of the connection and linearization coefficients of orthogonal polynomials of a discrete variable, we solve the duplication and linearization problem of all orthogonal polynomials of a discrete variable;

3. we propose a method to generate the connection, linearization and duplication coefficients for q-orthogonal polynomials;

4. we propose a unified method to obtain these coefficients in a generic way for orthogonal polynomials on the quadratic and q-quadratic lattices.

Our algorithmic approach to compute linearization, connection and duplication coefficients is based on the one used by Koepf and Schmersau [1998] and on the NaViMa algorithm (see e.g. [Ronveaux et al., 1995], [Godoy et al., 1997]). Our main technique is to use explicit formulas for structural identities of classical orthogonal polynomial systems. We find our results by an application of computer algebra. The major algorithmic tools for our development are Zeilberger's algorithm ([Petkovšek et al., 1996], [Koepf, 1998]), q-Zeilberger's algorithm ([Koornwinder, 1993], [Koepf, 1998], [Riese, 2003]), the Petkovšek-van-Hoeij algorithm ([Petkovšek, 1992], [van Hoeij, 1999]), the q-Petkovšek-van-Hoeij algorithm, Algorithm 2.2, p. 20 of Koepf [1998] and it q-analogue.

List of abbreviations

OP: orthogonal polynomials
CCOP: classical continuous orthogonal polynomials
CDOP: classical discrete orthogonal polynomials
q-COP: q-classical orthogonal polynomials
$\mathbb{N} = \{1, 2, 3, \ldots\}$
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{C}$= set of complex numbers
QQL: quadratic and q-quadratic lattices
I.C.L.D.: inversion, connection, linearization and duplication

Chapter 0

General Introduction

The addition formula for cosine given by

$$\cos m\theta \cos n\theta = \frac{1}{2}\cos(m+n)\theta + \frac{1}{2}\cos(m-n)\theta$$

pertains to Chebyshev polynomials of the first kind $T_n(x) = \cos n\theta$, where $x = \cos\theta$, $0 < \theta < \pi$. It is called a *linearization formula* since it represents a product of two polynomials as a linear combination of other polynomials of the same kind

$$T_n(x)T_m(x) = \frac{1}{2}T_{m+n}(x) + \frac{1}{2}T_{m-n}(x).$$

The *linearization problem* is the problem of finding the coefficients $L_k(m,n)$ in the expansion of the product $p_n(x)q_m(x)$ of two polynomial systems $\{p_n(x)\}_{n\in\mathbb{N}_0}$ and $\{q_m(x)\}_{m\in\mathbb{N}_0}$ in terms of a third sequence of polynomials $\{y_k(x)\}_{k\in\mathbb{N}_0}$

$$p_n(x)q_m(x) = \sum_{k=0}^{n+m} L_k(m,n)y_k(x). \tag{1}$$

These coefficients exist and are unique since $\deg p_n = n$, $\deg q_m = m$, $\deg y_k = k$ and the polynomials $\{y_k(x),\ k = 0,1,\ldots,n+m\}$ are linearly independent. Note that, in this setting, the polynomials $p_n(x)$, $q_m(x)$ and $y_k(x)$ may belong to three different polynomial families. When the polynomials p_n, q_m and y_k are solutions of the same differential equation, this is usually called the (standard) linearization [Askey, 1975] or Clebsch-Gordan-type problem for hypergeometric polynomials (the name Clebsch-Gordan is attached because the structure is similar to the Clebsch-Gordan series for spherical functions [Edmonds, 1957]). On the other hand, if $q_m(x) := 1$ in (1), we are faced with the so-called *connection problem*,

$$p_n(x) = \sum_{k=0}^{n} C_k(n)y_k(x), \tag{2}$$

which for $p_n(x) = x^n$ is known as the *inversion problem*

$$x^n = \sum_{k=0}^{n} I_k(n)y_k(x), \tag{3}$$

for the family $y_k(x)$. If we substitute x by ax in the left hand side of (2) and y_k by p_k, we get the *duplication problem*

$$p_n(ax) = \sum_{k=0}^{n} D_k(n,a)p_k(x). \tag{4}$$

Linearization, connection and duplication problems are not only important from a fundamental point of view, but also because they are used in the computation of physical and chemical properties of quantum-mechanical systems. As example, in the evaluation of the logarithmic potentials of orthogonal polynomials $V_n(t) = -\int [p_n(x)]^2 \log|x-t| dx$, which appears in the calculation of the position and momentum information entropies of quantum systems ([Dehesa et al., 1997a], [Sánchez-Ruiz, 1997]), the linearization formula $(p_n(x))^2 = \sum_{j=0}^{2n} L_j(m,n)p_j(x)$ is used to reduce the above integral into the form $V_n(t) = -\sum_{j=0}^{2n} L_j(m,n)\int p_j(x)\log|x-t|dx$ which can be easily computed.

In many applications of orthogonal polynomials, it is often important to know whether the linearization, connection or duplication coefficients are positive or non-negative (see e.g. [Askey, 1968], [Askey, 1975], [Gasper, 1975], [Ismail, 2005, Chapter 9]). This property has many important consequences. It gives rise to a convolution structure associated with the polynomial set $\{p_n(x)\}$ ([Gasper, 1970], [Askey and Gasper, 1971b], [Askey and Gasper, 1977], [Szwarc, 1992]). During the last decades, several sufficient conditions for these sign properties to hold have been derived (see e.g. [Askey, 1965], [Askey and Gasper, 1971a], [Gasper, 1975], [Askey, 1975], [Trench, 1976], [Koornwinder, 1978], [Szwarc, 1996], [Sánchez-Ruiz et al., 1999], [Szwarc, 2003], [Ismail, 2005]).

The literature on the standard linearization and connection problems is extremely vast, and a variety of methods and approaches for computing the coefficients have been developed for classical continuous, discrete, q-discrete orthogonal polynomials and also for orthogonal polynomials on a nonuniform lattice.

Classical orthogonal polynomials of a continuous, a discrete and a q-discrete variable, and on a nonuniform lattice $x = x(s)$ are known to satisfy, respectively, the following second-order holonomic differential, difference, q-difference and divided-difference equations (see e.g. [Nikiforov and Uvarov, 1988], [Foupouagnigni, 2008], [Koekoek et al., 2010]):

$$\begin{gathered}
\sigma(x)\,\tfrac{d^2}{dx^2}\,y(x) + \tau(x)\,\tfrac{d}{dx}\,y(x) + \lambda_n\,y(x) = 0,\\
\sigma(x)\,\Delta\,\nabla y(x) + \tau(x)\,\Delta\,y(x) + \lambda_n\,y(x) = 0,\\
\sigma(x)\,D_q\,D_{\frac{1}{q}} y(x) + \tau(x)\,D_q\,y(x) + \lambda_{n,q}\,y(x) = 0,\\
\sigma(x(s))\mathbb{D}_x^2 y_n(x(s)) + \tau(x(s))\mathbb{S}_x\mathbb{D}_x y_n(x(s)) + \lambda_n y_n(x(s)) = 0,
\end{gathered}$$

where Δ, ∇, and D_q are, respectively, the forward, the backward and the Hahn operators defined by

$$\Delta f(x) = f(x+1) - f(x),\ \nabla f(x) = f(x) - f(x-1),\ D_q f(x) = \frac{f(qx)-f(x)}{(q-1)x},\ q \neq 1,\ x \neq 0,$$

with $D_q f(0) = \lim_{x\to 0} D_q f(x) = f'(0)$, provided that $f'(0)$ exists, $\mathbb{D}_x$ and $\mathbb{S}_x$ are the operators defined by [Foupouagnigni, 2008]

$$\mathbb{D}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}, \quad \mathbb{S}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) + f(x(s-\frac{1}{2}))}{2},$$

$\sigma(x) = ax^2+bx+c$, $\tau(x) = dx+e$, are polynomials of maximum degree 2 and 1 respectively, and λ_n, $\lambda_{n,q}$ are constants.

For the classical continuous or discrete polynomials families, representations of linearization, connection and duplication coefficients have been obtained, usually in terms of generalized hypergeometric series or as a hypergeometric term (to be defined below), exploiting for this purpose several of their characterizing properties: Rodrigues' formula, generating functions, orthogonality weights, structure relations etc. (see for instance [Szegö, 1939], [Gasper, 1974], [Askey, 1975], [Rahman, 1981a], [Niukkanen, 1985], [Markett, 1994], [Area et al., 1998], [Koepf and Schmersau, 1998], [Lewanowicz, 2003a], [Sánchez-Ruiz et al., 1999]).

Definition 0.1. *The generalized hypergeometric series is defined by*

$$ {}_pF_q\left(\left.\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right| x\right) := \sum_{m=0}^{\infty} A_m x^m = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{x^m}{m!}, \tag{5}$$

where $(a)_m$ denotes the Pochhammer symbol (or shifted factorial) defined by

$$(a)_m = \begin{cases} 1 & \text{if } m = 0 \\ a(a+1)(a+2)\cdots(a+m-1) = \frac{\Gamma(a+m)}{\Gamma(a)} & \text{if } m \in \mathbb{N}. \end{cases}$$

We say that a term A_m is a hypergeometric term with respect to m if $\frac{A_{m+1}}{A_m} \in \mathbb{Q}(m)$, i.e. is a rational function in the variable m.

The summand $\alpha_m = A_m x^m$ of a generalized hypergeometric series is a hypergeometric term since

$$\frac{\alpha_{m+1}}{\alpha_m} = \frac{(m+a_1)\cdots(m+a_p)}{(m+b_1)\cdots(m+b_q)} \frac{x}{m+1}.$$

If any numerator parameter a_i is zero or a negative integer, the series terminates.

An application of the elementary ratio test to the power series on the right in (5) shows at once that:

a) If $p \le q$, the series converges for all $x \in \mathbb{C}$;

b) If $p = q+1$, the series converges for $|x| < 1$ and diverges for $|x| > 1$;

c) If $p > q+1$, the series diverges for $x \neq 0$.

If the series terminates, there is no question of convergence, and the conclusions (b) and (c) do not apply. If $p = q+1$, the series in (5) is absolutely convergent on the circle $|x| = 1$ if

$$\mathrm{Re}\Big(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\Big) > 0.$$

Ferrers [1877] and Adams [1878] found the linearization formula of the Legendre polynomials $P_n(x) = P_n^{(0,0)}(x)$[1]

$$P_n(x)P_m(x) = \sum_{k=0}^{\min(m,n)} \frac{m+n-2k+\frac{1}{2}}{m+n-k+\frac{1}{2}} \frac{A_k A_{m-k} A_{n-k}}{A_{m+n-k}} P_{m+n-2k}(x), \;\; A_m = \frac{\left(\frac{1}{2}\right)_m}{m!}$$

[1]The general notation for specific orthogonal systems are given in Chapter 1

by finding the coefficients for small values of m (Adams derived these coefficients for $m = 1, 2, 3, 4$ in his paper, guessing what the result would be for arbitrary m and n, and proving it by induction). Bailey [1933] gave the first proof of this formula by means of Whipple's transformation of a Saalschützian ${}_4F_3$ to a well-poised ${}_7F_6$ and Dougall [1953] gave a second proof. But a more systematic method should be found. Hylleraas [1962] computed a fourth order differential equation satisfied by the product $C_m^{(\alpha)}(x)C_n^{(\alpha)}(x)$ of ultraspherical polynomials and used it to set up a recurrence relation for the linearization coefficients. Then he solved this recurrence relation and obtained Dougall's linearization formula [Dougall, 1919]

$$
\begin{aligned}
C_n^{(\alpha)}(x)C_m^{(\alpha)}(x) = \sum_{k=0}^{\min(m,n)} & \frac{(n+m-2k+\alpha)(n+m-2k)!(\alpha)_k}{k!(n+m-k+\alpha)(n-k)!(m-k)!} \\
& \times \frac{(\alpha)_{n-k}(\alpha)_{m-k}(2\alpha)_{n+m-k}}{(\alpha)_{n+m-k}(2\alpha)_{n+m-2k}} C_{n+m-2k}^{(\alpha)}(x).
\end{aligned}
$$

For Jacobi polynomials the situation was far from satisfaction.

Rainville [1960] combined the hypergeometric representation and inversion formulas to get connection coefficients of continuous orthogonal polynomials. The technique he used was based on generating functions of the polynomials involved.

Recently, it has been shown by Lewanowicz [2003a] that the connection problem between two families of orthogonal polynomials can sometimes be solved by taking advantage of known theorems from the theory of generalized hypergeometric functions.

Koepf and Schmersau [1998] gave a general algorithmic method to solve connection problems for classical orthogonal polynomials of a continuous and a discrete variable. Their main technique was to use explicit formulas for structural identities of the given polynomial systems. Sánchez-Ruiz et al. [1999], by an integral evaluation, obtained general representations for the linearization coefficients, and the particular cases of the standard linearization and connection problems were singled out. Their method was based on the Rodrigues' formula and the orthogonality of the polynomial families. Alvarez-Nodarse et al. [1997] used the same approach to solve linearization and connection problems for discrete hypergeometric polynomials.

Another, rather general, approach allows the computation of the standard linearization and connection coefficients recursively (see e.g. [Markett, 1994], [Ronveaux et al., 1995], [Lewanowicz, 1996b], [Godoy et al., 1997]). For this purpose, an algorithm called NaViMa has been developed by Ronveaux et al. [1995], Godoy et al. [1997].

In contrast, the general linearization, connection and duplication problem has not yet been solved, to the best of our knowledge, for q-orthogonal polynomials and orthogonal polynomials on a quadratic and q-quadratic lattice, although some partial results are known for the linearization of the following families: little q-Jacobi [Andrews and Askey, 1977], Continuous q-Jacobi [Rahman, 1981b], q-Ultraspherical [Gasper and Rahman, 1990], q-Hermite [Markett, 1994]. Moreover for classical continuous othogonal polynomials the duplication problem is not completely solved whereas for classical discrete orthogonal polynomials, it is solved only for $a = -1$ and also the linearization problem is not completely solved.

In this work:

1. we provide a review of the main methods used for the computation of the connection, linearization and duplication coefficients between orthogonal polynomials of a

continuous variable, we solve the duplication problem of these polynomial families using a new approach. We recover known duplication formulas and moreover, we get new results for Jacobi and Gegenbauer polynomials (see Theorem 1.13).

2. we review the main methods used for the computation of the connection and linearization coefficients of orthogonal polynomials of a discrete variable. The duplication problem of the polynomials belonging to this family is solved for any value of a, therefore generalizing known results for $a = -1$ (see Theorem 2.13). Furthermore the linearization problem of all orthogonal polynomials of a discrete variable is solved and we get a hypergeometric series representation of their linearization coefficients (see Theorem 2.8); these results are new as far as we know.

3. we propose a method to generate the connection, linearization and duplication coefficients for q-orthogonal polynomials. To the best of our knowledge these results are new and we have already published some of them in [Foupouagnigni et al., 2012].

4. we propose a unified method to obtain the connection, linearization and duplication coefficients in a generic way for the Askey-Wilson polynomials. In [Foupouagnigni et al., 2013b], we have already published part of these results.

5. we use limiting and/or special cases to recover from the results obtained for the Askey-Wilson polynomials, the representation of connection, linearization and duplication coefficients for all the classical orthogonal polynomials on a quadratic and q-quadratic lattice. However, due to space limitation, we have provided these coefficients only for the q-Racah, Wilson and Racah orthogonal polynomials which are the most representative for the different types of quadratic and q-quadratic lattices.

Our algorithmic approach to compute linearization, connection and duplication coefficients is based on the one used by Koepf and Schmersau [1998] and on the NaViMa algorithm [Ronveaux et al., 1995], [Godoy et al., 1997]. We find our results by an application of the Maple and Mathematica computer algebra systems. Our main technique is to use explicit formulas for structural identities of classical orthogonal polynomial systems. The major algorithmic tools for our development are Zeilberger's algorithm, q-Zeilberger's algorithm, the Petkovšek-van-Hoeij algorithm, the q-Petkovšek-van-Hoeij algorithm, Algorithm 2.2, p. 20 of Koepf [1998] and it q-analogue.

Marko Petkovšek [Petkovšek, 1992] developed an algorithm to find all hypergeometric term solutions of a *holonomic recurrence equation*, i.e., homogeneous linear recurrence equation with polynomial coefficients. In some cases this algorithm is not very efficient. However Mark van Hoeij [van Hoeij, 1999] gave a very efficient version of such an algorithm. Cluzeau and van Hoeij [2006] described the complete algorithm to compute the hypergeometric term solutions of linear recurrence relations with rational function coefficients. An efficient version of this algorithm was implemented in Maple by van Hoeij. In the sequel, the Petkovšek-van-Hoeij algorithm refers to this efficient version of van Hoeij.

Zeilberger's algorithm (see e.g. [Petkovšek et al., 1996], [Koepf, 1998]) deals with sums of the form

$$S_n = \sum_{m=-\infty}^{\infty} A(n, m).$$

Zeilberger's algorithm applies if $A(n, m)$ is a hypergeometric term with respect to both n and m. It generates a holonomic recurrence equation for S_n. If the recurrence equation is of first order, then S_n (with n assumed to be an integer) can be converted to a hypergeometric term. Zeilberger's algorithm may not give a recurrence equation of first order, even if the sum is a hypergeometric term. In such a case, the combination of Zeilberger's with the Petkovšek-van-Hoeij algorithm guarantees to find out whether the given sum can be written as a hypergeometric term. If the recurrence equation doesn't have a hypergeometric term solution, it may be helpful to give a hypergeometric series representation of the sum S_n. This is done by Algorithm 2.2, p. 20 of Koepf [1998]. This algorithm converts hypergeometric sums into hypergeometric notation and is implemented in Maple in the package `hsum.mpl` by the procedure `sumtohyper`. Throughout this work, sumtohyper refers to this algorithm.

Concerning the other algorithms, we need the following definitions.

Definition 0.2. *1. The basic hypergeometric series ${}_r\phi_s$ is defined by*

$$ {}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}\middle| q;z\right) = \sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}\frac{z^k}{(q;q)_k}, $$

where the q-Pochhammer symbol $(a_1,a_2,\ldots,a_k;q)_n$ is defined by

$$ (a_1,\ldots,a_r;q)_k := (a_1;q)_k\cdots(a_r;q)_k, \quad \textit{with} \quad (a_i;q)_k = \begin{cases} \prod_{j=0}^{k-1}(1-a_iq^j) & \textit{if} \quad k=1,2,3,\ldots \\ 1 & \textit{if} \quad k=0. \end{cases} $$

If one of the numerator parameters a_i equals q^{-n}, where n is a nonnegative integer, the series terminates. Otherwise the radius of convergence of the hypergeometric series is given by

$$ \begin{cases} \infty & \textit{if} \quad r<s+1, \\ 1 & \textit{if} \quad r=s+1, \\ 0 & \textit{if} \quad r>s+1. \end{cases} $$

2. A term A_k is a q-hypergeometric term if $\frac{A_{k+1}}{A_k}\in\mathbb{Q}(q^k)$, i.e., is a rational function in the variable q^k.

3. A linear homogeneous recurrence equation

$$ \sum_{k=0}^{n}\alpha_k(q;q^m)C_{m+k}=0 $$

is called q-holonomic if the coefficients $\alpha_k(q;q^m)$ are rational w.r.t. q and polynomial functions w.r.t. the variable q^m.

The q-Petkovšek-van-Hoeij algorithm finds the q-hypergeometric term solutions of q-holonomic recurrence equations. A q-version of Petkovšek's algorithm was given by Abramov et al. [1998] and by Böing and Koepf [1999]. However, this algorithm is rather inefficient in some cases and therefore not at all suitable for many of our recurrence equations. Fortunately, Cluzeau and van Hoeij [2006] published a refined version which is much more efficient (see also Horn [2008], [Horn et al., 2012]). Sprenger [2009] presented a Maple implementation of this refined version `qHypergeomSolveRE` in his package `qFPS` (see also [Sprenger and Koepf, 2012]). It is worth noting that in the sequel, q-Petkovšek-van-Hoeij algorithm refers to this implementation.

The q-version of Zeilberger's algorithm (see e.g. [Koepf, 1998]) also deals with definite sums of the form

$$S_n = \sum_{m=-\infty}^{\infty} A(n, m)$$

and applies if $A(n, m)$ is a q-hypergeometric term with respect to both n and m. It generates a q-holonomic recurrence equation for S_n. If the recurrence equation is of first order, then S_n (with n assumed to be an integer) can be converted to a q-hypergeometric term. If the recurrence if of order greater than one, q-Petkovšek-van-Hoeij algorithm is used. If the recurrence equation doesn't have a q-hypergeometric term solution, it may be helpful to give a q-hypergeometric representation of the sum S_n. This is done by the q-analogue of Algorithm 2.2, p. 20 of Koepf [1998]. This algorithm converts q-hypergeometric sums into q-hypergeometric notation and is implemented in Maple in the package `qsum.mpl` by the procedure `sum2qhyper`. Throughout this work, sum2qhyper refers to this algorithm.

Chapter 1

Connection, Linearization and Duplication Coefficients of Classical Orthogonal Polynomials of a Continuous Variable

In this chapter, we recall known results on connection and linearization coefficients of classical orthogonal polynomials on the real line. We are interested in reviewing here some different general methods used to obtain these results. Furthermore we use a new approach to compute duplication coefficients of classical continuous orthogonal polynomials. We recover known results and get some new ones.

1.1 Introduction

Here, we recall some definitions and results which will be useful in the sequel.
Let $\mathcal{P}$ denotes the linear space of polynomials with coefficients in $\mathbb{C}$, the set of complex numbers.

Definition 1.1. 1. *A polynomial sequence $\{y_n(x)\}_{n\geq 0}$ in $\mathcal{P}$ is called a polynomial family (system or set) if $y_n(x)$ is of degree precisely n in x, $n = 0,\ 1,\ 2, \ldots$.*

2. *A positive function $\rho(x)$ defined on (A, B) (with $-\infty \leq A < B \leq +\infty$) is a weight function if $\rho(x)$ is continuous on (A, B) and $\int_A^B \rho(x)x^n dx \in \mathbb{R}$, for all $n \in \mathbb{N}_0$.*

3. *We say that a polynomial family $\{y_n(x)\}_{n\geq 0}$ of a continuous variable is orthogonal with respect to the weight function $\rho(x)$ defined on (A, B) if*

$$\int_A^B y_n(x)y_m(x)\rho(x)dx = h_n^2\delta_{mn}, \tag{1.1}$$

where h_n^2 is a nonnegative real and $\delta_{mn} = \begin{cases} 0 \text{ if } m \neq n \\ 1 \text{ if } m = n, \end{cases}$ designates the Kronecker symbol.

4. *An orthogonal polynomial family of a continuous variable* $\{y_n(x)\}_{n\geq 0}$ *is classical if the weight* $\rho(x)$ *is solution of the so-called Pearson equation*

$$(\sigma(x)\rho(x))' = \tau(x)\rho(x), \tag{1.2}$$

where $\sigma(x) = ax^2 + bx + c > 0$ *on* (A, B) *and* $\tau(x) = dx + e$ *are, respectively, polynomials of at most second order and first order and satisfy the boundary conditions*

$$\lim_{x\to A,\, x>A} \sigma(x)\rho(x)x^k = \lim_{x\to B,\, x<B} \sigma(x)\rho(x)x^k = 0,\ k \geq 0. \tag{1.3}$$

Let us mention that the classical orthogonal polynomials of a continuous variable $(y_n(x))$ satisfies a second-order differential equation of the form

$$\sigma(x)y_n''(x) + \tau(x)y_n'(x) + \lambda_n y_n(x) = 0, \tag{1.4}$$

with $\lambda_n = -n\Big[(n-1)a + d\Big]$. We shall refer to (1.4) as an *equation of hypergeometric type*, and its solutions as *polynomials of hypergeometric type* (see e.g. [Nikiforov and Uvarov, 1988]).

In [Nikiforov and Uvarov, 1988] (for example), it is shown that:

Theorem 1.2. *The polynomial solutions of* (1.4) *are defined up to a normalizing factor by the so-called Rodrigues formula*

$$y_n(x) = \frac{B_n}{\rho(x)}\frac{d^n}{dx^n}\Big[\sigma(x)^n\rho(x)\Big],\ n = 0, 1, \dots, \tag{1.5}$$

with

$$B_0 = k_0,\ \ B_n = k_n \prod_{j=0}^{n-1}\Big(d + (n+j-1)a\Big)^{-1},\ n = 1, 2, \dots,$$

where k_n *is the leading coefficient of the polynomial* $y_n(x) = k_n x^n + \dots$.

The norm h_n^2 is given in terms of B_n by

$$h_n^2 = (-1)^n n! k_n B_n \mu_n, \quad \text{with} \quad \hat{\mu}_n = \int_A^B \sigma(x)^n \rho(x)dx, \tag{1.6}$$

where $(\hat{\mu}_n)_{n\geq 0}$ defined above denotes the sequence of generalized moments of $\rho(x)$.

Theorem 1.3. *The* k^{th} *derivatives* $y_n^{(k)}(x)$ *of the classical orthogonal polynomials* $y_n(x)$*, orthogonal with weight* $\rho(x)$ *on* (A, B)*, are also classical polynomials, orthogonal with weight* $\rho_k(x) = \sigma(x)^k\rho(x)$ *on* (A, B)*:*

$$\int_A^B y_n^{(k)}(x)y_m^{(k)}(x)\rho_k(x)dx = h_{nk}^2\delta_{m,n}, \tag{1.7}$$

with

$$h_{n0}^2 = h_n^2,\ \ h_{nk}^2 = h_n^2(-1)^k\prod_{j=0}^{k-1}(n-j)\Big(d + (n+j-1)a\Big),\ k = 1, 2, \dots, n.$$

These derivatives are solutions of the second-order differential equation

$$\sigma(x)y''(x) + \tau_k(x)y'(x) + \lambda_{n,k}y_{n,k}(x) = 0, \tag{1.8}$$

with

$$\tau_k(x) = \tau(x) + k\sigma'(x) \text{ and } \lambda_{n,k} = -n\Big((n-1)a + d + 2ak\Big).$$

In this case, the Pearson equation is given by

$$[\sigma(x)\rho_k(x)]' = \tau_k(x)\rho_k(x).$$

Since the derivatives of all orders of polynomials $y_n(x)$ of hypergeometric type are also polynomials of hypergeometric type, the Rodrigues formula for $y_n^{(k)}(x)$ has the form (see e.g. [Nikiforov and Uvarov, 1988])

$$y_n^{(k)}(x) = \frac{A_{nk}B_n}{\sigma(x)^k\rho(x)}\frac{d^{n-k}}{dx^{n-k}}\Big[\sigma(x)^n\rho(x)\Big], \tag{1.9}$$

with

$$A_{n0} = 1,\ A_{nk} = \frac{n!}{(n-m)!}\prod_{j=0}^{k-1}\Big(d + (n+j-1)a\Big),\ 1 \le k \le n. \tag{1.10}$$

For much additional material on general orthogonal polynomials the reader should consult: [Szegö, 1939], [Jackson, 1941], [Nikiforov and Uvarov, 1988], [Ismail, 2005], [Koekoek et al., 2010].

Note that by $P_n^{(\alpha,\beta)}(x)$, $C_n^{(\alpha)}(x)$, $L_n^{(\alpha)}(x)$, $H_n(x)$, $B_n^{(\alpha)}(x)$, we denote, respectively, the Jacobi, Gegenbauer/Ultraspherical, Laguerre, Hermite and Bessel polynomials. Their hypergeometric representations are given in [Koekoek et al., 2010]

$$\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\left.\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}\,\right|\,\frac{1-x}{2}\right),\ \alpha > -1,\ \beta > -1\\
&= (-1)^n\frac{(\beta+1)_n}{n!}\,{}_2F_1\left(\left.\begin{matrix}-n, n+\alpha+\beta+1\\ \beta+1\end{matrix}\,\right|\,\frac{1+x}{2}\right),\\
C_n^{(\alpha)}(x) &= \frac{(\alpha)_n 2^n x^n}{n!}\,{}_2F_1\left(\left.\begin{matrix}-n/2, -n/2+1/2\\ -n-\alpha+1\end{matrix}\,\right|\,\frac{1}{x^2}\right),\ \alpha > -\frac{1}{2} \text{ and } \alpha \neq 0,\\
L_n^{(\alpha)}(x) &= \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\left.\begin{matrix}-n\\ \alpha+1\end{matrix}\,\right|\,x\right),\ \alpha > -1,\\
H_n(x) &= 2^n x^n\,{}_2F_0\left(\left.\begin{matrix}-n/2, -n/2+1/2\\ -\end{matrix}\,\right|\,-\frac{1}{x^2}\right),\\
B_n^{(\alpha)}(x) &= {}_2F_0\left(\left.\begin{matrix}-n, n+\alpha+1\\ -\end{matrix}\,\right|\,-\frac{x}{2}\right),\ n = 0, 1, \ldots, N,\ \alpha < -2N-1.
\end{aligned}$$

The data corresponding to each family are given in the following table [Koekoek et al., 2010]:

system	$P_n^{(\alpha,\beta)}(x)$	$C_n^{(\alpha)}(x)$	$L_n^{(\alpha)}(x)$	$H_n(x)$	$B_n^{(\alpha)}(x)$
$\sigma(x)$	$1-x^2$	$1-x^2$	x	1	x^2
$\tau(x)$	$\beta-\alpha-(\alpha+\beta+2)x$	$-(2\alpha+1)x$	$\alpha+1-x$	$-2x$	$2+(\alpha+2)x$
$\rho(x)$	$(1-x)^\alpha(1+x)^\beta$	$(1-x^2)^{\alpha-\frac{1}{2}}$	$x^\alpha e^{-x}$	e^{-x^2}	$x^\alpha e^{-\frac{2}{x}}$
(A,B)	$(-1,1)$	$(-1,1)$	$(0,\infty)$	$(-\infty,\infty)$	$(0,\infty)$
k_n	$\frac{(\alpha+\beta+n+1)_n}{2^n n!}$	$\frac{(\alpha)_n 2^n}{n!}$	$\frac{(-1)^n}{n!}$	2^n	$\frac{(n+\alpha+1)_n}{2^n}$

1.2 Connection and Linearization Coefficient Using Structural Relations

The main idea of the following methods is to determine recurrence equations for the linearization coefficients $L_k(m,n)$

$$p_n(x)q_m(x) = \sum_{k=0}^{n+m} L_k(m,n)y_k(x),$$

and the connection coefficients $C_m(n)$

$$p_n(x) = \sum_{m=0}^{n} C_m(n)q_m(x),$$

for the polynomial sequences p_n, q_n and y_n. These recurrence equations follow mainly from the structural formulas of the polynomials involved in the connection or linearization relations. The most interesting recurrence equations are those based only in one parameter which leave the other parameters fixed. The success of these methods will heavily depend on whether or not these recurrence equations are of the *lowest order*, i.e., whether or not no recurrence equations of lower orders for $L_k(m,n)$ or $C_m(n)$ are valid. In cases when the order of the resulting recurrence equation is one, it defines a hypergeometric term which can be given explicitly in terms of shifted factorials using the initial values $L_{n+m}(m,n)$ or $C_n(n)$.

1.2.1 Structural Formulas for Classical Orthogonal Polynomials of a Continuous Variable

Let $(p_n(x) = k_n x^n + \ldots)_{n\in\mathbb{N}_0}$ be a family of classical continuous orthogonal polynomials (in short CCOP). This family is solution of a differential equation of type (1.4). For every $p_n(x)$, $n \in \mathbb{N}_0$, with $p_{-1} \equiv 0$, the following structural relations are valid (see e.g. [Koepf and Schmersau, 1998]):

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \tag{1.11}$$
$$\sigma(x)p_n'(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \tag{1.12}$$
$$xp_n'(x) = \alpha_n^\star p_{n+1}'(x) + \beta_n^\star p_n'(x) + \gamma_n^\star p_{n-1}'(x), \tag{1.13}$$
$$p_n(x) = \hat{a}_n p_{n+1}'(x) + \hat{b}_n p_n'(x) + \hat{c}_n p_{n-1}'(x), \tag{1.14}$$

$$\sigma(x)p_n''(x) = a_n' p_{n+1}'(x) + b_n' p_n'(x) + c_n' p_{n-1}'(x). \tag{1.15}$$

Koepf and Schmersau [1998], using computer algebra, obtained explicitly in terms of a, b, c, d, e, k_{n-1}, k_n, and k_{n+1}, the coefficients involved in the above structural identities.

Theorem 1.4. *For orthogonal polynomial solutions of (1.4), the relations (1.11)–(1.15) are valid. The coefficients a_n, b_n, c_n, α_n, β_n, γ_n, $\alpha_n^\star$, $\beta_n^\star$, $\gamma_n^\star$, a_n', b_n', c_n' and $\hat{a}_n$, $\hat{b}_n$, $\hat{c}_n$, are given by*

$$\begin{aligned}
a_n &= \frac{k_n}{k_{n+1}},\\
b_n &= -\frac{2bn(an+d-a) - e(2a-d)}{(d+2an)(d-2a+2an)},\\
c_n &= -(n(an+d-2a)(4ac-b^2) + 4a^2c - ab^2 + ae^2 - 4acd + db^2 - bed + d^2c)\\
&\quad\times \frac{(an+d-2a)n}{(d-2a+2an)^2(2an-3a+d)(2an-a+d)} \cdot \frac{k_n}{k_{n-1}},\\
\alpha_n &= an\frac{k_n}{k_{n+1}},\\
\beta_n &= -\frac{n(an+d-a)(2ea-db)}{(d+2an)(d-2a+2an)},\\
\gamma_n &= ((n-1)(an+d-a)(4ca-b^2) + ae^2 + d^2c - bed)\\
&\quad\times \frac{(an+d-a)(an+d-2a)n}{(d-2a+2an)^2(2an-3a+d)(2an-a+d)} \cdot \frac{k_n}{k_{n-1}},\\
\alpha_n^\star &= \frac{n}{n+1} \cdot \frac{k_n}{k_{n+1}},\\
\beta_n^\star &= \frac{-2bn(an+d-a) + d(b-e)}{(d+2an)(d-2a+2an)},\\
\gamma_n^\star &= -\frac{n((n-1)(an+d-a)(4ac-b^2) + ae^2 + d^2c - bed)(an+d-a)}{(d-2a+2an)^2(2an-3a+d)(2an-a+d)} \cdot \frac{k_n}{k_{n-1}},\\
a_n' &= \frac{an(n-1)}{n+1} \cdot \frac{k_n}{k_{n+1}},\\
b_n' &= -\frac{(n-1)(an+d)(2ea-db)}{(d+2an)(d-2a+2an)},\\
c_n' &= \frac{n((n-1)(an+d-a)(4ca-b^2) + ae^2 + d^2c - bed)(an+d)(an+d-a)}{(d-2a+2an)^2(2an-3a+d)(2an-a+d)} \cdot \frac{k_n}{k_{n-1}},\\
\hat{a}_n &= \frac{1}{n+1}\frac{k_n}{k_{n+1}},\\
\hat{b}_n &= \frac{2ea-db}{(d+2an)(d-2a+2an)},\\
\hat{c}_n &= \frac{na((n-1)(an+d-a)(4ac-b^2) + ae^2 + d^2c - bed)}{(d-2a+2an)^2(2an-3a+d)(2an-a+d)} \cdot \frac{k_n}{k_{n-1}}.
\end{aligned}$$

Equation (1.14) is rather a special connection problem: it expresses the connection between the polynomial systems $\{p_n(x)\}$ and $\{p_{m+1}'(x)\}$. In this case the connection coefficients turn out to be rather simple: almost all of them (namely all with $m < n-2$) are zero.

Using the above structure relations, different authors derived the linearization and connection coefficients of classical continuous orthogonal polynomials using different methods.

1.2.2 First Method

Connection Formula

Here, we review the method presented by Koepf and Schmersau [1998]. We assume that $p_n(x)$ is a polynomial system given by (1.4) with $\sigma(x) = ax^2 + bx + c$, and $\tau(x) = dx + e$, and that $q_m(x)$ is a polynomial system given by (1.4) with $\sigma(x) := \bar{\sigma}(x) = \bar{a}x^2 + \bar{b}x + \bar{c}$, and $\tau(x) := \bar{\tau}(x) = \bar{d}x + \bar{e}$. Note that we will denote all coefficients connected with $q_m(x)$ by dashes. Hence we have

$$\begin{aligned} xp_n(x) &= a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \\ xq_m(x) &= \bar{a}_m q_{m+1}(x) + \bar{b}_m q_m(x) + \bar{c}_m q_{m-1}(x), \end{aligned}$$

with $a_n,\ b_n,\ c_n,\ \bar{a}_m,\ \bar{b}_m,\ \bar{c}_m$ given explicitly.

In three steps, we will now derive three linearly independent recurrence equations for $C_m(n)$. First, substituting $p_n(x) = \sum\limits_{m=0}^{n} C_m(n) q_m(x)$ in the three-term recurrence equation $xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x)$, we get (since $C_{n+s}(n) = 0,\ s = 1, 2, \dots$)

$$\sum_{m=0}^{n} C_m(n) x q_m(x) = \sum_{m=0}^{n+1} \Big(a_n C_m(n+1) q_m(x) + b_n C_m(n) q_m(x) + c_n C_m(n-1) q_m(x) \Big).$$

Then we substitute $xq_m(x)$ in the above equation using the three-term recurrence equation for $q_m(x)$. This yields

$$\begin{aligned} &\sum_{m=0}^{n} C_m(n) \Big(\bar{a}_m q_{m+1}(x) + \bar{b}_m q_m(x) + \bar{c}_m q_{m-1}(x) \Big) = \\ &\sum_{m=0}^{n+1} \Big(a_n C_m(n+1) + b_n C_m(n) + c_n C_m(n-1) \Big) q_m(x). \end{aligned}$$

By appropriate index shifts, we can equate the coefficient of $q_m(x)$ to get the "cross rule"

$$a_n C_m(n+1) + b_n C_m(n) + c_n C_m(n-1) = \bar{a}_{m-1} C_{m-1}(n) + \bar{b}_m C_m(n) + \bar{c}_{m+1} C_{m+1}(n). \quad (1.16)$$

To deduce a second cross rule in terms of the same variables $C_m(n+1)$, $C_m(n)$, $C_m(n-1)$, $C_{m-1}(n)$ and $C_{m+1}(n)$, we examine the term $xp'_n(x)$. Using both three-term recurrence equations for the derivatives

$$\begin{aligned} xp'_n(x) &= \alpha^\star_n p'_{n+1}(x) + \beta^\star_n p'_n(x) + \gamma^\star_n p'_{n-1}(x) \\ xq'_m(x) &= \bar{\alpha}^\star{}_m q'_{m+1}(x) + \bar{\beta}^\star{}_m q'_m(x) + \bar{\gamma}^\star{}_m q'_{m-1}(x), \end{aligned}$$

we get

$$xp_n'(x) = \alpha_n^\star p_{n+1}'(x) + \beta_n^\star p_n'(x) + \gamma_n^\star p_{n-1}'(x)$$
$$\Updownarrow$$
$$\sum_{m=0}^{n} C_m(n) x q_m'(x) = \sum_{m=0}^{n+1} \Big(\alpha_n^\star C_m(n+1) + \beta_n^\star C_m(n) + \gamma_n^\star C_m(n-1)\Big) q_m'(x)$$
$$\Updownarrow$$
$$\sum_{m=0}^{n} C_m(n)\Big(\bar{\alpha^\star}_m q_{m+1}'(x) + \bar{\beta^\star}_m q_m'(x) + \bar{\gamma^\star}_m q_{m-1}'(x)\Big) =$$
$$\sum_{m=0}^{n+1} \Big(\alpha_n^\star C_m(n+1) + \beta_n^\star C_m(n) + \gamma_n^\star C_m(n-1)\Big) q_m'(x).$$

Again, by appropriate index shifts, we can equate the coefficient of $q_m'(x)$ to get the cross rule

$$\alpha_n^\star C_m(n+1)+\beta_n^\star C_m(n)+\gamma_n^\star C_m(n-1) = \bar{\alpha}_{m-1}^\star C_{m-1}(n)+\bar{\beta}_m^\star C_m(n)+\bar{\gamma}_{m+1}^\star C_{m+1}(n). \quad (1.17)$$

In a similar way the cross rule

$$\hat{a}_n C_m(n+1) + \hat{b}_n C_m(n) + \hat{c}_n C_m(n-1) = \bar{\hat{a}}_{m-1} C_{m-1}(n) + \bar{\hat{b}}_m C_m(n) + \bar{\hat{c}}_{m+1} C_{m+1}(n) \quad (1.18)$$

can be obtained from (1.14). It turns out, however, that this relation is linearly dependent from (1.16) and (1.17), and hence does not yield new information. To obtain reasonably simple results, we now assume furthermore that $\bar{\sigma}(x) = \sigma(x)$.

Connection Formula with $\bar{\sigma}(x) = \sigma(x)$

Using both derivatives rules

$$\sigma(x)p_n'(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x),$$
$$\bar{\sigma}(x)q_m'(x) = \bar{\alpha}_m q_{m+1}(x) + \bar{\beta}_m q_m(x) + \bar{\gamma}_m q_{m-1}(x),$$

we get

$$\sigma(x)p_n'(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x)$$
$$\Updownarrow$$
$$\sum_{m=0}^{n} C_m(n)\sigma(x) q_m'(x) = \sum_{m=0}^{n+1} \Big(\alpha_n C_m(n+1) + \beta_n C_m(n) + \gamma_n C_m(n-1)\Big) q_m(x)$$
$$\Updownarrow$$
$$\sum_{m=0}^{n} C_m(n)\Big(\bar{\alpha}_m q_{m+1}(x) + \bar{\beta}_m q_m(x) + \bar{\gamma}_m q_{m-1}(x)\Big) =$$
$$\sum_{m=0}^{n+1} \Big(\alpha_n C_m(n+1) + \beta_n C_m(n) + \gamma_n C_m(n-1)\Big) q_m(x).$$

Again, by appropriate index shifts, this results in the cross rule

$$\alpha_n C_m(n+1)+\beta_n C_m(n)+\gamma_n C_m(n-1) = \bar{\alpha}_{m-1} C_{m-1}(n)+\bar{\beta}_m C_m(n)+\bar{\gamma}_{m-1} C_{m-1}(n). \quad (1.19)$$

To obtain a pure recurrence equation with respect to m, from the three cross rules (1.16), (1.17), and (1.19) by linear algebra we eliminate the variables $C_m(n+1)$ and $C_m(n-1)$, and to obtain a pure recurrence equation with respect to n, we eliminate the variables $C_{m-1}(n)$ and $C_{m+1}(n)$. This yields a second-order recurrence equation satisfied by the connection coefficients $C_m(n)$.

In different cases where $\bar{\sigma}(x) \neq \sigma(x)$, we need the power representation.

Power Representation or Inversion Formula

In many applications, one wants to develop a given polynomial in terms of a given orthogonal polynomial system. In this case handy formulas for the power x^n like

$$x^n = \sum_{m=0}^{n} I_m(n) q_m(x)$$

are very welcome. We remark that this formula is the connection formula for the specific case $p_n(x) = x^n$, and is called inversion formula.

For $q_m(x)$, we have the differential equation

$$\bar{\sigma}(x) q_m''(x) + \bar{\tau}(x) q_m'(x) + \bar{\lambda}_m q_m(x) = 0$$

with $\bar{\sigma}(x) = \bar{a}x^2 + \bar{b}x + \bar{c}$, and the derivative rule

$$\bar{\sigma}(x) q_m'(x) = \bar{\alpha}_m q_{m+1}(x) + \bar{\beta}_m q_m(x) + \bar{\gamma}_m q_{m-1}(x).$$

Our current $p_n(x) = x^n$ satisfies

$$\begin{aligned} \bar{\sigma}(x) p_n'(x) &= (\bar{a}x^2 + \bar{b}x + \bar{c}) n x^{n-1} \\ &= \bar{a} n x^{n+1} + \bar{b} n x^n + \bar{c} n x^{n-1} \\ &= \bar{a} n p_{n+1}(x) + \bar{b} n p_n(x) + \bar{c} n p_{n-1}(x), \end{aligned}$$

$$x p_n(x) = p_{n+1}(x), \;\; x p_n'(x) = \frac{n}{n+1} p_{n+1}'(x), \;\; p_n(x) = \frac{1}{n+1} p_{n+1}'(x).$$

Hence in our situation, we get the cross rule (1.16) with $a_n = 1$, $b_n = c_n = 0$

$$I_m(n+1) = \bar{a}_{m-1} I_{m-1}(n) + \bar{b}_m I_m(n) + \bar{c}_{m+1} I_{m+1}(n), \tag{1.20}$$

the cross rule (1.17) with $\alpha_n^\star = \frac{n}{n+1}$, $\beta_n^\star = \gamma_n^\star = 0$

$$\frac{n}{n+1} I_m(n+1) = \bar{\alpha^\star}_{m-1} I_{m-1}(n) + \bar{\beta^\star}_m I_m(n) + \bar{\gamma^\star}_{m+1} I_{m+1}(n), \tag{1.21}$$

the cross rule (1.18) with $\hat{a}_n = \frac{1}{n+1}$, $\hat{b}_n = \hat{c}_n = 0$

$$\frac{1}{n+1} I_m(n+1) = \bar{\hat{a}}_{m-1} I_{m-1}(n) + \bar{\hat{b}}_m I_m(n) + \bar{\hat{c}}_{m+1} I_{m+1}(n),$$

and the cross rule

$$\bar{a} n I_m(n+1) + \bar{b} n I_m(n) + \bar{c} n I_m(n-1) = \bar{\alpha}_{m-1} I_{m-1}(n) + \bar{\beta}_m I_m(n) + \bar{\gamma}_{m-1} I_{m-1}(n).$$

We substitute the cross rule (1.20) in (1.21), and then we obtain a pure recurrence equation with respect to m.

Remark 1.5. *In many instances the recurrence equations reduce to two terms. Then their hypergeometric term solutions are identified. If the recurrence equation is of order greater than 1, we use the Petkovšek-van-Hoeij algorithm to get its hypergeometric term solutions.*

Using the hypergeometric representation of the polynomial $p_n(x)$ and the inversion formula of $q_m(x)$, we get the general connection formula.

Connection Formula: General Case

In general, to find the coefficients $C_m(n)$ in the relation

$$p_n(x) = \sum_{m=0}^{n} C_m(n) q_m(x),$$

we combine

$$p_n(x) = \sum_{j=0}^{n} A_j(n) x^j \text{ and } x^j = \sum_{m=0}^{j} I_m(j) q_m(x),$$

which yields the representation

$$p_n(x) = \sum_{j=0}^{n} \sum_{m=0}^{j} A_j(n) I_m(j) q_m(x),$$

and then, interchanging the order of summation gives

$$C_m(n) = \sum_{j=0}^{n-m} A_{j+m}(n) I_m(j+m).$$

For orthogonal polynomials with even weight such as Hermite and Gegenbauer polynomials, we have the relations

$$p_n(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} A_j(n) x^{n-2j} \text{ and } x^j = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} I_m(j) q_{j-2m}(x),$$

from which we deduce

$$x^{n-2j} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor - j} I_m(n-2j) q_{n-2j-2m}(x).$$

Finally, we combine the above two expressions and substitute m by $m-j$ to get

$$C_m(n) = \sum_{j=0}^{m} A_j(n) I_{m-j}(n-2j),$$

with

$$p_n(x) = \sum_{m=0}^{n} C_m(n) q_{n-2m}(x).$$

Since the summand $F(j,m,n) := A_j(n) I_m(j)$ of $C_m(n)$ turns out to be a hypergeometric term with respect to (j,m,n), i.e., the term ratios $F(j+1,m,n)/F(j,m,n)$, $F(j,m+1,n)/F(j,m,n)$, and $F(j,m,n+1)/F(j,m,n)$ are rational functions, Zeilberger's (combined with the Petkovšek-van-Hoeij) algorithm applies. If a hypergeometric term solution exists, the representation of $C_m(n)$ follows then from the initial values $C_n(n) = k_n/\bar{k}_n$, $C_{n+s}(n) = 0$, $s = 1, 2, \ldots$, where k_n, $\bar{k}_n$ are, respectively, the leading coefficients of $p_n(x)$ and $q_n(x)$.

Linearization Formula

The linearization formula

$$p_n(x)q_m(x) = \sum_{k=0}^{n+m} L_k(m,n)y_k(x)$$

follows from the hypergeometric representation of the polynomials $p_n(x)$, $q_m(x)$ and the inversion formula of the polynomials $y_k(x)$.

In fact, if

$$p_n(x) = \sum_{i=0}^{n} A_i(n)x^i \text{ and } q_m(x) = \sum_{j=0}^{m} B_j(m)x^j,$$

then by the Cauchy product

$$p_n(x)q_m(x) = \sum_{l=0}^{n+m} G_l(m,n)x^l,$$

with

$$G_l(m,n) = \sum_{i=0}^{l} A_i(n)B_{l-i}(m).$$

Combining the preceding result with the inversion formula

$$x^l = \sum_{k=0}^{l} I_k(l)y_k(x),$$

we get

$$\begin{aligned} L_k(m,n) &= \sum_{l=0}^{n+m-k} G_{l+k}(m,n)I_k(l+k) \\ &= \sum_{l=0}^{n+m-k} \sum_{i=0}^{l+k} I_k(l+k)A_i(n)B_{l+k-i}(m). \end{aligned}$$

We note that we can apply Zeilberger's and the Petkovšek-van-Hoeij algorithm to reduce (if possible) $G_l(m,n)$ into a hypergeometric term and/or $L_k(m,n)$ into a single sum or a hypergeometric term.

1.2.3 Second Method: The NaViMa Algorithm

In this section, we describe a recurrent algorithm (called NaViMa) to compute recursively the connection and linearization coefficients of CCOP [Ronveaux et al., 1995], [Godoy et al., 1997]. We note that the name NaViMa comes from their authors' institutions which are Namur (in Belgium), Vigo and Madrid (in Spain). This method also uses the structural relations (1.11)-(1.15) of CCOP.

Connection Formula

The first step to obtain a recurrence relation for the connection coefficients consists in applying the differential operator $\mathcal{L}_{2,n}:\ \mathcal{P} \longrightarrow \mathcal{P}$, defined by

$$\mathcal{L}_{2,n}[p_n(x)] := \sigma(x)p_n''(x) + \tau(x)p_n'(x) + \lambda_n p_n(x) = 0 \tag{1.22}$$

to both sides of the connection identity $p_n(x) = \sum_{m=0}^{n} C_m(n)q_m(x)$. This gives

$$\sum_{m=0}^{n} C_m(n)\Big[\sigma(x)q_m''(x) + \tau(x)q_m'(x) + \lambda_n q_m(x)\Big] = 0. \tag{1.23}$$

Then, a recurrence relation for the connection coefficients $C_m(n)$ of maximum order 4 or 2 results if we expand the expression

$$\sigma(x)q_m''(x) + \tau(x)q_m'(x) + \lambda_n q_m(x). \tag{1.24}$$

in the basis $\{q_m''(x)\}$ (if $\sigma(x) \neq \bar{\sigma}(x)$) or in the basis $\{q_m'(x)\}$ (if $\sigma(x) = \bar{\sigma}(x)$), respectively. Note that we will denote all coefficients connected with $q_m(x)$ using dashes.

Using the $\{q_m''(x)\}$ basis

To consider $\{q_m''(x)\}$ as expanding basis one proceeds as follows. First, Equation (1.14) and its derivative allow to write

$$q_m(x) = \sum_{j=m-2}^{m+2} a_{m,j}q_j''(x), \tag{1.25}$$

with

$$\begin{aligned}
&a_{m,m+2} = \bar{\hat{a}}_m\bar{\hat{a}}_{m+1},\ \ a_{m,m+1} = \bar{\hat{a}}_m(\bar{\hat{b}}_m + \bar{\hat{b}}_{m+1}),\ \ a_{m,m} = \bar{\hat{a}}_m\bar{\hat{c}}_{m+1} + \bar{\hat{b}}_m^2 + \bar{\hat{c}}_m\bar{\hat{a}}_{m-1},\\
&a_{m,m-1} = \bar{\hat{c}}_m(\bar{\hat{b}}_m + \bar{\hat{b}}_{m-1}),\ \ a_{m,m-2} = \bar{\hat{c}}_m\bar{\hat{c}}_{m-1}.
\end{aligned}$$

Second, from (1.13) and the derivative of (1.14) one has

$$\tau(x)q_m'(x) = \sum_{j=m-2}^{m+2} a_{m,j}^{(1)}q_j''(x), \tag{1.26}$$

with

$$\begin{aligned}
&a_{m,m+2}^{(1)} = d\bar{\alpha}_m^\star\bar{\hat{a}}_{m+1},\ \ a_{m,m+1}^{(1)} = d\Big(\bar{\alpha}_m^\star\bar{\hat{b}}_{m+1} + \bar{\beta}_m^\star\bar{\hat{a}}_m\Big) + e\bar{\hat{a}}_m,\ \ a_{m,m-2}^{(1)} = d\bar{\gamma}_m^\star\bar{\hat{c}}_{m-1}\\
&a_{m,m-1}^{(1)} = d\Big(\bar{\beta}_m^\star\bar{\hat{c}}_m + \bar{\gamma}_m^\star\bar{\hat{b}}_{m-1}\Big) + e\bar{\hat{c}}_m,\ \ a_{m,m}^{(1)} = d\Big(\bar{\alpha}_m^\star\bar{\hat{c}}_{m+1} + \bar{\beta}_m^\star\bar{\hat{b}}_m + \bar{\gamma}_m^\star\bar{\hat{a}}_{m-1}\Big) + e\bar{\hat{b}}_m.
\end{aligned}$$

And third, from the derivative of (1.13) and the derivative of (1.14) we get

$$\sigma(x)q_m''(x) = \sum_{j=m-2}^{m+2} a_{m,j}^{(2)}q_j''(x), \tag{1.27}$$

with

$$a^{(2)}_{m,m} = a\Big((\bar{\beta}^\star_m - \bar{\bar{b}}_m)^2 + (\bar{\gamma}^\star_m - \bar{\hat{c}}_m)(\bar{\alpha}^\star_{m-1} - \bar{\hat{a}}_{m-1}) + (\bar{\alpha}^\star_m - \bar{\hat{a}}_m)(\bar{\gamma}^\star_{m+1} - \bar{\hat{c}}_{m+1})\Big)$$
$$+b(\bar{\beta}^\star_m - \bar{\bar{b}}_m) + c,\ a^{(2)}_{m,m-1} = (\bar{\hat{c}}_m - \bar{\gamma}^\star_m)\Big(a(\bar{\bar{b}}_m + \bar{\bar{b}}_{m-1} - \bar{\beta}^\star_m - \bar{\beta}^\star_{m-1}) - b\Big),$$
$$a^{(2)}_{m,m+1} = (\bar{\hat{a}}_m - \bar{\alpha}^\star_m)\Big(a(\bar{\bar{b}}_m + \bar{\bar{b}}_{m+1} - \bar{\beta}^\star_m - \bar{\beta}^\star_{m+1}) - b\Big),$$
$$a^{(2)}_{m,m-2} = a(\bar{\gamma}^\star_m - \bar{\hat{c}}_m)(\bar{\gamma}^\star_{m-1} - \bar{\hat{c}}_{m-1}),\ a^{(2)}_{m,m+2} = a(\bar{\alpha}^\star_m - \bar{\hat{a}}_m)(\bar{\alpha}^\star_{m+1} - \bar{\hat{a}}_{m+1}).$$

Now, inserting (1.25)-(1.27) in Equation (1.23) we obtain

$$\sum_{m=0}^{n} C_m(n)\Big\{\sum_{j=m-2}^{m+2} \Omega_{m,j}(n) q''_j(x)\Big\} = 0,\quad \Omega_{m,j}(n) = a^{(2)}_{m,j} + a^{(1)}_{m,j} + \lambda_n a_{m,j}.$$

Finally, after an appropriate shift of indices, this latter expression provides a recurrence relation of maximum order four which can be written as

$$\sum_{s=0}^{4} \Omega_{m+s,m+2}(n) C_{m+s}(n) = 0,\ 0 \le m \le n-1,$$

with the initial conditions given by $C_{n+s}(n) = 0$ $(s = 1, 2, 3)$ and $C_n(n) = k_n/\bar{k}_n$.

Using the $\{q'_m(x)\}$ Basis

If the differential equations satisfied by the polynomials $p_n(x)$ and $q_m(x)$ are such that $\sigma(x) = \bar{\sigma}(x)$, then the maximal order of the recurrence relation for the connection coefficients decreases from 4 to 2 when $\{q'_m(x)\}$ is the expanding basis for expression (1.24). For this reason, this basis should be used when dealing with the Jacobi-Jacobi, Gegenbauer-Gegenbauer, Laguerre-Laguerre, and Bessel-Bessel connection problems.

The algorithm in this case is as follows. First, Equation (1.14) for $q_m(x)$ can be written again as

$$q_m(x) = \sum_{j=m-1}^{m+1} b_{m,j} q'_j(x), \tag{1.28}$$

with $b_{m,m+1} = \bar{\hat{a}}_m$, $b_{m,m} = \bar{\bar{b}}_m$, $b_{m,m-1} = \bar{\hat{c}}_m$. Second, from the three-term recurrence relation (1.13) for the family $q'_m(x)$, we write

$$\tau(x) q'_m(x) = \sum_{j=m-1}^{m+1} b^{(1)}_{m,j} q'_j(x), \tag{1.29}$$

with $b^{(1)}_{m,m+1} = d\bar{\alpha}^\star_m$, $b^{(1)}_{m,m} = d\bar{\beta}^\star_m + e$, $b^{(1)}_{m,m-1} = d\bar{\gamma}^\star_m$. And third, since $\sigma(x) = \bar{\sigma}(x)$, Equation (1.15) satisfied by $q_m(x)$ can be written again

$$\sigma(x) q''_m(x) = \sum_{j=m-1}^{m+1} b^{(2)}_{m,j} q'_j(x), \tag{1.30}$$

with $b^{(2)}_{m,m+1} = \bar{a'_m}$, $b^{(2)}_{m,m} = \bar{b'_m}$, $b^{(2)}_{m,m-1} = \bar{c'_m}$.

Insertion of (1.28)-(1.30) into (1.23) gives

$$\sum_{m=0}^{n} C_m(n)\Big\{\sum_{j=m-1}^{m+1} \Lambda_{m,j}(n) q_j'(x)\Big\} = 0,\ \ \Lambda_{m,j}(n) = b_{m,j}^{(2)} + b_{m,j}^{(1)} + \lambda_n b_{m,j}.$$

Finally, after an appropriate shift of indices, this latter expression provides a recurrence relation of maximum order two which can be written as

$$\sum_{s=-1}^{1} \Lambda_{m+s,m}(n) C_{m+s}(n) = 0,\ \ 1 \le m \le n,$$

with initial conditions given by $C_{n+1}(n) = 0$ and $C_n(n) = k_n/\bar{k}_n$.

Generalized linearization problem

Stanley [1980], Ronveaux [1988] (see also [Ronveaux, 1991], [Salvy and Zimmermann, 1994]) showed that if the polynomials $p_n(x)$ and $q_m(x)$ are solutions of second order differential equations of type (1.4), then the product $p_n(x)q_m(x)$ is solution of a fourth order differential equation of the form

$$\mathbf{L}u(x) := r_4(x)u^{(4)}(x) + r_3(x)u^{(3)}(x) + r_2(x)u''(x) + r_1(x)u'(x) + r_0(x)u(x) = 0,$$

where $r_i(x)$ are polynomials of degree at most i, $i = 0, 1, 2, 3, 4$. Salvy and Zimmermann [1994] developed an algorithm to compute the differential equation satisfied by the product $p_n(x)q_m(x)$ if the ones of $p_n(x)$ and $q_m(x)$ are known. Here, we restrict ourselves to the special case when $p_n(x)$, $q_m(x)$ and $y_k(x)$ belong to the same polynomial family.

Using the differential equation of the product $p_n(x)q_m(x)$, we proceed as follows to get the recurrence relation of the linearization coefficients (see e.g. [Godoy et al., 1997]). First of all, we apply the operator $\mathbf{L}$ to both sides of the linearization formula

$$p_n(x)q_m(x) = \sum_{k=0}^{n+m} L_k(m,n) y_k(x),$$

and multiply the obtained equation by $\sigma(x)$ to get

$$\sum_{k=0}^{n+m} L_k(m,n)\sigma(x)\Big(r_4(x)y_k^{(4)}(x) + r_3(x)y^{(3)}(x) + r_2(x)y''(x) + r_1(x)y'(x) + r_0(x)y(x)\Big) = 0.$$

Since $y_k(x)$ is solution of a differential equation of type (1.4), we can write $\sigma(x)y_k^{(4)}(x)$, $\sigma(x)y_k^{(3)}(x)$, $\sigma(x)y''(x)$ as linear combinations of $y_k'(x)$ and $y_k(x)$ with polynomial coefficients, and therefore the previous equation can be written in the form

$$\sum_{k=0}^{n+m} L_k(m,n)\Big(f(x)y_k'(x) + g(x)y_k(x)\Big) = 0,$$

where $f(x)$ and $g(x)$ are polynomials. We use the structure relations (1.13) and (1.14) (for $y_k(x)$) to expand $f(x)y_k'(x) + g(x)y_k(x)$ in the basis $y_k'(x)$. By a shift of indices, we get a recurrence equation w.r.t. the variable k satisfied by the coefficients $L_k(m,n)$.

Remark 1.6. *Hylleraas [1962] computed the differential equation satisfied by the product of Jacobi polynomials and used it to set up a recurrence relation for the linearization coefficients of Gegenbauer polynomials. For Jacobi polynomials, the situation was far from satisfactory.*

Lewanowicz [1996b] obtained by a method which is alternative to NaViMa a second-order recurrence relation for the linearization coefficients.

1.3 Integral Evaluation of the Connection and Linearization Coefficients of CCOP

In this section, we review the work of Sánchez-Ruiz et al. [1999] who used integral evaluations to obtain the connection and linearization coefficients of continuous hypergeometric-type polynomials. We also notice that Ismail [2005, Chapter 9] used the same method to evaluate the connection relation for Jacobi polynomials.

Let $(y_n)_{n\geq 0}$ be a polynomial family, orthogonal with respect to the weight $\rho(x)$ defined on (A, B). Let us denote by $(p_n(x))_{n\geq 0}$ and $(q_n(x))_{n\geq 0}$ two (possibly different) polynomial sequences, not necessarily orthogonal or hypergeometric. Then we have:

Theorem 1.7 ([Sánchez-Ruiz et al., 1999]). *The linearization coefficients $L_k(m,n)$ of*

$$p_n(x)q_m(x) = \sum_{k=0}^{n+m} L_k(m,n)y_k(x) \tag{1.31}$$

are given by

$$L_k(m,n) = \frac{1}{k!a_k\hat{\mu}_k}\sum_{j=j_-}^{j_+}\binom{k}{j}\int_A^B \frac{d^j p_n(x)}{dx^j}\frac{d^{k-j}q_m(x)}{dx^{k-j}}\Big[\sigma(x)^k\rho(x)\Big]dx, \tag{1.32}$$

where $j_- = \max(0, k-m)$, $j_+ = \min(k,n)$.

Proof. If we multiply (1.31) by $y_l(x)\rho(x)$ (for a fixed l) and integrate from A to B, we get

$$\int_A^B p_n(x)q_m(x)y_l(x)\rho(x)dx = \sum_{k=0}^{n+m} L_k(m,n)\int_A^B y_k(x)y_l(x)\rho(x)dx.$$

It follows therefore from the orthogonality relation (1.1) of the family $(y_n(x))_{n\geq 0}$ that the linearization coefficients $L_k(m,n)$ are given by

$$L_k(m,n) = \frac{1}{h_k^2}\int_A^B p_n(x)q_m(x)y_k(x)\rho(x)dx. \tag{1.33}$$

Thus the integral evaluation of the product of three orthogonal polynomials of the same kind is equivalent to the linearization problem. One way of obtaining linearization coefficients would therefore be to look how the previous integral can be computed. Taking advantage of Equations (1.5) and (1.6), we rewrite (1.33) as

$$L_k(m,n) = \frac{(-1)^k}{k!a_k\hat{\mu}_k}\int_A^B p_n(x)q_m(x)\frac{d^k}{dx^k}\Big[\sigma(x)^k\rho(x)\Big]dx. \tag{1.34}$$

We show by induction on $j \in \mathbb{N}$ that for every $P \in \mathcal{P}$, there exists $Q \in \mathcal{P}$ such that

$$\frac{d^j}{dx^j}\Big[\sigma(x)^k \rho(x) P(x)\Big] = \sigma(x)^{k-j}\rho(x)Q(x),\ j \leq k. \tag{1.35}$$

Indeed, for $j = 1$, using Pearson's equation (1.2), we have

$$\begin{aligned}\frac{d}{dx}\Big[\sigma(x)^k \rho(x) P(x)\Big] &= \sigma(x)^{k-1}\rho(x)\Big(\tau(x)P(x) + (k-1)\sigma'(x)P(x) + \sigma(x)P'(x)\Big)\\ &= \sigma(x)^{k-1}\rho(x)Q_{1,k}(x)\end{aligned}$$

with $Q_{1,k}(x) = \tau(x)P(x) + (k-1)\sigma'(x)P(x) + \sigma(x)P'(x)$. We suppose that (1.35) is true for $j \geq 2$, and then it follows that

$$\begin{aligned}\frac{d^{j+1}}{dx^{j+1}}\Big[\sigma(x)^k \rho(x) P(x)\Big] &= \frac{d}{dx}\Big(\sigma(x)^{k-j}\rho(x)Q(x)\Big)\\ &= \sigma(x)^{k-j-1}\rho(x)Q_{1,k-j}(x).\end{aligned}$$

Thus, integrating by parts and taking into account, respectively, (1.35) and the boundary conditions (1.3), we rewrite (1.34) in the form

$$\begin{aligned}L_k(m,n) &= \frac{(-1)^k}{k!a_k\hat{\mu}_k}\Bigg(p_n(x)q_m(x)\frac{d^{k-1}}{dx^{k-1}}\Big[\sigma(x)^k\rho(x)\Big]\Big|_A^B\\ &\quad - \int_A^B \frac{d}{dx}\Big(p_n(x)q_m(x)\Big)\frac{d^{k-1}}{dx^{k-1}}\Big[\sigma(x)^k\rho(x)\Big]dx\Bigg)\\ &= \frac{(-1)^k}{k!a_k\hat{\mu}_k}\Bigg(p_n(x)q_m(x)\sigma(x)\rho(x)Q(x)\Big|_A^B\\ &\quad - \int_A^B \frac{d}{dx}\Big(p_n(x)q_m(x)\Big)\frac{d^{k-1}}{dx^{k-1}}\Big[\sigma(x)^k\rho(x)\Big]dx\Bigg)\\ &= \frac{(-1)^{k+1}}{k!a_k\hat{\mu}_k}\int_A^B \frac{d}{dx}\Big(p_n(x)q_m(x)\Big)\frac{d^{k-1}}{dx^{k-1}}\Big[\sigma(x)^k\rho(x)\Big]dx.\end{aligned}$$

We repeat the integration by parts $k-1$ times and use again (1.35) and (1.3) to obtain

$$L_k(m,n) = \frac{1}{k!a_k\hat{\mu}_k}\int_A^B \frac{d^k}{dx^k}\Big(p_n(x)q_m(x)\Big)\Big[\sigma(x)^k\rho(x)\Big]dx. \tag{1.36}$$

If $j > n$, $\dfrac{d^j}{dx^j}p_n(x) = 0$ and if $j < k-m$, $k-j > m$ and then $\dfrac{d^{k-j}}{dx^{k-j}}q_m(x) = 0$ such that from Leibniz's rule

$$\frac{d^k}{dx^k}(p_n(x)q_m(x)) = \sum_{j=0}^{k}\binom{k}{j}\frac{d^j}{dx^j}p_n(x)\frac{d^{k-j}}{dx^{k-j}}q_m(x)$$

the result follows. □

In particular, taking $p_n(x) = x^n$ and $q_m(x) = 1$, we obtain the solution of the inversion problem in terms of the moments of the weights $\rho_k(x)$:

Proposition 1.8 ([Sánchez-Ruiz et al., 1999]). *The coefficients $I_k(n)$ of the inversion problem*

$$x^n = \sum_{k=0}^{n} I_k(n) y_k(x)$$

are given by

$$I_k(n) = \binom{n}{k} \frac{1}{a_k \hat{\mu}_k} \int_A^B x^{n-k} \rho_k(x) dx.$$

Proof. Since $q_m(x) = 1$, $\frac{d^{k-j} q_m(x)}{dx^{k-j}} = \begin{cases} 0 \text{ if } j \neq k \\ 1 \text{ if } j = k. \end{cases}$ Thus for $P_n(x) = x^n$, we get

$$\begin{aligned} I_k(n) &= \frac{1}{k! a_k \hat{\mu}_k} \int_a^b \frac{d^k x^n}{dx^k} \sigma(x)^k \rho(x) dx \\ &= \binom{n}{k} \frac{1}{a_k \hat{\mu}_k} \int_a^b x^{n-k} \sigma(x)^k \rho(x) dx. \end{aligned}$$

□

We note that Sánchez-Ruiz and Dehesa [1997], Ismail [2005] used the above proposition to solve the inversion problem for CCOP.

Let us assume now that both $p_n(x)$ and $q_m(x)$ are also polynomials of hypergeometric type. Equation (1.32) is practical for the computation of the generalized linearization coefficients whenever the explicit expressions of the polynomials $\frac{d^j p_n(x)}{dx^j}$ and $\frac{d^{k-j} q_m(x)}{dx^{k-j}}$ are known, as is the case, e.g., for the classical hypergeometric families. For general families of polynomials, when only the coefficients of the corresponding differential operators are available, we can make one more step and find an equivalent expression for $L_k(m,n)$ that does not require the knowledge of the explicit expressions of the polynomials. We restrict our attention to the particular case when the three families of hypergeometric polynomials coincide $\Big(p_n(x) = q_n(x) = y_n(x)\Big)$, for which we have the following

Theorem 1.9 ([Sánchez-Ruiz et al., 1999]). *If $p_n(x) = q_n(x) = y_n(x)$, the linearization coefficients $L_k(m,n)$ in (1.31) are given by*

$$\begin{aligned} L_k(m,n) = &\frac{(B_n)^2}{k! a_k \hat{\mu}_k} \sum_{j=j_-}^{j_+} A_{nj} (-1)^{m+k+j} \binom{k}{j} \sum_{i=i_-}^{i_+} (-1)^i A_{m(n+k-i-2j)} \binom{n-j}{i} \times \\ &\int_A^B \rho_m(x) \frac{d^{m-n+i+2j-k}}{dx^{m-n+i+2j-k}} \left(\sigma(x)^{2j-k+i} \frac{d^i \sigma(x)^{k-i}}{dx^i} \right). \end{aligned} \tag{1.37}$$

Proof. Since $p_n(x) = q_n(x) = y_n(x)$, then

$$L_k(m,n) = \frac{1}{k! a_k \hat{\mu}_k} \sum_{j=j_-}^{j_+} \binom{k}{j} \int_A^B y_n^{(j)}(x) y_m^{k-j}(x) \rho_k(x) dx.$$

Using Equation (1.9) for $y_n^{(j)}(x)$, the above expression can be written as

$$L_k(m,n) = \frac{B_n}{k! a_k \hat{\mu}_k} \sum_{j=j_-}^{j_+} A_{nj} \binom{k}{j} \int_A^B \frac{d^{n-j} \rho_n(x)}{dx^{n-j}} y_m^{(k-j)}(x) \sigma(x)^{k-j} dx.$$

Observe that by (1.9) and (1.35), there exist two polynomials $Q_1(x)$ and $Q_2(x)$ such that for $0 \leq l \leq n-j$,

$$\begin{aligned} term &:= \frac{d^l}{dx^l}\left(y_m^{(k-j)}(x)\sigma(x)^{k-j}\right)\frac{d^{n-j-l-1}}{dx^{n-j-l-1}}(\rho_n(x)) \\ &= A_{m(k-j)}B_m\frac{d^l}{dx^l}\left(\frac{1}{\rho(x)}\frac{d^{m-k+j}\rho_m(x)}{dx^{m-k+j}}\right)\sigma(x)^{j+l+1}\rho(x)Q_2(x) \\ &= A_{m(k-j)}B_m\frac{d^l}{dx^l}\left(\sigma(x)^{k-j}Q_1(x)\right)\sigma(x)^{j+l+1}\rho(x)Q_2(x) \\ &= \sum_{i=0}^{N} a_i\sigma(x)\rho(x)x^i,\ a_i \in \mathbb{R},\ N \in \mathbb{N}. \end{aligned}$$

Thus, integrating by parts $n-j$ times and taking into account the boundary conditions (1.3),

$$L_k(m,n) = \frac{B_n}{k!a_k\hat{\mu}_k}\sum_{j=j_-}^{j_+} A_{nj}(-1)^{n-j}\binom{k}{j}\int_A^B \rho_n(x)\frac{d^{n-j}}{dx^{n-j}}(y_m^{(k-j)}(x)\sigma(x)^{k-j})dx.$$

Using the Leibniz rule,

$$L_k(m,n) = \frac{B_n}{k!a_k\hat{\mu}_k}\sum_{j=j_-}^{j_+} A_{nj}(-1)^{n-j}\binom{k}{j}\sum_{i=i_-}^{i_+}\binom{n-j}{i}\int_A^B \rho_n(x)\frac{d^i\sigma(x)^{k-j}}{dx^i}y_m^{(n+k-2j-i)}dx,$$

where $i_- = \max\{0, n-m+k-2j\}$, $i_+ = \min\{n-j, (k-j)\deg[\sigma(x)]\}$. In fact, $\frac{d^i\sigma(x)^{k-j}}{dx^i} \neq 0$ if and only if $i \leq (k-j)\deg[\sigma(x)]$ and $y_m^{(n+k-2j-i)} \neq 0$ if and only if $n+k-2j-i \leq m \Leftrightarrow i \geq n-m+k-2j$. We substitute the expression of $y_m^{(n+k-2j-i)}$ given by the Rodrigues formula (1.9), integrate by parts again, use Equation (1.35) and the boundary conditions (1.3) to get the result. □

In spite of its apparent complexity, this formula has the advantage that no derivatives of the weight functions are involved; it does not make use of the expressions of the polynomials either. In fact, if we know $\sigma(x)$ we can express the integrals appearing in (1.37) as a linear combination of the moments of the weight function $\rho_m(x)$, which makes this equation suitable for symbolic manipulation.

Let us consider now the connection problem

$$p_n(x) = \sum_{k=0}^{n} C_k(n)y_k(x) \tag{1.38}$$

where $\{p_n(x)\}$ is the sequence of polynomial of degree n, solution of the differential equation

$$\bar{\sigma}(x)p_n''(x) + \bar{\tau}(x)p_n'(x) + \bar{\lambda}_n p_n(x) = 0.$$

Taking $q_m(x) := 1$ in (1.31), we readily see that $C_k(n) = L_k(0,n)$, so that the connection coefficients $C_k(n)$ can be obtained as the particular case $m = 0$ of both (1.33) and (1.32). Again, it turns out to be much more convenient to use (1.32), which leads to

$$C_k(n) = \frac{1}{k!a_k\hat{\mu}_k}\int_A^B p_n^{(k)}(x)\rho_k(x)dx, \tag{1.39}$$

which does not require the use of the representation for $y_k(x)$. We note that the previous formula was already proved by [Rainville, 1960, Theorem 56, p. 151].

By Equation (1.9), the previous formula for $C_k(n)$ can be written as

$$C_k(n) = \frac{\bar{A}_{nk}\bar{B}_n}{k!a_k\hat{\mu}_k}\int_A^B \frac{\rho_k(x)}{\bar{\rho}_k(x)}\frac{d^{n-k}\bar{\rho}_n(x)}{dx^{n-k}}dx$$

or, equivalently, integrating by parts $n-k$ times (using (1.35), (1.3)),

$$C_k(n) = \frac{(-1)^{n-k}\bar{A}_{nk}\bar{B}_n}{k!a_k\hat{\mu}_k}\int_A^B \bar{\rho}_n(x)\frac{d^{n-k}}{dx^{n-k}}\Big(\frac{\rho_k(x)}{\bar{\rho}_k(x)}\Big)dx.$$

A common situation in connecting polynomials of the same family, but with different parameters, is when $\bar{\sigma}(x) = \sigma(x)$. In this case, if we put $\bar{\rho}(x) = f(x)\rho(x)$, the previous equation takes the form

$$C_k(n) = \frac{(-1)^{n-k}\bar{A}_{nk}\bar{B}_n}{k!a_k\hat{\mu}_k}\int_A^B f(x)\rho_n(x)\frac{d^{n-k}}{dx^{n-k}}\Big(\frac{1}{f(x)}\Big)dx,$$

which may be useful if the derivatives of $\frac{1}{f(x)}$ have simple representations.

It is a remarkable fact that, in the case when all the involved polynomials are of hypergeometric type, Equation (1.32) enables us to express the linearization coefficients in terms of two connection coefficients, namely those corresponding to the expansions of the polynomials $p_n^{(j)}(x)$ and $q_m^{(k-j)}(x)$ in series of $\{y_k^{(r)}(x)\}$,

$$p_n^{(j)}(x) = \sum_{r=0}^{n-j} C_{r,n}^{(j,k)}(p)y_r^{(k)}(x), \quad q_m^{(k-j)}(x) = \sum_{s=0}^{m+j-k} C_{s,m}^{(k-j,k)}(q)y_s^{(k)}.$$

Substituting these expressions into (1.32) and using the orthogonality relation (1.7), we obtain

$$\begin{aligned} L_k(m,n) &= \frac{1}{k!a_k\hat{\mu}_k}\sum_{j=j_-}^{j_+}\binom{k}{j}\sum_{r=0}^{n-j}\sum_{s=0}^{m+j-k} C_{r,n}^{(j,k)}(p)C_{s,m}^{(k-j,k)}(q)\int_A^B y_r^{(k)}(x)y_s^{(k)}\rho_k(x)dx \\ &= \frac{1}{k!a_k\hat{\mu}_k}\sum_{j=j_-}^{j_+}\binom{k}{j}\sum_{r=0}^{r_+} C_{r,n}^{(j,k)}(p)C_{r,m}^{(k-j,k)}(q)h_{rk}^2, \end{aligned}$$

where $r_+ = \min(n-j, m+j-k)$. In particular, for the standard linearization coefficients, the previous formula does apply with $p = q = y$, and we can omit the arguments of the connection coefficients to simplify the notation.

1.4 Other Methods

Besides the methods cited above, there exist some other methods regularly used in the literature to compute connection and inversion coefficients of CCOP. In this section, we recall two of them.

1.4.1 Using the Fields and Wimp Expansion Formula

One approach to evaluate the connection coefficients is to think of $p_n(x) = \sum_{m=0}^{n} C_m(n)q_m(x)$ as a polynomial expansion problem. One of the most important general expansion formulas for hypergeometric series is the Fields and Wimp [1961] expansion given by

$$
{}_{p+r}F_{q+s}\left(\left.\begin{matrix} a_1, \ldots, a_p, c_1, \ldots, c_r \\ b_1, \ldots, b_q, d_1, \ldots, d_s \end{matrix}\right| zw\right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_p)_n (\alpha)_n (-z)^n}{(b_1, \ldots, b_q)_n (\gamma+n)_n n!}
\tag{1.40}
$$

$$
\times {}_{p+1}F_{q+1}\left(\left.\begin{matrix} n+\alpha, n+a_1, \ldots, n+a_p \\ 2n+\gamma+1, n+b_1, \ldots, n+b_q \end{matrix}\right| z\right) {}_{r+2}F_{s+1}\left(\left.\begin{matrix} -n, n+\gamma, c_1, \ldots, c_r \\ \alpha, d_1, \ldots, d_s \end{matrix}\right| w\right).
$$

The letters p, q, r and s stand for nonnegative integers.
Proceeding as in [Njionou Sadjang, 2013], we choose $p = q = 0$, $w = x$ and $\gamma = 0$. We expand both sides of (1.40) in the basis $(z^n)_n$ and then equate the coefficients of z^n to obtain

$$
\frac{\prod_{j=1}^{r} (c_j)_n}{\prod_{j=1}^{s} (d_j)_n} x^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} {}_{r+1}F_s\left(\left.\begin{matrix} -k, c_1, \ldots, c_r \\ d_1, \ldots, d_s \end{matrix}\right| x\right).
$$

Using this relation, Njionou Sadjang [2013] derived inversion formulas of some CCOP. Sánchez-Ruiz [2001], by making the substitutions $r = s = 0$, $z = 1$, $w = 1 - x^2$ and by a suitable identification of the remaining parameters, derived from (1.40) connection and linearization formulas involving squares of Gegenbauer polynomials. Lewanowicz [2003a] used the formula (1.40) to obtain hypergeometric term representations and recurrence relations for the connection coefficients of CCOP.

1.4.2 Using Generating Functions

Andrews et al. [1999, p. 318] used a method based on generating functions to find the linearization coefficients of the Hermite polynomials. Rainville [1960] also used generating functions to get inversion and connection formulas of classical continuous orthogonal polynomials. Chaggara and Koepf [2010] starting from the generating function of the Jacobi polynomials and using symbolic computation, in particular Zeilberger's and Petkovšek-van-Hoeij algorithms, computed the linearization coefficients of Jacobi and Gegenbauer polynomials.

1.5 Connection and Linearization Coefficients of CCOP

As an immediate consequence of the above methods, we get the following connection, linearization and inversion coefficient of classical continuous orthogonal polynomials.

Theorem 1.10. *The following representations for the powers in terms of the classical continuous orthogonal polynomials are valid:*

$$
(1-x)^n = 2^n \Gamma(\alpha+n+1) \sum_{m=0}^{n} \frac{(\alpha+\beta+2m+1)\Gamma(\alpha+\beta+m+1)}{\Gamma(\alpha+m+1)\Gamma(\alpha+\beta+n+m+2)} (-n)_m P_m^{(\alpha,\beta)}(x)
$$

(see e.g. [Koepf and Schmersau, 1998], [Ismail, 2005]),

$$(1+x)^n = 2^n\Gamma(\beta+n+1)\sum_{m=0}^{n}(-1)^m(-n)_m\frac{(\alpha+\beta+2m+1)\Gamma(\alpha+\beta+m+1)}{\Gamma(\beta+m+1)\Gamma(\alpha+\beta+n+m+2)}P_m^{(\alpha,\beta)}(x)$$

(see e.g. [Askey, 1975], [Koepf and Schmersau, 1998], [Ismail, 2005]),

$$\begin{aligned}x^n &= \frac{n!}{(\alpha)_n 2^n}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-\frac{n}{2}-\frac{\alpha}{2}+1)_m(-n-\alpha)_m}{(-\frac{n}{2}-\frac{\alpha}{2})_m m!}(-1)^m C_{n-2m}^{\alpha}(x)\\ &= \frac{n!}{2^n}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n+\alpha-2m}{m!(\alpha)_{n+1-m}}C_{n-2m}^{(\alpha)}(x)\end{aligned}$$

(see e.g. [Rainville, 1960], [Koepf and Schmersau, 1998]),

$$x^n = (1+\alpha)_n\sum_{m=0}^{n}\frac{(-n)_m}{(1+\alpha)_m}L_m^{(\alpha)}(x) = n!\sum_{m=0}^{n}\binom{n+\alpha}{n-m}(-1)^m L_m^{(\alpha)}(x)$$

(see e.g. [Sánchez-Ruiz and Dehesa, 1997], [Koepf and Schmersau, 1998], [Ismail, 2005]),

$$x^n = \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-\frac{n}{2})_m(-\frac{n}{2}+\frac{1}{2})_m}{m!2^{n-2m}}H_{n-2m}(x) = \frac{n!}{2^n}\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{m!(n-2m)!}H_{n-2m}(x)$$

(see e.g. [Koepf and Schmersau, 1998], [Ismail, 2005]),

$$\begin{aligned}x^n &= \frac{(-2)^n}{(\alpha+2)_n}\sum_{m=0}^{n}\frac{(-n)_m(\alpha+1)_m(\frac{\alpha}{2}+\frac{3}{2})_m}{(n+2+\alpha)_m(\frac{\alpha}{2}+\frac{1}{2})_m m!}B_m^{(\alpha)}(x)\\ &= (-2)^n\sum_{m=0}^{n}(2m+\alpha+1)\frac{(-n)_m\Gamma(\alpha+m+1)}{m!\Gamma(n+m+\alpha+2)}B_m^{(\alpha)}(x)\end{aligned}$$

(see e.g. [Sánchez-Ruiz and Dehesa, 1997], [Koepf and Schmersau, 1998]).

Theorem 1.11. *The following connection relations between classical orthogonal polynomials are valid:*

$$\begin{aligned}P_n^{(\alpha,\beta)}(x) = &\sum_{m=0}^{n}(2m+\gamma+\beta+1)\frac{\Gamma(n+\beta+1)\Gamma(n+m+\alpha+\beta+1)}{\Gamma(m+\beta+1)\Gamma(n+\alpha+\beta+1)}\\ &\times\frac{\Gamma(m+\gamma+\beta+1)(\alpha-\gamma)_{n-m}}{\Gamma(n+m+\gamma+\beta+2)(n-m)!}P_m^{(\gamma,\beta)}(x)\end{aligned}$$

(see e.g. [Askey, 1975, p. 63], [Koepf and Schmersau, 1998]),

$$\begin{aligned}P_n^{(\alpha,\beta)} = &\sum_{m=0}^{n}(-1)^{n-m}(2m+\alpha+\delta+1)\frac{\Gamma(n+\alpha+1)\Gamma(n+m+\alpha+\beta+1)}{\Gamma(m+\alpha+1)\Gamma(n+\alpha+\beta+1)}\\ &\times\frac{\Gamma(m+\alpha+\delta+1)(\beta-\delta)_{n-m}}{\Gamma(n+m+\alpha+\delta+2)(n-m)!}P_m^{(\alpha,\delta)}(x)\end{aligned}$$

(see e.g. [Askey, 1975, p. 63], [Koepf and Schmersau, 1998], [Ismail, 2005, p. 258]),

$$P_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} \frac{(m+\alpha+1)_{n-m}(n+\alpha+\beta+1)_m}{(n-m)!(m+\gamma+\delta+1)_m} \times {}_3F_2\left(\left.\begin{matrix} m-n, n+m+\alpha+\beta+1, m+\gamma+1 \\ m+\alpha+1, 2m+\gamma+\delta+2 \end{matrix}\right| 1\right) P_m^{(\gamma,\delta)}(x)$$

(see e.g. [Gasper, 1974], [Lewanowicz, 2003a], [Ismail, 2005, p. 257], compare [Sánchez-Ruiz et al., 1999]),

$$C_n^{(\alpha)}(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\alpha-\beta)} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2m+\beta)\Gamma(m+\alpha-\beta)\Gamma(n-m+\alpha)}{m!\Gamma(n-m+\beta+1)} C_{n-2m}^{(\beta)}(x)$$

(see e.g. [Koepf and Schmersau, 1998], [Sánchez-Ruiz et al., 1999], [Ismail, 2005, p. 257]),

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \frac{(\alpha-\beta)_{n-m}}{(n-m)!} L_m^{(\beta)}(x)$$

(see e.g. [Koepf and Schmersau, 1998], [Sánchez-Ruiz et al., 1999]),

$$B_n^{(\alpha)}(x) = \sum_{m=0}^{n} (-1)^m (2m+\beta+1) \frac{(-n)_m (n+\alpha+1)_m \Gamma(m+\beta+1)\Gamma(\beta-\alpha+1)}{m!\Gamma(n+m+\beta+2)\Gamma(m-n+\beta-\alpha+1)} B_m^{(\beta)}(x)$$

(see e.g. [Godoy et al., 1997], [Koepf and Schmersau, 1998], [Sánchez-Ruiz et al., 1999]).

Theorem 1.12. *The following linearization formulas between classical orthogonal polynomials are valid:*

$$\begin{aligned} P_n^{(\lambda,\delta)}(x) P_m^{(\mu,\gamma)}(x) &= \sum_{k=0}^{n+m} \frac{(\alpha+\beta+1)_{n+m-k}(\alpha+1)_{n+m}(2(n+m-k)+\alpha+\beta+1)}{(\alpha+1)_{n+m-k}(\alpha+\beta+1)_{2(n+m)-k+1}} \\ &\times \frac{(-1)^k (n+m)!(\lambda+\delta+1)_{2n}(\mu+\gamma+1)_{2m}}{n!m!k!(\lambda+\delta+1)_n(\mu+\gamma+1)_m} \\ &\times \sum_{r,s=0}^{\infty} \frac{(-k, -\alpha-\beta-1-2(n+m)+k)_{r+s}(-n, -\lambda-n)_r(-m, -\mu-m)_s}{(-n-m, -\alpha-n-m)_{r+s}(-2n-\lambda-\delta)_r(-2m-\mu-\gamma)_s r!s!} P_{n+m-k}^{(\alpha,\beta)}(x), \end{aligned}$$

see [Chaggara and Koepf, 2010],

$$\begin{aligned} P_n^{(\lambda,\delta)}(x) P_m^{(\mu,\gamma)}(x) &= \sum_{k=0}^{n+m} \frac{(\mu+\lambda+\delta+\gamma+1)_{n+m-k}(2(n+m-k)+\mu+\lambda+\delta+\gamma+1)}{(\mu+\lambda+1)_{n+m-k}(\mu+\lambda+\delta+\gamma+1)_{2(n+m)-k+1}} \\ &\times \frac{(\mu+\lambda+1)_{n+m}(n+m)!(\lambda+\delta+1)_{2n}(\mu+\gamma+1)_{2m}(-2n-\lambda-\delta)_k}{n!m!k!(\lambda+\delta+1)_n(\mu+\gamma+1)_m(-2m-\mu-\gamma)_k} \\ &\times {}_3F_2\left(\left.\begin{matrix} -k, -\lambda-\mu-\delta-\gamma-1-2(n+m)+k, -n \\ -2n-\lambda-\delta, -n-m \end{matrix}\right| 1\right) \\ &\times {}_3F_2\left(\left.\begin{matrix} -k, -\lambda-\mu-\delta-\gamma-1-2(n+m)+k, -\lambda-n \\ -2n-\lambda-\delta, -\lambda-\mu-n-m \end{matrix}\right| 1\right) P_{n+m-k}^{(\lambda+\mu,\delta+\gamma)}(x), \end{aligned}$$

compare [Park and Kim, 2006], see [Chaggara and Koepf, 2010],

$$C_n^{(\alpha)}(x)C_m^{(\alpha)}(x) = \sum_{k=0}^{\min(m,n)} \frac{(n+m-2k+\alpha)(n+m-2k)!(\alpha)_k}{k!(n+m-k+\alpha)(n-k)!(m-k)!} \times \frac{(\alpha)_{n-k}(\alpha)_{m-k}(2\alpha)_{n+m-k}}{(\alpha)_{n+m-k}(2\alpha)_{n+m-2k}} C_{n+m-2k}^{(\alpha)}(x)$$

(see e.g. [Askey, 1975, p. 39], [Sánchez-Ruiz et al., 1999]),

$$H_n(x)H_m(x) = \sum_{k=0}^{\min(n,m)} \binom{n}{k}\binom{m}{k} 2^k k! H_{n+m-2k}(x)$$

(see e.g. [Watson, 1938], [Askey, 1975, p. 42], [Sánchez-Ruiz et al., 1999]),

$$L_n^{(\alpha)}(x)L_m^{(\alpha)}(x) = \sum_{k=|n-m|}^{n+m} \frac{(-2)^{n+m-k}k!}{(n+m-k)!(k-n)!(k-m)!} \times {}_3F_2\left(\left.\begin{array}{c} \frac{k-m-n}{2}, \frac{k-m-n+1}{2}, k+\alpha+1 \\ k-n+1, k-m+1 \end{array}\right| 1\right) L_k^{(\alpha)}(x)$$

(see e.g. [Watson, 1938]).

We note that in the above theorems, we gave only connection and linearization formulae between CCOP of the same family. In general we can obtain all the other connection and linearization coefficients using for example the first method of Section 1.2.2. In the case they aren't hypergeometric terms, we can use the sumtohyper algorithm if the coefficient is a single sum. If the coefficient is a double sum, Zeilberger's algorithm combined with Petkovšek-van-Hoeij algorithm may be used to simplify this double sum to a single sum or to a hypergeometric term (if possible).

1.6 Duplication Coefficients of CCOP

Given a polynomial system $\{p_n\}_{n\geq 0}$, we use a new approach to solve the so-called *duplication problem* associated to this system which consists in finding the coefficients $D_m(n,a)$ in the expansion

$$p_n(ax) = \sum_{m=0}^{n} D_m(n,a)p_m(x),$$

where a designates a nonzero complex number. In the following theorem, using the hypergeometric representations given in page 11 and the inversion formulas given in Theorem 1.10, we provide known duplication formulas and moreover, we get new results for Jacobi and Gegenbauer polynomials.

Theorem 1.13. *The following duplication formulas of orthogonal polynomials of a continuous variable are valid:*

$$P_n^{(\alpha,\beta)}(ax) = \sum_{m=0}^{n} \sum_{j=0}^{n-m} \frac{(-a)^m (1-a)^j (-n)_{m+j} (\alpha+1)_n (n+\alpha+\beta+1)_{m+j}}{2^j n! j! (\alpha+1)_{m+j} (\alpha+\beta+m+1)_m}$$
$$\times {}_2F_1\left(\left.\begin{matrix}\alpha+m+1, -j\\ \alpha+\beta+2m+2\end{matrix}\right| \frac{2a}{a-1}\right) P_m^{(\alpha,\beta)}(x),$$

$$C_n^{(\alpha)}(ax) = \Gamma(n+\alpha) a^n \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{n+\alpha-2m}{m!\Gamma(\alpha+n-m+1)} {}_2F_1\left(\left.\begin{matrix}-m, m-n-\alpha\\ -n-\alpha+1\end{matrix}\right| \frac{1}{a^2}\right) C_{n-2m}^{(\alpha)}(x),$$

$$L_n^{(\alpha)}(ax) = \sum_{m=0}^{n} \frac{(\alpha+1)_n a^m (1-a)^{n-m}}{(n-m)!(\alpha+1)_m} L_m^{(\alpha)}(x)$$

(see e.g. [Rainville, 1960, p. 209], [Chaggara and Koepf, 2007], [Ismail, 2005], compare [Lewanowicz, 2003a]),

$$H_n(ax) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{a^n n! (1-a^{-2})^m}{(n-2m)! m!} H_{n-2m}(x)$$

(see e.g. [Rainville, 1960, p. 209], [Chaggara and Koepf, 2007], [Ismail, 2005], compare [Godoy et al., 1997]),

$$B_n^{(\alpha)}(ax) = \sum_{m=0}^{n} \frac{(-a)^m (-n)_m (\alpha+n+1)_m}{m!(\alpha+m+1)_m} {}_2F_1\left(\left.\begin{matrix}m-n, \alpha+m+n+1\\ \alpha+2m+2\end{matrix}\right| a\right) B_m^{(\alpha)}(x)$$

(see e.g. [Lewanowicz, 2003a], [Doha and Ahmed, 2004]).

Remark 1.14. *The duplication coefficients given above for the Laguerre and Hermite polynomials were already given by Chaggara and Koepf [2007], Rainville [1960] and Ismail [2005] using generating functions. Those of Jacobi are new, as far as we know.*

Proof. In the proof, we consider three cases. In every case, the coefficients $A_j(n)$ and $I_m(n)$ are, respectively, those of the hypergeometric representations given in p. 11 and the inversion formulas given in Theorem 1.10.

1. Jacobi family $P_n^{(\alpha,\beta)}(x)$.
   We first prove the following variant of the binomial theorem

$$(1-ax)^j = \sum_{k=0}^{j} B_k(j,a)(1-x)^k \quad \text{with} \quad B_k(j,a) = a^k \binom{j}{k} (1-a)^{j-k}.$$

   Indeed, let f be a polynomial of degree j in the variable x. The Taylor formula for f at $x=1$ gives

$$f(x) = \sum_{k=0}^{n} B_k(j)(1-x)^k, \quad \text{with} \quad B_k(j) = \frac{(-1)^k}{k!}(D^k f)(1).$$

Applying this Taylor formula to $f(x) = (1-ax)^j$, we get

$$D^k f(x) = \frac{j!}{(j-k)!}(-a)^k(1-ax)^{j-k};$$

therefore

$$B_k(j) = \frac{(-1)^k}{k!}(D^k f)(1) = \frac{(-1)^k}{k!}\frac{j!}{(j-k)!}(-a)^k(1-ax)^{j-k} = \binom{j}{k}a^k(1-a)^{j-k}.$$

Combining

$$P_n^{(\alpha,\beta)}(ax) = \sum_{j=0}^{n} A_j(n)(1-ax)^j, \quad (1-ax)^j = \sum_{k=0}^{j} B_k(j,a)(1-x)^k$$

and

$$(1-x)^k = \sum_{m=0}^{k} I_m(k)P_m^{(\alpha,\beta)}(x)$$

and interchanging the order of summation yields the representation

$$P_n^{(\alpha,\beta)}(ax) = \sum_{m=0}^{n} D_m(n,a)P_m^{(\alpha,\beta)}(x)$$

with

$$D_m(n,a) = \sum_{j=0}^{n-m}\sum_{k=0}^{j} A_{j+m}(n)B_{m+k}(j+m,a)I_m(k+m).$$

We then use the sumtohyper algorithm to complete the proof.

2. Gegenbauer family $C_n^{(\alpha)}(x)$ and Hermite family $H_n(x)$.
In the Gegenbauer and Hermite cases, we combine

$$P_n(ax) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} A_j(n,a)x^{n-2j} \quad \text{and} \quad x^j = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} I_m(j)P_{j-2m}(x)$$

which yield

$$x^{n-2j} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor - j} I_m(n-2j)P_{n-2j-2m}(x)$$

and substitute m by $m-j$ to obtain

$$P_n(ax) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} D_m(n,a)P_{n-2m}(x)$$

with

$$D_m(n,a) = \sum_{j=0}^{m} A_j(n,a)I_{m-j}(n-2j).$$

In the Hermite case, Zeilberger's algorithm finds a recurrence equation of first order with respect to m from which the result follows. But in the Gegenbauer case, we get a recurrence equation of order 2 which according to the Petkovšek-van-Hoeij algorithm doesn't have a hypergeometric term solution.

3. Laguerre family $L_n^{(\alpha)}(x)$ and Bessel family $B_n^{(\alpha)}(x)$
   In both cases, we combine

$$P_n(ax) = \sum_{j=0}^{n} A_j(n,a)x^j \quad \text{and} \quad x^j = \sum_{m=0}^{j} I_m(j)P_m(x)$$

and interchange the order of sommation to get

$$D_m(n,a) = \sum_{j=0}^{n-m} A_{j+m}(n,a)I_m(j+m).$$

For the Laguerre family, Zeilberger's algorithm finds a recurrence equation of first order with respect to m from which the result follows. But in the Bessel case, we get a recurrence equation of order 2 which according to Petkovšek-van-Hoeij algorithm doesn't have a hypergeometric term solution.

$\square$

Proposition 1.15. *The duplication coefficients $D_m(n,a)$ of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are solutions of the fourth-order recurrence relation*

$$\begin{aligned}
&-4a^2(m+1+\beta)_2(m+1+\alpha)_2(\alpha+\beta+2m-3)_4(m-n+2)(m+n+\alpha+\beta+3)D_{m+2}(n,a)\\
&+2a(\alpha-\beta)(m+\beta+1)(m+\alpha+1)(\alpha+\beta+2m-3)_3(\alpha+\beta+m+2)(\alpha+\beta+2m+5)\times\\
&((\alpha+\beta+2m+4)(\alpha+\beta+2m)-4an(\beta+\alpha+n+1)+a(4m-\alpha^2-2\alpha\beta-\beta^2+8))D_{m+1}(n,a)\\
&+2a(\alpha-\beta)(\alpha+\beta+m)_3(\alpha+\beta+2m+3)_3(\alpha+\beta+2m-3)\Big(a(\alpha^2+\beta^2+2\alpha\beta+4\alpha+4\beta-4)\\
&+4a(m+n^2+n\alpha+n\beta+n)-(\alpha+\beta+2m+2)(\alpha+\beta+2m-2)\Big)D_{m-1}(n,a)\\
&-4a^2(\alpha+\beta+m-1)_4(\alpha+\beta+2m+2)_4(m-n-2)(m+n+\alpha+\beta-1)D_{m-2}(n,a)\\
&-(\alpha+\beta+2m-3)_2(\alpha+\beta+m+1)_2(\alpha+\beta+2m+4)_2\Big(-(\alpha+\beta+2m-1)_2(\alpha+\beta+2m+2)_2\\
&+2a(\alpha-\beta)^2(\alpha+\beta+2m-1)(\alpha+\beta+2m+3)+8m^4+16m^3(\alpha+\beta+1)\\
&+(8n^2+8n\alpha+8n\beta+8n+32\alpha\beta+12\beta^2-4+28\alpha+28\beta+12\alpha^2)m^2+4(\alpha+\beta+1)\times\\
&\Big(m(\alpha^2+3\alpha+2n\alpha+4\alpha\beta+\beta^2+2n^2+2n\beta+3\beta+2n-3)-n(\alpha^2-4\alpha\beta-\alpha+\beta^2-\beta)\Big)\\
&+4n^2(\alpha+\beta+4\alpha\beta-\alpha^2-\beta^2)-(\alpha+\beta-1)(\alpha+\beta+3)(\alpha^2-6\alpha\beta-2\alpha-2\beta+\beta^2)\Big)D_m(n,a),
\end{aligned}$$

with initial values $D_n(n,a) = a^n,\ D_{n+s}(n,a) = 0,\ s = 1,2,3.$

Proof. We apply the operator ε_a defined by $\varepsilon_a f(x) = f(ax)$ which fulfills the relations

$$\varepsilon_a(f(x)g(x)) = \varepsilon_a f(x)\varepsilon_a g(x);\ \varepsilon_a(\alpha f(x)+\beta g(x)) = \alpha\varepsilon_a f(x)+\beta\varepsilon_a g(x);\ \varepsilon_a f'(x) = \frac{1}{a}(\varepsilon_a f(x))'$$

to the differential equation

$$\sigma(x)p_n''(x) + \tau(x)p_n'(x) + \lambda_n p_n(x) = 0.$$

This yields (using the above properties of ε_a) the differential equation

$$\sigma(ax)\frac{d^2}{dx^2}p_n(ax) + a\tau(ax)\frac{d}{dx}p_n(ax) + a^2\lambda_n p_n(ax) = 0$$

satisfied by $p_n(ax)$. We substitute $p_n(ax) = \sum_{m=0}^{n} D_m(n,a)p_m(x)$ in the latter differential equation. This gives

$$\sum_{m=0}^{n} D_m(n,a)\Big(\sigma(ax)p''_m(x) + a\tau(ax)p'_m(x) + a^2\lambda_n p_m(x)\Big) = 0.$$

We proceed as in Section 1.2.3 using the $\{p''_m(x)\}$ basis to get the result. □

1.7 Applications of Connection and Linearization Formulae of CCOP

Linearization, connection and duplication problems are not only important from a fundamental point of view, but also because they are used in the computation of physical and chemical properties of quantum-mechanical systems. An example of an application of the linearization formula is the evaluation of the logarithmic potentials of orthogonal polynomials $V_n(t) = -\int [p_n(x)]^2 \log|x-t|dx$, which appear in the calculation of the position and momentum information entropies of quantum systems [Dehesa et al., 1997a], [Sánchez-Ruiz, 1997].

In many applications of orthogonal polynomials, it is often important to know whether the linearization, connection or duplication coefficients are positive or non-negative (see e.g. [Askey, 1968], [Askey, 1975], [Gasper, 1975], [Ismail, 2005]). This property has many important consequences. It gives rise to a convolution structure associated with the polynomial set $\{p_n(x)\}$ ([Gasper, 1970], [Askey and Gasper, 1971b], [Askey and Gasper, 1977], [Szwarc, 1992]). During the last decades, several sufficient conditions for these sign properties to hold have been derived (see e.g. [Askey, 1965], [Askey and Gasper, 1971a], [Askey, 1975], [Gasper, 1975], [Trench, 1976], [Koornwinder, 1978], [Szwarc, 1996], [Sánchez-Ruiz et al., 1999], [Szwarc, 2003], [Ismail, 2005]). In many cases these sign properties become obvious if the coefficients can be written as a sum of terms which are shown to have the same sign. Another application is given in the next section.

1.7.1 Parameter Derivatives

For some applications, it is important to know the rate of change in the direction of the parameters of the orthogonal systems, given in terms of the system itself. Fröhlich [1994] gives the following argument: for any family $p_n^{(\alpha)}(x) = \sum_{k=0}^{n} a_k(\alpha)x^k$ of orthogonal polynomials there is a representation of the form

$$\frac{\partial}{\partial\alpha}p_n^{(\alpha)}(x) = \sum_{k=0}^{n} c_k^{(\alpha)} p_k^{(\alpha)}(x) \tag{1.41}$$

for the α-derivative of $p_n^{(\alpha)}(x)$, since by termwise differentiation the expression

$$\frac{\partial}{\partial\alpha}p_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{\partial}{\partial\alpha}a_k(\alpha)x^k$$

is seen to be a polynomial of degree n with respect to x, and since any polynomial of degree n has a representation of the form (1.41). We call the derivative with respect to the parameter α *a parameter derivative* of $p_n^{(\alpha)}(x)$. These parameter derivative representations can be obtained from the connection formulas as shown by Koepf and Schmersau [1998].

Corollary 1.16. *The following representations for the parameter derivatives of the classical continuous orthogonal polynomials are valid:*

$$\frac{\partial}{\partial\alpha}P_n^{(\alpha,\beta)}(x)=\sum_{m=0}^{n-1}\frac{1}{\alpha+\beta+m+n+1}$$
$$\times\left(P_n^{(\alpha,\beta)}(x)+\frac{\alpha+\beta+1+2m}{n-m}\frac{(\beta+m+1)_{n-m}}{(\alpha+\beta+m+1)_{n-m}}P_m^{(\alpha,\beta)}(x)\right)$$

(see [Fröhlich, 1994, Theorem 3], [Koepf and Schmersau, 1998]),

$$\frac{\partial}{\partial\beta}P_n^{(\alpha,\beta)}(x)=\sum_{m=0}^{n-1}\frac{1}{\alpha+\beta+m+n+1}$$
$$\times\left(P_n^{(\alpha,\beta)}(x)+(-1)^{n-m}\frac{\alpha+\beta+1+2m}{n-m}\frac{(\alpha+m+1)_{n-m}}{(\alpha+\beta+m+1)_{n-m}}P_m^{(\alpha,\beta)}(x)\right)$$

(see [Fröhlich, 1994, Theorem 3], [Koepf and Schmersau, 1998]),

$$\frac{\partial}{\partial\alpha}C_n^{(\alpha)}(x)=\sum_{m=0}^{n-1}\left(\frac{2(m+1)}{(2\alpha+m)(2\alpha+1+2m)}+\frac{2}{2\alpha+m+n}\right)C_n^{(\alpha)}(x)$$
$$+\sum_{m=0}^{n-1}\frac{2(1+(-1)^{n-m})(\alpha+m)}{(2\alpha+m+n)(n-m)}C_m^{(\alpha)}(x)$$

(see [Koepf, 1997, Theorem 10], [Koepf and Schmersau, 1998]),

$$\frac{\partial}{\partial\alpha}L_n^{(\alpha)}(x)=\sum_{m=0}^{n-1}\frac{1}{n-m}L_m^{(\alpha)}(x)$$

(see [Koepf, 1997, Theorem 10], [Koepf and Schmersau, 1998]),

$$\frac{\partial}{\partial\alpha}B_n^{\alpha}(x)=\sum_{m=0}^{n-1}\frac{1}{\alpha+n+m+1}$$
$$\times\left(B_n^{\alpha}(x)+(-1)^{n-m}\frac{2m+\alpha+1}{n-m}\frac{n!}{(\alpha+m+1)_{n-m}m!}B_m^{\alpha}(x)\right)$$

(see [Koepf and Schmersau, 1998]).

Proof (See [Koepf and Schmersau, 1998]). Given the connection relation

$$p_n^{\alpha}(x)=\sum_{m=0}^{n}C_m(n;\alpha,\beta)p_m^{\beta}(x),$$

we build the difference quotient

$$\begin{aligned}\frac{p_n^\alpha(x)-p_n^\beta(x)}{\alpha-\beta} &= \sum_{m=0}^{n}\frac{C_m(n;\alpha,\beta)}{\alpha-\beta}p_m^\beta(x)-\frac{p_n^\beta(x)}{\alpha-\beta}\\ &= \frac{C_n(n;\alpha,\beta)-1}{\alpha-\beta}p_n^\beta(x)+\sum_{m=0}^{n-1}\frac{C_m(n;\alpha,\beta)}{\alpha-\beta}p_m^\beta(x)\end{aligned}$$

so that with $\beta\to\alpha$

$$\frac{\partial}{\partial\alpha}p_n^\alpha(x)=\lim_{\beta\to\alpha}\frac{C_n(n;\alpha,\beta)-1}{\alpha-\beta}p_n^\beta(x)+\sum_{m=0}^{n-1}\lim_{\beta\to\alpha}\frac{C_m(n;\alpha,\beta)}{\alpha-\beta}p_m^\beta(x)$$

since the systems $p_n^\alpha(x)$ are continuous with respect to α. This gives the results. □

Remark 1.17. *Lewanowicz [2002] also gave a general procedure to produce iteratively explicit parameter derivative representations for almost all the classical orthogonal polynomial families.*

1.7.2 Logarithmic Potential of Hermite Polynomials and Information Entropies of the Harmonic Oscillator Eigenstates (see [Sánchez-Ruiz, 1997] and References Therein)

For the n^{th} eigenstate of the one-dimensional harmonic oscillator Hamiltonian

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2,$$

the probability densities ρ for position and momentum γ are expressed in terms of the Hermite polynomials $H_n(x)$,

$$\rho(x)=\frac{\alpha}{2^n n!\sqrt{\pi}}(H_n(\alpha x))^2e^{-\alpha^2x^2},\ \ \gamma(p)=\frac{1}{2^n n!\sqrt{\pi}\alpha}(H_n(x))^2e^{-\frac{p^2}{\alpha^2}},$$

where $\alpha:=(m\omega)^2$. The corresponding entropies of position and momentum can be written as

$$S_\rho=-\ln\alpha+S_n,\ \ S_\gamma=\ln\alpha+S_n,$$

where

$$S_n=\ln(2^n n!\sqrt{\pi})+n+\frac{1}{2}+\frac{1}{2^n n!\sqrt{\pi}}E_n(H)$$

is given in terms of $E_n(H)$, the so-called entropy of the Hermite polynomials, whose representation is

$$E_n(H)=-\int_{-\infty}^{\infty}(H_n(x))^2\ln(H_n(x))^2e^{-x^2}dx.$$

Dehesa et al. [1997b] showed that $E_n(H)$ can be written in the form

$$E_n(H)=-2^n n!\sqrt{\pi}\ln(2^{2n})+2\sum_{i=1}^{n}V_n(x_{n,i}),$$

where $x_{n,i}$ $(i = 1, \ldots, n)$ is the i^{th} root of $H_n(x)$, and $V_n(t)$ is the logarithmic potential of the Hermite polynomial $H_n(x)$, defined as

$$V_n(t) = -\int_{-\infty}^{\infty} (H_n(x))^2 \ln |x - t| e^{-x^2} dx. \tag{1.42}$$

To calculate $V_n(t)$, Sánchez-Ruiz [1997] make use of the linearization formula for the Hermite polynomials

$$H_m(x)H_n(x) = \sum_{j=0}^{\min(m,n)} \frac{m!n!2^j}{(m-j)!(n-j)!j!} H_{m+n-2j}(x)$$

which in the particular case $m = n$ gives, writing $j = n - k$,

$$(H_n(x))^2 = 2^n n! \sum_{k=0}^{n} \binom{n}{k} \frac{H_{2k}(x)}{2^k k!}.$$

Substituting this equation in the expression (1.42) of the logarithmic potential $V_n(t)$ he gets

$$V_n(t) = 2^n n! \sum_{k=0}^{n} \binom{n}{k} \frac{W_{2k}(t)}{2^k k!}, \; W_{2k}(t) = -\int_{-\infty}^{\infty} H_{2k}(x) \ln |x - t| e^{-x^2} dx.$$

After some calculations, he obtains a representation for $V_n(t)$ which in turn yields a representation for the entropies when the exact location of the zeros of $H_n(x)$ are known.

Chapter 2

Connection, Linearization and Duplication Coefficients of Orthogonal Polynomials of a Discrete Variable

The first purpose of this chapter is to review some general methods used to compute linearization and connection coefficients of classical discrete orthogonal polynomials (in short CDOP). On the other hand, by an algorithmic approach, we solve the duplication problem for CDOP. Hypergeometric series representations of linearization coefficients of CDOP are also given.

2.1 Introduction

Let us consider the second-order difference equation of hypergeometric type, i.e. the equation [Nikiforov and Uvarov, 1988],

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda y(x) = 0, \tag{2.1}$$

where $\Delta y(x) = y(x+1) - y(x)$ and $\nabla y(x) = y(x) - y(x-1)$ denote the forward and backward difference operators, respectively, $\sigma(x) = ax^2 + bx + c$ and $\tau(x) = dx + e$ are polynomials of degree not greater than 2 and 1, respectively, and λ is a constant. This equation can be written in the form

$$\Delta\Big[\sigma(x)\rho(x)\nabla y(x)\Big] + \lambda\rho(x)y(x) = 0,$$

where the function $\rho(x)$ satisfies the Pearson-type difference equation

$$\Delta\Big[\sigma(x)\rho(x)\Big] = \tau(x)\rho(x).$$

The solutions of Equation (2.1) with

$$\lambda \equiv \lambda_n = -n\Big((n-1)a + d\Big)$$

are polynomials of degree n, usually called hypergeometric type "discrete" polynomials $y = y_n(x)$. These polynomials are orthogonal with respect to the weight function $\rho(x)$, $x = A,\ A+1,\ \ldots,\ B-1$ [Nikiforov et al., 1991], i.e.,

$$\sum_{x=A}^{B-1} y_n(x)y_m(x)\rho(x) = h_n^2\delta_{n,m}, \tag{2.2}$$

provided that $\rho(x) > 0$ for $A \leq x \leq B-1$ and

$$\sigma(x)\rho(x)x^k\Big|_{x=A,B} = 0, \ \forall k \geq 0. \tag{2.3}$$

The square of the norm of the polynomial $y_n(x)$ is given by (see e.g. [Nikiforov et al., 1991])

$$h_n^2 = (-1)^n k_n B_n \sum_{x=A}^{B-n-1} \rho_n(x), \tag{2.4}$$

where k_n is the leading coefficient of the polynomial $y_n(x) = k_n x^n + \ldots$, and B_n is the normalization constant of the Rodrigues-type formula

$$y_n(x) = \frac{B_n}{\rho(x)} \nabla^n \Big[\rho_n(x)\Big], \ n = 0, 1, 2, \ldots \ , \tag{2.5}$$

with

$$\rho_n(x) = \rho(x+n) \prod_{m=1}^{n} \sigma(x+m), n = 1, 2, \ldots \ , \ \rho_0(x) = \rho(x).$$

The use of Equation (2.5) together with the formula

$$\nabla^n[f(x)] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-k) \tag{2.6}$$

allows one to obtain an explicit expression for the polynomials [Nikiforov et al., 1991]. The constants k_n and B_n are related by

$$k_n = B_n \prod_{k=0}^{n-1} \Big[d + (n+k-1)a\Big], \ k = 1, 2, \ldots, \ k_0 = B_0.$$

The k^{th} difference derivative of the polynomials $y_n(x)$ also fulfills a Rodrigues-type formula [Nikiforov et al., 1991]

$$\Delta^k y_n(x) = \frac{A_{nk} B_n}{\rho_k(x)} \nabla^{n-k}[\rho_n(x)], \tag{2.7}$$

where

$$A_{nm} = \frac{n!}{(n-m)!} \prod_{k=0}^{m-1} [d + (n+k-1)a] \equiv \frac{n!}{(n-m)!} \frac{k_m}{B_m}, \ A_{n0} = 1.$$

The four referred families of discrete hypergeometric polynomials are the so-called CDOP. We denote by $Q_n(x; \alpha, \beta, N)$, $M_n(x; \gamma, \mu)$, $K_n(x; p, N)$ and $C_n(x; \mu)$, the Hahn, Meixner, Krawtchouk and Charlier polynomials, respectively. The hypergeometric representations of the above polynomials are given by Koekoek et al. [2010]

$$Q_n(x; \alpha, \beta, N) = {}_3F_2\left(\left. \begin{array}{c} -n, -x, n+1+\alpha+\beta \\ \alpha+1, -N \end{array} \right| 1 \right),$$

$$n, x = 0, 1, \ldots, N, \ \alpha > -1 \text{ and } \beta > -1, \text{ or } \alpha < -N \text{ and } \beta < -N,$$

$$M_n(x; \gamma, \mu) = {}_2F_1\left(\left. \begin{array}{c} -n, -x \\ \gamma \end{array} \right| 1 - \frac{1}{\mu} \right), \ \gamma > 0, \ 0 < \mu < 1, \ x = 0, 1, \ldots,$$

$$K_n(x;p,N) = {}_2F_1\left(\begin{matrix}-n,-x\\ -N\end{matrix}\,\middle|\,\frac{1}{p}\right),\ 0<p<1,\ n,x=0,1,\ldots,N,$$

$$C_n(x;\mu) = {}_2F_0\left(\begin{matrix}-n,-x\\ -\end{matrix}\,\middle|\,-\frac{1}{\mu}\right),\ \mu>0,\ x=0,1,\ldots.$$

The data corresponding to each family are given in the following table [Koekoek et al., 2010]:

system	$Q_n(x;\alpha,\beta,N)$	$M_n(x;\gamma,\mu)$	$K_n(x;p,N)$	$C_n(x;\mu)$
$\sigma(x)$	$x(N+\alpha-x)$	x	x	x
$\tau(x)$	$(\beta+1)(N-1)-(\alpha+\beta+2)x$	$(\mu-1)x+\mu\gamma$	$\frac{Np-x}{1-p}$	$\mu-x$
$\rho(x)$	$\frac{(\alpha+1)_x}{x!}\frac{(\beta+1)_{N-x}}{(N-x)!}$	$\frac{(\gamma)_x}{x!}\mu^x$	$\binom{N}{x}p^x(1-p)^{N-x}$	$\frac{e^{-\mu}\mu^x}{x!}$
k_n	$\frac{(\alpha+\beta+n+1)_n}{(-N)_n(\alpha+1)_n}$	$\frac{(1-\frac{1}{\mu})^n}{(\gamma)_n}$	$\frac{1}{(-N)_n p^n}$	$(-\frac{1}{\mu})^n$

In the continuous case, the polynomials were represented in terms of the powers x^n. The corresponding choice in the discrete case is a representation in terms of the falling factorials

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1) = (-1)^n(-x)_n.$$

The falling factorials satisfy the equations

$$\Delta^m x^{\underline{n}} = \frac{n!}{(n-m)!}x^{\underline{n-m}},\ \nabla^m x^{\underline{n}} = \frac{n!}{(n-m)!}(x-m)^{\underline{n-m}},\ m\le n. \tag{2.8}$$

The expansion of any arbitrary discrete polynomial $p_n(x)$ as a series w.r.t. a general set of discrete hypergeometric polynomials $q_n(x)$ is a matter of great interest. This is particulary true for the problem of linearization of a product of any two discrete polynomials. General and widely applicable strategies begin to appear about 20 years ago [Lewanowicz, 1996a], [Alvarez-Nodarse et al., 1997], [Koepf and Schmersau, 1998].

To solve the connection problem, Ronveaux et al. [1996] proposed a method which requires the knowledge of the three-term recurrence relation, the second-order difference equation, satisfied by the polynomials of the orthogonal set of the expansion problem in consideration, see [Lewanowicz and Ronveaux, 1996], [Zarzo et al., 1997], [Belmehdi et al., 1997], [Area et al., 1998], for further description and application of this method.

Koepf and Schmersau [1998] proposed a computer-algebra-based method which, starting from the second-order hypergeometric difference equation, produces by symbolic means and in a recurrent way the expansion coefficients of the CDOP in terms of the falling factorial polynomials as well as the expansion coefficients of the corresponding inverse problem. The combination of these two expansion problems allows these authors to solve the connection problems within each specific CDOP set.

All the previous methods provide the expansion coefficients via recursion relationships, which is very useful for the symbolic and/or numerical computation of their values. Starting from the second-order hypergeometric difference equation satisfied by the set of discrete orthogonal polynomials $\{y_n(x)\}$, Alvarez-Nodarse et al. [1997] found the representations of the expansion coefficients of any polynomial $p_n(x)$ and of the product $p_n(x)q_m(x)$ as a series w.r.t. the set $y_n(x)$.

In this chapter, we review some of the above general methods used to find the connection and linearization coefficients of CDOP. In addition as new results, we solve the general duplication problem for CDOP and we propose a method to generate hypergeometric series representations of the linearization coefficients of CDOP.

2.2 Evaluation of Connection and Linearization Coefficients

In this section, we review the method used by Alvarez-Nodarse et al. [1997] to find representations for connection and linearization coefficients of discrete orthogonal polynomials.

We want to find the expansion coefficients $L_k(m,n)$ of the relation

$$p_n(x)q_m(x) = \sum_{k=0}^{n+m} L_k(m,n)y_k(x), \tag{2.9}$$

where $\{y_k(x)\}$ is a discrete orthogonal set of hypergeometric polynomials which satisfy the difference equation (2.1), and $p_n(x)$ and $q_m(x)$ are arbitrary polynomial families.

Theorem 2.1 ([Alvarez-Nodarse et al., 1997]). *The coefficients $L_k(m,n)$ in the expansion (2.9) are given by*

$$L_k(m,n) = \frac{(-1)^k B_k}{h_k^2} \sum_{x=A}^{B-1} \rho_k(x-k)\nabla^k \Big[p_n(x)q_m(x)\Big]. \tag{2.10}$$

Proof. Multiplying both sides of Eq. (2.9) by $\rho(x)y_l(x)$, and summing between A and $B-1$, the orthogonality relation (2.2) gives

$$L_k(m,n) = \frac{1}{h_k^2} \sum_{x=A}^{B-1} p_n(x)q_m(x)y_k(x)\rho(x).$$

Using the Rodrigues formula (2.5) for $y_k(x)$ gives

$$L_k(m,n) = \frac{B_k}{h_k^2} \sum_{x=A}^{B-1} p_n(x)q_m(x)\nabla^k[\rho_k(x)].$$

From the following summation by parts formula

$$\sum_{x=A}^{B-1} f(x)\nabla g(x) = (f(x)g(x))\Big|_{A-1}^{B-1} - \sum_{x=A}^{B-1} g(x-1)\nabla f(x),$$

we get

$$L_k(m,n) = \frac{B_k}{h_k^2}\Big\{p_n(x)q_m(x)\nabla^{k-1}[\rho_k(x)]\Big|_{A-1}^{B-1} - \sum_{x=A}^{B-1} \nabla[p_n(x)q_m(x)]\nabla^{k-1}[\rho_k(x-1)]\Big\}.$$

For any integer j with $0 \leq j < k$, we have from Equation (2.6)

$$\begin{aligned}\nabla^j[\rho_k(x)] &= \sum_{i=0}^{j}(-1)^i\binom{j}{i}\rho_k(x-i)\\ &= \sum_{i=0}^{j}(-1)^i\binom{j}{i}\rho(x-i+k)\prod_{l=1}^{k}\sigma(x-i+l).\end{aligned}$$

We remark from the definition of the weight function of CDOP that for $k \geq 1$, $\rho(x+k) = \rho(x+1) \times f(x)$ where $f(x)$ is a rational function with finite limits at $A-1$ and $B-1$. It follows that for $j < k$, $\nabla^j[\rho_k(x)]$ is proportional to $\sigma(x+1)\rho(x+1)$. Thus taking into account the boundary condition (2.3) we get

$$L_k(m,n) = -\frac{B_k}{h_k^2}\sum_{x=A}^{B-1}\nabla[p_n(x)q_m(x)]\nabla^{k-1}[\rho_k(x-1)].$$

Repeating this process $k-1$ times, we obtain the desired expression (2.10). □

In the special case $p_n(x) = x^{\underline{n}}$ and $q_m(x) = 1$, we derive the following inversion formula from (2.10) and (2.8).

Corollary 2.2 ([Alvarez-Nodarse et al., 1997]).

$$x^{\underline{n}} = \sum_{k=0}^{n} I_k(n)y_k(x), \;\; \textit{with}\;\; I_k(n) = \frac{(-1)^k k! B_k}{(n-k)!h_k^2}\sum_{x=A}^{B-1}(x-k)^{\underline{n-k}}\rho_k(x-k). \tag{2.11}$$

In the special case when $p_n(x)$ is a hypergeometric polynomial satisfying the following second-order difference equation

$$\bar{\sigma}(x)\Delta\nabla p_n(x) + \bar{\tau}(x)\Delta p_n(x) + \bar{\lambda}_n p_n(x) = 0,$$

we get

Theorem 2.3 ([Alvarez-Nodarse et al., 1997]). *The coefficients $L_k(m,n)$ in the expansion (2.9) are given by*

$$\begin{aligned}L_k(m,n) = {} & \frac{(-1)^k B_k\bar{B}_n}{h_k^2}\sum_{j=j_-}^{j_+}\binom{k}{j}\bar{A}_{nj}\sum_{x=A}^{B-1}\sum_{l=0}^{n-j}(-1)^l\binom{n-j}{l}\times\\ & \frac{\rho_k(x-k)}{\bar{\rho}_j(x-j)}\bar{\rho}_n(x-j-l)\Big[\nabla^{k-j}q_m(x-j)\Big],\end{aligned} \tag{2.12}$$

where $j_- = \max(0, k-n)$ and $j_+ = \min(k,n)$.

Proof. Applying Leibniz's rule for the nth difference derivative of a product

$$\nabla^n\Big[f(x)g(x)\Big] = \sum_{j=0}^{n}\binom{n}{j}\Big[\nabla^j f(x)\Big]\Big[\nabla^{n-j}g(x-j)\Big]$$

to $p_n(x)q_m(x)$, it follows from Equation (2.10) that

$$L_k(m,n) = \frac{(-1)^k B_k}{h_k^2}\sum_{x=A}^{B-1}\rho_k(x-k)\sum_{j=j_-}^{j_+}\binom{k}{j}\Big[\nabla^j p_n(x)\Big]\Big[\nabla^{k-j}q_m(x-j)\Big].$$

From the Rodrigues-type formula (2.7), we deduce

$$\nabla^j p_n(x) \equiv \Delta^j p_n(x-j) = \frac{\bar{A}_{nj}\bar{B}_n}{\bar{\rho}_j(x-j)} \nabla^{n-j}\Big[\bar{\rho}_n(x-j)\Big],$$

where $\bar{A}_{nj}$ and $\bar{B}_n$ are defined in terms of the coefficients of $\bar{\sigma}(x)$ and $\bar{\tau}(x)$. It follows that

$$L_k(m,n) = \frac{(-1)^k B_k \bar{B}_n}{h_k^2} \sum_{j=j_-}^{j_+} \binom{k}{j} \bar{A}_{nj} \sum_{x=A}^{B-1} \frac{\rho_k(x-k)}{\bar{\rho}_j(x-j)} \nabla^{n-j}\Big[\bar{\rho}_n(x-j)\Big]\Big[\nabla^{k-j} q_m(x-j)\Big].$$

Using (2.6), we substitute the expression for $\nabla^{n-j}\Big[\bar{\rho}_n(x-j)\Big]$ in the previous equation and (2.12) follows. □

A very important particular case of the expansion (2.9) is the case $m = 0$, i.e. the connection problem

$$p_n(x) = \sum_{k=0}^{n} C_k(n) y_k(x).$$

It can be deduced that the connection coefficients are given by

$$C_k(n) = L_k(0,n) = \frac{(-1)^k B_k \bar{B}_n \bar{A}_{nk}}{h_k^2} \sum_{x=A}^{B-1} \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} \frac{\rho_k(x-k)}{\bar{\rho}_k(x-k)} \bar{\rho}_n(x-k-l). \quad (2.13)$$

If $\bar{\sigma}(x) = \sigma(x)$, the above expansion coefficients reduce to

$$C_k(n) = \frac{(-1)^k B_k \bar{B}_n \bar{A}_{nk}}{h_k^2} \sum_{x=A}^{B-1} \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} \frac{\rho(x)}{\bar{\rho}(x)} \bar{\rho}_n(x-k-l). \quad (2.14)$$

Using relations (2.13) and (2.14), Alvarez-Nodarse et al. [1997] provided formulas connecting the different monic families of CDOP.

2.3 Connection and Linearization Coefficients of CDOP Using Structural Relations

For any classical discrete orthogonal polynomial system $\{p_n(x)\}_{n\in\mathbb{N}_0}$ with $p_{-1} \equiv 0$, the following structure relations are valid (see e.g. [Nikiforov et al., 1991], [García et al., 1995], [Koepf and Schmersau, 1998]):

$$x p_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad (2.15)$$
$$\sigma(x)\nabla p_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x) \quad \text{or} \quad (2.16)$$
$$(\sigma(x)+\tau(x))\nabla p_n(x) = S_n p_{n+1}(x) + T_n p_n(x) + R_n p_{n-1}(x), \quad (2.17)$$
$$x\Delta p_n(x) = \alpha_n^\star \Delta p_{n+1}(x) + \beta_n^\star \Delta p_n(x) + \gamma_n^\star \Delta p_{n-1}(x), \quad (2.18)$$
$$p_n(x) = \hat{a}_n \Delta p_{n+1}(x) + \hat{b}_n \Delta p_n(x) + \hat{c}_n \Delta p_{n-1}(x), \quad (2.19)$$
$$\sigma(x)\Delta\nabla p_n(x) = a_n' \Delta p_{n+1}(x) + b_n' \Delta p_n(x) + c_n' \Delta p_{n-1}(x). \quad (2.20)$$

Koepf and Schmersau [1998] obtained explicitly the coefficients appearing in the previous equations.

Theorem 2.4 ([Koepf and Schmersau, 1998]). *For the polynomial solutions of* (2.1)*, the relations* (2.15)-(2.20) *are valid. The coefficients* a_n, b_n, c_n, α_n, β_n, γ_n, $\alpha_n^\star$, $\beta_n^\star$, $\gamma_n^\star$, $\hat{a}_n$, $\hat{b}_n$, $\hat{c}_n$, a_n', b_n', c_n', S_n, T_n, R_n, *are given in terms of the coefficients* a, b, c, d, e *of the difference equation* (2.1) *by*

$$
\begin{aligned}
a_n &= \frac{k_n}{k_{n+1}},\\
b_n &= -\frac{n(d+2b)(d+an-a)+e(d-2a)}{(2an-2a+d)(d+2an)},\\
c_n &= -\Big((and-db-ad+a^2n^2-2a^2n+4ac+a^2+2ea-b^2)\\
&\quad\times(n-1)(d+an-a)-dbe+d^2c+ae^2\Big)\\
&\quad\frac{(an+d-2a)n}{(d-a+2an)(d+2an-3a)(2an-2a+d)^2}\cdot\frac{k_n}{k_{n-1}},\\
\alpha_n &= an\frac{k_n}{k_{n+1}},\\
\beta_n &= -\frac{n(an+d-a)(2and-ad-db+2ea-2a^2n+2a^2n^2)}{(d+2an)(d-2a+2an)},\\
\gamma_n &= \Big((n-1)(an+d-a)(and-db-ad+a^2n^2-2a^2n+4ac+a^2+2ea-b^2)\\
&\quad-dbe+d^2c+ae^2\Big)\times\frac{(an+d-a)(an+d-2a)n}{(d-2a+2an)^2(2an-3a+d)(2an-a+d)}\cdot\frac{k_n}{k_{n-1}},\\
\alpha_n^\star &= \frac{n}{n+1}\frac{k_n}{k_{n+1}},\\
\beta_n^\star &= -\frac{n(d+2a+2b)(d+an-a)+d(e-a-b)}{(2an-2a+d)(d+2an)},\\
\gamma_n^\star &= -\Big((n-1)(d+an-a)(and-db-ad+a^2n^2-2a^2n+4ac+a^2+2ea-b^2)\\
&\quad-dbe+d^2c+ae^2\Big)\times\frac{(d+an-a)n}{(2an-2a+d)^2(2an-3a+d)(d-a+2an)}\cdot\frac{k_n}{k_{n-1}},\\
a_n' &= \frac{an(n-1)}{n+1}\cdot\frac{k_n}{k_{n+1}},\\
b_n' &= -\frac{(n-1)(an+d)(2and-ad-db+2ea-2a^2n+2a^2n^2)}{(2an-2a+d)(d+2an)}\\
c_n' &= \Big((n-1)(d+an-a)(and-db-ad+a^2n^2-2a^2n+4ac+a^2+2ea-b^2)\\
&\quad-dbe+d^2c+ae^2\Big)\times\frac{(d+an-a)(an+d)n}{(2an-2a+d)^2(2an-3a+d)(2an-a+d)}\cdot\frac{k_n}{k_{n-1}},\\
\hat{a}_n &= \frac{1}{n+1}\cdot\frac{k_n}{k_{n+1}},\\
\hat{b}_n &= \frac{-2an(d+an-a)-db+ad-d^2+2ea}{(2an-2a+d)(d+2an)},\\
\hat{c}_n &= \Big((n-1)(d+an-a)(and-db-ad+a^2n^2-2a^2n+4ac+a^2+2ea-b^2)\\
&\quad-dbe+d^2c+ae^2\Big)\times\frac{an}{(2an-2a+d)^2(2an-3a+d)(2an-a+d)}\cdot\frac{k_n}{k_{n-1}},
\end{aligned}
$$

$$S_n = \alpha_n,\ T_n = \beta_n - \lambda_n,\ R_n = \gamma_n.$$

2.3.1 First Method

We assume that $p_n(x)$ is a polynomial system given by (2.1) with $\sigma(x) = ax^2 + bx + c$, and $\tau(x) = dx + e$, and that $q_m(x)$ is a polynomial system given by (2.1) with $\bar{\sigma}(x) = \bar{a}x^2 + \bar{b}x + \bar{c}$, and $\bar{\tau}(x) = \bar{d}x + \bar{e}$.

If $\bar{\sigma}(x) = \sigma(x)$ or $\sigma(x) + \tau(x) = \bar{\sigma}(x) + \bar{\tau}(x)$ we proceed similarly as in the Section 1.2.1 by replacing, respectively, the structural formulas

$$\begin{aligned} xp_n'(x) &= \alpha_n^\star p_{n+1}'(x) + \beta_n^\star p_n'(x) + \gamma_n^\star p_{n-1}'(x) \\ \sigma(x)p_n'(x) &= \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x) \end{aligned}$$

by

$$\begin{aligned} x\Delta p_n(x) &= \alpha_n^\star \Delta p_{n+1}(x) + \beta_n^\star \Delta p_n(x) + \gamma_n^\star \Delta p_{n-1}(x), \\ \sigma(x)\nabla p_n(x) &= \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x). \end{aligned}$$

If $\bar{\sigma}(x) \neq \sigma(x)$ and $\sigma(x) + \tau(x) \neq \bar{\sigma}(x) + \bar{\tau}(x)$, similarly as in Section 1.2.1, we solve the falling factorial representation

$$x^{\underline{n}} = \sum_{m=0}^{n} I_m(n) q_m(x),$$

using the following structure relations for $p_n(x) = x^{\underline{n}}$:

$$xp_n(x) = p_{n+1}(x) + np_n(x),\ p_n(x) = \frac{1}{n+1}\Delta p_{n+1}(x),\ x\Delta p_n(x) = \frac{n}{n+1}\Delta p_{n+1}(x) + (n-1)\Delta p_n(x).$$

From the identity $x^{\underline{n}} = (-1)^n(-x)_n$ and the definition of the generalized hypergeometric function, we have for all CDOP

$$p_n(x) = \sum_{j=0}^{n} A_j(n) x^{\underline{j}}.$$

By the combination of the above formula with the inversion formula

$$x^{\underline{j}} = \sum_{m=0}^{j} I_m(j) q_m(x)$$

we get the connection formula as in Section 1.2.1.

2.3.2 Second Method: The NaViMa Algorithm

It is also possible to use the NaViMa algorithm to compute recursively the connection coefficients between two families of CDOP [Ronveaux et al., 1996], [Area et al., 1998].

In the first step, we apply the difference operator $\mathcal{L}_{2,n} :\ \mathcal{P} \longrightarrow \mathcal{P}$, defined by

$$\mathcal{L}_{2,n}[p_n(x)] := \sigma(x)\Delta\nabla p_n(x) + \tau(x)\Delta p_n(x) + \lambda_n p_n(x) = 0$$

to both sides of the connection identity $p_n(x) = \sum\limits_{m=0}^{n} C_m(n) q_m(x)$. This gives

$$\sum_{m=0}^{n} C_m(n) \Big(\sigma(x) \Delta\nabla q_m(x) + \tau(x) \Delta q_m(x) + \lambda_n q_m(x) \Big) = 0. \tag{2.21}$$

To obtain a recurrence relation for the coefficients $C_m(n)$ of maximum order 4 or 2, we expand the expression

$$\sigma(x) \Delta\nabla q_m(x) + \tau(x) \Delta q_m(x) + \lambda_n q_m(x) \tag{2.22}$$

in the basis $\Delta\nabla q_m(x)$ if $\sigma(x) \neq \bar{\sigma}(x)$, or in the basis $\Delta q_m(x)$ if $\sigma(x) = \bar{\sigma}(x)$, respectively. Note that we denote all coefficients connected with $q_m(x)$ by dashes.

Using the $\{\Delta\nabla q_m(x)\}$ basis ($\sigma(x) \neq \bar{\sigma}(x)$)

We proceed as follows. In the first step, we apply ∇ to (2.19) and use the properties

$$\Delta\nabla = \nabla\Delta, \;\; \Delta = \nabla + \Delta\nabla$$

to get

$$\Delta q_m(x) = \bar{\hat{a}}_m \Delta\nabla q_{m+1}(x) + (\bar{\hat{b}}_m + 1) \Delta\nabla q_m(x) + \bar{\hat{c}}_m \Delta\nabla q_{m-1}(x). \tag{2.23}$$

Then, this expression together with (2.19) yields

$$q_m(x) = \sum_{j=m-2}^{m+2} a_{m,j} \Delta\nabla q_j(x), \tag{2.24}$$

with

$$a_{m,m+2} = \bar{\hat{a}}_m \bar{\hat{a}}_{m+1}, \;\; a_{m,m+1} = \bar{\hat{a}}_m (\bar{\hat{b}}_m + \bar{\hat{b}}_{m+1} + 1), \;\; a_{m,m} = \bar{\hat{a}}_m \bar{\hat{c}}_{m+1} + \bar{\hat{b}}_m^2 + \bar{\hat{b}}_m + \bar{\hat{c}}_m \bar{\hat{a}}_{m-1},$$
$$a_{m,m-1} = \bar{\hat{c}}_m (\bar{\hat{b}}_m + \bar{\hat{b}}_{m-1} + 1), \;\; a_{m,m-2} = \bar{\hat{c}}_m \bar{\hat{c}}_{m-1}.$$

Second, from (2.18) we have

$$\tau(x) \Delta q_m(x) = \bar{\alpha}_m^\star d \Delta q_{m+1}(x) + (e + d\bar{\beta}_m^\star) \Delta q_m(x) + \bar{\gamma}_m^\star d \Delta q_{m-1}(x),$$

and using (2.23) this yields

$$\tau(x) \Delta q_m(x) = \sum_{j=m-2}^{m+2} a_{m,j}^{(1)} \Delta\nabla q_j(x), \tag{2.25}$$

with

$$a_{m,m+2}^{(1)} = d\bar{\alpha}_m^\star \bar{\hat{a}}_{m+1}, \;\; a_{m,m+1}^{(1)} = d\Big(\bar{\alpha}_m^\star (\bar{\hat{b}}_{m+1} + 1) + \bar{\beta}_m^\star \bar{\hat{a}}_m \Big) + e\bar{\hat{a}}_m,$$
$$a_{m,m-2}^{(1)} = d\bar{\gamma}_m^\star \bar{\hat{c}}_{m-1}, \;\; a_{m,m-1}^{(1)} = d\Big(\bar{\beta}_m^\star \bar{\hat{c}}_m + \bar{\gamma}_m^\star (\bar{\hat{b}}_{m-1} + 1) \Big) + e\bar{\hat{c}}_m,$$
$$a_{m,m}^{(1)} = d\Big(\bar{\alpha}_m^\star \bar{\hat{c}}_{m+1} + \bar{\beta}_m^\star (\bar{\hat{b}}_m + 1) + \bar{\gamma}_m^\star \bar{\hat{a}}_{m-1} \Big) + e(\bar{\hat{b}}_m + 1).$$

And third, we apply ∇ to both sides of (2.18) and use the properties

$$\nabla(p(x)q(x)) = \nabla p(x) \cdot q(x) + p(x-1)\nabla q(x), \;\; \Delta\nabla = \nabla\Delta$$

and (2.23) to get

$$\begin{aligned} x\Delta\nabla q_m(x) &= \bar{\alpha}_m^{\star}\Delta\nabla q_{m+1}(x) + (\bar{\beta}_m^{\star}+1)\Delta\nabla q_m(x) + \bar{\gamma}_m^{\star}\Delta\nabla q_{m-1}(x) - \Delta q_m(x) \\ &= (\bar{\alpha}_m^{\star} - \bar{\hat{a}}_m)\Delta\nabla q_{m+1}(x) + (\bar{\beta}_m^{\star} - \bar{\hat{b}}_m)\Delta\nabla q_m(x) + (\bar{\gamma}_m^{\star} - \bar{\hat{c}}_m)\Delta\nabla q_{m-1}(x). \end{aligned}$$

It follows that

$$\sigma(x)\Delta\nabla q_m(x) = \sum_{j=m-2}^{m+2} a_{m,j}^{(2)}\Delta\nabla q_j(x), \tag{2.26}$$

with the coefficients $a_{m,j}^{(2)}$ given as in (1.27).

Now, we substitute (2.24)-(2.26) in Equation (2.21) and proceed as in Section 1.2.3.

Using the $\{\Delta q_m(x)\}$ basis ($\sigma(x) = \bar{\sigma}(x)$)

The procedure is the same as in Section 1.2.3 with Equations (1.28), (1.29) and (1.30) replaced, respectively, by

$$q_m(x) = \sum_{j=m-1}^{m+1} b_{m,j}\Delta q_j(x),$$

$$\tau(x)\Delta q_m(x) = \sum_{j=m-1}^{m+1} b_{m,j}^{(1)}\Delta q_j(x),$$

and

$$\sigma(x)\Delta\nabla q_m(x) = \bar{\sigma}(x)\Delta\nabla q_m(x) = \sum_{j=m-1}^{m+1} b_{m,j}^{(2)}\Delta q_j(x).$$

with the same coefficients.

2.3.3 Linearization Problem for CDOP: the NaViMa Algorithm

Every CDOP family satisfies a difference equation of type (2.1) which can be easily rewritten as

$$A(x)y(x+1) + B(x)y(x) + C(x)y(x-1) = 0, \tag{2.27}$$

where $A(x) = \sigma(x) + \tau(x)$, $B(x) = \lambda_n - 2\sigma(x) - \tau(x)$ and $C(x) = \sigma(x)$ are polynomials of maximum degree 2. Salvy and Zimmermann [1994] (see also [Stanley, 1980]) developed an algorithm to compute the difference equation of the product of two CDOP $p_n(x)q_m(x)$. This algorithm is implemented in Maple in the package `gfun` by the procedure `rec*rec`. If $p_n(x)$ and $q_m(x)$ are solutions of a difference equation of type (2.27), then using this algorithm, we obtain a fourth order difference equation

$$r_4(x)y(x+4) + r_3(x)y(x+3) + r_2(x)y(x+2) + r_1(x)y(x+1) + r_0(x)y(x) = 0, \tag{2.28}$$

satisfies by the product $y(x) = p_n(x)p_m(x)$. To solve recursively the linearization problem

$$p_n(x)p_m(x) = \sum_{k=0}^{n+m} L_k(m,n)p_k(x),$$

we substitute the above expression in (2.28), and then from (2.27), $p_k(x+4)$, $p_k(x+3)$, $p_k(x+2)$ are given as linear combinations of $p_k(x+1)$ and $p_k(x)$. This yields a difference equation of the form

$$\sum_{k=0}^{n+m} f(x)p_k(x+1) + g(x)p_k(x) = 0,$$

where $f(x)$ and $g(x)$ are polynomials. Substituting $p_k(x+1)$ by $\Delta p_k(x)+p_k(x)$ and then using structural formulae (2.18) and (2.19), we obtain after an appropriate shift of indices a recurrence relation in the variable k satisfied by the linearization coefficients $L_k(m,n)$.

Remark 2.5. 1. *Ronveaux et al. [1996], Lewanowicz [1996a] also proposed an algorithmic method of obtaining recurrence relations satisfied by the connection coefficients between two families of the CDOP, and using the same approach, Belmehdi et al. [1997] gave a recurrence relation for the linearization coefficients of CDOP. Their method uses again structure formulas of CDOP.*

2. *Zarzo et al. [1997] used the NaViMa algorithm to solve the inversion problem of monic CDOP.*

3. *Using theorems from the theory of generalized hypergeometric functions of Section 1.4, Chapter 1, Lewanowicz [2003a] obtained a representation of the connection coefficients of some CDOP.*

2.4 Connection and Linearization Coefficients of CDOP

Gasper [1974], Ronveaux et al. [1996], Koepf and Schmersau [1998] (compare [Zarzo et al., 1997], [Alvarez-Nodarse et al., 1997]) proved that

Theorem 2.6. *The following representations for the falling factorials in terms of the classical discrete orthogonal polynomials are valid:*

$$x^{\underline{n}} = \sum_{m=0}^{n} \frac{(1+\alpha)_n(-N)_n(-1)^n}{(\alpha+\beta+2)_n}\frac{(\alpha+\beta+1+2m)}{(\alpha+\beta+1)}\frac{(-n)_m(\alpha+\beta+1)_m}{(n+2+\alpha+\beta)_m m!}Q_m(x;\alpha,\beta,N),$$
$$x^{\underline{n}} = \sum_{m=0}^{n} \frac{(-1)^n(\gamma)_n(\frac{\mu}{\mu-1})^n(-n)_m}{m!}M_m(x;\gamma,\mu),$$
$$x^{\underline{n}} = \sum_{m=0}^{n} \frac{(-1)^n(-N)_n p^n(-n)_m}{m!}K_m(x;p,N),$$
$$x^{\underline{n}} = \sum_{m=0}^{n} \frac{\mu^n(-n)_m}{m!}C_m(x;\mu).$$

In [Gasper, 1974], [Koepf and Schmersau, 1998], [Lewanowicz, 2003a] (compare [Lewanowicz, 1996a], [Area et al., 1998]), the connection coefficients within the same family of CDOP were given.

Theorem 2.7. *The following connection relations between the classical discrete orthogonal polynomials are valid:*

$$Q_n(x;\alpha,\beta,N)=\sum_{m=0}^{n}\frac{(\beta-\delta)_n(-1)^n}{(2+\alpha+\delta)_n}\frac{(\alpha+\delta+1+2m)}{(\alpha+\delta+1)}$$
$$\times\frac{(-n)_m(1+\alpha+\delta)_m(n+1+\alpha+\beta)_m(-1)^m}{(\alpha+2+n+\delta)_m(1-\beta+\delta-n)_m m!}Q_m(x;\alpha,\delta,N),$$

$$Q_n(x;\alpha,\beta,N)=\sum_{m=0}^{n}\frac{(\alpha-\gamma)_n(\beta+1)_n}{(\alpha+1)_n(2+\beta+\gamma)_n}\frac{(\beta+\gamma+1+2m)}{(\beta+\gamma+1)}$$
$$\times\frac{(-n)_m(1+\beta+\gamma)_m(\gamma+1)_m(n+1+\alpha+\beta)_m}{(\beta+1)_m(\beta+\gamma+n+2)_m(\gamma-\alpha-n+1)_m m!}Q_m(x;\gamma,\beta,N),$$

$$Q_n(x;\alpha,\beta,N)=\sum_{m=0}^{n}\frac{n!(\gamma+1)_m(n+\alpha+\beta+1)_m}{m!(n-m)!(\alpha+1)_m(m+\gamma+\delta+1)_m}$$
$$\times{}_3F_2\left(\left.\begin{matrix}m-n,m+\gamma+1,n+m+\alpha+\beta+1\\ m+\alpha+1,2m+\gamma+\delta+2\end{matrix}\right|1\right)Q_m(x;\gamma,\delta,N),$$

if $n=0,1,\ldots,M$ *and* $x=0,1,\ldots,\min(M,N)$, *then*

$$Q_n(x;\alpha,\beta,M)=\sum_{m=0}^{\min(n,N)}\frac{n!(-N)_m(\gamma+1)_m(n+\alpha+\beta+1)_m}{m!(n-m)!(-M)_m(\alpha+1)_m(m+\gamma+\delta+1)_m}$$
$$\times{}_4F_3\left(\left.\begin{matrix}m-n,m-N,m+\gamma+1,n+m+\alpha+\beta+1\\ m-M,m+\alpha+1,2m+\gamma+\delta+2\end{matrix}\right|1\right)Q_m(x;\gamma,\delta,N),$$

$$M_n(x;\gamma,\mu)=\sum_{m=0}^{n}\frac{(\gamma-\delta)_n(-n)_m(\delta)_m}{(\delta-n+1-\gamma)_m(\gamma)_n m!}M_m(x;\delta,\mu),$$
$$M_n(x;\gamma,\mu)=\sum_{m=0}^{n}\Big(\frac{\nu-\mu}{\mu(\nu-1)}\Big)^n\frac{(-n)_m}{m!}\Big(-\frac{\nu(\mu-1)}{\nu-\mu}\Big)^m M_m(x;\gamma,\nu),$$
$$M_n(x;\gamma,\mu)=\sum_{m=0}^{n}\frac{n!(\delta)_m}{m!(n-m)!(\gamma)_m}\Big[\frac{\nu(1-\mu)}{\mu(1-\nu)}\Big]^m{}_2F_1\left(\left.\begin{matrix}m-n,m+\delta\\ m+\gamma\end{matrix}\right|\frac{\nu(1-\mu)}{\mu(1-\nu)}\right)M_m(x;\delta,\nu),$$
$$K_n(x;p,N)=\sum_{m=0}^{n}\frac{q^m(p-q)^{n-m}\binom{n}{m}}{p^n}K_m(x;q,N),$$
$$K_n(x;p,N)=\sum_{m=0}^{n}\frac{(-M)_m(M-N)_{n-m}\binom{n}{m}}{(-N)_n}K_m(x;p,M),$$

if $n=0,1,\ldots,M$ *and* $x=0,1,\ldots,\min(M,N)$, *then*

$$K_n(x;q,M)=\sum_{m=0}^{\min(n,N)}\frac{(-N)_m p^m\binom{n}{m}}{(-M)_m q^m}{}_2F_1\left(\left.\begin{matrix}m-n,m-N\\ m-M\end{matrix}\right|\frac{p}{q}\right)K_m(x;p,N),$$

$$C_n(x;\mu) = \sum_{m=0}^{n} (-1)^n \frac{\nu^m}{\mu^n} (\nu-\mu)^{n-m} \frac{(-n)_m}{m!} C_m(x;\nu).$$

Belmehdi et al. [1997] gave a procedure to get a recurrence relation for the linearization coefficients of CDOP. Here, using a systematic approach, we provide hypergeometric series representation of these coefficients. These results, as far as we know, are new.

Theorem 2.8. *The following linearization formulae hold*
1. for the Hahn polynomial family

$$Q_n(x;\alpha,\beta,N)Q_m(x;\alpha,\beta,N) = \sum_{r=0}^{n+m} L_r(m,n) Q_r(x;\alpha,\beta,N), \quad m+n \leq N,$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-n)_j(n+\alpha+\beta+1)_j(-m)_{l+r-j}(m+\alpha+\beta+1)_{l+r-j}}{(\alpha+1)_j(-N)_j(\alpha+1)_{l+r-j}(-N)_{l+r-j}(l+r-j)!j!r!}$$
$$\times \frac{(\alpha+1)_{l+r}(-N)_{l+r}(\alpha+\beta+1+2r)(-l-r)_r(\alpha+\beta+1)_r}{(\alpha+\beta+2)_{l+r}(\alpha+\beta+1)(\alpha+\beta+2+l+r)_r}$$
$$\times {}_3F_2\left(\left.\begin{array}{c} -j, l+r-j+m+\alpha+\beta+1, l+r-j-m \\ l+r-j-N, l+r-j+\alpha+1 \end{array}\right| 1\right),$$

2. for the Meixner polynomial family

$$M_n(x;\gamma,\mu)M_m(x;\gamma,\mu) = \sum_{r=0}^{n+m} L_r(m,n) M_r(x;\gamma,\mu),$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-n)_j(-m)_{l+r-j}(\gamma)_{l+r}(-l-r)_r}{(\gamma)_j(\gamma)_{l+r-j}(l+r-j)!j!r!}$$
$$\times {}_2F_1\left(\left.\begin{array}{c} -j, l+r-j-m \\ l+r-j+\gamma \end{array}\right| \frac{\mu-1}{\mu}\right),$$

3. for the Krawtchouk polynomial family

$$K_n(x;p,N)K_m(x;p,N) = \sum_{r=0}^{n+m} L_r(m,n) K_r(x;p,N),$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-n)_j(-m)_{l+r-j}(-N)_{l+r}(-l-r)_r}{(-N)_j(-N)_{l+r-j}(l+r-j)!j!r!}$$
$$\times {}_2F_1\left(\left.\begin{array}{c} l+r-j-m, -j \\ l+r-j-N \end{array}\right| \frac{1}{p}\right),$$

4. for the Charlier polynomial family

$$C_n(x;\mu)C_m(x;\mu) = \sum_{r=0}^{n+m} L_r(m,n)C_r(x;\mu),$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=max(0,l+r-m)}^{\min(n,l+r)} \frac{(-n)_j(-m)_{l+r-j}(-l-r)_r}{(l+r-j)!j!r!}$$
$$\times {}_2F_0\left(\left.\begin{matrix} l+r-j-m, -j \\ - \end{matrix}\right| -\frac{1}{\mu}\right).$$

The proof of Theorem 2.8 uses the following lemma.

Lemma 2.9. *If $m<n$,*

$$x^{\underline{n}}x^{\underline{m}} = \sum_{k=0}^{m} \frac{m!n!}{k!(m-k)!(n-m+k)!} x^{\underline{n+k}}. \tag{2.29}$$

Proof. From the definition of $x^{\underline{n}}$, $x^{\underline{n}}=0 \;\Leftrightarrow\; x=0,1,\ldots,n-1$ and $(n+s)^{\underline{n}}\neq 0,\; s=0,1,\ldots$. We suppose that

$$x^{\underline{n}}x^{\underline{m}} = \sum_{k=0}^{n+m} H_k(m,n)x^{\underline{k}} = H_0(m,n) + \sum_{k=1}^{n+m} H_k(m,n)x^{\underline{k}}.$$

For $x=0$, $x^{\underline{k}}=0$, $k=1,2,\ldots$ and then the previous equation yields $H_0(m,n)=0$, $m\geq 1$. Therefore

$$x^{\underline{n}}x^{\underline{m}} = H_1(m,n)x + \sum_{k=2}^{n+m} H_k(m,n)x^{\underline{k}},$$

and for $x=1$ and $m\geq 2$, it follows that $H_1(m,n)=0$. Progressively, in a similar way, we obtain

$$H_0(m,n) = H_1(m,n) = \ldots = H_{n-1}(m,n) = 0.$$

Consequently,

$$x^{\underline{n}}x^{\underline{m}} = \sum_{k=n}^{n+m} H_k(m,n)x^{\underline{k}} = \sum_{k=0}^{m} H_{n+k}(m,n)x^{\underline{n+k}}.$$

Since $x^{\underline{n+k}} = x^{\underline{n}}(x-n)^{\underline{k}}$, it follows that

$$x^{\underline{m}} = \sum_{k=0}^{m} H_{n+k}(m,n)(x-n)^{\underline{k}}.$$

We apply Δ^l, $l=0,1,\ldots,m$ to both sides of the previous equation and use (2.8) to get

$$\frac{m!}{(m-l)!}x^{\underline{m-l}} = \sum_{k=l}^{m} H_{n+k}(m,n)\frac{k!}{(k-l)!}(x-n)^{\underline{k-l}}$$
$$= H_{n+l}(m,n)l! + \sum_{k=l+1}^{m} H_{n+k}(m,n)\frac{k!}{(k-l)!}(x-n)^{\underline{k-l}}.$$

For $x = n$, we obtain
$$H_{n+l}(m,n) = \frac{m!n!}{l!(m-l)!(n-m+l)!}.$$
$\square$

Remark 2.10. *If we substitute k by $m-k$ in (2.29), we get*
$$x^{\underline{n}}x^{\underline{m}} = \sum_{k=0}^{\min(m,n)} k!\binom{n}{k}\binom{m}{k} x^{\underline{n+m-k}}.$$

Having derived the linearization relation for $x^{\underline{n}}$, we now prove Theorem 2.8.

Proof (**of Theorem 2.8**). We have
$$p_n(x) = \sum_{j=0}^{n} A_j(n)x^{\underline{j}} \text{ and } p_m(x) = \sum_{k=0}^{m} A_k(m)x^{\underline{k}},$$
so that
$$\begin{aligned} p_n(x)p_m(x) &= \sum_{j=0}^{n}\sum_{k=0}^{m} A_j(n)A_k(m)x^{\underline{j}}x^{\underline{k}} \\ &\overset{(2.29)}{=} \sum_{j=0}^{n}\sum_{k=0}^{m} A_j(n)A_k(m)\Big(\sum_{l=0}^{k} H_{j+l}(k,j)x^{\underline{j+l}}\Big) \\ &= \sum_{l=0}^{n+m} B_l(m,n)x^{\underline{l}}, \end{aligned}$$
with
$$B_l(m,n) = \sum_{j=\max(0,l-m)}^{\min(n,l)} \sum_{k=l-j}^{\min(m,l)} A_j(n)A_k(m)H_l(k,j),$$
where $H_l(k,j) = 0$ if $l < \max(k,j)$ or $k+j < l$.

Since
$$x^{\underline{l}} = \sum_{r=0}^{l} I_r(l)p_r(x),$$
we get
$$p_n(x)p_m(x) = \sum_{r=0}^{n+m} L_r(m,n)p_r(x)$$
with
$$\begin{aligned} L_r(m,n) &= \sum_{l=0}^{n+m-r} B_{l+r}(m,n)I_r(l+r) \\ &= \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \sum_{k=l+r-j}^{\min(m,l+r)} A_j(n)A_k(m)H_{l+r}(k,j)I_r(l+r) \\ &= \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \sum_{k=0}^{\min(m-l-r+j,j)} A_j(n)A_{l+r-j+k}(m)H_{l+r}(l+r-j+k,j)I_r(l+r). \end{aligned}$$

To get our final results, we use the sumtohyper algorithm. □

Some simpler results were obtained by Askey and Gasper [1977], using generating functions, and are given by

Theorem 2.11 ([Askey and Gasper, 1977]). *For Meixner and Krawtchouk orthogonal polynomials, the following results hold, respectively:*

$$M_n(x;\gamma,\mu)M_m(x;\gamma,\mu)=\sum_{i=0}^{n+m}\frac{n!m!(1+\mu)^{n+m+i}(-1)^{n+m+i}}{\mu^{n+m}(\gamma)_n(\gamma)_m}\times\sum_{j\geq 0}\frac{(\gamma)_j\mu^j(1+\mu)^{-2j}}{(j-n)!(j-m)!(j-i)!(n+m+i-2j)!}M_i(x;\gamma,\mu),$$

$$K_n(x;p,N)K_m(x;p,N)=\sum_{i=0}^{n+m}\frac{N!}{p^{n+m}(1-p)^i\binom{N}{n}\binom{N}{m}}\times\sum_{j\geq 0}\frac{p^j(1-p)^j(2p-1)^{i+m+n-2j}}{(j-i)!(j-m)!(j-n)!(i+m+n-2j)!(N-j)!}K_i(x;p,N).$$

To conclude this section, we deduce the linearization of the product of two falling factorials in terms of CDOP.

Proposition 2.12. *The following linearization formulae are valid:*

$$x^{\underline{n}}x^{\underline{m}}=\sum_{j=0}^{n+m}\frac{m!n!(\alpha+\beta+2j+1)(\alpha+\beta+1)_j}{(\alpha+\beta+1)(\alpha+\beta+k+2)_j j!}\times\sum_{k=\max(m,n,j)}^{n+m}\frac{(\alpha+1)_k(-N)_k(-1)^k(-k)_j}{(k-n)!(k-m)!(n+m-k)!(\alpha+\beta+2)_k}Q_j(x;\alpha,\beta,N),$$

$$x^{\underline{n}}x^{\underline{m}}=\sum_{j=0}^{n+m}\sum_{k=\max(m,n,j)}^{n+m}\frac{m!n!(-1)^k(\gamma)_k(-k)_j\mu^k}{(k-n)!(k-m)!(n+m-k)!j!(\mu-1)^k}M_j(x;\gamma,\mu),$$

$$x^{\underline{n}}x^{\underline{m}}=\sum_{j=0}^{n+m}\sum_{k=\max(m,n,j)}^{n+m}\frac{m!n!(-1)^k(-N)_kp^k(-k)_j}{(k-n)!(k-m)!(n+m-k)!j!}K_j(x;p,N),$$

$$x^{\underline{n}}x^{\underline{m}}=\sum_{j=0}^{n+m}\sum_{k=\max(m,n,j)}^{n+m}\frac{m!n!\mu^k(-k)_j}{(k-n)!(k-m)!(n+m-k)!j!}C_j(x;\mu).$$

Proof. In fact, the combination of

$$x^{\underline{n}}x^{\underline{m}}=\sum_{k=\max(m,n)}^{n+m}H_k(m,n)x^{\underline{k}}\quad\text{and}\quad x^{\underline{k}}=\sum_{j=0}^{k}I_j(k)p_j(x)$$

yields the linearization formula

$$x^{\underline{n}}x^{\underline{m}}=\sum_{j=0}^{n+m}L_j(m,n)p_j(x),\quad\text{with}\quad L_j(m,n)=\sum_{k=\max(m,n,j)}^{n+m}H_k(m,n)I_j(k).$$

□

2.5 Duplication Problem for CDOP

Using generating functions, Chaggara and Koepf [2007] solved the duplication problem

$$p_n(ax) = \sum_{m=0}^{n} D_m(n,a)p_m(x)$$

for the Charlier, Meixner and Krawtchouk polynomials for the specific case $a = -1$. Recurrence relations satisfied by the duplication coefficients were also given. Area et al. [2003] presented an algorithmic approach to obtain these recurrence relations. Their approach was based on the NaViMa algorithm. In this section the general duplication problem for CDOP is solved.

Theorem 2.13. *For the classical discrete orthogonal polynomials, the following duplication relations are valid.*

$$\begin{aligned} Q_n(ax;\alpha,\beta,N) &= \sum_{m=0}^{n} \frac{(\alpha+\beta+1+2m)(\alpha+\beta+1)_m}{(\alpha+\beta+1)m!} \\ &\times \sum_{j=0}^{n-m} \frac{(\alpha+1,-N)_{n-j}(-n+j)_m}{(\alpha+\beta+2)_{n-j}(\alpha+\beta+n-j+2)_m(n-j)!} \\ &\times \sum_{k=0}^{j} \frac{(-n,n+\alpha+\beta+1)_{k+n-j}}{(\alpha+1,-N)_{k+n-j}(k+n-j)!} \times \sum_{l=0}^{n-j}(-1)^l \binom{n-j}{l} (-al)_{k+n-j} Q_m(x;\alpha,\beta,N), \end{aligned}$$

$$\begin{aligned} M_n(ax;\gamma,\mu) &= \sum_{m=0}^{n}\sum_{j=0}^{n-m} \frac{(\gamma)_{n-j}(-n+j)_m}{m!(n-j)!} \times \sum_{k=0}^{j} \frac{(-n)_{n+k-j}}{(n+k-j)!(\gamma)_{n+k-j}} \left(\frac{\mu-1}{\mu}\right)^k \\ &\times \sum_{l=0}^{n-j}(-1)^l \binom{n-j}{l} (-al)_{n+k-j} M_m(x;\gamma,\mu), \end{aligned}$$

$$\begin{aligned} K_n(ax;p,N) &= \sum_{m=0}^{n}\sum_{j=0}^{n-m} \frac{(-N)_{n-j}(-n+j)_m}{m!(n-j)!} \\ &\times \sum_{k=0}^{j} \frac{(-n)_{n+k-j}}{p^k(-N)_{n+k-j}(n+k-j)!} \sum_{l=0}^{n-j}(-1)^l \binom{n-j}{l} (-al)_{n+k-j} K_m(x;p,N), \end{aligned}$$

$$\begin{aligned} C_n(ax;\mu) &= \sum_{m=0}^{n}\sum_{j=0}^{n-m} \frac{(-n+j)_m}{m!(n-j)!} \\ &\times \sum_{k=0}^{j} \frac{(-n)_{n+k-j}}{(n+k-j)!(-\mu)^k} \sum_{l=0}^{n-j}(-1)^l \binom{n-j}{l} (-al)_{n+k-j} C_m(x;\mu). \end{aligned}$$

The proof of this theorem uses the following result.

Lemma 2.14. *If f is a polynomial w.r.t. x of degree n, then*

$$f(x) = \sum_{k=0}^{n} f_k x^{\underline{k}}, \tag{2.30}$$

where

$$f_k = \frac{1}{k!}\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}f(l).$$

Proof. If we apply the operator Δ^j to Equation (2.30), we get for any positive integer j

$$\Delta^j f(x) = \sum_{k=0}^{n} f_k \Delta^j x^{\underline{k}}.$$

Since $\Delta^j x^{\underline{k}} = \frac{k!}{(k-j)!}x^{\underline{k-j}}$, it follows that

$$\Delta^j f(x) = \sum_{k=0, k\neq j}^{n} f_k \frac{k!}{(k-j)!}x^{\underline{k-j}} + f_j j!.$$

For $x = 0$, this yields

$$f_j = \frac{\Delta^j f(0)}{j!}.$$

The result follows from the relation

$$\Delta^k f(x) = \sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}f(x+l).$$

□

Replacing $f(x)$ by $(ax)^{\underline{n}}$ in (2.30), we are led to

Corollary 2.15. *The following duplication formula holds:*

$$(ax)^{\underline{n}} = \sum_{k=0}^{n}\frac{(-1)^{n+k}}{k!}\sum_{l=0}^{k}(-1)^l\binom{k}{l}(-al)_n x^{\underline{k}}.$$

Proof (of Theorem 2.13). Combining $p_n(ax) = \sum_{k=0}^{n} A_k(n)(ax)^{\underline{k}}$, $(ax)^{\underline{k}} = \sum_{i=0}^{k} E_i(k,a)x^{\underline{i}}$ with $E_i(k,a) = \sum_{l=0}^{i} F_l(i,k,a)$, $x^{\underline{i}} = \sum_{m=0}^{i} I_m(i)p_m(x)$, interchanging the order of summation and substituting i by $n-m-j$ yields the duplication relation $p_n(ax) = \sum_{m=0}^{n} D_m(n,a)p_m(x)$ with

$$D_m(n,a) = \sum_{j=0}^{n-m}\sum_{k=0}^{j}\sum_{l=0}^{n-j} A_{k+n-j}(n)F_l(n-j,k+n-j,a)I_m(n-j).$$

□

2.6 Application of Connection Formulae of CDOP: Parameter Derivatives

The following results obtained by Koepf and Schmersau [1998] are deduced from Theorem 2.7, following the method used in the proof of Corollary 1.16.

Corollary 2.16. *The following representations for the parameter derivatives of the classical discrete orthogonal polynomials are valid:*

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}Q_n(x;\alpha,\beta,N) &= \sum_{m=0}^{n-1}\Bigg(\Big(\frac{1}{\alpha+\beta+m+n+1}-\frac{1}{\alpha+m+1}\Big)Q_n(x;\alpha,\beta,N)\\
&\quad+\frac{(\alpha+\beta+1+2m)(\beta+1+m)_{n-m}n!}{(n-m)(\alpha+\beta+n+m+1)(\alpha+1+m)_{n-m}(\alpha+\beta+1+m)_{n-m}m!}Q_m(x;\alpha,\beta,N)\Bigg),\\
\frac{\partial}{\partial\beta}Q_n(x;\alpha,\beta,N) &= \sum_{m=0}^{n-1}\frac{1}{\alpha+\beta+m+n+1}\cdot\Big(Q_n(x;\alpha,\beta,N)\\
&\qquad+\frac{(-1)^{n-m}(\alpha+\beta+1+2m)n!}{(n-m)(\alpha+\beta+1+m)_{n-m}m!}Q_m(x;\alpha,\beta,N)\Big),\\
\frac{\partial}{\partial\mu}M_n(x;\gamma,\mu) &= \frac{n}{\mu(\mu-1)}(M_n(x;\gamma,\mu)-M_{n-1}(x;\gamma,\mu)),\\
\frac{\partial}{\partial\gamma}M_n(x;\gamma,\mu) &= \sum_{m=0}^{n-1}\Big(-\frac{1}{\gamma+m}M_n(x;\gamma,\mu)+\frac{n!}{m!(n-m)(m+\gamma)_{n-m}}M_m(x;\gamma,\mu)\Big),\\
\frac{\partial}{\partial p}K_n(x;p,N) &= \frac{n}{p}\Big(K_{n-1}(x;p,N)-K_n(x;p,N)\Big),\\
\frac{\partial}{\partial N}K_n(x;p,N) &= \sum_{m=0}^{n-1}\Big(\frac{1}{m-N}K_n(x;p,N)-\frac{n!}{m!(n-m)(m-N)_{n-m}}K_m(x;p,N)\Big),\\
\frac{\partial}{\partial\mu}C_n(x;\mu) &= \frac{n}{\mu}\Big(C_{n-1}(x;\mu)-C_n(x;\mu)\Big).
\end{aligned}
$$

We note that $\frac{\partial}{\partial\mu}M_n(x;\gamma,\mu)$, $\frac{\partial}{\partial\gamma}M_n(x;\gamma,\mu)$, $\frac{\partial}{\partial p}K_n(x;p,N)$ are different from the results of Koepf [1998] because the polynomial families are different by a normalizing constant. We also note that in [Koepf, 1998], the parameter derivative $\frac{\partial}{\partial\alpha}Q_n(x;\alpha,\beta,N)$ has a misprint.

Chapter 3

Connection, Linearization and Duplication Coefficients of q-Orthogonal Polynomials of the q-Hahn Class

Area et al. [1999], Foupouagnigni et al. [2012] solved the inversion problem of q-classical orthogonal polynomials. An algorithmic approach was given by Lewanowicz [2003b] to construct recurrence relations for the coefficients of the series expansion of a solution of a linear q-difference equation in q-classical orthogonal polnomials (of the q-Hahn class). Area et al. [2001] proposed a method to solve the connection and linearization problem of q-classical orthogonal polynomials from which they deduced the connection coefficients between the little q-Jacobi and shifted Jacobi polynomials. In this chapter, using this method and another algorithmic approach we solve the connection and linearization problems of all the q-classical orthogonal polynomial families of the q-Hahn class. Furthermore, using two algorithmic approaches, the duplication problem for q-classical orthogonal polynomials is also solved.

3.1 Introduction

A family

$$y(x) = p_n(x) = k_n x^n + k'_n x^{n-1} + k''_n x^{n-2} + \dots \qquad (n \in \mathbb{N}_0,\ k_n \neq 0) \tag{3.1}$$

of polynomials of degree exactly n is a family of q-classical orthogonal polynomials (in short q-COP) of the q-Hahn class if it is solution of a q-differential equation of the type (see e.g. [Koepf and Schmersau, 2001], [Medem et al., 2001])

$$\sigma(x) D_q D_{\frac{1}{q}} p_n(x) + \tau(x) D_q p_n(x) + \lambda_{n,q} p_n(x) = 0, \tag{3.2}$$

which is equivalent to

$$\tilde{\sigma}(x) D_{\frac{1}{q}} D_q p_n(x) + \tau(x) D_{\frac{1}{q}} p_n(x) + \lambda_{n,q} p_n(x) = 0,$$

where $\sigma(x) = ax^2 + bx + c$, $\tilde{\sigma}(x) = \sigma(x) + (q-1)x\tau(x)$ are polynomials of at most second order and $\tau(x) = dx + e$ is a polynomial of first order and the q-differential operator D_q

is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0,\, q \neq 1$$

and $D_q f(0) := f'(0)$ by continuity, provided $f'(0)$ exists. In the sequel, we shall always assume that $0 < q < 1$.

As we can read in the preface of Koekoek et al. [2010], Hahn [1949] found orthogonal polynomial solutions of second order q-difference equations. This class of orthogonal polynomials is known as the q-Hahn class. Many other families of orthogonal polynomials such as the discrete classical orthogonal polynomials have been very well known for a long time, but a classification of all these families did not exist. A first attempt to combine both the continuous and discrete classical orthogonal polynomials was made by Askey and Wilson [1985] by introducing the so-called Askey scheme of hypergeometric orthogonal polynomials. All known q-analogues of the families of orthogonal polynomials belonging to the Askey scheme were arranged into a q-analogue of this Askey-scheme. The polynomial systems which are solution of (3.2) form the q-Hahn tableau. The following systems are members of the q-Hahn tableau [Koekoek et al., 2010]:

1. the big q-Jacobi polynomials

$$P_n(x;\alpha,\beta,\gamma;q) = {}_3\phi_2\left(\left.\begin{array}{c} q^{-n}, \alpha\beta q^{n+1}, x \\ \alpha q, \gamma q \end{array}\right| q; q\right),$$

   which for $\alpha = \beta = 1$ are the big q-Legendre polynomials

$$P_n(x;\gamma;q) = {}_3\phi_2\left(\left.\begin{array}{c} q^{-n}, q^{n+1}, x \\ q, \gamma q \end{array}\right| q; q\right),$$

2. the q-Hahn polynomials

$$Q_n(\bar{x};\alpha,\beta,N|q) = {}_3\phi_2\left(\left.\begin{array}{c} q^{-n}, \alpha\beta q^{n+1}, \bar{x} \\ \alpha q, q^{-N} \end{array}\right| q; q\right), \text{ with } \bar{x} = q^{-x} \text{ and } n = 0, 1, \ \ldots, \ N,$$

3. the big q-Laguerre polynomials

$$P_n(x;\alpha,\beta;q) = {}_3\phi_2\left(\left.\begin{array}{c} q^{-n}, x, 0 \\ \alpha q, \beta q \end{array}\right| q; q\right),$$

4. the little q-Jacobi polynomials

$$p_n(x;\alpha,\beta|q) = {}_2\phi_1\left(\left.\begin{array}{c} q^{-n}, \alpha\beta q^{n+1} \\ \alpha q \end{array}\right| q; qx\right),$$

   which for $\alpha = \beta = 1$ are the little q-Legendre polynomials

$$p_n(x|q) = {}_2\phi_1\left(\left.\begin{array}{c} q^{-n}, q^{n+1} \\ q \end{array}\right| q; qx\right),$$

5. the q-Meixner polynomials

$$M_n(\bar{x};\beta,\gamma;q) = {}_2\phi_1\left(\left.\begin{matrix} q^{-n},\bar{x} \\ \beta q \end{matrix}\right| q;-\frac{q^{n+1}}{\gamma}\right), \quad \text{with} \quad \bar{x}=q^{-x},$$

6. the quantum q-Krawtchouk polynomials

$$K_n^{\mathrm{qtm}}(\bar{x};p,N;q) = {}_2\phi_1\left(\left.\begin{matrix} q^{-n},\bar{x} \\ q^{-N} \end{matrix}\right| q;pq^{n+1}\right), \text{ with } \bar{x}=q^{-x} \text{ and } n=0,1,\ \ldots,\ N,$$

7. the q-Krawtchouk polynomials

$$K_n(\bar{x};p,N;q) = {}_3\phi_2\left(\left.\begin{matrix} q^{-n},\bar{x},-pq^n \\ q^{-N},0 \end{matrix}\right| q;q\right) \text{ with } \bar{x}=q^{-x} \text{ and } n=0,1,\ \ldots,\ N,$$

8. the affine q-Krawtchouk polynomials

$$K_n^{\mathrm{Aff}}(\bar{x};p,N;q) = {}_3\phi_2\left(\left.\begin{matrix} q^{-n},\bar{x},0 \\ pq,q^{-N} \end{matrix}\right| q;q\right) \text{ with } \bar{x}=q^{-x} \text{ and } n=0,1,\ \ldots,\ N,$$

9. the little q-Laguerre / Wall polynomials

$$p_n(x;\alpha|q) = {}_2\phi_1\left(\left.\begin{matrix} q^{-n},0 \\ \alpha q \end{matrix}\right| q;qx\right),$$

10. the q-Laguerre polynomials

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_1\phi_1\left(\left.\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix}\right| q;-q^{n+\alpha+1}x\right),$$

11. the alternative q-Charlier or q-Bessel polynomials

$$y_n(x;\alpha;q) = {}_2\phi_1\left(\left.\begin{matrix} q^{-n},-\alpha q^n \\ 0 \end{matrix}\right| q;qx\right),$$

12. the q-Charlier polynomials

$$C_n(\bar{x};\alpha;q) = {}_2\phi_1\left(\left.\begin{matrix} q^{-n},\bar{x} \\ 0 \end{matrix}\right| q;-\frac{q^{n+1}}{\alpha}\right),$$

13. the Al-Salam-Carlitz I polynomials

$$U_n^{(\alpha)}(x;q) = (-\alpha)^n q^{\binom{n}{2}} {}_2\phi_1 \left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array} \middle| q; \frac{qx}{\alpha} \right),$$

14. the Al-Salam-Carlitz II polynomials

$$V_n^{(\alpha)}(x;q) = (-\alpha)^n q^{-\binom{n}{2}} {}_2\phi_0 \left(\begin{array}{c} q^{-n}, x \\ - \end{array} \middle| q; \frac{q^n}{\alpha} \right),$$

15. the Stieltjes-Wigert polynomials

$$S_n(x;q) = \frac{1}{(q;q)_n} {}_1\phi_1 \left(\begin{array}{c} q^{-n} \\ 0 \end{array} \middle| q; -q^{n+1}x \right),$$

16. the discrete q-Hermite I polynomials

$$h_n(x;q) = q^{\binom{n}{2}} {}_2\phi_1 \left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array} \middle| q; -qx \right),$$

the discrete q-Hermite I polynomials are the Al-Salam-Carlitz I polynomials with $\alpha = -1$ i.e. $h_n(x;q) = U_n^{(-1)}(x;q)$,

17. the discrete q-Hermite II polynomials

$$\tilde{h}_n(x;q) = i^{-n} q^{-\binom{n}{2}} {}_2\phi_0 \left(\begin{array}{c} q^{-n}, ix \\ - \end{array} \middle| q; -q^n \right),$$

the discrete q-Hermite II polynomials are related to the Al-Salam-Carlitz II polynomials with $\alpha = -1$ by $\tilde{h}_n(x;q) = i^{-n} V_n^{(-1)}(ix;q)$.

The representation of the polynomials $p_n(x)$ belonging to the q-Hahn tableau as basic hypergeometric series (see Definition 0.2) suggests four natural bases $\{\mathcal{V}_m\}$ to obtain expansions of the form

$$p_n(x) = \sum_{m=0}^{n} A_m(n) \mathcal{V}_m(x).$$

These expansion bases are the q-shifted factorials (i.e. $\mathcal{V}_m(x) = (x;q)_m$), the powers of x (i.e. $\mathcal{V}_m(x) = x^m$), $\mathcal{V}_m(x) = (ix;q)_m$ and $\mathcal{V}_m(x) = (x-1)(x-q) \cdots (x-q^{m-1}) = (x^{-1};q)_m x^m$. These four bases can be generalized to the basis [Sprenger, 2009]

$$(b \ominus a)_q^n = \begin{cases} (b-a)(b-aq) \cdots (b-aq^{n-1}), & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases}$$

where $a,\ b \in \mathbb{C}$. Indeed, we have

$$(x;q)_n = (1 \ominus x)_q^n,\ x^n = (x \ominus 0)_q^n,\ (ix;q)_n = (1 \ominus ix)_q^n \text{ and } (x^{-1};q)_n x^n = (x \ominus 1)_q^n.$$

It is easy to see that (3.2) is equivalent to

$$B(x)p_n(qx) - (B(x)+D(x))p_n(x) + D(x)p_n\left(\frac{x}{q}\right) = -(q-1)^2 x^2 \lambda_{n,q} p_n(x) \tag{3.3}$$

with $B(x) = \sigma(x) + (q-1)x\tau(x)$ and $D(x) = q\sigma(x)$. From the above equivalent form of Equation (3.2) given in [Koekoek and Swarttouw, 1998], [Koekoek et al., 2010], we get $\sigma(x) = \frac{D(x)}{q}$ and $\tau(x) = \frac{qB(x)-D(x)}{q(q-1)x}$. The coefficients $a,\ b,\ c,\ d,\ e$ of the polynomials $\sigma(x)$ and $\tau(x)$, the leading coefficients of the q-COP and their bases are given in Table 3.1 appearing on the next page.

To compute the connection, linearization, inversion and duplication coefficients of q-COP, we proceed algorithmically using the structure formulae satisfied by these polynomials.

3.2 Structural Formulas for q-Orthogonal Polynomials of the q-Hahn Class

Using computer algebra, the following method gives the coefficients a_n, b_n and c_n of the desired structure relation in terms of a, b, c, d, e, n, q, q^n, k_{n-1}, k_n, and k_{n+1} and using the following algorithm (see [Koepf and Schmersau, 2002]).

1. Substitute
$$p_n(x) = k_n x^n + k_n' x^{n-1} + k_n'' x^{n-2} + \ldots$$
in the q-differential equation (3.2).

2. Equate the coefficients of x^n to determine $\lambda_{n,q}$.

3. Equate the coefficients of x^{n-1} and x^{n-2}. This gives k_n' and k_n'', respectively, in terms of k_n.

4. Substitute $p_n(x)$ in the proposed structure relation, and equate again the three highest coefficients. This computes the three unknowns a_n, b_n and c_n successively.

1. First step: We substitute $p_n(x)$ in the q-differential equation (3.2), use the relations

$$D_q x^n = [n]_q x^{n-1},\ D_{\frac{1}{q}} x^n = \frac{[n]_q}{q^{n-1}} x^{n-1}$$

and by equating the coefficients of x^n, one gets

$$\lambda_{n,q} = -[n]_{\frac{1}{q}}[n-1]_q a - [n]_q d, \tag{3.4}$$

where the abbreviation

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}$$

q-orthogonal polynomials family		a	b	c	d	e	k_n	basis
$\tilde{h}_n(x;q)$	discrete q-Hermite II	0	0	$q-1$	1	0	1	$(ix;q)_n$
$V_n^{(\alpha)}(x;q)$	Al-Salam-Carlitz II	0	0	$(q-1)\alpha$	1	$-\alpha-1$	1	$(x;q)_n$
$C_n(x;\alpha;q)$	q-Charlier	0	$(q-1)\alpha$	0	q	$-\alpha-q$	$(-1)^n\frac{q^{n^2}}{\alpha^n}$	$(x;q)_n$
$L_n^{(\alpha)}(x;q)$	q-Laguerre	0	$q-1$	0	qq^{α}	$qq^{\alpha}-1$	$(-1)^n\frac{q^{n(n+\alpha)}}{(q;q)_n}$	x^n
$S_n(x;q)$	Stieltjes-Wigert	0	$q-1$	0	q	-1	$(-1)^n\frac{q^{n^2}}{(q;q)_n}$	x^n
$M_n(x;\beta,\gamma;q)$	q-Meixner	0	$(q-1)\gamma$	$q(1-q)\beta\gamma$	q	$\beta\gamma q-q-\gamma$	$(-1)^n\frac{q^{n^2}}{\gamma^n(\beta q;q)_n}$	$(x;q)_n$
$K_n^{\mathrm{qtm}}(x;p,N;q)$	Quantum-q-Krawtchouk	0	$(q-1)q^N$	$1-q$	$-pqq^N$	$1-(1-pq)q^N$	$\frac{p^nq^{n^2}}{(q^{-N};q)_n}$	$(x;q)_n$
$K_n(x;p,N;q)$	q-Krawtchouk	$(1-q)q^N$	$q-1$	0	$(1+pq)q^N$	$-(1+pqq^N)$	$\frac{(-pq^n;q)_n}{(q^{-N};q)_n}$	$(x;q)_n$
$p_n(x;\alpha\|q)$	little q-Laguerre	$1-q$	$q-1$	0	1	$\alpha q-1$	$(-1)^n\frac{q^{-\binom{n}{2}}}{(\alpha q;q)_n}$	x^n
$y_n(x;\alpha\|q)$	alternative q-Charlier	$1-q$	$q-1$	0	$1+\alpha q$	-1	$(-1)^nq^{-\binom{n}{2}}(-\alpha q^n;q)_n$	x^n
$p_n(x;\alpha,\beta\|q)$	little q-Jacobi	$1-q$	$q-1$	0	$1-\alpha\beta q^2$	$\alpha q-1$	$(-1)^n\frac{q^{-\binom{n}{2}}(\alpha\beta qq^n;q)_n}{(\alpha q;q)_n}$	x^n
$h_n(x;q)$	discrete q-Hermite I	$1-q$	0	$q-1$	1	0	1	$(x\ominus 1)_q^n$
$U_n^{(\alpha)}(x;q)$	Al-Salam-Carlitz I	$1-q$	$(q-1)(1+\alpha)$	$(1-q)\alpha$	1	$-\alpha-1$	1	$(x\ominus 1)_q^n$
$P_n(x;\alpha,\beta;q)$	big q-Laguerre	$1-q$	$q(q-1)(\alpha+\beta)$	$q^2(1-q)\alpha\beta$	1	$q(\alpha\beta q-\alpha-\beta)$	$\frac{1}{(\alpha q;q)_n(\beta q;q)_n}$	$(x;q)_n$
$K_n^{\mathrm{aff}}(x;p,N;q)$	affine q-Krawtchouk	$(1-q)q^N$	$(q-1)(1+pqq^N)$	$(1-q)pq$	q^N	$-(1+pq(q^N-1))$	$\frac{1}{(pq;q)_n(q^{-N};q)_n}$	$(x;q)_n$
$Q_n(x;\alpha,\beta;N\|q)$	q-Hahn	$(1-q)q^N$	$(q-1)(1+\alpha qq^N)$	$q(1-q)\alpha$	$(1-\alpha\beta q^2)q^N$	$\alpha q(1+q^N(\beta q-1))-1$	$\frac{(\alpha\beta qq^n;q)_n}{(\alpha q;q)_n(q^{-N};q)_n}$	$(x;q)_n$
$P_n(x;\alpha,\beta,\gamma;q)$	big q-Jacobi	$1-q$	$q(q-1)(\alpha+\gamma)$	$q^2(1-q)\alpha\gamma$	$1-\alpha\beta q^2$	$q(\alpha q(\beta+\gamma)-\alpha-\gamma)$	$\frac{(\alpha\beta qq^n;q)_n}{(\alpha q;q)_n(\gamma q;q)_n}$	$(x;q)_n$

Table 3.1: Data of classical q-orthogonal polynomials families

denotes the so-called q-brackets. Note that $\lim_{q\to 1}[n]_q = n$.

2. Second step: Equating the coefficients of x^{n-1} and x^{n-2} gives k_n', and k_n'', respectively, as rational multiples w.r.t. $N = q^n$ of k_n:

$$k_n' = \frac{(N-1)(-Ne+bN+Neq-bq)\,q}{(q-1)(-aq^2+N^2dq+aN^2-N^2d)}k_n \tag{3.5}$$

$$\begin{aligned} k_n'' = {} & k_n q^2 \Big((N-1)(N-q)\Big(b^2N^2 - 2N^2e^2q + 2bN^2qe - caN^2 + cN^2d \\ & +cqaN^2 + N^2e^2q^2 + N^2e^2 + cq^2N^2d - 2cqN^2d \\ & -2bN^2e - Nb^2q - b^2Nq^2 + Nbqe - Neq^3b - cq^3a + b^2q^3 + caq^2\Big)\Big) \Big/ \\ & \Big((q+1)(q-1)^2\left(-aq^3+N^2dq+aN^2-N^2d\right)\left(-aq^2+N^2dq+aN^2-N^2d\right)\Big). \end{aligned} \tag{3.6}$$

Here and throughout this chapter[1], we will use the notations $N = q^n$ and $M = q^m$.

Now we determine the coefficients of the structure relations. Koepf and Schmersau [2001], Koepf and Schmersau [2002], Medem et al. [2001], Area et al. [2006], Koekoek et al. [2010] showed that any solution of (3.2) satisfies a recurrence equation

$$p_{n+1}(x) = (A_nx + B_n)p_n(x) - C_np_{n-1}(x) \qquad (n \in \mathbb{N}_0,\ P_{-1} \equiv 0)$$

or equivalently

$$xp_n(x) = a_np_{n+1} + b_np_n(x) + c_np_{n-1}(x) \tag{3.7}$$

with

$$a_n = \frac{1}{A_n} \quad b_n = -\frac{B_n}{A_n}, \quad c_n = \frac{C_n}{A_n}.$$

Theorem 3.1 (See e.g. [Koepf and Schmersau, 2001], [Koepf and Schmersau, 2002], [Medem et al., 2001], [Area et al., 2006], [Koekoek et al., 2010], [Foupouagnigni et al., 2012])**.** *For any q-classical orthogonal polynomial family, the relation (3.7) is valid. The coefficients a_n, b_n and c_n are given by*

$$\begin{aligned} a_n = {} & \frac{k_n}{k_{n+1}}, \\ b_n = {} & -N\Big(-Neq^3a - bNaq^2 + N^2eq^2d - bq^2Nd + eaq^2 + baq^2 + bq^2N^2d + N^2eqa - 2N^2edq \\ & +Neqa - 2bNqa - eqa + bqaN^2 + bqa - N^2ea - bN^2d + bNd - bNa + N^2ed + baN^2\Big) \Big/ \\ & \Big(\left(-aq^2+N^2dq+aN^2-N^2d\right)\left(-a+N^2dq+aN^2-N^2d\right)\Big), \end{aligned}$$

$$\begin{aligned} c_n = {} & -\frac{k_n}{k_{n-1}}\Big(Nq(N-1)\left(Ndq+Na-Nd-aq^2\right)\Big(-N^3ebdq + cq^4a^2 - 2cN^2daq^3 + 2bN^2eaq^3 \\ & +2caN^4dq - b^2Naq^3 + b^2N^3dq - 2N^2e^2aq^3 - b^2N^3q^2d + N^2e^2q^4a - 2caN^4d - 2cN^4d^2q \\ & +cq^2N^4d^2 - b^2q^2N^2d - N^3b^2qa + 2b^2N^2q^2a + N^2e^2q^2a - 2cq^2a^2N^2 + b^2q^3N^2d + cN^4d^2 \\ & +ca^2N^4 + 2cq^2aN^2d - N^3eq^3bd - N^3eq^2ba + 2N^3eq^2bd + bN^3eaq - 2bN^2q^2ea - Neq^4ab \\ & +Nebaq^3\Big)\Big) \Big/ \Big(\left(-aq^2+N^2dq+aN^2-N^2d\right)^2\left(-aq^3+N^2dq+aN^2-N^2d\right) \\ & \times\left(-aq+N^2dq+aN^2-N^2d\right)\Big). \end{aligned}$$

[1]The use of $N = q^n$ should not be confused with the parameter N of the q-Hahn polynomials

Proof. Substitute $p_n(x)$ in the proposed equation (3.7) and equate the three highest coefficients. This yields a_n, b_n, and c_n in terms of a, b, c, d, e, q, q^n, k_{n-1}, k_n, k_{n+1}, k'_{n-1}, k'_n, k'_{n+1}, k''_{n-1}, k''_n, k''_{n+1} by linear algebra.

Substituting the values of k'_{n-1}, k'_n, k'_{n+1}, k''_{n-1}, k''_n, and k''_{n+1} given by (3.5) and (3.6) yields the above formulas. □

Theorem 3.2 (See [Koepf and Schmersau, 2001], [Medem et al., 2001], [Foupouagnigni et al., 2012]). *Orthogonal polynomials of the q-Hahn class satisfy the structure formula*

$$\sigma(x)D_{\frac{1}{q}}p_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \tag{3.8}$$

where the coefficients α_n, β_n and γ_n are given by the explicit formulas

$$\begin{aligned}
\alpha_n &= a[n]_{1/q}\frac{k_n}{k_{n+1}},\\
\beta_n &= \Big(q(N-1)(aN+dNq-aq-Nd)(eNaq^2-bdN^2q+Nqbq\\
&\quad -bqa+bdN^2-baN^2+bNa-eNa)\Big)\Big/\\
&\quad \Big((-dN^2q+dN^2+aq^2-aN^2)(q-1)(dN^2q-a+aN^2-dN^2)\Big),\\
\gamma_n &= \Big(Nq(N-1)(aN+dNq-dN-aq)(aq^2-dNq+dN-aN)(2b^2N^2aq^2\\
&\quad +q^4ca^2+b^2q^3dN^2-b^2q^2dN^2+e^2N^2aq^2+cq^2N^4+cN^4a^2-2cq^3adN^2\\
&\quad +eNbaq^3-2eN^2baq^2-eq^4Nba-eq^2N^3ba+2eq^2N^3bd+2bq^3N^2ea-eq^3N^3bd\\
&\quad +eN^3baq+2cqaN^4d+e^2q^4N^2a-2e^2q^3N^2a+qb^2N^3d-q^3b^2Na-qb^2N^3a\\
&\quad -b^2q^2N^3d-2cqd^2N^4-2q^2N^2ca^2-2dN^4ca+cq^2d^2N^4+2q^2dcaN^2-eN^3bdq)\Big)\Big/\\
&\quad \Big((-aq+dN^2q+aN^2-dN^2)(-dN^2q+dN^2+aq^3-aN^2)\\
&\quad \times(-dN^2q+dN^2+aq^2-aN^2)^2(q-1)\Big)\frac{k_n}{k_{n-1}}.
\end{aligned}$$

Proposition 3.3 (See e.g. [Hahn, 1949], [Medem et al., 2001], [Foupouagnigni et al., 2012]). *If a function $y(x)$ is solution of (3.2), then $Y(x) = D_q y(x)$ satisfies*

$$\sigma_1(x)D_qD_{\frac{1}{q}}Y(x) + \tau_1(x)D_qY(x) + \lambda_{n,q,1}Y(x) = 0 \tag{3.9}$$

with $\sigma_1(x) = \sigma(x)$, $\tau_1(x) = D_q\sigma(x) + q\tau(qx)$ and $\lambda_{n,q,1} = q(D_q\tau(x) + \lambda_{n,q})$.

Proof. We recall the following identities

$$D_{\frac{1}{q}}D_q = qD_qD_{\frac{1}{q}}, \tag{3.10}$$

$$D_q(f(x)g(x)) = f(qx)D_qg(x) + g(x)D_qf(x). \tag{3.11}$$

Applying identity (3.11) with $g(x) = \sigma(x)$ and $f(x) = D_qD_{\frac{1}{q}}y(x)$, we obtain

$$D_q\left(\sigma(x)D_qD_{\frac{1}{q}}y(x)\right) = \sigma(x)D_q\left(D_qD_{\frac{1}{q}}y(x)\right) + D_q\sigma(x)\left(D_qD_{\frac{1}{q}}y\right)(qx).$$

Using

$$\left(D_qD_{\frac{1}{q}}y\right)(qx) = \frac{1}{q}D_qD_qy(x),$$

we obtain

$$D_q\left(\sigma(x)D_qD_{\frac{1}{q}}y(x)\right)=\sigma(x)D_q\left(D_qD_{\frac{1}{q}}y(x)\right)+\frac{1}{q}D_q\sigma(x)D_q\left(D_qy(x)\right).$$

Now, using identity (3.10) for the first term of the right hand side of the previous relation, we get

$$D_q\left(\sigma(x)D_qD_{\frac{1}{q}}y(x)\right)=\frac{1}{q}\sigma(x)D_qD_{\frac{1}{q}}\left(D_qy(x)\right)+\frac{1}{q}D_q\sigma(x)D_q\left(D_qy(x)\right).$$

Identity (3.11) with $g(x)=D_qy(x)$ and $f(x)=\tau(x)$ yields

$$D_q(\tau(x)D_qy(x))=\tau(qx)D_q\left(D_qy(x)\right)+D_q\tau(x)D_qy(x).$$

Finally,

$$\begin{aligned}(3.2)\Rightarrow\, & D_q\left(\sigma(x)D_qD_{\frac{1}{q}}y(x)+\tau(x)D_qy(x)+\lambda_{n,q}y(x)\right)=0\\ \Rightarrow\, & \frac{1}{q}\sigma(x)D_qD_{\frac{1}{q}}\left(D_qy(x)\right)+\frac{1}{q}D_q\sigma(x)D_q\left(D_qy(x)\right)\\ & +\tau(qx)D_q\left(D_qy(x)\right)+D_q\tau(x)D_qy(x)+\lambda_{n,q}D_qy(x)=0\\ \Rightarrow\, & \sigma(x)D_qD_{\frac{1}{q}}\left(D_qy(x)\right)+\left(D_q\sigma(x)+q\tau(qx)\right)D_q\left(D_qy(x)\right)\\ & +q\left(D_q\tau(x)+\lambda_{n,q}\right)D_qy(x)=0,\end{aligned}$$

which proves the assertion. □

A computation shows that

$$\sigma_1(x)=a'x^2+b'x+c',\quad \tau_1(x)=d'x+c',\quad \lambda_{n,q,1}=q(\lambda_{n,q}+d) \tag{3.12}$$

where

$$a'=a,\ b'=b,\ c'=c\quad d'=a(q+1)+dq^2,\ \ e'=b+eq. \tag{3.13}$$

From this, we deduce that the equation

$$xD_qp_n(x)=\alpha_n^\star D_qp_{n+1}(x)+\beta_n^\star D_qp_n(x)+\gamma_n^\star D_qp_{n-1}(x) \tag{3.14}$$

which is the recurrence equation for the family $D_qp_n(x)$, is valid, and from (3.13) it follows that

$$\alpha_n^\star=a_n(a,b,c,a(q+1)+dq^2,b+eq),\qquad \beta_n^\star=b_n(a,b,c,a(q+1)+dq^2,b+eq),$$

and

$$\gamma_n^\star=c_n(a,b,c,a(q+1)+dq^2,b+eq)$$

where $a_n(a,b,c,d,e)$, $b_n(a,b,c,d,e)$ and $c_n(a,b,c,d,e)$, are given in Theorem 3.1.

Theorem 3.4 (See e.g. [Area et al., 1999], [Koepf and Schmersau, 2001], [Medem et al., 2001]). *Assume $p_n(x)$ is a solution family of (3.2). Then a structure formula of the type*

$$p_n(x)=\hat{a}_nD_qp_{n+1}(x)+\hat{b}_nD_qp_n(x)+\hat{c}_nD_qp_{n-1}(x) \tag{3.15}$$

is valid for $p_n(x)$.

In order to obtain those coefficients $\hat{a}_n$, $\hat{b}_n$ and $\hat{c}_n$, we use the same algorithm described for the determination of a_n, b_n and c_n where (3.7) is substituted now by (3.15). This gives the following

Theorem 3.5 (See e.g. [Medem et al., 2001], [Area et al., 2006], [Foupouagnigni et al., 2012]). *Assume $p_n(x)$ is a solution family of (3.2), then the coefficients $\hat{a}_n$, $\hat{b}_n$, and $\hat{c}_n$ of (3.15) are given by:*

$$
\begin{aligned}
\hat{a}_n &= \frac{1}{[n+1]_q}\frac{k_n}{k_{n+1}},\\
\hat{b}_n &= \Big(N(q-1)(-bdNq^2+eaq^2+edN^2q^2+eN^2qa+bdN^2q-2\,edN^2q\\
&\quad -bNqa+bqa-eaq-eaN^2-bdN^2+baN^2+bNd+edN^2-bNa)\Big)\Big/\\
&\quad \Big(\left(-a+dN^2q-dN^2+aN^2\right)\left(-aq^2+dN^2q-dN^2+aN^2\right)\Big),\\
\hat{c}_n &= \frac{k_n}{k_{n-1}}\Big(\left(q-1\right)\left(2\,dN^2q^2ac+ebNq^3a-ebNq^4a+2\,dN^4qac+ebN^3qa-ebN^3q^2a\right.\\
&\quad +2\,bedN^3q^2-bedN^3q^3-bedN^3q-2\,ebN^2q^2a-2\,dN^2q^3ac+2\,ebN^2q^3a\\
&\quad +cq^4a^2+N^4a^2c+N^4cd^2-b^2dN^2q^2+b^2dN^2q^3+d^2N^4q^2c-2\,d^2N^4qc\\
&\quad -b^2N^3qa-2\,N^4acd-b^2dN^3q^2+b^2dN^3q-2\,a^2N^2q^2c-2\,e^2N^2q^3a+2\,b^2N^2q^2a\\
&\quad \left.+e^2q^4N^2a+e^2N^2q^2a-b^2Nq^3a\right)\left(a+dq-d\right)N^2q\left(N-1\right)\Big)\Big/\\
&\quad \Big(\left(aN^2+dN^2q-dN^2-aq^3\right)\left(-aq^2+dN^2q-dN^2+aN^2\right)^2\left(aN^2+dN^2q-dN^2-aq\right)\Big).
\end{aligned}
$$

Note that, applying the operator D_q to the equation (3.7), we obtain the following

$$D_q(xp_n(x)) = a_nD_qp_{n+1}(x)+b_nD_qp_n(x)+c_nD_qp_{n-1}(x). \tag{3.16}$$

Using (3.11) for the left hand side of (3.16), we get

$$qxD_qp_n(x) = a_nD_qp_{n+1}(x)+b_nD_qp_n(x)+c_nD_qp_{n-1}(x)-p_n(x).$$

Now, we use the structure relation (3.15) to get the following

Proposition 3.6 (See e.g. [Medem et al., 2001], [Foupouagnigni et al., 2012]). *The coefficients $\alpha_n^\star$, $\beta_n^\star$ and $\gamma_n^\star$ of the relation (3.14) are linked to the coefficients a_n, b_n and c_n of the three-term recurrence relation (3.7) and the coefficients $\hat{a}_n$, $\hat{b}_n$, and $\hat{c}_n$ of (3.15) by the following formulas:*

$$\alpha_n^\star = \frac{a_n-\hat{a}_n}{q}, \qquad \beta_n^\star = \frac{b_n-\hat{b}_n}{q}, \qquad \gamma_n^\star = \frac{c_n-\hat{c}_n}{q},$$

and are given explicitly by:

$$
\begin{aligned}
\alpha_n^\star &= \frac{1-N}{1-qN}\frac{k_n}{k_{n+1}},\\
\beta_n^\star &= -N\Big(-eNq^2a+2\,bdN^2q-2\,edN^2q-2\,bNqa-eaN^2+eNa+2\,baN^2-2\,bNa\\
&\quad +edN^2-eaq+2\,bqa+eaq^2-2\,bdN^2-bdNq^2+edN^2q^2+eN^2qa+bNd\Big)\Big/\\
&\quad \left(-a+dN^2q-dN^2+aN^2\right)\left(-aq^2+dN^2q-dN^2+aN^2\right),
\end{aligned}
$$

$$\gamma_n^{\star} = -\frac{k_n}{k_{n-1}}\Big(2\,dN^2q^2ac + ebNq^3a - ebNq^4a + 2\,dN^4qac + ebN^3qa - ebN^3q^2a + 2\,bedN^3q^2$$
$$-bedN^3q^3 - 2\,dN^2q^3ac + 2\,ebN^2q^3a + cq^4a^2 + N^4a^2c + N^4cd^2 - b^2dN^2q^2 + b^2dN^2q^3$$
$$-bedN^3q + d^2N^4q^2c - 2\,d^2N^4qc - b^2N^3qa - 2\,N^4acd - b^2dN^3q^2 + b^2dN^3q - 2\,a^2N^2q^2c$$
$$-2\,e^2N^2q^3a + 2\,b^2N^2q^2a + e^2q^4N^2a + e^2N^2q^2a - b^2Nq^3a - 2\,ebN^2q^2a\Big)$$
$$\times Nq\,(N-1)\,(-dN - aq + aN + dNq)\Big/$$
$$\left(aN^2 + dN^2q - dN^2 - aq^3\right)\left(-aq^2 + dN^2q - dN^2 + aN^2\right)^2\left(aN^2 + dN^2q - dN^2 - aq\right).$$

Proposition 3.7 (See [Foupouagnigni et al., 2012]). *Assume a family $p_n(x)$ is a solution of (3.2). Then a structure formula of the type*

$$(\sigma(x) + (q-1)x\tau(x))D_q p_n(x) = S_n p_{n+1}(x) + T_n p_n(x) + R_n p_{n-1}(x), \tag{3.17}$$

is valid for $p_n(x)$ where

$$S_n = \alpha_n + (1-q)a_n\lambda_n,\;\; T_n = \beta_n + (1-q)b_n\lambda_n,\;\; R_n = \gamma_n + (1-q)c_n\lambda_n,$$

and are given explicitly by

$$S_n = \frac{(a + dq - d)\,(q^n - 1)}{q-1}\frac{k_n}{k_{n+1}},$$
$$T_n = \Big(-\,(eq^2N^2d - q^2Nbd + eaq^2 - 2\,eN^2qd - aqNb + N^2qbd + eaN^2q - eqa$$
$$+bqa - bdN^2 - bNa - eN^2a + eN^2d + bdN + baN^2)\,(N-1)\,(aN + qNd - dN - aq)\Big)\Big/$$
$$\Big((q-1)\left(-a + N^2qd + aN^2 - dN^2\right)\left(-aq^2 + N^2qd + aN^2 - dN^2\right)\Big),$$
$$R_n = \Big((-2\,caN^4d - 2\,N^4qd^2c - 2\,e^2aN^2q^3 + 2\,aq^2N^2b^2 + e^2aq^2N^2 + N^2q^3b^2d - 2\,a^2q^2N^2c$$
$$-aN^3qb^2 + N^3qb^2d + q^2N^4d^2c - N^3q^2b^2d + e^2aN^2q^4 - q^2N^2b^2d - aNq^3b^2 + ca^2N^4$$
$$+cd^2N^4 + cq^4a^2 + eaNq^3b - eN^3qbd + 2\,eN^3q^2bd + 2\,aN^4qdc - 2\,eaq^2N^2b$$
$$-2\,aN^2q^3dc + 2\,eaN^2q^3b + 2\,aq^2N^2dc - eN^3q^3bd - eaN^3q^2b + eaN^3qb - eaNq^4b)$$
$$(N-1)\left(aN - aq^2 - dN + qNd\right)(aN + qNd - dN - aq)\,q\Big)\Big/$$
$$\Big((q-1)\left(-aq + N^2qd + aN^2 - dN^2\right)\left(-aq^3 + N^2qd + aN^2 - dN^2\right)$$
$$\left(-aq^2 + N^2qd + aN^2 - dN^2\right)^2\Big) \times \frac{k_n}{k_{n-1}}.$$

Proof. The q-differential operator obeys the identity

$$D_q D_{1/q} f(x) = \frac{D_q f(x) - D_{1/q} f(x)}{(q-1)x},$$

so that the q-differential equation (3.2) can be rewritten in the form

$$(\sigma(x) + (q-1)x\tau(x))D_q P_n(x)(x) - \sigma(x)D_{1/q}P_n(x) + (q-1)\lambda_n x P_n(x) = 0.$$

Next, we use the three-term recurrence relation (3.7) to get rid of the $xp_n(x)$ term and the structure formula (3.8) to get rid of the $\sigma(x)D_{1/q}p_n(x)$ term. The structure formula (3.17) is obtained by simplification. □

We will also need the following structural formula.

Proposition 3.8. *If a family $p_n(x)$ is a solution of (3.2), then a structure formula of the type [Koepf and Schmersau, 2001]*

$$\sigma(x)D_qD_{\frac{1}{q}}p_n(x) = a'_nD_qp_{n+1}(x) + b'_nD_qp_n(x) + c'_nD_qp_{n-1}(x), \tag{3.18}$$

holds for $p_n(x)$ where

$$a'_n = -d\alpha^\star_n - \lambda_n\hat{a}_n,\ b'_n = -d\beta^\star_n - e - \hat{b}_n\lambda_n,\ c'_n = -\gamma^\star_n - \hat{c}_n\lambda_n,$$

and are given explicitly by

$$\begin{aligned}
a'_n &= \frac{a(N-1)(N-q)k_n}{N(1-q)(1-qN)k_{1+n}},\\
b'_n &= \Big((N-q)(Ndq + Na - Nd - a(baN^2 + bqN^2d - bN^2d - bNa - q^2Nae + Nea\\
&\quad -bNqa + bqa)\Big)\Big/\Big((-a + N^2dq + aN^2 - N^2d)(-aq^2 + N^2dq + aN^2 - N^2d)(q-1)\Big),\\
c'_n &= \frac{k_n}{k_{n-1}}\Big(Nq(N-1)(Ndq + Na - Nd - a)(Na - aq + Ndq - Nd)\Big(-N^3ebdq + cq^4a^2\\
&\quad +2\,bN^2eaq^3 + 2\,caN^4dq - b^2Naq^3 + b^2N^3dq - 2\,N^2e^2aq^3 - b^2N^3q^2d + N^2e^2q^4a - 2\,caN^4d\\
&\quad -2\,cN^4d^2q + cq^2N^4d^2 - b^2q^2N^2d - N^3b^2qa + 2\,b^2N^2q^2a + N^2e^2q^2a - 2\,cq^2a^2N^2 + b^2q^3N^2d\\
&\quad +cN^4d^2 + ca^2N^4 + 2\,cq^2aN^2d - N^3eq^3bd - N^3eq^2ba + 2\,N^3eq^2bd + bN^3eaq - 2\,bN^2q^2ea\\
&\quad -Neq^4ab + Nebaq^3 - 2\,cN^2daq^3\Big)\Big)\Big/\Big((-aq^2 + N^2dq + aN^2 - N^2d)^2\\
&\quad (-aq + N^2dq + aN^2 - N^2d)(-aq^3 + N^2dq + aN^2 - N^2d)(q-1)\Big).
\end{aligned}$$

Proof. From (3.2), we have

$$\sigma(x)D_qD_{\frac{1}{q}}p_n(x) = -(dx+e)D_qp_n(x) - \lambda_np_n(x) = -dxD_qp_n(x) - eD_qp_n(x) - \lambda_np_n(x).$$

Using structure relations (3.14) and (3.15), the previous equation gives (3.18) with

$$a'_n = -d\alpha^\star_n - \lambda_n\hat{a}_n,\ b'_n = -d\beta^\star_n - e - \hat{b}_n\lambda_n,\ c'_n = -\gamma^\star_n - \hat{c}_n\lambda_n.$$

The result follows from the representations of the coefficients given in Theorem 3.5 and Proposition 3.6. ☐

3.3 Inversion Problem of q-COP

Every q-classical orthogonal polynomial family $(p_n(x))_{n\in\mathbb{N}_0}$ is represented as series in one of the four bases $\mathcal{V}_m(x) = (x;q)_m$, $\mathcal{V}_m(x) = x^m$, $\mathcal{V}_m(x) = (ix;q)_m$ and $\mathcal{V}_m(x) = (x-1)(x-q)\ \cdots\ (x-q^{m-1}) = (x^{-1};q)_mx^m$, i.e.

$$p_n(x) = \sum_{m=0}^{n} A_m(n)\mathcal{V}_m(x).$$

In this section, we solve the inverse problem, i.e., the problem of determining the coefficients $I_m(n)$ of the expansion

$$\mathcal{V}_n(x) = \sum_{m=0}^{n} I_m(n)p_m(x). \tag{3.19}$$

3.3.1 The case $\mathcal{V}_n(x) = x^n$

Since $D_q x^n = \frac{[n]_q}{x} x^n$, the polynomial $\mathcal{V}_n(x) = x^n$ is solution of the q-differential equation

$$xD_q x^n - [n]_q x^n = 0.$$

Using this q-differential equation, we prove that

Theorem 3.9. *The coefficients $I_m(n)$ of the inversion problem*

$$x^n = \sum_{m=0}^{n} I_m(n) p_m(x)$$

are solution of the recurrence equation

$$(\alpha^{\star}_{m-1} - [n]_q \hat{a}_{m-1}) I_{m-1}(n) + (\beta^{\star}_m - [n]_q \hat{b}_m) I_m(n) + (\gamma^{\star}_{m+1} - [n]_q \hat{c}_{m+1}) I_{m+1}(n) = 0,$$

with initial values $I_n(n) = \frac{1}{k_n}$ and $I_{n+1}(n) = 0$, where the coefficients are given in Theorem 3.5 and Proposition 3.6.

Proof. Substituting $x^n = \sum\limits_{m=0}^{n} I_m(n) p_m(x)$ in the q-differential equation $xD_q x^n - [n]_q x^n = 0$ gives

$$\sum_{m=0}^{n} I_m(n) x D_q p_m(x) - \sum_{m=0}^{n} [n]_q I_m(n) p_m(x) = 0.$$

Then we use the structure formulas (3.14) and (3.15) to substitute $xD_q p_m(x)$ and $p_m(x)$. The result follows after a shift of indices. □

We note that here and in some theorems below, we didn't write out the recurrence relations containing a, b, c, d and e because they are too large.

3.3.2 The cases $\mathcal{V}_n(x) = (x; q)_n$ and $\mathcal{V}_n(x) = (ix; q)_n$

Theorem 3.10. *The coefficients $I_m(n)$ of the inversion problem*

$$(x; q)_n = \sum_{m=0}^{n} I_m(n) p_n(x)$$

are solution of the recurrence equation

$$(\alpha^{*}_{m-1} - [n]_q \hat{a}_{m-1}) I_{m-1}(n) + (\beta^{*}_m - [n]_q \hat{b}_m - 1) I_m(n) + (\gamma^{*}_{m+1} - [n]_q \hat{c}_{m+1}) I_{m+1}(n) = 0$$

with initial values $I_n(n) = \frac{(-1)^n q^{\binom{n}{2}}}{k_n}$ and $I_{n+1}(n) = 0$; and the coefficients $I_m(n)$ of the inversion problem

$$(ix; q)_n = \sum_{m=0}^{n} I_m(n) p_n(x)$$

are solution of the recurrence equation

$$i(\alpha^{*}_{m-1} - [n]_q \hat{a}_{m-1}) I_{m-1}(n) + (i\beta^{*}_m - i[n]_q \hat{b}_m - 1) I_m(n) + i(\gamma^{*}_{m+1} - [n]_q \hat{c}_{m+1}) I_{m+1}(n) = 0$$

with initial values $I_n(n) = \frac{(-i)^n q^{\binom{n}{2}}}{k_n}$ and $I_{n+1}(n) = 0$, where the coefficients are given in Theorem 3.5 and Proposition 3.6.

Proof. We substitute $(x;q)_n = \sum\limits_{m=0}^{n} I_m(n)p_n(x)$ and $(ix;q)_n = \sum\limits_{m=0}^{n} I_m(n)p_n(x)$, respectively, in the q-differential equations $(x-1)D_q(x;q)_n - [n]_q(x;q)_n = 0$ (since $D_q(x;q)_n = \frac{[n]_q}{x-1}(x;q)_n$) and $(ix-1)D_q(ix;q)_n - i[n]_q(x;q)_n = 0$ (since $D_q(ix;q)_n = \frac{i[n]_q}{ix-1}(ix;q)_n$). This gives, respectively,

$$\sum_{m=0}^{n} I_m(n)(xD_q p_m(x) - D_q p_m(x)) - \sum_{m=0}^{n}[n]_q I_m(n)p_m(x) = 0,$$

and

$$\sum_{m=0}^{n} I_m(n)(ixD_q p_m(x) - D_q p_m(x)) - \sum_{m=0}^{n} i[n]_q I_m(n)p_m(x) = 0.$$

Then we use the structure formulas (3.14) and (3.15) to substitute $xD_q p_m(x)$ and $p_m(x)$. After an index shift, we obtain the results. The initial values follow from the expansions

$$(x;q)_n = (-1)^n q^{\binom{n}{2}} x^n + \ldots, \quad (ix;q)_n = (-i)^n q^{\binom{n}{2}} x^n + \ldots \tag{3.20}$$

by equating the coefficients of x^n on both sides of the inversion formula. □

3.3.3 The case $\mathcal{V}_n(x) = (x \ominus 1)_q^n$

Theorem 3.11. *The coefficients $I_m(n)$ of the inversion problem*

$$(x \ominus 1)_q^n = \sum_{m=0}^{n} I_m(n)p_n(x)$$

are solution of the recurrence equation

$$(\alpha^*_{m-1} - [n]_q\hat{a}_{m-1})I_{m-1}(n) + (\beta^*_m - [n]_q\hat{b}_m - q^{n-1})I_m(n) + (\gamma^*_{m+1} - [n]_q\hat{c}_{m+1})I_{m+1}(n) = 0,$$

with initial values $I_n(n) = \frac{1}{k_n}$ and $I_{n+1}(n) = 0$, with the coefficients given in Theorem 3.5 and Proposition 3.6.

Proof. We substitute $(x \ominus 1)_q^n = \sum\limits_{m=0}^{n} I_m(n)p_n(x)$ in the q-differential equation $(x - q^{n-1})D_q(x \ominus 1)_q^n - [n]_q(x \ominus 1)_q^n = 0$ (since $D_q(x \ominus 1)_q^n = \frac{[n]_q}{x-q^{n-1}}(x \ominus 1)_q^n$). This yields

$$\sum_{m=0}^{n} I_m(n)(xD_q p_m(x) - q^{n-1}D_q p_m(x)) - \sum_{m=0}^{n}[n]_q I_m(n)p_m(x) = 0.$$

Then we use the structure formulas (3.14) and (3.15) to substitute $xD_q p_m(x)$ and $p_m(x)$. After an index shift, we obtain the result. The initial value derives from $(x \ominus 1)_q^n = x^n + \ldots$ by equating the coefficients of x^n in the inversion formula. □

The above computations show that in the generic case, the inversion coefficients are solutions of a q-holonomic recurrence equation of order 2.

In order to find solutions which are q-hypergeometric terms—hence satisfying a first-order q-holonomic recurrence—in some specific situations, we can use a q-version of the Petkovšek-van-Hoeij algorithm, referred in the sequel as the q-Petkovšek-van-Hoeij algorithm. All our results are derived using this algorithm.

In particular, we solve the recurrence equations of Theorems 3.9–3.11 for all particular systems and therefore obtain up to a multiplicative constant K_n the q-hypergeometric term representations of the inversion coefficients of q-COP. The constant K_n follows from the initial value $I_n(n)$. This method yields

Corollary 3.12 (Compare [Area et al., 1999], [Foupouagnigni et al., 2012]). *The inversion coefficients of the polynomial systems of the q-Hahn class are given in Table 3.2 of the next page, where* $\begin{bmatrix} n \\ m \end{bmatrix}_q$ *denotes the q-binomial coefficient, defined by*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, \quad m = 0, 1, 2, \ldots, n, \quad n \in \mathbb{N}_0.$$

Remark 3.13. 1. *The inversion coefficients of Table 3.2 can be obtained from [Area et al., 1999, Table 3], [Foupouagnigni et al., 2012, Table 2] by multiplying their inversion coefficients by $\frac{1}{k_m}$ since their works dealt with monic polynomial families.*

2. *The inversion coefficients given in Table 3.2 are already known, except for the q-Krawtchouk, Affine q-Krawtchouk, quantum q-Krawtchouk and Al-Salam-Carlitz I polynomials, which are new, as far as we know.*

3.4 Connection Problem of q-COP

Here, we assume that $(p_n(x) = k_n x^n + \ldots)_{n\geq 0}$ denotes a family of q-COP of degree exactly n and $(q_m(x) = \bar{k}_m x^m + \ldots)_{m\geq 0}$ denotes a family of q-COP of degree exactly m. We want to determine the connection coefficients $C_m(n), (n \in \mathbb{N}, m = 0, \ldots, n)$, between the systems $p_n(x)$ and $q_m(x)$,

$$p_n(x) = \sum_{m=0}^{n} C_m(n) q_m(x). \tag{3.21}$$

We assume that $C_m(n) = 0$ outside the above $n \times m$ region. We will denote all coefficients connected with $q_m(x)$ by dashes. If $\sigma(x) \neq \bar{\sigma}(x)$, we use in general the inversion formula to solve the connection problem and if $\sigma(x) = \bar{\sigma}(x)$ structural formulae of Section 3.2 are used.

3.4.1 Connection Coefficients for $\sigma(x) = \bar{\sigma}(x)$

Whereas in [Foupouagnigni et al., 2012] we proceeded as in [Koepf and Schmersau, 1998] to solve the connection problem of q-COP, here we show the same results using the NaViMa algorithm.

We substitute (3.21) in (3.2) to get

$$\sum_{m=0}^{n} C_m(n) \Big(\sigma(x) D_q D_{\frac{1}{q}} q_m(x) + \tau(x) D_q q_m(x) + \lambda_n q_m(x) = 0 \Big). \tag{3.22}$$

First, Equation (3.15) for $q_m(x)$ can be rewritten as

$$q_m(x) = \sum_{j=m-1}^{m+1} a_{m,j} D_q q_m(x) \tag{3.23}$$

Family	Basis	$I_m(n)$
big q-Jacobi	$\{(x;q)_n\}_n$	$(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m-1)}{2}} \frac{(\alpha q, \gamma q;q)_n (1-\alpha\beta q^{2m+1})}{(\alpha\beta q^{m+1};q)_n (1-\alpha\beta q^{n+m+1})}$
q-Hahn	$\{(x;q)_n\}_n$	$(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m-1)}{2}} \frac{(\alpha q, q^{-N};q)_n (1-\alpha\beta q^{2m+1})}{(\alpha\beta q^{m+1};q)_n (1-\alpha\beta q^{n+m+1})}$
big q-Laguerre	$\{(x;q)_n\}_n$	$(-1)^m q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q (\alpha q, \beta q;q)_n$
q-Meixner	$\{(x;q)_n\}_n$	$(-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{1}{2}(m+1)(m-2n)} \gamma^n (\beta q;q)_n$
q-Charlier	$\{(x;q)_n\}_n$	$(-1)^{n-m} \alpha^n \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{1}{2}(m+1)(m-2n)}$
Al-Salam-Carlitz II	$\{(x;q)_n\}_n$	$(-1)^n \alpha^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{m(m-n)+\frac{n(n-1)}{2}}$
discrete q-Hermite II	$\{(ix;q)_n\}_n$	$(-i)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{m(m-n)+\frac{n(n-1)}{2}}$
affine q-Krawtchouk	$\{(x;q)_n\}$	$(-1)^m q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q (q^{-N};q)_n (pq;q)_n$
q-Krawtchouk	$\{(x;q)_n\}$	$(-1)^m q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(q^{-N};q)_n}{(-pq^m;q)_m (-pq^{2m+1};q)_{n-m}}$
quantum q-Krawtchouk	$\{(x;q)_n\}$	$(-1)^m q^{\frac{1}{2}(m+1)(m-2n)} p^{-n} \begin{bmatrix} n \\ m \end{bmatrix}_q (q^{-N};q)_n$
little q-Jacobi	$\{x^n\}_n$	$(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m-1)}{2}} \frac{(\alpha q;q)_n (1-\alpha\beta q^{2m+1})}{(\alpha\beta q^{m+1};q)_n (1-\alpha\beta q^{n+m+1})}$
alternative q-Charlier	$\{x^n\}_n$	$(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{q^{\frac{m(m-1)}{2}} (1+\alpha q^{2m})}{(-\alpha q^{m+1};q)_n (1+\alpha q^m)}$
little q-Laguerre Wall	$\{x^n\}_n$	$(-1)^m q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q (\alpha q;q)_n$
q-Laguerre	$\{x^n\}_n$	$(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{q^{\frac{1}{2}(m-n)(m+n+1)}}{q^{n(m+\alpha)} (1-q)^{n-m}} (q^{\alpha+m+1};q)_{n-m} (q;q)_m$
Stieltjes-Wigert	$\{x^n\}_n$	$(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{1}{2}(m-n)(m+n+1)-mn} (q;q)_m$
discrete q-Hermite I	$\{(x \ominus 1)_q^n\}$	$(-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q$
Al-Salam-Carlitz I	$\{(x \ominus 1)_q^n\}$	$\alpha^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q$

Table 3.2: Inversion coefficients for q-COP

with $a_{m,m+1} = \bar{\hat{a}}_m$, $a_{m,m} = \bar{\hat{b}}_m$ and $a_{m,m-1} = \bar{\hat{c}}_m$.

Second, the three-term recurrence equation (3.14) for the family $D_q q_m(x)$ yields

$$\tau(x)D_q q_m(x) = \sum_{j=m-1}^{m+1} a^{(1)}_{m,j} D_q q_m(x) \tag{3.24}$$

with $a^{(1)}_{m,m+1} = d\bar{\alpha}^{\star}_m$, $a^{(1)}_{m,m} = d\bar{\beta}^{\star}_m + e$, $a^{(1)}_{m,m-1} = d\bar{\gamma}^{\star}_m$.

Third, since $\sigma(x) = \bar{\sigma}(x)$, from (3.18) we have

$$\sigma(x)D_q D_{\frac{1}{q}} q_m(x) = \bar{\sigma}(x)D_q D_{\frac{1}{q}} q_m(x) = \sum_{j=m-1}^{m+1} a^{(2)}_{m,j} D_q q_m(x) \tag{3.25}$$

with $a^{(2)}_{m,m+1} = \bar{a'}_m$, $a^{(2)}_{m,m} = \bar{b'}_m$ and $a^{(2)}_{m,m-1} = \bar{c'}_m$.

Insertion of (3.23)-(3.25) into (3.22) gives

$$\sum_{m=0}^{n} C_m(n)\Big\{ \sum_{j=m-1}^{m+1} \Lambda_{m,j}(n) D_q q_j(x) \Big\} = 0,\ \ \Lambda_{m,j}(n) = a^{(2)}_{m,j} + a^{(1)}_{m,j} + \lambda_n a_{m,j}.$$

Finally, after an appropriate shift of indices, this latter expression provides a recurrence relation of maximum order two which can be written as

$$\sum_{s=-1}^{1} \Lambda_{m+s,m}(n) C_{m+s}(n) = 0,\ \ 1 \le m \le n.$$

Theorem 3.14. *Let $p_n(x)$ be a polynomial system given by the q-differential equation (3.2) with $\sigma(x) = ax^2 + bx + c$, and $\tau(x) = dx + e$ and $q_m(x)$ be a polynomial system given by (3.2) with $\bar{\sigma}(x) = \sigma(x)$, and $\tau(x) = \bar{d}x + \bar{e}$. Then the relation (3.21) is valid, $C_m(n)$ satisfying the second order recurrence equation with respect to m*

$$\begin{aligned}&(\bar{a'}_{m-1} + d\bar{\alpha}^{\star}_{m-1} + \lambda_n \bar{\hat{a}}_{m-1})C_{m-1}(n) + (\bar{b'}_m + d\bar{\beta}^{\star}_m + e + \lambda_n \bar{\hat{b}}_m)C_m(n)\\ &+(\bar{c'}_{m+1} + d\bar{\gamma}^{\star}_{m+1} + \lambda_n \bar{\hat{c}}_{m+1})C_{m+1}(n) = 0,\end{aligned}$$

with initial conditions given by $C_{n+1}(n) = 0$ and $C_n(n) = k_n/\bar{k}_n$, where the coefficients are given in Theorem 3.5, Propositions 3.6 and 3.8.

To solve this recurrence equation, we use the q-Petkovšek-van-Hoeij algorithm. With the aid of this algorithm, we obtain the following connection formulae.

Corollary 3.15. *The following connection relations between the orthogonal polynomial systems of the q-Hahn class are valid:*

1. big q-Jacobi

$$P_n(x;\alpha,\beta,\gamma;q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha\beta q^{n+1})^{n-m}(\alpha\beta q^{n+1};q)_m(\frac{\beta_1 q^{m+1}}{\beta q^n};q)_{n-m}}{(\alpha\beta_1 q^{2(m+1)};q)_{n-m}(\alpha\beta_1 q^{m+1};q)_m} P_m(x;\alpha,\beta_1,\gamma;q);$$

2. q-Hahn

$$Q_n(\bar{x};\alpha,\beta,N|q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha\beta q^{(n+1)})^{n-m}(\alpha\beta q^{n+1};q)_m(\frac{\beta_1 q^{m+1}}{\beta q^n};q)_{n-m}}{(\alpha\beta_1 q^{2(m+1)};q)_{n-m}(\alpha\beta_1 q^{m+1};q)_m} Q_m(\bar{x};\alpha,\beta_1,N|q);$$

3. little q-Jacobi

$$p_n(x;\alpha,\beta|q)=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_q(-\alpha q^{\frac{n-m+1}{2}})^{n-m}\frac{(\alpha_1 q,\alpha\beta q^{n+1};q)_m(\beta q^{m+1},\frac{\alpha_1 q^{m+1}}{\alpha q^n};q)_{n-m}}{(\alpha_1\beta q^{2(m+1)};q)_{n-m}(\alpha_1\beta q^{m+1};q)_m(\alpha q;q)_n}p_m(x;\alpha_1,\beta|q);$$

$$p_n(x;\alpha,\beta|q)=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_q(\alpha\beta q^{n+1})^{n-m}\frac{(\alpha\beta q^{n+1};q)_m(\frac{\beta_1 q^{m+1}}{\beta q^n};q)_{n-m}}{(\alpha\beta_1 q^{2(m+1)};q)_{n-m}(\alpha\beta_1 q^{m+1};q)_m}p_m(x;\alpha,\beta_1|q);$$

4. alternative q-Charlier

$$y_n(x;\alpha;q)=\sum_{m=0}^{n}(-\alpha q^n)^{n-m}\begin{bmatrix}n\\m\end{bmatrix}_q\frac{(-\alpha q^n;q)_m(\frac{\alpha_1 q^{m+1}}{\alpha q^n};q)_{n-m}}{(-\alpha_1 q^{2m+1};q)_{n-m}(-\alpha_1 q^m;q)_m}y_m(x;\alpha_1;q);$$

5. little q-Laguerre

$$p_n(x;\alpha|q)=\sum_{m=0}^{n}(-\alpha q^{\frac{n-m+1}{2}})^{n-m}\begin{bmatrix}n\\m\end{bmatrix}_q\frac{(\alpha_1 q;q)_m(\frac{\alpha_1 q^{m+1}}{\alpha q^n};q)_{n-m}}{(\alpha q;q)_n}p_m(x;\alpha_1|q),$$

6. q-Krawtchouk

$$K_n(\bar{x};p,N;q)=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_q\frac{(-pq^n)^{n-m}(-pq^n;q)_m(\frac{p_1 q^{m+1}}{pq^n};q)_{n-m}}{(-p_1 q^m;q)_m(-p_1 q^{2m+1};q)_{n-m}}K_m(\bar{x};p_1,N;q),$$

7. quantum q-Krawtchouk

$$K_n^{qtm}(\bar{x};p,N;q)=\sum_{m=0}^{n}(-1)^{n-m}\left(\frac{p}{p_1}\right)^n q^{\frac{1}{2}(m-n)(m-n+1)}\begin{bmatrix}n\\m\end{bmatrix}_q\left(\frac{p_1 q^{m+1}}{pq^n};q\right)_{n-m}K_m^{qtm}(\bar{x};p_1,N;q),$$

8. q-Laguerre

$$L_n^{(\alpha)}(x;q)=\sum_{m=0}^{n}(-1)^{n-m}\frac{\left(\frac{q^{m+\beta+1}}{q^{n+\alpha}};q\right)_{n-m}q^{\frac{1}{2}m(m+1)+\frac{1}{2}n(n-1)}}{(q;q)_{n-m}q^{n(m+\beta-\alpha)}}L_m^{(\beta)}(x;q).$$

3.4.2 Connection Coefficients Using Inversion Formulas

This method is generally used when $\sigma(x)\neq\bar{\sigma}(x)$. If

$$p_n(x)=\sum_{j=0}^{n}A_j(n)\mathcal{V}_j(x)\text{ and }\mathcal{V}_j(x)=\sum_{m=0}^{j}I_m(j)q_m(x)$$

where $\mathcal{V}_j(x)=x^j$, $\mathcal{V}_j(x)=(x;q)_j$, $\mathcal{V}_j(x)=(ix;q)_j$ or $\mathcal{V}_j(x)=(x\ominus 1)_q^j$ then

$$p_n(x)=\sum_{j=0}^{n}A_j(n)\Big(\sum_{m=0}^{j}I_m(j)q_m(x)\Big),$$

and by rearranging the order of summation gives

$$p_n(x)=\sum_{m=0}^{n}C_m(n)q_m(x),\ \text{with } C_m(n)=\sum_{j=m}^{n}A_j(n)I_m(j)=\sum_{j=0}^{n-m}A_{j+m}(n)I_m(j+m).$$

q-Zeilberger's algorithm combined with the q-Petkovšek-van-Hoeij algorithm yields in the following cases the q-hypergeometric term representation of $C_m(n)$.

Theorem 3.16. *The following connection formulae of classical q-orthogonal polynomials are valid:*
1. big q-Jacobi

$$P_n(x;\alpha,\beta,\gamma;q)=\sum_{m=0}^{n}(\alpha_1 q)^{n-m}\begin{bmatrix} n\\ m\end{bmatrix}_q(1-\alpha_1\beta q^{2m+1}) \times\frac{\left(\alpha_1 q,\alpha\beta q^{n+1};q\right)_m(\beta q;q)_n\left(\frac{\alpha}{\alpha_1};q\right)_{n-m}}{(\beta q;q)_m(\alpha_1\beta q^{m+1};q)_{n+1}(\alpha q;q)_n}P_m(x;\alpha_1,\beta,\gamma;q),$$

2. q-Hahn

$$Q_n(\bar{x};\alpha,\beta,N;q)=\sum_{m=0}^{n}(\alpha_1 q)^{n-m}\begin{bmatrix} n\\ m\end{bmatrix}_q(1-\alpha_1\beta q^{2m+1}) \times\frac{\left(\alpha_1 q,\alpha\beta q^{n+1};q\right)_m(\beta q;q)_n\left(\frac{\alpha}{\alpha_1};q\right)_{n-m}}{(\beta q;q)_m(\alpha_1\beta q^{m+1};q)_{n+1}(\alpha q;q)_n}Q_m(\bar{x};\alpha_1,\beta,N;q),$$

3. affine q-Krawtchouk

$$K_n^{aff}(\bar{x};p,N;q)=\sum_{m=0}^{n}(p_1 q)^{n-m}\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{(p_1 q;q)_m\left(\frac{p}{p_1};q\right)_{n-m}}{(pq;q)_n}K_m^{Aff}(\bar{x};p_1,N;q),$$

$$K_n^{aff}(\bar{x};p,N;q)=\sum_{m=0}^{n}\left(-q^{\frac{m+n-1}{2}}\right)^{m-n}\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{(q^{-M},q)_n(q^{M-N};q)_{n-m}}{(q^{-N},q)_n(q^{M+1-n},q)_{n-m}}K_m^{Aff}(\bar{x};p,M;q),$$

4. big q-Laguerre

$$P_n(x;\alpha,\beta;q)=\sum_{m=0}^{n}(\alpha_1 q)^{n-m}\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{(\alpha_1 q;q)_m(\frac{\alpha}{\alpha_1};q)_{n-m}}{(\alpha q;q)_n}P_m(x;\alpha_1,\beta;q),$$

$$P_n(x;\alpha,\beta;q)=\sum_{m=0}^{n}(\beta_1 q)^{n-m}\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{(\beta_1 q;q)_m(\frac{\beta}{\beta_1};q)_{n-m}}{(\beta q;q)_n}P_m(x;\alpha,\beta_1;q),$$

5. Al-Salam-Carlitz I

$$U_n^{(\alpha)}(x;q)=\sum_{m=0}^{n}\alpha_1^{n-m}\begin{bmatrix} n\\ m\end{bmatrix}_q(\frac{\alpha}{\alpha_1};q)_{n-m}U_m^{(\alpha_1)}(x;q),$$

6. q-Meixner

$$M_n(x;\beta,\gamma;q)=\sum_{m=0}^{n}\left(\frac{\gamma_1}{\gamma}\right)^m\begin{bmatrix} n\\ m\end{bmatrix}_q\left(\frac{\gamma_1}{\gamma};q\right)_{n-m}M_m(x;\beta,\gamma_1;q),$$

7. q-Charlier

$$C_n(x;\alpha;q)=\sum_{m=0}^{n}\left(\frac{\alpha_1}{\alpha}\right)^m\begin{bmatrix} n\\ m\end{bmatrix}_q\left(\frac{\alpha_1}{\alpha};q\right)_{n-m}C_m(x;\alpha_1;q),$$

8. Al-Salam-Carlitz II

$$V_n^{(\alpha)}(x;q)=\sum_{m=0}^{n}(-\alpha)^{n-m}q^{\frac{1}{2}(m-n)(m+n-1)}\begin{bmatrix}n\\m\end{bmatrix}_q\left(\frac{\alpha_1}{\alpha};q\right)_{n-m}V_m^{(\alpha_1)}(x;q).$$

In those cases in which the connection coefficients are no q-hypergeometric terms, we use the sum2qhyper algorithm. This yields

Theorem 3.17. *For classical q-orthogonal polynomials, the following connection formulae hold:*
1. big q-Jacobi

$$\begin{aligned}P_n(x;\alpha,\beta,\gamma;q)=&\sum_{m=0}^{n}(-1)^m q^{\frac{1}{2}m(m+1)}\frac{(\alpha\beta q^{n+1};q)_m(\gamma_1 q;q)_m(q^{-n};q)_m}{(\alpha\beta q^{m+1};q)_m(\gamma q;q)_m(q;q)_m}\\&\times{}_3\phi_2\left(\left.\begin{matrix}q^{m-n},\gamma_1 q^{m+1},\alpha\beta q^{m+n+1}\\ \gamma q^{m+1},\alpha\beta q^{2m+2}\end{matrix}\right|q;q\right)P_m(x;\alpha,\beta,\gamma_1;q),\end{aligned}$$

$$\begin{aligned}P_n(x;\alpha,\beta,\gamma;q)=&\sum_{m=0}^{n}(-1)^m q^{\frac{1}{2}m(m+1)}\frac{(\alpha\beta q^{n+1};q)_m(\alpha_1 q;q)_m(\gamma_1 q;q)_m(q^{-n};q)_m}{(\alpha_1\beta_1 q^{m+1};q)_m(\alpha q;q)_m(\gamma q;q)_m(q;q)_m}\\&\times{}_4\phi_3\left(\left.\begin{matrix}q^{m-n},\alpha_1 q^{m+1},\gamma_1 q^{m+1},\alpha\beta q^{m+n+1}\\ \alpha q^{m+1},\gamma q^{m+1},\alpha_1\beta_1 q^{2m+2}\end{matrix}\right|q;q\right)P_m(x;\alpha_1,\beta_1,\gamma_1;q),\end{aligned}$$

2. q-Hahn

$$\begin{aligned}Q_n(x;\alpha,\beta,N|q)=&\sum_{m=0}^{n}(-1)^m q^{\frac{1}{2}m(m+1)}\frac{(\alpha\beta q^{n+1};q)_m(q^{-M};q)_m(q^{-n};q)_m}{(\alpha\beta q^{m+1};q)_m(q^{-N};q)_m(q;q)_m}\\&\times{}_3\phi_2\left(\left.\begin{matrix}q^{m-n},q^{-M+m},\alpha\beta q^{m+n+1}\\ q^{-N+m},\alpha\beta q^{2m+2}\end{matrix}\right|q;q\right)Q_m(x;\alpha,\beta,M|q),\end{aligned}$$

$$\begin{aligned}Q_n(x;\alpha,\beta,N|q)=&\sum_{m=0}^{n}(-1)^m q^{\frac{1}{2}m(m+1)}\frac{(\alpha\beta q^{n+1};q)_m(\alpha_1 q;q)_m(q^{-M};q)_m(q^{-n};q)_m}{(\alpha_1\beta_1 q^{m+1};q)_m(\alpha q;q)_m(q^{-N};q)_m(q;q)_m}\\&\times{}_4\phi_3\left(\left.\begin{matrix}q^{m-n},\alpha_1 q^{m+1},q^{-M+m},\alpha\beta q^{m+n+1}\\ \alpha q^{m+1},q^{-N+m},\alpha_1\beta_1 q^{2m+2}\end{matrix}\right|q;q\right)Q_m(x;\alpha_1,\beta_1,M|q),\end{aligned}$$

3. affine q-Krawtchouk

$$\begin{aligned}K_n^{\mathit{aff}}(\bar{x};p,N;q)=&\sum_{m=0}^{n}(-1)^m q^{\frac{1}{2}m(m+2N+1-2M)}\frac{(p_1 q;q)_m(q^{-n};q)_m(q^{1-m};q)_N(q;q)_M}{(pq;q)_m(q;q)_m(q;q)_N(q^{1-m};q)_M}\\&\times{}_3\phi_2\left(\left.\begin{matrix}q^{m-M},q^{m-n},p_1 q^{m+1}\\ q^{m-N},pq^{m+1}\end{matrix}\right|q;q\right)K_m^{\mathit{Aff}}(\bar{x};p_1,M;q),\end{aligned}$$

4. big q-Laguerre

$$P_n(x;\alpha,\beta;q)=\sum_{m=0}^{n}(-1)^m q^{\frac{1}{2}m(m+1)}\frac{(\alpha_1 q;q)_m(\beta_1 q;q)_m(q^{-n};q)_m}{(\alpha q;q)_m(\beta q;q)_m(q;q)_m}$$
$$\times {}_3\phi_2\left(\begin{matrix} q^{m-n},\alpha_1 q^{m+1},\beta_1 q^{m+1} \\ \alpha q^{m+1},\beta q^{m+1}\end{matrix}\middle| q;q\right)P_m(x;\alpha_1,\beta_1;q),$$

5. little q-Jacobi

$$p_n(x;\alpha,\beta|q)=\sum_{m=0}^{n}(-1)^m q^{\frac{1}{2}m(m+1)}\frac{(\alpha_1 q;q)_m(\alpha\beta q^{n+1};q)_m(q^{-n};q)_m}{(\alpha q;q)_m(q;q)_m(\alpha_1\beta_1 q^{m+1};q)_m}$$
$$\times {}_3\phi_2\left(\begin{matrix} q^{m-n},\alpha_1 q^{m+1},\alpha\beta q^{m+n+1} \\ \alpha q^{m+1},\alpha_1\beta_1 q^{2m+2}\end{matrix}\middle| q;q\right)p_m(x;\alpha_1,\beta_1|q),$$

6. q-Krawtchouk

$$K_n(\bar{x};p,N;q)=\sum_{m=0}^{n}(-1)^m q^{\frac{1}{2}m(m+2N-2M+1)}\frac{(-pq^n;q)_m(q;q)_M(q^{1-m};q)_N(q^{-n};q)_m}{(q;q)_m(q^{1-m};q)_M(q;q)_N(-pq^m;q)_m}$$
$$\times {}_3\phi_2\left(\begin{matrix} q^{m-M},q^{m-n},-pq^{n+m} \\ q^{m-N},-pq^{2m+1}\end{matrix}\middle| q;q\right)K_m(\bar{x};p,M;q),$$

$$K_n(\bar{x};p,N;q)=\sum_{m=0}^{n}(-1)^m q^{\frac{1}{2}m(m+2N-2M+1)}\frac{(-pq^n;q)_m(q;q)_M(q^{1-m};q)_N(q^{-n};q)_m}{(q;q)_m(q^{1-m};q)_M(q;q)_N(-p_1q^m;q)_m}$$
$$\times {}_3\phi_2\left(\begin{matrix} q^{m-M},q^{m-n},-pq^{n+m} \\ q^{m-N},-p_1q^{2m+1}\end{matrix}\middle| q;q\right)K_m(\bar{x};p_1,M;q),$$

7. quantum q-Krawtchouk

$$K_n^{qtm}(\bar{x};p,N;q)=\sum_{m=0}^{n}(-1)^m q^{-\frac{1}{2}m(m+2M-2N-2n-1)}\frac{(q;q)_M(q^{1-m};q)_N}{(q;q)_m(q;q)_N(q^{1-m};q)_M}$$
$$\times {}_2\phi_1\left(\begin{matrix} q^{m-M},q^{m-n} \\ q^{m-N}\end{matrix}\middle| q;\frac{q^n}{q^m}\right)K_n^{qtm}(\bar{x};p,M;q),$$

$$K_n^{qtm}(\bar{x};p,N;q)=\sum_{m=0}^{n}(-1)^m(\frac{p}{p_1})^m q^{-\frac{1}{2}m(m+2M-2N-2n-1)}\frac{(q;q)_M(q^{1-m};q)_N}{(q;q)_m(q;q)_N(q^{1-m};q)_M}$$
$$\times {}_2\phi_1\left(\begin{matrix} q^{m-M},q^{m-n} \\ q^{m-N}\end{matrix}\middle| q;\frac{pq^n}{p_1q^m}\right)K_n^{qtm}(\bar{x};p_1,M;q),$$

8. q-Meixner

$$M_n(\bar{x};\beta,\gamma;q)=\sum_{m=0}^{n}(-1)^m q^{\frac{1}{2}m(2n-m+1)}\frac{(\beta_1 q;q)_m(q^{-n};q)_m}{(\beta q;q)_m(q;q)_m}$$
$$\times {}_2\phi_1\left(\begin{array}{c} q^{m-n},\beta_1 q^{m+1}\\ \beta q^{m+1}\end{array}\middle|\, q;\frac{q^n}{q^m}\right)M_m(\bar{x};\beta_1,\gamma;q),$$

$$M_n(\bar{x};\beta,\gamma;q)=\sum_{m=0}^{n}(-1)^m(\frac{\gamma_1}{\gamma})^m q^{\frac{1}{2}m(2n-m+1)}\frac{(\beta_1 q;q)_m(q^{-n};q)_m}{(\beta q;q)_m(q;q)_m}$$
$$\times {}_2\phi_1\left(\begin{array}{c} q^{m-n},\beta_1 q^{m+1}\\ \beta q^{m+1}\end{array}\middle|\, q;\frac{\gamma_1 q^n}{\gamma q^m}\right)M_m(\bar{x};\beta_1,\gamma_1;q).$$

If $p_n(x)$ and $q_m(x)$ are not represented in the same basis, we need the following connection formulas between the four aforementioned bases of q-COP.

Lemma 3.18. *The following connection formulas between the bases $\mathcal{V}_j(x)=x^j$, $\mathcal{V}_j(x)=(x;q)_j$, $\mathcal{V}_j(x)=(ix;q)_j$ and $\mathcal{V}_j(x)=(x\ominus 1)_q^j$ are valid:*

$$x^n=\sum_{m=0}^{n}\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{(-1)^m q^{\frac{m(m+1)}{2}}}{q^{mn}}(x;q)_m$$

see e.g. [Area et al., 2001],

$$x^n=\sum_{m=0}^{n}\begin{bmatrix} n\\ m\end{bmatrix}_q(x\ominus 1)_q^m$$

see [Kac, 2002, p. 12],

$$(x;q)_n=\sum_{m=0}^{n}(-1)^m q^{\frac{m(m-1)}{2}}\begin{bmatrix} n\\ m\end{bmatrix}_q x^m$$

see e.g. [Area et al., 2001],

$$(x;q)_n=\sum_{m=0}^{n}(-1)^m q^{\binom{m}{2}}\begin{bmatrix} n\\ m\end{bmatrix}_q(q^m;q)_{n-m}(x\ominus 1)_q^m,$$

$$(x;q)_n=\sum_{m=0}^{n}\begin{bmatrix} n\\ m\end{bmatrix}_q(-i)^m(-i;q)_{n-m}(ix;q)_m,$$

$$(ix;q)_n=\sum_{m=0}^{n}\begin{bmatrix} n\\ m\end{bmatrix}_q(i)^m(i;q)_{n-m}(x;q)_m,$$

$$(ix;q)_n=\sum_{m=0}^{n}\begin{bmatrix} n\\ m\end{bmatrix}_q(-i)^m q^{\binom{m}{2}}(iq^m;q)_{n-m}(x\ominus 1)_q^m,$$

$$(x \ominus 1)_q^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (-1)^{n-m} q^{\binom{n-m}{2}} x^m$$

see [Kac, 2002, p. 14],

$$(x \ominus 1)_q^n = \sum_{m=0}^{n} (-1)^m q^{\frac{1}{2}m(m+1-2n)} \begin{bmatrix} n \\ m \end{bmatrix}_q (q^m; q)_{n-m} (x; q)_m,$$

$$(x \ominus 1)_q^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (-i)^{n-2m} q^{\frac{1}{2}m(m+1-2n)} (iq^m; q)_{n-m} (ix; q)_m.$$

Proof. The proof uses the following properties which are valid for $j = 1, 2, \ldots, n$:

$$D_q^j x^n = [n]_q [n-1]_q \cdots [n-j+1]_q x^{n-j}, \tag{3.26}$$
$$D_q^j (x; q)_n = (-1)^j q^{\binom{j}{2}} [n]_q [n-1]_q \cdots [n-j+1]_q (xq^j; q)_{n-j}, \tag{3.27}$$
$$D_q^j (ix; q)_n = (-i)^j q^{\binom{j}{2}} [n]_q [n-1]_q \cdots [n-j+1]_q (ixq^j; q)_{n-j}, \tag{3.28}$$
$$D_q^j (x \ominus 1)_q^n = [n]_q [n-1]_q \cdots [n-j+1]_q (x \ominus 1)_q^{n-j}. \tag{3.29}$$

We suppose that $\mathcal{V}_j(x)$ and $\theta_j(x)$ are any of the polynomials x^j, $(x;q)_j$, $(ix;q)_j$, $(x \ominus 1)_q^j$ and that

$$\mathcal{V}_n(x) = \sum_{m=0}^{n} C_m(n) \theta_m(x).$$

We apply the operator D_q^j $(j = 1, \ldots, n)$ to both sides of the above equation, use properties (3.26)–(3.29) and substitute x by 0, q^{-j}, $-iq^{-j}$ or 1 (since for $k \neq 0$, $x^k = 0 \Leftrightarrow x = 0$, $(xq^j; q)_k = 0 \Leftrightarrow x = q^{-j}$, $(ixq^j; q)_k = 0 \Leftrightarrow x = -iq^{-j}$, $(x \ominus 1)_q^k = 0 \Leftrightarrow x = 1$) to get the result. □

Now we suppose that $p_n(x)$ and $q_m(x)$ are represented in two different bases $\mathcal{V}_j(x)$ and $\theta_j(x)$, i.e.

$$p_n(x) = \sum_{j=0}^{n} A_j(n) \mathcal{V}_j(x), \;\; q_m(x) = \sum_{k=0}^{m} d_k(m) \theta_k(x),$$

$$\mathcal{V}_j(x) = \sum_{k=0}^{j} B_k(j) \theta_k(x), \text{and } \theta_k(x) = \sum_{m=0}^{k} I_m(k) q_m(x).$$

From these expansions, we obtain the connection coefficient representation as a double sum

$$p_n(x) = \sum_{m=0}^{n} C_m(n) q_m(x) \text{ with } C_m(n) = \sum_{j=0}^{n-m} \sum_{k=0}^{j} A_{j+m}(n) B_{k+m}(j+m) I_m(k+m).$$

For example the connection formula between the big q-Jacobi polynomials $P_n(x; \alpha, \beta, \gamma; q)$ represented in the basis $((x;q)_n)_n$ and the little q-Jacobi polynomials $p_n(x; \alpha, \beta | q)$ represented in the basis $(x^n)_n$ is given by

$$P_n(x; \alpha, \beta, \gamma; q) = \sum_{m=0}^{n} C_m(n) p_m(x; \alpha, \beta | q)$$

with

$$C_m(n)=\sum_{j=0}^{n-m}\frac{q^{m^2+j}(q^{-n},\alpha\beta q^{n+1};q)_{m+j}}{(q,\alpha q,\gamma q;q)_{m+j}(q;q)_m(q;q)_k}\sum_{k=0}^{j}\frac{\begin{bmatrix}m+j\\m+k\end{bmatrix}_q q^{mk}(q,\alpha q;q)_{m+k}(1-\alpha\beta q^{2m+1})}{(-1)^k q^{-\binom{k}{2}}(\alpha\beta q^{m+1};q)_{m+k}(1-\alpha\beta q^{2m+k+1})}.$$

3.4.3 Parameter Derivatives

By a limiting process, the parameter derivatives of q-COP can be obtained from the results of Corollary 3.15 and Theorem 3.16.

Corollary 3.19. *The following representations for the parameter derivatives of the classical q-orthogonal polynomials are valid:*
1. big q-Jacobi

$$\frac{\partial}{\partial\alpha}P_n(x;\alpha,\beta,\gamma;q)=\sum_{m=0}^{n-1}\Big(\Big(\frac{q^{m+1}}{1-\alpha q^{m+1}}-\frac{\beta q^{m+n+1}}{1-\alpha\beta q^{m+n+1}}\Big)P_n(x;\alpha,\beta,\gamma;q)$$
$$-\frac{(\alpha q)^{n-m}(q;q)_{n-m-1}\begin{bmatrix}n\\m\end{bmatrix}_q(1-\alpha\beta q^{2m+1})(\alpha q,\alpha\beta q^{n+1};q)_m(\beta q;q)_n}{\alpha(\beta q;q)_m(\alpha\beta q^{m+1};q)_{n+1}(\alpha q;q)_n}P_m(x;\alpha,\beta,\gamma;q)\Big),$$

$$\frac{\partial}{\partial\beta}P_n(x;\alpha,\beta,\gamma;q)=\sum_{m=0}^{n-1}\Bigg(\frac{-\alpha q^{n+m+1}}{1-\alpha\beta q^{n+m+1}}P_n(x;\alpha,\beta,\gamma;q)$$
$$+\begin{bmatrix}n\\m\end{bmatrix}_q\frac{(\alpha\beta q^{n+1})^{n-m}(\alpha\beta q^{n+1};q)_m(q^{m-n+1};q)_{n-m-1}}{\beta(\alpha\beta q^{2(m+1)};q)_{n-m}(\alpha\beta q^{m+1};q)_m}P_m(x;\alpha,\beta,\gamma;q)\Bigg),$$

2. q-Hahn

$$\frac{\partial}{\partial\alpha}\;Q_n(x;\alpha,\beta,N;q)=\sum_{m=0}^{n-1}\Big(\Big(\frac{q^{m+1}}{1-\alpha q^{m+1}}-\frac{\beta q^{m+n+1}}{1-\alpha\beta q^{m+n+1}}\Big)Q_n(x;\alpha,\beta,N;q)$$
$$-\frac{(\alpha q)^{n-m}(q;q)_{n-m-1}\begin{bmatrix}n\\m\end{bmatrix}_q(1-\alpha\beta q^{2m+1})(\alpha q,\alpha\beta q^{n+1};q)_m(\beta q;q)_n}{\alpha(\beta q;q)_m(\alpha\beta q^{m+1};q)_{n+1}(\alpha q;q)_n}Q_m(x;\alpha,\beta,N;q)\Big),$$

$$\frac{\partial}{\partial\beta}Q_n(x;\alpha,\beta,N;q)=\sum_{m=0}^{n-1}\Bigg(\frac{-\alpha q^{n+m+1}}{1-\alpha\beta q^{n+m+1}}Q_n(x;\alpha,\beta,N;q)$$
$$+\begin{bmatrix}n\\m\end{bmatrix}_q\frac{(\alpha\beta q^{n+1})^{n-m}(\alpha\beta q^{n+1};q)_m(q^{m-n+1};q)_{n-m-1}}{\beta(\alpha\beta q^{2(m+1)};q)_{n-m}(\alpha\beta q^{m+1};q)_m}Q_m(x;\alpha,\beta,N;q)\Bigg),$$

3. little q-Jacobi

$$\frac{\partial}{\partial\alpha}p_n(x;\alpha,\beta|q)=\sum_{m=0}^{n-1}\Bigg(\Big(\frac{q^{m+1}}{1-\alpha q^{m+1}}-\frac{\beta q^{n+m+1}}{1-\alpha\beta q^{n+m+1}}\Big)p_n(x;\alpha,\beta|q)+\begin{bmatrix}n\\m\end{bmatrix}_q(-\alpha q^{\frac{n-m+1}{2}})^{n-m}$$
$$\times\frac{(\alpha q,\alpha\beta q^{n+1};q)_m(\beta q^{m+1};q)_{n-m}(q^{m-n+1};q)_{n-m-1}}{\alpha(\alpha\beta q^{2(m+1)};q)_{n-m}(\alpha\beta q^{m+1};q)_m(\alpha q;q)_n}p_m(x;\alpha,\beta|q)\Bigg),$$

$$\frac{\partial}{\partial\beta}p_n(x;\alpha,\beta|q)=\sum_{m=0}^{n-1}\Bigg(\frac{-\alpha q^{n+m+1}}{1-\alpha\beta q^{n+m+1}}p_n(x;\alpha,\beta|q)$$
$$+\begin{bmatrix}n\\m\end{bmatrix}_q(\alpha\beta q^{n+1})^{n-m}\frac{(\alpha\beta q^{n+1};q)_m(q^{m-n+1};q)_{n-m-1}}{\beta(\alpha\beta q^{2(m+1)};q)_{n-m}(\alpha\beta q^{m+1};q)_m}p_m(x;\alpha,\beta|q)\Bigg),$$

4. alternative q-Charlier

$$\frac{\partial}{\partial\alpha}y_n(x;\alpha|q)=\sum_{m=0}^{n-1}\Bigg(\frac{q^{n+m}}{1+\alpha q^{n+m}}y_n(x;\alpha|q)$$
$$+(-\alpha q^n)^{n-m}\begin{bmatrix}n\\m\end{bmatrix}_q\frac{(-\alpha q^n;q)_m(q^{m-n+1};q)_{n-m-1}}{\alpha(-\alpha q^{2m+1};q)_{n-m}(-\alpha q^m;q)_m}y_m(x;\alpha|q)\Bigg),$$

5. little q-Laguerre

$$\frac{\partial}{\partial\alpha}p_n(x;\alpha|q)=\sum_{m=0}^{n-1}\Bigg(\frac{q^{m+1}}{1-\alpha q^{m+1}}p_n(x;\alpha|q)$$
$$+(-\alpha q^{\frac{n-m+1}{2}})^{n-m}\begin{bmatrix}n\\m\end{bmatrix}_q\frac{(\alpha q;q)_m(q^{m-n+1};q)_{n-m-1}}{\alpha(\alpha q;q)_n}p_m(x;\alpha|q)\Bigg),$$

6. q-Krawtchouk

$$\frac{\partial}{\partial p}K_n(\bar{x};p,N;q)=\sum_{m=0}^{n-1}\Big(\frac{q^{m+n}}{1+pq^{m+n}}K_n(\bar{x};p,N;q)$$
$$+\begin{bmatrix}n\\m\end{bmatrix}_q\frac{(-pq^n)^{n-m}(-pq^n;q)_m(q^{m-n+1};q)_{n-m-1}}{p(-pq^m;q)_m(-pq^{2m+1};q)_{n-m}}K_m(\bar{x};p,N;q)\Big),$$

7. quantum q-Krawtchouk

$$\frac{\partial}{\partial p}K_n^{qtm}(\bar{x};p,N;q)=\frac{n}{p}K_n^{qtm}(\bar{x};p,N;q)$$
$$+\sum_{m=0}^{n-1}(-q^{\frac{m-n+1}{2}})^{m-n}\begin{bmatrix}n\\m\end{bmatrix}_q\frac{(q^{m-n+1};q)_{n-m-1}}{p}K_m^{qtm}(\bar{x};p,N;q),$$

8. q-Laguerre

$$\frac{\partial}{\partial\alpha}L_n^{(\alpha)}(x;q)=n\ln qL_n^{(\alpha)}(x;q)-\sum_{m=0}^{n-1}\frac{\ln q}{(1-q^{n-m})}L_m^{(\alpha)}(x;q),$$

9. affine q-Krawtchouk

$$\frac{\partial}{\partial p}K_n^{aff}(\bar{x};p,N;q)=\sum_{m=0}^{n-1}\Big(\frac{q^{m+1}}{1-pq^{m+1}}K_n^{aff}(\bar{x};p,N;q)$$
$$-\begin{bmatrix}n\\m\end{bmatrix}_q\frac{(pq)^{n-m}(q;q)_{n-m-1}(pq;q)_m}{p(pq;q)_n}K_m^{Aff}(\bar{x};p,N;q)\Big),$$

$$\frac{\partial}{\partial N}K_n^{aff}(\bar{x};p,N;q)=\sum_{m=0}^{n-1}\Big(\frac{-q^{m-N}\ln q}{1-q^{m-N}}K_n^{Aff}(\bar{x};p,N;q)$$
$$+(-q^{\frac{m+n-1}{2}})^{m-n}\ln q\begin{bmatrix} n \\ m\end{bmatrix}_q\frac{(q;q)_{n-m-1}}{(q^{N+1-n};q)_{n-m}}K_m^{aff}(\bar{x};p,N;q)\Big),$$

10. big q-Laguerre

$$\frac{\partial}{\partial\alpha}P_n(x;\alpha,\beta;q)=\sum_{m=0}^{n-1}\Big(\frac{q^{m+1}}{1-\alpha q^{m+1}}P_n(x;\alpha,\beta;q)$$
$$-\begin{bmatrix} n \\ m\end{bmatrix}_q\frac{(\alpha q)^{n-m}(\alpha q;q)_m(q;q)_{n-m-1}}{\alpha(\alpha q;q)_n}P_m(x;\alpha,\beta;q)\Big),$$

$$\frac{\partial}{\partial\beta}P_n(x;\alpha,\beta;q)=\sum_{m=0}^{n-1}\Big(\frac{q^{m+1}}{1-\beta q^{m+1}}P_n(x;\alpha,\beta;q)$$
$$-\begin{bmatrix} n \\ m\end{bmatrix}_q\frac{(\beta q)^{n-m}(\beta q;q)_m(q;q)_{n-m-1}}{\beta(\beta q;q)_n}P_m(x;\alpha,\beta;q)\Big),$$

11. Al-Salam-Carlitz I

$$\frac{\partial}{\partial\alpha}U_n^{(\alpha)}(x;q)=-\sum_{m=0}^{n-1}\alpha^{n-m-1}(q;q)_{n-m-1}\begin{bmatrix} n \\ m\end{bmatrix}_q U_m^{(\alpha)}(x;q),$$

12. q-Meixner

$$\frac{\partial}{\partial\gamma}M_n(x;\beta,\gamma;q)=-\frac{n}{\gamma}M_n(x;\beta,\gamma;q)+\frac{1}{\gamma}\sum_{m=0}^{n-1}(q;q)_{n-m-1}\begin{bmatrix} n \\ m\end{bmatrix}_q M_m(x;\beta,\gamma;q),$$

13. q-Charlier

$$\frac{\partial}{\partial\alpha}C_n(x;\alpha;q)=-\frac{n}{\alpha}C_n(x;\alpha;q)+\frac{1}{\alpha}\sum_{m=0}^{n-1}(q;q)_{n-m-1}\begin{bmatrix} n \\ m\end{bmatrix}_q C_m(x;\alpha;q),$$

14. Al-Salam-CArlitz II

$$\frac{\partial}{\partial\alpha}V_n^{(\alpha)}(x;q)=\frac{1}{\alpha}\sum_{m=0}^{n-1}(-\alpha)^{n-m}q^{\frac{(m-n)(m+n-1)}{2}}(q;q)_{n-m-1}\begin{bmatrix} n \\ m\end{bmatrix}_q V_n^{(\alpha)}(x;q).$$

Proof. If

$$p_n^{\alpha}(x)=\sum_{m=0}^{n}C_m(n,\alpha,\beta)p_m^{\beta}(x),$$

then

$$\frac{\partial}{\partial\alpha}p_n^{\alpha}(x)=\lim_{\beta\to\alpha}\frac{C_n(n,\alpha,\beta)-1}{\alpha-\beta}p_n^{\beta}(x)+\sum_{m=0}^{n-1}\lim_{\beta\to\alpha}\frac{C_m(n,\alpha,\beta)}{\alpha-\beta}P_m^{\beta}(x).$$

We know that

$$\frac{d}{dx}\prod_{j=1}^{n} f_j(x) = \sum_{m=1}^{n} \frac{f'_m(x)}{f_m(x)} \prod_{j=1}^{n} f_j(x),$$

from which we get for a parameter γ

$$\frac{\partial}{\partial\alpha}(\alpha\gamma q^{n+1};q)_n = \sum_{m=0}^{n-1} \frac{-\gamma q^{n+m+1}}{1-\alpha\gamma q^{n+m+1}}(\alpha\gamma q^{n+1};q)_n.$$

Using l'Hospital's rule, we obtain

$$\lim_{\beta\to\alpha} \frac{(\alpha\gamma q^{n+1};q)_n - (\beta\gamma q^{n+1};q)_n}{(\beta\gamma q^{n+1};q)_n(\alpha-\beta)} = \sum_{m=0}^{n-1} \frac{-\gamma q^{n+m+1}}{1-\alpha\gamma q^{n+m+1}}.$$

We also have

$$\lim_{\beta\to\alpha} \frac{(\frac{\beta q^{m+1}}{\alpha q^n};q)_{n-m}}{\alpha-\beta} = \frac{1}{\alpha}(q^{m-n+1};q)_{n-m-1}$$

and

$$\lim_{\beta\to\alpha} \frac{(\frac{\alpha}{\beta};q)_{n-m}}{\alpha-\beta} = -\frac{1}{\alpha}(q;q)_{n-m-1}.$$

From all the preceding, the result follows. □

Remark 3.20. *Lewanowicz [2003b] gave an algorithmic approach to construct recurrence relations for the coefficients C_k of the expansion of the form $\frac{\partial}{\partial\alpha}P_n(x;\alpha;q) = \sum_{k=0}^{n-1} C_k P_k(x;\alpha;q)$, where $P_n(x;\alpha;q)$ is a q-COP. He solved this recurrence relation for the big q-Jacobi polynomials $P_k(x;\alpha,\beta,\gamma;q)$ and gave explicitly the coefficients C_k.*

3.5 Linearization Problem of q-COP

The linearization problem of the q-COP family $(p_n(x))_n$ depends on the basis $\{x^n\}$, $\{(x;q)_n\}$, $\{(ix;q)_n\}$ or $\{(x\ominus 1)_q^n\}$ in which $p_n(x)$ is represented.

3.5.1 Representation Basis $\{x^n\}$

Theorem 3.21. *The following linearization relations between the orthogonal polynomial systems of the q-Hahn class represented in the basis $\{x^n\}$ are valid:*
1. little q-Jacobi

$$p_n(x;\alpha,\beta|q)p_m(x;\alpha_1,\beta_1|q) = \sum_{l=0}^{n+m}\sum_{k=0}^{n+m-l} \frac{(-1)^l q^{k+\frac{l(l+1)}{2}}(q,\alpha_2 q;q)_{k+l}(1-\alpha_2\beta_2 q^{2l+1})}{(q;q)_l(q;q)_k(\alpha_2\beta_2 q^{l+1};q)_{k+l}(1-\alpha_2\beta_2 q^{k+2l+1})}$$
$$\times \sum_{r=0}^{k+l} \frac{(q^{-n},\alpha\beta q^{n+1};q)_r(q^{-m},\alpha_1\beta_1 q^{m+1};q)_{k+l-r}}{(q,\alpha q;q)_r(q,\alpha_1 q;q)_{k+l-r}} p_l(x;\alpha_2,\beta_2|q),$$

2. alternative q-Charlier

$$y_n(x;\alpha|q)y_m(x;\alpha_1|q)=\sum_{l=0}^{n+m}\sum_{k=0}^{n+m-l}\frac{(-1)^l q^{k+\frac{l(l+1)}{2}}(q;q)_{k+l}(1+\alpha_2 q^{2l})}{(q;q)_k(q;q)_l(-\alpha_2 q^{l+1};q)_{k+l}(1+\alpha_2 q^l)}$$
$$\times\sum_{r=0}^{k+l}\frac{(q^{-n},-\alpha q^n;q)_r(q^{-m},-\alpha_1 q^m;q)_{k+l-r}}{(q;q)_r(q;q)_{k+l-r}}y_l(x;\alpha_2|q),$$

3. little q-Laguerre/Wall

$$p_n(x;\alpha|q)p_m(x;\alpha_1|q)=\sum_{l=0}^{n+m}\sum_{k=0}^{n+m-l}\frac{(-1)^l q^{k+\frac{l(l+1)}{2}}(q;q)_{k+l}(\alpha_2 q;q)_{k+l}}{(q;q)_k(q;q)_l}$$
$$\times\sum_{r=0}^{k+l}\frac{(q^{-n};q)_r(q^{-m};q)_{k+l-r}}{(q,\alpha q;q)_r(q,\alpha_1 q;q)_{k+l-r}}p_l(x;\alpha_2|q),$$

4. q-Laguerre

$$L_n^{(\alpha)}(x;q)L_m^{(\beta)}(x;q)=\sum_{l=0}^{n+m}\sum_{k=0}^{n+m-l}\frac{(-1)^l q^{(m+\beta-\gamma)(k+l)}(q^{\alpha+1};q)_n(q^{\beta+1};q)_m(q;q)_{k+l}(q^{\gamma+l+1};q)_k}{q^{\frac{1}{2}l(l+2k-1)}(q;q)_n(q;q)_m(q;q)_k}$$
$$\times\sum_{r=0}^{k+l}\frac{q^{r(n+\alpha+r)}(q^{-n};q)_r(q^{-m};q)_{k+l-r}}{q^{r(m+k+l+\beta)}(q,q^{\alpha+1};q)_r(q,q^{\beta+1};q)_{k+l-r}}L_l^{(\gamma)}(x;q),$$

5. Stieltjes-Wigert

$$S_n(x;q)S_m(x;q)=\sum_{l=0}^{n+m}\sum_{k=0}^{n+m-l}\sum_{r=0}^{k+l}\frac{(-1)^l q^{m(k+l)+r(n+r)}(q;q)_{k+l}(q^{-n};q)_r(q^{-m};q)_{k+l-r}}{q^{\frac{1}{2}l(l+2k-1)+r(m+k+l)}(q;q)_n(q;q)_m(q;q)_k(q;q)_r(q;q)_{k+l-r}}S_l(x;q).$$

Proof. We suppose

$$p_n(x)=\sum_{i=0}^{n}A_i(n)x^i,\ \ q_m(x)=\sum_{j=0}^{m}B_j(m)x^j,\ \text{ and } x^k=\sum_{l=0}^{k}I_l(k)y_l(x)$$

where $p_n(x), q_m(x), y_l(x)$ are three families of classical q-orthogonal polynomials.

We then obtain the Cauchy product

$$p_n(x)q_m(x)=\sum_{k=0}^{n+m}C_k(m,n)x^k,\ \text{ where } C_k(m,n)=\sum_{r=0}^{k}A_r(n)B_{k-r}(m).$$

Using the inversion formula, this can be rewritten as

$$p_n(x)q_m(x)=\sum_{l=0}^{n+m}L_l(m,n)y_l(x)$$

with

$$L_l(m,n)=\sum_{k=0}^{n+m-l}C_{k+l}(m,n)I_l(k+l)=\sum_{k=0}^{n+m-l}\sum_{r=0}^{k+l}A_r(n)B_{k+l-r}(m)I_l(k+l).$$

We consider here the standard linearization problem of Clebsch-Gordan-type where $p_n(x)$, $q_m(x)$, $y_l(x)$ belong to the *same* q-orthogonal family. □

3.5.2 Representation Basis $\{(x;q)_n\}$ or $\{(ix;q)_n\}$

In these cases, we need the following linearization coefficients of the basis $\{(x;q)_n\}$ or $\{(ix;q)_n\}$.

Proposition 3.22. *The basis $\{(x;q)_n\}$ fulfils the linearization relation*

$$(x;q)_n(x;q)_m = \sum_{k=\max(m,n)}^{n+m} \frac{(q;q)_n(q;q)_m(-1)^{n+m-k}q^{\binom{n+m-k}{2}-nm}}{(q;q)_{k-n}(q;q)_{k-m}(q;q)_{n+m-k}}(x;q)_k. \tag{3.30}$$

Proof. Without loss of generality, we suppose $n \geq m$.
$(x;q)_n = 0 \Leftrightarrow x = q^{-j},\ j = 0,1,\ldots,n-1$. For $x = 1$, the equation

$$(x;q)_n(x;q)_m = \sum_{k=0}^{n+m} H_k(m,n)(x;q)_k \tag{3.31}$$

yields $H_0(m,n) = 0$, $m \geq 1$ so that

$$(x;q)_n(x;q)_m = \sum_{k=1}^{n+m} H_k(m,n)(x;q)_k.$$

Next for $x = q^{-1}$, the above equation gives $H_1(m,n)(q^{-1},q)_1 = 0$, $m \geq 2$. Since $(q^{-n};q)_n \neq 0$ therefore $H_1(m,n) = 0$.
Progressively, it follows in a similar way that

$$H_0(m,n) = H_1(m,n) = \ldots = H_{n-1}(m,n) = 0.$$

Consequently

$$(x;q)_n(x;q)_m = \sum_{k=n}^{n+m} H_k(m,n)(x;q)_k = \sum_{k=0}^{m} H_{n+k}(m,n)(x;q)_{n+k}.$$

Since $(x;q)_{n+k} = (x;q)_n(xq^n;q)_k$ it follows that

$$(x;q)_m = \sum_{k=0}^{m} H_{n+k}(m,n)(xq^n;q)_k. \tag{3.32}$$

For $x = q^{-n}$, (3.32) gives $H_n(m,n) = (q^{-n};q)_m$, $m \geq 1$. From the property

$$D_q(xq^n;q)_k = -[k]_q q^n (xq^{n+1};q)_{k-1}$$

it follows that for any $j = 1,2,\ldots,m$

$$D_q^j(xq^n;q)_k = (-1)^j[k]_q[k-1]_q\cdots[k-j+1]_q q^{nj+\binom{j}{2}}(xq^{n+j};q)_{k-j}.$$

We apply D_q^j to both sides of (3.32) to get

$$\begin{aligned}
[m]_q[m-1]_q\cdots[m-j+1]_q(xq^j;q)_{m-j} &= \sum_{k=0}^{m}[k]_q[k-1]_q\cdots[k-j+1]_q q^{nj}(xq^{n+j};q)_{k-j} \\
&= \sum_{k=0,k\neq j}^{m} [k]_q[k-1]_q\cdots[k-j+1]_q q^{nj}(xq^{n+j};q)_{k-j} \\
&\quad + q^{nj}[j]_q!H_{n+j}(m,n)
\end{aligned}$$

where

$$[j]_q! = \begin{cases} [j]_q[j-1]_q \cdots [1]_q, j \in \mathbb{N} \\ 1, & j = 0. \end{cases}$$

For $x = q^{-n-j}$ the latter equation gives

$$H_{n+j}(m,n) = \frac{[m]_q[m-1]_q \cdots [m-j+1]_q q^{\binom{j}{2}}(q^{-n};q)_{m-j}}{q^{nj+\binom{j}{2}}[j]_q!} = \frac{[m]_q!(q^{-n};q)_{m-j}}{[j]_q![m-j]_q! q^{nj}}, \; j = 0, 1, \dots m.$$

Using the relations

$$\frac{[n]_q!}{[m]_q![n-m]_q!} = \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}$$

and

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, \tag{3.33}$$

the representation of $H_{n+j}(m,n)$ can be rewritten as

$$H_{n+j} = \frac{(q;q)_n(q;q)_m(-1)^{m-j}q^{\binom{m-j}{2}-nm}}{(q;q)_j(q;q)_{m-j}(q;q)_{n-m+j}}.$$

For $j = k - n$, the result follows. □

Using the basic hypergeometric series representation of $p_n(x)$, the inversion problem and the linearization formula (3.30), we prove

Theorem 3.23. *The following linearization formulae of the q-COP represented in the basis $\{(x;q)_n\}$ are valid:*

1. discrete q-Hermite II

$$\tilde{h}_n(x;q)\tilde{h}_m(x;q) = \sum_{r=0}^{n+m} L_r(m,n)\tilde{h}_r(x;q)$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-i)^{m+n+r}q^{nj+m(l+r)}(q;q)_{l+r}(q^{-n};q)_j(q^{-m};q)_{l+r-j}}{q^{\binom{n}{2}+\binom{m}{2}+mj+rl}(q;q)_j(q;q)_r(q;q)_l(q;q)_{l+r-j}}$$
$$\times {}_2\phi_0\left(\left.\begin{matrix} q^{-j}, q^{l+r-j-m} \\ - \end{matrix}\right| q; -q^{m+j-l-r}\right),$$

2. Al-Salam-Carlitz II

$$V_n^{(\alpha)}(x;q)V_m^{(\alpha)}(x;q) = \sum_{r=0}^{n+m} L_r(m,n)V_r^{(\alpha)}(x;q)$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-1)^{m+n}\alpha^{m+n-r}q^{nj+m(l+r)}(q;q)_{l+r}(q^{-n};q)_j(q^{-m};q)_{l+r-j}}{q^{\binom{n}{2}+\binom{m}{2}+mj+rl}(q;q)_j(q;q)_{l+r-j}(q;q)_r(q;q)_l}$$
$$\times {}_2\phi_0\left(\left.\begin{matrix} q^{-j}, q^{l+r-j-m} \\ - \end{matrix}\right| q; \frac{q^{m+j-l-r}}{\alpha}\right),$$

3. q-Charlier

$$C_n(x;\alpha;q)C_m(x;\alpha;q) = \sum_{r=0}^{n+m} L_r(m,n)C_r(x;\alpha;q)$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-1)^r q^{m(l+r)+j(n+j)}(q^{-n};q)_j(q^{-m};q)_{l+r-j}(q;q)_{l+r}}{q^{\binom{r}{2}+j(m+l+r)+rl}(q;q)_{l+r-j}(q;q)_j(q;q)_r(q;q)_l} \times {}_2\phi_1\left(\begin{matrix} q^{-j}, q^{l+r-m-j} \\ 0 \end{matrix} \middle| q; -\frac{q^{m+1}}{\alpha}\right),$$

4. q-Meixner

$$M_n(x;\beta,\gamma;q)M_m(x;\beta,\gamma;q) = \sum_{r=0}^{n+m} L_r(m,n)M_r(x;\beta,\gamma;q)$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-1)^r q^{j(n+j)+m(l+r)}(q^{-n};q)_j(q^{-m};q)_{l+r-j}(q;q)_{l+r}}{q^{\binom{r}{2}+j(m+l+r)+rl}(q;q)_j(q;q)_r(q;q)_l(q;q)_{l+r-j}} \times \frac{(\beta q;q)_{l+r}}{(\beta q;q)_j(\beta q;q)_{l+r-j}} {}_2\phi_1\left(\begin{matrix} q^{-j}, q^{l+r-m-j} \\ \beta q^{l+r-j+1} \end{matrix} \middle| q; -\frac{q^{m+1}}{\gamma}\right),$$

5. quantum q-Krawtchouk

$$K_n^{qtm}(\bar{x};p,N;q)K_m^{qtm}(\bar{x};p,N;q) = \sum_{r=0}^{n+m} L_r(m,n)K_r^{qtm}(\bar{x};p,N;q),\ \ m+n \leq N$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-1)^r q^{j(n+j)+m(l+r)}(q^{-n};q)_j(q^{-m};q)_{l+r-j}(q;q)_{l+r}}{q^{\binom{r}{2}+j(m+l+r)+rl}(q;q)_j(q;q)_r(q;q)_l(q;q)_{l+r-j}} \times \frac{(q^{-N};q)_{l+r}}{(q^{-N};q)_j(q^{-N};q)_{l+r-j}} {}_2\phi_1\left(\begin{matrix} q^{-j}, q^{l+r-m-j} \\ q^{l+r-N-j} \end{matrix} \middle| q; pq^{m+1}\right),$$

6. q-Krawtchouk

$$K_n(\bar{x};p,N;q)K_m(\bar{x};p,N;q) = \sum_{r=0}^{n+m} L_r(m,n)K_r(\bar{x};p,N;q),\ \ m+n \leq N$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-1)^r q^{\frac{r(r+1)}{2}+j^2+l}(q;q)_{l+r}(-pq^m;q)_{l+r-j}(q^{-m};q)_{l+r-j}(q^{-n};q)_j}{q^{j(r+l)}(q;q)_j(q;q)_r(q;q)_l(q;q)_{l+r-j}(-pq^{2r+1};q)_l(-pq^r;q)_r} \times \frac{(q^{-N};q)_{l+r}(-pq^n;q)_j}{(q^{-N};q)_j(q^{-N};q)_{l+r-j}} {}_3\phi_2\left(\begin{matrix} q^{-j}, q^{l+r-m-j}, -pq^{m+l+r-j} \\ q^{l+r-N-j}, 0 \end{matrix} \middle| q; q\right),$$

7. big q-Laguerre

$$P_n(x;\alpha,\beta;q)P_m(x;\alpha,\beta;q) = \sum_{r=0}^{n+m} L_r(m,n)P_r(x;\alpha,\beta;q)$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-1)^r q^{\frac{r(r+1)}{2}+j^2+l}(q;q)_{l+r}(q^{-m};q)_{l+r-j}(q^{-n};q)_j}{q^{j(r+l)}(q;q)_j(q;q)_r(q;q)_l(q;q)_{l+r-j}(\alpha q;q)_j(\beta q;q)_j}$$
$$\times \frac{(\alpha q;q)_{l+r}(\beta q;q)_{l+r}}{(\alpha q;q)_{l+r-j}(\beta q;q)_{l+r-j}} {}_3\phi_2\left(\left.\begin{matrix} q^{-j}, q^{l+r-m-j}, 0 \\ \alpha q^{l+r-j+1}, \beta q^{l+r-j+1}\end{matrix}\right| q;q\right),$$

8. affine q-Krawtchouk

$$K_n^{aff}(\bar{x};p,N;q)K_m^{aff}(\bar{x};p,N;q) = \sum_{r=0}^{n+m} L_r(m,n)K_r^{aff}(\bar{x};p,N;q),\ \ m+n\le N$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-1)^r q^{\frac{r(r+1)}{2}+j^2+l}(q;q)_{l+r}(pq;q)_{l+r}(q^{-m};q)_{l+r-j}(q^{-n};q)_j}{q^{j(l+r)}(q;q)_j(q;q)_r(q;q)_l(q;q)_{l+r-j}(pq;q)_j(pq;q)_{l+r-j}}$$
$$\times \frac{(q^{-N};q)_{l+r}}{(q^{-N};q)_{l+r-j}(q^{-N};q)_j} {}_3\phi_2\left(\left.\begin{matrix} q^{-j}, q^{l+r-m-j}, 0 \\ q^{l+r-N-j}, pq^{l+r-j+1}\end{matrix}\right| q;q\right),$$

9. q-Hahn

$$Q_n(x;\alpha,\beta,N|q)Q_m(x;\alpha,\beta,N|q) = \sum_{r=0}^{n+m} L_r(m,n)Q_r(x;\alpha,\beta,N|q),\ \ m+n\le N$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-1)^r q^{\frac{r(r+1)}{2}+j^2+l}(q;q)_{l+r}(q^{-m};q)_{l+r-j}(q^{-n};q)_j(\alpha q;q)_{l+r}}{q^{j(r+l)}(q;q)_j(q;q)_{l+r-j}(q;q)_r(q;q)_l(\alpha q;q)_j(q^{-N};q)_j}$$
$$\times \frac{(q^{-N};q)_{l+r}(\alpha\beta q^{m+1};q)_{l+r-j}(\alpha\beta q^{n+1};q)_j}{(\alpha q;q)_{l+r-j}(q^{-N};q)_{l+r-j}(\alpha\beta q^{r+1};q)_r(\alpha\beta q^{2r+2};q)_l} {}_3\phi_2\left(\left.\begin{matrix} q^{-j}, q^{l+r-m-j}, \alpha\beta q^{m+l+r-j+1} \\ \alpha q^{l+r-j+1}, q^{l+r-N-j}\end{matrix}\right| q;q\right),$$

10. big q-Jacobi

$$P_n(x;\alpha,\beta,\gamma;q)P_m(x;\alpha,\beta,\gamma;q) = \sum_{r=0}^{n+m} L_r(m,n)P_r(x;\alpha,\beta,\gamma;q)$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{(-1)^r q^{\frac{r(r+1)}{2}+j^2+l}(q;q)_{l+r}(q^{-m};q)_{l+r-j}(q^{-n};q)_j(\alpha q;q)_{l+r}}{q^{j(r+l)}(q;q)_j(q;q)_{l+r-j}(q;q)_r(q;q)_l(\alpha q;q)_j(\gamma q;q)_j}$$
$$\times \frac{(\gamma q;q)_{l+r}(\alpha\beta q^{m+1};q)_{l+r-j}(\alpha\beta q^{n+1};q)_j}{(\alpha q;q)_{l+r-j}(\gamma q;q)_{l+r-j}(\alpha\beta q^{r+1};q)_r(\alpha\beta q^{2r+2};q)_l} {}_3\phi_2\left(\left.\begin{matrix} q^{-j}, q^{l+r-m-j}, \alpha\beta q^{m+l+r-j+1} \\ \alpha q^{l+r-j+1}, \gamma q^{l+r-j+1}\end{matrix}\right| q;q\right).$$

Proof. The proof follows the same procedure as the one of Theorem 2.8 with $x^{\underline{n}}$ replaced by $(x;q)_n$ and the final results are deduced using the sum2qhyper algorithm. □

Theorem 3.24. *For the classical q-orthogonal polynomials represented in the basis $\{(x;q)_n\}$, the following linearization formulae are valid:*
1. big q-Jacobi

$$(x;q)_n(x,q)_m = \sum_{j=0}^{n+m} \frac{(-1)^{n+m+j} q^{-nm+\frac{j(j-1)}{2}} (q;q)_n (q;q)_m (1-\alpha\beta q^{2j+1})}{(q;q)_j} \times$$
$$\sum_{k=\max(m,n,j)}^{n+m} \frac{(-1)^k q^{\binom{n+m-k}{2}} (q;q)_k (\alpha q;q)_k (\gamma q;q)_k P_j(x;\alpha,\beta,\gamma;q)}{(q;q)_{k-n}(q;q)_{k-m}(q;q)_{k-j}(q;q)_{n+m-k}(\alpha\beta q^{j+1};q)_k (1-\alpha\beta q^{k+j+1})},$$

2. q-Hahn

$$(x;q)_n(x,q)_m = \sum_{j=0}^{n+m} \frac{(-1)^{n+m+j} q^{-nm+\frac{j(j-1)}{2}} (q;q)_n (q;q)_m (1-\alpha\beta q^{2j+1})}{(q;q)_j} \times$$
$$\sum_{k=\max(m,n,j)}^{n+m} \frac{(-1)^k q^{\binom{n+m-k}{2}} (q;q)_k (\alpha q;q)_k (q^{-N};q)_k Q_j(x;\alpha,\beta,N|q)}{(q;q)_{k-n}(q;q)_{k-m}(q;q)_{k-j}(q;q)_{n+m-k}(\alpha\beta q^{j+1};q)_k (1-\alpha\beta q^{k+j+1})},$$

3. affine q-Krawtchouk

$$(x;q)_n(x,q)_m = \sum_{j=0}^{n+m} \sum_{k=\max(m,n,j)}^{n+m} \frac{q^{\binom{n+m-k}{2}+\frac{j^2}{2}} (q;q)_n (q;q)_m (q;q)_k (pq;q)_k (q^{-N};q)_k K_j^{aff}(x;p,N;q)}{(-1)^{n+m+k+j} q^{nm+\frac{j}{2}} (q;q)_{k-n}(q;q)_{k-m}(q;q)_{k-j}(q;q)_j (q;q)_{n+m-k}},$$

4. big q-Laguerre

$$(x;q)_n(x,q)_m = \sum_{j=0}^{n+m} \sum_{k=\max(m,n,j)}^{n+m} \frac{q^{\binom{n+m-k}{2}+\frac{j^2}{2}} (q;q)_n (q;q)_m (q;q)_k (\alpha q;q)_k (\beta q;q)_k P_j(x;\alpha,\beta;q)}{(-1)^{n+m+k+j} q^{nm+\frac{j}{2}} (q;q)_{k-n}(q;q)_{k-m}(q;q)_{k-j}(q;q)_j (q;q)_{n+m-k}},$$

5. q-Krawtchouk

$$(x;q)_n(x,q)_m = \sum_{j=0}^{n+m} \frac{(-1)^{n+m+j} q^{\frac{1}{2}j(j-1)-nm} (q;q)_n (q;q)_m}{(q;q)_j (-pq^j;q)_j} \times$$
$$\sum_{k=\max(m,n,j)}^{n+m} \frac{(-1)^k q^{\binom{n+m-k}{2}} (q;q)_k (q^{-N};q)_k K_j(x;p,N;q)}{(q;q)_{k-n}(q;q)_{k-m}(q;q)_{k-j}(q;q)_{n+m-k}(-pq^{2j+1};q)_{k-j}},$$

6. quantum q-Krawtchouk

$$(x;q)_n(x,q)_m = \sum_{j=0}^{n+m} \sum_{k=\max(m,n,j)}^{n+m} \frac{q^{\binom{n+m-k}{2}+\frac{1}{2}(j+1)(j-2k)} (q;q)_n (q;q)_m (q;q)_k (q^{-N};q)_k K_j^{qtm}(x;p,N;q)}{(-1)^{n+m+k+j} q^{nm} (q;q)_{k-n}(q;q)_{k-m}(q;q)_{k-j}(q;q)_j (q;q)_{n+m-k} p^k},$$

7. q-Meixner

$$(x;q)_n(x,q)_m = \sum_{j=0}^{n+m} \sum_{k=\max(m,n,j)}^{n+m} \frac{q^{\binom{n+m-k}{2}+\frac{1}{2}(j+1)(j-2k)} (q;q)_n (q;q)_m (q;q)_k (\beta q;q)_k \gamma^k M_j(x;\beta,\gamma;q)}{(-1)^{n+m+j} q^{nm} (q;q)_{k-n}(q;q)_{k-m}(q;q)_{k-j}(q;q)_j (q;q)_{n+m-k}},$$

8. q-Charlier

$$(x;q)_n(x,q)_m = \sum_{j=0}^{n+m} \sum_{k=\max(m,n,j)}^{n+m} \frac{q^{\binom{n+m-k}{2}+\frac{1}{2}(j+1)(j-2k)}(q;q)_n(q;q)_m(q;q)_k \alpha^k C_j(x;\alpha;q)}{(-1)^{n+m+j} q^{nm}(q;q)_{k-n}(q;q)_{k-m}(q;q)_{k-j}(q;q)_j(q;q)_{n+m-k}},$$

9. Al-Salam-Carlitz II

$$(x;q)_n(x,q)_m = \sum_{j=0}^{n+m} \sum_{k=\max(m,n,j)}^{n+m} \frac{q^{\binom{n+m-k}{2}+\frac{1}{2}k(k-1)+j(j-k)}(q;q)_n(q;q)_m(q;q)_k \alpha^{k-j} V_j^{(\alpha)}(x;q)}{(-1)^{n+m} q^{nm}(q;q)_{k-n}(q;q)_{k-m}(q;q)_{k-j}(q;q)_j(q;q)_{n+m-k}},$$

10. discrete q-Hermite II

$$(x;q)_n(x,q)_m = \sum_{j=0}^{n+m} \sum_{k=\max(m,n,j)}^{n+m} \frac{q^{\binom{n+m-k}{2}+\frac{1}{2}k(k-1)+j(j-k)}(q;q)_n(q;q)_m(q;q)_k (-i)^j \tilde{h}_j(x;q)}{(-1)^{n+m-k} q^{nm}(q;q)_{k-n}(q;q)_{k-m}(q;q)_{k-j}(q;q)_j(q;q)_{n+m-k}}.$$

Proof. The combination of

$$(x;q)_n(x;q)_m = \sum_{k=\max(m,n)}^{n+m} H_k(m,n)(x;q)_k \quad \text{and} \quad (x;q)_k = \sum_{j=0}^{k} I_j(k)p_j(x)$$

yields the linearization formula

$$(x;q)_n(x;q)_m = \sum_{j=0}^{n+m} L_j(m,n)p_j(x), \quad \text{with} \quad L_j(m,n) = \sum_{k=\max(m,n,j)}^{n+m} H_k(m,n)I_j(k).$$

□

3.5.3 Representation Basis $\{(x \ominus 1)_q^n\}$

To solve the linearization problem of q-COP represented in the basis $\{(x \ominus 1)_q^n\}$, the following linearization formula for this basis is necessary.

Proposition 3.25. *For the basis $\{(x \ominus a)_q^n\}$, the linearization formula*

$$(x \ominus a)_q^n (x \ominus a)_q^m = \sum_{k=\max(m,n)}^{n+m} \frac{(q;q)_m(q;q)_n(-a)^{n+m-k} q^{\binom{n+m-k}{2}}}{(q;q)_{k-n}(q;q)_{k-m}(q;q)_{n+m-k}} (x \ominus a)_q^k \tag{3.34}$$

holds.

Proof. Without loss of generality, we suppose $n \geq m$. For every positive integer n

$$(x \ominus a)_q^n = \prod_{j=0}^{n-1}(x - aq^j) = 0 \Leftrightarrow x = aq^j, \ j = 0, 1, \ldots, n-1. \tag{3.35}$$

For $x = a$ and $m \geq 1$, the linearization formula

$$(x \ominus a)_q^n (x \ominus a)_q^m = \sum_{k=0}^{n+m} H_k(m,n)(x \ominus a)_q^k$$

gives thanks to (3.35) $H_0(m,n)=0$, so that

$$(x\ominus a)_q^n(x\ominus a)_q^m=\sum_{k=1}^{n+m}H_k(m,n)(x\ominus a)_q^k.$$

For $x=aq$ and $m\geq 2$ the above equation yields (using (3.35)) $H_1(m,n)(aq\ominus a)_q^1=0$. Since by definition $(aq^n\ominus a)_q^n\neq 0$, it follows that $H_1(m,n)=0$. In a similar way, we can show that

$$H_0(m,n)=H_1(m,n)=\cdots=H_{n-1}(m,n)=0.$$

Therefore

$$(x\ominus a)_q^n(x\ominus a)_q^m=\sum_{k=n}^{n+m}H_k(m,n)(x\ominus a)_q^k=\sum_{k=0}^{m}H_{n+k}(m,n)(x\ominus a)_q^{n+k}.$$

Since

$$(x\ominus a)_q^{n+k}=(x\ominus a)_q^n(x\ominus aq^n)_q^k$$

the latter equation becomes

$$(x\ominus a)_q^m=\sum_{k=0}^{m}H_{n+k}(m,n)(x\ominus aq^n)_q^k. \tag{3.36}$$

For $x=aq^n$ and using (3.35), (3.36) gives $H_n(m,n)=(aq^n\ominus a)_q^m$.
We have

$$D_q(x\ominus a)_q^m=[m]_q(x\ominus a)_q^{m-1}$$

from which for any integer $j=1,2,\ldots,m$,

$$D_q^j(x\ominus a)_q^m=[m]_q[m-1]_q\cdots[m-j+1]_q(x\ominus a)_q^{m-j}.$$

Applying D_q^j to both sides of (3.36), we get

$$\begin{aligned}[m]_q[m-1]_q\cdots[m-j+1]_q(x\ominus a)_q^{m-j}&=\sum_{k=0}^{m}H_{n+k}(m,n)[k]_q[k-1]_q\cdots[k-j+1]_q(x\ominus aq^n)_q^{k-j}\\&=\sum_{k=0,k\neq j}^{m}H_{n+k}(m,n)\frac{[k]_q!}{[k-j]_q!}(x\ominus aq^n)_q^{k-j}+H_{n+j}(m,n)[j]_q!.\end{aligned}$$

For $x=aq^n$, the previous equation yields thanks to (3.35)

$$\begin{aligned}H_{n+j}(m,n)&=\frac{[m]_q[m-1]_q\cdots[m-j+1]_q(aq^n\ominus a)_q^{m-j}}{[j]_q!}\\&=\frac{(q;q)_m(aq^n\ominus a)_q^{m-j}}{(q;q)_j(q;q)_{m-j}}.\end{aligned}$$

We show that

$$(aq^n\ominus a)_q^{m-j}=(aq^n)^{m-j}(q^{-n};q)_{m-j}\overset{(3.33)}{=}\frac{(-a)^{m-j}q^{\binom{m-j}{2}}(q;q)_n}{(q;q)_{n-m+j}},$$

therefore

$$H_{n+j}(m,n)=\frac{(q;q)_m(-a)^{m-j}q^{\binom{m-j}{2}}(q;q)_n}{(q;q)_j(q;q)_{m-j}(q;q)_{n-m+j}}.$$

Substituting j by $k-n$, the result follows. □

Proceeding as in the proof of Theorem 3.23, we get

Theorem 3.26. *For the q-COP represented in the basis $(x\ominus 1)_q^n$, the following linearization formulae are valid:*
1. Al-Salam-Carlitz I

$$U_n^{(\alpha)}(x;q)U_m^{(\alpha)}(x;q)=\sum_{r=0}^{n+m}\sum_{l=0}^{n+m-r}\sum_{j=\max(0,l+r-m)}^{\min(n,l+r)}\frac{(-1)^{n+m}\alpha^{m+n-r}q^{\binom{n}{2}+\binom{m}{2}+l+r}(q^{-n};q)_j}{(q;q)_j(q;q)_r(q;q)_l}$$
$$\times\frac{(q^{-m};q)_{l+r-j}(q;q)_{l+r}}{(q;q)_{l+r-j}}{}_2\phi_1\left(\begin{matrix}q^{-j},q^{-m-j+l+r}\\0\end{matrix}\middle|\,q;\frac{q^{j+1}}{\alpha}\right)U_r^{(\alpha)}(x;q),$$

2. discrete q-Hermite I polynomials

$$h_n(x;q)h_m(x;q)=\sum_{r=0}^{n+m}\sum_{l=0}^{n+m-r}\sum_{j=\max(0,l+r-m)}^{\min(n,l+r)}\frac{(-1)^{r}q^{\binom{n}{2}+\binom{m}{2}+l+r}(q^{-n};q)_j}{(q;q)_j(q;q)_r(q;q)_l}$$
$$\times\frac{(q^{-m};q)_{l+r-j}(q;q)_{l+r}}{(q;q)_{l+r-j}}{}_2\phi_1\left(\begin{matrix}q^{-j},q^{-m-j+l+r}\\0\end{matrix}\middle|\,q;-q^{j+1}\right)h_r(x;q).$$

Theorem 3.27. *For the classical q-orthogonal polynomials represented in the basis $\{(x\ominus 1)_q^n\}$, the following linearization formulae are valid:*
1. Al-Salam-Carlitz I

$$(x\ominus 1)_q^n(x\ominus 1)_q^m=\sum_{j=0}^{n+m}\sum_{k=\max(m,n,j)}^{n+m}\frac{(-1)^{n+m-k}\alpha^{k-j}q^{\binom{n+m-k}{2}}(q,q)_n(q,q)_m(q,q)_k}{(q,q)_{k-n}(q,q)_{k-m}(q,q)_{k-j}(q,q)_{n+m-k}(q,q)_j}U_j^{(\alpha)}(x;q),$$

2. discrete q-Hermite I

$$(x\ominus 1)_q^n(x\ominus 1)_q^m=\sum_{j=0}^{n+m}\sum_{k=\max(m,n,j)}^{n+m}\frac{(-1)^{n+m-j}q^{\binom{n+m-k}{2}}(q,q)_n(q,q)_m(q,q)_k}{(q,q)_{k-n}(q,q)_{k-m}(q,q)_{k-j}(q,q)_{n+m-k}(q,q)_j}h_j(x;q).$$

Proof. The combination of

$$(x\ominus 1)_q^n(x\ominus 1)_q^m=\sum_{k=\max(m,n)}^{n+m}H_k(m,n)(x\ominus 1)_q^k\quad\text{and}\quad(x\ominus 1)_q^k=\sum_{j=0}^{k}I_j(k)p_j(x)$$

yields the linearization formula

$$(x\ominus 1)_q^n(x\ominus 1)_q^m=\sum_{j=0}^{n+m}L_j(m,n)p_j(x),\quad\text{with}\quad L_j(m,n)=\sum_{k=\max(m,n,j)}^{n+m}H_k(m,n)I_j(k).$$

□

3.6 Duplication Problem of q-COP

For q-COP, we propose two different methods to solve the duplication problem

$$p_n(ax)=\sum_{m=0}^{n}D_m(n,a)p_m(x).\tag{3.37}$$

3.6.1 First Method

First we determine a q-differential equation satisfied by $p_n(ax)$. For this purpose, we need the operator ε_a acting on a function f defined by

$$\varepsilon_a f(x) := f(ax).$$

Sprenger [2009] showed that the operator ε_a has the following properties for any constant b, c:

$$\varepsilon_a(bf(x) + cg(x)) = b\varepsilon_a f(x) + c\varepsilon_a g(x), \tag{3.38}$$
$$\varepsilon_a(f(x) \cdot g(x)) = \varepsilon_a f(x) \cdot \varepsilon_a g(x), \tag{3.39}$$
$$\varepsilon_a D_q = \frac{1}{a} D_q \varepsilon_a. \tag{3.40}$$

Using the operator ε_a, we prove

Proposition 3.28. *If a q-COP family $p_n(x)$ is a solution of the q-differential equation (3.2), then $p_n(ax) = \varepsilon_a p_n(x)$ is a solution of the q-differential equation*

$$\sigma(ax) D_q D_{\frac{1}{q}} p_n(ax) + a\tau(ax) D_q p_n(ax) + a^2 \lambda_{n,q} p_n(ax) = 0. \tag{3.41}$$

Proof. We apply the operator ε_a to Equation (3.2) and use the properties (3.38)–(3.40) to get the result. □

Using this q-differential equation, we show

Theorem 3.29. *Let $p_n(x)$ be a q-COP solution of Equation (3.2) with $\deg(\sigma) \leq 1$. The duplication coefficients $D_m(n, A)$ of the duplication problem (3.37) fulfill the recurrence relations*

$$\begin{aligned}&(a'_{m-1} + A^2(d\alpha^\star_{m-1} + \lambda_{n,q}\hat{a}_{m-1}))D_{m-1}(n, A) + (b'_m + A(Ad\beta^\star_m + e + A\lambda_{n,q}\hat{b}_m))D_m(n, A)\\ &+(c'_{m+1} + A^2(d\gamma^\star_{m+1} + \lambda_{n,q}\hat{c}_{m+1}))D_{m+1}(n, A) = 0\end{aligned}$$

if σ is a constant and

$$\begin{aligned}&(a'_{m-1} + A(d\alpha^\star_{m-1} + \lambda_n\hat{a}_{m-1}))D_{m-1}(n, A) + (b'_m + Ad\beta^\star_m + e + A\lambda_n\hat{b}_m)D_m(n, A)\\ &+(c'_{m+1} + A(d\gamma^\star_{m+1} + \lambda_n\hat{c}_{m+1}))D_{m+1}(n, A) = 0\end{aligned}$$

if σ is linear, with initial conditions given by $D_{n+1}(n, A) = 0$ and $D_n(n, A) = A^n$, where the coefficients are given in Theorem 3.5 and Propositions 3.6 and 3.8.

Proof. We substitute (3.37) with $a = A$ in (3.41) to get

$$\sum_{m=0}^{n} D_m(n, A)\Big(\sigma(Ax) D_q D_{\frac{1}{q}} p_m(x) + A\tau(Ax) D_q p_m(x) + A^2 \lambda_n p_m(x)\Big) = 0 \tag{3.42}$$

and proceed as in Subsection 3.4.1.

First, for $p_m(x)$ Equation (3.15) can be rewritten as

$$p_m(x) = \sum_{j=m-1}^{m+1} a_{m,j} D_q p_m(x) \tag{3.43}$$

with $a_{m,m+1} = \hat{a}_m$, $a_{m,m} = \hat{b}_m$ and $a_{m,m-1} = \hat{c}_m$.

Second, the three-term recurrence equation for the family $D_q p_m(x)$ yields

$$\tau(Ax) D_q p_m(x) = \sum_{j=m-1}^{m+1} a^{(1)}_{m,j} D_q p_m(x) \tag{3.44}$$

with $a^{(1)}_{m,m+1} = Ad\alpha^\star_m$, $a^{(1)}_{m,m} = Ad\beta^\star_m + e$, $a^{(1)}_{m,m-1} = Ad\gamma^\star_m$.

At this stage, only two cases arise for which we can use the structure relation (3.18).

1. **Case σ is a constant**

   In this case, since $\sigma(Ax) = \sigma(x)$, from (3.18) we have

$$\sigma(Ax) D_q D_{\frac{1}{q}} p_m(x) = \sigma(x) D_q D_{\frac{1}{q}} p_m(x) = \sum_{j=m-1}^{m+1} a^{(2)}_{m,j} D_q q_m(x) \tag{3.45}$$

   with $a^{(2)}_{m,m+1} = a'_m$, $a^{(2)}_{m,m} = b'_m$ and $a^{(2)}_{m,m-1} = c'_m$.

2. **Case σ is linear**

   In this case, since $\sigma(Ax) = A\sigma(x)$, from (3.18) we have

$$\sigma(Ax) D_q D_{\frac{1}{q}} p_m(x) = A\sigma(x) D_q D_{\frac{1}{q}} p_m(x) = \sum_{j=m-1}^{m+1} a^{(2)}_{m,j} D_q q_m(x) \tag{3.46}$$

   with $a^{(2)}_{m,m+1} = Aa'_m$, $a^{(2)}_{m,m} = Ab'_m$ and $a^{(2)}_{m,m-1} = Ac'_m$.

Substitution of (3.43), (3.44) and (3.45) or (3.46) into (3.42) gives

$$\sum_{m=0}^{n} D_m(n,A) \Big\{ \sum_{j=m-1}^{m+1} \Lambda_{m,j}(n) D_q q_j(x) \Big\} = 0, \;\; \Lambda_{m,j}(n) = a^{(2)}_{m,j} + Aa^{(1)}_{m,j} + A^2 \lambda_n a_{m,j}.$$

Finally, after an appropriate shift of indices, the results follow. □

The q-Petkovšek-van-Hoeij algorithm yields for some of these recurrence equations the following results.

Corollary 3.30. *The duplication formulae of the q-Charlier, q-Laguerre, Stieltjes-Wigert and discrete q-Hermite II polynomials are given, respectively, by*

$$C_n(ax;\alpha;q) = \sum_{m=0}^{n} a^n \Big(-q^{\frac{m-n+1}{2}} \Big)^{m-n} \begin{bmatrix} n \\ m \end{bmatrix}_q \Big(\frac{q^{m+1}}{aq^n}; q \Big)_{n-m} \Big(-\frac{q^{m+1}}{\alpha}; q \Big)_{n-m} C_m(x;\alpha;q),$$

$$L^{(\alpha)}_n(ax;q) = \sum_{m=0}^{n} a^n \Big(-q^{\frac{m-n+1}{2}} \Big)^{m-n} \frac{(q^{m+\alpha+1};q)_{n-m} \Big(\frac{q^{m+1}}{aq^n}; q \Big)_{n-m}}{(q;q)_{n-m}} L^{(\alpha)}_m(x;q),$$

$$S_n(ax;q) = \sum_{m=0}^{n} \frac{a^n}{(q;q)_{n-m}} \Big(-q^{\frac{m-n+1}{2}} \Big)^{m-n} \Big(\frac{q^{m+1}}{aq^n}; q \Big)_{n-m} S_m(x;q),$$

$$\tilde{h}_{2n}(ax;q) = \sum_{k=0}^{n} \frac{(-1)^{n-k} a^{2n} q^{(k-n)(2k+2n+1)} (q;q)_{2n} (q^{-2n};q^2)_k \left(\frac{q^2}{a^2 q^{2n}};q^2\right)_n}{(q;q)_{2k} (q^{-2n};q^2)_n \left(\frac{q^2}{a^2 q^{2n}};q^2\right)_k} \tilde{h}_{2k}(x;q),$$

$$\tilde{h}_{2n+1}(ax;q) = \sum_{k=0}^{n} \frac{(-1)^{n-k} a^{2n+1} q^{(k-n)(2k+2n+3)} (q^2;q)_{2n} (q^{-2n};q^2)_k \left(\frac{q^2}{a^2 q^{2n}};q^2\right)_n}{(q^2;q)_{2k} (q^{-2n};q^2)_n \left(\frac{q^2}{a^2 q^{2n}};q^2\right)_k} \tilde{h}_{2k+1}(x;q).$$

For the Al-Salam-Carlitz II polynomials for which $\sigma(x) = (q-1)\alpha$, we proved that the duplication coefficient $D_m(n,a)$ satisfies the recurrence equation

$$\begin{aligned}(q^m - q^n) a^2 q^{2m+3} D_m(n,a) + a q^{m+2} (\alpha+1)(q^{m+1}-1)(-q^{m+1} + a q^n) D_{m+1}(n,a) \\ -\alpha\,(q^{2+m}-1)(q^{m+1}-1)(-q^{m+2} + a^2 q^n) D_{m+2}(n,a) = 0,\end{aligned} \tag{3.47}$$

with the initial values $D_n(n,a) = a^n,\ D_{n+s}(n,a) = 0,\ s = 1,2$, which according to the q-Petkovšek-van-Hoeij algorithm doesn't have a q-hypergeometric term solution.

3.6.2 Second Method

Due to the four bases in which the q-COP are represented, we consider here three cases.

Duplication Coefficients of q-Orthogonal Polynomials Expanded in the Basis $\{x^n\}$

We suppose that the q-COP $p_n(x)$ are expanded in the basis x^n, i.e.

$$p_n(x) = \sum_{j=0}^{n} A_j(n) x^j,$$

so that

$$p_n(ax) = \sum_{j=0}^{n} A_j(n) a^j x^j.$$

We combine the latter expression with the inversion formula

$$x^j = \sum_{m=0}^{j} I_m(j) p_m(x)$$

to get

$$p_n(ax) = \sum_{m=0}^{n} D_m(n,a) p_m(x) \quad \text{with} \quad D_m(n,a) = \sum_{j=0}^{n-m} a^{j+m} A_{j+m}(n) I_m(j+m).$$

By interchanging the order of summation, this relation becomes

$$p_n(ax) = \sum_{k=0}^{n} D_k(n,a) p_{n-k}(x) \quad \text{with} \quad D_k(n,a) = \sum_{j=0}^{k} a^{j+n-k} A_{j+n-k}(n) B_{n-k}(j+n-k).$$

It follows that

Theorem 3.31. *The duplication formulae of the little q-Jacobi, alternative q-Charlier and little q-Laguerre polynomials represented in the basis $\{x^n\}$ are given, respectively, by:*

$$p_n(ax;\alpha,\beta|q)=\sum_{m=0}^{n}\frac{(-a)^m q^{\frac{m(m+1)}{2}}(\alpha\beta q^{n+1};q)_m(q^{-n};q)_m}{(\alpha\beta q^{m+1};q)_m(q;q)_m}\,{}_2\phi_1\left(\left.\begin{matrix}q^{m-n},\alpha\beta q^{m+n+1}\\ \alpha\beta q^{2m+2}\end{matrix}\right| q;aq\right)p_m(x;\alpha,\beta|q),$$

$$y_n(ax;\alpha;q)=\sum_{m=0}^{n}\frac{(-a)^m q^{\frac{m(m+1)}{2}}(q^{-n};q)_m(-\alpha q^{n};q)_m}{(q;q)_m(-\alpha q^{m};q)_m}\,{}_2\phi_1\left(\left.\begin{matrix}q^{m-n},-\alpha q^{m+n}\\ -\alpha q^{2m+1}\end{matrix}\right| q;aq\right)y_m(x;\alpha;q),$$

$$p_n(ax;\alpha|q)=\sum_{m=0}^{n}a^n q^{n-m}\frac{(q^{-n};q)_{n-m}(a^{-1};q)_{n-m}}{(q;q)_{n-m}}p_m(x;\alpha|q).$$

Duplication Coefficients of q-Orthogonal Polynomials Expanded in the Bases $\{(x;q)_n\}$ or $\{(ix;q)_n\}$

To solve the duplication problem in these cases, we need the following result given in [Gasper and Rahman, 1990, p. 20] as exercice.

Lemma 3.32. *The duplication formula of the basis $\{(x;q)_n\}$ is given by*

$$(ax;q)_n=\sum_{m=0}^{n}\begin{bmatrix}n\\ m\end{bmatrix}_q a^m(a;q)_{n-m}(x;q)_m. \tag{3.48}$$

Proof. Since

$$D_q(ax;q)_n=-a[n]_q(aqx;q)_{n-1},$$

it follows by iteration that

$$D_q^k(ax;q)_n=(-a)^k q^{\binom{k}{2}}\frac{(q^{n-k+1};q)_k}{(1-q)^k}(aq^kx;q)_{n-k}.$$

In order to obtain (3.48) we apply the operator D_q^k to both sides of the relation $(ax;q)_n=\sum\limits_{m=0}^{n}D_m(n,a)(x;q)_m$. This yields

$$a^k(q^{n-k+1};q)_k(aq^kx;q)_{n-k}=\sum_{m=k}^{n}D_m(n,a)(q^{m-k+1};q)_k(q^kx;q)_{m-k}.$$

For $x=q^{-k}$, since $(1;q)_k=0,\ k\neq 0$, the latter equation gives the result. □

Using the representation of $p_n(x)$ in the basis $(x;q)_n$, Equation (3.48) and the inversion formulae of these polynomial systems, we prove

Theorem 3.33. *For the classical q-orthogonal polynomials represented in the basis $\{(x;q)_n\}$, the following duplication formulae are valid:*
1. Al-Salam-Carlitz II

$$V_n^{(\alpha)}(ax;q)=\sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-1)^{n-j}a^m\alpha^{n-m-j}q^{nj+nm-jm}(q^{-n};q)_{m+j}(a;q)_j}{q^{\binom{n}{2}+\binom{j}{2}}(q;q)_j(q;q)_m}\,{}_1\phi_1\left(\left.\begin{matrix}q^{-j}\\ \frac{q^{1-j}}{a}\end{matrix}\right| q;\alpha q\right)V_m^{(\alpha)}(x;q),$$

2. q-Meixner

$$M_n(ax;\beta,\gamma;q)=\sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-1)^{m+j}a^m q^{mn+nj+j}(a;q)_j(q^{-n};q)_m(q^{m-n};q)_j}{q^{\binom{m}{2}}\gamma^j(\beta q^{m+1};q)_j(q;q)_j(q;q)_m}$$
$$\times{}_2\phi_1\left(\begin{array}{c}q^{-j},\beta q^{m+1}\\ \frac{q}{aq^j}\end{array}\middle|\,q;-\frac{\gamma}{q^m}\right)M_m(x;\beta,\gamma;q),$$

3. quantum q-Krawtchouk

$$K_n^{qtm}(ax;p,N;q)=\sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-a)^m p^j q^{mn+nj+j}(q^{-n};q)_{j+m}(a;q)_j(q^{-N};q)_m}{q^{\binom{m}{2}}(q^{-N};q)_{j+m}(q;q)_j(q;q)_m}$$
$$\times{}_2\phi_1\left(\begin{array}{c}q^{-j},q^{m-N}\\ \frac{q^{1-j}}{a}\end{array}\middle|\,q;\frac{1}{pq^m}\right)K_m^{qtm}(x;p,N;q),$$

4. q-Krawtchouk

$$K_n(ax;p,N;q)=\sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-a)^m q^{j+\frac{m(m+1)}{2}}(q^{-n};q)_{m+j}(-pq^n;q)_{m+j}(a;q)_j(q^{-N};q)_m}{(q^{-N};q)_{m+j}(q;q)_j(q;q)_m(-pq^m;q)_m}$$
$$\times{}_3\phi_2\left(\begin{array}{c}q^{-j},q^{m-N},0\\ \frac{q^{1-j}}{a},-pq^{2m+1}\end{array}\middle|\,q;q\right)K_m(x;p,N;q),$$

5. big q-Laguerre

$$P_n(ax;\alpha,\beta;q)=\sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-a)^m q^{j+\frac{m(m+1)}{2}}(a;q)_j(q^{-n};q)_m(q^{m-n};q)_j}{(\alpha q^{m+1};q)_j(\beta q^{m+1};q)_j(q;q)_j(q;q)_m}$$
$$\times{}_3\phi_2\left(\begin{array}{c}q^{-j},\alpha q^{m+1},\beta q^{m+1}\\ \frac{q}{aq^j},0\end{array}\middle|\,q;q\right)P_m(x;\alpha,\beta;q),$$

6. affine q-Krawtchouk

$$K_n^{aff}(ax;p,N;q)=\sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-a)^m q^{j(N+1)+\frac{m(m+1)}{2}}(\frac{q}{q^{m+j}};q)_N(a;q)_j(q^{-n};q)_m(q^{m-n};q)_j}{(pq^{m+1};q)_j(q;q)_j(q;q)_m(q^m;q)_j(q^{1-m};q)_N}$$
$$\times{}_3\phi_2\left(\begin{array}{c}q^{-j},q^{m-N},pq^{m+1}\\ \frac{q}{aq^j},0\end{array}\middle|\,q;q\right)K_m^{aff}(x;p,N;q),$$

7. q-Hahn

$$Q_n(ax;\alpha,\beta,N|q)=\sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-a)^m q^{j+\frac{m(m+1)}{2}}(q^{m-n};q)_j(q^{-n};q)_m(a;q)_j(\alpha\beta q^{n+1};q)_{m+j}}{(q;q)_j(q;q)_m(q^{m-N};q)_j(\alpha q^{m+1};q)_j(\alpha\beta q^{m+1};q)_m}$$
$$\times{}_3\phi_2\left(\begin{array}{c}q^{-j},q^{m-N},\alpha q^{m+1}\\ \frac{q}{aq^j},\alpha\beta q^{2m+2}\end{array}\middle|\,q;q\right)Q_m(x;\alpha,\beta,N|q),$$

8. big q-Jacobi

$$P_n(ax;\alpha,\beta,\gamma;q)=\sum_{m=0}^{n}\sum_{j=0}^{n-m}\frac{(-a)^m q^{j+\frac{m(m+1)}{2}}(q^{m-n};q)_j(q^{-n};q)_m(a;q)_j(\alpha\beta q^{n+1};q)_{m+j}}{(q;q)_j(q;q)_m(\alpha q^{m+1};q)_j(\gamma q^{m+1};q)_j(\alpha\beta q^{m+1};q)_m}$$
$$\times{}_3\phi_2\left(\left.\begin{matrix}q^{-j},\alpha q^{m+1},\gamma q^{m+1}\\ \frac{q}{aq^j},\alpha\beta q^{2m+2}\end{matrix}\right|q;q\right)P_m(x;\alpha,\beta,\gamma;q).$$

Proof. Combining

$$p_n(ax)=\sum_{j=0}^{n}A_j(n)(ax;q)_j,\ \ (ax;q)_j=\sum_{k=0}^{j}B_k(j)(x;q)_k\ \text{ and }\ (x;q)_k=\sum_{m=0}^{k}I_m(k)p_m(x)$$

and interchanging the order of summation gives

$$p_n(ax)=\sum_{m=0}^{n}D_m(n,a)p_m(x)$$

with

$$D_m(n,a)=\sum_{j=0}^{n-m}\sum_{k=0}^{j}A_{j+m}(n)B_{m+k}(j+m)I_m(k+m).$$

□

Riese [2003] developed an algorithm which finds recurrences for q-hypergeometric multiple sums and implemented it in Mathematica in his package *qMultisum*. Using this algorithm, we recover recurrence relation (3.47) satisfied by the duplication coefficients of the Al-Salam-Carlitz II polynomials. Moreover we get

Proposition 3.34. *The following recurrence relations are satisfied by the duplication coefficients of:*
1. q-Meixner polynomials

$$\begin{aligned}&-\gamma a^2q^6\left(q^n-q^m\right)\left(1-q^{1+m}\beta\right)\left(1-q^{2+m}\beta\right)D_m(n,a)-aq^3\left(1-q^{1+m}\right)\left(1-q^{2+m}\beta\right)\Big(q^{2m+4}\\&-aq^{m+n+3}+a\gamma q^{m+1}(1+q)+\gamma q^{m+3}-a\gamma q^n(1+q+q^2)-\beta\gamma q^{2m+4}+a\beta\gamma q^{m+n+3}\Big)D_{m+1}(n,a)\\&+q\left(1-q^{1+m}\right)\left(1-q^{2+m}\right)\Big((aq^{5+2m}-a^2q^{3+m+n}+a\gamma q^{m+3}-a\beta\gamma q^{2m+5}+a^2\beta\gamma q^{m+n+3})(1+q)\\&-q^{8+3m}\beta+a^2q^{6+2m+n}\beta+a^2q^{2+m}\gamma-a^2\gamma q^n(1+q+q^2)\Big)D_{m+2}(n,a)\\&-a\left(-1+q^{1+m}\right)\left(-1+q^{2+m}\right)\left(-1+q^{3+m}\right)\left(-q^{3+m}+aq^n\right)\left(q^{3+m}+\gamma\right)D_{3+m}(n,a)=0,\end{aligned}$$

2. quantum q-Krawtchouk polynomials

$$\begin{aligned}&a^2q^6\left(q^m-q^n\right)\left(q^{1+m}-q^N\right)\left(q^N-q^m\right)D_m(n,a)+aq^3\left(q^{1+m}-1\right)\left(q^{1+m}-q^N\right)\Big(apq^{3+m+n+N}\\&+aq^{2+m+n}-q^{3+2m}+q^{3+m+N}-pq^{4+2m+N}-aq^{n+N}(1+q+q^2)+aq^{1+m+N}(1+q)\Big)D_{m+1}(n,a)\\&+q^{1+N}\left(-1+q^{1+m}\right)\left(-1+q^{2+m}\right)\Big(-pq^{7+3m}+a^2pq^{5+2m+n}-a^2q^{2+m+N}+a^2q^{n+N}(1+q+q^2)\\&+(aq^{4+2m}-a^2q^{2+m+n}-aq^{3+m+N}+apq^{5+2m+N}-a^2pq^{3+m+n+N})(1+q)\Big)D_{m+2}(n,a)\\&-aq^{2N}\left(-1+q^{1+m}\right)\left(-1+q^{2+m}\right)\left(-1+q^{3+m}\right)\left(-1+pq^{3+m}\right)\left(-q^{3+m}+aq^n\right)D_{m+3}(n,a)=0,\end{aligned}$$

with the initial conditions $D_n(n,a)=a^n,\ D_{n+s}(n,a)=0,\ \ s=1,2,3.$

Duplication Coefficients of q-Orthogonal Polynomials Represented in the Basis $\{(x \ominus 1)_q^n\}$

Theorem 3.35. *For the classical q-orthogonal polynomial represented in the basis $(x\ominus 1)_q^n$, the following duplication formulae hold:*
1. Al-Salam-Carlitz I

$$U_n^{(\alpha)}(ax;q) = \sum_{m=0}^{n}\sum_{j=0}^{n-m} \frac{(-1)^n a^{m+j}\alpha^{n-m-j}q^{\binom{n}{2}+m+j}(q^{-n};q)_{m+j}(\frac{1}{a};q)_j}{(q;q)_j(q;q)_m}\, {}_2\phi_1\left(\left.\begin{matrix} q^{-j},0 \\ aq^{1-j}\end{matrix}\right| q;a\alpha q\right) U_m^{(\alpha)}(x;q),$$

2. q-Hermite I

$$h_{2n}(ax;q) = \sum_{k=0}^{n} \frac{a^{2k}q^{2(k-n)}(q;q)_{2n}(q^{-2n};q^2)_k(a^2q^{-2(n-1)};q^2)_n}{(q;q)_{2k}(q^{-2n};q^2)_n(a^2q^{-2(n-1)};q^2)_k} h_{2k}(x;q),$$

$$h_{2n+1}(ax;q) = \sum_{k=0}^{n} \frac{a^{2k+1}q^{2(k-n)}(q^2;q)_{2n}(q^{-2n};q^2)_k(a^2q^{-2(n-1)};q^2)_n}{(q^2;q)_{2k}(q^{-2n};q^2)_n(a^2q^{-2(n-1)};q^2)_k} h_{2k+1}(x;q).$$

For the Al-Salam-Carlitz I polynomials $U_n^{(\alpha)}(x;q)$ the duplication coefficients are solutions of the recurrence equation

$$\begin{aligned} &a^2q^2(q^n-q^m)D_m(n,a) + aq(\alpha+1)(1-q^{m+1})(q^n-aq^{m+1})D_{m+1}(n,a) \\ &\quad +\alpha(1-q^{m+1})(1-q^{m+2})(q^n-a^2q^{m+2})D_{m+2}(n,a) = 0, \end{aligned} \tag{3.49}$$

with the initial conditions $D_n(n,a) = a^n$, $D_{n+s}(n,a) = 0$, $s = 1,2$.

The proof of this theorem needs the following

Lemma 3.36. *The duplication formula*

$$(ax\ominus 1)_q^n = \sum_{m=0}^{n} a^n\left(\frac{1}{a};q\right)_{n-m}\begin{bmatrix} n \\ m\end{bmatrix}_q (x\ominus 1)_q^m$$

for the basis $\{(x\ominus 1)_q^n\}$ is valid.

Proof. We have

$$D_q(ax\ominus 1)_q^n = a[n]_q(ax\ominus 1)_q^{n-1}$$

so that

$$D_q^k(ax\ominus 1)_q^n = a^k[n]_q[n-1]_q\cdots[n-k+1]_q(ax\ominus 1)_q^{n-k},\ k\le n.$$

We apply the operator D_q^k to both sides of the relation

$$(ax\ominus 1)_q^n = \sum_{m=0}^{n} D_m(n,a)(x\ominus 1)_q^m$$

and use the previous relation to get

$$\begin{aligned} a^k[n]_q[n-1]_q\cdots[n-k+1]_q(ax\ominus 1)_q^{n-k} &= \sum_{m=0}^{n} D_m(n,a)[m]_q[m-1]_q\cdots[m-k+1]_q(x\ominus 1)_q^{m-k} \\ &= \sum_{m=0,m\neq k}^{n} D_m(n,a)\frac{[m]_q!}{[m-k]_q!}(x\ominus 1)_q^{m-k} + D_k(n,a)[k]_q!. \end{aligned}$$

For $x = 1$, since $(1 \ominus 1)_q^k = 0,\ k \neq 0$, it follows that

$$D_k(n,a) = a^k (a \ominus 1)_q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

The result follows from $(a \ominus 1)_q^{n-k} = a^{n-k}(\frac{1}{a}; q)_{n-k}$. □

Proof (of Theorem 3.35). Combining

$$p_n(ax) = \sum_{j=0}^{n} A_j(n)(ax \ominus 1)_q^j,\ (ax \ominus 1)_q^j = \sum_{k=0}^{j} B_k(j)(x \ominus 1)_q^k \text{ and } (x \ominus 1)_q^k = \sum_{m=0}^{k} I_m(k) p_m(x)$$

and interchanging the order of summation gives

$$p_n(ax) = \sum_{m=0}^{n} D_m(n,a) p_m(x)$$

with

$$D_m(n,a) = \sum_{j=0}^{n-m} \sum_{k=0}^{j} A_{j+m}(n) B_{m+k}(j+m) I_m(k+m).$$

Using once more Riese's algorithm [Riese, 2003], we get the recurrence relations (3.49) and

$$a^2 q^2 (q^n - q^m) D_m(n,a) + (1 - q^{m+1})(1 - q^{m+2})(a^2 q^{m+2} - q^n) D_{m+2}(n,a) = 0$$

satisfied, respectively, by the Al-Salam-Carlitz I and the discrete q-Hermite I polynomials. We solve the latter recurrence relation for m odd and even to get the duplication coefficients of the discrete q-Hermite I polynomials. According to the q-Petkovšek-van-Hoeij algorithm, the recurrence relation (3.49) doesn't have a q-hypergeometric term solution. □

Chapter 4

Connection, Linearization and Duplication Coefficients of Orthogonal Polynomials on Quadratic and q-Quadratic Lattices

Area et al. [2001] used the formula

$$
{}_r\phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\middle| q; yx\right) = \sum_{j=0}^{\infty} \frac{[(-1)^j q^{\binom{j}{2}}]^2 y^j}{(q;q)_j}\, {}_1\phi_1\left(\begin{matrix}0\\ 0\end{matrix}\middle| q; q^j y\right) {}_{r+1}\phi_s\left(\begin{matrix}q^{-j}, a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\middle| q; qx\right), \tag{4.1}
$$

obtained from Verma's q-extension [Verma, 1966] of the Fields and Wimp [Fields and Wimp, 1961] expansion of

$$
{}_{r+t}\phi_{s+u}\left(\begin{matrix}a_1,\ldots,a_r, c_1,\ldots,c_t\\ b_1,\ldots,b_s, d_1,\ldots,d_u\end{matrix}\middle| q; yw\right)
$$

in powers of yw as given in [Gasper and Rahman, 1990, Eq. (3.7.9)], to derive the inversion formula of the Askey-Wilson polynomials from which the Askey-Wilson connection formula with the same first parameter follows. Using an integral evaluation, Ismail and Rahman [2011] solved the Askey-Wilson connection problem in a more general setting (with no need for the first parameters to be identical). Using the Fields and Wimp [1961] expansion formula (1.40), Sánchez-Ruiz and Dehesa [2001] derived the connection formulae of the Wilson and Racah polynomials. In this chapter, we use an algorithmic approach to derive the inversion, the connection, the linearization and the duplication coefficients of the classical orthogonal polynomials on quadratic and q-quadratic lattices. We recover known connection formulae and moreover we get new connection and linearization relations. From these connection formulae, we get the parameter derivatives of the above mentioned polynomial families. The duplication formulae, the parameter derivatives are also completely new, as far as we know.

4.1 Introduction

Classical orthogonal polynomials on a quadratic and q-quadratic lattice [Nikiforov and Uvarov, 1988], [Magnus, 1995], [Koekoek et al., 2010] are known to satisfy a divided-difference equation of the type [Nikiforov and Uvarov, 1988], [Suslov, 1989], [Atakishiyev et al., 1995], [Foupouagnigni, 2008],

$$\left\{\phi(x(s))\frac{\Delta}{\nabla x_1(s)}\frac{\nabla}{\nabla x(s)}+\frac{\psi(x(s))}{2}\left[\frac{\Delta}{\Delta x(s)}+\frac{\nabla}{\nabla x(s)}\right]+\lambda_n\right\}p_n(x(s))=0,\ n\geq 0,\quad (4.2)$$

where $\phi(x(s))=\phi_2x^2(s)+\phi_1x(s)+\phi_0$ and $\psi(x(s))=\psi_1x(s)+\psi_0$ are polynomials of maximal degree two and one respectively, λ_n is a constant depending on the integer n and the leading coefficients ϕ_2 and ψ_1 of ϕ and ψ respectively, and $x(s)$ is a quadratic or q-quadratic lattice defined by [Magnus, 1995]

$$x(s)=\begin{cases}c_1q^s+c_2q^{-s}+c_3 & \text{if } q\neq 1,\\ c_4s^2+c_5s+c_6 & \text{if } q=1,\end{cases}$$

with

$$x_\mu(s)=x\left(s+\frac{\mu}{2}\right),\qquad \mu,\ c_1,\ \ldots,\ c_6\in\mathbb{C}.$$

Foupouagnigni [2008] showed by means of the companion operators $\mathbb{D}_x$ and $\mathbb{S}_x$, given by

$$\mathbb{D}_xf(x(s))=\frac{f(x(s+\frac{1}{2}))-f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2})-x(s-\frac{1}{2})},\quad \mathbb{S}_xf(x(s))=\frac{f(x(s+\frac{1}{2}))+f(x(s-\frac{1}{2}))}{2},$$

that Equation (4.2), which characterizes the classical orthogonal polynomials on a quadratic or q-quadratic lattice [Foupouagnigni et al., 2011], can be rewritten as

$$\phi(x(s))\mathbb{D}_x^2p_n(x(s))+\psi(x(s))\mathbb{S}_x\mathbb{D}_xp_n(x(s))+\lambda_np_n(x(s))=0.\qquad (4.3)$$

We note that the divided-difference operator $\mathbb{D}_x$ is equal to the Askey-Wilson operator $\mathcal{D}_q$ (see e.g. [Askey and Wilson, 1985])

$$\mathbb{D}_xf(x(s))=\mathcal{D}_qf(x):=\frac{\breve{f}(q^{1/2}e^{i\theta})-\breve{f}(q^{-1/2}e^{i\theta})}{\breve{e}(q^{1/2}e^{i\theta})-\breve{e}(q^{-1/2}e^{i\theta})},$$

with $x=x(s)=\cos\theta=\dfrac{q^s+q^{-s}}{2}$, $q^s=e^{i\theta}$, where for a function f defined on $(-1,1)$ we have $\breve{f}(e^{i\theta}):=f(x)$, that is

$$\breve{f}(z)=f((z+1/z)/2),\quad z=e^{i\theta},$$

and $\breve{e}(x)=x$. Throughout this chapter, we assume that $0<q<1$. It is not difficult to see that if f is differentiable at x, then

$$\lim_{q\to 1}(\mathcal{D}_qf)(x)=f'(x).$$

The hypergeometric and the basic hypergeometric representations of classical orthogonal polynomials on a quadratic or q-quadratic lattice are given below (see [Koekoek et al.,

2010]):
1. Askey-Wilson

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left(\left.\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}\right| q;q\right), \quad x=\cos\theta,$$

2. q-Racah

$$R_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = {}_4\phi_3\left(\left.\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \gamma\delta q^{x+1} \\ \alpha q, \beta\delta q, \gamma q \end{matrix}\right| q;q\right), \quad n=0,1,\ldots,N,$$

where

$$\mu(x) := q^{-x} + \gamma\delta q^{x+1}$$

and

$$\alpha q = q^{-N} \text{ or } \beta\delta q = q^{-N} \text{ or } \gamma q = q^{-N},$$

with a nonnegative integer N,
3. Continuous dual q-Hahn

$$p_n(x;a,b,c|q) = \frac{(ab,ac;q)_n}{a^n}\,{}_3\phi_2\left(\left.\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac \end{matrix}\right| q;q\right), \quad x=\cos\theta,$$

4. Continuous q-Hahn

$$p_n(x;a,b,c,d;q) = \frac{(abe^{2i\hat{\theta}},ac,ad;q)_n}{(ae^{i\hat{\theta}})^n}\,{}_4\phi_3\left(\left.\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i(\theta+2\hat{\theta})}, ae^{-i\theta} \\ abe^{2i\hat{\theta}}, ac, ad \end{matrix}\right| q;q\right), \quad x=\cos(\theta+\hat{\theta}),$$

5. Dual q-Hahn

$$R_n(\mu(x);\gamma,\delta,N|q) = {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, q^{-x}, \gamma\delta q^{x+1} \\ \gamma q, q^{-N} \end{matrix}\right| q;q\right), \quad n=0,1,\ldots,N,$$

where $\mu(x) := q^{-x} + \gamma\delta q^{x+1}$,
6. Al-Salam-Chihara

$$Q_n(x;a,b|q) = \frac{(ab;q)_n}{a^n}\,{}_3\phi_2\left(\left.\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix}\right| q;q\right), \quad x=\cos\theta,$$

7. q-Meixner-Pollaczek

$$P_n(x;a|q) = a^{-n}e^{-in\hat{\theta}}\frac{(a^2;q)_n}{(q;q)_n}\,{}_3\phi_2\left(\left.\begin{matrix} q^{-n}, ae^{i(\theta+2\hat{\theta})}, ae^{-i\theta} \\ a^2, 0 \end{matrix}\right| q;q\right), \quad x=\cos(\theta+\hat{\theta}),$$

8. Continuous q-Jacobi

$$P_n^{(\alpha,\beta)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_4\phi_3\left(\left.\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{\alpha}{2}+\frac{1}{4}}e^{i\theta}, q^{\frac{\alpha}{2}+\frac{1}{4}}e^{-i\theta} \\ q^{\alpha+1}, -q^{\frac{\alpha+\beta+1}{2}}, -q^{\frac{\alpha+\beta+2}{2}} \end{matrix}\right| q;q\right), \quad x=\cos\theta,$$

9. Continuous q-Ultraspherical / Rogers

$$C_n(x;\beta|q)=\frac{(\beta^2;q)_n}{(q;q)_n}\beta^{-\frac{n}{2}}{}_4\phi_3\left(\begin{matrix}q^{-n},\beta^2q^n,\beta^{\frac{1}{2}}e^{i\theta},\beta^{\frac{1}{2}}e^{-i\theta}\\ \beta q^{\frac{1}{2}},-\beta,-\beta q^{\frac{1}{2}}\end{matrix}\middle| q;q\right),\ x=\cos\theta,$$

10. Dual q-Krawtchouk

$$K_n(\lambda(x);c,N|q)={}_3\phi_2\left(\begin{matrix}q^{-n},q^{-x},cq^{x-N}\\ q^{-N},0\end{matrix}\middle| q;q\right),\ n=0,1,\ldots,N,$$

where $\lambda(x):=q^{-x}+cq^{x-N}$,
11. Continuous big q-Hermite

$$H_n(x;a|q)=a^{-n}{}_3\phi_2\left(\begin{matrix}q^{-n},ae^{i\theta},ae^{-i\theta}\\ 0,0\end{matrix}\middle| q;q\right),\ x=\cos\theta,$$

12. Continuous q-Laguerre

$$P_n^{(\alpha)}(x|q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_3\phi_2\left(\begin{matrix}q^{-n},q^{\frac{\alpha}{2}+\frac{1}{4}}e^{i\theta},q^{\frac{\alpha}{2}+\frac{1}{4}}e^{-i\theta}\\ q^{\alpha+1},0\end{matrix}\middle| q;q\right),\ x=\cos\theta,$$

13. Wilson

$$W_n(x^2;a,b,c,d)=(a+b)_n(a+c)_n(a+d)_n{}_4F_3\left(\begin{matrix}-n,n+a+b+c+d-1,a+ix,a-ix\\ a+b,a+c,a+d\end{matrix}\middle| 1\right),$$

14. Racah

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta)={}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-x,x+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix}\middle| 1\right),\ n=0,1,\ldots,N,$$

where

$$\lambda(x)=x(x+\gamma+\delta+1)$$

and

$$\alpha+1=-N\text{ or }\beta+\delta+1=-N\text{ or }\gamma+1=-N$$

with a nonnegative integer N,
15. Continuous dual Hahn

$$S_n(x^2;a,b,c)=(a+b)_n(a+c)_n{}_3F_2\left(\begin{matrix}-n,a+ix,a-ix\\ a+b,a+c\end{matrix}\middle| 1\right),$$

16. Continuous Hahn

$$p_n(x;a,b,c,d)=i^n\frac{(a+c)_n(a+d)_n}{n!}{}_3F_2\left(\begin{matrix}-n,n+a+b+c+d-1,a+ix\\ a+c,a+d\end{matrix}\middle| 1\right),$$

17. Dual Hahn

$$R_n(\lambda(x);\gamma,\delta,N) = {}_3F_2\left(\begin{matrix}-n,-x,x+\gamma+\delta+1\\ \gamma+1,-N\end{matrix}\middle|\,1\right),\ n=0,1,\dots,N,$$

where $\lambda(x)=x(x+\gamma+\delta+1)$,
18. Meixner-Pollaczek

$$P_n^{(\lambda)}(x;\theta) = \frac{(2\lambda)_n}{n!}e^{in\theta}{}_2F_1\left(\begin{matrix}-n,\lambda+ix\\ 2\lambda\end{matrix}\middle|\,1-e^{-2i\theta}\right).$$

Let us set

$$B_n(a,x) = (aq^s;q)_n(aq^{-s};q)_n = \prod_{k=0}^{n-1}(1-2axq^k+a^2q^{2k}),\ n\geq 1,\ B_0(a,x)\equiv 1, \tag{4.4}$$

where $x=x(s)=\cos\theta = \dfrac{q^s+q^{-s}}{2}$, $q^s=e^{i\theta}$;

$$\vartheta_n(a,x) = (a+ix)_n(a-ix)_n; \tag{4.5}$$

$$\begin{cases}\xi_n(\gamma,\delta,\mu(x)) = (q^{-x};q)_n(\gamma\delta q^{x+1};q)_n = \prod\limits_{k=0}^{n-1}(1+\gamma\delta q^{2k+1}-\mu(x)q^k),\ n\geq 1,\\ \xi_0(\gamma,\delta,\mu(x))\equiv 1,\end{cases} \tag{4.6}$$

with $\mu(x)=q^{-x}+\gamma\delta q^{x+1}$;

$$\begin{cases}\chi_n(\gamma,\delta,\lambda(x)) = (-x)_n(x+\gamma+\delta+1)_n = \prod\limits_{k=0}^{n-1}\Big(k(\gamma+\delta+k+1)-\lambda(x)\Big),\ n\geq 1,\\ \chi_0(\gamma,\delta,\lambda(x))\equiv 1,\end{cases} \tag{4.7}$$

for $\lambda(x)=x(x+\gamma+\delta+1)$. The hypergeometric and the basic hypergeometric representations of classical orthogonal polynomials on a quadratic or q-quadratic lattice suggest to use the natural bases $\{B_n(a,x)\}$, $\{(a+ix)_n\}$, $\{\xi_n(\gamma,\delta,\mu(x))\}$ or $\{\chi_n(\gamma,\delta,\lambda(x))\}$ which are polynomials of degree n in the variables x, x, $\mu(x)$ or $\lambda(x)$, respectively, and the bases $\{\vartheta_n(a,x)\}$ which are polynomials of degree n in the variable x^2. The operator $\mathbb{D}_x$ is appropriate for $B_n(a,x)$, $\xi_n(\gamma,\delta,\mu(x))$ and $\chi_n(\gamma,\delta,\lambda(x))$ whereas the corresponding operator for the basis $\{\vartheta_n(a,x)\}$ is the Wilson operator [Cooper, 2012], [Ismail and Stanton, 2012] defined by

$$\mathbf{D}f(x) = \frac{f\left(x+\frac{i}{2}\right)-f\left(x-\frac{i}{2}\right)}{2ix}. \tag{4.8}$$

This operator satisfies $\mathbf{D}x^2=1$.
The bases $\{\vartheta_n(a,x)\}$, $\{\xi_n(\gamma,\delta,\mu(x))\}$ and $\{\chi_n(\gamma,\delta,\lambda(x))\}$ are related to the Askey-Wilson basis $\{B_n(a,x)\}$ in the following way.

Remark 4.1. *1. If we substitute $a\to q^a$, $q^s=e^{i\theta}\to q^{ix}$ in $\frac{B_n(a,x)}{(1-q)^{2n}}$ and take the limit when $q\to 1$, we get $\vartheta_n(a,x)$.*

2. *If we substitute* $a \to (\gamma\delta q)^{\frac{1}{2}}$ *and* $q^s = e^{i\theta} \to (\gamma\delta q)^{\frac{1}{2}} q^x$ *in* $B_n(a,x)$ *we obtain* $\xi_n(\gamma,\delta,\mu(x))$.

3. *If we substitute* $\gamma \to q^\gamma$, $\delta \to q^\delta$ *in* $\frac{\xi_n(\gamma,\delta,\mu(x))}{(1-q)^{2n}}$ *and take the limit as* $q \to 1$, *we get* $\chi_n(\gamma,\delta,\lambda(x))$.

4. *If we set* $a = \frac{1}{2}(\gamma+\delta+1)$ *and* $ix \to x + \frac{1}{2}(\gamma+\delta+1)$ *in* $\vartheta_n(a,x)$ *we get* $\chi_n(\gamma,\delta,\lambda(x))$.

5. *We also have*
$$\lim_{t\to\infty} \frac{\vartheta_n(a-it,x+t)}{(-2it)^n} = (a+ix)_n.$$

In this chapter we find the inversion, connection, duplication and linearization formulae for the Askey-Wilson polynomials. The results can be extended to other families of classical orthogonal polynomials on a quadratic or q-quadratic lattice by means of specialization and/or by limiting processes following the Askey scheme and it q-analogue. To illustrate this, we solve the inversion, connection, duplication and linearization problem for the q-Racah, Wilson and Racah polynomials.

The Askey-Wilson orthogonal polynomial family $p_n(x;a,b,c,d|q)$ satisfies the divided-difference equation of type (4.3) with [Foupouagnigni, 2008]

$$\begin{aligned}\phi(x(s)) = {} & 2(abcd+1)x^2(s) - (a+b+c+d+abc+abd+acd+bcd)x(s)\\ & +ab+ac+ad+bc+bd+cd-abcd-1,\\ \psi(x(s)) = {} & \frac{4(abcd-1)q^{\frac{1}{2}}x(s)}{q-1} + \frac{2(a+b+c+d-abc-abd-acd-bcd)q^{\frac{1}{2}}}{q-1}.\end{aligned}$$

The Askey-Wilson basis $\{B_n(a,x)\}$ has the following properties.

Proposition 4.2 ([Foupouagnigni et al., 2013a], [Foupouagnigni et al., 2013b]). *The* q*-quadratic lattice* $x(s) = \frac{q^s+q^{-s}}{2}$ *and the corresponding polynomial basis*

$$B_n(a,x) = (aq^s;q)_n(aq^{-s};q)_n,\ n \geq 1,\ B_0(a,x) \equiv 1,$$

fulfil the relations

$$\mathbb{D}_x B_n(a,x) = \eta(a,n) B_{n-1}(a\sqrt{q},x); \tag{4.9}$$
$$\mathbb{S}_x B_n(a,x) = \beta_1(a,n) B_{n-1}(a\sqrt{q},x) + \beta_2(n) B_n(a\sqrt{q},x);$$
$$B_1(a,x)\mathbb{D}_x^2 B_n(a,x) = \eta(a,n)\eta(a\sqrt{q},n-1)B_{n-1}(a,x); \tag{4.10}$$
$$B_1(a,x)\mathbb{S}_x\mathbb{D}_x B_n(a,x) = \eta(a,n)\left(\beta_1(a\sqrt{q},n-1)B_{n-1}(a,x) + \beta_2(n-1)B_n(a,x)\right); \tag{4.11}$$
$$xB_n(a,x) = \mu_1(a,n)B_n(a,x) + \mu_2(a,n)B_{n+1}(a,x); \tag{4.12}$$
$$B_1(a,x)B_n(a,x) = \nu_1(a,n)B_n(a,x) + \nu_2(n)B_{n+1}(a,x); \tag{4.13}$$
$$B_1(a,x)B_n(aq,x) = B_{n+1}(a,x),$$

where

$$\eta(a,n) = \tfrac{2a(1-q^n)}{q-1},\quad \beta_1(a,n) = \tfrac{1}{2}(1-a^2q^{2n-1})(1-q^{-n}),\quad \beta_2(n) = \tfrac{1}{2} + \tfrac{1}{2q^n},$$
$$\mu_1(a,n) = \tfrac{1+a^2q^{2n}}{2aq^n},\quad \mu_2(a,n) = \tfrac{-1}{2aq^n},\quad \nu_1(a,n) = (1-q^{-n})(1-a^2q^n),\quad \nu_2(n) = q^{-n}.$$

Using these properties of the basis $\{B_n(a,x)\}$, we will derive the inversion formulae of Askey-Wilson polynomials.

4.2 Inversion Formula of Askey-Wilson Polynomials

To solve the inversion problem of the Askey-Wilson polynomials we need the coefficients of the three-term recurrence relation of the Askey-Wilson polynomials $p_n(x;a,b,c,d|q)$, and the coefficients of the three-term recurrence relation of the second divided-derivative of the Askey-Wilson polynomials $\mathbb{D}_x^2 p_n(x;a,b,c,d|q)$.

4.2.1 Three-Term Recurrence Equation of the Family $(p_n(x;a,b,c,d|q))_n$

Proposition 4.3 (See e.g. [Koekoek et al., 2010]). *The Askey-Wilson polynomial family satisfies a three-term recurrence equation of the form*

$$x p_n(x;a,b,c,d|q) = \alpha_n p_{n+1}(x;a,b,c,d|q) + \beta_n p_n(x;a,b,c,d|q) + \gamma_n p_{n-1}(x;a,b,c,d|q), \tag{4.14}$$

with

$$\alpha_n = -\frac{1}{2aq^n}\frac{k_n}{k_{n+1}}, \tag{4.15}$$

$$\begin{aligned}\beta_n &= \frac{1}{2}\frac{q^n}{(-q^2+dcbaq^{2n})(-1+dcbaq^{2n})}\Big((q^2+abcdq^{2n+1}+abcdq^{n+1}-abcdq^n)\times \\ &\quad (a+b+c+d)-(q^{n+1}+q^{n+2}-q-abcdq^{2n})(abc+abd+acd+bcd)\Big),\end{aligned} \tag{4.16}$$

$$\begin{aligned}\gamma_n &= -\frac{k_n}{2k_{n-1}}\frac{(q^n dc-q)(dbq^n-q)(q^n bc-q)(dcbaq^n-q^2)\,q^n a\,(baq^n-q)(caq^n-q)}{q\,(-q^2+dcbaq^{2n})^2(-q^3+dcbaq^{2n})} \\ &\quad \times\frac{(aq^n d-q)(-1+q^n)}{(-q+dcbaq^{2n})},\end{aligned}$$

where k_n is the leading coefficient of the polynomial $p_n(x;a,b,c,d|q)$ represented in the basis $(B_n(a,x))_n$

$$p_n(x;a,b,c,d|q) = k_n B_n(a,x) + k_n' B_{n-1}(a,x) + k_n'' B_{n-2}(a,x) + \dots \tag{4.17}$$

and is given explicitly by $k_n = \dfrac{(abcdq^{n-1};q)_n}{(-a)^n q^{\binom{n}{2}}}$.

Proof. We will first need to compute, in terms of the leading coefficient k_n, the coefficients k_n' and k_n'' of the expansion (4.17) of $p_n(x;a,b,c,d|q)$ in the appropriate basis $(B_n(a,x))_n$. In order to compute these coefficients, first we substitute (4.17) in the divided-difference equation (4.3). Next we multiply this equation by $B_1(a,x)$ and use Relations (4.10), (4.11) and (4.13). To eliminate the terms $xB_k(a,x)$ and $x^2B_k(a,x)$, we use Relations (4.12) and

$$\begin{aligned}x^2 B_n(a,x) = {}& \mu_1^2(a,n)B_n(a,x) + \mu_2(a,n)(\mu_1(a,n)+\mu_1(a,n+1))B_{n+1}(a,x) \\ & + \mu_2(a,n)\mu_2(a,n+1)B_{n+2}(a,x).\end{aligned} \tag{4.18}$$

Equating the coefficients of $B_{n+1}(a,x)$ gives (compare [Foupouagnigni, 2008, p. 158, (104)])

$$\lambda_n = -4\,\frac{(-1+q^n)\sqrt{q}\,(abcdq^n-q)}{(q-1)^2 q^n}. \tag{4.19}$$

Equating the coefficients of $B_n(a,x)$, we deduce using (4.19) that

$$k_n' = -\frac{(-1+q^n)(-q+aq^nd)(aq^nc-q)(aq^nb-q)}{q(q-1)(-q^2+abcdq^{2n})}k_n. \tag{4.20}$$

By equating the coefficients of $B_{n-1}(a,x)$ and using (4.19)-(4.20), one gets

$$\begin{aligned} k_n'' = &\frac{(-1+q^n)(-q+aq^nd)(aq^nc-q)(aq^nb-q)(-q+q^n)(aq^nd-q^2)}{(1+q)(q-1)^2(-q^3+abcdq^{2n})(-q^2+abcdq^{2n})} \\ &\times\frac{(aq^nc-q^2)(aq^nb-q^2)}{q^4}k_n. \end{aligned} \tag{4.21}$$

To get the coefficients α_n, β_n and γ_n of Equation (4.14), we substitute the expression of p_n given by (4.17) in the recurrence equation (4.14) and use Equations (4.12), (4.19)-(4.21). By equating the coefficients of $B_{n+1}(a,x)$, one gets α_n. Equating the coefficients of $B_n(a,x)$ and using (4.15) yields β_n. Similarly, equating the coefficients of $B_{n-1}(a,x)$ and using (4.15)-(4.16) gives γ_n. □

Remark 4.4. *The recurrence equation (4.14) was also given in [Koekoek et al., 2010, p. 417] in the form*

$$2x\tilde{p}_n(x) = A_n\tilde{p}_{n+1}(x) + [a+a^{-1}-(A_n+C_n)]\tilde{p}_n(x) + C_n\tilde{p}_{n-1}(x), \tag{4.22}$$

where

$$\tilde{p}_n(x) := \frac{a^np_n(x;a,b,c,d|q)}{(ab,ac,ad:q)_n}$$

and

$$\begin{aligned} A_n &= \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})} \\ C_n &= \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}. \end{aligned}$$

It can be proved by direct computation that Equation (4.22) is equivalent to Equation (4.14) with

$$\alpha_n = A_n\frac{a(ab,ac,ad;q)_n}{2(ab,ac,ad;q)_{n+1}},\ \beta_n = \frac{[a+a^{-1}-(A_n+C_n)]}{2},\ \gamma_n = C_n\frac{(ab,ac,ad;q)_n}{2a(ab,ac,ad;q)_{n-1}}.$$

4.2.2 Three-Term Recurrence Equation of the Family $(\mathbb{D}_x^2p_n(x;a,b,c,d|q))_n$

Foupouagnigni et al. [2011] proved that if $(p_n)_n$ is an orthogonal polynomial family satisfying (4.3), then the families $(\mathbb{D}_x^mp_n)_n$ are also orthogonal, for all $m\in\mathbb{N}$. Consequently, they also satisfy a three-term recurrence equation.

Proposition 4.5. *The second-order divided-difference $\mathbb{D}_x^2p_n$ of the Askey-Wilson orthogonal polynomials defined by (4.17) satisfies the following recurrence equation*

$$\begin{aligned} x\mathbb{D}_x^2p_n(x;a,b,c,d|q) = &\alpha_n^\star\mathbb{D}_x^2p_{n+1}(x;a,b,c,d|q) + \beta_n^\star\mathbb{D}_x^2p_n(x;a,b,c,d|q) \\ &+\gamma_n^\star\mathbb{D}_x^2p_{n-1}(x;a,b,c,d|q), \end{aligned} \tag{4.23}$$

with

$$\alpha_n^\star = -\frac{1}{2}\frac{-q+q^n}{aq^n\left(-1+q^{n+1}\right)}\frac{k_n}{k_{n+1}}, \tag{4.24}$$

$$\begin{aligned}\beta_n^\star =& \frac{1}{2}\frac{q^n}{\left(-q^2+abcdq^{2n}\right)\left(-1+abcdq^{2n}\right)}\Big((a+b+c+d)(q-abcdq^{n+1}-abcdq^{n+2}+abcdq^{2n})\\ &+(abc+abd+acd+bcd)(q^2-q^n-q^{n+1}+abcdq^{2n+1})\Big),\end{aligned} \tag{4.25}$$

$$\begin{aligned}\gamma_n^\star =& -\frac{k_n}{2k_{n-1}}\frac{\left(-q+q^n dc\right)\left(q^n bd-q\right)\left(q^n bc-q\right)\left(abcdq^n-1\right)q^n a\left(-1+q^n\right)\left(baq^n-q\right)}{\left(-q^2+abcdq^{2n}\right)^2\left(-q^3+abcdq^{2n}\right)}\\ &\times\frac{\left(acq^n-q\right)\left(-q+aq^n d\right)}{\left(-q+abcdq^{2n}\right)}.\end{aligned}$$

Proof. We substitute the expression of p_n given by (4.17) in the recurrence equation (4.23), and then multiply the equation obtained by $B_1(a,x)$. Next we use (4.10) and (4.12), respectively, to eliminate $B_1(a,x)\mathbb{D}_x^2 B_n(a,x)$ and $xB_n(a,x)$. By equating the coefficients of $B_n(a,x)$, one gets $\alpha_n^\star$. Equating the coefficients of $B_{n-1}(a,x)$ and using (4.24) yields $\beta_n^\star$. Similarly, equating the coefficients of $B_{n-2}(a,x)$ and using (4.24) and (4.25), we obtain $\gamma_n^\star$. □

4.2.3 Inversion Formula of Askey-Wilson Polynomials

Since the family $(p_k(x;a,b,c,d|q))_{k=0,\ldots,n}$ is a basis of polynomials of degree less than or equal to n, and since $B_n(a,x)$ is a polynomial of degree n w.r.t. x, it follows that

$$B_n(a,x) = \sum_{m=0}^{n} I_m(n) p_m(x;a,b,c,d|q), \tag{4.26}$$

which is called the inversion formula for the family $(p_n(x;a,b,c,d|q))_n$ represented in the basis $(B_n(a,x))_n$. We prove that

Proposition 4.6 (See e.g. [Area et al., 2001], [Foupouagnigni et al., 2013b])**.** *The inversion formula of the Askey-Wilson orthogonal polynomial family is given by*

$$B_n(a,x) = \sum_{m=0}^{n} {n \brack m}_q q^{\frac{m(m-1)}{2}} \frac{(-a)^m (abq^m, acq^m, adq^m; q)_{n-m}}{(abcdq^{m-1};q)_m (abcdq^{2m};q)_{n-m}} p_m(x;a,b,c,d|q). \tag{4.27}$$

The proof of this proposition uses the following

Lemma 4.7. *The second $\mathbb{D}_x$-derivative of the basis $(B_n(a,x))_n$ satisfies the recurrence relation*

$$x\mathbb{D}_x^2 B_n(a,x) = \mu_1(a,n-1)\mathbb{D}_x^2 B_n(a,x) + \mu_2(a,n-1)\frac{\eta(a,n)\eta(a\sqrt{q},n-1)}{\eta(a,n+1)\eta(a\sqrt{q},n)}\mathbb{D}_x^2 B_{n+1}(a,x), \tag{4.28}$$

with the coefficients given in Proposition 4.2.

Proof (of the lemma). From (4.10) we obtain $B_n(a,x)$ in terms of $\mathbb{D}_x^2 B_{n+1}(a,x)$. If we substitute this in the recurrence equation (4.12), the result follows. □

Proof (of Proposition 4.6). We substitute $B_n(a,x)$ given by (4.26) in the recurrence equation (4.12) and replace $xp_m(x;a,b,c,d|q)$ by the expression given by (4.14). This yields the equation

$$\begin{aligned}&\mu_1(a,n)\sum_{m=0}^{n} I_m(n)p_m(x;a,b,c,d|q)+\mu_2(a,n)\sum_{m=0}^{n+1} I_m(n+1)p_m(x;a,b,c,d|q)\\ &=\sum_{m=0}^{n} I_m(n)\Big(\alpha_m p_{m+1}(x;a,b,c,d|q)+\beta_m p_m(x;a,b,c,d|q)+\gamma_m p_{m-1}(x;a,b,c,d|q)\Big).\end{aligned}$$

By an appropriate shift of indices, and equating the coefficients of $p_m(x;a,b,c,d|q)$, we get

$$\mu_1(a,n)I_m(n)+\mu_2(a,n)I_m(n+1)=\alpha_{m-1}I_{m-1}(n)+\beta_m I_m(n)+\gamma_{m+1}I_{m+1}(n). \tag{4.29}$$

Similarly, we substitute $B_n(a,x)$ given by (4.26) in (4.28) and replace $x\mathbb{D}_x^2 p_m(x;a,b,c,d|q)$ by the expression given in (4.23). By an appropriate shift of indices, and equating the coefficients of $p_m(x;a,b,c,d|q)$, we get

$$\begin{aligned}&\mu_1(a,n-1)I_m(n)+\mu_2(a,n-1)\frac{\eta(a,n)\eta(a\sqrt{q},n-1)}{\eta(a,n+1)\eta(a\sqrt{q},n)}I_m(n+1)\\ &=\alpha^\star_{m-1}I_{m-1}(n)+\beta^\star_m I_m(n)+\gamma^\star_{m+1}I_{m+1}(n).\end{aligned} \tag{4.30}$$

We substitute $I_m(n+1)$ obtained from (4.29) in (4.30) and get a second-order recurrence equation w.r.t. the variable m:

$$\begin{aligned}&q(q^n-q^m)(1-abcdq^{2m+2})(abcdq^{2m};q)_4 I_m(n)+q^m\left(1-q^{m+1}\right)\left(abcdq^{2m+1};q\right)_3\\ &\times\Big(q+a^2q^n+a^2b^2c^2d^2q^{4m+3}+a^4b^2c^2d^2q^{4m+n+2}+q^{2m+n+2}a^2bcd(b+c+d)\\ &+a^2q^{2m+1}(db+dc+bc)(1+q)+(a^2bcdq^{2m+n+1}-aq^{m+1}-a^2bcdq^{3m+2})(d+a+b+c)\\ &-abcdq^{2m}(q^3-q^2+aq^n)-(aq^{m+n+1}+a^2bdcq^{3m+n+2})(abc+abd+acd+bcd)\Big)I_{m+1}(n)\\ &+q^{2m+1}a^2(q^{m+1};q)_2\left(q^mqdc-1\right)\left(q^mqdb-1\right)\left(q^mqbc-1\right)\left(q^mqad-1\right)\left(q^mqac-1\right)\\ &\times\left(abq^{m+1}-1\right)\left(abcdq^{2m}-1\right)\left(-1+abcdq^{m+n+1}\right)I_{m+2}(n)=0.\end{aligned}$$

To solve this recurrence equation, we use the q-Petkovšek-van-Hoeij algorithm. With this algorithm, we get up to a multiplicative constant the solution of this recurrence equation. Equating the coefficients of $B_n(a,x)$ in (4.26) gives the constant and (4.27) follows. □

Remark 4.8. *The inversion formula (4.27) was already obtained in [Area et al., 2001], but using (4.1).*

4.3 Connection, Duplication, Linearization Formulae of Askey-Wilson Polynomials

4.3.1 Connection Coefficients Between $(p_n(x;a,b,c,d|q))_n$ and $(p_m(x;a,\beta,\gamma,\delta|q))_m$

From the basic hypergeometric representation and the inversion problem of the Askey-Wilson polynomials, we determine the coefficients $C_m(n)$ of the expansion

$$p_n(x;a,b,c,d|q)=\sum_{m=0}^{n} C_m(n)p_m(x;a,\beta,\gamma,\delta|q). \tag{4.31}$$

These connection coefficients are given by

Proposition 4.9 (See [Askey and Wilson, 1985], [Area et al., 2001], Foupouagnigni et al. [2013b]). *The Askey-Wilson orthogonal polynomial family satisfies the following connection formula*

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \frac{a^{m-n}q^{m(m-n)}}{(q;q)_{n-m}} \frac{(q,ab,ac,ad;q)_n(abcdq^{n-1};q)_m}{(q,ab,ac,ad;q)_m(\alpha\beta\gamma\delta q^{m-1};q)_m} \times$$
$${}_5\phi_4\left(\left.\begin{matrix} q^{m-n}, a\beta q^m, a\gamma q^m, a\delta q^m, abcdq^{m+n-1} \\ abq^m, acq^m, adq^m, a\beta\gamma\delta q^{2m}\end{matrix}\,\right|\, q;q\right) p_m(x;a,\beta,\gamma,\delta|q). \quad (4.32)$$

Proof. From the basic hypergeometric representation of the Askey-Wilson polynomials, we have

$$p_n(x;a,b,c,d|q) = \sum_{j=0}^{n} A_j(n)B_j(a,x);$$

with $A_j(n) = \dfrac{(ab,ac,ad;q)_n(q^{-n},abcdq^{n-1},q)_jq^j}{a^n(q,ab,ac,ad;q)_j}$ and $B_j(a,x) = (ae^{i\theta}, ae^{-i\theta};q)_j$. It follows from the inversion formula (4.27) that

$$B_j(a,x) = \sum_{m=0}^{j} I_m(j)p_m(x;a,\beta,\gamma,\delta|q).$$

Combining the last two relations, one gets (4.31) with

$$C_m(n) = \sum_{j=0}^{n-m} A_{j+m}(n)I_m(j+m).$$

To get the result we apply the sum2qhyper algorithm. □

q-Zeilberger's algorithm combined with the q-Petkovšek-van-Hoeij algorithm yields the specialized cases given in

Proposition 4.10. *The following connection formulae for the Askey-Wilson polynomials are valid:*

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \frac{(-1)^m b_1^n q^{\frac{m(m-2n+1)}{2}}}{b^m(q;q)_{n-m}} \frac{(q,ac,ad,cd,\frac{b}{b_1};q)_n(abcdq^{n-1};q)_m}{(q,ac,ad,cd,ab_1cdq^{m-1},ab_1cdq^n,\frac{b_1q}{bq^n};q)_m}$$
$$\times \frac{(ab_1cd;q)_{2m}}{(ab_1cd;q)_n} p_m(x;a,b_1,c,d|q), \quad (4.33)$$

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \frac{(-1)^m c_1^n q^{\frac{m(m-2n+1)}{2}}}{c^m(q;q)_{n-m}} \frac{(q,ab,ad,bd,\frac{c}{c_1};q)_n(abcdq^{n-1};q)_m}{(q,ab,ad,bd,abc_1dq^{m-1},abc_1dq^n,\frac{c_1q}{cq^n};q)_m}$$
$$\times \frac{(abc_1d;q)_{2m}}{(abc_1d;q)_n} p_m(x;a,b,c_1,d|q), \quad (4.34)$$

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \frac{(-1)^m d_1^n q^{\frac{m(m-2n+1)}{2}}}{d^m (q;q)_{n-m}} \frac{(q,ab,ac,bc,\frac{d}{d_1};q)_n (abcdq^{n-1};q)_m}{(q,ab,ac,bc,abcd_1q^{m-1},abcd_1q^n,\frac{d_1q}{dq^n};q)_m}$$
$$\times \frac{(abcd_1;q)_{2m}}{(abcd_1;q)_n} p_m(x;a,b,c,d_1|q). \tag{4.35}$$

From these connection formulae, we proceed as in the proof of Corollary 3.19 to derive the parameter derivatives of the Askey-Wilson polynomials.

Corollary 4.11. *The following parameter derivatives of the Askey-Wilson polynomials are valid:*

$$\frac{\partial}{\partial b} p_n(x;a,b,c,d|q) = -\sum_{m=0}^{n-1} \Big(\frac{acdq^{m+n-1}}{1-abcdq^{m+n-1}} p_n(x;a,b,c,d|q)$$
$$+ \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{b^n (ac,ad,cd;q)_n (q;q)_{n-m-1} (abcdq^{n-1};q)_m (abcd;q)_{2m}}{b^{m+1} (ac,ad,cd,abcdq^{m-1},abcdq^n;q)_m (abcd;q)_n} p_m(x;a,b,c,d|q) \Big),$$

$$\frac{\partial}{\partial c} p_n(x;a,b,c,d|q) = -\sum_{m=0}^{n-1} \Big(\frac{abdq^{m+n-1}}{1-abcdq^{m+n-1}} p_n(x;a,b,c,d|q)$$
$$+ \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{c^n (ab,ad,bd;q)_n (q;q)_{n-m-1} (abcdq^{n-1};q)_m (abcd;q)_{2m}}{c^{m+1} (ab,ad,bd,abcdq^{m-1},abcdq^n;q)_m (abcd;q)_n} p_m(x;a,b,c,d|q) \Big),$$

$$\frac{\partial}{\partial d} p_n(x;a,b,c,d|q) = -\sum_{m=0}^{n-1} \Big(\frac{abcq^{m+n-1}}{1-abcdq^{m+n-1}} p_n(x;a,b,c,d|q)$$
$$+ \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{d^n (ab,ac,bc;q)_n (q;q)_{n-m-1} (abcdq^{n-1};q)_m (abcd;q)_{2m}}{d^{m+1} (ab,ac,bc,abcdq^{m-1},abcdq^n;q)_m (abcd;q)_n} p_m(x;a,b,c,d|q) \Big).$$

4.3.2 Connection Coefficients Between $(p_n(x;a,b,c,d|q))_n$ and $(p_m(x;\alpha,\beta,\gamma,\delta|q))_m$

We remark that in the connection formula (4.32), the parameter a is kept identical on both sides of the formula. We would now like to get a similar formula for different a. For this purpose, we need the following connection formula of $B_n(a,x)$.

We expand $B_n(a,x)$ in terms of $B_m(\alpha,x)$

$$B_n(a,x) = \sum_{m=0}^{n} F_m(n) B_m(\alpha,x), \tag{4.36}$$

and obtain

Theorem 4.12 (See e.g. [Ismail, 1995]). *The connection between the basis $(B_n(a,x))_n$ and $(B_n(\alpha,x))_n$ is given by*

$$B_n(a,x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \left(\frac{a}{\alpha}\right)^m (a\alpha q^m;q)_{n-m} \left(\frac{a}{\alpha};q\right)_{n-m} B_m(\alpha,x). \tag{4.37}$$

To get the coefficients $F_m(n)$ of the expansion (4.36), we need the following intermediate result which gives a second-order divided-difference equation for $B_n(a,x)$.

Proposition 4.13. *The polynomial family $(B_n(a,x))_n$ satisfies the second order divided-difference equation*

$$\left(a^2q^{2\,n-2}-1\right)(q-1)^2\,B_1(a,x)\,\mathbb{D}_x^2B_n(a,x)-4\;a\sqrt{q}q^{n-1}\,(q-1)B_1(a,x)\,\mathbb{S}_x\mathbb{D}_xB_n(a,x)$$
$$+4\,a^2\,\sqrt{q}\,\left(1+q^{n-1}\right)(1-q^n)\,B_n\,(a,x)=0. \tag{4.38}$$

Proof. We substitute the expression of $B_{n-1}(a,x)$ obtained from (4.10) in Equation (4.11) and the result follows. □

Proof (of Theorem 4.12). We substitute $B_n(a,x)$ by the sum in (4.36) in the divided-difference equation (4.38) and multiply the obtained equation by $B_1(\alpha,x)$ for the purpose to use Equations (4.10), (4.11) and (4.13) (with a replaced by α). Next, we substitute $B_1(a,x)$ by its representation in terms of x, that is $B_1(a,x)=1+a^2-2ax$, and use Equation (4.12) with $a=\alpha$. Finally we collect the coefficients of $B_m(\alpha,x)$, $B_{m-1}(\alpha,x)$, $B_{m+1}(\alpha,x)$ and by an appropriate shift of indices, we rewrite all the summands in terms of $B_m(\alpha,x)$. Since the family $(B_m(\alpha,x))_m$ is linearly independent, it follows after simplification that $F_m(n)$ satisfies the following second-order recurrence equation

$$\begin{aligned}
&-a\left(q^{m+1}-1\right)\Big(\alpha\,q^{m+4}-\alpha\,q^{2m+4}-\alpha\,q^{m+2}a^2q^{2n}+\alpha\,q^{2m+2}a^2q^{2n}+aq^{m+3}\\
&-aq^{2m+4}\alpha^2-aq^{m+2}q^{2n}+aq^{2m+3}q^n\alpha^2-aq^3q^n-aq^n\alpha^2q^{3m+3}-aq^2q^n\\
&-aq^n\alpha^2q^{3m+4}+\alpha\,q^nq^{m+3}+\alpha\,q^nq^{2m+3}+\alpha\,q^nq^{m+3}a^2+\alpha\,q^nq^{2m+3}a^2\Big)F_{m+1}(n)\\
&+a^2q\left(-q^{m+1}+q^nq\right)\left(q^nq^{m+1}+q^2\right)F_m(n)+\left(q^{m+1}-1\right)\left(-a+\alpha\,q^{m+1}\right)\\
&\times\left(\alpha\,q^{m+1}a-1\right)\left(q^{m+1}q-1\right)\left(aq^n-\alpha\,q^{m+1}q\right)\left(\alpha\,q^{m+1}aq^n+q\right)F_{m+2}(n)=0.
\end{aligned}$$

Using again the q-Petkovšek-van-Hoeij algorithm, we solve this recurrence equation and the result follows using the formulas

$$(aq^{-n};q)_n=(a^{-1}q;q)_n(-a)^nq^{-n-\binom{n}{2}},\;\;a\neq 0,$$

$$(aq^k;q)_{n-k}=\frac{(a;q)_n}{(a;q)_k},\qquad k=0,1,2,\ldots,n,$$

$$(q^{-n};q)_k=\frac{(q;q)_n}{(q;q)_{n-k}}(-1)^kq^{\binom{k}{2}-nk},\;\;k=0,1,\ldots,n, \tag{4.39}$$

given, respectively, by Equations (1.8.12), (1.8.14), (1.8.18) of [Koekoek et al., 2010, p. 12–13]. □

We can now use the connection formula (4.37) between $B_n(a,x)$ and $B_n(\alpha,x)$ to derive the representation of the Askey-Wilson polynomials in the basis $(B_n(\alpha,x))_n$. From

$$p_n(x;a,b,c,d|q)=\sum_{j=0}^{n}A_j(n)B_j(a,x)\;\;\text{and}\;\;B_j(a,x)=\sum_{m=0}^{j}F_m(j)B_m(\alpha,x),$$

we get

$$p_n(x;a,b,c,d|q)=\sum_{m=0}^{n}G_m(n)B_m(\alpha,x),$$

with

$$G_m(n) = \sum_{j=0}^{n-m} A_{j+m}(n) F_m(j+m).$$

Using the sum2qhyper algorithm, one gets the following representation.

Proposition 4.14. *The element $p_n(x;a,b,c,d|q)$ of the Askey-Wilson orthogonal polynomial family has the following representation in the basis $(B_n(\alpha,x))_n$*

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \left(-\frac{a}{\alpha}\right)^m \frac{q^{\frac{m(m-2n+1)}{2}}}{a^n (q;q)_{n-m}} \frac{(q,ab,ac,ad;q)_n (abcdq^{n-1};q)_m}{(q,ab,ac,ad;q)_m} \times$$
$$ {}_4\phi_3\left(\begin{matrix} q^{m-n}, a\alpha q^m, abcdq^{m+n-1}, \frac{a}{\alpha} \\ abq^m, acq^m, adq^m \end{matrix} \middle| q;q\right) B_m(\alpha,x).$$

Remark 4.15. *For $\alpha=b$, $\alpha=c$, $\alpha=d$ the latter representation of $p_n(x;a,b,c,d|q)$ of Proposition 4.14 is, respectively, reduced using q-Zeilberger's algorithm to*

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \frac{q^m (ab,bc,bd;q)_n (q^{-n}, abcdq^{n-1};q)_m}{b^n (q,ab,bc,bd;q)_m} B_m(b,x) = p_n(x;b,a,c,d|q),$$

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \frac{q^m (ac,bc,cd;q)_n (q^{-n}, abcdq^{n-1};q)_m}{c^n (q,ac,bc,cd;q)_m} B_m(c,x) = p_n(x;c,b,a,d|q),$$

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \frac{q^m (ad,bd,cd;q)_n (q^{-n}, abcdq^{n-1};q)_m}{d^n (q,ad,bd,cd;q)_m} B_m(d,x) = p_n(x;d,b,c,a|q).$$

These relations show what happens when we interchange the first parameter of $p_n(x;a,b,c,d|q)$ with any of the other three and were given in Rahman [1981b] for the family $p_n(x;a,b,c,d) = \dfrac{a^n p_n(x;a,b,c,d|q)}{(ab,ac,ad;q)_n}$.

Since $p_n(x;a,b,c,d|q)$ is represented, respectively, in the bases $B_n(b,x)$, $B_n(c,x)$ and $B_n(d,x)$ as shown in the preceding remark, we can represent $B_n(b,x)$, $B_n(c,x)$ and $B_n(d,x)$ in the basis $p_n(x;a,b,c,d|q)$. This is given by

Proposition 4.16. *For the Askey-Wilson polynomials, the following inversion formulae hold:*

$$B_n(b,x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} \frac{(-b)^m (baq^m, bcq^m, bdq^m;q)_{n-m}}{(abcdq^{m-1};q)_m (abcdq^{2m};q)_{n-m}} p_m(x;a,b,c,d|q),$$

$$B_n(c,x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} \frac{(-c)^m (caq^m, cbq^m, cdq^m;q)_{n-m}}{(abcdq^{m-1};q)_m (abcdq^{2m};q)_{n-m}} p_m(x;a,b,c,d|q),$$

$$B_n(d,x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} \frac{(-d)^m (daq^m, dbq^m, dcq^m;q)_{n-m}}{(abcdq^{m-1};q)_m (abcdq^{2m};q)_{n-m}} p_m(x;a,b,c,d|q).$$

Proof. The proof follows from Proposition 4.6 and Remark 4.15. □

The combination of the basic q-hypergeometric representations of $p_n(x;a,b,c,d|q)$ of Remark 4.15 and the inversion formulae of Proposition 4.16 gives using the q-Zeilberger or sum2qhyper algorithm the following connection formulae.

Proposition 4.17. *The following connection formulae are valid for the Askey-Wilson polynomials:*

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \frac{(-1)^m a_1^n q^{\frac{m(m-2n+1)}{2}}}{a^m (q;q)_{n-m}} \frac{(q,bc,bd,cd,\frac{a}{a_1};q)_n (abcdq^{n-1};q)_m}{(q,bc,bd,cd,a_1bcdq^n,\frac{a_1 q}{aq^n};q)_m (a_1bcd;q)_n} \times \frac{(a_1bcd;q)_{2m}}{(a_1bcdq^{m-1};q)_m} p_m(x;a_1,b,c,d|q), \tag{4.40}$$

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \frac{b^{m-n} q^{m(m-n)}}{(q;q)_{n-m}} \frac{(q,ba,bc,bd;q)_n (abcdq^{n-1};q)_m}{(q,ba,bc,bd;q)_m (\alpha b\gamma\delta q^{m-1};q)_m} \times {}_5\phi_4\left(\begin{matrix} q^{m-n}, b\alpha q^m, b\gamma q^m, b\delta q^m, abcdq^{m+n-1} \\ baq^m, bcq^m, bdq^m, \alpha b\gamma\delta q^{2m} \end{matrix}\middle| q;q\right) p_m(x;\alpha,b,\gamma,\delta|q), \tag{4.41}$$

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \frac{c^{m-n} q^{m(m-n)}}{(q;q)_{n-m}} \frac{(q,ca,cb,cd;q)_n (abcdq^{n-1};q)_m}{(q,ca,cb,cd;q)_m (\alpha\beta c\delta q^{m-1};q)_m} \times {}_5\phi_4\left(\begin{matrix} q^{m-n}, c\alpha q^m, c\beta q^m, c\delta q^m, abcdq^{m+n-1} \\ caq^m, cbq^m, cdq^m, \alpha\beta c\delta q^{2m} \end{matrix}\middle| q;q\right) p_m(x;\alpha,\beta,c,\delta|q), \tag{4.42}$$

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} \frac{d^{m-n} q^{m(m-n)}}{(q;q)_{n-m}} \frac{(q,da,db,dc;q)_n (abcdq^{n-1};q)_m}{(q,da,db,dc;q)_m (\alpha\beta\gamma dq^{m-1};q)_m} \times {}_5\phi_4\left(\begin{matrix} q^{m-n}, d\alpha q^m, d\beta q^m, d\gamma q^m, abcdq^{m+n-1} \\ daq^m, dbq^m, dcq^m, \alpha\beta\gamma dq^{2m} \end{matrix}\middle| q;q\right) p_m(x;\alpha,\beta,\gamma,d|q). \tag{4.43}$$

A consequence of connection formula (4.40) is the parameter derivative

$$\frac{\partial}{\partial a} p_n(x;a,b,c,d|q) = -\sum_{m=0}^{n-1} \Big(\frac{bcdq^{m+n-1}}{1-abcdq^{m+n-1}} p_n(x;a,b,c,d|q) + \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{a^n (bc,bd,cd;q)_n (q;q)_{n-m-1} (abcdq^{n-1};q)_m (abcd;q)_{2m}}{a^{m+1} (bc,bd,cd,abcdq^{m-1},abcdq^n;q)_m (abcd;q)_n} p_m(x;a,b,c,d|q) \Big).$$

We note that the above parameter derivative follows easily from Corollary 4.11 and Remark 4.15.
From the representation of $p_n(x;a,b,c,d|q)$ of Proposition 4.14 and the inversion formula (4.27), we have

$$p_n(x;a,b,c,d|q) = \sum_{j=0}^{n} G_j(n) B_j(\alpha,x) \text{ and } B_j(\alpha,x) = \sum_{m=0}^{j} I_m(j) p_m(x;\alpha,\beta,\gamma,\delta|q),$$

from which we get

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} C_m(n) p_m(x;\alpha,\beta,\gamma,\delta|q),$$

with

$$C_m(n) = \sum_{j=0}^{n-m} G_{j+m}(n) I_m(j+m).$$

Using once more the sum2qhyper algorithm, one gets the following

Theorem 4.18 (See e.g. [Ismail and Rahman, 2011], [Foupouagnigni et al., 2013b]). *The following connection formula is satisfied by the Askey-Wilson polynomial family*

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} C_m(n) p_m(x;\alpha,\beta,\gamma,\delta|q), \tag{4.44}$$

with

$$C_m(n) = \sum_{k=0}^{n-m} \left(\frac{a}{\alpha}\right)^k \frac{(q^{m-n};q)_k q^k}{(q;q)_k} \frac{(\alpha\beta q^m, \alpha\gamma q^m, \alpha\delta q^m;q)_k (abcdq^{n-1};q)_{k+m}}{(ab,ac,ad;q)_{k+m}(\alpha\beta\gamma\delta q^{2m};q)_k}$$
$$\times \frac{(aq^m)^{m-n}(q;q)_n(ab,ac,ad;q)_n}{(q;q)_m(q;q)_{n-m}(\alpha\beta\gamma\delta q^{m-1};q)_m} {}_4\phi_3\left(\begin{matrix} q^{k+m-n}, a\alpha q^{k+m}, abcdq^{m+n+k-1}, \frac{a}{\alpha} \\ abq^{k+m}, acq^{k+m}, adq^{k+m}\end{matrix}\middle| q;q\right).$$

4.3.3 Duplication Formula of Askey-Wilson Polynomials

In order to get the duplication coefficients $D_m(n,\alpha)$ of the relation

$$p_n(\alpha x;a,b,c,d|q) = \sum_{m=0}^{n} D_m(n,\alpha) p_m(x;a,b,c,d|q)$$

we need the duplication coefficients of the basis family $(B_n(a,x))_n$ given by the following

Proposition 4.19. *For the Askey-Wilson polynomial basis $(B_n(a,x))_n$, the duplication formula*

$$B_n(a,\alpha x) = \sum_{k=0}^{n} E_k(n) B_k(a,x) \tag{4.45}$$

holds with

$$E_k(n) = q^k \sum_{j=0}^{k} \frac{q^{-j^2} a^{-2j} \prod_{l=0}^{n-1}(1-\alpha a^2 q^{l+j} - \alpha q^{l-j} + a^2 q^{2l})}{(q, a^2 q^{1+2j};q)_{k-j}(q, a^{-2} q^{1-2j};q)_j}.$$

The proof of this proposition uses the following

Theorem 4.20 (Expansion theorem, see e.g. [Ismail, 1995], [Ismail and Stanton, 2003]). *Let f be a polynomial of degree n, then*

$$f(x) = \sum_{k=0}^{n} f_k B_k(a,x),$$

where

$$f_k = \frac{(q-1)^k}{(2a)^k (q;q)_k} q^{-\frac{k(k-1)}{4}} (\mathcal{D}_q^k f)(x_k) \tag{4.46}$$

with

$$x_k := \frac{1}{2}(aq^{k/2} + a^{-1}q^{-k/2}).$$

Proof. Let $m = 0, 1, \ldots, k$, we apply $\mathcal{D}_q^m$ to both sides of $f(x) = \sum_{k=0}^{n} f_k B_k(a, x)$ to get

$$\mathcal{D}_q^m f(x) = \sum_{k=0}^{n} f_k \mathcal{D}_q^m B_k(a, x) = f_m \mathcal{D}_q^m B_m(a, x) + \sum_{k=m+1}^{n} f_k \mathcal{D}_q^m B_k(a, x). \tag{4.47}$$

By iteration of Equation (4.9), we have

$$\mathcal{D}_q^m B_k(a, x) = \frac{(2a)^m (q^{k-m+1}; q)_m q^{\frac{m(m-1)}{4}}}{(q-1)^m} B_{k-m}(aq^{\frac{m}{2}}, x). \tag{4.48}$$

For all $k \neq 0$, $B_k(aq^{\frac{m}{2}}, x) = 0 \Leftrightarrow x = x_m = \frac{1}{2}(aq^{m/2} + a^{-1}q^{-m/2})$. We substitute x by x_m in (4.47) and use (4.48) to get the result. $\square$

We also need the following q-derivative rule due to Cooper [2012] which is a generalization of Relation (4.48).

Theorem 4.21 ([Cooper, 2012]). *The action of $\mathcal{D}_q^n$ on a function f is given by*

$$\mathcal{D}_q^n f(x) = \frac{2^n q^{\frac{n(1-n)}{4}}}{(q^{1/2} - q^{-1/2})^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(n-k)} z^{2k-n} \breve{f}(q^{\frac{(n-2k)}{2}} z)}{(q^{1+n-2k}z^2; q)_k (q^{2k-n+1}z^{-2}; q)_{n-k}}, \tag{4.49}$$

where $x = \cos\theta = \frac{1}{2}(z + z^{-1})$ and $\breve{f}(z) = f\Big((z + 1/z)/2\Big)$ with $z = e^{i\theta}$.

Proof (of Proposition 4.19). We have $x_k = \frac{1}{2}(aq^{\frac{k}{2}} + a^{-1}q^{\frac{-k}{2}}) = \frac{1}{2}(z + z^{-1})$ with $z = aq^{\frac{k}{2}}$. The combination of Equations (4.46) and (4.49) with $x = x_k$ and $z = aq^{\frac{k}{2}}$ yields

$$f_k = q^{k(1-k)} \sum_{j=0}^{k} \frac{q^{j(2k-j)} a^{2(j-k)} \breve{f}(aq^{k-j})}{(q, a^2q^{1+2(k-j)}; q)_j (q, a^{-2}q^{1-2(k-j)}; q)_{k-j}}.$$

If we substitute j by $k - j$, $f(x)$ by $B_n(a, \alpha x)$, the result follows. $\square$

We can now state and prove the duplication formula of the Askey-Wilson polynomials.

Theorem 4.22. *The following duplication formula is valid for the Askey-Wilson polynomials:*

$$p_n(\alpha x; a, b, c, d|q) = \sum_{m=0}^{n} D_m(n, \alpha) p_m(x; a, b, c, d|q), \tag{4.50}$$

with

$$D_m(n,\alpha) = \frac{(-1)^m a^{m-n} q^{\binom{m}{2}+2n} (ab, ac, ad; q)_n}{(abcdq^{m-1}; q)_m} \sum_{s=0}^{n-m} \begin{bmatrix} n-s \\ m \end{bmatrix}_q \frac{q^{-2s}(abq^m, acq^m, adq^m; q)_{n-m-s}}{(abcdq^{2m}; q)_{n-m-s}} \times$$
$$\sum_{i=0}^{n-s} \frac{a^{-2i}q^{-i^2}}{(q, a^2q^{2i+1}; q)_{n-s-i} (q, \frac{q}{a^2q^{2i}}; q)_i} \times$$
$$\sum_{j=0}^{s} \frac{q^j \left(q^{-n}, abcdq^{n-1}; q\right)_{n+j-s} \prod_{l=0}^{n+j-s-1} (1 - \alpha a^2 q^{l+i} - \alpha q^{l-i} + a^2 q^{2l})}{(q, ab, ac, ad; q)_{n+j-s}}.$$

Proof. From the basic hypergeometric representation of the Askey-Wilson polynomials given on p. 105, since $p_n(x;a,b,c,d|q) = \sum_{j=0}^{n} A_j(n)B_j(a,x)$ with $B_j(a,x)$ defined by (4.4), we have

$$p_n(\alpha x;a,b,c,d|q) = \sum_{j=0}^{n} A_j(n)B_j(a,\alpha x).$$

From (4.45) and from the inversion formula (4.27), we have

$$B_j(a,\alpha x) = \sum_{k=0}^{j} E_k(j)B_k(a,x) \quad \text{and} \quad B_k(a,x) = \sum_{m=0}^{k} I_m(k)p_m(x;a,b,c,d|q),$$

respectively. The combination of the above representations yields

$$p_n(\alpha x;a,b,c,d|q) = \sum_{m=0}^{n} D_m(n,\alpha)p_m(x;a,b,c,d|q)$$

with

$$D_m(n,\alpha) = \sum_{k=0}^{n-m}\sum_{j=0}^{n-m-k} I_m(k+m)A_{j+k+m}(n)E_{k+m}(j+k+m).$$

Since

$$E_k(n) = \sum_{i=0}^{k} F_i(k,n) \quad \text{with} \quad F_i(k,n) = q^k \frac{q^{-i^2}a^{-2i}\prod_{l=0}^{n-1}(1-\alpha a^2q^{l+i}-\alpha q^{l-i}+a^2q^{2l})}{(q,a^2q^{1+2i};q)_{k-i}(q,a^{-2}q^{1-2i};q)_i},$$

it follows that

$$D_m(n,\alpha) = \sum_{k=0}^{n-m}\sum_{j=0}^{n-m-k} I_m(k+m)A_{j+k+m}(n)\sum_{i=0}^{k+m} F_i(k+m,j+k+m).$$

From the substitution $n-m-k=s$, we get

$$\begin{aligned} D_m(n,\alpha) &= \sum_{s=0}^{n-m} I_m(n-s)\sum_{j=0}^{s}\sum_{i=0}^{n-s} A_{j+n-s}(n)F_i(n-s,j+n-s) \\ &= \sum_{s=0}^{n-m} I_m(n-s)\sum_{i=0}^{n-s}\sum_{j=0}^{s} A_{j+n-s}(n)F_i(n-s,j+n-s). \end{aligned}$$

□

4.3.4 Linearization Formula of Askey-Wilson Polynomials

We want to determine the linearization coefficients $L_k(m,n)$ of the formula

$$p_n(x;a,b,c,d|q)p_m(x;a,b,c,d|q) = \sum_{k=0}^{n+m} L_k(m,n)p_k(x;a,b,c,d|q).$$

For this purpose we need to derive the linearization relation for the basis $(B_n(a,x))_n$.

Proposition 4.23 (See [Ismail and Stanton, 2003]). *The basis $(B_n(a,x))_n$ of the Askey-Wilson orthogonal polynomial family satisfies the following linearization formula*

$$B_n(a,x)B_m(a_1,x) = \sum_{k=0}^{m} H_{n+k}(m,n)B_{n+k}(a,x), \ \ m,n \in \mathbb{N}, \tag{4.51}$$

with

$$H_{n+k}(m,n) = \begin{bmatrix} m \\ k \end{bmatrix}_q q^{-nk} \left(\frac{a_1}{a}\right)^k \left(\frac{a_1}{aq^n};q\right)_{m-k} \left(aa_1q^{n+k};q\right)_{m-k}, \ \ k = 0,1,\ldots,m.$$

Proof. We recall that

$$B_n(a,x) = \prod_{j=0}^{n-1}(1-2aq^jx+a^2q^{2j}).$$

For

$$x = \varepsilon_j(a) \equiv \frac{1+a^2q^{2j}}{2aq^j} = \frac{1}{2}(aq^j + a^{-1}q^{-j}),$$

we get

$$B_n(a,\varepsilon_j(a)) = 0, \ \ n \geq 1, \ \ j = 0,1,\ldots,n-1 \tag{4.52}$$

and for $j = n$,

$$B_n(a,\varepsilon_n(a)) \neq 0. \tag{4.53}$$

In the first step, we expand $B_n(a,x)B_m(a_1,x)$, $n \geq 1$ in the basis $(B_k(a,x))_k$

$$B_n(a,x)B_m(a_1,x) = \sum_{k=0}^{n+m} H_k(m,n)B_k(a,x), \tag{4.54}$$

and use Relation (4.52) to get $H_0(m,n) = B_n(a,\varepsilon_0(a))B_m(a_1,\varepsilon_0(a)) = 0$, $n \geq 1$.

Considering (4.54) for $x = \varepsilon_1(a)$ and observing that $H_0(m,n) = 0$, we get using again (4.52) that

$$H_1(m,n)B_1(a,\varepsilon_1(a)) = B_n(a,\varepsilon_1(a))B_m(a_1,\varepsilon_1(a)) = 0, \ \ n \geq 2.$$

Therefore, $H_1(m,n) = 0$ thanks to (4.53). Progressively, we obtain in a similar way for a fixed integer j using (4.52), (4.53) and (4.54) that

$$H_0(m,n) = H_1(m,n) = \cdots = H_j(m,n) = 0, \ \ j \leq n-1.$$

In the second step, we rewrite Relation (4.54) accordingly with the previous result

$$B_n(a,x)B_m(a_1,x) = \sum_{k=0}^{m} H_{n+k}(m,n)B_{n+k}(a,x). \tag{4.55}$$

Using the relation

$$B_{n+k}(a,x) = B_n(a,x)B_k(aq^n,x),$$

derived from the definition

$$B_n(a,x) = \prod_{j=0}^{n-1}(1-aq^sq^j)(1-aq^{-s}q^j),$$

we get from (4.55) that

$$B_m(a_1,x)=\sum_{k=0}^{m}H_{n+k}(m,n)\frac{B_{n+k}(a,x)}{B_n(a,x)}=\sum_{k=0}^{m}H_{n+k}(m,n)B_k(aq^n,x). \tag{4.56}$$

For $x=\varepsilon_0(aq^n)$, Equation (4.56) gives $H_n(m,n)=B_m(a_1,\varepsilon_0(aq^n))$.

Iterating Relation (4.9), we have for fixed l

$$\mathbb{D}_x^l B_n(a,x)=\prod_{j=0}^{l-1}\eta(aq^{\frac{j}{2}},n-j)B_{n-l}(aq^{\frac{l}{2}},x),\ 1\le l\le n.$$

Applying the operator $\mathbb{D}_x^l$ to Relation (4.56), we get

$$\prod_{j=0}^{l-1}\eta(a_1q^{\frac{j}{2}},m-j)B_{m-l}(a_1q^{\frac{l}{2}},x)=\sum_{k=l}^{m}H_{n+k}(m,n)\prod_{j=0}^{l-1}\eta(aq^{n+\frac{j}{2}},k-j)B_{k-l}(aq^{n+\frac{l}{2}},x),\ 1\le l\le m.$$

For $x=\varepsilon_0(aq^{n+\frac{l}{2}})$, the preceding relation gives, taking into account (4.52)

$$H_{n+l}(m,n)=\frac{\prod\limits_{j=0}^{l-1}\eta(a_1q^{\frac{j}{2}},m-j)}{\prod\limits_{j=0}^{l-1}\eta(aq^{n+\frac{j}{2}},l-j)}B_{m-l}(a_1q^{\frac{l}{2}},\varepsilon_0(aq^{n+\frac{l}{2}})),\ l=1,2,\ldots,m.$$

We replace $\eta(a_1q^{n+\frac{j}{2}},l-j)$, $\eta(a_1q^{\frac{j}{2}},m-j)$ and $B_{m-l}(a_1q^{\frac{l}{2}},\varepsilon_0(aq^{n+\frac{l}{2}}))$ by their values obtained from Proposition 4.2 and from the definition $B_n(a,x)=\prod\limits_{j=0}^{n-1}(1-2aq^jx+a^2q^{2j})$. This yields

$$H_{n+k}(m,n)=\left(\frac{a_1}{a}\right)^k q^{(m-n-k)k}\frac{(q^{-m};q)_k}{(q^{-k};q)_k}\left(\frac{a_1}{aq^n};q\right)_{m-k}(aa_1q^{n+k};q)_{m-k}.$$

The result follows by using the transformation (4.39). □

Remark 4.24. *1. Substituting $a_1=a$ in Relation (4.51) yields*

$$\begin{aligned}B_n(a,x)B_m(a,x)&=\sum_{k=0}^{m}\begin{bmatrix}m\\k\end{bmatrix}_q q^{-nk}(q^{-n};q)_{m-k}(a^2q^{n+k};q)_{m-k}B_{n+k}(a,x)\\&=\sum_{k=n}^{n+m}\begin{bmatrix}m\\k-n\end{bmatrix}_q q^{-n(k-n)}(q^{-n};q)_{m+n-k}(a^2q^k;q)_{m+n-k}B_k(a,x).\end{aligned}\tag{4.57}$$

For negative subscripts we define [Koekoek et al., 2010, Eq. (1.8.5)]

$$(a;q)_{-n}=\frac{1}{\prod_{k=1}^{n}(1-aq^{-k})},\ a\neq q,q^2,\ldots,q^n,\ n=1,2,3,\ldots,$$

so that

$$\frac{1}{(q;q)_{-n}}=0,\ n=1,2,3,\ldots.$$

It follows from this remark that we can write again

$$B_n(a,x)B_m(a,x) = \sum_{k=0}^{n+m} \begin{bmatrix} m \\ k-n \end{bmatrix}_q q^{-n(k-n)}(q^{-n};q)_{m+n-k}(a^2q^k;q)_{m+n-k}B_k(a,x). \tag{4.58}$$

2. *If we take $m=1$ in (4.57), we recover Relation (4.13). We also remark that for $n=0$, (4.51) is the connection formula (4.37) with $a=a_1$ and $\alpha=a$.*

Having derived the linearization relation for $B_n(a,x)$, we now state and prove:

Theorem 4.25. *The Askey-Wilson orthogonal polynomial family satisfies the linearization formula*

$$p_n(x;a,b,c,d|q)p_m(x;a,b,c,d|q) = \sum_{r=0}^{n+m} L_r(m,n)p_r(x;a,b,c,d|q), \tag{4.59}$$

with

$$L_r(m,n) = \frac{q^{\frac{1}{2}r(r+1)}(ab,ac,ad;q)_n(ab,ac,ad;q)_m}{(-1)^r a^{m+n-r}(abcdq^{r-1};q)_r} \sum_{l=0}^{n+m-r} \begin{bmatrix} l+r \\ r \end{bmatrix}_q \frac{q^l(abq^r,acq^r,adq^r;q)_l}{(abcdq^{2r};q)_l} \times$$
$$\sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \frac{q^{j(j-l-r)}(q^{-n},abcdq^{n-1};q)_j}{(q;q)_{l+r-j}(q,ab,ac,ad;q)_j} \sum_{k=0}^{\min(j,m-l-r+j)} \frac{q^k(q^{-m},abcdq^{m-1};q)_{k+l+r-j}(q^{-j},a^2q^{l+r};q)_k}{(ab,ac,ad;q)_{k+l+r-j}(q;q)_k}.$$

Proof. We proceed as in the proof of Theorem 2.8 with $x^{\underline{n}}$ replaced by $B_n(a,x)$. □

Remark 4.26. *By an integral evaluation, Rahman [1981b] solved the linearization problem of the polynomial family*

$$p_n(x;a,b,c,d) = {}_4\phi_3\left(\left.\begin{matrix} q^{-n},abcdq^{n-1},ae^{i\theta},ae^{-i\theta} \\ ab,ac,ad \end{matrix}\right| q;q\right)$$

and got the linearization coefficients as a triple sum, see [Rahman, 1981b, Eq. (1.27)]. From these coefficients, he solved the linearization problem

$$P_n^{(\alpha,\beta)}(x;q)P_{n-s}^{(\alpha,\beta)}(x;q) = \sum_{j=0}^{2n-2s} b_j P_{s+j}^{(\alpha,\beta)}(x;q)$$

for the continuous q-Jacobi polynomials defined by

$$P_n^{(\alpha,\beta)}(x;q) = \frac{(q^{\alpha+1};q)_n(-q^{\beta+1};q)_n}{(q;q)_n(-q;q)_n} p_n(x;\sqrt{q},q^{\alpha+\frac{1}{2}},-q^{\beta+\frac{1}{2}},-\sqrt{q})$$

and gave the linearization coefficients as a ${}_{10}\phi_9$ series.

4.4 Inversion, Connection, Linearization and Duplication Formulae for the q-Racah, Wilson and Racah Polynomials

The inversion, connection, duplication and linearization formulae for the Askey-Wilson polynomials can be extended to other families of classical orthogonal polynomials on a quadratic or q-quadratic lattice by means of specialization and/or by limiting processes following the Askey scheme and it q-analogue. In this section we consider as example the cases of the q-Racah, Wilson and Racah polynomials. The results are derived from Equations (4.27), (4.32)-(4.35), (4.40)–(4.44), (4.50) and (4.59). For the inversion formulae, we can also use the limiting relations given in Remark 4.1. The parameter derivatives are derived from the connection formulae.

4.4.1 Inversion, Connection, Duplication and Linearization Coefficients for the q-Racah Orthogonal Polynomials

The q-Racah polynomials $R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$ are related to the Askey-Wilson polynomials in the following way. If we substitute [Koekoek et al., 2010, p. 421]

$$a^2=\gamma\delta q,\ b^2=\alpha^2\gamma^{-1}\delta^{-1}q,\ c^2=\beta^2\gamma^{-1}\delta q,\ d^2=\gamma\delta^{-1}q \text{ and } e^{2i\theta}=\gamma\delta q^{2x+1}$$

in the definition of the Askey-Wilson polynomials, we find

$$R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)=\frac{(\gamma\delta q)^{\frac{1}{2}n}}{(\alpha q,\beta\delta q,\gamma q;q)_n}\,p_n(\nu(x);(\gamma\delta q)^{\frac{1}{2}},\alpha(\gamma\delta)^{-\frac{1}{2}}q^{\frac{1}{2}},\beta\gamma^{-\frac{1}{2}}(\delta q)^{\frac{1}{2}},(\gamma q)^{\frac{1}{2}}\delta^{-\frac{1}{2}}|q),$$

where

$$\nu(x)=\frac{1}{2}\gamma^{\frac{1}{2}}\delta^{\frac{1}{2}}q^{x+\frac{1}{2}}+\frac{1}{2}\gamma^{-\frac{1}{2}}\delta^{-\frac{1}{2}}q^{-x-\frac{1}{2}}.$$

For the q-Racah polynomial family, the following relations are valid:
Inversion:

$$\xi_n(\gamma,\delta,\mu(x))=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_q\frac{(-1)^mq^{\binom{m}{2}}(\alpha q,\beta\delta q,\gamma q;q)_n}{(\alpha\beta q^{m+1};q)_m(\alpha\beta q^{2m+2};q)_{n-m}}R_m(\mu(x);\alpha,\beta,\gamma,\delta|q),$$

Connection:

$$\begin{aligned}R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)=&\sum_{m=0}^{n}\frac{(\alpha_1q)^n(q^{-n},\alpha\beta q^{n+1},\alpha_1q,\alpha_1\beta q;q)_m(1-\alpha_1\beta q^{2m+1})}{\alpha^m(q,\beta q,\frac{\alpha_1q}{\alpha q^n},\alpha_1\beta q^{n+2};q)_m(1-\alpha_1\beta q)}\\&\times\frac{(\frac{\alpha}{\alpha_1},\beta q;q)_n}{(\alpha q,\alpha_1\beta q^2;q)_n}R_m(\mu(x);\alpha_1,\beta,\gamma,\delta|q),\end{aligned}$$

$$\begin{aligned}R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)=&\sum_{m=0}^{n}\frac{(\beta_1\delta q)^n(\beta\delta)^{-m}(q^{-n},\alpha\beta q^{n+1},\beta_1\delta q,\alpha\beta_1q;q)_m(1-\alpha\beta_1q^{2m+1})}{(q,\frac{\alpha q}{\delta},\frac{\beta_1q}{\beta q^n},\alpha\beta_1q^{n+2};q)_m(1-\alpha\beta_1q)}\\&\times\frac{(\frac{\alpha q}{\delta},\frac{\beta}{\beta_1};q)_n}{(\beta\delta q,\alpha\beta_1q^2;q)_n}R_m(\mu(x),\alpha,\beta_1,\gamma,\delta|q),\end{aligned}$$

$$R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)=\sum_{m=0}^{n}\frac{q^{m(m-n)}(q;q)_n(\alpha_1 q,\beta_1\delta q,\alpha\beta q^{n+1};q)_m}{(q;q)_m(q;q)_{n-m}(\alpha q,\beta\delta q,\alpha_1\beta_1 q^{m+1};q)_m}$$
$$\times{}_4\phi_3\left(\left.\begin{array}{c}q^{m-n},\alpha_1 q^{m+1},\alpha\beta q^{m+n+1},\beta_1\delta q^{m+1}\\ \alpha q^{m+1},\beta\delta q^{m+1},\alpha_1\beta_1 q^{2m+2}\end{array}\right|q,q\right)R_n(\mu(x);\alpha_1,\beta_1,\gamma,\delta|q),$$

or more generally if $\gamma\delta=\gamma_1\delta_1$, then

$$R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)=\sum_{m=0}^{n}\frac{q^{m(m-n)}(q;q)_n(\alpha_1 q,\gamma_1 q,\beta_1\delta_1 q,\alpha\beta q^{n+1};q)_m}{(q;q)_m(q;q)_{n-m}(\alpha q,\gamma q,\beta\delta q,\alpha_1\beta_1 q^{m+1};q)_m}$$
$$\times{}_5\phi_4\left(\left.\begin{array}{c}q^{m-n},\alpha_1 q^{m+1},\gamma_1 q^{m+1},\alpha\beta q^{m+n+1},\beta_1\delta_1 q^{m+1}\\ \alpha q^{m+1},\gamma q^{m+1},\beta\delta q^{m+1},\alpha_1\beta_1 q^{2m+2}\end{array}\right|q,q\right)R_n(\mu(x);\alpha_1,\beta_1,\gamma_1,\delta_1|q),$$

Parameter derivative:

$$\frac{\partial}{\partial\alpha}R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)=\sum_{m=0}^{n-1}\Bigg(\Big(\frac{q^{m+1}}{1-\alpha q^{m+1}}-\frac{\beta q^{m+n+1}}{1-\alpha\beta q^{m+n+1}}\Big)R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$$
$$-\frac{(\alpha q)^n(q^{-n},\alpha\beta q^{n+1},\alpha q,\alpha\beta q;q)_m(1-\alpha\beta q^{2m+1})(\beta q;q)_n(q;q)_{n-1}}{\alpha^{m+1}(q,\beta q,q^{1-n},\alpha\beta q^{n+2};q)_m(1-\alpha\beta q)(\alpha q,\alpha\beta q^2;q)_n}R_m(\mu(x);\alpha,\beta,\gamma,\delta|q)\Bigg),$$

$$\frac{\partial}{\partial\beta}R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)=\sum_{m=0}^{n-1}\Bigg(\Big(\frac{\delta q^{m+1}}{1-\beta\delta q^{m+1}}-\frac{\alpha q^{m+n+1}}{1-\alpha\beta q^{m+n+1}}\Big)R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$$
$$-\frac{(\beta\delta q)^n(q^{-n},\alpha\beta q^{n+1},\beta\delta q,\alpha\beta q;q)_m(1-\alpha\beta q^{2m+1})(\frac{\alpha q}{\delta};q)_n(q;q)_{n-1}}{\beta(\beta\delta)^m(q,\frac{\alpha q}{\delta},q^{1-n},\alpha\beta q^{n+2};q)_m(1-\alpha\beta q)(\beta\delta q,\alpha\beta q^2;q)_n}R_m(\mu(x);\alpha,\beta,\gamma,\delta|q)\Bigg),$$

Duplication:

$$R_n(A\cdot\mu(x);\alpha,\beta,\gamma,\delta|q)=\sum_{m=0}^{n}D_m(n,A)R_m(\mu(x);\alpha,\beta,\gamma,\delta|q)$$

with

$$D_m(n,A)=\frac{(-1)^m q^{\binom{m}{2}+2n}(\alpha q,\beta\delta q,\gamma q;q)_m}{(\alpha\beta q^{m+1};q)_m}\sum_{s=0}^{n-m}\begin{bmatrix}n-s\\ m\end{bmatrix}_q\frac{(\alpha q^{m+1},\beta\delta q^{m+1},\gamma q^{m+1};q)_{n-m-s}}{q^{2s}(\alpha\beta q^{2m+2};q)_{n-m-s}}\times$$
$$\sum_{i=0}^{n-s}\frac{(\gamma\delta q)^{-i}q^{-i^2}}{(q,\gamma\delta q^{2i+2};q)_{n-s-i}(q,\frac{q^{-2i}}{\gamma\delta};q)_i}\times$$
$$\sum_{j=0}^{s}\frac{q^j(q^{-n},\alpha\beta q^{n+1};q)_{n+j-s}\prod_{l=0}^{j+n-s-1}(1-A\gamma\delta q^{l+i+1}-Aq^{l-i}+\gamma\delta q^{2l+1})}{(q,\alpha q,\beta\delta q,\gamma q;q)_{n+j-s}},$$

Linearization:

$$R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)R_m(\mu(x);\alpha,\beta,\gamma,\delta|q)=\sum_{r=0}^{n+m}L_r(m,n)R_r(\mu(x);\alpha,\beta,\gamma,\delta|q)$$

with

$$L_r(m,n)=\frac{(-1)^r q^{\frac{1}{2}r(r+1)}(\alpha q,\beta\delta q,\gamma q;q)_r}{(\alpha\beta q^{r+1};q)_r}\sum_{l=0}^{n+m-r}\begin{bmatrix} l+r \\ r\end{bmatrix}_q \frac{q^l(\alpha q^{r+1},\beta\delta q^{r+1},\gamma q^{r+1};q)_l}{(\alpha\beta q^{2r+2};q)_l}\times$$
$$\sum_{j=\max(0,l+r-m)}^{\min(n,l+r)}\frac{q^{j(j-l-r)}(q^{-n},\alpha\beta q^{n+1};q)_j}{(q,\alpha q,\beta\delta q,\gamma q;q)_j(q;q)_{l+r-j}}\sum_{k=0}^{\min(j,m-l-r+j)}\frac{q^k(q^{-m},\alpha\beta q^{m+1};q)_{k+l+r-j}(q^{-j},\gamma\delta q^{l+r+1};q)_k}{(\alpha q,\beta\delta q,\gamma q;q)_{k+l+r-j}(q;q)_k}.$$

4.4.2 Inversion, Connection, Duplication and Linearization Coefficients for the Wilson Orthogonal Polynomials

To construct the Wilson polynomials from the Askey-Wilson polynomials, we set [Koekoek et al., 2010, p. 421] $a\to q^a$, $b\to q^b$, $c\to q^c$, $d\to q^d$, and $e^{i\theta}=q^{ix}$ and take the limit $q\to 1$:

$$W_n(x^2;a,b,c,d)=\lim_{q\to 1}\frac{p_n(\frac{1}{2}(q^{ix}+q^{-ix});q^a,q^b,q^c,q^d|q)}{(1-q)^{3n}}.$$

From the above substitutions and the relations

$$\lim_{q\to 1}\frac{(q^\alpha;q)_n}{(1-q)^n}=(\alpha)_n,\ \lim_{q\to 1}\begin{bmatrix} n \\ m\end{bmatrix}_q=\binom{n}{m},$$

we get the following results:
Inversion:

$$\vartheta_n(a,x)=\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m(m+a+b,m+a+c,m+a+d)_{n-m}}{(a+b+c+d+m-1)_m(a+b+c+d+2m)_{n-m}}W_m(x^2;a,b,c,d),$$

Connection:

$$W_n(x^2;a,b,c,d)=\sum_{m=0}^{n}\frac{(-1)^m\binom{n}{m}(b+c,b+d,c+d,a-a_1)_n(a_1+b+c+d)_{2m}}{(b+c,b+d,c+d,a_1+b+c+d+n,a_1+1-n-a)_m}\times$$
$$\frac{(a+b+c+d+n-1)_m}{(a_1+b+c+d)_n(a_1+b+c+d+m-1)_m}W_m(x^2;a_1,b,c,d),$$

$$W_n(x^2;a,b,c,d)=\sum_{m=0}^{n}\frac{(-1)^m\binom{n}{m}(m+a+c,m+a+d,m+c+d)_{n-m}(b-b_1)_n}{(m+a+b_1+c+d-1,n+a+b_1+c+d,b_1+1-n-b)_m}\times$$
$$\frac{(n+a+b+c+d-1)_m(a+b_1+c+d)_{2m}}{(a+b_1+c+d)_n}W_m(x^2;a,b_1,c,d),$$

$$W_n(x^2;a,b,c,d)=\sum_{m=0}^{n}\frac{(-1)^m\binom{n}{m}(m+a+b,m+a+d,m+b+d)_{n-m}(c-c_1)_n}{(m+a+b+c_1+d-1,n+a+b+c_1+d,c_1+1-n-c)_m}\times$$
$$\frac{(n+a+b+c+d-1)_m(a+b+c_1+d)_{2m}}{(a+b+c_1+d)_n}W_m(x^2;a,b,c_1,d),$$

$$W_n(x^2;a,b,c,d)=\sum_{m=0}^{n}\frac{(-1)^m\binom{n}{m}(m+a+b,m+a+c,m+b+c)_{n-m}(d-d_1)_n}{(m+a+b+c+d_1-1,n+a+b+c+d_1,d_1+1-n-d)_m}\times$$
$$\frac{(n+a+b+c+d-1)_m(a+b+c+d_1)_{2m}}{(a+b+c+d_1)_n}W_m(x^2;a,b,c,d_1)$$

compare [Sánchez-Ruiz and Dehesa, 2001, p. 583],

$$W_n(x^2;a,b,c,d)=\sum_{m=0}^{n}\frac{\binom{n}{m}(n+a+b+c+d-1)_m(a+b,a+c,a+d)_n}{(m+a+\beta+\gamma+\delta-1)_m(a+b,a+c,a+d)_m}\times$$
$${}_5F_4\left(\left.\begin{matrix}m-n,m+n+a+b+c+d-1,m+a+\beta,m+a+\gamma,m+a+\delta\\2m+a+\beta+\gamma+\delta,m+a+b,m+a+c,m+a+d\end{matrix}\right|1\right)W_m(x^2;a,\beta,\gamma,\delta)$$

see [Sánchez-Ruiz and Dehesa, 2001],

$$W_n(x^2;a,b,c,d)=\sum_{m=0}^{n}\frac{\binom{n}{m}(n+a+b+c+d-1)_m(b+a,b+c,b+d)_n}{(m+\alpha+b+\gamma+\delta-1)_m(b+a,b+c,b+d)_m}\times$$
$${}_5F_4\left(\left.\begin{matrix}m-n,m+n+a+b+c+d-1,m+b+\alpha,m+b+\gamma,m+b+\delta\\2m+\alpha+b+\gamma+\delta,m+b+a,m+b+c,m+b+d\end{matrix}\right|1\right)W_m(x^2;\alpha,b,\gamma,\delta),$$

$$W_n(x^2;a,b,c,d)=\sum_{m=0}^{n}\frac{\binom{n}{m}(n+a+b+c+d-1)_m(c+a,c+b,c+d)_n}{(m+\alpha+\beta+c+\delta-1)_m(c+a,c+b,c+d)_m}\times$$
$${}_5F_4\left(\left.\begin{matrix}m-n,m+n+a+b+c+d-1,m+c+\alpha,m+c+\beta,m+c+\delta\\2m+\alpha+\beta+c+\delta,m+c+a,m+c+b,m+c+d\end{matrix}\right|1\right)W_m(x^2;\alpha,beta,c,\delta),$$

$$W_n(x^2;a,b,c,d)=\sum_{m=0}^{n}\frac{\binom{n}{m}(n+a+b+c+d-1)_m(d+a,d+b,d+c)_n}{(m+\alpha+\beta+\gamma+d-1)_m(d+a,d+b,d+c)_m}\times$$
$${}_5F_4\left(\left.\begin{matrix}m-n,m+n+a+b+c+d-1,m+d+\alpha,m+d+\beta,m+d+\gamma\\2m+\alpha+\beta+\gamma+d,m+d+a,m+d+b,m+d+c\end{matrix}\right|1\right)W_m(x^2;\alpha,\beta,\gamma,d),$$

$$W_n(x^2;a,b,c,d)=\sum_{m=0}^{n}\binom{n}{m}\frac{(n+a+b+c+d-1)_m(m+a+b,m+a+c,m+a+d)_{n-m}}{(m+\alpha+\beta+\gamma+\delta-1)_m}\times$$
$$\sum_{k=0}^{n-m}\frac{(m-n)_k(m+\alpha+\beta,m+\alpha+\gamma,m+\alpha+\delta)_k(m+n+a+b+c+d-1)_k}{k!(m+a+b,m+a+c,m+a+d)_k(2m+\alpha+\beta+\gamma+\delta)_k}\times \tag{4.60}$$
$${}_4F_3\left(\left.\begin{matrix}k+m-n,k+m+a+\alpha,m+n+k+a+b+c+d-1,a-\alpha\\m+k+a+b,m+k+a+c,m+k+a+d\end{matrix}\right|1\right)W_m(x^2;\alpha,\beta,\gamma,\delta),$$

Parameter derivative:

$$\frac{\partial}{\partial a}W_n(x^2;a,b,c,d)=\sum_{m=0}^{n-1}\Big(\frac{1}{n+m+a+b+c+d-1}W_n(x^2;a,b,c,d)+\frac{(a+b+c+d)_{2m}}{(a+b+c+d)_n}\times$$
$$\frac{(-1)^m\binom{n}{m}(n-1)!(b+c,b+d,c+d)_n(n+a+b+c+d-1)_m}{(b+c,b+d,c+d,m+a+b+c+d-1,n+a+b+c+d,1-n)_m}W_m(x^2;a,b,c,d)\Big),$$

$$\frac{\partial}{\partial b}W_n(x^2;a,b,c,d)=\sum_{m=0}^{n-1}\Big(\frac{1}{n+m+a+b+c+d-1}W_n(x^2;a,b,c,d)+\frac{(a+b+c+d)_{2m}}{(a+b+c+d)_n}\times$$
$$\frac{(-1)^m\binom{n}{m}(n-1)!(a+c,a+d,c+d)_n(n+a+b+c+d-1)_m}{(a+c,a+d,c+d,m+a+b+c+d-1,n+a+b+c+d,1-n)_m}W_m(x^2;a,b,c,d)\Big),$$

$$\frac{\partial}{\partial c}W_n(x^2;a,b,c,d)=\sum_{m=0}^{n-1}\Big(\frac{1}{n+m+a+b+c+d-1}W_n(x^2;a,b,c,d)+\frac{(a+b+c+d)_{2m}}{(a+b+c+d)_n}\times$$
$$\frac{(-1)^m\binom{n}{m}(n-1)!(a+b,a+d,b+d)_n(n+a+b+c+d-1)_m}{(a+b,a+d,b+d,m+a+b+c+d-1,n+a+b+c+d,1-n)_m}W_m(x^2;a,b,c,d)\Big),$$

$$\frac{\partial}{\partial d}W_n(x^2;a,b,c,d)=\sum_{m=0}^{n-1}\Big(\frac{1}{n+m+a+b+c+d-1}W_n(x^2;a,b,c,d)+\frac{(a+b+c+d)_{2m}}{(a+b+c+d)_n}\times$$
$$\frac{(-1)^m\binom{n}{m}(n-1)!(a+b,a+c,b+c)_n(n+a+b+c+d-1)_m}{(a+b,a+c,b+c,m+a+b+c+d-1,n+a+b+c+d,1-n)_m}W_m(x^2;a,b,c,d)\Big),$$

Linearization:

$$W_n(x^2;a,b,c,d)W_m(x^2;a,b,c,d)=\sum_{r=0}^{n+m}L_r(m,n)W_r(x^2;a,b,c,d)$$

with

$$L_r(m,n)=\frac{(-1)^r(a+b,a+c,a+d)_n(a+b,a+c,a+d)_m}{(a+b+c+d+r-1)_r}\times$$
$$\sum_{l=0}^{n+m-r}\frac{\binom{l+r}{r}(a+b+r,a+c+r,a+d+r)_l}{(a+b+c+d+2r)_l}\sum_{j=\max(0,l+r-m)}^{\min(n,l+r)}\frac{(-n,a+b+c+d+n-1)_j}{j!(l+r-j)!(a+b,a+c,a+d)_j}\times$$
$$\sum_{k=0}^{\min(j,m-l-r+j)}\frac{(-m,a+b+c+d+m-1)_{k+l+r-j}(-j,2a+l+r)_k}{(a+b,a+c,a+d)_{k+l+r-j}(k)!}.$$

To get the duplication formula for the Wilson polynomials, it is possible to proceed by a limiting process. However, we use here the Wilson operator defined by Equation (4.8). Furthermore we need the following duplication formula of the Wilson basis $(\vartheta_n(a,x))_n$.

Proposition 4.27. *The following duplication formula is valid for the Wilson basis* $(\vartheta_n(a,x))_n$*:*

$$\vartheta_n(a,\alpha x)=\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(-k)_l}{k!l!}\frac{(2a+2l)(a-\alpha a-\alpha l)_n(a+\alpha a+\alpha l)_n}{(2a+l)_{k+1}}\vartheta_k(a,x). \tag{4.61}$$

The proof of this proposition needs the following theorems which are the analogues of Theorems 4.20 and 4.21.

Theorem 4.28 (See e.g. [Ismail and Stanton, 2012]). *Let*

$$y_k=i\Big(a+\frac{k}{2}\Big),$$

and assume that $f(x)$ is a polynomial of degree n in x^2. Then

$$f(x)=\sum_{k=0}^{n} f_k\vartheta_k(a,x), \quad \textit{with} \quad f_k=\frac{1}{k!}(\mathbf{D}^k f)(y_k). \tag{4.62}$$

Proof. Let $j=0,1,\dots,k$. We apply $\mathbf{D}^j$ to both sides of $f(x)=\sum_{k=0}^{n} f_k\vartheta_k(a,x)$ and use the relation

$$\mathbf{D}^j\vartheta_k(a,x)=\frac{k!}{(k-j)!}\vartheta_{k-j}\left(a+\frac{j}{2},x\right),\ 0\le j\le k$$

to get

$$\mathbf{D}^j f(x)=f_j j!+\sum_{k=j+1}^{n} f_k\frac{k!}{(k-j)!}\vartheta_{k-j}(a+\frac{j}{2},x).$$

For $x=i\left(a+\frac{j}{2}\right)$, since $\vartheta_k(a,ai)=0,\ \forall k\ge 1$, we obtain

$$\mathbf{D}^j f\left(i\left(a+\frac{j}{2}\right)\right)=f_j j!.$$

This proves the theorem. □

Theorem 4.29 (See [Cooper, 2012]). *Let k be a nonnegative integer. Then*

$$\mathbf{D}^k f(x)=\sum_{l=0}^{k}\frac{(-k)_l}{l!}\frac{(2ix-k+2l)}{(2ix-k+l)_{k+1}}f\left(x+\frac{k-2l}{2}i\right). \tag{4.63}$$

Proof (of Proposition 4.27). We combine (4.62) and (4.63) with $x=i(a+\frac{k}{2})$ to get

$$f_k=\sum_{l=0}^{k}\frac{(-k)_l}{l!k!}\frac{(-2a-2k+2l)}{(-2a-2k+l)_{k+1}}f(i(a+k-l)).$$

If we substitute l by $k-l$ and $f(x)$ by $\vartheta_n(a,\alpha x)$, using $(-1)^{k+1}(-2a-k-l)_{k+1}=(2a+l)_{k+1}$, the result follows. □

Combining the representation of the Wilson polynomial w.r.t. the basis $(\vartheta_n(a,x))_n$, the duplication formula (4.61) and the inversion formula of the Wilson polynomials and proceeding as in the proof of Theorem 4.22, we get

Theorem 4.30. *The following duplication formula is valid for the Wilson polynomials:*

$$W_n((\alpha x)^2;a,b,c,d)=\sum_{m=0}^{n} D_m(n,\alpha)W_m(x^2;a,b,c,d) \quad \textit{with}$$

$$D_m(n,\alpha)=\frac{(-1)^m(a+b,a+c,a+d)_n}{(a+b+c+d+m-1)_m}\sum_{s=0}^{n-m}\frac{\binom{n-s}{m}(m+a+b,m+a+c,m+a+d)_{n-m-s}}{(n-s)!(2m+a+b+c+d)_{n-m-s}}$$
$$\sum_{l=0}^{n-s}\frac{(-n+s)_l(2a+2l)}{l!(2a+l)_{n-s+1}}\sum_{j=0}^{s}\frac{(-n,n+a+b+c+d-1)_{n+j-s}(a-\alpha a-\alpha l,a+\alpha a+\alpha l)_{j+n-s}}{(j+n-s)!(a+b,a+c,a+d)_{j+n-s}}.$$

4.4.3 Inversion, Connection and Linearization Coefficients for the Racah Orthogonal Polynomials

If we set [Koekoek et al., 2010, p. 190]

$$a=\frac{1}{2}(\gamma+\delta+1),\ b=\frac{1}{2}(2\alpha-\gamma-\delta+1),\ c=\frac{1}{2}(2\beta-\gamma+\delta+1),\ d=\frac{1}{2}(\gamma-\delta+1)$$

and replace

$$ix\to x+\frac{1}{2}(\gamma+\delta+1)$$

in

$$\tilde{W}_n(x^2;a,b,c,d)=\frac{W_n(x^2;a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n},$$

and take

$$\alpha+1=-N\quad\text{or}\quad\beta+\delta+1=-N\quad\text{or}\quad\gamma+1=-N$$

with a nonnegative integer N, we obtain the Racah polynomials $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$. Therefore, for the Racah polynomials, the following relations are valid:
Inversion:

$$\chi_n(\gamma,\delta,\lambda(x))=\sum_{m=0}^{n}\frac{(-1)^m\binom{n}{m}(\alpha+1,\gamma+1,\beta+\delta+1)_n}{(m+\alpha+\beta+1)_m(2m+\alpha+\beta+2)_{n-m}}R_m(\lambda(x);\alpha,\beta,\gamma,\delta)$$

see e.g. [Area et al., 2001, Eq. (3.7)],
Connection:

$$\begin{aligned}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)=&\sum_{m=0}^{n}\frac{(-n,\alpha+\beta+n+1,\alpha_1+1,\alpha_1+\beta+1)_m(\alpha_1+\beta+2m+1)}{m!(\beta+1,\alpha_1+1-n-\alpha,\alpha_1+\beta+n+2)_m(1+\alpha_1+\beta)}\\&\times\frac{(\beta+1,\alpha-\alpha_1)_n}{(\alpha+1,\alpha_1+\beta+2)_n}R_m(\lambda(x);\alpha_1,\beta,\gamma,\delta),\end{aligned}$$

$$\begin{aligned}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)=&\sum_{m=0}^{n}\frac{(-n,\alpha+\beta+n+1,\alpha+\beta_1+1,\beta_1+\delta+1)_m(\alpha+\beta_1+2m+1)}{m!(\alpha-\delta+1,\beta_1+1-n-\beta,\beta_1+\alpha+n+2)_m(1+\alpha+\beta_1)}\\&\times\frac{(\alpha+1-\delta,\beta-\beta_1)_n}{(\beta+\delta+1,\alpha+\beta_1+2)_n}R_m(\lambda(x);\alpha,\beta_1,\gamma,\delta).\end{aligned}$$

When we perform the above substitutions in (4.60), the ${}_4F_3$ term is reduced to 1 if $\gamma+\delta=c+d$ and therefore we get (see e.g. [Sánchez-Ruiz and Dehesa, 2001, Eq. (13)])

$$\begin{aligned}R_n(\lambda(x),\alpha,\beta,\gamma,\delta)=&\sum_{m=0}^{n}\binom{n}{m}\frac{(n+\alpha+\beta+1)_m(a+1,b+d+1,c+1)_m}{(\alpha+1,\beta+\delta+1,\gamma+1,m+a+b+1)_m}\times\\&{}_5F_4\left(\left.\begin{matrix}m-n,n+m+\alpha+\beta+1,m+a+1,m+b+d+1,m+c+1\\m+\alpha+1,m+\beta+\delta+1,2m+a+b+2,m+\gamma+1\end{matrix}\right|1\right)R_m(\lambda(x),a,b,c,d)\end{aligned}$$

under the condition $\gamma+\delta=c+d$.
Parameter derivatives:

$$\begin{aligned}\frac{\partial}{\partial\alpha}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)=&\sum_{m=0}^{n-1}\Big(\Big(\frac{1}{m+n+\alpha+\beta+1}-\frac{1}{m+\alpha+1}\Big)R_n(\lambda(x);\alpha,\beta,\gamma,\delta)\\&+\frac{(-n,\alpha+\beta+n+1,\alpha+1,\alpha+\beta+1)_m(\alpha+\beta+2m+1)(\beta+1)_n(n-1)!}{m!(\beta+1,1-n,\alpha+\beta+n+2)_m(1+\alpha+\beta)(\alpha+1,\alpha+\beta+2)_n}R_m(\lambda(x);\alpha,\beta,\gamma,\delta)\Big),\end{aligned}$$

$$\frac{\partial}{\partial\alpha}R_n(\lambda(x);\alpha,\beta,\gamma,\delta)=\sum_{m=0}^{n-1}\Bigg(\Big(\frac{1}{m+n+\alpha+\beta+1}-\frac{1}{m+\beta+\delta+1}\Big)R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$$
$$+\frac{(-n,\alpha+\beta+n+1,\alpha+\beta+1,\beta+\delta+1)_m(\alpha+\beta+2m+1)(\alpha+1-\delta)_n(n-1)!}{m!(\alpha-\delta+1,1-n,\beta+\alpha+n+2)_m(1+\alpha+\beta)(\beta+\delta+1,\alpha+\beta+2)_n}R_m(\lambda(x);\alpha,\beta,\gamma,\delta)\Bigg),$$

Linearization:

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta)R_m(\lambda(x);\alpha,\beta,\gamma,\delta)=\sum_{r=0}^{n+m}L_r(m,n)R_r(\lambda(x);\alpha,\beta,\gamma,\delta)$$

with

$$L_r(m,n)=\sum_{l=0}^{n+m-r}\frac{(-1)^r(\alpha+1,\beta+\delta+1,\gamma+1)_r}{(\alpha+\beta+r+1)_r}\frac{\binom{l+r}{r}(\alpha+r+1,\beta+\delta+r+1,\gamma+r+1)_l}{(\alpha+\beta+2r+2)_l}$$
$$\sum_{j=\max(0,l+r-m)}^{\min(n,l+r)}\frac{(-n,n+\alpha+\beta+1)_j}{j!(l+r-j)!(\alpha+1,\beta+\delta+1,\gamma+1)_j}\times$$
$$\sum_{k=0}^{\min(j,m+j-l-r)}\frac{(-m,m+\alpha+\beta+1)_{k+l+r-j}(-j,\gamma+\delta+l+r+1)_k}{(\alpha+1,\beta+\delta+1,\gamma+1)_{k+l+r-j}k!}.$$

Chapter 5

Conclusion and Perspectives

We have provided in this thesis representations for the inversion, connection, linearization and duplication coefficients of all classical orthogonal polynomials of a continuous, discrete and q-discrete variable listed in [Koekoek et al., 2010]. We also solved the inversion, connection, linearization and duplication problem for the Askey-Wilson polynomials and showed how the results can be extended to other families of classical orthogonal polynomials on quadratic or q-quadratic lattice. From the various connection formulae parameter derivatives have been derived.

Using our algorithmic approach, we recovered known results and obtained many new ones.

As perspectives, it would be interesting:

1. to find conditions on the parameters to simplify the coefficients appearing in double and triple summation,

2. to compute by means of limit and/or specialization the inversion, connection, linearization and duplication coefficients for other families of classical orthogonal polynomials on a quadratic or q-quadratic lattice,

3. to extend our method to other classes of orthogonal polynomials that are obtained by a modification of the three-term recurrence relations of the classical orthogonal polynomials, for example [Foupouagnigni, 2006]: the associated orthogonal polynomials, the co-recursive orthogonal polynomials,

4. to implement in Maple for example an algorithm which solves the inversion, connection, linearization or duplication problem for the classical orthogonal polynomials.

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