Deterministic Genericity and the Computation of homological Invariants

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Summary

The main goal of this thesis is to discuss the determination of homological invariants of polynomial ideals. Thereby we consider different coordinate systems and analyze their meaning for the computation of certain invariants. In particular, we provide an algorithm that transforms any ideal into strongly stable position if char $\mathbb{k} = 0$. With a slight modification, this algorithm can also be used to achieve a stable or quasi-stable position. If our field has positive characteristic, the Borel-fixed position is the maximum we can obtain with our method. Further, we present some applications of Pommaret bases, where we focus on how to directly read off invariants from this basis.

In the second half of this dissertation we take a closer look at another homological invariant, namely the (absolute) reduction number. It is a known fact that one immediately receives the reduction number from the basis of the generic initial ideal. However, we show that it is not possible to formulate an algorithm – based on analyzing only the leading ideal – that transforms an ideal into a position, which allows us to directly receive this invariant from the leading ideal. So in general we can not read off the reduction number of a Pommaret basis. This result motivates a deeper investigation of which properties a coordinate system must possess so that we can determine the reduction number easily, i.e. by analyzing the leading ideal. This approach leads to the introduction of some generalized versions of the mentioned stable positions, such as the *weakly D-stable* or *weakly D-minimal stable* position. The latter represents a coordinate system that allows to determine the reduction number without any further computations. Finally, we introduce the notion of *β-maximal* position, which provides lots of interesting algebraic properties. In particular, this position is in combination with weakly D-stable sufficient for the weakly D-minimal stable position and so possesses a connection to the reduction number.

Keywords: (Strongly) Stable Ideals, Quasi-stable Ideals, Borel-fixed Ideals, Pommaret Basis, Reduction Number, Weakly D-Minimal Stable Ideals, weakly D-stable, β-Maximality

Notations

$\Bbbk$	infinite field
$\mathrm{Gl}(n, \Bbbk)$	general linear group of degree n over $\Bbbk$
$\mathcal{P}$	the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ in n variables
$\mathfrak{m}$	the maximal homogeneous ideal $\langle x_1, \ldots, x_n \rangle \lhd \mathcal{P}$
$\mathbf{H}^i_{\mathfrak{m}}(\mathcal{M})$	ith local cohomology of $\mathcal{M}$ with respect to $\mathfrak{m}$
$\mathbf{x}^\mu$	$x_1^{\mu_1} \cdots x_n^{\mu_n}$, $\mu \in \mathbb{N}^n$
$\mathbb{T}$	$\{\mathbf{x}^\mu \mid \mu \in \mathbb{N}^n\}$, set of all terms
$\mathcal{I}$	homogeneous polynomial ideal
$\dim(\mathcal{P}/\mathcal{I})$	Krull dimension of $\mathcal{P}/\mathcal{I}$
$\dim_\Bbbk$	$\Bbbk$-vector space dimension
$\mathcal{I}_q$	set of all homogeneous elements of $\mathcal{I}$ with degree q
$\mathcal{I} : \tilde{\mathcal{I}}$, $\mathcal{I} : f$	ideal quotient, $\mathcal{I} : \tilde{\mathcal{I}} = \{f \in \mathcal{P} \mid f\tilde{\mathcal{I}} \subseteq \mathcal{I}\}$, $\mathcal{I} : f = \mathcal{I} : \langle f \rangle$
$\mathcal{I} : \tilde{\mathcal{I}}^\infty$	$\bigcup_k \mathcal{I} : \tilde{\mathcal{I}}^k$
$\prec$	degree reverse lexicographical term order, see Chapter 1 on page 5
$\prec_{\text{revlex}}$	reverse lexicographical order, see Definition 2.2.7 on page 20

$\prec_{\text{lex}}$	lexicographical term order, see Definition 6.1.3 on page 92
$\operatorname{lt} f$, $\operatorname{lc} f$	leading term of a polynomial f with respect to $\prec$, leading coefficient of a polynomial f
$\operatorname{lt} \mathcal{I}$	leading ideal of $\mathcal{I}$ with respect to $\prec$
$\deg\{f_1, \dots, f_\ell\}$	$\max\{\deg f_1, \dots, \deg f_\ell\}$
$\operatorname{supp}(f)$	support of a polynomial f (e.g. $\operatorname{supp}(\sum_{i=1}^{\ell} a_i t_i) = \{t_1, \dots, t_\ell\}$, $a_i \in \Bbbk$, $t_i \in \mathbb{T}$)
$\#\{\dots\}$	number of elements of the set $\{\dots\}$
$\beta_{ij}^{\mathcal{M}}$, $\beta_i^{\mathcal{M}}$	graded and total Betti number, see Definition 1.1.1 on page 5
$\operatorname{Syz}_i(\mathcal{M})$	ith Module of Syzygies of $\mathcal{M}$, see Remark 1.1.5 on page 7
$\operatorname{reg}(\mathcal{I})$	Castelnuovo-Mumford regularity of $\mathcal{I}$, see Definition 1.1.6 on page 7
$\operatorname{pd}(\mathcal{I})$	projective dimension of $\mathcal{I}$, see Definition 1.1.8 on page 7
$\operatorname{depth}(\mathcal{I})$	depth of $\mathcal{I}$, see Definition 1.1.12 on page 8
$\operatorname{Ext}^i_{\mathcal{P}}(\mathcal{M}, \mathcal{N})$	the ith Ext-module of $\mathcal{P}$-modules $\mathcal{M}, \mathcal{N}$, see Remark 1.1.13 on page 8
$A \cdot \mathcal{I}, \Psi(\mathcal{I})$	coordinate transformation, see Notation 1.3.3 on page 11
$\mathrm{m}(f)$, $\mathrm{m}(F)$	maximal index, see Notation 1.3.7 on page 13
$\mathcal{L}_i$	see Remark 2.1.2 on page 16
$\mathscr{L}(F)$, $\prec_{\mathscr{L}}$	see Definition 2.2.7 on page 20

$F^{\blacktriangle}$, F^{Δ}	(head) autoreduced set, see Definition 2.2.11 on page 22
$\mathtt{C}_f(t)$	see proof of Lemma 2.2.13 on page 23
$\mathfrak{B}$, $\mathfrak{B}_\ell^w$, $\mathfrak{B}_\ell$	Borel group, see Definition 2.4.1 on page 35, weak ℓ-Borel group, see Definition 5.3.1 on page 86, ℓ-Borel group, see Definition 5.3.8 on page 88
$\langle F\rangle_{\mathscr{P}}$	Pommaret span, see Definition 3.1.1 on page 39
$\vert_{\mathscr{P}}$	Pommaret division, see Definition 3.1.4 on page 40
$\beta_0^{(k)}$, $\beta_{0,j}^{(k)}$	see Lemma 3.2.4 on page 42, see Remark 3.2.13 on page 45
$S_{(\alpha,j)}$	see Notation 3.2.7 on page 44
$\mathcal{I}_{\langle q\rangle}$	the component ideal $\mathcal{I}_{\langle q\rangle} = \langle \mathcal{I}_q\rangle$, see Chapter 3 on page 46
$\mathcal{I}_{\geq q}$	the truncated ideal $\mathcal{I}_{\geq q} = \bigoplus_{p\geq q}\mathcal{I}_p$, see Chapter 3 on page 47
$\mathcal{I}_{[q]}$	$\langle \bigcup_{p\leq q}\mathcal{I}_p\rangle$, see Chapter 3 on page 49
$H^{(i)}$	$\{h\in H \mid \mathrm{m}(h) = i\}$, see Proposition 3.2.39 on page 58
$a_i(\mathcal{P}/\mathcal{I})$, $\mathrm{reg}_t(\mathcal{P}/\mathcal{I})$, $a_t^*(\mathcal{P}/\mathcal{I})$	see Chapter 3 on page 59
$r_{\mathcal{R}}(\mathcal{P}/\mathcal{I})$, $r(\mathcal{P}/\mathcal{I})$	(absolute) reduction number, see Definition 4.1.2 on page 61
$\mathrm{rSet}(\mathcal{P}/\mathcal{I})$	set of all reduction numbers, see Definition 4.1.2 on page 61
$br(\mathcal{P}/\mathcal{I})$	big reduction number, see Definition 4.2.2 on page 65
$\deg_{x_k}\mathcal{J}$	maximal x_k-degree of a minimal generator of $\mathcal{J}$, see Notation 5.2.12 on page 84

$B_q(\mathcal{I})$	see Definition 6.1.1 on page 91
$\beta_q^{(k)}(\mathcal{I})$, $\beta_q(\mathcal{I})$	β-vector, see Definition 6.1.1 on page 91
$h_{\mathcal{I}}$, $h_{F,\mathscr{P}}$	Hilbert function, see Notation 6.1.4 on page 92
$\beta_{F_q}^{(k)}$	$\#\{f \in F_q \mid \mathrm{m}(f) = k\}$, see Remark 6.1.5 on page 92
$\mathtt{C}_{ik}$	see Remark 6.1.5 on page 92

CHAPTER 0

Introduction

Commutative algebra is an area of mathematics that focuses mainly on the study of commutative rings and their ideals. The special case of polynomial ideals will be in the center of our attention throughout this thesis. In order to analyze, characterize and compare given ideals one considers homological invariants. These invariants can be interpreted as a kind of complexity measure. To understand what "complexity measure" means in this context one has to consider a given ideal $\mathcal{I}$ as a module. As free modules present the simplest class of modules one can say that the more a module differs from being a free module the higher is its complexity. To measure the "distance" from $\mathcal{I}$ to a free module one takes a look at the *minimal free resolution*. In Section 1.1 we provide a short review of the concept of free resolutions, in particular we repeat the fact that any module we consider in this thesis has a unique minimal free resolution. In Remark 1.1.3 we also offer a brief insight into a method to construct a free resolution. This already gives a first impression on how costly it is to compute the minimal free resolution. However, the determination of the minimal free resolution is a classical method to retrieve to already mentioned homological invariants. Well-known examples of such invariants are the *Castelnuovo-Mumford regularity*, the *projective dimension* and the *depth*. While the first two can directly be read off from the minimal free resolution, the latter one can be derived indirectly. Therefore one uses a famous result from Auslander and Buchsbaum that the sum of the projective dimension and the depth equals the number of variables of the considered polynomial ring. Hence the determination of one of those two invariants immediately yields to the other one. An alternative computation of the depth is based on the determination of several Ext-modules that we describe in Remark 1.1.13. Unfortunately, it is not possible to predict whether either the projective dimension or the depth is easier to determine in general.

Anyway this is not the approach we want to consider in this thesis. We rather would like to investigate another ansatz that is motivated by some properties of the *generic initial ideal* $\operatorname{gin}\mathcal{I}$, which was introduced by Galligo and Bayer/Stillman[1]. Thereby we consider the fact that the determination of the mentioned invariants

[1] Green's article [**Gre98**] offers a summary for the various attributes of the generic initial ideal.

becomes trivial whenever the chosen coordinate system is in gin-*position*, i.e. the leading ideal of $\mathcal{I}$ equals $\operatorname{gin}\mathcal{I}$. Bayer and Stillman showed that in this case a lot of invariants of $\mathcal{I}$ coincide with those of $\operatorname{gin}\mathcal{I}$ and can directly be read off the minimal monomial basis of $\operatorname{gin}\mathcal{I}$ as formulated in Theorem 1.3.9. For the computation of $\operatorname{gin}\mathcal{I}$ there exist mainly two different approaches, which are both rather expensive.

The first one is based on performing a random change of coordinates. It is a theoretical result of probability theory that then after this transformation the leading ideal of our considered ideal is $\operatorname{gin}\mathcal{I}$. This method is even used by some computer algebra systems such as CoCoA . As this approach is not a deterministic but a probabilistic method it would be interesting to verify the final result, but there is no ansatz known that provides this opportunity. Further, a random coordinate change leads to dense polynomials, which obviously cause a more costly computation especially when computing a Gröbner basis.

The second approach is to compute a *Gröbner system*[2] of $\mathcal{I}$, which was introduced by Weispfenning [**Wei92**] in the context of his research on comprehensive Gröbner basis. This ansatz represents – contrary to the method above – a deterministic way to compute $\operatorname{gin}\mathcal{I}$ so that a verification of the result is not necessary. However, the determination of a Gröbner system requires parametric computations, which makes this method even more expensive than the first one.

Hence we want to discover alternative coordinate systems that might be easier respectively cheaper to reach but still provide properties similar to a coordinate system in gin-position. To implement this plan we start with the consideration of ideals in *quasi-stable*, *stable* and *strongly stable* position. In Algorithm 1 we provide – as one of the main results of this dissertation – a deterministic way to put any ideal into strongly stable position. Thereby we use sparse coordinate changes so that in most cases this method is much cheaper than transforming into gin-position. With slight modifications this algorithm can also be used to achieve a quasi-stable or stable position. Especially the quasi-stable position is of great interest since ideals in this position possess a finite *Pommaret basis*, which is a special kind of involutive bases. Seiler shows in [**Sei09b**] that the above mentioned invariants can be read off the Pommaret basis and so he generalizes some results of Bayer/Stillman concerning the generic initial ideal.

However, in the *reduction number* we found an invariant that in general can not be obtained from the Pommaret basis. The main focus of the second half of this dissertation will lie on the study of this invariant. As another important result we even prove that it is not possible to formulate a *simple* algorithm that transforms a given ideal into a position from which the reduction number can be determined easily. Thereby we call an algorithm simple if it is based on analyzing only the leading ideal. This result motivates to develop generalized versions of the stable

[2] We recall its definition in Definition 2.1.1

positions, in order to construct a position that allows a simple determination of the reduction number. In this context we establish the theory of β*-maximality* as the final highlight of this thesis. As we are able to directly derive the reduction number from ideals in β-maximal position under certain assumptions, this ideal class represent a generalization of the generic position in the considered context.

Most of the results of this thesis have already been published in the papers [**HSS12**] and [**HSS14**]. A third paper with the same authors and the title *Deterministic Genericity for Polynomial Ideals* is already in preparation but was not published at the date of submitting this dissertation.

We now present short summaries of every chapter of this thesis:

Chapter 1. We repeat the definition of the minimal free resolution of an ideal and show its relation to the homological invariants Castelnuovo-Mumford regularity, projective dimension and depth. Further, we compare the invariants of an ideal with the one of its leading ideal and introduce the generic initial ideal $\operatorname{gin}\mathcal{I}$.

Chapter 2. We present a way to deterministically compute $\operatorname{gin}\mathcal{I}$ via Gröbner systems. Then different stable positions are introduced and – as one of the main results of this thesis – Algorithm 1 describes a method to put any ideal into one of the stable positions under the assumption $\operatorname{char}\Bbbk = 0$. Afterwards we discuss the case $\operatorname{char}\Bbbk > 0$, where a slightly modified version of the mentioned algorithm leads at least to a Borel-fixed position.

Chapter 3. We provide a short overview of the theory of Pommaret basis – which are a special kind of involutive basis – and explain its relation to quasi-stability. Afterwards we repeat some results of [**Sei09b**], concerning how to read off several invariants of a Pommaret basis. We are able to deliver more such invariants before we present further applications of Pommaret bases. Thereby another stable position – which we call *componentwise quasi-stable* – is introduced. For this position the component ideals $\mathcal{I}_{\langle q\rangle}$ for all integers $q \geq 0$ are considered.

Chapter 4. We study the reduction number and present an algorithm for its computation. An example provided by Green [**Gre98**] shows us that this invariant can not be read off a Pommaret basis in general. Even more, it is another consequence of this example that it is not possible to transform an ideal into a coordinate system, from which the reduction number can be determined by analyzing the corresponding leading ideal, with a simple[3] algorithm.

[3]See explanation above.

Chapter 5. In this chapter we establish some generalized versions of the quasi-stable and stable position. Thereby we especially outline the *weakly D-stable* and *weakly D-quasi-stable* position. Weakly D-stable ideals are monomial ideals with the property that their reduction number can be read off their monomial basis. This position plays also a decisive role when we introduce the notion of *weakly D-minimal stable*, which describes a position with properties similar to the gin-position in terms of the reduction number. Further, we are able to provide an alternative definition of the well-known Noether position by showing its equivalence to weakly D-quasi-stable position. Remarkably, we thereby present a combinatorial characterization of Noether position. Moreover, since weakly D-quasi-stable position is achievable by a modified version of Algorithm 1, we thus deliver a deterministic algorithm that transforms a given ideal into Noether position.

Chapter 6. With the concept of β-maximal position we introduce another class of ideals that possesses interesting algebraic properties. For example, we will see that in this position the length of the Pommaret basis is minimal under the restriction that we only consider polynomial rings with at most three variables. Further, we also have a connection to the reduction number since β-maximality implies weak D-minimal stability if we assume that the considered ideal is in weakly D-stable position.

Chapter 7. Finally, we provide several examples that allow us to clearly separate all of the discussed positions and enables us to draw the *map of positions*.

CHAPTER 1

Preliminaries

At the beginning of this thesis we repeat some well-known facts that are a required for the following chapters. Thereby we denote throughout this dissertation by $\mathcal{I}$ a homogeneous ideal of the polynomial ring $\mathcal{P} = \Bbbk[x_1, \ldots, x_n]$ in n variables over an infinite field $\Bbbk$. Further, we use the multi-index notation $\mathbf{x}^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$, where $\mu \in \mathbb{N}^n$ is a n-tuple. As term order $\prec$ we use the *degree reverse lexicographical*[1] order defined by $\mathbf{x}^\mu \prec \mathbf{x}^\nu$, if and only if $\deg \mathbf{x}^\mu < \deg \mathbf{x}^\nu$ or $\mu_m > \nu_m$ with $m = \max\{i \mid \mu_i \neq \nu_i\}$ (compare [**LA94**, Def. 1.4.4]) . In particular, we denote by $\operatorname{lt} \mathcal{I}$ the leading ideal of $\mathcal{I}$ with respect to the term order $\prec$.

As most of the invariants that we want to analyze in this thesis are related to the minimal free resolution, we first present a short overview of the most important results, which are associated with this concept. Afterwards we will talk about the difference between the invariants of an ideal $\mathcal{I}$ and its leading ideal $\operatorname{lt} \mathcal{I}$, before we finally take a closer look at the notion of genericity.

1.1. Free Resolution

In the following $\mathcal{M}$ will denote a finitely generated graded $\mathcal{P}$-module.

Definition 1.1.1.
A *free resolution* of $\mathcal{M}$ is an exact sequence

$$\cdots \xrightarrow{\varphi_{i+1}} \mathcal{F}_i \xrightarrow{\varphi_i} \mathcal{F}_{i-1} \xrightarrow{\varphi_{i-1}} \cdots \xrightarrow{\varphi_2} \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} \mathcal{M} \longrightarrow 0$$

where the $\mathcal{F}_i$ are free graded $\mathcal{P}$-Modules of the form

$$\mathcal{F}_i = \bigoplus_j \mathcal{P}(-j)^{r_{i,j}}$$

with $\mathcal{P}(-j) = \bigoplus_\nu \mathcal{P}_{\nu - j}$ and φ_i are graded homomorphisms with $\varphi_i((\mathcal{F}_i)_r) \subseteq \mathcal{P}_r$. If there is an index ℓ such that $\mathcal{F}_\ell \neq 0$ and $\mathcal{F}_j = 0$ for all $j > \ell$, we call the resolution *finite*. If the smallest ℓ with this property is denoted by ℓ_0, the *length* of the resolution is $\ell_0 + 1$. The resolution is called *minimal* if the maps φ_i satisfy $\operatorname{im} \varphi_i \subseteq \mathfrak{m} \bigoplus_j \mathcal{P}(-j)^{r_{i,j}-1}$, where $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ is the maximal graded ideal.

[1] As we often cite the references [**Sei09a**], [**Sei09b**], [**Sei10**], [**Sei12**] and [**HSS12**], we want to mention that in these papers the degree reverse lexicographical ordering is defined by $\mathbf{x}^\mu \prec \mathbf{x}^\nu$, if and only if $\deg \mathbf{x}^\mu < \deg \mathbf{x}^\nu$ or $\mu_m > \nu_m$ with $m = \min\{i \mid \mu_i \neq \nu_i\}$.

In this case the ranks $r_{i,j}$ are the *graded Betti numbers* $\beta_{i,j}^{\mathcal{M}}$. Finally, we define the *(total) Betti numbers* $\beta_i^{\mathcal{M}}$ by $\beta_i^{\mathcal{M}} = \sum_j \beta_{i,j}^{\mathcal{M}}$.

THEOREM 1.1.2 (Graded Hilbert Syzygy Theorem ([**Eis95**, Thm. 1.13])).
Every finitely generated graded $\mathcal{P}$-module has a finite graded free resolution of length at most n.

REMARK 1.1.3 ([**Sei10**, Prop. B.2.31, et. seq.]).
One way to construct a free resolution of $\mathcal{M}$ is based on the fact that any $\mathcal{P}$-module is *finitely presented*, i.e. there is a $\mathcal{P}$-module $\mathcal{M}_0$ and an integer r_0 such that

$$0 \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{P}^{r_0} \longrightarrow \mathcal{M} \longrightarrow 0$$

is an exact sequence. Indeed, if $\{m_1, \dots, m_\ell\}$ is a generating set of $\mathcal{M}$ we can define a homomorphism $\varphi : \mathcal{P}^\ell \to \mathcal{M}$ by $\varphi(f_1, \dots, f_\ell) = \sum_{i=1}^{\ell} f_i m_i$ so that a finite presentation of $\mathcal{M}$ is given by:

$$0 \longrightarrow \ker\varphi \hookrightarrow \mathcal{P}^\ell \xrightarrow{\varphi} \mathcal{M} \longrightarrow 0$$

Now, as the $\mathcal{P}$-module $\mathcal{M}_0 = \ker\varphi$ is also finitely presented, we find another $\mathcal{P}$-module $\mathcal{M}_1$ and an integer r_1 such that

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{P}^{r_1} \longrightarrow \mathcal{M}_0 \longrightarrow 0$$

is an exact sequence. Going on like this leads to free resolution of $\mathcal{M}$ as the following diagram demonstrates:

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & \mathcal{M}_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[rr] \arrow[dr] & & \mathcal{P}^{r_2} \arrow[ur] \arrow[rr] & & \mathcal{P}^{r_1} \arrow[rr] \arrow[dr] & & \mathcal{P}^{r^0} \arrow[r] & \mathcal{M} \arrow[r] & 0 \\
 & \mathcal{M}_2 \arrow[ur] \arrow[dr] & & & & \mathcal{M}_0 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

THEOREM 1.1.4 ([**Eis95**, Thm. 20.2]).
The minimal free resolution of $\mathcal{M}$ is unique up to isomorphism.

Remark 1.1.5 ([**GP08**, Rem. 2.5.2, et. seq.]).
Let the following sequence be the minimal free resolution of the module $\mathcal{M}$:

$$0 \longrightarrow \mathcal{F}_\ell \xrightarrow{\varphi_\ell} \mathcal{F}_{\ell-1} \xrightarrow{\varphi_{\ell-1}} \cdots \xrightarrow{\varphi_2} \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} \mathcal{M} \longrightarrow 0 \tag{1.1}$$

Then we should note that the number of minimal generators of the $\mathcal{P}$-module $\ker \varphi_i$ is $\beta_i^{\mathcal{M}}$ and $\beta_{i,j}^{\mathcal{M}}$ is the number of minimal generators of $\ker \varphi_i$ in degree $i+j$. In the literature $\ker \varphi_i$ is called the ith *module of syzygies of* $\mathcal{M}$ and denoted by $\mathrm{Syz}_i(\mathcal{M})$. Thereby we understand under a *syzygy* of k elements $f_1, \ldots, f_k \in \mathcal{M}$ a k-tuple $(g_1, \ldots, g_k) \in \mathcal{P}^k$ with $\sum_{i=0}^k g_i f_i = 0$.

Definition 1.1.6 ([**HH11**, p.48]).
The *Castelnuovo-Mumford regularity* $\mathrm{reg}(\mathcal{M})$ of $\mathcal{M}$ is defined by

$$\mathrm{reg}(\mathcal{M}) = \max\{j \mid \beta_{i,i+j}^{\mathcal{M}} \neq 0 \text{ for some } i\}.$$

Remark 1.1.7 ([**Cha07**, §1]).
An alternative definition that connects $\mathrm{reg}(\mathcal{M})$ with the ith local cohomology $\mathbf{H}_{\mathfrak{m}}^i(\mathcal{M})$ of $\mathcal{M}$ with respect to $\mathfrak{m}$ is

$$\mathrm{reg}(\mathcal{M}) = \max\{a_i(\mathcal{M}) + i \mid i \geq 0\},$$

where $a_i(\mathcal{M}) = \max\{q \mid \mathbf{H}_{\mathfrak{m}}^i(\mathcal{M})_q \neq 0\}$.

Definition 1.1.8 ([**HH11**, p.48]).
The *projective dimension* $\mathrm{pd}(\mathcal{M})$ of $\mathcal{M}$ is defined by

$$\mathrm{pd}(\mathcal{M}) = \max\{i \mid \beta_{i,j}^{\mathcal{M}} \neq 0 \text{ for some } j\}.$$

With other words, $\mathrm{pd}(\mathcal{M})$ is the length of the minimal free resolution of $\mathcal{M}$.

Remark 1.1.9 ([**Eis05**, page 7, et. seq.]).
If again (1.1) is the minimal free resolution of $\mathcal{M}$ the tabular

$$\begin{array}{rr|cccccc}
 & & & & & \mathrm{pd}(\mathcal{M}) & & \\
 & & & & & \shortparallel & & \\
 & & 0 & 1 & \cdots & \ell & & \\
\hline
 & & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & j & \beta_{0,j}^{\mathcal{M}} & \beta_{1,1+j}^{\mathcal{M}} & \cdots & \beta_{\ell,\ell+j}^{\mathcal{M}} & 0 & \cdots \\
 & j+1 & \beta_{0,j+1}^{\mathcal{M}} & \beta_{1,1+j+1}^{\mathcal{M}} & \cdots & \beta_{\ell,\ell+j+1}^{\mathcal{M}} & 0 & \cdots \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
\mathrm{reg}(\mathcal{M}) = & j+k & \beta_{0,j+k}^{\mathcal{M}} & \beta_{1,1+j+k}^{\mathcal{M}} & \cdots & \beta_{\ell,\ell+j+k}^{\mathcal{M}} & 0 & \cdots \\
 & & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

is called the *Betti diagram* of $\mathcal{M}$. Thereby is j the smallest degree of a minimal generator of $\mathcal{M}$.

It is clear that we can determine the entries of this diagram by computing the minimal free resolution. But, as Albert, Fetzer, Saenz-de-Cabezon and Seiler showed in [**MFdCS15**], it also possible to compute the Betti numbers without explicitly determine the minimal free resolution. Further, we see that knowing $\operatorname{reg}(\mathcal{M})$ and $\operatorname{pd}(\mathcal{M})$ leads to a first rough description of the Betti diagram as they provide certain bounds.

Definition 1.1.10.
We say that the elements $r_1, \dots, r_d \in \mathcal{P}$ form a *$\mathcal{M}$-regular sequence* if $\mathcal{M} \neq \langle r_1, \dots, r_d \rangle \mathcal{M}$ and if r_i is a nonzero divisor in $\mathcal{M}/\langle r_1, \dots, r_{i-1} \rangle \mathcal{M}$ for all $i \in \{1, \dots, d\}$.

Further, we say that a $\mathcal{M}$-regular sequence $r_1, \dots, r_d \in \mathcal{P}$ is *maximal* if it is impossible to find an element $r_{d+1} \in \mathcal{P}$ such that $r_1, \dots, r_{d+1}$ form a $\mathcal{M}$-regular sequence.

Theorem 1.1.11 ([**Sha01**, Thm. 16.13]).
All maximal $\mathcal{M}$-regular sequences are of the same length.

Definition 1.1.12 ([**Sha01**, Rem. 16.16]).
The length of a maximal $\mathcal{M}$-regular sequence is called the *depth* of $\mathcal{M}$ and is denoted by $\operatorname{depth}(\mathcal{M})$.

Remark 1.1.13 ([**Eis05**, Thm. A2.14.]).
Two alternative definitions that connect $\operatorname{depth}(\mathcal{M})$ with Ext-modules[2] respectively local cohomology are presented in the following:

$$\operatorname{depth}(\mathcal{M}) = \min\{i \mid \operatorname{Ext}^i_{\mathcal{P}}(\mathcal{P}/\mathfrak{m}, \mathcal{M}) \neq 0\} = \min\{i \mid \mathbf{H}^i_{\mathfrak{m}}(\mathcal{M}) \neq 0\}$$

The following famous result of Auslander-Buchsbaum shows the relationship between projective dimension and depth.

Theorem 1.1.14 (Auslander-Buchsbaum Formula ([**Eis95**, Thm. 19.9])).
$\operatorname{pd}(\mathcal{M}) + \operatorname{depth}(\mathcal{M}) = n$

Hence in practice it is enough to determine only one of those two invariants since one receives to other one immediately by this formula. In general, it is not predictable whether the computation of the projective dimension or the one of the depth is cheaper. The classical method to determine $\operatorname{pd}(\mathcal{M})$ is to compute the minimal free resolution while $\operatorname{depth}(\mathcal{M})$ is received by computing the Ext-modules described in Remark 1.1.13. If $\operatorname{pd}(\mathcal{M}) \ll \operatorname{depth}(\mathcal{M})$, then only a few syzygy modules have to be computed to receive the projective dimension while the determination of the depth costs several Ext-computations, hence in this case it is recommended to compute $\operatorname{pd}(\mathcal{M})$ instead of $\operatorname{depth}(\mathcal{M})$. Analogously, we can argue that mostly it should be cheaper to compute $\operatorname{depth}(\mathcal{M})$ if $\operatorname{depth}(\mathcal{M}) \ll \operatorname{pd}(\mathcal{M})$.

[2] Recall that the ith Ext-module of given $\mathcal{P}$-modules $\mathcal{M}, \mathcal{N}$ is defined by $\operatorname{Ext}^i_{\mathcal{P}}(\mathcal{M}, \mathcal{N}) = R^i \operatorname{Hom}_{\mathcal{P}}(\mathcal{M}, -)(\mathcal{N})$, where $R^i \operatorname{Hom}_{\mathcal{P}}(\mathcal{M}, -)$ denotes the ith right derivative of the Hom functor (see e.g. [**Eis95**, A3.11]).

Remark 1.1.15.
We consider in this thesis only the two cases $\mathcal{M} = \mathcal{I}$ and $\mathcal{M} = \mathcal{P}/\mathcal{I}$, where $\mathcal{I} \lhd \mathcal{P}$ is a homogeneous ideal. The relation between those two cases, in terms of the considered invariants, is presented in the following:

- $\operatorname{depth}(\mathcal{I}) = \operatorname{depth}(\mathcal{P}/\mathcal{I}) + 1$
- $\operatorname{pd}(\mathcal{I}) = \operatorname{pd}(\mathcal{P}/\mathcal{I}) - 1$
- $\operatorname{reg}(\mathcal{I}) = \operatorname{reg}(\mathcal{P}/\mathcal{I}) + 1$
- $\beta^{\mathcal{I}}_{i,j} = \beta^{\mathcal{P}/\mathcal{I}}_{i+1,j}$

This is a simple consequence of the fact that if

$$0 \longrightarrow \mathcal{F}_\ell \xrightarrow{\varphi_\ell} \cdots \xrightarrow{\varphi_2} \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} \mathcal{I} \longrightarrow 0$$

is the minimal free resolution of $\mathcal{I}$, then

$$0 \longrightarrow \mathcal{F}_\ell \xrightarrow{\varphi_\ell} \cdots \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} \mathcal{P} \xrightarrow{\pi} \mathcal{P}/\mathcal{I} \longrightarrow 0$$

is the minimal free resolution of $\mathcal{P}/\mathcal{I}$ (where π denotes the canonical projection).

1.2. Invariants of $\mathcal{I}$ and $\operatorname{lt}\mathcal{I}$

Before we present a theorem that delivers a decisive description of the relationship between the invariants of $\mathcal{I}$ and $\operatorname{lt}\mathcal{I}$, we first should discuss why we are interested in comparing them. The most important and obvious difference between these two ideals is that $\operatorname{lt}\mathcal{I}$ is monomial while $\mathcal{I}$ is assumed to be a polynomial ideal. Since monomial ideals are combinatorial objects, they are much easier to handle in terms of computing invariants. So knowing the connection between the invariants of $\mathcal{I}$ and $\operatorname{lt}\mathcal{I}$ allows us to reduce the computation of the invariants of the polynomial ideal $\mathcal{I}$ to the case of the monomial ideal $\operatorname{lt}\mathcal{I}$.

Theorem 1.2.1 ([**HH11**, Thm. 3.3.4]).
For a polynomial ideal $\mathcal{I}$ the following holds:

- $\operatorname{reg}(\mathcal{I}) \leq \operatorname{reg}(\operatorname{lt}\mathcal{I})$
- $\operatorname{pd}(\mathcal{I}) \leq \operatorname{pd}(\operatorname{lt}\mathcal{I})$
- $\operatorname{depth}(\mathcal{I}) \geq \operatorname{depth}(\operatorname{lt}\mathcal{I})$

Thereby the assertions of Theorem 1.2.1 are a consequence of the following proposition.

Proposition 1.2.2 ([**HH11**, Cor. 3.3.3]).
Let $\mathcal{I}$ be a polynomial ideal then $\beta^{\mathcal{I}}_{i,j} \leq \beta^{\operatorname{lt}\mathcal{I}}_{i,j}$.

Example 1.2.3.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_2+x_3^2\rangle$ be an ideal in $\Bbbk[x_1, x_2, x_3]$. Then its minimal free resolution is

$$0 \longrightarrow \mathcal{P}(-4) \longrightarrow \mathcal{P}(-2)^2 \longrightarrow \mathcal{I} \longrightarrow 0\ ,$$

while the minimal free resolution of $\operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_1x_3^2,\ x_3^4\rangle$ is

$$0 \longrightarrow \mathcal{P}(-5) \longrightarrow \mathcal{P}(-3) \oplus \mathcal{P}(-4)^2 \oplus \mathcal{P}(-5) \longrightarrow \mathcal{P}(-2)^2 \oplus \mathcal{P}(-3) \oplus \mathcal{P}(-4) \longrightarrow \operatorname{lt}\mathcal{I} \longrightarrow 0\ .$$

The two corresponding Betti diagrams are presented in the following:

	0	1
2	$\beta_{0,2}^{\mathcal{I}} = 2$	$\beta_{1,3}^{\mathcal{I}} = 0$
3	$\beta_{0,3}^{\mathcal{I}} = 0$	$\beta_{1,4}^{\mathcal{I}} = 1$

	0	1	2
2	$\beta_{0,2}^{\operatorname{lt}\mathcal{I}} = 2$	$\beta_{1,3}^{\operatorname{lt}\mathcal{I}} = 1$	$\beta_{2,4}^{\operatorname{lt}\mathcal{I}} = 0$
3	$\beta_{0,3}^{\operatorname{lt}\mathcal{I}} = 1$	$\beta_{1,4}^{\operatorname{lt}\mathcal{I}} = 2$	$\beta_{2,5}^{\operatorname{lt}\mathcal{I}} = 1$
4	$\beta_{0,4}^{\operatorname{lt}\mathcal{I}} = 1$	$\beta_{1,5}^{\operatorname{lt}\mathcal{I}} = 1$	$\beta_{2,6}^{\operatorname{lt}\mathcal{I}} = 0$

With this information we can now directly derive that:

$$\begin{aligned}
\operatorname{reg}(\mathcal{I}) = 3 \quad &< \quad 4 = \operatorname{reg}(\operatorname{lt}\mathcal{I}) \\
\operatorname{pd}(\mathcal{I}) = 1 \quad &< \quad 2 = \operatorname{pd}(\operatorname{lt}\mathcal{I}) \\
\operatorname{depth}(\mathcal{I}) = 2 \quad &> \quad 1 = \operatorname{depth}(\operatorname{lt}\mathcal{I}) \\
\beta_{1,3}^{\mathcal{I}} = 0 \quad &< \quad 1 = \beta_{1,3}^{\operatorname{lt}\mathcal{I}}
\end{aligned}$$

This explicitly shows that the considered invariants of an ideal can differ from those of its leading ideal.

Theorem 1.2.1 delivers two important aspects. On the one hand we see that the invariants of the leading ideal provide an upper respectively lower bound for those of the ideal itself. On the other hand a new question arises from these statements:

Under which circumstances do the invariants of $\mathcal{I}$ coincide with those of $\operatorname{lt}\mathcal{I}$?

1.3. Genericity

The answer to this question lies in the notion of genericity. Since the concept of genericity is used in many different contexts in the literature, we give a concrete explanation of what we mean by a generic property. Therefore we firstly recall the definition of the well-known *Zariski topology*. Since it is enough to describe a topology by naming its closed sets, we only have to remember that a set $X \subseteq \Bbbk^n$ is *Zariski closed* if there exists an ideal $\mathcal{I} \lhd \mathcal{P}$ such that X equals the variety $\mathcal{V}(\mathcal{I})$.

Definition 1.3.1.
We call a property *generic* if it holds on a nonempty Zariski open set.

REMARK 1.3.2.
Let us consider the case $\Bbbk = \mathbb{Q}$.

Then, in sense of measure theory, a generic property is one that holds *almost everywhere*. Further, a property holds almost everywhere if the set of elements for which the property does not hold is a set of Lebesgue measure zero. Since proper Zariski closed sets have a dimension lower than the whole space, they are of measure zero. So this definition is equivalent to the one above.

In probability theory the to "almost everywhere" equivalent concept is *almost surely*. Thereby a property holds almost surely if it happens with probability one.

NOTATION 1.3.3.
Let $\mathcal{I} = \langle f_1, \ldots, f_\ell \rangle \lhd \mathcal{P}$. We will use the following two different notations for performing a coordinate transformation on $\mathcal{I}$.

- Let $A = (a_{ij}) \in \mathrm{Gl}(n, \Bbbk)$ then we define the following notations:
  - $A \cdot \mathbf{x}^\mu = (\sum_{j=1}^{n} a_{1j} x_j)^{\mu_1} \cdots (\sum_{j=1}^{n} a_{nj} x_j)^{\mu_n}$
  - $A \cdot \sum_\mu \mathtt{C}_\mu \mathbf{x}^\mu = \sum_\mu \mathtt{C}_\mu (A \cdot \mathbf{x}^\mu)$, $\mathtt{C}_\mu \in \Bbbk$
  - $A \cdot \{f_1, \ldots, f_\ell\} = \{A \cdot f_1, \ldots, A \cdot f_\ell\}$
  - $A \cdot \mathcal{I} = \langle A \cdot f_1, \ldots, A \cdot f_\ell \rangle$
- Let $a_1, \ldots, a_n \in \Bbbk$ then we understand under $\Psi : (x_j \mapsto x_j + \sum_{i \neq j} a_i x_i)$ a coordinate transformation that maps x_j to $(x_j + \sum_{i \neq j} a_i x_i)$ and x_i to x_i for all $i \neq j$. Therefore:
  - $\Psi(\mathbf{x}^\mu) = (x_j + \sum_{i \neq j} a_i x_i)^{\mu_j} \prod_{i \neq j} x_i^{\mu_i}$
  - $\Psi(\sum_\mu \mathtt{C}_\mu \mathbf{x}^\mu) = \sum_\mu \mathtt{C}_\mu \Psi(\mathbf{x}^\mu)$, $\mathtt{C}_\mu \in \Bbbk$
  - $\Psi(\{f_1, \ldots, f_\ell\}) = \{\Psi(f_1), \ldots, \Psi(f_\ell)\}$
  - $\Psi(\mathcal{I}) = \langle \Psi(f_1), \ldots, \Psi(f_\ell) \rangle$

The next theorem illustrates the for us most important statement in the context of genericity.

THEOREM 1.3.4 (Galligo, Bayer-Stillman ([**Gre98**, Thm. 1.27])).
There exists an open Zariski subset $\mathcal{U} \subseteq \mathrm{Gl}(n, \Bbbk)$ and a monomial ideal $\mathcal{J} \lhd \mathcal{P}$ such that for all $A \in \mathcal{U}$:

$$\mathrm{lt}(A \cdot \mathcal{I}) = \mathcal{J}$$

Proof. For some integer q we consider $\langle \mathcal{I}_q \rangle_{\Bbbk}$ as $\Bbbk$-linear subspace of $\mathcal{P}_q$. Let $B = \{t_1, \ldots, t_s\}$ be the monomial basis of $\mathcal{P}_q$ with $t_1 \succ \cdots \succ t_s$, then $s = \dim_{\Bbbk}(\mathcal{P}_q) = \binom{n-1+q}{q}$. Further, let $r = \dim_{\Bbbk}(\langle \mathcal{I}_q \rangle_{\Bbbk})$ and we define the $r \times s$ matrix $M(\mathcal{I}_q) = (m_{ij})$ such that the set

$$\left\{ \sum_{j=1}^{s} m_{ij} t_j \mid 1 \leq i \leq r \right\}$$

is a $\Bbbk$-basis of $\langle \mathcal{I}_q \rangle_{\Bbbk}$. If $M_k(\mathcal{I}_q)$ is the submatrix of $M(\mathcal{I}_q)$ consisting of the first k columns, then

$$\dim_{\Bbbk}(\langle (\operatorname{lt} \mathcal{I})_q \rangle_{\Bbbk} \cap \langle t_1, \ldots, t_k \rangle_{\Bbbk}) = \operatorname{rank}(M_k(\mathcal{I}_q)). \tag{1.2}$$

Since the rank is the size of the largest minor having a nonzero determinant, there exists for every k a Zariski open subset $\mathcal{U}_{qk} \subset \mathrm{Gl}(n, \Bbbk)$ such that $\operatorname{rank}(M_k((A \cdot \mathcal{I})_q))$ is constant for all $A \in \mathcal{U}_{qk}$. Hence by (1.2) $\langle (\operatorname{lt}(A \cdot \mathcal{I}))_q \rangle_{\Bbbk}$ is constant for all A from the Zariski open set $\mathcal{U}_q = \mathcal{U}_{q1} \cap \ldots \cap \mathcal{U}_{qs}$. We set $\mathcal{U}'_0 = \mathcal{U}_0$ and $\mathcal{U}'_{q+1} = \mathcal{U}_q \cap \mathcal{U}_{q+1}$ for all $q \geq 0$ and so consequently $\mathcal{U}'_{q+1} \subseteq \mathcal{U}'_q$. Now the homogeneous component $(\operatorname{lt}(A \cdot \mathcal{I}))_q$ of the ideal $\operatorname{lt}(A \cdot \mathcal{I})$ is constant for any $A \in \mathcal{U}'_q$ so that we can define an ideal $\mathcal{J}$ by $\mathcal{J}_q = (\operatorname{lt}(A \cdot \mathcal{I}))_q$. As the ideal $\mathcal{J}$ is generated by a finite set F we can set $\hat{q} = \deg F$. The Zariski open set with the desired property is $\mathcal{U}'_{\hat{q}}$. ☐

Remark 1.3.5.
We showed in the proof of Theorem 1.3.4 that there is an integer c such that:

$$\operatorname{rank}(M_k((A \cdot \mathcal{I})_q)) = c, \quad \text{for all } A \in \mathcal{U}_{qk} \tag{1.3}$$

As we will use this fact in the proof of Proposition 6.1.13, we want to note here that it is not only constant, but also maximal with respect to all $A \in \mathrm{Gl}(n, \Bbbk)$. This is a consequence of the semicontinuity[3] of rank. To understand this argument we assume that there is a matrix A_1 such that:

$$\underset{\substack{\shortparallel \\ c_1}}{\operatorname{rank}(M_k((A_1 \cdot \mathcal{I})_q))} \quad > \quad c$$

Now choose $\varepsilon > 0$ such that $\varepsilon < c_1 - c$. It follows from the semicontinuity that there exits a neighborhood $\mathcal{W}$ of A_1 such that:

$$\operatorname{rank}(M_k((B \cdot \mathcal{I})_q)) \geq c_1 - \varepsilon, \quad \text{for all } B \in \mathcal{W} \tag{1.4}$$

[3] Let X be a topological space and $f : X \to \mathbb{R}$ a function. Then f is *lower semicontinuous* at $x_0 \in X$ if for every $\varepsilon > 0$ there exists a neighborhood $\mathcal{W}$ of x_0 such that $f(x) \geq f(x_0) - \varepsilon$ for all $x \in \mathcal{W}$.

Since $\mathcal{U}_{qk}$ is Zariski open, the intersection $\mathcal{U}_{qk} \cap \mathcal{W}$ is nonempty. So let $A_2 \in \mathcal{U}_{qk} \cap \mathcal{W}$ then

$$\begin{array}{ccc} \operatorname{rank}(M_k((A_2 \cdot \mathcal{I})_q)) & \overset{(1.4)}{\geq} & c_1 - \varepsilon, \\ \shortparallel\,(1.3) & & \\ c & & \end{array}$$

which leads to a contradiction to the choice of ε.

Definition 1.3.6.
The ideal $\mathcal{J}$ from Theorem 1.3.4 above is called *generic initial ideal* of $\mathcal{I}$ and is denoted by $\operatorname{gin} \mathcal{I}$. If $\mathcal{I}$ is an ideal such that $\operatorname{lt} \mathcal{I} = \operatorname{gin} \mathcal{I}$, then we say that $\mathcal{I}$ is in gin-*position*.

Notation 1.3.7.
We denote the *(maximal) index* of a term, polynomial and finite set of polynomials as follows:

- $\mathrm{m}(\mathbf{x}^\mu) = \max\{i \mid \mu_i \neq 0\}$
- $\mathrm{m}(f) = \mathrm{m}(\operatorname{lt} f)$ for all $f \in \mathcal{P} \setminus \{0\}$
- $\mathrm{m}(F) = \max\{\mathrm{m}(f) \mid f \in F\}$ for all finite sets $F \subseteq \mathcal{P} \setminus \{0\}$

Remark 1.3.8 ([**Sei10**, Lem. A.1.8]).
The degree reverse lexicographic term order $\prec$ is *index respecting*. This means that if $\mathbf{x}^\mu, \mathbf{x}^\nu \in \mathbb{T}$ are two terms with $\deg \mathbf{x}^\mu = \deg \mathbf{x}^\nu$ and $\mathbf{x}^\mu \succ \mathbf{x}^\nu$ then $\mathrm{m}(\mathbf{x}^\mu) \leq \mathrm{m}(\mathbf{x}^\nu)$. Moreover, for two terms $\mathbf{x}^\lambda, \mathbf{x}^\kappa \in \mathbb{T}$ with $\mathbf{x}^\lambda | \mathbf{x}^\kappa$, we also have $\mathrm{m}(\mathbf{x}^\lambda) \leq \mathrm{m}(\mathbf{x}^\kappa)$.

Now we can come back the question of the last section. Analogous to Theorem 1.2.1 we present the following assertion:

Theorem 1.3.9 (Bayer-Stillman ([**Eis95**, Cor. 19.11, Cor. 20.21]).
Let H be the minimal monomial basis of $\operatorname{gin} \mathcal{I}$. *Then:*

- $\operatorname{reg}(\mathcal{I}) = \operatorname{reg}(\operatorname{gin} \mathcal{I}) = \deg H$
- $\operatorname{pd}(\mathcal{I}) = \operatorname{pd}(\operatorname{gin} \mathcal{I}) = \mathrm{m}(H) - 1$
- $\operatorname{depth}(\mathcal{I}) = \operatorname{depth}(\operatorname{gin} \mathcal{I}) = n - \mathrm{m}(H) + 1$

This theorem does not only show that if our ideal is in gin-position the invariants of $\mathcal{I}$ and $\operatorname{lt} \mathcal{I}$ coincide, but also that they can directly be read off the basis of $\operatorname{gin} \mathcal{I}$ without any further computation.

CHAPTER 2

gin-Position vs. stable Positions

We have seen in Theorem 1.3.9 that the gin-position provides several useful properties. Naturally, it is our next step to investigate how we can put a given ideal $\mathcal{I}$ into this position or alternatively how we can compute $\operatorname{gin}\mathcal{I}$. This investigation will be done in the first section of this chapter, while the following sections discuss different notions of *stable* positions. Thereby we will develop an algorithm that allows us to put any ideal into *strongly stable* position if $\operatorname{char}\mathbb{k} = 0$. Further, we examine possible strategies for the implementation of this algorithm and consider the case of positive characteristic.

2.1. Gröbner System

Following Weispfenning [**Wei92**][1], we denote by $\hat{\mathcal{P}} = \mathbb{k}[\mathbf{a},\mathbf{x}]$ a parametric polynomial ring where $\mathbf{a} = a_1,\dots,a_m$ represents the parameters and $\mathbf{x} = x_1,\dots,x_n$ the variables. Let $\prec_{\mathbf{x}}$ (resp. $\prec_{\mathbf{a}}$) be a term order for the power products of the variables x_i (resp. the parameters a_i). Then we introduce the block elimination term order $\prec_{\mathbf{x},\mathbf{a}}$ in the usual manner:

For all $\kappa,\mu \in \mathbb{N}_0^n$ and all $\lambda,\nu \in \mathbb{N}_0^m$, we define $\mathbf{a}^\nu\mathbf{x}^\mu \prec_{\mathbf{x},\mathbf{a}} \mathbf{a}^\lambda\mathbf{x}^\kappa$ if either $\mathbf{x}^\mu \prec_{\mathbf{x}} \mathbf{x}^\kappa$ or $\mathbf{x}^\mu = \mathbf{x}^\kappa$ and $\mathbf{a}^\nu \prec_{\mathbf{a}} \mathbf{a}^\lambda$.
Further, we call a homomorphism $\sigma : \mathbb{k}[\mathbf{a}] \to \mathbb{k}$ with $\sigma|_{\mathbb{k}} = \mathrm{id}_{\mathbb{k}}$ a *specialization*[2] of $\hat{\mathcal{P}}$. So any specialization is uniquely determined by its restriction to $\mathbb{k}$ and the images $\sigma(a_i)$ of the parameters in $\mathbb{k}[\mathbf{a}]$.

Definition 2.1.1.
A finite set of triples $\{(\hat{G}_i, N_i, W_i)\}_{i=1}^{\ell}$ with finite sets $\hat{G}_i \subseteq \hat{\mathcal{P}}$ and $N_i, W_i \subseteq \mathbb{k}[\mathbf{a}]$ is a *Gröbner system* for a parametric ideal $\hat{\mathcal{I}} \lhd \hat{\mathcal{P}}$ with respect to the block order $\prec_{\mathbf{x},\mathbf{a}}$ if for every index $1 \le i \le \ell$ and every specialization σ of $\hat{\mathcal{P}}$ with

(i) $\forall g \in N_i : \sigma(g) = 0$
(ii) $\forall h \in W_i : \sigma(h) \neq 0$

$\sigma(\hat{G}_i)$ is a Gröbner basis of $\sigma(\hat{\mathcal{I}}) \lhd \mathcal{P}$ with respect to the order $\prec_{\mathbf{x}}$ and if for any point $\boldsymbol{p} \in \mathbb{k}^m$ an index $1 \le i \le \ell$ exists such that $\boldsymbol{p} \in \mathcal{V}(N_i) \setminus \mathcal{V}(\prod_{h\in W_i} h)$.

[1] Weispfenning introduced the notion of Gröbner system in the context of his research on comprehensive Gröbner bases.
[2] σ has a canonical extension $\overline{\sigma} : \hat{\mathcal{P}} = \mathbb{k}[\mathbf{a},\mathbf{x}] \to \mathbb{k}[\mathbf{x}]$.

REMARK 2.1.2.
A Gröbner systems yields a Gröbner basis for all possible values of the parameters **a**. Basically every algorithm (in particular the algorithm[3] used by us) produces Gröbner systems such that given one specific triple $(\hat{G}_i, N_i, W_i)$ all specializations σ satisfying (2.1.1) yield the same leading terms $\operatorname{lt}\sigma(\hat{G}_i)$. Hence we can speak of a monomial ideal $\mathcal{L}_i \lhd \mathcal{P}$ determined by the conditions (N_i, W_i). In the sequel, we will always assume that a Gröbner system with this property is used.

THEOREM 2.1.3 ([**Wei92**, Theorem 2.7]).
Every parametric ideal $\hat{\mathcal{I}} \lhd \hat{\mathcal{P}}$ possesses a Gröbner system.

To get a better feeling for these abstract definitions, we now explain to concept of Gröbner systems based on a pretty simple example.

EXAMPLE 2.1.4.
Let $\mathcal{I} = \langle x_3 \rangle \lhd \Bbbk[x_1, x_2, x_3]$ and $A = (a_{ij})$ a parametric 3×3 matrix. We consider the parametric ideal

$$\hat{\mathcal{I}} = A \cdot \mathcal{I} = \langle a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \rangle \lhd \Bbbk[a_{11}, a_{12}, a_{13}, \ldots, a_{31}, a_{32}, a_{33}, x_1, x_2, x_3].$$

With the notations of Definition 2.1.1 and Remark 2.1.2 we get:

i	$\mathcal{L}_i$	N_i	W_i
1	$\langle x_1 \rangle$	$\{\}$	$\{a_{31}\}$
2	$\langle x_2 \rangle$	$\{a_{31}\}$	$\{a_{32}\}$
3	$\langle x_3 \rangle$	$\{a_{31}, a_{32}\}$	$\{a_{33}\}$
4	$\langle 0 \rangle$	$\{a_{31}, a_{32}, a_{33}\}$	$\{\}$

Finally, we have to verify whether the equation

$$\begin{aligned} \Bbbk^9 &= \bigcup_i \mathcal{V}(N_i) \setminus \mathcal{V}(\prod_{h \in W_i} h) \\ &=^4 \Bbbk^9 \setminus \mathcal{V}(a_{31}) \ \cup\ \mathcal{V}(a_{31}) \setminus \mathcal{V}(a_{32}) \ \cup\ \mathcal{V}(a_{3,1}, a_{32}) \setminus \mathcal{V}(a_{33}) \cup\ \mathcal{V}(a_{31}, a_{32}, a_{33}) \end{aligned}$$

holds. Therefore we choose a point $\boldsymbol{p} = (a_{11}, a_{12}, a_{13}, \ldots, a_{31}, a_{32}, a_{33}) \in \Bbbk^9$. The following case distinction shows the above equation:

$$\begin{array}{rcl} a_{31} \neq 0 & \Rightarrow & \boldsymbol{p} \in \Bbbk^9 \setminus \mathcal{V}(a_{31}) \\ a_{31} = 0 \wedge a_{32} \neq 0 & \Rightarrow & \boldsymbol{p} \in \mathcal{V}(a_{31}) \setminus \mathcal{V}(a_{32}) \\ a_{31} = 0 \wedge a_{32} = 0 \wedge a_{33} \neq 0 & \Rightarrow & \boldsymbol{p} \in \mathcal{V}(a_{31}, a_{32}) \setminus \mathcal{V}(a_{33}) \\ a_{31} = 0 \wedge a_{32} = 0 \wedge a_{33} = 0 & \Rightarrow & \boldsymbol{p} \in \mathcal{V}(a_{31}, a_{32}, a_{33}) \end{array}$$

[3] Weispfenning [**Wei92**] provided a first algorithm for computing Gröbner systems. Subsequently, improvements and alternatives were presented by many authors [**KSW10, KSW13, Mon02, MW10, SS06**]. Our calculations were done using a MAPLE implementation of the DISPGB algorithm of Montes which is available at `http://amirhashemi.iut.ac.ir/softwares`

[4] Remember that by convention $\prod_{h \in \{\}} h = 1$ and $\mathcal{V}(1) = \emptyset$.

Remark 2.1.5.
Gröbner systems are defined for parametric ideals. In this thesis we will always construct these ideals by transforming a given ideal $\mathcal{I} \lhd \mathcal{P}$ with a parametric matrix, as we have done it in the preceding example. So for us the case $i = 4$ of this example is not relevant, since then the matrix A is singular and so it does not represent a coordinate change.

Remark 2.1.6.
Every Gröbner system $\left\{(\hat{G}_i, N_i, W_i)\right\}_{i=1}^{\ell}$ has one branch j with $N_j = \{\}$ and so $\mathcal{V}(N_j) = \Bbbk^m$. Otherwise it would not be possible to find a finite number of tuples (N_i, W_i) with

$$\Bbbk^m = \bigcup_i \mathcal{V}(N_i) \setminus \mathcal{V}(\prod_{h \in W_i} h).$$

This branch j is called *generic branch* and the set $\mathcal{V}(N_j) \setminus \mathcal{V}(\prod_{h \in W_j} h)$ is Zariski open.

If we now compute the generic branch j of the parametric ideal $A \cdot \mathcal{I} \lhd \Bbbk[\mathbf{a}, \mathbf{x}]$, where A is a $n \times n$ parametric matrix, then

$$\mathcal{L}_j = \operatorname{lt} \sigma(\hat{G}_j) = \operatorname{gin} \mathcal{I}.$$

If $(b_{11}, \ldots, b_{1n}, b_{21}, \ldots, b_{2n}, \ldots, b_{n1}, \ldots, b_{nn})$ is an element of the Zariski open set $\Bbbk^{n^2} \setminus \mathcal{V}(\prod_{h \in W_j} h)$ such that $B = (b_{ij}) \in \mathrm{Gl}(n, \Bbbk)$, then we have $\operatorname{lt}(B \cdot \mathcal{I}) = \operatorname{gin} \mathcal{I}$. With other words B transforms $\mathcal{I}$ into gin-position.

Now we know a way to determine the generic initial ideal. But as we have to compute a Gröbner basis of a parametric ideal with n^2 parameters this method is obviously rather expensive. The next lemma brings a slight optimization to this problem by reducing the number of parameters from n^2 to $\frac{n^2-n}{2}$.

Lemma 2.1.7.
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal and $A \in \mathrm{Gl}(n, \Bbbk)$ matrix. There exists a matrix L with $\operatorname{lt}(A \cdot \mathcal{I}) = \operatorname{lt}(L \cdot \mathcal{I})$ and

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ l_{n1} & \cdots & l_{n,n-1} & 1 \end{pmatrix}. \tag{2.1}$$

Proof. Indeed, any regular matrix A can be written as a product[5] $A = UDL$ where L is a lower triangular, U an upper triangular and D a diagonal matrix and where both L and U have only ones on the diagonal. As we are considering

[5]Classically, one uses decompositions $A = LDU$. But such a decomposition for the inverse A^{-1} yields immediately a decomposition of our form for A.

a polynomial ring over a field the transformation induced by D does trivially not change the leading term of any polynomial for arbitrary term orders, i.e.:

$$\operatorname{lt}(D \cdot \mathcal{I}') = \operatorname{lt} \mathcal{I}', \quad \text{for any } \mathcal{I}' \lhd \mathcal{P} \tag{2.2}$$

Now we want to analyze the effect of the matrix U on the leading terms. Let $s \in \mathbb{T}$ be a term then it follows from the definition of the reverse lexicographic order that $s \succeq r$ for any $r \in \operatorname{supp}(U \cdot s)$. In particular, for any polynomial $f \in \mathcal{P}$ we have $\operatorname{lt}(U \cdot f) = \operatorname{lt} f$ so that:

$$\operatorname{lt}(U \cdot \mathcal{I}') = \operatorname{lt} \mathcal{I}', \quad \text{for any } \mathcal{I}' \lhd \mathcal{P} \tag{2.3}$$

Hence also the transformation induced by U does not affect any leading term and we can finally conclude:

$$\operatorname{lt}(A \cdot \mathcal{I}) = \operatorname{lt}(U(DL \cdot \mathcal{I})) \overset{(2.3)}{=} \operatorname{lt}(D(L \cdot \mathcal{I})) \overset{(2.2)}{=} \operatorname{lt}(L \cdot \mathcal{I})$$

□

As already mentioned it is a consequence of this lemma that we can reduce the number of parameters to determine $\operatorname{gin} \mathcal{I}$ compared to the method described in Remark 2.1.6. Since instead of using a full parametric $n \times n$ matrix A it is enough to consider a parametric matrix of the form (2.1) that contains only $\frac{n^2-n}{2}$ parameters.

Remark 2.1.8.
In practice, one tries to avoid such expensive parametric computations by using a different ansatz to determine $\operatorname{gin} \mathcal{I}$. For example the computer algebra system CoCoA has a function called `Gin`, which is described in the manual as follows:

> *These functions return the [probabilistic] gin (generic initial ideal) of the ideal I. It is obtained by computing the leading term ideal of g(I), where g is a random change of coordinates. While* `Gin` *uses integer coefficients in [-Range, Range], with default value [-100, 100] (repeated until 4 consecutive random changes of coordinates give the same result)(...)*[6]

This approach leads to a probabilistic algorithm (Monte Carlo algorithm) which is based on the following idea. We have seen in Theorem 1.3.4 that there is an Zariski open set $\mathcal{U}$ such that $\operatorname{lt}(A \cdot \mathcal{I}) = \operatorname{gin} \mathcal{I}$ for every $A \in \mathcal{U}$. Further, we know by Remark 1.3.2[7] that a randomly chosen matrix lies almost surely in this set $\mathcal{U}$.

[6] See `http://cocoa.dima.unige.it/download/CoCoAManual/html/cmdGinGin5.html` for more details.

[7] CoCoA uses $\Bbbk = \mathbb{Q}$ by default.

2.2. Quasi-stable, stable and strongly stable Ideals

So far we studied the gin-position and presented its interesting properties concerning homological invariants in Theorem 1.3.9. Furthermore, we found in the previous section a deterministic method to compute the generic initial ideal $\operatorname{gin}\mathcal{I}$ by using the theory of Gröbner systems (see Remark 2.1.6).

In this section we will take a look at certain combinatorial properties that ideals can have. These different properties are summarized under the term "*stable*-positions" which will be in the center of our attention for the rest of this dissertation. Thereby we now deliver a deterministic algorithm that allows us to put any ideal into a desired stable position (if the characteristic of the considered field $\Bbbk$ is zero).

DEFINITION 2.2.1.
Let $\mathcal{J}$ be a monomial ideal and B its minimal basis.

- $\mathcal{J}$ is *quasi-stable* if for every term $\mathbf{x}^\mu \in \mathcal{J}$ and all $i < k = \mathrm{m}(\mathbf{x}^\mu)$ the term $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}}$ also lies in $\mathcal{J}$.
- $\mathcal{J}$ is *stable* if for every term $\mathbf{x}^\mu \in \mathcal{J}$ and all $i < k = \mathrm{m}(\mathbf{x}^\mu)$ the term $x_i \frac{\mathbf{x}^\mu}{x_k}$ also lies in $\mathcal{J}$.
- $\mathcal{J}$ is *strongly stable* if for every term $\mathbf{x}^\mu \in \mathcal{J}$, all indices j with $\mu_j > 0$ and all $i < j$ the term $x_i \frac{\mathbf{x}^\mu}{x_j}$ also lies in $\mathcal{J}$.

REMARK 2.2.2.
We can directly derive from this definition the following hierarchy:

$$\mathcal{J} \text{ strongly stable } \Rightarrow \mathcal{J} \text{ stable } \Rightarrow \mathcal{J} \text{ quasi-stable}$$

EXAMPLE 2.2.3.
Let $\mathcal{J}_1 = \langle x_1^2,\ x_2^2\rangle$, $\mathcal{J}_2 = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_2x_3\rangle$, $\mathcal{J}_3 = \langle x_1^2,\ x_1x_2,\ x_1x_3\rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then $\mathcal{J}_1$ is quasi-stable but not stable since $x_1x_2 \notin \mathcal{J}_1$, $\mathcal{J}_2$ is stable but not strongly stable since $x_1x_3 \notin \mathcal{J}_2$ and finally $\mathcal{J}_3$ represents an example of a strongly stable ideal.

LEMMA 2.2.4 ([**HH11**, Lem. 4.2.3]).
To verify, whether an ideal has one of the properties defined in Definition 2.2.1, it is enough to check if the elements of the minimal basis fulfill the desired property.

PROOF. We will show this only for the case of strong stability, since the proof for the other notions is similar. So let us assume that every element of the minimal basis B of a monomial ideal $\mathcal{J}$ fulfills the property for strongly stable. Further let $\mathbf{x}^\mu \in \mathcal{J}$ be a term with $\mu_j > 0$ for some index j. Since there must be an element $t \in B$ that divides $\mathbf{x}^\mu$, we can write $\mathbf{x}^\mu = st$ for some $s \in \mathbb{T}$. Therefore either s or t must be divisible by x_j. In the first case $\frac{s}{x_j} \in \mathbb{T}$ and so $\frac{s}{x_j}t = \frac{\mathbf{x}^\mu}{x_j} \in \mathcal{J}$, hence $x_i \frac{\mathbf{x}^\mu}{x_j} \in \mathcal{J}$ for all indices i. In the second case it follows by our assumption that $x_i \frac{t}{x_j} \in \mathcal{J}$ for all $i < j$. But then of course the same holds for $x_i \frac{st}{x_j} = x_i \frac{\mathbf{x}^\mu}{x_j}$. $\square$

Definition 2.2.5.
We say that a polynomial ideal is in *quasi-stable/stable/strongly stable* position if its leading ideal is quasi-stable/stable/strongly stable.

Remark 2.2.6.
By Lemma 2.2.4 we can see that it is very easy to verify whether an ideal is in one of the positions described in Definition 2.2.1. This is an important difference to the gin-position for which there exists no simple test to verify that one has really obtained this position.

Analogous to Remark 2.1.6 it is natural to ask whether we are able to put an ideal into one of the stable positions. Indeed, for the case $\operatorname{char}\Bbbk = 0$, we can present an algorithm in the following that outputs a coordinate transformation Ψ such that $\Psi(\mathcal{I})$ is strongly stable. Before we are able to present this algorithm we must first define a special type of a list of terms and an associated ordering on these lists. This definition will play a fundamental role when we prove the termination of the mentioned algorithm.

Definition 2.2.7.
Let $F \subseteq \mathcal{P}$ be a finite set of polynomials with $\operatorname{lt} F = \{t_1, \dots, t_\ell\}$ where the terms t_i are ordered by the *reverse lexicographical*[8] ordering such that $t_1 \succ_{\text{revlex}} \cdots \succ_{\text{revlex}} t_\ell$. Then we define:

$$\mathscr{L}(F) = (t_1, \dots, t_\ell)$$

Let $F, \tilde{F} \subseteq \mathcal{P}$ be two finite sets of polynomials with $\mathscr{L}(F) = (t_1, \dots, t_\ell)$ and $\mathscr{L}(\tilde{F}) = (\tilde{t}_1, \dots, \tilde{t}_{\tilde{\ell}})$. We define:

$$\mathscr{L}(F) \prec_{\mathscr{L}} \mathscr{L}(\tilde{F}) \text{ if } \begin{cases} \exists j < \min(\ell, \tilde{\ell}), \forall i < j : t_i = \tilde{t}_i \text{ and } t_j \prec_{\text{revlex}} \tilde{t}_j & \text{or} \\ \forall j \leq \min(\ell, \tilde{\ell}) : t_j = \tilde{t}_j \text{ and } \ell < \tilde{\ell}. \end{cases}$$

[8] Following [**LA94**, Sec. 1.4] we set $\mathbf{x}^\mu \prec_{\text{revlex}} \mathbf{x}^\nu$, if and only if $\mu_m > \nu_m$ with $m = \max\{i \mid \mu_i \neq \nu_i\}$.

Algorithm 1 SS-TRAFO: Transformation to strongly stable position in char $\Bbbk = 0$

Input: Reduced Gröbner basis G of homogeneous ideal $\mathcal{I} \lhd \mathcal{P}$
Output: a linear change of coordinates Ψ such that $\operatorname{lt}\Psi(\mathcal{I})$ is strongly stable

1: $\Psi := \mathrm{id}$;
2: **while** $\exists\, g \in G,\ 1 \leq j \leq n,\ 1 \leq i < j\ :\ x_j | \operatorname{lt} g \wedge x_i \frac{\operatorname{lt} g}{x_j} \notin \langle \operatorname{lt} G\rangle$ **do**
3:     $\psi := (x_j \mapsto x_j + x_i)$; $\Psi = \psi \circ \Psi$;
4:     $\tilde{G} := \textsc{ReducedGröbnerBasis}\big(\psi(G)\big)$;
5:     **while** $\mathscr{L}(G) \succeq_{\mathscr{L}} \mathscr{L}(\tilde{G})$ **do**
6:         $\psi := (x_j \mapsto x_j + x_i)$; $\Psi = \psi \circ \Psi$;
7:         $\tilde{G} := \textsc{ReducedGröbnerBasis}\big(\psi(\tilde{G})\big)$;
8:     **end while**
9:     $G := \tilde{G}$;
10: **end while**
11: **return** Ψ

REMARK 2.2.8.
Line 2 of Algorithm 1 reflects the definition of strongly stable. We will see later (in Remark 2.2.17) that replacing it by the corresponding condition for quasi-stable/stable leads to a coordinate transformation Ψ that puts the ideal in quasi-stable/stable position. In particular, we will discuss in Section 2.4 that for quasi-stability a slightly modified version of this algorithm also works if the considered field[9] $\Bbbk$ has positive characteristic (see Theorem 2.4.11).

EXAMPLE 2.2.9.
In this example we want to perform Algorithm 1 on the ideal $\mathcal{I} = \langle x_1^3,\ x_2^3,\ x_2^2 x_3\rangle \lhd \Bbbk[x_1, x_2, x_3]$. $\mathcal{I}$ is not strongly stable since $x_1 \frac{x_2^2 x_3}{x_3} = x_1 x_2^2 \notin \mathcal{I}$. So according to the algorithm we perform a coordinate transformation $\Psi_1 : (x_3 \mapsto x_3 + x_1)$ and get

$$\operatorname{lt}\Psi_1(\mathcal{I}) = \langle x_1^3,\ x_1 x_2^2,\ x_2^3,\ x_2^2 x_3^2\rangle.$$

As $(x_1^3,\ x_2^3,\ x_2^2 x_3) \prec_{\mathscr{L}} (x_1^3,\ x_1 x_2^2,\ x_2^3,\ x_2^2 x_3^2)$, we do not enter the while loop of line 5. But $\operatorname{lt}\Psi_1(\mathcal{I})$ is still not strongly stable since $x_1 \frac{x_1 x_2^2}{x_2} = x_1^2 x_2 \notin \operatorname{lt}(\Psi_1(\mathcal{I}))$. So we transform the coordinates again, this time by $\Psi_2 : (x_2 \mapsto x_2 + x_1)$ which leads to

$$\operatorname{lt}\Psi_2(\Psi_1(\mathcal{I})) = \langle x_1^3,\ x_1^2 x_2,\ x_1 x_2^2,\ x_2^4,\ x_1^2 x_3^3\rangle.$$

Again we do not enter the while loop of line 5 since:

$$(x_1^3,\ x_1 x_2^2,\ x_2^3,\ x_2^2 x_3^2) \prec_{\mathscr{L}} (x_1^3,\ x_1^2 x_2,\ x_1 x_2^2,\ x_2^4,\ x_1^2 x_3^3)$$

[9] Remember that throughout this thesis, we assume $\Bbbk$ to be an infinite field if nothing different is mentioned.

Now there are no obstructions[10] left, i.e. $\operatorname{lt}\Psi_2(\Psi_1(\mathcal{I}))$ is strongly stable (in this case we even have $\operatorname{lt}\Psi_2(\Psi_1(\mathcal{I})) = \operatorname{gin}\mathcal{I}$).

The next example will show that the result of Algorithm 1 is not unique, i.e. there may exist more than one transformation Ψ that transforms an ideal into strongly stable position.

Example 2.2.10.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2x_3,\ x_2^3\rangle \lhd \Bbbk[x_1,x_2,x_3]$. Since both terms $x_1\frac{x_2x_3}{x_2} = x_1x_3$ and $x_2\frac{x_2x_3}{x_3} = x_2^2$ are not in $\mathcal{I}$, we have the choice to perform either $\Psi_1 : (x_2 \mapsto x_2 + x_1)$ or $\Psi_2 : (x_3 \mapsto x_3 + x_1)$. Because

$$\begin{aligned} \operatorname{lt}\Psi_1(\mathcal{I}) &= \langle x_1^2,\ x_1x_2,\ x_1x_3,\ x_2^3,\ x_2^2x_3\rangle \\ \operatorname{lt}\Psi_2(\mathcal{I}) &= \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_2x_3^2\rangle \end{aligned}$$

we see that applying Ψ_1 directly leads to a strongly stable ideal, while $\operatorname{lt}\Psi_2(\mathcal{I})$ is still not strongly stable since $x_1\frac{x_2x_3^2}{x_2} = x_1x_3^2$ does not lie in $\operatorname{lt}\Psi_2(\mathcal{I})$. However,

$$\operatorname{lt}\Psi_1(\Psi_2(\mathcal{I})) = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3^2\rangle$$

is strongly stable but nevertheless not equal to $\operatorname{lt}\Psi_1(\mathcal{I})$.

The termination of Algorithm 1 is one of the main results of this thesis and we will have to do a lot of preparation before we are able to deliver the corresponding proof at the end of this section.

Definition 2.2.11.
Let $F = \{f_1, \ldots, f_\ell\} \subseteq \mathcal{P}$ be a set of polynomials.

- We call F *completely autoreduced* if for every index i holds:

$$\forall t \in \operatorname{supp}(f_i) \quad \forall j \neq i \ :\ \operatorname{lt} f_j \nmid t$$

- We call F *head autoreduced* if for every index i holds:

$$\forall j \neq i \ :\ \operatorname{lt} f_j \nmid \operatorname{lt} f_i$$

We denote the complete autoreduction of F by $F^{\blacktriangle}$ and the head autoreduction by $F^{\triangle}$.

Remark 2.2.12.
We will in the following often use the expression:

for a generic choice of $a \in \Bbbk$

According to our definition of generic (see Definition 1.3.1) this expressions means that we choose the element $a \in \Bbbk$ from a Zariski open set, or with other words we choose $a \in \Bbbk$ such that it does not lie on a variety. In this context it becomes more clear why we have assumed our field $\Bbbk$ to be infinite. Because then the concept of *generic choice* means that the desired property holds for almost all choices.

[10] By obstructions we always mean elements of the ideal that cause an obstruction to strong stability (resp. quasi-stability/stability).

Indeed one could generalize the following assertions by using a "sufficiently large" field such that there exits an element $a \in \mathbb{k}$ with the requested property.

LEMMA 2.2.13.
Let $F \subseteq \mathcal{P}$ be a finite and completely autoreduced set of polynomials. Further, let $\Psi : (x_j \mapsto x_j + ax_i)$ be a coordinate transformation with $i < j$ and $a \in \mathbb{k}$. Then for a generic choice of $a \in \mathbb{k}$ holds

$$\mathscr{L}(F) \preceq_{\mathscr{L}} \mathscr{L}(\Psi(F)^{\triangle}).$$

PROOF. Before we start with the proof we want to introduce a notation that we will use in the following. For a polynomial $f \in \mathcal{P}$ and $t \in \operatorname{supp}(f)$ we denote the coefficient of t in f with $\mathtt{C}_f(t)$.

Let $F = \{f_1, \ldots, f_\ell\}$ with $\operatorname{lt} f_k \prec_{\mathrm{revlex}} \operatorname{lt} f_l$ if $k > l$. Further, let $t_k = \operatorname{lt} f_k$ and $s_k = \operatorname{lt} \Psi(f_k)$ for each k. Without loss of generality suppose that $\operatorname{lc} f_k = 1$ for each k. It is easy to see that $t_k \preceq_{\mathrm{revlex}} s_k$ for all k. If $t_k = s_k$ for all k, then there is nothing to prove since then $\operatorname{lt} F = \operatorname{lt} \Psi(F) = \operatorname{lt} \Psi(F)^{\triangle}$ and consequently $\mathscr{L}(F) = \mathscr{L}(\Psi(F)^{\triangle})$. Otherwise, let α be minimal such that $t_\alpha \neq s_\alpha$. In other words:

$$\begin{array}{ll} t_k = s_k & \forall k < \alpha \\ t_\alpha \prec_{\mathrm{revlex}} s_\alpha & \\ t_k \preceq_{\mathrm{revlex}} s_k & \forall k > \alpha \end{array}$$

Let h_α be the remainder of $\Psi(f_\alpha)$ after reducing it by the set $\{\Psi(f_1), \ldots, \Psi(f_{\alpha-1})\}$ – note that this set is head autoreduced, but in general not completely autoreduced. As a first step we want to show that t_α is still in the support of h_α since then $\operatorname{lt} h_\alpha \succeq_{\mathrm{revlex}} t_\alpha$ holds.

Claim 1: $t_\alpha \in \operatorname{supp}(h_\alpha)$.
If $h_\alpha = \Psi(f_\alpha)$ we are done since $t_\alpha \in \operatorname{supp}(\Psi(f_\alpha))$, otherwise there is an index $\beta < \alpha$ such that $s_\beta = t_\beta$ divides s_α. So the question is whether t_α remains in the support of

$$h_\beta = \Psi(f_\alpha) - \frac{\mathtt{C}_{\Psi(f_\alpha)}(s_\alpha) s_\alpha}{\mathtt{C}_{\Psi(f_\beta)}(t_\beta) t_\beta} \Psi(f_\beta)$$

or not.

Claim 1.1: $t_\alpha \in \operatorname{supp}(h_\beta)$.
Let us assume that this is not the case. Hence there must be a monomial $m_\beta = \mathtt{C}_{\Psi(f_\beta)}(t_{m_\beta}) t_{m_\beta}$ in $\Psi(f_\beta)$ that causes the cancellation of t_α. This means the following equation must hold:

$$\begin{aligned} \mathtt{C}_{\Psi(f_\alpha)}(t_\alpha) t_\alpha &= \frac{\mathtt{C}_{\Psi(f_\alpha)}(s_\alpha) s_\alpha}{\mathtt{C}_{\Psi(f_\beta)}(t_\beta) t_\beta} \mathtt{C}_{\Psi(f_\beta)}(t_{m_\beta}) t_{m_\beta} \\ \Leftrightarrow \quad \mathtt{C}_{\Psi(f_\alpha)}(t_\alpha) \mathtt{C}_{\Psi(f_\beta)}(t_\beta) t_\alpha t_\beta &= \mathtt{C}_{\Psi(f_\alpha)}(s_\alpha) \mathtt{C}_{\Psi(f_\beta)}(t_{m_\beta}) s_\alpha t_{m_\beta} \end{aligned} \tag{2.4}$$

Let us now have a look at the coefficients which we can interpret as elements of $\Bbbk[a]$, i.e. polynomials in the parameter a. Because of the form of the transformation Ψ we have $1 \in \operatorname{supp}(\mathtt{C}_{\Psi(f_\beta)}(t_\alpha))$, $\operatorname{supp}(\mathtt{C}_{\Psi(f_\beta)}(t_\beta))$ and hence

$$1 \in \operatorname{supp}(\mathtt{C}_{\Psi(f_\alpha)}(t_\alpha)\mathtt{C}_{\Psi(f_\beta)}(t_\beta)).$$

But since $s_\alpha \succ_{\text{revlex}} t_\alpha$ it follows that $1 \notin \operatorname{supp}(\mathtt{C}_{\Psi(f_\alpha)}(s_\alpha))$ and therefore

$$1 \notin \operatorname{supp}(\mathtt{C}_{\Psi(f_\alpha)}(s_\alpha)\mathtt{C}_{\Psi(f_\beta)}(t_{m_\beta})).$$

This shows that the polynomials $\mathtt{C}_{\Psi(f_\alpha)}(t_\alpha)\mathtt{C}_{\Psi(f_\beta)}(t_\beta)$ and $\mathtt{C}_{\Psi(f_\alpha)}(s_\alpha)\mathtt{C}_{\Psi(f_\beta)}(t_{m_\beta})$ are not equal. For any element $\boldsymbol{p}$ of the Zariski open set

$$\Bbbk \setminus \mathcal{V}(\mathtt{C}_{\Psi(f_\alpha)}(t_\alpha)\mathtt{C}_{\Psi(f_\beta)}(t_\beta) - \mathtt{C}_{\Psi(f_\alpha)}(s_\alpha)\mathtt{C}_{\Psi(f_\beta)}(t_{m_\beta})) \tag{2.5}$$

obviously holds $\left(\mathtt{C}_{\Psi(f_\alpha)}(t_\alpha)\mathtt{C}_{\Psi(f_\beta)}(t_\beta)\right)(\boldsymbol{p}) \neq \left(\mathtt{C}_{\Psi(f_\alpha)}(s_\alpha)\mathtt{C}_{\Psi(f_\beta)}(t_{m_\beta})\right)(\boldsymbol{p})$. As the parameter a is chosen generically, we know from Remark 2.2.12, that a is chosen from a Zariski open set. Since two nonempty Zariski open sets always have a nonempty intersection, we may assume that a is an element of the set presented in (2.5). Therefore equation (2.4) does not hold in this case which leads to a contradiction of our assumption that $t_\alpha \notin \operatorname{supp}(h_\beta)$. Hence claim 1.1 is true.

Additionally we can also see that the coefficient of t_α in h_β is:

$$\begin{aligned} \mathtt{C}_{h_\beta}(t_\alpha) &= \mathtt{C}_{\Psi(f_\alpha)}(t_\alpha) - \frac{\mathtt{C}_{\Psi(f_\alpha)}(s_\alpha)}{\mathtt{C}_{\Psi(f_\beta)}(t_\beta)}\mathtt{C}_{\Psi(f_\beta)}(t_{m_\beta}) \\ \Leftrightarrow \quad \mathtt{C}_{\Psi(f_\beta)}(t_\beta)\mathtt{C}_{h_\beta}(t_\alpha) &= \mathtt{C}_{\Psi(f_\beta)}(t_\beta)\mathtt{C}_{\Psi(f_\alpha)}(t_\alpha) - \mathtt{C}_{\Psi(f_\alpha)}(s_\alpha)\mathtt{C}_{\Psi(f_\beta)}(t_{m_\beta}) \end{aligned}$$

With the arguments from above we have $1 \in \operatorname{supp}(\mathtt{C}_{\Psi(f_\beta)}(t_\beta)\mathtt{C}_{h_\beta}(t_\alpha))$ and therefore

$$1 \in \operatorname{supp}(\mathtt{C}_{h_\beta}(t_\alpha)). \tag{2.6}$$

If now already $h_\beta = h_\alpha$ holds then immediately claim 1 follows, otherwise there is an index $\gamma < \alpha$ such that the term $s_\gamma = t_\gamma$ divides $\operatorname{lt} h_\beta = t_{h_\beta}$. The existence of such a divisor shows that t_{h_β} can not be equal to t_α since F is a completely autoreduced set – note that we could not argue like this if F was only head autoreduced – and therefore

$$t_{h_\beta} \succ_{\text{revlex}} t_\alpha. \tag{2.7}$$

As above we want to check whether t_α remains in the support of

$$h_\gamma = h_\beta - \frac{\mathtt{C}_{h_\beta}(t_{h_\beta})t_{h_\beta}}{\mathtt{C}_{\Psi(f_\gamma)}(t_\gamma)t_\gamma}\Psi(f_\gamma)$$

or not.

Claim 1.2: $t_\alpha \in \operatorname{supp}(h_\gamma)$.
Let us assume that this is not the case. Hence there is a monomial $m_\gamma = \mathtt{C}_{\Psi(f_\gamma)}(t_{m_\gamma})t_{m_\gamma}$ in $\Psi(f_\gamma)$ such that:

$$\begin{aligned} \mathtt{C}_{h_\beta}(t_\alpha)t_\alpha &= \frac{\mathtt{C}_{h_\beta}(t_{h_\beta})t_{h_\beta}}{\mathtt{C}_{\Psi(f_\gamma)}(t_\gamma)t_\gamma}\mathtt{C}_{\Psi(f_\gamma)}(t_{m_\gamma})t_{m_\gamma} \\ \Leftrightarrow \quad \mathtt{C}_{h_\beta}(t_\alpha)\mathtt{C}_{\Psi(f_\gamma)}(t_\gamma)t_\alpha t_\gamma &= \mathtt{C}_{h_\beta}(t_{h_\beta})\mathtt{C}_{\Psi(f_\gamma)}(t_{m_\gamma})t_{h_\beta}t_{m_\gamma} \end{aligned} \tag{2.8}$$

Let us now again have a look at the coefficients. As above we immediately get $1 \in \operatorname{supp}(\mathtt{C}_{\Psi(f_\gamma)}(t_\gamma))$ because of the form of Ψ. In (2.6) we already saw that $1 \in \operatorname{supp}(\mathtt{C}_{h_\beta}(t_\alpha))$, hence $1 \in \operatorname{supp}(\mathtt{C}_{h_\beta}(t_\alpha)\mathtt{C}_{\Psi(f_\gamma)}(t_\gamma))$.
We are done if we are able to show that

$$1 \notin \operatorname{supp}(\mathtt{C}_{h_\beta}(t_{h_\beta})). \tag{2.9}$$

Because then $1 \notin \operatorname{supp}(\mathtt{C}_{h_\beta}(t_{h_\beta})\mathtt{C}_{\Psi(f_\gamma)}(t_{m_\gamma}))$ and so equation (2.8) does not hold for a generic choice of the parameter a. This would finally lead to a contradiction of our assumption that $t_\alpha \notin \operatorname{supp}(h_\gamma)$.
To show (2.9) we should remember the construction of h_β:

$$h_\beta = \Psi(f_\alpha) - \frac{\mathtt{C}_{\Psi(f_\alpha)}(s_\alpha)s_\alpha}{\mathtt{C}_{\Psi(f_\beta)}(t_\beta)t_\beta}\Psi(f_\beta)$$

From this we can derive:

$$\begin{aligned} \mathtt{C}_{h_\beta}(t_{h_\beta}) &= \mathtt{C}_{\Psi(f_\alpha)}(t_{h_\beta}) - \frac{\mathtt{C}_{\Psi(f_\alpha)}(s_a)}{\mathtt{C}_{\Psi(f_\beta)}(t_\beta)}\mathtt{C}_{\Psi(f_\beta)}(t_{h_\beta}) \\ \Leftrightarrow \quad \mathtt{C}_{h_\beta}(t_{h_\beta})\mathtt{C}_{\Psi(f_\beta)}(t_\beta) &= \mathtt{C}_{\Psi(f_\alpha)}(t_{h_\beta})\mathtt{C}_{\Psi(f_\beta)}(t_\beta) - \mathtt{C}_{\Psi(f_\alpha)}(s_a)\mathtt{C}_{\Psi(f_\beta)}(t_{h_\beta}) \end{aligned} \tag{2.10}$$

On the one hand we should notice that $1 \notin \operatorname{supp}(\mathtt{C}_{\Psi(f_\alpha)}(t))$ for all terms $t \in \operatorname{supp}(\Psi(f_\alpha))$ with $t \succ_{\text{revlex}} t_\alpha$. So, since $t_{h_\beta} \succ_{\text{revlex}} t_\alpha$ by (2.7), we can follow that if $t_{h_\beta} \in \operatorname{supp}(\Psi(f_\alpha))$ then $1 \notin \operatorname{supp}(\mathtt{C}_{\Psi(f_\alpha)}(t_{h_\beta}))$ and therefore

$$1 \notin \operatorname{supp}(\mathtt{C}_{\Psi(f_\alpha)}(t_{h_\beta})\mathtt{C}_{\Psi(f_\beta)}(t_\beta))$$

On the other hand we have already seen above that $1 \notin \operatorname{supp}(\mathtt{C}_{\Psi(f_\alpha)}(s_\alpha))$ and so:

$$1 \notin \operatorname{supp}(\mathtt{C}_{\Psi(f_\alpha)}(s_a)\mathtt{C}_{\Psi(f_\beta)}(t_{h_\beta}))$$

Since at least one of the coefficients $\mathtt{C}_{\Psi(f_\alpha)}(t_{h_\beta})$ and $\mathtt{C}_{\Psi(f_\beta)}(t_{h_\beta})$ must be nonzero (otherwise t_{h_β} would not occur in the support of h_β), we can follow from (2.10) that $1 \notin \operatorname{supp}(\mathtt{C}_{h_\beta}(t_{h_\beta})\mathtt{C}_{\Psi(f_\beta)}(t_\beta))$. Now (2.9) follows from the fact that $1 \in \operatorname{supp}(\mathtt{C}_{\Psi(f_\beta)}(t_\beta))$ and so we proved claim 1.2.

We can repeat this procedure until we end up at h_α and with the arguments from above we know that $t_\alpha \in \operatorname{supp}(h_\alpha)$, which finally proves claim 1.

This means either $t_\alpha \prec_{\text{revlex}} \operatorname{lt} h_\alpha = t_{h_\alpha}$ or $t_\alpha = t_{h_\alpha}$.

Case I: $t_\alpha \prec_{\text{revlex}} t_{h_\alpha}$.
It is not clear that the set

$$\{\Psi(f_1), \dots, \Psi(f_{\alpha-1}), h_\alpha\}$$

is head autoreduced since it could happen that there is an index $\delta < \alpha$ such that t_{h_α} divides $s_\delta = t_\delta$. Since $t_{h_\alpha} \neq t_\delta$ by the construction of h_α, we know that, $t_{h_\alpha} \succ_{\text{revlex}} t_\delta$. In this case we check whether the set $\{\Psi(f_1), \dots, \Psi(f_{\delta-1}), h_\alpha\}$ is head autoreduced or not. If it is not, then there is an index $\varepsilon < \delta$ such that t_{h_α} divides $s_\varepsilon = t_\varepsilon$. Again we check whether the set $\{\Psi(f_1), \dots, \Psi(f_{\varepsilon-1}), h_\alpha\}$ is head autoreduced or not and go on like this, until we reach an index $\zeta < \varepsilon$ such that the set $\{\Psi(f_1), \dots, \Psi(f_{\zeta-1}), h_\alpha\}$ is head autoreduced. Still it is not clear if this set is a subset of $\Psi(F)^\Delta$, but we know that $\operatorname{lt} f_\zeta \prec_{\text{revlex}} \operatorname{lt} h_\alpha$ and so

$$\mathscr{L}(f_1, \dots, f_\zeta) \prec_{\mathscr{L}} \mathscr{L}(\Psi(f_1), \dots, \Psi(f_{\zeta-1}), h_\alpha).$$

Let $\Psi(F)^\Delta = \{\hat{f}_1, \dots \hat{f}_{\hat{m}}\}$ then of course

$$\mathscr{L}(\Psi(f_1), \dots, \Psi(f_{\zeta-1}), h_\alpha) \preceq_{\mathscr{L}} \mathscr{L}(\hat{f}_1, \dots, \hat{f}_\zeta),$$

which proves the lemma since then $\mathscr{L}(F) \prec_{\mathscr{L}} \mathscr{L}(\Psi(F)^\Delta)$.

Case II: $t_\alpha = t_{h_\alpha}$.
In this case we have to look for the smallest index $\alpha' > \alpha$ such that $t_{\alpha'} \neq s_{\alpha'}$. Then we reduce $\Psi(f_{\alpha'})$ by the set

$$\{\Psi(f_1), \dots, \Psi(f_{\alpha-1}), h_\alpha, \Psi(f_{\alpha+1}), \dots, \Psi(f_{\alpha'-1})\} \tag{2.11}$$

to $h_{\alpha'}$ in the same way as above – note that (2.11) is head autoreduced since the leading terms did not change in comparison to the completely autoreduced set F. It is clear that if we go on like this we will either end up by $\Psi(F)^\Delta$ with $\operatorname{lt} \hat{f}_k = \operatorname{lt} f_k$ for all k, which would mean that $\mathscr{L}(F) = \mathscr{L}(\Psi(F)^\Delta)$ or we find a h_ω with $t_\omega \prec_{\text{revlex}} \operatorname{lt} h_\omega$ which would lead us back to case I. $\square$

LEMMA 2.2.14.
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal and G its reduced Gröbner basis. Let $\Psi : (x_j \mapsto x_j + ax_i)$ be a coordinate transformation with $i < j$ and $a \in \Bbbk$. Further, let $\tilde{G}$ be the reduced Gröbner basis of $\Psi(\mathcal{I})$. Then, for a generic choice of $a \in \Bbbk$, we have

$$\mathscr{L}(\Psi(G)^\Delta) \preceq_{\mathscr{L}} \mathscr{L}(\tilde{G}).$$

PROOF. Suppose that $\mathscr{L}(\Psi(G)^\Delta) = (t_1, \dots, t_\ell)$ and $\mathscr{L}(\tilde{G}) = (\tilde{t}_1, \dots, \tilde{t}_{\tilde{\ell}})$. Since

$$t_k \in \operatorname{lt} \Psi(G)^\Delta \subseteq \operatorname{lt}\langle \Psi(G)^\Delta \rangle = \operatorname{lt} \Psi(\mathcal{I}) = \langle \operatorname{lt} \tilde{G} \rangle$$

for all k, there is a $\tilde{g}_k \in \tilde{G}$ such that $\operatorname{lt} \tilde{g}_k$ divides t_k and therefore $\operatorname{lt} \tilde{g}_k \succeq_{\text{revlex}} t_k$. Now we start to compare the two lists beginning with the first entry.

Let $\operatorname{lt} \tilde{g}_1 = \tilde{t}_\alpha$, if $\alpha > 1$ we are done because then

$$\tilde{t}_1 \succ_{\text{revlex}} \tilde{t}_\alpha = \operatorname{lt} \tilde{g}_1 \succeq_{\text{revlex}} t_1.$$

So we assume $\operatorname{lt} \tilde{g}_1 = \tilde{t}_1$. Of course we are done if $\tilde{t}_1 \succ_{\text{revlex}} t_1$, so we additionally assume that $t_1 = \tilde{t}_1$ and go on with the next entry.
First we should mention that $\tilde{g}_1 \neq \tilde{g}_2$ since otherwise $t_1 = \tilde{t}_1 = \operatorname{lt} \tilde{g}_1 = \operatorname{lt} \tilde{g}_2$ divides t_2 which is a contradiction to $\Psi(G)^\Delta$ being head autoreduced. Now we have to check which position $\operatorname{lt} \tilde{g}_2 = \tilde{t}_\beta$ has in the list $\mathscr{L}(\tilde{G})$. Since $\tilde{G}$ is reduced $\operatorname{lt} \tilde{g}_1 \neq \operatorname{lt} \tilde{g}_2$ and therefore $\beta > 1$. If $\beta > 2$ we have – analogous to above – the situation

$$\tilde{t}_2 \succ_{\text{revlex}} \tilde{t}_\beta = \operatorname{lt} \tilde{g}_2 \succeq_{\text{revlex}} t_2$$

and hence we are done. Otherwise $\beta = 2$ and so either $\tilde{t}_2 \succ_{\text{revlex}} t_2$ or $\tilde{t}_2 = t_2$. In the first case our assertion follows and in the second one, we go on with the next entry. So sooner or later we either find an index ω with $\tilde{t}_\omega \succ_{\text{revlex}} t_\omega$, which shows that $\mathscr{L}(\Psi(G)^\Delta) \prec_{\mathscr{L}} \mathscr{L}(\tilde{G})$, or we have

$$\tilde{t}_k = t_k \text{ for all } k \leq \min(\tilde{\ell}, \ell). \tag{2.12}$$

In the first case we are done so let us assume that (2.12) holds. Since $\tilde{G}$ is a Gröbner basis of $\langle \Psi(G)^\Delta \rangle$ and both $\Psi(G)^\Delta$ and $\tilde{G}$ are reduced sets we must have $\ell \leq \tilde{\ell}$. Hence it follows from the definitions of $\prec_{\mathscr{L}}$ that:

$$\begin{array}{rcl} \mathscr{L}(\Psi(G)^\Delta) & = & \mathscr{L}(\tilde{G}), \text{ if } \ell = \tilde{\ell} \\ \mathscr{L}(\Psi(G)^\Delta) & \prec_{\mathscr{L}} & \mathscr{L}(\tilde{G}), \text{ if } \ell < \tilde{\ell} \end{array}$$

□

Until now we did not have to care about the characteristic of our field $\Bbbk$. But the following Lemma clearly shows its importance and will also play a decisive role in Section 2.4.

Lemma 2.2.15.
Let $f \in \mathcal{P}$ be a polynomial and $\Psi : (x_j \mapsto x_j + ax_i)$ a coordinate transformation with $i < j$ and $a \in \Bbbk \setminus \{0\}$. Further, let $\mathbf{x}^\mu \in \operatorname{supp}(f)$ be a term such that $\mu_j > 0$. Then for a generic choice of $a \in \Bbbk$ holds:

$$x_i^{\mu_j - u} \frac{\mathbf{x}^\mu}{x_j^{\mu_j - u}} \in \operatorname{supp}(\Psi(f)),$$

for all integers u with $\begin{cases} \binom{\mu_j}{u} \neq 0, & \textit{if } \operatorname{char} \Bbbk = 0 \\ \binom{\mu_j}{u} \not\equiv 0 \mod p, & \textit{if } \operatorname{char} \Bbbk = p > 0 \end{cases}$.

Proof. Let $\mathtt{C}_\mu \in \Bbbk$ be the coefficient of $\mathbf{x}^\mu$ in f and $g = f - \mathtt{C}_\mu \mathbf{x}^\mu$. Then $\Psi(f) = \Psi(\mathtt{C}_\mu \mathbf{x}^\mu) + \Psi(g)$ and

$$\begin{aligned}\Psi(\mathtt{C}_\mu \mathbf{x}^\mu) &= \mathtt{C}_\mu (x_j + a x_i)^{\mu_j} \frac{\mathbf{x}^\mu}{x_j^{\mu_j}} \\ &= \mathtt{C}_\mu \left(\sum_{u=0}^{\mu_j} \binom{\mu_j}{u} x_j^u (a x_i)^{\mu_j - u} \right) \frac{\mathbf{x}^\mu}{x_j^{\mu_j}} \\ &= \mathtt{C}_\mu \sum_{u=0}^{\mu_j} \binom{\mu_j}{u} a^{\mu_j - u} x_i^{\mu_j - u} \frac{\mathbf{x}^\mu}{x_j^{\mu_j - u}}\end{aligned}$$

So we see that every term $t_u = x_i^{\mu_j - u} \dfrac{\mathbf{x}^\mu}{x_j^{\mu_j - u}}$ lies in the support $\Psi(\mathtt{C}_\mu \mathbf{x}^\mu)$ if $\binom{\mu_j}{u}$ is nonzero. Now we have to clarify whether these terms can be cancelled out by $\Psi(g)$.

Since $t_{\mu_j} = \mathbf{x}^\mu \notin \operatorname{supp}(g)$ we see that if $\mathbf{x}^\mu$ lies in $\operatorname{supp}(\Psi(g))$ it must have a coefficient that can be interpreted as an element of $\Bbbk[a] \setminus \Bbbk$. But the coefficient of t_{μ_j} in $\Psi(\mathtt{C}_\mu \mathbf{x}^\mu)$ is $\mathtt{C}_\mu$ – which is an element of $\Bbbk$ – and so, because of the generic choice of a, this term can not be deleted by a term of g (for more details see our argumentation in the proof of Lemma 2.2.13). Therefore we have

$$t_{\mu_j} \in \operatorname{supp}(\Psi(f)).$$

Now we consider the terms t_u with $u < \mu_j$. Every one of them has a coefficient in $\Bbbk[a] \setminus \Bbbk$. Let us now assume that $t_u \notin \operatorname{supp}(\Psi(f))$ then t_u must be removed by a term of $\Psi(g)$ that does not appear in g. Because – as a consequence of the form of Ψ – other terms of $\Psi(g)$ can not have a coefficient in $\Bbbk[a] \setminus \Bbbk$. So let $\mathbf{x}^{\nu^{(1)}}, \ldots, \mathbf{x}^{\nu^{(\ell)}} \in \operatorname{supp}(g)$ with $t_u \in \operatorname{supp}(\Psi(\mathbf{x}^{\nu^{(k)}}))$ for all $k = 1, \ldots, \ell$. The elements of $\operatorname{supp}(\Psi(\mathbf{x}^{\nu^{(k)}}))$ are of the form:

$$s_v^{(k)} = x_i^{\nu_j^{(k)} - v} \frac{\mathbf{x}^{\nu^{(k)}}}{x_j^{\nu_j^{(k)} - v}}, \quad 0 \le v \le \nu_j^{(k)},\ 1 \le k \le \ell$$

So for each k there must be a $\tilde{v}_k$ such that $s_{\tilde{v}_k}^{(k)} = t_u$. Since the exponent of x_j is v in $s_v^{(k)}$ and u in t_u it follows that $\tilde{v}_k = u$ for every k. In particular, $s_u^{(k)} = t_u$ and so:

$$\nu_j^{(k)} + \nu_i^{(k)} = \mu_j + \mu_i \text{ and } \nu_l^{(k)} = \mu_l \text{ for all } l \neq i, j. \tag{2.13}$$

If $\mathtt{C}_{\nu^{(k)}} \in \Bbbk$ denotes the coefficient of $\mathbf{x}^{\nu^{(k)}}$ in g it follows from our assumption that

$$\sum_{k=1}^{\ell} \mathtt{C}_{\nu^{(k)}} \binom{\nu_j^{(k)}}{u} a^{\nu_j^{(k)} - u} = \mathtt{C}_\mu \binom{\mu_j}{u} a^{\mu_j - u} \tag{2.14}$$

since the left-hand side is the coefficient of t_u in $\Psi(g)$ while the right-hand side is the coefficient of t_u in $\Psi(\mathtt{C}_\mu \mathbf{x}^\mu)$. Because $\mathtt{C}_{\nu^{(k)}}$ and $\mathtt{C}_\mu$ are not in $\Bbbk[a]$, it follows from (2.14) that $\nu_j^{(k)} = \mu_j$ for all k. Hence $\nu^{(k)} = \mu$ for all k by (2.13) but this is a contradiction since $\mathbf{x}^\mu \notin \operatorname{supp}(g)$ by construction of g and so we have

$$t_u \in \operatorname{supp}(\Psi(f))$$

for all $u < \mu_j$. □

Proposition 2.2.16.
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal. Further, let G be the reduced Gröbner basis of $\mathcal{I}$ and $g \in G$ with $\operatorname{lt} g = \mathbf{x}^\mu$. Assume there are indices i, j with $i < j$ and $\mu_j > 0$ so that

$$x_i^{\mu_j - \tilde{u}} \frac{\operatorname{lt} g}{x_j^{\mu_j - \tilde{u}}} \notin \operatorname{lt} \mathcal{I} \tag{2.15}$$

for an integers $\tilde{u}$ with $\begin{cases} \binom{\mu_j}{\tilde{u}} \neq 0, & \textit{if } \operatorname{char} \Bbbk = 0 \\ \binom{\mu_j}{\tilde{u}} \not\equiv 0 \mod p, & \textit{if } \operatorname{char} \Bbbk = p > 0 \end{cases}$.
Further, let $\Psi : (x_j \mapsto x_j + a x_i)$ be a coordinate transformation and $\tilde{G}$ the reduced Gröbner basis of $\Psi(\mathcal{I})$. Then for a generic choice of $a \in \Bbbk$, we have

$$\mathscr{L}(G) \prec_{\mathscr{L}} \mathscr{L}(\tilde{G}).$$

Proof. From the Lemmas 2.2.13 and 2.2.14 we can see that

$$\mathscr{L}(G) \preceq_{\mathscr{L}} \mathscr{L}(\Psi(G)^\Delta) \preceq_{\mathscr{L}} \mathscr{L}(\tilde{G})$$

To prove our assertion we will now show that (2.15) causes the inequality $\mathscr{L}(G) \neq \mathscr{L}(\Psi(G)^\Delta)$. Let us assume that this was not the case. Further, let $G = \{g_1, \dots, g_\ell\}$ and $\Psi(G)^\Delta = \{\hat{g}_1, \dots, \hat{g}_\ell\}$ - note that $\#G = \#\Psi(G)^\Delta$ since both sets are reduced. Without loss of generality suppose that $\operatorname{lt} g_k \prec_{\text{revlex}} \operatorname{lt} g_l$ and $\operatorname{lt} \hat{g}_k \prec_{\text{revlex}} \operatorname{lt} \hat{g}_l$ if $k > l$. By our assumption follows:

$$\operatorname{lt} g_k = \operatorname{lt} \hat{g}_k, \quad k \leq \ell \tag{2.16}$$

There must be an index r such that $g = g_r$ and so $\operatorname{lt} g_r = \mathbf{x}^\mu$. Let $t_u = x_i^{\mu_j - u} \dfrac{\operatorname{lt} g_r}{x_j^{\mu_j - u}}$ then $t_{\mu_j} = \operatorname{lt} g_r \in \operatorname{lt} \mathcal{I}$ and so it follows from (2.15) that $\tilde{u} < \mu_j$. Hence because of the reverse lexicographical ordering:

$$\operatorname{lt} g_r \prec_{\text{revlex}} t_{\tilde{u}} \tag{2.17}$$

We know from Lemma 2.2.15 that every term t_u, where $\binom{\mu_j}{u}$ does not vanish, is in the support of $\Psi(g_r)$, so in particular $t_{\tilde{u}} \in \operatorname{supp}(\Psi(g_r))$. Since $\operatorname{lt} g_r = \operatorname{lt} \hat{g}_r$ by (2.16), we know that every term of $\Psi(g_r)$ that is greater than $\operatorname{lt} g_r$ will be reduced. Since $t_{\tilde{u}}$ is one of these terms because of (2.17), there must be an element in $\{\operatorname{lt} g_1, \dots, \operatorname{lt} g_\ell\}$ that divides $t_{\tilde{u}}$. But this means that $t_{\tilde{u}} \in \langle \operatorname{lt} g_1, \dots, \operatorname{lt} g_\ell \rangle = \operatorname{lt} \mathcal{I}$ which is a contradiction to (2.15). □

Remark 2.2.17.
Proposition 2.2.16 formulates the central point for our termination proof and – as already mentioned before – we will now discuss that we can also use this result to achieve a stable or quasi-stable position.

Stability. By Definition 2.2.1 an obstruction to stability caused by a monomial $\mathbf{x}^\mu \in \mathcal{I}$ has the form $x_i \frac{\mathbf{x}^\mu}{x_k}$ with $k = \mathrm{m}(\mathbf{x}^\mu)$ and $i < k$. But this corresponds to (2.15) with $\tilde{u} = \mu_k - 1$.

Quasi-Stability. In the quasi-stable case we have an obstruction if $\mathbf{x}^\mu \in \operatorname{lt}\mathcal{I}$ but $t = x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}} \notin \operatorname{lt}\mathcal{I}$ with $k = \mathrm{m}(\mathbf{x}^\mu)$, $i < k$ and B denotes the minimal basis of $\operatorname{lt}\mathcal{I}$. But if the term t is not contained in the leading ideal, then of course the same holds for its divisor $x_i^{\mu_k} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}}$ which is again corresponds to (2.15) with $\tilde{u} = 0$.

In the preceding statements we always assumed the choice of the parameter a to be generic. To make this concept clearer we now want to provide an example that shows the effect of a nongeneric choice.

Example 2.2.18.
Let $\mathcal{I} = \langle x_1^2 - x_1x_2,\ x_2^3\rangle \lhd \Bbbk[x_1, x_2]$ and $\Psi : (x_2 \mapsto x_2 + ax_1)$. For $a = 1$ the reduced Gröbner basis of $\mathcal{I}$ is $G = \{x_1^2 - x_1x_2,\ x_2^3\}$ and $\tilde{G} = \{x_1x_2,\ x_1^3 + x_2^3,\ x_2^4\}$ is the one of $\Psi(\mathcal{I})$. Therefore

$$\mathscr{L}(G) = (x_1^2,\ x_2^3) \succ_{\mathscr{L}} (x_1^3,\ x_1x_2,\ x_2^4) = \mathscr{L}(\tilde{G}).$$

So obviously $a = 1$ is not a generic choice which can easily be seen if one considers a is a parameter and takes a look at $\Psi(\mathcal{I}) \lhd \Bbbk[a, x_1, x_2]$:

$$\Psi(\mathcal{I}) = \langle (1-a)x_1^2 - x_1x_2,\ a^3x_1^3 + 3a^2x_1^2x_2 + 3ax_1x_2^2 + x_2^3\rangle$$

Now we can see that $a = 1 \in \mathcal{V}(1-a)$ is a root of a "leading coefficient polynomial" of the first generator and therefore not a generic choice.

Finally, we are now able to prove the termination and correctness of Algorithm 1 which is the main result of this chapter.

Theorem 2.2.19.
If $\operatorname{char}\Bbbk = 0$ *Algorithm 1 terminates in finitely many steps and returns a coordinate transformation* Ψ *so that* $\Psi(\mathcal{I})$ *is in strongly stable position.*

Proof. Let $\mathcal{I}$ be the given homogeneous ideal and G its reduced Gröbner basis. Further, let $A = (a_{ij}) \in \mathrm{Gl}(n, \Bbbk)$ be a matrix. We interpret the elements of A as parameters and compute a Gröbner system $\{(\hat{G}_i, N_i, W_i)\}_{i=1}^{\ell}$ of the parametric ideal $A \cdot \mathcal{I} \lhd \Bbbk[\mathbf{a}, \mathbf{x}]$. According to Remark 2.1.2, we denote by $\mathcal{L}_i$ the monomial ideals that are determined by N_i and W_i. This means that the set $\mathfrak{L} = \{\mathcal{L}_i\}_{i=1}^{\ell}$ contains all leading ideals of $\mathcal{I}$ under any possible coordinate transformations. Let B_i be the monomial basis of $\mathcal{L}_i$ and without loss of generality we assume:

$$\mathscr{L}(B_1) \prec_{\mathscr{L}} \cdots \prec_{\mathscr{L}} \mathscr{L}(B_\ell)$$

In particular, there must be an index $\alpha \le \ell$ such that $\operatorname{lt}\mathcal{I} = \mathcal{L}_\alpha \in \mathfrak{L}$ and – since G is the reduced Gröbner basis – we have $\operatorname{lt} G = B_\alpha$ so that $\mathscr{L}(G) = \mathscr{L}(B_\alpha)$.
If $\operatorname{lt}\mathcal{I}$ is not strongly stable, there exists a polynomial $g \in G$ and integers $i, j \in \{1, \ldots, n\}$ with $i < j$ such that x_j divides $\operatorname{lt} g = \mathbf{x}^\mu$ and

$$x_i \frac{\operatorname{lt} g}{x_j} = x_i^{\mu_j-(\mu_j-1)} \frac{\mathbf{x}^\mu}{x_j^{\mu_j-(\mu_j-1)}} \notin \operatorname{lt}\mathcal{I}.$$

Let $\Psi_1 : (x_j \mapsto x_j + x_i)$ and $\hat{G}_1$ the reduced Gröbner basis of $\Psi_1(\mathcal{I})$. There is an index $\beta \le \ell$ such that $\operatorname{lt}\Psi_1(\mathcal{I}) = \mathcal{L}_\beta \in \mathfrak{L}$ and so $\mathscr{L}(\hat{G}_1) = \mathscr{L}(B_\beta)$. Now if $a = 1$ is a generic choice, we know by Proposition 2.2.16 that $\alpha < \beta$. Otherwise, we enter the while loop of line 5 and perform again the same transformation Ψ_1, which is altogether equivalent to the transformation $(x_j \mapsto x_j + 2x_i)$. We know from Proposition 2.2.16 that the list of monomials must increase if we choose a generic value for a. As there are only finitely many nongeneric values for a, we will reach a generic one after a finite number of iterations. Hence there is an integer r such that for the reduced Gröbner basis $\hat{G}_r$ of $\Psi_1^r(\mathcal{I})$ holds $\mathscr{L}(\hat{G}_r) = \mathscr{L}(B_\gamma)$ with $\alpha < \gamma \le \ell$.
Since $\mathfrak{L}$ is finite, it is clear that we can not repeat this process infinitely many times. This means that after a finite number of steps we end up at an ideal

$$\mathcal{I}^{(\omega)} = \Psi_\omega^{r_\omega} \cdots \Psi_1^{r_1}(\mathcal{I})$$

with reduced Gröbner basis $\hat{G}_\omega$ that does not contain an element which causes an obstruction to strong stability. In particular, $\mathcal{I}^{(\omega)}$ is strongly stable. ☐

The last thing we want to mention in this section is that once we have transformed an ideal into strongly stable position, doing another coordinate change of the form $\Psi : (x_j \mapsto x_j + a x_i)$ can result in a leading ideal that is not strongly stable although the new list of generators has increased with respect to $\prec_{\mathscr{L}}$. This effect is illustrated in the next example.

Example 2.2.20.
Let $\mathcal{I} = \langle x_1^3,\ x_1^2x_2 + x_1x_2^2 + x_1x_3^2,\ x_1^2x_3,\ x_1^2x_4\rangle \lhd \Bbbk[x_1, x_2, x_3, x_4]$ then

$$\operatorname{lt}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1^2x_4,\ x_1x_2^3,\ x_1x_2^2x_3,\ x_1x_2^2x_4\rangle$$

is strongly stable. If we perform the transformation $\Psi : (x_3 \mapsto x_3 + x_2)$ then

$$\operatorname{lt}\Psi(\mathcal{I}) = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_1^2x_4,\ x_1^2x_3^2\rangle$$

is not strongly stable since $x_3 \frac{x_1^2x_4}{x_4} = x_1^2x_3 \notin \operatorname{lt}\Psi(\mathcal{I})$. However:

$$(x_1^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1^2x_4,\ x_1x_2^3,\ x_1x_2^2x_3,\ x_1x_2^2x_4) \prec_{\mathscr{L}} (x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_1^2x_4,\ x_1^2x_3^2)$$

2.3. Some thoughts about Efficiency

Although we will not discuss the implementation of Algorithm 1 in this thesis, we now want to take a short look at possible strategies to implement line 2 of the algorithm. Thereby we focus on approaches that could possibly decrease either the number of checks for obstructions or the number of transformations we have to apply to our ideal until it is in strongly stable position.

Reduce Number of Obstruction-Checks.

Degree-by-Degree. The strategy of the *degree-by-degree* method is to remove all obstructions starting in the lowest respectively highest degree that occurs in the basis of the leading ideal. Afterwards one continues with the next higher respectively lower degree without checking the lower respectively higher degrees for obstructions. Obviously, this method would reduce the number of obstruction-checks since every degree is only considered once.
The following two examples show that this procedure is not working in general since it can happen that after removing all obstructions in one degree another one in a lower respectively higher degree can appear.

Example 2.3.1 (Starting from the bottom).
Let $\mathcal{I} = \langle x_1^3,\ x_1^2x_2 + x_2^3,\ x_1^2x_3\rangle \lhd \Bbbk[x_1, x_2, x_3]$ then

$$\operatorname{lt}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1x_2^3,\ x_2^3x_3,\ x_2^5\rangle$$

has no obstruction in degree 3, which is the lowest degree of a minimal generator. But since $x_2\frac{x_2^3x_3}{x_3} = x_2^4 \notin \operatorname{lt}\mathcal{I}$ there is one in degree 4. We can remove this obstruction by applying the transformation $\Psi : (x_3 \mapsto x_3 + x_2)$ to $\mathcal{I}$. The new leading ideal

$$\operatorname{lt}\Psi(\mathcal{I}) = \langle x_1^3,\ x_1^2x_2,\ x_2^3,\ x_1^2x_3^3\rangle$$

has no obstruction in degree 4 or 5, which is the highest degree of a minimal generator. But $\operatorname{lt}\Psi(\mathcal{I})$ is not strongly stable since $x_1\frac{x_2^3}{x_2} = x_1x_2^2 \notin \operatorname{lt}\Psi(\mathcal{I})$.

Example 2.3.2 (Starting from the top).
Let $\mathcal{I} = \langle x_3,\ x_1^2\rangle \lhd \Bbbk[x_1, x_2, x_3]$. $\mathcal{I}$ has no obstruction in degree 2, but there are some in degree 1 since $x_1\frac{x_3}{x_3} = x_1$ and $x_2\frac{x_3}{x_3} = x_2$ are not in $\operatorname{lt}\mathcal{I}$. If we apply the transformation $\Psi : (x_3 \mapsto x_3 + x_1)$ to $\mathcal{I}$, the new leading ideal $\operatorname{lt}\Psi(\mathcal{I}) = \langle x_1,\ x_3^2\rangle$ has no obstruction in degree 1. But $\operatorname{lt}\Psi(\mathcal{I})$ is still not strongly stable since $x_2\frac{x_3^2}{x_3} = x_2x_3 \notin \operatorname{lt}\Psi(\mathcal{I})$.

Index-by-Index. Analogous to the above idea, the *index-by-index* method eliminates all obstructions that appear in one index and then moves on to the next higher respectively lower index. Again it is clear that this strategy leads to a reduction of obstruction-checks since every index is considered only once.
If we start with the lowest index, we will see in the following example that removing obstructions in one index can cause the occurrence of an obstruction in a lower

index. So we have to conclude that this strategy is also not working in general.

Example 2.3.3 (From low to high index).
Let $\mathcal{I} = \langle x_2x_3\rangle \lhd \Bbbk[x_1,x_2,x_3]$. Since $\mathcal{I}$ has only one minimal generator, this generator provides the lowest (and highest) occurring index of the monomial basis of $\mathcal{I}$. The generator x_2x_3 causes 3 obstructions since $x_1\frac{x_2x_3}{x_3} = x_1x_2$, $x_2\frac{x_2x_3}{x_3} = x_2^2$ and $x_1\frac{x_2x_3}{x_2} = x_1x_3$ are not in $\mathcal{I}$. Following the prescribed strategy, we would transform the ideal $\mathcal{I}$ either by $\Psi_1 : (x_3 \mapsto x_3 + x_1)$ or by $\Psi_2 : (x_3 \mapsto x_3 + x_2)$ so that no obstruction with index 3 is left. But both of the new leading ideals $\operatorname{lt}\Psi_1(\mathcal{I}) = \langle x_1x_2\rangle$ and $\operatorname{lt}\Psi_2(\mathcal{I}) = \langle x_2^2\rangle$ are not strongly stable. As 3 is the maximal possible index that can appear in $\Bbbk[x_1,x_2,x_3]$, we do not check further indices according to our strategy.

The next example shows that beginning with the highest index can also cause the appearance of obstructions in higher indices at least during single steps. We can not deliver a counter example for this case but Example 2.3.4 makes clear that it is nontrivial to verify whether this method provides a real optimization. To give a final answer to this question a more detailed investigation would be necessary.

Example 2.3.4.
Let

$$\mathcal{I} = \left\langle \begin{array}{l} x_1^2x_2x_3,\ x_1x_2^2x_3,\ x_1x_2x_3^2,\ x_1x_2x_3x_4 + x_1x_3^2x_4 \\ \qquad x_1^5,\ x_2^5,\ x_1^3x_3^2,\ x_1^2x_3^3,\ x_1x_3^4,\ x_3^5 \end{array} \right\rangle \lhd \Bbbk[x_1,x_2,x_3,x_4]$$

then

$$\operatorname{lt}\mathcal{I} = \left\langle \begin{array}{c} x_1^2x_2x_3,\ x_1x_2^2x_3,\ x_1x_2x_3^2,\ x_1x_2x_3x_4, \\ x_1^5,\ x_2^5,\ x_1^3x_3^2,\ x_1^2x_3^3,\ x_1x_3^4,\ x_3^5,\ x_1^2x_3^2x_4,\ x_1x_3^3x_4 \end{array} \right\rangle$$

has no obstruction with index 4. But there are many obstructions with index 3, for example $x_1\frac{x_1^2x_2x_3}{x_3} = x_1^3x_2$, $x_2\frac{x_1^2x_2x_3}{x_3} = x_1^2x_2^2$, $x_1\frac{x_1^2x_2x_3}{x_2} = x_1^3x_3 \notin \operatorname{lt}\mathcal{I}$. Applying the transformations $\Psi_1 : (x_3 \mapsto x_3 + x_1)$ and $\Psi_2 : (x_3 \mapsto x_3 + x_2)$ to $\mathcal{I}$ leads to the leading ideal

$$\operatorname{lt}\Psi_2(\Psi_1(\mathcal{I})) = \left\langle \begin{array}{l} x_1^3x_2,\ x_1^2x_2^2,\ x_1^2x_2x_3,\ x_1^3x_4,\ x_1^5,\ x_1x_2^4,\ x_2^5,\ x_1^4x_3,\ x_2^4x_3,\ x_1^3x_3^2, \\ x_1x_2^3x_3^2,\ x_2^3x_3^3,\ x_1^2x_3^4,\ x_1x_2^2x_3^4,\ x_2^2x_3^5,\ x_1x_2x_3^6,\ x_2x_3^7,\ x_1x_3^8,\ x_3^9 \end{array} \right\rangle$$

which has no obstruction with index 3. Nevertheless there is one with index 4 now, since $x_1\frac{x_1^3x_4}{x_4} = x_1^4 \notin \operatorname{lt}\Psi_2(\Psi_1(\mathcal{I}))$. However, if we go on and search for obstructions with index 2 we find $x_1\frac{x_1^3x_2}{x_2} = x_1^4 \notin \operatorname{lt}\Psi_1(\mathcal{I})$. After performing the coordinate change $\Psi_3 : (x_2 \mapsto x_2 + x_1)$, we finally arrive at the strongly stable ideal

$$\operatorname{lt}\Psi_3(\Psi_2(\Psi_1(\mathcal{I}))) = \left\langle \begin{array}{c} x_1^4,\ x_1^3x_2,\ x_1^3x_3,\ x_1^3x_4, \\ x_1^2x_2^3,\ x_1x_2^4,\ x_2^5,\ x_1^2x_2^2x_3,\ x_1x_2^3x_3,\ x_1^2x_2x_3^2,\ x_1^2x_2^2x_4,\ x_1^2x_2x_3x_4, \\ x_2^4x_3^2,\ x_1x_2^2x_3^3,\ x_1^2x_3^4,\ x_2^3x_3^4,\ x_1x_2x_3^5,\ x_2^2x_3^6,\ x_1x_3^7,\ x_2x_3^8,\ x_3^9 \end{array} \right\rangle$$

and so particularly there are no obstructions with index 4 left.

Reduce Number of Transformations. In order to investigate a strategy that reduces the number of transformations, we first state that we always choose our coordinate changes such that j is minimal. Looking for example at the simple ideal $\mathcal{I} = \langle x_3 \rangle$ one immediately realizes that this postulation does make sense since the transformation $\Psi : (x_3 \mapsto x_3 + x_1)$ directly puts $\mathcal{I}$ in to strongly stable position while $\Psi : (x_3 \mapsto x_3 + x_2)$ does not.
Under this assumption the question is, whether it is better (in the sense of reducing the number of transformations) to start removing the obstructions caused by generators with the highest or lowest index. Again we can provide examples to illustrate that this question can not be answered in general.

Example 2.3.5.
Let $\mathcal{I} = \langle x_3^2,\ x_4x_5 \rangle + \langle x_1, x_2, x_3, x_4, x_5 \rangle^3 \lhd \Bbbk[x_1, x_2, x_3, x_4, x_5]$. If we begin to delete the obstructions induced by the minimal generator x_3^2, we would first transform $\mathcal{I}$ by $\Psi_1 : (x_3 \mapsto x_3 + x_1)$. Afterwards we could remove the obstructions caused by x_4x_5 with the transformations $\Psi_2 : (x_5 \mapsto x_5 + x_1)$ and $\Psi_3 : (x_4 \mapsto x_4 + x_2)$. After this three transformations we arrive at the strongly stable ideal

$$\operatorname{lt} \Psi_3(\Psi_2(\Psi_1(\mathcal{I}))) = \langle x_1^2,\ x_1x_2 \rangle + \langle x_1, x_2, x_3, x_4, x_5 \rangle^3.$$

While starting with the generator of highest index, namely x_4x_5, would force us to do four transformations. At first we have to apply $\tilde{\Psi}_1 : (x_5 \mapsto x_5 + x_1)$ and $\tilde{\Psi}_2 : (x_4 \mapsto x_4 + x_2)$. Then, considering now the generator x_3^2, we have to perform $\tilde{\Psi}_3 : (x_3 \mapsto x_3 + x_1)$ and $\tilde{\Psi}_4 : (x_3 \mapsto x_3 + x_2)$ until we have:

$$\operatorname{lt} \tilde{\Psi}_4(\tilde{\Psi}_3(\tilde{\Psi}_2(\tilde{\Psi}_1(\mathcal{I})))) = \operatorname{lt} \Psi_3(\Psi_2(\Psi_1(\mathcal{I})))$$

Example 2.3.6.
Let $\mathcal{I} = \langle x_1x_3, x_4^2 \rangle + \langle x_1, x_2, x_3, x_4 \rangle^3 \lhd \Bbbk[x_1, x_2, x_3, x_4]$. As above we first consider the generator with lowest index which is x_1x_3. To remove its induced obstructions we perform $\Psi_1 : (x_3 \mapsto x_3 + x_1)$ on $\mathcal{I}$. Now we eliminate the obstructions that are produced by the generator x_4^2 with the coordinate changes $\Psi_2 : (x_4 \mapsto x_4 + x_1)$ and $\Psi_3 : (x_4 \mapsto x_4 + x_2)$ so that we finally arrive at the strongly stable ideal

$$\operatorname{lt} \Psi_3(\Psi_2(\Psi_1(\mathcal{I}))) = \langle x_1^2,\ x_1x_2 \rangle + \langle x_1, x_2, x_3, x_4 \rangle^3.$$

Compared to the previous example it is this time better to begin with the generator of highest index. Because if we first transform $\mathcal{I}$ by $\tilde{\Psi}_1 : (x_4 \mapsto x_4 + x_1)$ and then – in order to manage the obstructions induced by x_1x_3 – perform the coordinate change $\tilde{\Psi}_2 : (x_3 \mapsto x_3 + x_2)$, we already reach the leading ideal

$$\operatorname{lt} \tilde{\Psi}_2(\tilde{\Psi}_1(\mathcal{I})) = \operatorname{lt} \Psi_3(\Psi_2(\Psi_1(\mathcal{I})))$$

with only two transformations.

Conclusion and Alternatives. In summary we must notice that in the majority of the considered cases we could provide a counter example so that non of the discussed strategies does seem to work in general. The only way to get an answer to the question which one of these methods is (at least mostly) better is to do a larger experimental research.

Further strategies that would be interesting to analyze are on the one hand performing all transformation that are induced by obstruction at once and on the other hand optimizing the Gröbner basis computation in line 4 by considering that we started with a reduced Gröbner basis and performed only a sparse transformation. Unfortunately, the investigation of these ideas would go beyond the scope of this dissertation.

2.4. Borel-fixed and positive Characteristic

For the last section of this chapter we want to draw our attention to the case of positive characteristic. In this context another position, which is called *Borel-fixed*, gains more importance and will be introduced in the following. Finally, we discuss how to modify Algorithm 1 so that we can also use it if $\operatorname{char} \Bbbk > 0$.

Definition 2.4.1.
The subgroup $\mathfrak{B} \subseteq \mathrm{Gl}(n, \Bbbk)$ consisting of all lower triangular invertible $n \times n$ matrices is called *Borel group.*
A matrix $A = (a_{ij}) \in \mathfrak{B}$ is called *lower elementary matrix*, if $a_{ii} = 1$ for all i and if there exists integers $1 \leq l < k \leq n$ such that $a_{kl} \neq 0$ while $a_{ij} = 0$ for all $i \neq j$ with $\{i, j\} \neq \{k, l\}$. Recall from linear algebra that the subgroup of all nonsingular diagonal matrices together with the set of all lower elementary matrices generates $\mathfrak{B}$.

Remark 2.4.2.
Every coordinate transformation of the form $\Psi : (x_j \mapsto x_j + ax_i)$ with $i < j$ corresponds to a transformation by a lower elementary matrix (see Notations 1.3.3).

Definition 2.4.3.
A monomial ideal $\mathcal{J}$ is *Borel-fixed* if $\mathcal{J} = A \cdot \mathcal{J}$ for all $A \in \mathfrak{B}$. We say that a polynomial ideal $\mathcal{I}$ is in *Borel-fixed position* if $\operatorname{lt} \mathcal{I}$ is Borel-fixed.

Theorem 2.4.4 ([**HH11**, Thm. 4.2.1, Prop. 4.2.4, Prop. 4.2.6, Cor. 4.2.7, Thm. 4.2.10]).
The following assertions hold for arbitrary characteristic:

- *(i)* $\operatorname{gin} \mathcal{I}$ *is Borel-fixed.*
- *(ii) If $\mathcal{I}$ is in Borel-fixed position, then $\mathcal{I}$ is in quasi-stable position.*
- *(iii) If $\mathcal{I}$ is in strongly stable position, then $\mathcal{I}$ is in Borel-fixed position.*
- *(iv)* $\operatorname{gin} \mathcal{I} = \mathcal{I}$, *if and only if $\mathcal{I}$ is monomial and Borel-fixed.*
- *(v)* $\operatorname{gin}(\operatorname{gin} \mathcal{I}) = \operatorname{gin} \mathcal{I}$.

PROPOSITION 2.4.5 ([**HH11**, Prop. 4.2.6]).
If $\operatorname{char} \Bbbk = 0$ *then an ideal is strongly stable, if and only if it is Borel-fixed.*

The previous proposition makes clear that in the characteristic zero case we can use Algorithm 1 to put an ideal into Borel-fixed position. To analyze the positive characteristic case we need an alternative definition of Borel-fixed which is presented in the next proposition. The form of this characterization is similar to our definition of strongly stable and is therefore essential in order to adapt Algorithm 1 to the case of positive characteristic.

PROPOSITION 2.4.6 ([**Eis95**, Thm. 15.23]).
Let $\operatorname{char} \Bbbk = p > 0$*,* $\mathcal{J}$ *a monomial ideal and* B *its monomial basis then the following statements are equivalent:*

(i) $\mathcal{J}$ *is Borel-fixed.*
(ii) For every $\mathbf{x}^\mu \in B$ *and all indices* i, j *with* $\mu_j > 0$ *and* $i < j$ *the term* $x_i^{\mu_j - u} \frac{\mathbf{x}^\mu}{x_j^{\mu_j - u}}$ *also lies in* $\mathcal{J}$ *for all integers* u *that satisfy* $\binom{\mu_j}{u} \not\equiv 0 \mod p$*.*

EXAMPLE 2.4.7.
The ideal $\mathcal{J} = \langle x_1^p,\ x_2^p \rangle \lhd \Bbbk[x_1, x_2]$ with $\operatorname{char} \Bbbk = p > 0$ is not stable since $x_1 \frac{x_2^p}{x_2} = x_1 x_2^{p-1} \notin \mathcal{J}$ and so in particular not strongly stable. But it is Borel-fixed because

$$\binom{p}{u} \equiv 0 \mod p, \text{ for all integer } u \text{ with } u \notin \{0, p\} \tag{2.18}$$

and so we only have to verify that the two terms $x_1^p \frac{x_2^p}{x_2^p} = x_1^p,\ x_1^0 \frac{x_2^p}{x_2^0} = x_2^p$ lie in $\mathcal{J}$. By the definition of Borel-fixed and Remark 2.4.2 we know that $\mathcal{J}$ does not change under any coordinate transformation of the from $(x_j \mapsto x_j + a x_i)$ with $i < j$. Hence we can not transform $\mathcal{J}$ into strongly stable or stable position. In particular, if $\Psi : (x_2 \mapsto x_2 + a x_1)$, then:

$$\Psi(\mathcal{J}) = \langle x_1^p,\ (x_2 + a x_1)^p \rangle = \langle x_1^p,\ \sum_{u=0}^{p} \binom{p}{u} x_2^{p-u} (a x_1)^u \rangle \overset{(2.18)}{=} \langle x_1^p,\ x_2^p + (a x_1)^p \rangle = \mathcal{J}$$

So this example clearly shows that we can not use Algorithm 1 to transform every ideal into strongly stable or stable position if $\operatorname{char} \Bbbk > 0$. However, we will now present two corollaries to Proposition 2.2.16 and with their help we are able to formulate the corresponding analogons to Theorem 2.2.19 for Borel-fixed and quasi-stable position.

Corollary 2.4.8.
Let $\operatorname{char}\Bbbk = p > 0$ *and* $\mathcal{I} \lhd \mathcal{P}$ *be an ideal. Further, let* G *be the reduced Gröbner basis of* $\mathcal{I}$ *and* $g \in G$ *with* $\operatorname{lt} g = \mathbf{x}^\mu$. *Assume there are indices* i, j *with* $i < j$ *and* $\mu_j > 0$ *so that*

$$x_i^{\mu_j} \frac{\mathbf{x}^\mu}{x_j^{\mu_j}} \notin \operatorname{lt}\mathcal{I}. \tag{2.19}$$

Further, let $\Psi : (x_j \mapsto x_j + a x_i)$ *be a coordinate transformation and* $\tilde{G}$ *the reduced Gröbner basis of* $\Psi(\mathcal{I})$. *Then for a generic choice of* $a \in \Bbbk$, *we have*

$$\mathscr{L}(G) \prec_{\mathscr{L}} \mathscr{L}(\tilde{G}).$$

Proof. This assertion is a simple consequence of Proposition 2.2.16 and the fact that:

$$\binom{\mu_j}{0} = 1 \not\equiv 0 \mod p, \quad \text{for all } p \geq 0$$

$\square$

Corollary 2.4.9.
Let $\mathcal{I} \lhd \mathcal{P}$ *be an ideal and* $\mathcal{L}_1, \dots, \mathcal{L}_\ell \lhd \mathcal{P}$ *the monomial ideals according to Remark 2.1.2 received from the Gröbner system of the ideal* $A \cdot \mathcal{I} \lhd \hat{\mathcal{P}} = \Bbbk[\mathbf{a}, \mathbf{x}]$ *where* A *is a* $n \times n$ *parametric matrix. Further, let* B_i *be the monomial basis of* $\mathcal{L}_i$. *If* m *is the index such that* $\mathscr{L}(B_m)$ *is maximal with respect to* $\prec_{\mathscr{L}}$, *then* $\mathcal{L}_m$ *is Borel-fixed.*

Proof. Let $A_m \in \mathrm{Gl}(n, \Bbbk)$ be a matrix such that for the reduced Gröbner basis G_m of $A_m \cdot \mathcal{I}$ holds $\operatorname{lt} G_m = B_m$.
Now let us assume that $\mathcal{L}_m$ is not Borel-fixed. By Proposition 2.4.6 there must be an obstruction of the form (2.15) and so it follows from Proposition 2.2.16 that there exists a coordinate transformation $\Psi : (x_j \mapsto x_j + a x_i)$ such that $\mathscr{L}(G_m) \prec_{\mathscr{L}} \mathscr{L}(\tilde{G}_m)$, where $\tilde{G}_m$ is the reduced Gröbner basis of $\Psi(A_m \cdot \mathcal{I})$. But again by Remark 2.1.2 there must be an index l with $\mathcal{L}_l = \langle \operatorname{lt} \tilde{G}_m \rangle$. Hence

$$\mathscr{L}(B_m) = \mathscr{L}(G_m) \prec_{\mathscr{L}} \mathscr{L}(\tilde{G}_m) = \mathscr{L}(B_l),$$

which is a contradiction to the choice of the index m. $\square$

Summarizing the above results we can now present a modified version of Algorithm 1 that transforms a given ideal into Borel-fixed position if $\operatorname{char}\Bbbk = p > 0$.

Algorithm 2 BF-Trafo: Transformation to Borel-fixed position in $\operatorname{char} \Bbbk = p > 0$

Input: Reduced Gröbner basis G of homogeneous ideal $\mathcal{I} \lhd \mathcal{P}$
Output: a linear change of coordinates Ψ such that $\operatorname{lt} \Psi(\mathcal{I})$ is Borel-fixed

1: $\Psi := \mathrm{id}$;
2: **while** $\exists\, g \in G,\ 1 \leq j \leq n,\ 1 \leq i < j,\ 1 \leq u \leq \mu_j$:
$$x_j |\operatorname{lt} g = \mathbf{x}^\mu \wedge \binom{\mu_j}{u} \not\equiv 0 \mod p \wedge x_i \frac{\operatorname{lt} g}{x_j} \notin \langle \operatorname{lt} G \rangle \textbf{ do}$$
3: $\quad \psi := (x_j \mapsto x_j + a x_i)$ with $a \in \Bbbk$ randomly chosen; $\Psi = \psi \circ \Psi$;
4: $\quad \tilde{G} := \textsc{ReducedGröbnerBasis}\big(\psi(G)\big)$;
5: $\quad$ **while** $\mathscr{L}(G) \succeq_{\mathscr{L}} \mathscr{L}(\tilde{G})$ **do**
6: $\quad\quad \psi := (x_j \mapsto x_j + a x_i)$with $a \in \Bbbk$ randomly chosen; $\Psi = \psi \circ \Psi$;
7: $\quad\quad \tilde{G} := \textsc{ReducedGröbnerBasis}\big(\psi(\tilde{G})\big)$;
8: $\quad$ **end while**
9: $\quad G := \tilde{G}$;
10: **end while**
11: **return** Ψ

Theorem 2.4.10.
If $\operatorname{char} \Bbbk = p > 0$ *Algorithm 2 terminates in finitely many steps and returns a coordinate transformation* Ψ *so that* $\Psi(\mathcal{I})$ *is in Borel-fixed position.*

Proof. Compared to the proof of Theorem 2.2.19, we only have to change the way we look for a generic value for the coefficient $a \in \Bbbk$ of the coordinate transformation. Since $\Bbbk$ is infinite we know that a generic value for a does exist. But – as $\operatorname{char} \Bbbk > 0$ – there is no canonical way to reach all elements of $\Bbbk$ by simply adding 1 to the current coefficient[11]. So in this case, we use the strategy of random choice (see Line 3 and 6) to achieve a generic value for a. □

Theorem 2.4.11.
Let $\operatorname{char} \Bbbk = p > 0$. *Replacing line 2 of Algorithm 2 by*

$$\textbf{while } \exists\, g \in G,\ 1 \leq i < j = \mathrm{m}(g) \,:\, x_i^{\deg G} \frac{\operatorname{lt} g}{x_j^{\mu_j}} \notin \langle \operatorname{lt} G \rangle \textit{ with } \operatorname{lt} g = \mathbf{x}^\mu \textbf{ do}$$

leads to an algorithm that terminates in finitely many steps and returns a coordinate transformation Ψ *so that* $\Psi(\mathcal{I})$ *is in quasi-stable position.*

Proof. Mainly, we can use the proofs of the Theorems 2.2.19 and 2.4.10. Thereby we have to note that whenever Proposition 2.2.16 occurs in these proofs we have to replace it by Proposition 2.4.8. □

[11] This is basically what we did in the proof of Theorem 2.2.19 by transforming iteratively with $(x_j \mapsto x_j + x_i)$.

CHAPTER 3

Pommaret Basis

Theorem 1.3.9 motivated us to take a closer look at the generic initial ideal $\operatorname{gin}\mathcal{I}$. Hence we were particularly interested in methods to compute $\operatorname{gin}\mathcal{I}$ and saw that one possible approach is provided by the concept of Gröbner systems which we introduced in Section 2.1 of the preceding chapter. Because of the parametric computations the determination of $\operatorname{gin}\mathcal{I}$ is obviously very expensive and so it is worth to search for possible alternatives. One can be found by investigating a special generating set of quasi-stable ideals which is called *Pommaret basis.* With the help of this basis we are able to present an alternative version of Theorem 1.3.9 that does not require the computation of $\operatorname{gin}\mathcal{I}$.

Afterwards we provide a short overview of some applications of Pommaret basis in the areas of free resolutions, componentwise linearity, linear quotients and local cohomology.

3.1. The Main Feature of Quasi-Stability

We now give a brief introduction to the theory of Pommaret basis. Thereby we especially examine its connection to quasi-stable position.

DEFINITION 3.1.1.
Let $F \subseteq \mathcal{P}$ be a set of polynomials. We define the *Pommaret span* of F as the $\Bbbk$-linear space

$$\langle F\rangle_{\mathscr{P}} = \bigoplus_{f\in F} \Bbbk[x_{\mathrm{m}(f)}, \dots, x_n] \cdot f.$$

Further, for a polynomial $f \in F$, we call the set $\Bbbk[x_{\mathrm{m}(f)}, \dots, x_n] \cdot f$ the *Pommaret cone* of f.

DEFINITION 3.1.2.

- Let $H \subseteq \mathcal{P}$ be a set of terms. H is a *Pommaret basis* of the monomial ideal $\mathcal{J} = \langle H\rangle$ if $\langle H\rangle_{\mathscr{P}} = \mathcal{J}$.
- Let $H \subseteq \mathcal{P}$ be a set of polynomials. H is a *Pommaret basis* of the polynomial ideal $\mathcal{I} = \langle H\rangle$ if all elements of H possess distinct leading terms and $\operatorname{lt} H$ forms a Pommaret basis of $\operatorname{lt}\mathcal{I}$.

Pommaret bases are a special form of *involutive bases.* A general survey can be found in [**Sei09a**]. The algebraic theory of Pommaret bases was developed in [**Sei09b**] (see also [**Sei10**, Chpts. 3-5]).

In the following we will see a criterion for the existence of a finite Pommaret basis[1]. Further, we present a proposition concerning its uniqueness and describe how a finite Pommaret basis can be determined.

Theorem 3.1.3 ([**Sei09b**, Def. 4.3, Prop. 4.4]).
An ideal $\mathcal{I} \lhd \mathcal{P}$ possesses a finite Pommaret basis, if and only if $\operatorname{lt} \mathcal{I}$ is quasi-stable.

Definition 3.1.4.
We say that $\mathbf{x}^\mu$ is a *Pommaret divisor* of another term $\mathbf{x}^\nu$ if $\mathbf{x}^\mu \mid \mathbf{x}^\nu$ and $\mathbf{x}^{\nu-\mu} \in \Bbbk[x_{\mathrm{m}(\mathbf{x}^\mu)}, \dots, x_n]$. In this case we write $\mathbf{x}^\mu|_{\mathscr{P}}\mathbf{x}^\nu$.

A set $F \subseteq \mathcal{P}$ is *Pommaret autoreduced* if no polynomial $f \in F$ contains a term $\mathbf{x}^\mu \in \operatorname{supp}(f)$ such that another polynomial $f' \in F \setminus \{f\}$ exits with $\operatorname{lt} f'|_{\mathscr{P}}\mathbf{x}^\mu$.

Remark 3.1.5.
Let $\mathcal{I}$ be an ideal in quasi-stable position and H its Pommaret basis. Then any term of $\operatorname{lt} \mathcal{I}$ has a unique Pommaret divisor in $\operatorname{lt} H$. In particular, any Pommaret basis is a Gröbner basis and any two distinct elements of H have disjoint Pommaret cones.

Proposition 3.1.6 ([**Sei09a**, Prop. 2.11, Prop. 5.16]).
The Pommaret basis of a monomial ideal is unique. In particular, different Pommaret basis of a polynomial ideal have the same number of elements.

Further, the Pommaret basis of a polynomial ideal is unique if it is monic and Pommaret autoreduced.

Proposition 3.1.7.
Let $H \subseteq \mathcal{P}$ be a set of terms. H is a Pommaret basis of the monomial ideal $\mathcal{J} = \langle H \rangle$ if $x_j h \in \langle H \rangle_{\mathscr{P}}$ for all $h \in H$ and all $j < \mathrm{m}(h)$.

Remark 3.1.8.
The criterion described in Proposition 3.1.7 is called *local involution* (see [**Sei09a**, Def. 6.1, Prop. 6.3, Lem. 6.4] for more details).

We know by Remark 2.2.17 that a modified version of Algorithm 1 let us put any ideal in quasi-stable position. Once we are in this position, we can determine a Pommaret basis by using Proposition 3.1.7 in the following way:

Let G be the reduced Gröbner basis of $\mathcal{I}$. Then we first verify whether $\operatorname{lt} G$ is already a Pommaret basis of $\operatorname{lt} \mathcal{I}$ by checking if $x_j \operatorname{lt} g \in \langle \operatorname{lt} G \rangle_{\mathscr{P}}$ for all $g \in G$ and $j < \mathrm{m}(g)$. If this is the case then $H = G$ is a Pommaret basis of $\mathcal{I}$. Otherwise, there is a polynomial $g \in G$ and an index $j < \mathrm{m}(g)$ with $x_j \operatorname{lt} g \notin \langle \operatorname{lt} G \rangle_{\mathscr{P}}$. We then set $H = G \cup \{x_j g\}$ and repeat this until $x_j \operatorname{lt} h \in \langle \operatorname{lt} H \rangle_{\mathscr{P}}$ for all $h \in H$ and $j < \mathrm{m}(h)$ (see [**Sei10**, Alg. 4.5] for more details).

[1] Whenever we talk about the existence of a Pommaret basis, we always mean the existence of a finite Pommaret basis.

EXAMPLE 3.1.9.
Let $\mathcal{J} = \langle x_1x_2\rangle \lhd \Bbbk[x_1,x_2]$ then $\mathcal{J}$ is not quasi-stable since $x_1\frac{x_1x_2}{x_2} = x_1^2 \notin \mathcal{J}$. To determine the Pommaret basis of $\mathcal{J}$, we follow the above description and set $H_1 = \{x_1x_2\} \cup \{x_1x_2^2\}$ since $x_1x_2^2 \notin \langle x_1x_2\rangle_{\mathscr{P}}$. As $x_1x_2^3 \notin \langle H_1\rangle_{\mathscr{P}}$ we define $H_2 = H_1 \cup \{x_1x_2^3\}$. Going on like this leads to the infinite Pommaret basis $H = \{x_1^kx_2 \mid k \geq 0\}$ of $\mathcal{J}$ with:

$$\langle H\rangle_{\mathscr{P}} = \bigoplus_{k\geq 0} \Bbbk[x_2]\cdot x_1^kx_2 = \mathcal{J}$$

Analogous to Algorithm 1 we apply the transformation $\Psi : (x_2 \mapsto x_2 + x_1)$ to $\mathcal{J}$ in order to make $\mathcal{J}$ quasi-stable. This leads to the quasi-stable ideal $\Psi(\mathcal{J}) = \langle x_1^2 + x_1x_2\rangle$ and because of

$$\langle \operatorname{lt}\{x_1^2 + x_1x_2\}\rangle_{\mathscr{P}} = \Bbbk[x_1,x_2]\cdot x_1^2 = \langle x_1^2\rangle = \operatorname{lt}\Psi(\mathcal{J}),$$

the Pommaret basis of $\Psi(\mathcal{J})$ is the finite set $\{x_1^2 + x_1x_2\}$.

THEOREM 3.1.10 ([**Sei09b**, Prop. 2.20, Thm. 8.11, Prop. 9.2]).
Let H be a finite Pommaret basis of $\mathcal{I}$. Then:

- $\operatorname{reg}(\mathcal{I}) = \operatorname{reg}(\operatorname{lt}\mathcal{I}) = \deg H$
- $\operatorname{pd}(\mathcal{I}) = \operatorname{pd}(\operatorname{lt}\mathcal{I}) = \operatorname{m}(H) - 1$
- $\operatorname{depth}(\mathcal{I}) = \operatorname{depth}(\operatorname{lt}\mathcal{I}) = n - \operatorname{m}(H) + 1$

The statements of this theorem are similar to the ones of Theorem 1.3.9, but instead off computing $\operatorname{gin}\mathcal{I}$ we only need to determine a finite Pommaret basis of $\mathcal{I}$. So we can avoid the costly parametric computations without loosing the possibility to directly read off the considered invariants from a generating set of $\operatorname{lt}\mathcal{I}$.

EXAMPLE 3.1.11.
Let $\mathcal{J} = \langle x_1x_2^2,\ x_1^4\rangle \lhd \Bbbk[x_1,x_2]$. $\mathcal{J}$ is quasi-stable, but $\{x_1x_2^2,\ x_1^4\}$ is not a Pommaret basis of $\mathcal{J}$ since for example $x_1\cdot x_1x_2^2 \notin \langle x_1x_2^2,\ x_1^4\rangle_{\mathscr{P}}$. However, the set $H = \{x_1x_2^2,\ x_1^2x_2^2,\ x_1^3x_2^2,\ x_1^4\}$ is a Pommaret basis of $\mathcal{J}$ and therefore we can immediately derive from the theorem above:

$$\operatorname{reg}(\mathcal{J}) = 5, \quad \operatorname{pd}(\mathcal{J}) = 2-2+1 = 1, \quad \operatorname{depth}(\mathcal{J}) = 2-1 = 1$$

Indeed, the minimal free resolution of $\mathcal{J}$ and its Betti diagram is:

$$0 \longrightarrow \mathcal{P}(-6) \longrightarrow \mathcal{P}(-3)\oplus\mathcal{P}(-4) \longrightarrow \mathcal{J} \longrightarrow 0\,,$$

	0	1
3	$\beta_{0,3}^{\mathcal{J}} = 1$	$\beta_{1,4}^{\mathcal{J}} = 0$
4	$\beta_{0,4}^{\mathcal{J}} = 1$	$\beta_{1,5}^{\mathcal{J}} = 0$
5	$\beta_{0,5}^{\mathcal{J}} = 0$	$\beta_{1,6}^{\mathcal{J}} = 1$

3.2. Further Properties

After presenting some well-known basic results, we will define the *Pommaret resolution*, which describes a free resolution of quasi-stable ideals. Further, we consider a necessary condition for the minimality of this resolution that called *componentwise linear*. This property induces a new kind of stable position, the *componentwise quasi-stable* position. Also for this position we are able to provide a corresponding transformation algorithm analogous to Algorithm 1. Finally, we introduce the *$\mathscr{P}$-graph* and the *$\mathscr{P}$-ordering* in the context of *linear quotients*, before we end this section with a short excursion to local cohomology.

Basics.

LEMMA 3.2.1.
Any zero-dimensional ideal possesses a finite Pommaret basis.

PROOF. It is enough to show that any zero-dimensional ideal is in quasi-stable position because then the assertion follows from Theorem 3.1.3.

Let $\mathcal{I}$ be a zero-dimensional ideal. It is a well-known fact that in this case $\operatorname{lt}\mathcal{I}$ contains a pure power of any variable (see e.g. [**KR00**, Prop. 3.7.1, Def. 3.7.2]). Assume that $\operatorname{lt}\mathcal{I}$ is not quasi-stable. Hence there is a monomial $\mathbf{x}^\mu \in \operatorname{lt}\mathcal{I}$ and an index $i < k = \mathrm{m}(\mathbf{x}^\mu)$ such that $t = x_i^{\deg B}\frac{\mathbf{x}^\mu}{x_k^{\mu_k}} \notin \operatorname{lt}\mathcal{I}$, where B denotes the monomial basis of $\operatorname{lt}\mathcal{I}$. But this is not possible since $\operatorname{lt}\mathcal{I}$ contains a pure power of x_i and so there is an integer $e \leq \deg B$ such that $x_i^e \in B$, which is a divisor of the term t. $\square$

PROPOSITION 3.2.2 ([**Sei09a**, Thm. 5.4]).
The finite set $H \subseteq \mathcal{I}$ is a Pommaret basis of the ideal $\mathcal{I} \lhd \mathcal{P}$, if and only if every polynomial $0 \neq f \in \mathcal{I}$ possesses a unique involutive standard representation

$$f = \sum_{h \in H} P_h h,$$

where each nonzero coefficient $P_h \in \Bbbk[x_{\mathrm{m}(h)}, \dots, x_n]$ satisfies $\operatorname{lt}(P_h h) \preceq \operatorname{lt} f$.

LEMMA 3.2.3 ([**Mal98**, Lemma 2.13] ,[**Sei10**, Prop. 5.5.6]).
A monomial ideal is stable, if and only if its minimal basis is a Pommaret basis.

LEMMA 3.2.4.
Let $\mathcal{I}$ be in quasi-stable position and H its Pommaret basis. If we denote by $\beta_0^{(k)}$ the number of generators $h \in H$ with $\mathrm{m}(h) = k$, then $\beta_0^{(k)} > 0$ for all $k \leq \mathrm{m}(H)$.

PROOF. It follows from the definition of $\mathrm{m}(H)$ that $\beta_0^{(\mathrm{m}(H))} > 0$. Assume now that there is an integer $k < \mathrm{m}(H)$ such that $\beta_0^{(k)} = 0$. Let $h \in H$ with $\mathrm{m}(h) = \mathrm{m}(H)$ and $\operatorname{lt} h = \mathbf{x}^\mu$. Then

$$t = x_k^{(\mathrm{m}(H)-k)\deg H} \frac{\mathbf{x}^\mu}{x_{k+1}^{\mu_{k+1}} \cdots x_{\mathrm{m}(H)}^{\mu_{\mathrm{m}(H)}}} \in \operatorname{lt}\mathcal{I}$$

since $\operatorname{lt}\mathcal{I}$ is quasi-stable. Hence there must be an element $h' \in H \setminus \{h\}$ such that $t \in \Bbbk[x_{\mathrm{m}(h')}, \dots, x_n] \cdot \operatorname{lt} h'$. As a consequence of $\mathrm{m}(t) = k$ and our assumption, we must have $\mathrm{m}(h') < k$. But this means that $\operatorname{lt} h'$ is a Pommaret divisor of $\frac{\mathbf{x}^\mu}{x_{k+1}^{\mu_{k+1}} \cdots x_{\mathrm{m}(H)}^{\mu_{\mathrm{m}(H)}}}$ and therefore a Pommaret divisor of $\mathbf{x}^\mu = \operatorname{lt} h$, which leads by Remark 3.1.5 to a contradiction since h and h' are distinct elements of the Pommaret basis H. ☐

Free Resolution. Eliahou and Kervaire provided in [**EK90**] an explicit representation of the minimal free resolution of monomial ideals in stable position. Analogously, we will now define the *Pommaret resolution* for polynomial ideals in quasi-stable position in the following theorem.

Theorem 3.2.5 ([**Sei09b**, Thm. 6.1]).
Let H be a Pommaret basis of the ideal $\mathcal{I} \subseteq \mathcal{P}$. If we denote by $\beta_0^{(k)}$ the number of generators $h \in H$ with $\mathrm{m}(h) = k$ and set $d = n - \mathrm{m}(H) + 1$, then $\mathcal{I}$ possesses a finite free resolution

$$0 \longrightarrow \mathcal{P}^{r_{n-d}} \longrightarrow \cdots \longrightarrow \mathcal{P}^{r_1} \longrightarrow \mathcal{P}^{r_0} \longrightarrow \mathcal{I} \longrightarrow 0 \tag{3.1}$$

of length $n - d$ where the ranks of the free modules are given by

$$r_i = \sum_{k=d}^{n-i} \binom{n-k}{i} \beta_0^{(n-k+1)} . \tag{3.2}$$

We call the resolution presented in (3.1) Pommaret resolution*.*

Note that a Pommaret resolution is not minimal in general, which the following example shows.

Example 3.2.6.
Let $\mathcal{J} = \langle x_1^2,\ x_2^2 \rangle \lhd \Bbbk[x_1, x_2]$ then $H = \{x_1^2,\ x_1x_2^2,\ x_2^2\}$ is a Pommaret basis of $\mathcal{J}$. With the notations of Theorem 3.2.5 we have $d = 2 - 2 + 1 = 1$, $\beta_0^{(1)} = 1$ and $\beta_0^{(2)} = 2$. Hence we have

$$\begin{aligned} r_0 &= \binom{2-1}{0}\beta_0^{(2-1+1)} + \binom{2-2}{0}\beta_0^{(2-2+1)} = 3 \\ r_1 &= \binom{2-1}{1}\beta_0^{(2-1+1)} = 2 \end{aligned}$$

and therefore $\mathcal{J}$ possesses a free resolution of the form:

$$0 \longrightarrow \mathcal{P}^2 \longrightarrow \mathcal{P}^3 \longrightarrow \mathcal{J} \longrightarrow 0$$

However, the minimal free resolution of $\mathcal{J}$ is:

$$0 \longrightarrow \mathcal{P}^1 \longrightarrow \mathcal{P}^2 \longrightarrow \mathcal{J} \longrightarrow 0$$

NOTATION 3.2.7.
Let $H = \{h_1, \dots, h_\ell\}$ be a Pommaret basis of $\mathcal{I}$. Then $x_j h_\alpha \in \mathcal{I}$ for all $j < \mathrm{m}(h_\alpha)$ and so, by Proposition 3.2.2, there exists a unique involutive standard representation $x_j h_\alpha = \sum_{\beta=1}^{\ell} P_\beta^{(\alpha,j)} h_\beta$ with $P_\beta^{(\alpha,j)} \in \Bbbk[x_{\mathrm{m}(h_\beta)}, \dots, x_n]$. We define the *syzygy*[2] $S_{(\alpha,j)}$ by

$$S_{(\alpha,j)} = x_j e_\alpha - \sum_{\beta=1}^{\ell} P_\beta^{(\alpha,j)} e_\beta \in \mathcal{P}^\ell, \tag{3.3}$$

where $e_1, \dots, e_\ell$ denotes the standard basis of $\mathcal{P}^\ell$. Further, we call $S_{(\alpha,j)}$ *free of constant terms* if $P_\beta^{(\alpha,j)} \notin \Bbbk$ for all β.

LEMMA 3.2.8 ([**Sei09b**, Lem. 8.1]).
The resolution (3.1) *is minimal, if and only if all syzygies* $S_{(\alpha,j)}$ *are free of constant terms.*

EXAMPLE 3.2.9.
Recalling Example 3.2.6, the Pommaret resolution of the ideal $\mathcal{J} = \langle x_1^2,\ x_2^2 \rangle$ is not minimal. If $H = \{h_1, h_2, h_3\}$ denotes the Pommaret basis of $\mathcal{J}$ with $h_1 = x_1^2$, $h_2 = x_1 x_2^2$, $h_3 = x_2^2$, then we have the involutive standard representation $x_1 h_3 = x_1 x_2^2 = h_2$ which entails that:

$$S_{(3,1)} = x_1 e_3 - (0 \cdot e_1 + 1 \cdot e_2 + 0 \cdot e_3) = \begin{pmatrix} 0 \\ -1 \\ x_1 \end{pmatrix}$$

Therefore the syzygy $S_{(3,1)}$ is obviously not free of constant terms.

THEOREM 3.2.10 ([**Sei09b**, Thm. 8.9]).
Let $\mathcal{J} \lhd \mathcal{P}$ *be a monomial, quasi-stable ideal. Then the resolution* (3.1) *is minimal, if and only if* $\mathcal{J}$ *is stable.*

EXAMPLE 3.2.11.
Let $\mathcal{I} = \langle h_1, \dots, h_5 \rangle \lhd \Bbbk[x_1, x_2, x_3]$ with $h_1 = x_1^2 + x_2 x_3$, $h_2 = x_1 x_2 - x_1 x_3$, $h_3 = x_2^2 + x_1 x_3$, $h_4 = x_1 x_3^2$, $h_5 = x_2 x_3^2$. Then $H = \{h_1, \dots, h_5\}$ is a Pommaret basis of $\mathcal{I}$ and $x_1 h_2 = (-x_2 + x_3) h_1 + x_3 h_3 - h_4 - h_5$, so that the syzygy

$$S_{(2,1)} = x_1 e_2 - ((-x_2 + x_3) e_1 + x_3 e_3 - e_4 - e_5) = \begin{pmatrix} x_2 - x_3 \\ x_1 \\ -x_3 \\ 1 \\ 1 \end{pmatrix}$$

[2] To understand the analogy to Remark 1.1.5, we should note that the right hand side of (3.3) vanishes if one substitutes e_β by h_β for all β.

contains constant terms. Hence the Pommaret resolution of $\mathcal{I}$ is not minimal by Lemma 3.2.8. On the other hand it follows from Theorem 3.2.10 that the Pommaret resolution of $\operatorname{lt}\mathcal{I}$ is minimal since the leading ideal

$$\operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3^2,\ x_2x_3^2\rangle$$

is stable. Indeed, since $x_1 \operatorname{lt} h_2 = x_1^2 x_2 = x_2 \operatorname{lt} h_1$ the corresponding syzygy of the leading ideal is $x_1e_2 - x_2e_1$ and therefore free of constant terms.

Lemma 3.2.12.
Let $\mathcal{I} \lhd \mathcal{P}$ be in quasi-stable position and H its Pommaret basis. Then all syzygies of $\operatorname{lt} H$ are free of constant terms if all syzygies of H are free of constant terms.

Proof. Assume that there is a syzygies $S_{(\alpha,j)}$ of $\operatorname{lt} H$ that contains a constant term. This means that the coefficient $\hat{P}_{\hat{\beta}}^{(\alpha,j)}$ of the involutive standard representation $x_j \operatorname{lt} h_\alpha = \hat{P}_{\hat{\beta}}^{(\alpha,j)} \operatorname{lt} h_{\hat{\beta}}$ lies in $\mathbb{k}$. Considering now the involutive standard representation $x_j h_\alpha = \sum_{\beta=1}^{\ell} P_\beta^{(\alpha,j)} h_\beta$ leads to:

$$\hat{P}_{\hat{\beta}}^{(\alpha,j)} \operatorname{lt} h_{\hat{\beta}} = x_j \operatorname{lt} h_\alpha = \operatorname{lt}(x_j h_\alpha) = \operatorname{lt}\left(\sum_{\beta=1}^{\ell} P_\beta^{(\alpha,j)} h_\beta\right) = \operatorname{lt} P_{\hat{\beta}}^{(\alpha,j)} \operatorname{lt} h_{\hat{\beta}}$$

So $\operatorname{lt} P_{\hat{\beta}}^{(\alpha,j)} = \hat{P}_{\hat{\beta}}^{(\alpha,j)} \in \mathbb{k}$, which implies $P_{\hat{\beta}}^{(\alpha,j)} \in \mathbb{k}$ and thus leads to a contradiction since we know that the syzygies of H are free of constant terms. $\square$

Remark 3.2.13.
To formulate the graded version of Theorem 3.2.5, we first provide the graded analogon the (3.1):

$$0 \longrightarrow \bigoplus_j \mathcal{P}(-j)^{r_{n-d,j}} \longrightarrow \cdots \longrightarrow \bigoplus_j \mathcal{P}(-j)^{r_{1,j}} \longrightarrow \bigoplus_j \mathcal{P}(-j)^{r_{0,j}} \longrightarrow \mathcal{I} \longrightarrow 0$$

Using the arguments that lead to Theorem 3.2.5 degree by degree it is trivial to derive that the ranks of the free modules are given by

$$r_{i,j} = \sum_{k=d}^{n-i} \binom{n-k}{i} \beta_{0,j-i}^{(n-k+1)}\ , \tag{3.4}$$

where $\beta_{0,j}^{(k)}$ denotes the number of generators $h \in H$ with $\operatorname{m}(h) = k$ and $\deg h = j$.

The next corollary is a classical result [**Gre98**, Cor. 1.33] for which we provide an alternative proof.

Corollary 3.2.14 ([**HSS12**, Cor. 15]).
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal. Then all graded Betti numbers satisfy the inequality:

$$\beta_{i,j}^{\mathcal{P}/\mathcal{I}} \leq \beta_{i,j}^{\mathcal{P}/\operatorname{gin}\mathcal{I}}$$

PROOF. First we should note that the Betti numbers are invariant under coordinate transformation, so $\beta_{i,j}^{\mathcal{P}/\mathcal{I}} = \beta_{i,j}^{\mathcal{P}/A\cdot\mathcal{I}}$ for all $A \in \mathrm{Gl}(n, \Bbbk)$. We know from Remark 2.1.6 that there is a matrix $B \in \mathrm{Gl}(n, \Bbbk)$ such that $\mathrm{lt}(B \cdot \mathcal{I}) = \mathrm{gin}\,\mathcal{I}$. Hence

$$\beta_{i,j}^{\mathcal{P}/\mathcal{I}} = \beta_{i,j}^{\mathcal{P}/B\cdot\mathcal{I}} \leq \beta_{i,j}^{\mathcal{P}/\,\mathrm{lt}(B\cdot\mathcal{I})} = \beta_{i,j}^{\mathcal{P}/\,\mathrm{gin}\,\mathcal{I}},$$

where the inequality is a consequence of Proposition 1.2.2 and Remark 1.1.15. $\square$

EXAMPLE 3.2.15.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_2 + x_2^2\rangle \lhd \Bbbk[x_1, x_2]$, then its minimal free resolution is

$$0 \longrightarrow \mathcal{P}(-4) \longrightarrow \mathcal{P}(-2)^2 \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}/\mathcal{I} \longrightarrow 0\ .$$

Further, the minimal free resolution of $\mathrm{lt}\,\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^3\rangle = \mathrm{gin}\,\mathcal{I}$ is

$$0 \longrightarrow \mathcal{P}(-3) \oplus \mathcal{P}(-4) \longrightarrow \mathcal{P}(-2)^2 \oplus \mathcal{P}(-3) \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}/\,\mathrm{gin}\,\mathcal{I} \longrightarrow 0\ .$$

The corresponding Betti diagrams are of the following form:

	0	1	2
0	$\beta_{0,0}^{\mathcal{P}/\mathcal{I}} = 1$	$\beta_{1,1}^{\mathcal{P}/\mathcal{I}} = 0$	$\beta_{2,2}^{\mathcal{P}/\mathcal{I}} = 0$
1	$\beta_{0,1}^{\mathcal{P}/\mathcal{I}} = 0$	$\beta_{1,2}^{\mathcal{P}/\mathcal{I}} = 2$	$\beta_{2,3}^{\mathcal{P}/\mathcal{I}} = 0$
2	$\beta_{0,2}^{\mathcal{P}/\mathcal{I}} = 0$	$\beta_{1,3}^{\mathcal{P}/\mathcal{I}} = 0$	$\beta_{2,4}^{\mathcal{P}/\mathcal{I}} = 1$

	0	1	2
0	$\beta_{0,0}^{\mathcal{P}/\,\mathrm{gin}\,\mathcal{I}} = 1$	$\beta_{1,1}^{\mathcal{P}/\,\mathrm{gin}\,\mathcal{I}} = 0$	$\beta_{2,2}^{\mathcal{P}/\,\mathrm{gin}\,\mathcal{I}} = 0$
1	$\beta_{0,1}^{\mathcal{P}/\,\mathrm{gin}\,\mathcal{I}} = 0$	$\beta_{1,2}^{\mathcal{P}/\,\mathrm{gin}\,\mathcal{I}} = 2$	$\beta_{2,3}^{\mathcal{P}/\,\mathrm{gin}\,\mathcal{I}} = 1$
2	$\beta_{0,2}^{\mathcal{P}/\,\mathrm{gin}\,\mathcal{I}} = 0$	$\beta_{1,3}^{\mathcal{P}/\,\mathrm{gin}\,\mathcal{I}} = 1$	$\beta_{2,4}^{\mathcal{P}/\,\mathrm{gin}\,\mathcal{I}} = 1$

Hence we can conclude that although the ideal $\mathcal{I}$ is in gin-position some of the graded Betti numbers do not coincide, e.g. $\beta_{1,3}^{\mathcal{P}/\mathcal{I}} = 0 < 1 = \beta_{1,3}^{\mathcal{P}/\,\mathrm{gin}\,\mathcal{I}}$.

Componentwise Linear Ideals. Given an ideal $\mathcal{I} \lhd \mathcal{P}$, we denote by $\mathcal{I}_{\langle q\rangle} = \langle \mathcal{I}_q \rangle$ the ideal generated by the homogeneous component $\mathcal{I}_q$ of degree q. Herzog and Hibi [**HH99**] called $\mathcal{I}$ *componentwise linear* if for every degree $q \geq 0$ the ideal $\mathcal{I}_{\langle q\rangle} = \langle \mathcal{I}_q \rangle$ has a *linear resolution*[3]. For a connection with Pommaret bases, we need a refinement of quasi-stability.

DEFINITION 3.2.16.
An ideal $\mathcal{I}$ is in *componentwise quasi-stable position* if all ideals $\mathcal{I}_{\langle q\rangle}$ for $q \geq 0$ are in quasi-stable position.

[3]We have a linear resolution if there exists a value $\hat{q}$ such that the jth module of the minimal resolution is generated in degree $\hat{q} + j$.

Example 3.2.17.
A simple example for an ideal that is in quasi-stable but not in componentwise quasi-stable position is the zero-dimensional ideal $\mathcal{J} = \langle x_2,\ x_1^2\rangle \lhd \Bbbk[x_1, x_2]$. Since obviously $\mathcal{J}_{\langle 2\rangle} = \langle x_2\rangle$ is not quasi-stable.

Remark 3.2.18.
Briefly, we now want discuss the difference between a component ideal $\mathcal{I}_{\langle q\rangle}$ and a truncated ideal $\mathcal{I}_{\geq q} = \bigoplus_{p\geq q}\mathcal{I}_p$. Considering for example the ideal $\mathcal{I} = \langle x_2,\ x_1^2\rangle$ it is easy to see that $\mathcal{I}_{\langle 1\rangle} = \langle x_2\rangle \neq \mathcal{I} = \mathcal{I}_{\geq 1}$. Hence for arbitrary values of q we can not assume that $\mathcal{I}_{\langle q\rangle}$ and $\mathcal{I}_{\geq q}$ are equal. But if F is a head autoreduced generating set of $\mathcal{I}$, then we have:

$$\mathcal{I}_{\langle q\rangle} = \mathcal{I}_{\geq q}, \quad \text{for all } q \geq \deg F \tag{3.5}$$

In particular, (3.5) holds for all $q \geq \operatorname{reg}(\mathcal{I})$ since $\deg F \leq \operatorname{reg}(\mathcal{I})$ by [**BS07**, Thm. 16.3.1].

Lemma 3.2.19 ([**Sei09b**, Lem. 2.2][4]).
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal and $q \geq \operatorname{reg}(\mathcal{I})$. Then $\mathcal{I}$ is in quasi-stable position, if and only if the ideal $\mathcal{I}_{\geq q}$ is in quasi-stable position.

Proposition 3.2.20.
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal such that $\mathcal{I}_{\langle q\rangle}$ is in quasi-stable position for all $q \leq \operatorname{reg}(\mathcal{I})$. Then $\mathcal{I}$ is in componentwise quasi-stable position.

Proof. By assumption $\mathcal{I}_{\langle \operatorname{reg}(\mathcal{I})\rangle}$ is in quasi-stable position, hence by Lemma 3.2.19 and Remark 3.2.18 $\mathcal{I}$ itself is in quasi-stable position. Using Lemma 3.2.19 again, we can now derive that $\mathcal{I}_{\langle q\rangle}$ is in quasi-stable position for all $q \geq \operatorname{reg}(\mathcal{I})$. □

This proposition shows that for the definition of componentwise quasi-stability it suffices to consider the finitely many degrees $q \leq \operatorname{reg}(\mathcal{I})$. The following modified version of Algorithm 1 allows us to put any ideal into componentwise quasi-stable position.

[4]Indeed, it follows from its proof that this lemma also holds for all $q \geq 0$ (compare Lemma 3.2.24)

Algorithm 3 CQS-TRAFO: Transformation to componentwise quasi-stable position for char $\Bbbk = 0$

Input: Reduced Gröbner bases G_q of homogeneous ideals $\mathcal{I}_{\langle q\rangle} \lhd \mathcal{P}$ for all $q \le \operatorname{reg}(\mathcal{I})$

Output: a linear change of coordinates Ψ such that $\operatorname{lt}\Psi(\mathcal{I})$ is componentwise quasi-stable

1: $\Psi := \mathrm{id}$;
2: **while** $\exists\, q \le \operatorname{reg}(\mathcal{I}),\ g \in G_q,\ 1 \le i < j = \mathrm{m}(g)\ :\ x_i^{\deg G_q} \frac{\operatorname{lt} g}{x_j^{\mu_j}} \notin \langle \operatorname{lt} G_q\rangle$ with $\operatorname{lt} g = \mathbf{x}^\mu$ **do**
3: $\quad\psi := (x_j \mapsto x_j + x_i)$; $\Psi = \psi \circ \Psi$;
4: $\quad$**for** $\alpha = 0$ **to** $\operatorname{reg}(\mathcal{I})$ **do**
5: $\quad\quad\tilde{G}_\alpha :=$ REDUCEDGRÖBNERBASIS$\big(\psi(G_\alpha)\big)$;
6: $\quad$**end for**
7: $\quad$**while** $(\exists\, \beta \le \operatorname{reg}(\mathcal{I})\ : \mathscr{L}(G_\beta) \succ_{\mathscr{L}} \mathscr{L}(\tilde{G}_\beta))$ **or** $(\mathscr{L}(G_q) = \mathscr{L}(\tilde{G}_q))$ **do**
8: $\quad\quad\psi := (x_j \mapsto x_j + x_i)$; $\Psi = \psi \circ \Psi$;
9: $\quad\quad$**for** $\alpha = 0$ **to** $\operatorname{reg}(\mathcal{I})$ **do**
10: $\quad\quad\quad\tilde{G}_\alpha :=$ REDUCEDGRÖBNERBASIS$\big(\psi(G_\alpha)\big)$;
11: $\quad\quad$**end for**
12: $\quad$**end while**
13: $\quad$**for** $q = 0$ **to** $\operatorname{reg}(\mathcal{I})$ **do**
14: $\quad\quad G_q := \tilde{G}_q$;
15: $\quad$**end for**
16: **end while**
17: **return** Ψ

THEOREM 3.2.21.
Algorithm 3 terminates in finitely many steps and returns a coordinate transformation Ψ so that $\Psi(\mathcal{I})$ is in componentwise quasi-stable position.

PROOF. The main structure of the algorithm is close to Algorithm 1 and we already know from Theorem 2.2.19/2.4.11, that we can transform any ideal into quasi-stable position. So if we choose a $q_1 \le \operatorname{reg}(\mathcal{I})$, we can determine a transformation Ψ_1 such that $\Psi_1(\mathcal{I}_{\langle q_1\rangle})$ is in quasi-stable position. But now the question arises, which effect this transformation might have for the other component ideals. With other words let $\mathcal{I}_{\langle q_2\rangle}$ be already in quasi-stable position for some $q_1 \neq q_2 \le \operatorname{reg}(\mathcal{I})$, it is not clear whether $\Psi_1(\mathcal{I}_{\langle q_2\rangle})$ is still in this position or not.

Therefore we first ensure with the while loop in line 7 that we perform a transformation $\psi : (x_j \mapsto x_j + a x_i)$ where $a \in \Bbbk$ is of generic choice for all G_α. The existence of such an element a is clear, since for every set G_q with $q \le \operatorname{reg}(\mathcal{I})$, there are only finitely many nongeneric values for a and so – as our field is infinite – we will find an integer a that is generic for every G_α.

So we will leave the while loop of line 2 at the latest when every G_α has reached its maximum with respect to $\prec_{\mathscr{L}}$. Since then we know by Corollary 2.4.9 and Theorem 2.4.4(ii) that every G_α is quasi-stable. $\square$

REMARK 3.2.22.
It is possible to make Algorithm 3 work for arbitrary characteristic. Therefore we have to replace the transformation $\psi := (x_j \mapsto x_j + x_i)$ of Line 3 and 8 with:

$$\psi := (x_j \mapsto x_j + ax_i) \text{ with } a \in \Bbbk \text{ randomly chosen}$$

We already discussed the reason for that in the proof of Theorem 2.4.10.

One disadvantage of Algorithm 3 is that one has to compute a reduced Gröbner basis for each component ideal $\mathcal{I}_{\langle q\rangle}$, which is rather expensive. Therefore we now develop a sufficient criterion for an ideal $\mathcal{I}$ to be in componentwise quasi-stable position which does not require the consideration of the component ideals $\mathcal{I}_{\langle q\rangle}$. Assuming that the ideal $\mathcal{I}$ is already in quasi-stable position, we can derive such a criterion based on the first syzygies of $\mathcal{I}$.

Before we start we want to introduce another notation that we use in the following:

$$\mathcal{I}_{[q]} = \langle \bigcup_{p\leq q} \mathcal{I}_p \rangle$$

LEMMA 3.2.23.
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal and $q \geq 0$. Then $\mathcal{I}_{\langle q\rangle} = (\mathcal{I}_{[q]})_{\geq q}$.

PROOF. This assertion is a simple consequence of:

$$\mathcal{I}_{\langle q\rangle} = \langle \mathcal{I}_q \rangle_{\geq q} = \langle \bigcup_{p\leq q} \mathcal{I}_p \rangle_{\geq q} = (\mathcal{I}_{[q]})_{\geq q}$$

$\square$

LEMMA 3.2.24.
If an ideal $\mathcal{I}$ is in quasi-stable/stable/strongly stable position, then $\mathcal{I}_{\geq q}$ is in quasi-stable/stable/strongly stable position for any $q \geq 0$.

Conversely, if $\mathcal{I}_{\geq q}$ is in quasi-stable position for some $q \geq 0$, then already $\mathcal{I}$ is in quasi-stable position.

PROOF. Firstly, let $\mathcal{I}$ be in quasi-stable/stable/strongly stable position and assume that $\mathcal{I}_{\geq \hat{q}}$ is not in quasi-stable/stable/strongly stable position for some integer $\hat{q} \geq 0$. Then there is a monomial $\mathbf{x}^\mu \in \operatorname{lt}(\mathcal{I}_{\geq \hat{q}}) \subseteq \operatorname{lt}\mathcal{I}$ that causes an obstruction to the respective position, i.e. there is some monomial t – induced by $\mathbf{x}^\mu$ – that is "missing" in $\operatorname{lt}(\mathcal{I}_{\geq \hat{q}})$. However, as by assumption $\mathcal{I}$ is in the respective position we know that

$$t \in \operatorname{lt}\mathcal{I}. \tag{3.6}$$

We saw in Lemma 2.2.4 that it is enough to verify the condition for the desired position on the elements of the corresponding monomial basis. Hence without loss of generality we may assume that $\mathbf{x}^\mu \in B$, where B is the monomial basis of $\operatorname{lt}(\mathcal{I}_{\geq \hat{q}})$. Now[5] the definition of quasi-stable/stable/strongly stable entails that $\deg t \geq \deg \mathbf{x}^\mu = \hat{q} = \deg B$ and so by (3.6) $t \in (\operatorname{lt} \mathcal{I})_{\geq \hat{q}} = \operatorname{lt}(\mathcal{I}_{\geq \hat{q}})$. This contradicts our assumption and shows the first half of the lemma.

Now let $\mathcal{I}_{\geq \hat{q}}$ be in quasi-stable position for some $\hat{q} \geq 0$. Assume $\mathcal{I}$ was not in quasi-stable position. Hence there is a monomial $\mathbf{x}^\mu \in \operatorname{lt} \mathcal{I}$ such that

$$x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}} \notin \operatorname{lt} \mathcal{I} \tag{3.7}$$

for some $i < k = \operatorname{m}(\mathbf{x}^\mu)$, where B is the monomial basis of $\operatorname{lt} \mathcal{I}$. Let $\deg \mathbf{x}^\mu = p$ and $f \in \mathcal{I}$ with $\operatorname{lt} f = \mathbf{x}^\mu$. Then obviously $f \in \mathcal{I}_p$ and therefore $f \in \mathcal{I}_{\geq q}$ for all $q \leq p$. In particular, f causes an obstruction to quasi-stability in $\mathcal{I}_{\geq \hat{q}}$ if $\hat{q} \leq p$, which contradicts our assumption that $\mathcal{I}_{\geq \hat{q}}$ is in quasi-stable position. So let $\hat{q} > p$ but then obviously $x_k^{\hat{q}-p} f$ lies in $\mathcal{I}_{\geq \hat{q}}$ and $\operatorname{lt}(x_k^{\hat{q}-p} f) = x_k^{\hat{q}-p} \mathbf{x}^\mu$. Since $\mathcal{I}_{\geq \hat{q}}$ is in quasi-stable position the term

$$x_i^{\deg B} \frac{x_k^{\hat{q}-p} \mathbf{x}^\mu}{x_k^{\hat{q}-p} x_k^{\mu_k}} = x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}}$$

lies in $\operatorname{lt}(\mathcal{I}_{\geq \hat{q}}) \subseteq \operatorname{lt} \mathcal{I}$, which again leads to a contradiction because of (3.7). $\square$

The following example shows that the "converse"-part of Lemma 3.2.24 is not true for (strongly) stable position.

Example 3.2.25.
Let $\mathcal{J} = \langle x_1^2,\ x_2^2 \rangle \lhd \Bbbk[x_1, x_2]$. Then $H = \{x_1^2,\ x_2^2,\ x_1 x_2^2\}$ is a Pommaret basis of $\mathcal{J}$ and so $\operatorname{reg}(\mathcal{J}) = 3$ by Theorem 3.1.10. Hence

$$\mathcal{J}_{\geq 3} \overset{\text{Rem. 3.2.18}}{=} \mathcal{J}_{\langle 3 \rangle} = \langle x_1^3,\ x_1^2 x_2,\ x_1 x_2^2,\ x_2^3 \rangle$$

so that it is easy to see that $\mathcal{J}_{\geq 3}$ is strongly stable. However, $\mathcal{J}$ is not (strongly) stable since $x_1 \frac{x_2^2}{x_2} = x_1 x_2 \notin \mathcal{J}$.

From Lemma 3.2.23 and Lemma 3.2.24 we can immediately derive the following corollary:

Corollary 3.2.26.
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal and $q \geq 0$. Then $\mathcal{I}_{\langle q \rangle}$ is in quasi-stable position, if and only if $\mathcal{I}_{[q]}$ is in quasi-stable position.

[5] In the stable or strongly stable case we have $\deg t = \deg \mathbf{x}^\mu$. For quasi-stability we need the assumption that $\deg \mathbf{x}^\mu \leq \deg B$ to ensure that $\deg t$ can not be less than $\deg \mathbf{x}^\mu$.

If the set $H = \{h_1, \dots, h_\ell\}$ is a Pommaret basis of $\mathcal{I}$, then by Proposition 3.2.2 the product $x_j h_\alpha$ with $j < \mathrm{m}(h_\alpha)$ possesses a unique involutive standard representation

$$x_j h_\alpha = \sum_{\beta=1}^{\ell} P_\beta^{(\alpha,j)} h_\beta, \tag{3.8}$$

where each nonvanishing coefficient $P_\beta^{(\alpha,j)} \in \Bbbk[x_{\mathrm{m}(h_\beta)}, \dots, x_n]$ satisfies $\operatorname{lt}(P_\beta^{(\alpha,j)} h_\beta) \preceq \operatorname{lt} x_j h_\alpha$. Given a degree $q \geq 0$ such that $\mathcal{I}_q \neq 0$, we introduce two subsets of the Pommaret basis H:

$$H(q) = \{h \in H \mid \deg h \leq q\}, \quad \hat{H}(q) = \{\hat{h} \in H \mid \exists h \in H(q) : \operatorname{lt} h | \operatorname{lt} \hat{h}\} \tag{3.9}$$

Proposition 3.2.27.
Let $\mathcal{I} \lhd \mathcal{P}$ be a homogeneous ideal in quasi-stable position and $q \geq 0$ a degree such that $\mathcal{I}_q \neq 0$. The ideal $\mathcal{I}_{[q]}$ is in quasi-stable position if in every involutive standard representation (3.8) *with $h_\alpha \in \hat{H}(q)$ all generators h_β with $P_\beta^{(\alpha,j)} \neq 0$ also lie in $\hat{H}(q)$. In this case, $\hat{H}(q)$ is the Pommaret basis of $\mathcal{I}_{[q]}$.*

Proof. We first note that since $\langle H \rangle = \mathcal{I}$ we obviously have:

$$\langle H(q) \rangle = \mathcal{I}_{[q]} \tag{3.10}$$

By our assumption, any element of the ideal $\hat{\mathcal{I}} = \langle \hat{H}(q) \rangle$ possesses an involutive standard representation with respect to $\hat{H}(q)$. Hence it follows from Proposition 3.2.2 that $\hat{H}(q)$ is the Pommaret basis of $\hat{\mathcal{I}}$. Now the assertion immediately follows if we prove the following claim:

Claim: $\mathcal{I}_{[q]} = \hat{\mathcal{I}}$.

Since obviously $H(q) \subseteq \hat{H}(q)$ we consequently have $\mathcal{I}_{[q]} \overset{(3.10)}{=} \langle H(q) \rangle \subseteq \langle \hat{H}(q) \rangle = \hat{\mathcal{I}}$. If we now assume that $\mathcal{I}_{[q]}$ is a proper subset of $\hat{\mathcal{I}}$, then there must exist a generator $\hat{h} \in \hat{H}(q)$ which is not contained in $\mathcal{I}_{[q]}$. Let $\hat{h}$ be among all such generators the one with the smallest leading term with respect to the used term order, i.e.:

$$h' \in \mathcal{I}_{[q]}, \quad \text{for all } h' \in \hat{H}(q) \text{ with } \operatorname{lt} h' \prec \operatorname{lt} \hat{h} \tag{3.11}$$

Since $\hat{h} \in \hat{H}(q)$, there exists $h \in H(q)$ such that $\operatorname{lt} \hat{h} = \mathbf{x}^\nu \operatorname{lt} h$ for some term $\mathbf{x}^\nu$. We consider the polynomial $g = \operatorname{lc} h \cdot \hat{h} - \operatorname{lc} \hat{h} \cdot \mathbf{x}^\nu h \in \hat{\mathcal{I}}$. It possesses an involutive standard representation with respect to the Pommaret basis $\hat{H}(q)$ of the form $g = \sum_{\hat{f} \in \hat{H}(q)} P_{\hat{f}} \hat{f}$. Every generator $\hat{f}$ with $P_{\hat{f}} \neq 0$ must have a leading term smaller than $\hat{h}$ as it follows from the construction of g that:

$$\operatorname{lt} \hat{f} \preceq \operatorname{lt}(P_{\hat{f}} \hat{f}) \preceq \operatorname{lt} g \prec \operatorname{lt} \hat{h}$$

Thus all of these polynomials $\hat{f}$ must lie in $\mathcal{I}_{[q]}$ according to (3.11). But this implies that $g \in \mathcal{I}_{[q]}$ and – as $h \in H(q) \subseteq \mathcal{I}_{[q]}$ – therefore $\hat{h} \in \mathcal{I}_{[q]}$ contradicting our assumption. □

Remark 3.2.28.
By Lemma 3.2.24 we can replace in the assertion of Proposition 3.2.27 both appearances of "quasi-stable" with "(strongly) stable". The criterion itself does not change. As in this case the leading terms $\operatorname{lt} H$ form even the minimal basis of $\operatorname{lt} \mathcal{I}$, we find that $\hat{H}(q) = H(q)$ which simplifies the application of the criterion.

We now present two examples where the first one proves that the criterion described in Proposition 3.2.27 is not necessary for an ideal to be in componentwise quasi-stable position. With the second example we just check that the criterion is not satisfied for an ideal which is not in componentwise quasi-stable position.

Example 3.2.29.
Let $\mathcal{J} = \langle x_1^5,\ x_1x_2^4,\ x_1^3x_2^3\rangle \lhd \Bbbk[x_1, x_2]$. Then $\mathcal{J}$ is quasi-stable and a Pommaret basis is given by:

$$H = \{h_1 = x_1^5,\ h_2 = x_1x_2^4,\ h_3 = x_1^3x_2^3,\ h_4 = x_1^2x_2^4,\ h_5 = x_1^4x_2^3\}$$

According to (3.9) we have $H(5) = \{h_1, h_2\}$, $\hat{H}(5) = \{h_1, h_2, h_4\}$. Now let us consider the involutive standard representation of x_1h_4:

$$x_1h_4 \quad = \quad x_1^3x_2^4 \quad = \quad x_2\boldsymbol{h_3}$$

We should note that the bold marked element h_3 is not in $\hat{H}(5)$ and therefore the criterion described in Proposition 3.2.27 is not fulfilled. However, $\mathcal{J}_{\langle 5\rangle} = \mathcal{J}_{[5]} = \langle x_1^5,\ x_1x_2^4\rangle$ is in quasi-stable position.

Example 3.2.30.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_2{+}x_2x_3,\ x_2^3,\ x_2^2x_3\rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then $\operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^3,\ x_2^2x_3,\ x_2x_3^2\rangle$ is stable and therefore

$$H = \{h_1 = x_1^2,\ h_2 = x_1x_2 + x_2x_3,\ h_3 = x_2^3,\ h_4 = x_2^2x_3,\ h_5 = x_2x_3^2\}$$

is a Pommaret basis of $\mathcal{I}$ by Lemma 3.2.3. Further, $\operatorname{lt}\mathcal{I}_{\langle 2\rangle} = \langle x_1^2,\ x_1x_2,\ x_2x_3^2\rangle$ is not quasi-stable since $x_2^3\frac{x_2x_3^2}{x_3^2} = x_2^4 \notin \operatorname{lt}\mathcal{I}_{\langle 2\rangle}$. Hence the criterion described in Proposition 3.2.27 is not fulfilled.

According to (3.9) we have $H(2) = \{h_1, h_2\} = \hat{H}(2)$. Now let us consider the involutive standard representation of x_1h_2:

$$x_1h_2 \quad = x_1^2x_2 + x_1x_2x_3 \quad = \quad x_2h_1 + x_3h_2 - \boldsymbol{h_5}$$

Indeed, we can see that – in consistence with Proposition 3.2.27 – the bold marked element h_5 does not lie in $\hat{H}(2)$.

Theorem 3.2.31 ([**Sei09b**, Thm. 8.2, Thm. 9.12]).
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal in quasi-stable position and H its Pommaret basis. Then the following statements are equivalent:

(i) $\mathcal{I}$ is componentwise linear and in componentwise quasi-stable position.
(ii) The free resolution (3.1) of $\mathcal{I}$ induced by the Pommaret basis H is minimal and the Betti numbers of $\mathcal{I}$ are given by (3.2).

As a corollary, we obtain a simple proof of an estimate given by Aramova et al. [**AHH00**, Cor. 1.5] (based on [**HK84**, Thm. 2]).

Corollary 3.2.32 ([**HSS12**, Cor. 21]).
Let $\mathcal{I} \lhd \mathcal{P}$ be a componentwise linear ideal with $\operatorname{pd}(\mathcal{I}) = p$. Then the Betti numbers satisfy $\beta_i^{\mathcal{I}} \geq \binom{p+1}{i+1}$.

Proof. Let $\mathcal{I}$ be in componentwise quasi-stable position, H its Pommaret basis and $d = n - \mathrm{m}(H) + 1$. By Theorem 3.2.31, (3.1) is the minimal resolution of $\mathcal{I}$ and hence $\beta_i^{\mathcal{I}}$ is given by (3.2). Further, $p = n - d$ by Theorem 3.1.10. We know from Lemma 3.2.4 that $\beta_0^{(k)} > 0$ for all $1 \leq k \leq n - d + 1$. Now we compute

$$\begin{aligned}
\beta_i^{\mathcal{I}} &= \sum_{k=d}^{n-i} \binom{n-k}{i} \beta_0^{(n-k+1)} &&=^6 \sum_{l=i}^{n-d} \binom{n-d-l+i}{i} \beta_0^{(n-d-l+i+1)} \\
&= \sum_{l=i}^{n-d} \binom{l}{i} \beta_0^{(l+1)} &&\geq \sum_{l=i}^{p} \binom{l}{i} \\
&= \binom{p+1}{i+1}
\end{aligned}$$

by a well-known identity for binomial coefficients. □

The estimate in Corollary 3.2.32 is sharp. Equality is realized by any componentwise linear ideal whose Pommaret basis satisfies $\beta_0^{(k)} = 0$ for $k > n - d + 1$ and $\beta_0^{(k)} = 1$ for $k \leq n - d + 1$.

Example 3.2.33.
As a simple monomial example, consider the ideal $\mathcal{J} = \langle x_1^2,\ x_1x_2 \rangle \lhd \Bbbk[x_1, x_2]$. Since $\mathcal{J}$ is stable $H = \{x_1^2,\ x_1x_2\}$ is indeed simultaneously the Pommaret and the minimal basis of $\mathcal{J}$ (compare Lemma 3.2.3). Hence in this case $d = 1$, $n = 2$, $\beta_0^{(1)} = \beta_0^{(2)} = 1$ and so:

$$\begin{aligned}
\beta_0^{\mathcal{J}} &= \binom{1}{0}\beta_0^{(2)} + \binom{0}{0}\beta_0^{(1)} &&= 2 = \binom{1+1}{0+1} \\
\beta_1^{\mathcal{J}} &= \binom{1}{1}\beta_0^{(1)} &&= 1 = \binom{1+1}{1+1}
\end{aligned}$$

[6] In this step we substitute k with $d + l - i$.

Recently, Nagel and Römer [**NR15**, Thm. 2.5] provided some criteria for componentwise linearity based on $\operatorname{gin}\mathcal{I}$ (see also [**AHH00**, Thm 1.1] where the case $\operatorname{char}\Bbbk = 0$ is treated). We will now show that again $\operatorname{gin}\mathcal{I}$ may be replaced by $\operatorname{lt}\mathcal{I}$ if it is in componentwise quasi-stable position. Furthermore, our proof is considerably simpler than the one by Nagel and Römer.

Theorem 3.2.34 ([**HSS12**, Thm. 23]).
Let the ideal $\mathcal{I} \lhd \mathcal{P}$ be in componentwise quasi-stable position. Then the following statements are equivalent:

(i) $\mathcal{I}$ *is componentwise linear.*
(ii) $\operatorname{lt}\mathcal{I}$ *is stable and all graded Betti numbers* $\beta_{i,j}^{\mathcal{I}}$ *and* $\beta_{i,j}^{\operatorname{lt}\mathcal{I}}$ *coincide.*
(iii) $\operatorname{lt}\mathcal{I}$ *is stable and all total Betti numbers* $\beta_{i}^{\mathcal{I}}$ *and* $\beta_{i}^{\operatorname{lt}\mathcal{I}}$ *coincide.*
(iv) $\operatorname{lt}\mathcal{I}$ *is stable and* $\beta_{0}^{\mathcal{I}}$ *and* $\beta_{0}^{\operatorname{lt}\mathcal{I}}$ *coincide.*

Proof.
“*(i)* ⇒ *(ii)*”.
Since $\mathcal{I}$ is in componentwise quasi-stable position the resolution (3.1) of $\mathcal{I}$ is minimal by assumption (*i*) and Theorem 3.2.31. This implies immediately (compare Lemma 3.2.8, Lemma 3.2.12) that the resolution (3.1) of $\operatorname{lt}\mathcal{I}$ is minimal. Therefore $\operatorname{lt}\mathcal{I}$ is stable by Theorem 3.2.10 and componentwise linear by Theorem 3.2.31. So we know from Remark 3.2.13 that $\beta_{i,j}^{\mathcal{I}}$ and $\beta_{i,j}^{\operatorname{lt}\mathcal{I}}$ are obtained from equation (3.4). As the numbers $\beta_{0,j-i}^{(n-k+1)}$ of the formula (3.4) do only depend on the leading terms of the corresponding Pommaret basis, there is no difference whether we consider $\mathcal{I}$ or $\operatorname{lt}\mathcal{I}$. Hence all graded Betti numbers $\beta_{i,j}^{\mathcal{I}}$ and $\beta_{i,j}^{\operatorname{lt}\mathcal{I}}$ must coincide.

The implications “(*ii*) ⇒ (*iii*)” and “(*iii*) ⇒ (*iv*)” are trivial. Thus there only remains to prove:
“*(iv)* ⇒ *(i)*”.
Let H be the Pommaret basis of $\mathcal{I}$. Since $\operatorname{lt}\mathcal{I}$ is stable by assumption, $\operatorname{lt} H$ is its minimal basis by Lemma 3.2.3 and so by Remark 1.1.5/1.1.3

$$\beta_0^{\operatorname{lt}\mathcal{I}} = \#(\operatorname{lt} H) = \#H.$$

Hence the assumption that $\beta_0^{\mathcal{I}}$ and $\beta_0^{\operatorname{lt}\mathcal{I}}$ are equal implies that H is a minimal[7] generating system of $\mathcal{I}$. Let $x_j h'$ be a polynomial with $h' \in H$ and $j < \operatorname{m}(h')$. Now let us assume its involutive standard representation $x_j h' = \sum_{h \in H} P_h h$ contains a coefficient $P_{\hat{h}}$ that lies in $\Bbbk \setminus \{0\}$. Hence $\hat{h} = \frac{1}{P_{\hat{h}}}(x_j h' - \sum_{h \in H \setminus \{\hat{h}\}} P_h h) \in \langle H \setminus \{\hat{h}\}\rangle$ and so $\langle H \setminus \{\hat{h}\}\rangle = \langle H \rangle = \mathcal{I}$ which is a contradiction to the minimality of H. Therefore none of the syzygies may contain a nonvanishing constant coefficient. By Lemma 3.2.8, this observation implies that the resolution (3.1) induced by H is minimal and so the ideal $\mathcal{I}$ is componentwise linear by Theorem 3.2.31. □

[7] With minimal generating system we mean minimal with respect to inclusion, i.e. for all $\hat{h} \in H$ holds $\langle H \setminus \{\hat{h}\}\rangle \neq \mathcal{I}$.

Linear Quotients. Linear quotients were introduced by Herzog and Takayama [**HT02**] in the context of constructing iteratively a free resolution via mapping cones. As a special case, they considered monomial ideals where certain colon ideals defined by an ordered minimal basis are generated by variables. Their definition was generalized by Sharifan and Varabaro [**SV08**] to arbitrary ideals.

DEFINITION 3.2.35.
Let $\mathcal{I} = \langle F \rangle \lhd \mathcal{P}$ be an ideal with $F = \{f_1, \dots, f_\ell\} \subseteq \mathcal{P}$. Then $\mathcal{I}$ has *linear quotients* with respect to F if for each $1 < k \leq \ell$ the ideal $\langle f_1, \dots, f_{k-1} \rangle : f_k$ is generated by a subset of $\{x_1, \dots, x_n\}$.

We show first that in the monomial case this concept captures the essence of a Pommaret basis. For this purpose, we "invert" some notions introduced in [**Sei09b**]. We associate with a monomial Pommaret basis H a directed graph, its *$\mathscr{P}$-graph.* Its vertices are the elements of H. For every term $x_j h$, where $h \in H$ is a generator and $j < \mathrm{m}(h)$, there exists a unique Pommaret divisor $\hat{h} \in H$ (see Remark 3.1.5) and we include a directed edge from h to $\hat{h}$.

An ordering of the elements of H is called an *inverse $\mathscr{P}$-ordering*, if $\alpha > \beta$ whenever the $\mathscr{P}$-graph contains a path from h_α to h_β. It is straightforward to describe explicitly an inverse $\mathscr{P}$-ordering: we set $\alpha > \beta$, if $\mathrm{m}(h_\alpha) > \mathrm{m}(h_\beta)$ or if $\mathrm{m}(h_\alpha) = \mathrm{m}(h_\beta)$ and $h_\alpha \prec_{\mathrm{lex}} h_\beta$, i. e. we sort the generators h_α first by their index and then within each index lexicographically. One easily verifies that this defines an inverse $\mathscr{P}$-ordering.

EXAMPLE 3.2.36.
Consider the monomial ideal $\mathcal{J} \lhd \Bbbk[x_1, x_2, x_3]$ generated by the six terms $h_1 = x_1^2$, $h_2 = x_1x_2$, $h_3 = x_2^2$, $h_4 = x_1x_3$, $h_5 = x_2x_3$ and $h_6 = x_3^2$. It is easy to see that $\mathcal{J}$ is stable and therefore these terms form a Pommaret basis of $\mathcal{J}$ by Lemma 3.2.3.

The $\mathscr{P}$-graph in (3.12) shows that the generators are already inversely $\mathscr{P}$-ordered, according to the description above.

$$
\begin{tikzcd}
 & h_5 \arrow[r, "4"] \arrow[dr, "3"] & h_3 \arrow[d, "7"] \\
h_6 \arrow[ur, "2"] \arrow[dr, "1"'] & & h_2 \arrow[d, "8"] \\
 & h_4 \arrow[r, "5"'] \arrow[ur, "6"] & h_1
\end{tikzcd}
\qquad
\begin{array}{ll}
1:\ h_4 |_{\mathscr{P}} x_1 h_6 & 5:\ h_1 |_{\mathscr{P}} x_1 h_4 \\
2:\ h_5 |_{\mathscr{P}} x_2 h_6 & 6:\ h_2 |_{\mathscr{P}} x_2 h_4 \\
3:\ h_2 |_{\mathscr{P}} x_1 h_5 & 7:\ h_2 |_{\mathscr{P}} x_1 h_3 \\
4:\ h_3 |_{\mathscr{P}} x_2 h_5 & 8:\ h_1 |_{\mathscr{P}} x_1 h_2
\end{array}
\tag{3.12}
$$

Proposition 3.2.37 ([**HSS12**, Prop. 26]).
Let $H = \{h_1, \dots, h_\ell\}$ be an inversely $\mathscr{P}$-ordered monomial Pommaret basis of the quasi-stable monomial ideal $\mathcal{J} \lhd \mathcal{P}$. Then the ideal $\mathcal{J}$ possesses linear quotients with respect to the basis H and

$$\langle h_1, \dots, h_{k-1} \rangle : h_k = \langle x_1, \dots, x_{\mathrm{m}(h_k)-1} \rangle \qquad k = 2, \dots, \ell \, . \tag{3.13}$$

Conversely, assume that $H = \{h_1, \dots, h_\ell\}$ with $\mathrm{m}(h_1) = 1$ is a monomial generating set of the monomial ideal $\mathcal{J} \lhd \mathcal{P}$ such that (3.13) is satisfied. Then $\mathcal{J}$ is quasi-stable and H its Pommaret basis.

Proof. Since H is a Pommaret basis the product $x_j h_k$ with $j < \mathrm{m}(h_k)$ possesses a Pommaret divisor $h_i \in H$ (see Remark 3.1.5) and therefore $x_j \in \langle h_i \rangle : h_k$. By definition, it also follows that the $\mathscr{P}$-graph of H contains an edge from k to i and so $i < k$ because H is inversely $\mathscr{P}$-ordered. Thus $x_j \in \langle h_1, \dots, h_{k-1} \rangle : h_k$ which proves the inclusion "$\supseteq$".

The following argument shows that the inclusion cannot be strict. Consider a term $t \notin \langle x_1, \dots, x_{\mathrm{m}(h_k)-1} \rangle$, i.e. $t \in \Bbbk[x_{\mathrm{m}(h_k)}, \dots, x_n]$ and assume:

Assumption: $th_k \in \langle h_1, \dots, h_{k-1} \rangle$.

Hence $th_k = s_1 h_{i_1}$ – recall that every h_i is monomial – for some term $s_1 \in \Bbbk[x_1, \dots, x_n]$ and some index $i_1 < k$. By definition of a Pommaret basis we must have $s_1 \notin \Bbbk[x_{\mathrm{m}(h_{i_1})}, \dots, x_n]$ since otherwise the Pommaret cones $\Bbbk[x_{\mathrm{m}(h_{i_1})}, \dots, x_n] \cdot h_{i_1}$ and $\Bbbk[x_{\mathrm{m}(h_k)}, \dots, x_n] \cdot h_k$ would intersect (see Remark 3.1.5). So there is an index $j_1 < \mathrm{m}(h_{i_1})$ such that $x_{j_1} | s_1$. Let h_{i_2} be the Pommaret divisor of $x_{j_1} h_{i_1}$, this implies on the one hand that there is a term $s_2 \in \Bbbk[x_{\mathrm{m}(h_{i_2})}, \dots, x_n]$ with $x_{j_1} h_{i_1} = s_2 h_{i_2}$ and on the other hand that $\mathrm{m}(h_{i_2}) \leq m(h_{i_1})$. Because of the $\mathscr{P}$-ordering and since $i_1 \neq i_2$ we have $i_1 > i_2$. We can not repeat this procedure infinitely many times since the sequence $i_1 > i_2 > \cdots$ is strictly decreasing and $\mathrm{m}(h_1) = 1$[8]. Hence after a finite number of iterations we arrive at a representation

$$th_k = \frac{s_1}{x_{j_1}} x_{j_1} h_{i_1} = \frac{s_1}{x_{j_1}} s_2 h_{i_2} = \cdots = \hat{s} h_{i_\omega}$$

where $\hat{s} \in \Bbbk[x_{\mathrm{m}(h_{i_\omega})}, \dots, x_n]$. This leads to a contradiction by Remark 3.1.5 since then the Pommaret cones of h_k and h_{i_ω} would intersect, which is not possible as H is a Pommaret basis.

For the converse, we show by a finite induction over k that every product $x_j h_k$ with $j < \mathrm{m}(h_k)$ possesses a Pommaret divisor h_i with $i < k$ which implies our assertion by Proposition 3.1.7 and Theorem 3.1.3.

$\boldsymbol{k = 2}$.

For $k = 2$ it follows from (3.13) that $x_j h_2 \in \langle h_1 \rangle$. This implies that h_1 is a Pommaret divisor of $x_j h_2$ since $\mathrm{m}(h_1) = 1$.

[8] Any Pommaret basis contains an element of index 1 (see Lemma 3.2.4). Because of the $\mathscr{P}$-ordering it must be h_1 in this case.

$(k-1) \to k$.
Assume now that our claim was true for $h_2, h_3, \dots, h_{k-1}$. Because of (3.13) $x_j h_k \in \langle h_1, \dots, h_{k-1} \rangle$, so we may write $x_j h_k = t_1 h_{i_1}$ for some $t_1 \in \mathcal{P}$ and $i_1 < k$. If $t_1 \in \Bbbk[x_{\mathrm{m}(h_{i_1})}, \dots, x_n]$ we are done because then h_{i_1} is the Pommaret divisor of h_k. Otherwise, there exists an index $j_1 < \mathrm{m}(h_{i_1})$ such that $x_{j_1} | t_1$. By our induction assumption, $x_{j_1} h_{i_1}$ has a Pommaret divisor h_{i_2} with $i_2 < i_1$ and so there is a $t_2 \in \Bbbk[x_{\mathrm{m}(h_{i_2})}, \dots, x_n]$ such that $x_{j_1} h_{i_1} = t_2 h_{i_2}$. With the same arguments like above, we find after finitely many steps a representation

$$x_j h_k = \frac{t_1}{x_{j_1}} x_{j_1} h_{i_1} = \frac{t_1}{x_{j_1}} t_2 h_{i_2} = \cdots = \hat{t} h_{i_\omega}$$

where h_{i_ω} is the Pommaret divisor of $x_j h_k$. □

In general, we cannot expect that the second part of Proposition 3.2.37 remains true, when we consider arbitrary polynomial ideals. However, for the first part we find the following variation of [**SV08**, Thm. 2.3].

Proposition 3.2.38 ([**HSS12**, Prop. 28]).
Let H be a Pommaret basis of the polynomial ideal $\mathcal{I} \lhd \mathcal{P}$ and $h' \in \mathcal{P}$ a polynomial with $\operatorname{lt} h' \notin \operatorname{lt} H$. If $\mathcal{I} : h' = \langle x_1, \dots, x_{\mathrm{m}(h')-1} \rangle$, then $H' = H \cup \{h'\}$ is a Pommaret basis of $\mathcal{I}' = \mathcal{I} + \langle h' \rangle$. If furthermore $\mathcal{I}$ is in componentwise quasi-stable position, componentwise linear and H' is a minimal basis[9] of $\mathcal{I}'$, then $\mathcal{I}'$ is componentwise linear, too.

Proof. If $\mathcal{I} : h' = \langle x_1, \dots, x_{\mathrm{m}(h')-1} \rangle$, then all products of $x_j h'$ with $j < \mathrm{m}(h')$ lie in $\mathcal{I} = \langle H \rangle_{\mathscr{P}} \subseteq \langle H' \rangle_{\mathscr{P}}$. This immediately implies the first assertion by Proposition 3.1.7.

Let $x_j h_\alpha$ be a product with $h_\alpha \in H$ and $j < \mathrm{m}(h_\alpha)$. Further, let $x_j h_\alpha = \sum_{h \in H} P_h^{(\alpha,j)} h$ be the involutive standard representation of $x_j h_\alpha$. We know from Theorem 3.2.5 that the Pommaret resolution is minimal since $\mathcal{I}$ is componentwise linear and in componentwise quasi-stable position. Hence all syzygies $S_{(\alpha,j)}$ are free of constant coefficients by Lemma 3.2.8. Let us now assume that there is an index $j < \mathrm{m}(h')$ such that the involutive standard representation of the product $x_j h' = \sum_{h \in H} P'_h h$ contains a constant coefficient $P'_{\hat{h}} \in \Bbbk \setminus \{0\}$. But this means that

$$\hat{h} = \frac{1}{P'_{\hat{h}}} \left(x_j h' - \sum_{h \in H \setminus \{\hat{h}\}} P_h h \right) \in \langle H' \setminus \{\hat{h}\} \rangle$$

---

[9] With minimal basis we mean minimal with respect to inclusion, i.e. for all $\hat{h} \in H'$ holds $\langle H' \setminus \{\hat{h}\} \rangle \neq \mathcal{I}'$.

and so $\mathcal{I}' = \langle H' \rangle = \langle H' \setminus \{\hat{h}\} \rangle$, which is a contradiction to the minimality of H'. Therefore all syzygies obtained from products $x_j h'$ with $j < \mathrm{m}(h')$ are free of constant terms. Finally, we can again conclude with Lemma 3.2.8 that the resolution of $\mathcal{I}'$ induced by H' is minimal and $\mathcal{I}'$ componentwise linear by Theorem 3.2.31. □

Local Cohomology. Finally, a last application field of Pommaret bases we want to consider is induced by some results of Trung [**Tru01**] related to invariants that are connected to local cohomology. Using Pommaret basis we are able to generalize these results.

Further, we provide an alternative proof for Grothendieck's results concerning the vanishing and nonvanishing of local cohomology.

Proposition 3.2.39 ([**HSS12**, Prop. 9]).
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal in quasi-stable position. If H denotes the corresponding Pommaret basis and

$$H^{(i)} = \{h \in H \mid \mathrm{m}(h) = i\} \subseteq H$$

the subset of generators of index i, then the number

$$q_{n-i+1} = \max\left\{ q \mid (\langle \mathcal{I}, x_{n-i+2}, \dots, x_n \rangle : x_{n-i+1})_q \neq \langle \mathcal{I}, x_{n-i+2}, \dots, x_n \rangle_q \right\} \tag{3.14}$$

satisfies $q_{n-i+1} = \deg H^{(n-i+1)} - 1$ (with the convention that $\deg \emptyset = \max \emptyset = -\infty$).

Proof. Set $\hat{\mathcal{P}} = \Bbbk[x_1, \dots, x_{n-i+1}]$ and $\hat{\mathcal{I}} = \mathcal{I}|_{x_{n-i+2}=\cdots=x_n=0} \lhd \hat{\mathcal{P}}$. Then it is easy to see that $q_{n-i+1} = \max\{q \mid (\hat{\mathcal{I}} : x_{n-i+1})_q \neq \hat{\mathcal{I}}_q\}$. It follows from [**Sei12**, Lemma 3.1] that the Pommaret basis of $\hat{\mathcal{I}}$ is given by:

$$\hat{H} = \{h|_{x_{n-i+2}=\cdots=x_n=0} \mid h \in H \wedge \mathrm{m}(h) < n-i+2\}$$

We should note that obviously $\deg \hat{H}^{(n-i+1)} = \deg H^{(n-i+1)}$.

Case I: $\hat{H}^{(n-i+1)} = \emptyset$**.**
For this case we will now show that $(\hat{\mathcal{I}} : x_{n-i+1}) = \hat{\mathcal{I}}$. Therefore let $\hat{f}_\alpha \in (\hat{\mathcal{I}} : x_{n-i+1})$ be a polynomial and $x_{n-i+1}\hat{f}_\alpha = \sum_{\hat{h} \in \hat{H}} P^{(\alpha)}_{\hat{h}} \hat{h} \in \hat{\mathcal{I}}$ its involutive standard representation with respect to $\hat{H}$. Then one immediately sees that all $P^{(\alpha)}_{\hat{h}} \hat{h}$ must lie in $\langle x_{n-i+1} \rangle$ and since we assumed that any element of $\hat{H}$ has an index lower than $n-i+1$ we must have $P^{(\alpha)}_{\hat{h}} \in \langle x_{n-i+1} \rangle$. Hence $\hat{f}_\alpha = \sum_{\hat{h} \in \hat{H}} \frac{P^{(\alpha)}_{\hat{h}}}{x_{n-i+1}} \hat{h}$, is an involutive standard representation, which implies $\hat{f}_\alpha \in \hat{\mathcal{I}}$ by Proposition 3.2.2. Thus $(\hat{\mathcal{I}} : x_{n-i+1}) = \hat{\mathcal{I}}$ and therefore $q_{n-i+1} = -\infty = \deg \hat{H}^{(n-i+1)} - 1$.

Case II: $\hat{H}^{(n-i+1)} \neq \emptyset$.
In this case we can choose a generator $\hat{h}_{\max} \in \hat{H}^{(n-i+1)}$ of maximal degree, i.e. $\deg \hat{h}_{\max} = \deg \hat{H}^{(n-i+1)}$. Obviously, we find $\hat{h}_{\max} \in \langle x_{n-i+1} \rangle$ and hence may write $\hat{h}_{\max} = x_{n-i+1}\hat{g}$ for some $\hat{g} \in \hat{\mathcal{P}}$. By definition of a Pommaret basis $\hat{g} \notin \hat{\mathcal{I}}$ since otherwise $\hat{g}$ would be a Pommaret divisor of $\hat{h}_{\max}$. Hence $\hat{g} \in (\hat{\mathcal{I}} : x_{n-i+1}) \setminus \hat{\mathcal{I}}$ and so $(\hat{\mathcal{I}} : x_{n-i+1})_{\deg \hat{g}} \neq \hat{\mathcal{I}}_{\deg \hat{g}}$. Thus $q_{n-i+1} \geq \deg \hat{g} = \deg \hat{H}^{(n-i+1)} - 1$.

Assume now that $q_{n-i+1} > \deg \hat{H}^{(n-i+1)} - 1$ then $(\hat{\mathcal{I}} : x_{n-i+1})_{\deg \hat{H}^{(n-i+1)}} \neq \hat{\mathcal{I}}_{\deg \hat{H}^{(n-i+1)}}$. So there is a polynomial $\hat{f}_\beta \in (\hat{\mathcal{I}} : x_{n-i+1})_{\deg \hat{H}^{(n-i+1)}}$ with $\hat{f}_\beta \notin \hat{\mathcal{I}}$. Consider the involutive standard representation $x_{n-i+1}\hat{f}_\beta = \sum_{\hat{h}\in\hat{H}} P^{(\beta)}_{\hat{h}}\hat{h}$ with respect to $\hat{H}$. Analogous to above we have $P^{(\beta)}_{\hat{h}}\hat{h} \in \langle x_{n-i+1} \rangle$. If $\mathrm{m}(\hat{h}) < n-i+1$, then we must have $P^{(\beta)}_{\hat{h}} \in \langle x_{n-i+1} \rangle$. If $\mathrm{m}(\hat{h}) = n-i+1$, then the definition of the involutive standard representation entails that $P^{(\beta)}_{\hat{h}} \in \Bbbk[x_{n-i+1}]$. Since $\deg(x_{n-i+1}\hat{f}_\beta) = \deg \hat{H}^{(n-i+1)} + 1 > \deg \hat{H}^{(n-i+1)}$, any nonvanishing coefficient $P^{(\beta)}_{\hat{h}}$ must be of positive degree in this case. Thus we can conclude that in both cases all nonvanishing coefficients $P^{(\beta)}_{\hat{h}}$ lie in $\langle x_{n-i+1} \rangle$. But then $\hat{f}_\beta = \sum_{\hat{h}\in\hat{H}} \frac{P^{(\beta)}_{\hat{h}}}{x_{n-i+1}}\hat{h}$ is an involutive standard representation of $\hat{f}_\beta$ itself so that, $\hat{f}_\beta \in \hat{\mathcal{I}}$ by Proposition 3.2.2 in contradiction to the assumptions we made. □

Consider the following invariants related to $\mathbf{H}^i_{\mathfrak{m}}$, the local cohomology of $\mathcal{P}/\mathcal{I}$ with respect to the maximal graded ideal $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$:

$$\begin{aligned}
a_i(\mathcal{P}/\mathcal{I}) &= \max\{q \mid \mathbf{H}^i_{\mathfrak{m}}(\mathcal{P}/\mathcal{I})_q \neq 0\}, && 0 \leq i \leq \dim(\mathcal{P}/\mathcal{I}),\\
\mathrm{reg}_t(\mathcal{P}/\mathcal{I}) &= \max\{a_i(\mathcal{P}/\mathcal{I}) + i \mid 0 \leq i \leq t\}, && 0 \leq t \leq \dim(\mathcal{P}/\mathcal{I}),\\
a^*_t(\mathcal{P}/\mathcal{I}) &= \max\{a_i(\mathcal{P}/\mathcal{I}) \mid 0 \leq i \leq t\}, && 0 \leq t \leq \dim(\mathcal{P}/\mathcal{I}).
\end{aligned}$$

Trung [**Tru01**, Thm. 2.4] related them for monomial Borel-fixed ideals and $\mathrm{char}\,\Bbbk = 0$ to the degrees of the minimal generators. We can now generalize this result to arbitrary homogeneous polynomial ideals.

Corollary 3.2.40 ([**HSS12**, Cor. 10]).
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal in quasi-stable position. Denote again by $H^{(i)}$ the subset of the Pommaret basis H of $\mathcal{I}$ consisting of the generators of index i and set $q_i = \deg H^{(i)} - 1$. Then

$$\begin{aligned}
\mathrm{reg}_t(\mathcal{P}/\mathcal{I}) &= \max\{q_{n-t}, q_{n-t+1}, \dots, q_n\}, && 0 \leq t \leq \dim(\mathcal{P}/\mathcal{I}),\\
a^*_t(\mathcal{P}/\mathcal{I}) &= \max\{q_{n-t} - t, q_{n-(t-1)} - (t-1), \dots, q_n\}, && 0 \leq t \leq \dim(\mathcal{P}/\mathcal{I}).
\end{aligned}$$

Proof. This follows immediately from [**Tru01**, Thm. 1.1] and Proposition 3.2.39. □

Grothendieck's vanishing and nonvanishing theorems[10] present a very famous result in the theory of local cohomology. The assertion of these theorems are on the one hand that

$$\mathbf{H}^i_{\mathfrak{m}}(\mathcal{P}/\mathcal{I}) = 0, \quad \text{if } i < \operatorname{depth}(\mathcal{P}/\mathcal{I}) \text{ or } i > \dim(\mathcal{P}/\mathcal{I})$$

and on the other hand that

$$\mathbf{H}^i_{\mathfrak{m}}(\mathcal{P}/\mathcal{I}) \neq 0, \quad \text{if } i = \operatorname{depth}(\mathcal{P}/\mathcal{I}) \text{ or } i = \dim(\mathcal{P}/\mathcal{I}).$$

With the results from above we can now present an alternative proof for those statements concerning $\operatorname{depth}(\mathcal{P}/\mathcal{I})$.

Proposition 3.2.41.
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal and $d = \operatorname{depth}(\mathcal{P}/\mathcal{I})$, then:

(i) $\mathbf{H}^i_{\mathfrak{m}}(\mathcal{P}/\mathcal{I}) = 0$ *for all* $i < d$
(ii) $\mathbf{H}^d_{\mathfrak{m}}(\mathcal{P}/\mathcal{I}) \neq 0$

Proof. We know from the previous chapter[11] that we can transform any ideal into quasi-stable position. Since $\mathbf{H}^i_{\mathfrak{m}}(\mathcal{P}/\mathcal{I})$ is invariant under coordinate transformation, we can assume without loss of generality that $\mathcal{I}$ is already in quasi-stable position. Further, let H be its Pommaret basis and $q_i = \deg H^{(i)} - 1$.

Since $H^{(i)} = \emptyset$ for all $i > \mathrm{m}(H)$ it follows that $q_{\mathrm{m}(H)+1}, \ldots, q_n = -\infty$ and so $a^*_{n-\mathrm{m}(H)-1}(\mathcal{P}/\mathcal{I}) = -\infty$ by Corollary 3.2.40. Hence $a_i(\mathcal{P}/\mathcal{I}) = -\infty$ for all $0 \leq i \leq n - \mathrm{m}(H) - 1$ by definition of $a^*_t(\mathcal{P}/\mathcal{I})$. Therefore $\mathbf{H}^i_{\mathfrak{m}}(\mathcal{P}/\mathcal{I}) = 0$ for all $0 \leq i \leq n - \mathrm{m}(H) - 1$. Since Theorem 3.1.10 and Remark 1.1.15 entail that $d = n - \mathrm{m}(H)$, we have already proven (i).

Obviously, $H^{(\mathrm{m}(H))} \neq \emptyset$ and so $q_{\mathrm{m}(H)} = q_{n-d} \neq -\infty$. Hence with the arguments from above

$$a_d(\mathcal{P}/\mathcal{I}) = a^*_d(\mathcal{P}/\mathcal{I}) = q_{n-d} \neq -\infty,$$

which finally proves (ii). □

[10] See for example Proposition A1.16 in the appendix of [**Eis05**].
[11] See Remark 2.2.17 and Theorem 2.4.11.

CHAPTER 4

The Reduction Number

In the first part of this thesis we saw that the costs for the determination of certain invariants depend on the coordinate system. In particular, we presented in Theorem 1.3.9 that the invariants $\operatorname{reg}(\mathcal{I})$, $\operatorname{pd}(\mathcal{I})$ and $\operatorname{depth}(\mathcal{I})$ are easy to determine whenever the considered ideal $\mathcal{I}$ is in gin-position. As a next step we took a look at Seiler's generalized version of this results in Theorem 3.1.10. Thereby he makes use of Pommaret bases which only require that $\mathcal{I}$ is in quasi-stable position. With the intention to find further homological invariants that are easy to compute not only in gin-position but also in quasi-stable position we will now introduce and analyze the *absolute reduction number*.

Furthermore, we give a brief overview of the *big reduction number* – which is related to the absolute reduction number – and consider two different methods to compute it.

4.1. Computing the absolute Reduction Number

After presenting its definition, our main goal of this section will be to introduce an algorithm to compute the absolute reduction number. Thereby we do not care about the given coordinate system, since this will be discussed in more detail in Section 4.3.

Definition 4.1.1.
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal and $y_1, \dots, y_\ell$ linear forms. We call $\mathcal{R} = \mathcal{I} + \langle y_1, \dots, y_\ell \rangle$ a *reduction* of $\mathcal{I}$ if $\dim(\mathcal{P}/\mathcal{R}) = 0$. In particular, a reduction $\mathcal{R}$ of $\mathcal{I}$ is *minimal* if $\ell = \dim(\mathcal{P}/\mathcal{I})$.

Definition 4.1.2.
Let $\mathcal{R}$ be a minimal reduction of an ideal $\mathcal{I}$, then

$$r_{\mathcal{R}}(\mathcal{P}/\mathcal{I}) = \max\{q \mid (\mathcal{P}/\mathcal{R})_q \neq 0\}$$

is the *reduction number of $\mathcal{I}$ with respect to* $\mathcal{R}$. Further, we denote the set of all reduction numbers by

$$\operatorname{rSet}(\mathcal{P}/\mathcal{I}) = \{r_{\mathcal{R}}(\mathcal{P}/\mathcal{I}) \mid \mathcal{R} \text{ minimal reduction of } \mathcal{I}\}.$$

Finally, we define the *absolute reduction number* of $\mathcal{I}$ by

$$r(\mathcal{P}/\mathcal{I}) = \min \operatorname{rSet}(\mathcal{P}/\mathcal{I}).$$

LEMMA 4.1.3.
Let $\mathcal{R} \lhd \mathcal{P}$ a zero-dimensional ideal, then $\operatorname{reg}(\mathcal{P}/\mathcal{R}) = \max\{q \mid (\mathcal{P}/\mathcal{R})_q \neq 0\}$*. In particular, if $\mathcal{R}$ is a minimal reduction of an ideal $\mathcal{I}$, then* $r_{\mathcal{R}}(\mathcal{P}/\mathcal{I}) = \operatorname{reg}(\mathcal{P}/\mathcal{R})$*.*

PROOF. First we should remember that $\mathcal{P}$ is Noetherian by Hilbert's Basis Theorem. Further, $\mathcal{P}/\mathcal{R}$ is Artinian since $\dim(\mathcal{P}/\mathcal{R}) = 0$ (see e.g.[**AM69**, Thm. 8.5]). Hence the $\mathcal{P}$-module $\mathcal{P}/\mathcal{R}$ is of finite length by [**Eis95**, Cor. 2.17] and so as a consequence of [**Eis05**, Cor. A1.5] we have:

$$\mathbf{H}^i_{\mathfrak{m}}(\mathcal{P}/\mathcal{R}) = 0 \text{ for all } i > 0 \tag{4.1}$$

$$\mathbf{H}^0_{\mathfrak{m}}(\mathcal{P}/\mathcal{R}) = \mathcal{P}/\mathcal{R} \tag{4.2}$$

With the notations of Remark 1.1.7 we can now conclude:

$$\begin{aligned} \operatorname{reg}(\mathcal{P}/\mathcal{R}) &\overset{\text{Rem. 1.1.7}}{=} \max\{a_i(\mathcal{P}/\mathcal{R}) + i \mid i \geq 0\} \\ &\overset{(4.1)}{=} a_0(\mathcal{P}/\mathcal{R}) \\ &\overset{(4.2)}{=} \max\{q \mid (\mathcal{P}/\mathcal{R})_q \neq 0\} \\ &\overset{\text{Def. 4.1.2}}{=} r_{\mathcal{R}}(\mathcal{P}/\mathcal{I}) \end{aligned}$$

□

We present now Algorithm 4 for the computation of $r(\mathcal{P}/\mathcal{I})$. Instead of a coordinate transformation, it is based on a parametric computation. The main point will be to keep the number of parameters as small as possible.

Algorithm 4 REDNUM: (Absolute) Reduction Number

Input: Gröbner basis G of a homogeneous ideal $\mathcal{I} \lhd \mathcal{P}$
Output: the absolute reduction number $r(\mathcal{P}/\mathcal{I})$
1: $D := \dim(\mathcal{P}/\mathcal{I})$
2: $\tilde{G} := G$ with x_{n-D+i} replaced by $-\sum_{j=1}^{n-D} b_{ij}x_j$ for all $i = 1, \dots, D$
3: $\tilde{\mathcal{R}} := \langle \tilde{G} \rangle_{\tilde{\mathcal{P}}}$
4: $H :=$ POMMARETBASIS $(\tilde{\mathcal{R}})$
5: **return** $\deg H - 1$

The algorithm simply adds $D = \dim(\mathcal{P}/\mathcal{I})$ linear forms z_i of the special form

$$z_i = x_{n-D+i} + \sum_{j=1}^{n-D} b_{ij}x_j.$$

The occurring coefficients b_{ij} are then considered as undetermined parameters. Replacing in the ideal $\mathcal{I}$ every variable x_{n-D+i} with $i > 0$ by $-\sum_{j=1}^{n-D} b_{ij}x_j$, we obtain a new homogeneous ideal $\tilde{\mathcal{R}}$ in the polynomial ring $\tilde{\mathcal{P}} = \Bbbk(\mathbf{b})[x_1, \dots, x_{n-D}]$ over the field of rational functions in the $D(n-D)$ parameters b_{ij} and compute its Pommaret basis.

In order to prove the correctness of Algorithm 4 we have to recall some results of Trung first.

THEOREM 4.1.4 ([**Tru03**, Thm. 1.2][**Tru01**, Lem. 4.2]).
Let $\hat{\mathcal{R}} = \mathcal{I} + \langle y_1, \ldots, y_D\rangle \lhd \hat{\mathcal{P}} = \Bbbk(\mathbf{a})[x_1, \ldots, x_n]$ *be an ideal with* $y_i = \sum_{j=1}^{n} a_{ij}x_j$ *for* $i = 1, \ldots, D$*. Then* $\hat{\mathcal{R}}$ *is a minimal reduction of the ideal* $\hat{\mathcal{I}} \lhd \hat{\mathcal{P}}$ *obtained by considering* $\mathcal{I}$ *in* $\hat{\mathcal{P}}$ *and*

$$r_{\hat{\mathcal{R}}}(\hat{\mathcal{P}}/\hat{\mathcal{I}}) = r(\mathcal{P}/\mathcal{I}).$$

COROLLARY 4.1.5 ([**Tru03**, Cor. 1.3]).
Let $\mathcal{I}\lhd\mathcal{P}$ *be an ideal. Further, let* $\tilde{\mathcal{R}}\lhd\tilde{\mathcal{P}} = \Bbbk(\mathbf{b})[x_1, \ldots, x_{n-D}]$ *be the ideal obtained by substituting every variable* x_{n-D+i} *with* $-\sum_{j=1}^{n-D} b_{ij}x_j$ *for all* $i > 0$ *in the ideal* $\mathcal{I}$ *. Then*

$$r(\mathcal{P}/\mathcal{I}) = \max\{q \mid (\tilde{\mathcal{P}}/\tilde{\mathcal{R}})_q \neq 0\}.$$

THEOREM 4.1.6 ([**HSS14**, Thm. 5.6]).
Algorithm 4 correctly determines $r(\mathcal{P}/\mathcal{I})$*.*

PROOF. Using the notations of Corollary 4.1.5, the only thing we have to prove is that:

$$\dim(\tilde{\mathcal{P}}/\tilde{\mathcal{R}}) = 0 \tag{4.3}$$

Because then $\tilde{\mathcal{R}}$ possesses a finite Pommaret basis H by Lemma 3.2.1 and as a consequence of Corollary 4.1.5, Lemma 4.1.3, Remark 1.1.15 and Theorem 3.1.10 we get:

$$r(\mathcal{P}/\mathcal{I}) = \operatorname{reg}(\tilde{\mathcal{P}}/\tilde{\mathcal{R}}) = \operatorname{reg}(\tilde{\mathcal{R}}) - 1 = \deg H - 1$$

To prove (4.3) we consider the ideal $\hat{\mathcal{R}} = \mathcal{I}+\langle y_1, \ldots, y_D\rangle$ in the polynomial ring $\hat{\mathcal{P}} = \Bbbk(\mathbf{a})[x_1, \ldots, x_n]$, where the $y_i = \sum_{j=1}^{n} a_{ij}x_j$ for $i = 1, \ldots, D = \dim(\mathcal{P}/\mathcal{I})$ are generic linear forms. It follows from Theorem 4.1.4 that $\hat{\mathcal{R}}$ is a minimal reduction of the ideal $\hat{\mathcal{I}} \lhd \hat{\mathcal{P}}$, which we obtain by considering $\mathcal{I}$ in $\hat{\mathcal{P}}$. Therefore

$$\dim(\hat{\mathcal{P}}/\hat{\mathcal{R}}) = 0. \tag{4.4}$$

Now consider the $D \times n$ matrix (a_{ij}): as we consider the a_{ij} as parameters, the determinant of the submatrix composed of the last D columns does not vanish. Then by a Gaussian elimination we obtain a set of linear forms z_i in the “reduced” triangular form

$$z_i = x_{n-D+i} + \sum_{j=1}^{n-D} b_{ij}x_j, \quad b_{ij} \in \Bbbk(\mathbf{a})$$

such that $\hat{\mathcal{R}} = \mathcal{I} + \langle z_1, \ldots, z_D\rangle \lhd \hat{\mathcal{P}}$. Therefore $\tilde{\mathcal{P}}/\tilde{\mathcal{R}} \cong \hat{\mathcal{P}}/\hat{\mathcal{R}}$ and so (4.3) follows from (4.4). $\square$

EXAMPLE 4.1.7.
The homogenized Weispfenning94 ideal $\mathcal{I} \lhd \Bbbk[x_1, x_2, x_3, x_4]$ is generated by the polynomials

$$\begin{aligned} f_1 &= x_2^4 + x_1 x_2^2 x_3 + x_1^2 x_4^2 - 2x_1 x_2 x_4^2 + x_2^2 x_4^2 + x_3^2 x_4^2 , \\ f_2 &= x_1 x_2^4 + x_2 x_3^4 - 2x_1^2 x_2 x_4^2 - 3x_4^5 , \\ f_3 &= -x_1^3 x_2^2 + x_1 x_2 x_3^3 + x_2^4 x_4 + x_1 x_2^2 x_3 x_4 - 2x_1 x_2 x_4^3 . \end{aligned}$$

Here $\dim(\mathcal{P}/\mathcal{I}) = 2$ and we replace x_4 by $-(b_{41}x_1 + b_{42}x_2)$ and x_3 by $-(b_{31}x_1 + b_{32}x_2)$ in $\mathcal{I}$ to obtain the new ideal $\tilde{\mathcal{R}} \lhd \Bbbk(b_{31}, b_{32}, b_{41}, b_{42})[x_1, x_2]$. We compute a Pommaret basis H of $\tilde{\mathcal{R}}$ and get as leading terms

$$\operatorname{lt} H = \left\{ x_1^4,\ x_1^3 x_2^2,\ x_1^2 x_2^3,\ x_1 x_2^5,\ x_2^6 \right\} .$$

Therefore $r(\mathcal{P}/\mathcal{I}) = \deg H - 1 = 5$.

4.2. Computing the big Reduction Number

Analogous to Trungs paper [**Tru03**] we will now investigate the *big reduction number* which is related to absolute reduction number that we got to know in the previous section. In particular, we provide an alternative method for its computation and compare it with the one of Trung by performing each method on the same example.

THEOREM 4.2.1 ([**HSS14**, Thm. 6.2]).
Let $\mathcal{I} \lhd \mathcal{P}$ be a homogeneous ideal. Then its reduction number set $\operatorname{rSet}(\mathcal{P}/\mathcal{I})$ *is finite.*

PROOF. By definition, any minimal reduction of $\mathcal{I}$ is induced by $D = \dim(\mathcal{P}/\mathcal{I})$ linear forms

$$y_i = \sum_{j=1}^{n} a_{ij} x_j , \qquad i = 1, \dots, D \tag{4.5}$$

with $a_{ij} \in \Bbbk$ such that $\mathcal{R} = \mathcal{I} + \langle y_1, \dots, y_D \rangle$ is a zero-dimensional ideal. Considering the coefficients a_{ij} as parameters, we may identify $\mathcal{R}$ with a parametric ideal $\hat{\mathcal{R}} \lhd \Bbbk(\mathbf{a})[x_1, \dots, x_n]$. Let $\left\{ (\hat{G}_i, N_i, W_i) \right\}_{i=1}^{\ell}$ be a Gröbner system[1] for $\hat{\mathcal{R}}$. Without loss of generality, we may assume that for the first s triples the ideals $\langle \hat{G}_i \rangle \lhd \mathcal{P}$ are zero-dimensional, whereas all other triples lead to ideals of positive dimension. Hence precisely the parameter values satisfying one of the conditions (N_i, W_i) with $1 \leq i \leq s$ define minimal reductions. Let $d_i = r_{\langle \hat{G}_i \rangle}(\mathcal{P}/\mathcal{I})$ then it follows that $\operatorname{rSet}(\mathcal{P}/\mathcal{I}) = \{d_1, \dots, d_s\}$. □

[1] Recall that any parametric ideal possess a finite Gröbner system (see Theorem 2.1.3).

As our field $\Bbbk$ is infinite it is clear that for any ideal $\mathcal{I} \lhd \mathcal{P}$ there is an infinite number of minimal reductions. But – as we have seen in the proof of Theorem 4.2.1 – these minimal reductions lead only to finitely many different reduction numbers.

Trung showed that a generic minimal reduction leads to the absolute reduction number (see Theorem 4.1.4). Since almost all choices of the parameters a_{ij} induce a generic minimal reduction of the form

$$\mathcal{I} + \langle \sum_{j=1}^{n} a_{1j}x_j, \dots, \sum_{j=1}^{n} a_{Dj}x_j \rangle,$$

for almost all minimal reductions $\mathcal{R}$ we find $r_{\mathcal{R}}(\mathcal{P}/\mathcal{I}) = r(\mathcal{P}/\mathcal{I})$ (see [**Tru03**, Cor. 2.2]).

Definition 4.2.2.
We define the *big reduction number* of $\mathcal{I}$ by:

$$br(\mathcal{P}/\mathcal{I}) = \max \mathrm{rSet}(\mathcal{P}/\mathcal{I})$$

The proof of Theorem 4.2.1 describes a method to determine $\mathrm{rSet}(\mathcal{P}/\mathcal{I})$. It is based on computing a Gröbner system for the parametric ideal:

$$\mathcal{I} + \langle \sum_{j=1}^{n} a_{1j}x_j, \dots, \sum_{j=1}^{n} a_{Dj}x_j \rangle \lhd \Bbbk(\mathbf{a})[x_1, \dots, x_n]$$

Any branch of this Gröbner system that yields to a zero-dimensional ideal leads us to an element of $\mathrm{rSet}(\mathcal{P}/\mathcal{I})$. Once we have determined the whole $\mathrm{rSet}(\mathcal{P}/\mathcal{I})$ we can directly read off the big reduction number $br(\mathcal{P}/\mathcal{I})$.

Example 4.2.3.
Consider the ideal $\mathcal{I} = \langle x_1^2,\ x_1x_2 + x_2^2,\ x_1x_3 \rangle \lhd \Bbbk[x_1, x_2, x_3]$ introduced by Green [**Gre98**]. Since $\dim(\mathcal{P}/\mathcal{I}) = 1$ we set:

$$\hat{\mathcal{R}} = \mathcal{I} + \langle a_1x_1 + a_2x_2 + a_3x_3 \rangle \lhd \Bbbk(a_1, a_2, a_3)[x_1, x_2, x_3]$$

The Gröbner system for $\hat{\mathcal{R}}$ consists of 4 triples. For simplicity, we present in the following list for each branch as first entry only the corresponding[2] leading ideal $\mathcal{L}_i$; the other two entries are the equations N_i and the inequations W_i, respectively.

i	$\mathcal{L}_i$	N_i	W_i
1	$\langle x_1,\ x_2^2,\ x_2x_3,\ x_3^2 \rangle$	$\{\}$	$\{a_1,\ a_2,\ a_1 - a_2\}$
2	$\langle x_1,\ x_2^2,\ x_2x_3,\ x_3^2 \rangle$	$\{a_1 - a_2\}$	$\{a_2\}$
3	$\langle x_1,\ x_2^2,\ x_3^2 \rangle$	$\{a_2\}$	$\{a_1\}$
4	$\langle x_2,\ x_1^2,\ x_1x_3,\ x_3^2 \rangle$	$\{a_1\}$	$\{\}$

[2] Compare Remark 2.1.2.

We observe that all four branches lead to zero-dimensional leading ideals which are therefore quasi-stable. As $\mathcal{L}_1 = \mathcal{L}_2$ is even stable its monomial basis $H_1 = \{x_1,\ x_2^2,\ x_2x_3,\ x_3^2\}$ represents simultaneously the Pommaret basis by Lemma 3.2.3. Moreover, $H_3 = \{x_1,\ x_2^2,\ x_3^2,\ x_2x_3^2\}$ is the Pommaret basis of $\mathcal{L}_3$ while $H_4 = \{x_2,\ x_1^2,\ x_1x_2,\ x_1x_3,\ x_3^2\}$ is the one of $\mathcal{L}_4$. As $\deg H_1 = \deg H_4 = 2$ and $\deg H_3 = 3$ we have $\mathrm{rSet}(\mathcal{P}/\mathcal{I}) = \{1,2\}$ according to Algorithm 4 and so $br(\mathcal{P}/\mathcal{I}) = 2$.

Remark 4.2.4.
For comparison, we briefly outline Trung's constructive characterization [**Tru03**] of the big reduction number of an ideal. He also takes $D = \dim(\mathcal{P}/\mathcal{I})$ linear forms (4.5) with undetermined coefficients a_{ij} and proceeds with the ideal $\hat{\mathcal{R}} = \mathcal{I} + \langle y_1, \dots, y_D\rangle \lhd \Bbbk[\mathbf{a}, x_1, \dots, x_n]$. Then he introduces the matrix M_q of the coefficients of the generators in a $\Bbbk$-linear basis of $\hat{\mathcal{R}}_q$ (which are elements in $\Bbbk[\mathbf{a}]$). Let $\mathcal{V}_q$ be the variety of the ideal generated in $\Bbbk[\mathbf{a}]$ by all the minors of M_q of the size of the number of terms of degree q, i.e. the $\binom{q+n-1}{n-1} \times \binom{q+n-1}{n-1}$ minors. Then, $br(\mathcal{P}/\mathcal{I})$ is the largest q such that $\mathcal{V}_q \neq \mathcal{V}_{q+1}$ [**Tru03**, Cor. 2.3].

Note, however that a priori it is unclear how to detect that one has obtained the largest q with this property. Thus his approach becomes truly algorithmic only by combining it with another result of his, namely that $br(\mathcal{P}/\mathcal{I}) + 1$ is bounded by the Castelnuovo-Mumford regularity $\mathrm{reg}(\mathcal{I})$ [**Tru87**, Prop. 3.2]. Now one can check all degrees q until $\mathrm{reg}(\mathcal{I})$ – which has to be computed first – and then finally decide on the value of $br(\mathcal{P}/\mathcal{I})$.

Example 4.2.5.
Using the same ideal as in Example 4.2.3 we now want to determine $br(\mathcal{P}/\mathcal{I})$ with the method of Trung described in Remark 4.2.4.

So let $\mathcal{I} = \langle f_1, f_2, f_3\rangle \lhd \Bbbk[x_1, x_2, x_3]$ with $f_1 = x_1^2$, $f_2 = x_1x_2 + x_2^2$ and $f_3 = x_1x_3$. Further, let $y = a_1x_1 + a_2x_2 + a_3x_3$ be a parametric linear form in $\Bbbk[\mathbf{a}, \mathbf{x}]$ and $\hat{\mathcal{R}} = \mathcal{I} + \langle y\rangle \lhd \Bbbk[a_1, a_2, a_3, x_1, x_2, x_3]$. According to Trung we now have to determine the matrix M_2. Therefore we must consider the $\Bbbk$-linear basis of $\hat{\mathcal{R}}_2$:

$$\begin{array}{c|cccccc} & x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 \\ \hline f_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ f_2 & 0 & 1 & 1 & 0 & 0 & 0 \\ f_3 & 0 & 0 & 0 & 1 & 0 & 0 \\ x_1y & a_1 & a_2 & 0 & a_3 & 0 & 0 \\ x_2y & 0 & a_1 & a_2 & 0 & a_3 & 0 \\ x_3y & 0 & 0 & 0 & a_1 & a_2 & a_3 \end{array} \rightsquigarrow M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a_1 & a_2 & 0 & a_3 & 0 & 0 \\ 0 & a_1 & a_2 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 \end{pmatrix}$$

Then $\mathcal{V}_2 = \mathcal{V}(a_2a_3^2)$ since $\binom{2+3-1}{3-1} = 6$ and $a_2a_3^2$ is the 6×6 minor of M_2. Going on to degree 3 leads to:

	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	$x_1^2x_3$	$x_1x_2x_3$	$x_2^2x_3$	$x_1x_3^2$	$x_2x_3^2$	x_3^3
x_1f_1	1	0	0	0	0	0	0	0	0	0
x_2f_1	0	1	0	0	0	0	0	0	0	0
x_3f_1	0	0	0	0	1	0	0	0	0	0
x_1f_2	0	1	1	0	0	0	0	0	0	0
x_2f_2	0	0	1	1	0	0	0	0	0	0
x_3f_2	0	0	0	0	0	1	1	0	0	0
x_1f_3	0	0	0	0	1	0	0	0	0	0
x_2f_3	0	0	0	0	0	1	0	0	0	0
x_3f_3	0	0	0	0	0	0	0	1	0	0
x_1^2y	a_1	a_2	0	0	a_3	0	0	0	0	0
x_1x_2y	0	a_1	a_2	0	0	a_3	0	0	0	0
x_2^2y	0	0	a_1	a_2	0	0	a_3	0	0	0
x_1x_3y	0	0	0	0	a_1	a_2	0	a_3	0	0
x_2x_3y	0	0	0	0	0	a_1	a_2	0	a_3	0
x_3^2y	0	0	0	0	0	0	0	a_1	a_2	a_3

Bringing the resulting matrix M_3 in the reduced row echelon form[3], yields to the following $\binom{3+3-1}{3-1} = 10 \times 10$ matrix $\tilde{M}_3$:

$$\tilde{M}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & a_3 \end{pmatrix}$$

Hence $\mathcal{V}_3 = \mathcal{V}(a_3^2) \neq \mathcal{V}(a_2a_3^2) = \mathcal{V}_2$ showing that $br(\mathcal{P}/\mathcal{I}) = 2$ since $\operatorname{reg}(\mathcal{I}) = 3$.

The considered example is pretty simple, since it is “nearly” monomial and generated in one degree, which leads to sparsely populated matrices M_i. So a final answer whether Trungs or our method to compute $br(\mathcal{P}/\mathcal{I})$ is better can not be given on the basis of this example. But although the computation of a Gröbner system is surely a rather expensive operation, we strongly believe that it is much more efficient than the determination and subsequent analysis of large determinantal ideals. Furthermore, our approach yields directly all possible values for the

[3] By performing the row operations we should observe that the parameters might possibly stand for 0. Therefore one has to check which operations are allowed under this circumstances.

reduction number, whereas Trung must consider one determinantal ideal after the other (of increasing size).

Finally, we note that Trung [**Tru03**] proved that $br(\mathcal{P}/\mathcal{I}) \leq br(\mathcal{P}/\operatorname{lt}\mathcal{I})$ if $\mathcal{P}/\mathcal{I}$ is *Cohen-Macaulay*, i.e. $\operatorname{depth}(\mathcal{P}/\mathcal{I}) = \dim(\mathcal{P}/\mathcal{I})$. He also claimed that generally one cannot compare $br(\mathcal{P}/\mathcal{I})$ and $br(\mathcal{P}/\operatorname{lt}\mathcal{I})$. However, he did not provide a concrete example where the above inequality is violated – which we will do now.

EXAMPLE 4.2.6.
Consider the ideal $\mathcal{I} \lhd \Bbbk[x_1, x_2, x_3]$ with

$$\mathcal{I} = \langle x_1^3 - x_1x_2^2,\ x_1^2x_2 + x_1x_2^2,\ x_2^3 + x_2^2x_3,\ x_1^2x_3 + x_1x_2x_3,\ x_1x_3^5,\ x_2^2x_3^5\rangle.$$

The given generators form already a Gröbner basis and $\operatorname{lt}\mathcal{I}$ is quasi-stable. So we find a Pommaret basis H of $\mathcal{I}$ with

$$\operatorname{lt} H = \{x_1^3,\ x_1^2x_2,\ x_2^3,\ x_1^2x_3,\ x_1x_2^3,\ x_1x_3^5,\ x_1x_2x_3^5,\ x_2^2x_3^5,\ x_1x_2^2x_3^5\}.$$

Therefore $\operatorname{depth}(\mathcal{P}/\mathcal{I}) = 0$ by Theorem 3.1.10 and Remark 1.1.15. Hence $\mathcal{P}/\mathcal{I}$ is not Cohen-Macaulay since $\dim(\mathcal{P}/\mathcal{I}) = 1$. To compute $br(\mathcal{P}/\mathcal{I})$, we set $\hat{\mathcal{R}} = \mathcal{I} + \langle a_1x_1 + a_2x_2 + a_3x_3\rangle \lhd \Bbbk(\mathbf{a})[x_1, x_2, x_3]$. From the zero-dimensional branches of the Gröbner system of this ideal we can derive that $\operatorname{rSet}(\mathcal{P}/\mathcal{I}) = \{3, 5, 6\}$ and therefore $br(\mathcal{P}/\mathcal{I}) = 6$.

The following table lists a possible choice for the coefficients a_1, a_2, a_3 that leads to the mentioned reduction numbers. Thereby $\sigma : \Bbbk[\mathbf{a}] \to \Bbbk$ is a specialization homomorphism that evaluates the parameters a_i so that $\mathcal{R} = \sigma(\hat{\mathcal{R}})$ is an ideal of $\mathcal{P}$.

(a_1, a_2, a_3)	$\operatorname{lt}(\text{POMMARETBASIS}(\mathcal{R}))$	$r_{\mathcal{R}}(\mathcal{P}/\mathcal{I})$
$(0,0,1)$	$\{x_3,\ x_1x_3,\ x_2x_3,\ x_1^3,\ x_1^2x_2,\ x_2^3,\ x_1^2x_3,\ x_1x_2x_3,\ x_2^2x_3,\ x_1x_2^3,\ x_1x_2^2x_3\}$	3
$(1,2,1)$	$\{x_1,\ x_2^3,\ x_2^2x_3,\ x_2x_3^2,\ x_3^6\}$	5
$(1,1,1)$	$\{x_1,\ x_2^3,\ x_2^2x_3,\ x_2x_3^2,\ x_3^7\}$	6

On the other hand, we set $\hat{\mathcal{R}}' = \operatorname{lt}\mathcal{I} + \langle a_1x_1 + a_2x_2 + a_3x_3\rangle$ and compute its Gröbner system. Only three branches are zero-dimensional and they all have as reduction number 3 (e.g. if you choose $a_1 = a_2 = a_3 = 1$ the set of leading terms of the Pommaret basis of $\sigma(\hat{\mathcal{R}}')$ is $\{x_1,\ x_2^3,\ x_2^2x_3,\ x_2x_3^2,\ x_3^4\}$). This shows that $br(\mathcal{P}/\operatorname{lt}\mathcal{I}) = 3 < 6 = br(\mathcal{P}/\mathcal{I})$.

4.3. Relation with strong Stability

Now we continue to study the absolute reduction number. We will see in this section that although this invariant shows parallels to our previously discussed invariants, we can only find an ansatz to directly read off the absolute reduction number in the monomial case. Moreover, an example provided by Green even shows that it is not possible to formulate a statement similar to Theorem 3.1.10 for this invariant.

But first we want to have a look at the following results provided by Trung and Conca concerning the relationship between ideal and leading ideal in terms of the absolute reduction number.

Theorem 4.3.1 ([**Con02**, Thm. 1.1], [**Tru03**, Cor. 3.4]).
For any ideal $\mathcal{I} \lhd \mathcal{P}$ the inequality $r(\mathcal{P}/\mathcal{I}) \leq r(\mathcal{P}/\operatorname{lt}\mathcal{I})$ holds.

Theorem 4.3.2 ([**Tru01**, Thm. 4.3]).
For any ideal $\mathcal{I} \lhd \mathcal{P}$ we always find $r(\mathcal{P}/\mathcal{I}) = r(\mathcal{P}/\operatorname{gin}\mathcal{I})$.

Since these results are similar to the ones of Theorem 1.2.1 and Theorem 1.3.9 it appears that the absolute reduction number also behaves like the other invariants which we discussed in the first half of this thesis.

The next question that arises is how we can find a minimal reduction that leads to the absolute reduction number. We already saw in Theorem 4.1.4 that a generic minimal reduction does yield to the absolute reduction number. Another characterization of the different minimal reductions is presented in the following lemma.

Lemma 4.3.3 ([**BH99**, Lem. 5]).
Let $\mathcal{J} \lhd \mathcal{P}$ be a monomial ideal such that the variables $x_{n-D+1}, \dots, x_n$ induce a minimal reduction, where $D = \dim(\mathcal{P}/\mathcal{J})$. Then every minimal reduction is induced by linear forms

$$z_i = x_{n-D+i} + \sum_{j=1}^{n-D} b_{ij} x_j \,, \quad b_{ij} \in \Bbbk \,. \tag{4.6}$$

If the ideal $\mathcal{J}$ from the above Lemma is quasi-stable, then we will see later (Proposition 5.1.10) that $\mathcal{J}$ contains a power of x_i for all $i \leq n - D$. Hence $\mathcal{R} = \mathcal{J} + \langle x_{n-D+1}, \dots, x_n \rangle$ is a minimal reduction of $\mathcal{J}$ since obviously[4] $\dim(\mathcal{P}/\mathcal{R}) = 0$.

The following theorem from Bresinsky and Hoa shows, how to read off the absolute reduction number from a strongly stable monomial ideal.

[4]See for example [**KR00**, Prop. 3.7.1, Def. 3.7.2].

THEOREM 4.3.4 ([**BH99**, Thm. 11]).
Let $\mathcal{J} \lhd \mathcal{P}$ be a Borel-fixed monomial ideal and $D = \dim(\mathcal{P}/\mathcal{J})$. Then $\mathcal{J}$ has a minimal generator x_{n-D}^s and $r(\mathcal{P}/\mathcal{J}) = r_{\mathcal{R}}(\mathcal{P}/\mathcal{J})$ for any minimal reduction $\mathcal{R}$ of $\mathcal{J}$. Moreover, if $\mathcal{J}$ is strongly stable we have $r(\mathcal{P}/\mathcal{J}) = s - 1$.

REMARK 4.3.5.
A combination of Trungs Theorem 4.3.2 and the above theorem from Bresinsky and Hoa delivers a way to directly read off the absolute reduction number under the in the following described conditions.

Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal that is in gin-position and $\operatorname{char} \Bbbk = 0$. Hence $\operatorname{lt} \mathcal{I} = \operatorname{gin} \mathcal{I}$ is strongly stable[5] and it follows from Theorem 4.3.4 that there is an integer s such that x_{n-D}^s is a minimal generator of $\operatorname{lt} \mathcal{I}$. Finally, $r(\mathcal{P}/\mathcal{I}) = r(\mathcal{P}/\operatorname{lt} \mathcal{I}) = s - 1$ by Theorem 4.3.2 and Theorem 4.3.4.

EXAMPLE 4.3.6.
Let us consider again Green's Example 4.2.3. Since $\operatorname{lt} \mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_1x_3,\ x_2^3,\ x_2^2x_3 \rangle$ is strongly stable and $\dim(\mathcal{P}/\mathcal{I}) = 1$, we can use Theorem 4.3.4 to receive $r(\mathcal{P}/\operatorname{lt} \mathcal{I}) = 3 - 1 = 2$. But since $\operatorname{gin} \mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3^2 \rangle$ it follows from Theorem 4.3.2 that:

$$r(\mathcal{P}/\mathcal{I}) = r(\mathcal{P}/\operatorname{gin} \mathcal{I}) = 2 - 1 = 1 < r(\mathcal{P}/\operatorname{lt} \mathcal{I}) \tag{4.7}$$

REMARK 4.3.7.
Example 4.2.3 shows that there can not exist an algorithm for the computation of the absolute reduction number that is based on the analysis of the leading ideal. Because if we set $\mathcal{I}' = \operatorname{lt} \mathcal{I}$, then we have two distinct ideals $\mathcal{I}$ and $\mathcal{I}'$ with the same leading ideal, however (4.7) shows that they have different absolute reduction numbers.

[5] As $\operatorname{char} \Bbbk = 0$ $\operatorname{gin} \mathcal{I}$ is strongly stable by Proposition 2.4.5 and Theorem 2.4.4.

CHAPTER 5

Generalization of stable Positions

We discussed in Remark 4.3.7 that the absolute reduction number behaves differently than the other invariants we considered in the first part of this thesis. This motivated us to search for a generalized version of Bresinsky and Hoa's Theorem 4.3.4. This intention leads to several new variants of the different stable positions from Definition 2.2.1 which we will introduce in this chapter. So that we are finally able to present the desired generalization of the result of Bresinsky/Hoa in Theorem 5.2.17.

Afterwards we turn our attention to the Borel-fixed position, which – as already described in Section 2.4 – is especially relevant in the case of positive characteristic, and generalize this position analogues to the stable ones.

5.1. DQS-Test and an alternative Characterization of Noether Position

The first position we want to generalize is the quasi-stable position. Thereby we illustrate its connection to *Noether position*, which will be defined in Definition 5.1.7. Further, we provide an algorithm that checks whether a given ideal is in one of the generalized quasi-stable positions.

Definition 5.1.1.
Let $\mathcal{J}$ be a monomial ideal, B its minimal basis and $0 \leq \ell < n$ an integer.

- $\mathcal{J}$ is *ℓ-quasi-stable* if for every term $\mathbf{x}^\mu \in \mathcal{J}$ with $\mathrm{m}(\mathbf{x}^\mu) = k \geq n - \ell$ and all $i < k$ the term $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}}$ also lies in $\mathcal{J}$.
- $\mathcal{J}$ is *weakly ℓ-quasi-stable* if for every term $\mathbf{x}^\mu \in \mathcal{J}$ with $\mathrm{m}(\mathbf{x}^\mu) = k \geq n - \ell$ and every $i \leq n - \ell$ the term $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}}$ also lies in $\mathcal{J}$.

Further, we call $\mathcal{J}$ *(weakly) D-quasi-stable* if $\mathcal{J}$ is (weakly) $\dim(\mathcal{P}/\mathcal{J})$-quasi-stable. Finally, a polynomial ideal is in *(weakly) ℓ-quasi-stable position* if its leading ideal is (weakly) ℓ-quasi-stable.

Remark 5.1.2.
It is easy to derive the following hierarchy immediately from the definition:

$$\mathcal{J} \text{ quasi-stable } \Rightarrow \mathcal{J}\ \ell\text{-quasi-stable } \Rightarrow \mathcal{J} \text{ weakly } \ell\text{-quasi-stable}$$

With the same arguments of the proof of Lemma 2.2.4, we see that it is enough to check the defining condition on the elements of the minimal basis of the ideal.

Before we begin to analyze the properties of the (weakly) ℓ-quasi-stable position, we first want to recall some well-known statements that are equivalent to the notion quasi-stable.

Proposition 5.1.3 ([**BG06**, Prop. 3.2],[**HPV03**, Prop. 2.2],[**Sei09b**, Prop. 4.4],[**Sei12**, Lem. 3.4]).
Let $\mathcal{J} \lhd \mathcal{P}$ be a monomial ideal, B its minimal basis and $\dim(\mathcal{P}/\mathcal{J}) = D$. Then the following statements are equivalent:

(i) $\mathcal{J}$ is quasi-stable.
(ii) Let $\mathbf{x}^\mu \in \mathcal{J}$ with $\mu_j > 0$ for some $1 < j \leq n$, then for each $r \leq \mu_j$ and $i < j$ an integer $s \geq 0$ exists such that $x_i^s \frac{\mathbf{x}^\mu}{x_j^r}$ lies in $\mathcal{J}$.
(iii) For all $0 \leq j \leq n-1$ holds:

$$\mathcal{J} : x_{n-j}^\infty = \mathcal{J} : \langle x_1, \ldots, x_{n-j} \rangle^\infty \tag{5.1}$$

(iv) x_n is not a zero divisor on $\mathcal{P}/(\mathcal{J} : \mathfrak{m}^\infty)$ and x_{n-j} is not a zero divisor on $\mathcal{P}/(\langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle : \mathfrak{m}^\infty)$ for all $0 < j < D$.
(v) $\mathcal{J} : x_n^\infty = \mathcal{J} : \mathfrak{m}^\infty$ and for all $0 < j < D$ holds:

$$\langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle : x_{n-j}^\infty = \langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle : \mathfrak{m}^\infty \tag{5.2}$$

Proof. For the equivalences $(i) - (iv)$ see the citations from above. To prove the equivalence between (v) and the other four statements, we will now show that "$(iii) \Rightarrow (v)$" and "$(v) \Rightarrow (iv)$".

"$(iii) \Rightarrow (v)$".
Let $f \in \mathcal{P}$ with $f x_{n-j}^s \in \langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle$ for some integer $s > 0$ and $0 < j < D$. If $\mathrm{m}(f) > n - j$ then by the definition of the degree reverse lexicographic term order we have $\mathrm{m}(t) > n - j$ for all $t \in \mathrm{supp}(f)$, so that obviously:

$$f \in \langle x_n, \ldots, x_{n-j+1} \rangle \subseteq \langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle \subseteq \langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle : \mathfrak{m}^\infty$$

Otherwise, $f x_{n-j}^s \in \mathcal{J}$ and so $f \in \mathcal{J} : x_{n-j}^\infty = \mathcal{J} : \langle x_1, \ldots, x_{n-j} \rangle^\infty$ by (iii). Hence in both cases we have $f \in \langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle : \mathfrak{m}^\infty$ which shows (v).

"$(v) \Rightarrow (iv)$".
Assume that x_{n-j} is a zero divisor on $\mathcal{P}/(\langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle : \mathfrak{m}^\infty)$ for some $0 < j < D$. This means that there exists an element

$$f \in \mathcal{P} \setminus (\langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle : \mathfrak{m}^\infty)$$

such that $x_{n-j} f \in \langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle : \mathfrak{m}^\infty$. Hence there is an integer s such that $x_{n-j}^{s+1} f \in \langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle$. This entails by (v) that

$$f \in \langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle : x_{n-j}^\infty = \langle \mathcal{J}, x_n, \ldots, x_{n-j+1} \rangle : \mathfrak{m}^\infty,$$

which is a contradiction to the choice of f. Analogously, we can show that x_n is not a zero divisor of $\mathcal{P}/(\mathcal{J} : \mathfrak{m}^\infty)$ if $\mathcal{J} : x_n^\infty = \mathcal{J} : \mathfrak{m}^\infty$. $\square$

Now we are going to present the corresponding version of this proposition for the notions ℓ-quasi-stable and weakly ℓ-quasi-stable.

PROPOSITION 5.1.4.
Let $\mathcal{J} \lhd \mathcal{P}$ be a monomial ideal, B its minimal basis and ℓ an integer. Then the following statements are equivalent:

(i) *$\mathcal{J}$ is ℓ-quasi-stable.*
(ii) *Let $\mathbf{x}^\mu \in \mathcal{J}$ with $\mathrm{m}(\mathbf{x}^\mu) \geq n-\ell$ and $\mu_j > 0$ for some $n-\ell \leq j \leq n$, then for each $r \leq \mu_j$ and $i < j$ an integer $s \geq 0$ exists such that $x_i^s \frac{\mathbf{x}^\mu}{x_j^r}$ lies in $\mathcal{J}$.*
(iii) *For all $0 \leq j \leq \ell$ holds:*

$$\mathcal{J} : x_{n-j}^\infty = \mathcal{J} : \langle x_1, \ldots, x_{n-j}\rangle^\infty \tag{5.3}$$

PROOF.
"(*i*) ⇒ (*ii*)".
Assume first that $\mathcal{J}$ is ℓ-quasi-stable. Let $\mathbf{x}^\mu \in \mathcal{J}$ be a term with $\mu_j > 0$ for some $n-\ell \leq j \leq n$, hence $k = \mathrm{m}(\mathbf{x}^\mu) \geq j$. Further, let r be an integer with $r \leq \mu_j$. We want to prove *(ii)* by showing that for all integers $i < j$ there is a term $\mathbf{x}^\nu \in \mathcal{J}$ which divides $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_j^r}$. By the definition of ℓ-quasi-stability $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}} \in \mathcal{J}$ for $i < k$. Therefore there is a term $\mathbf{x}^{\nu^{(1)}} \in B$ with

$$\mathbf{x}^{\nu^{(1)}} \mid x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}} \tag{5.4}$$

and so $k_1 = \mathrm{m}(\mathbf{x}^{\nu^{(1)}}) \leq \mathrm{m}(x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}}) < k$. Obviously, $\nu_\alpha^{(1)} \leq \mu_\alpha$ for all $i \neq \alpha < k$ and $\nu_i^{(1)} \leq \mu_i + \deg B$. If $k_1 \leq i$, then $\mathbf{x}^{\nu^{(1)}}$ divides $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_{i+1}^{\mu_{i+1}} \cdots x_k^{\mu_k}}$, but since the latter monomial is a divisor of $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_j^r}$, we are already done in this case. Otherwise, we know by ℓ-quasi-stability that $x_i^{\deg B} \frac{\mathbf{x}^{\nu^{(1)}}}{x_{k_1}^{\nu_{k_1}^{(1)}}} \in \mathcal{J}$ and so there is a term $\mathbf{x}^{\nu^{(2)}} \in B$ with $\mathbf{x}^{\nu^{(2)}} \mid x_i^{\deg B} \frac{\mathbf{x}^{\nu^{(1)}}}{x_{k_1}^{\nu_{k_1}^{(1)}}}$ and $\mathrm{m}(\mathbf{x}^{\nu^{(2)}}) = k_2 < k_1$. Therefore $\mathbf{x}^{\nu^{(2)}} \mid x_i^{2\cdot\deg B} \frac{\mathbf{x}^\mu}{x_{k_1}^{\nu_{k_1}^{(1)}} x_k^{\mu_k}}$ since by (5.4):

$$\mathbf{x}^{\nu^{(2)}} \mid \left(\frac{x_i^{\deg B}}{x_{k_1}^{\nu_{k_1}^{(1)}}}\right) \mathbf{x}^{\nu^{(1)}} \mid \left(\frac{x_i^{\deg B}}{x_{k_1}^{\nu_{k_1}^{(1)}}}\right) x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}} = x_i^{2\cdot\deg B} \frac{\mathbf{x}^\mu}{x_{k_1}^{\nu_{k_1}^{(1)}} x_k^{\mu_k}}$$

This entails that

$$\mathbf{x}^{\nu^{(2)}} \mid x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_{k_2+1}^{\mu_{k_2+1}} \cdots x_k^{\mu_k}} \tag{5.5}$$

because $\deg \mathbf{x}^{\nu^{(2)}} \leq \deg B$ and $\mathrm{m}(\mathbf{x}^{\nu^{(2)}}) = k_2$. We go on like this until we end up at a term $\mathbf{x}^{\nu^{(\omega)}} \in B$ with $\mathbf{x}^{\nu^{(\omega)}} \mid x_i^{\deg B} \frac{\mathbf{x}^{\nu^{(\omega-1)}}}{x_{k_{\omega-1}}^{\nu_{k_{\omega-1}}^{(\omega-1)}}}$ and $\mathrm{m}(\mathbf{x}^{\nu^{(\omega)}}) = k_\omega < \cdots < k_1 < k$ such that $k_\omega < j$. Then the following holds:

- $\nu_\alpha^{(\omega)} = 0$ for all $\alpha \geq j > k_\omega$
- $\nu_\alpha^{(\omega)} \leq \nu_\alpha^{(\omega-1)} \leq \cdots \leq \nu_\alpha^{(1)} \leq \mu_\alpha$ for all $i \neq \alpha < k_\omega < j$
- $\nu_i^{(\omega)} \leq \nu_i^{(\omega-1)} \leq \cdots \leq \nu_i^{(1)} \leq \mu_i + \deg B$

Again with same arguments like above we are immediately done if $k_\omega \leq i$, so we may assume $k_\omega > i$ and analogous to (5.4) and (5.5) we get:

$$\mathbf{x}^{\nu^{(\omega)}} \mid x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_{k_\omega+1}^{\mu_{k_\omega+1}} \cdots x_k^{\mu_k}}$$

Since $k_\omega < j \leq k$ this entails that $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_{k_\omega+1}^{\mu_{k_\omega+1}} \cdots x_k^{\mu_k}}$ divides $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_j^r}$ and so in particular $\mathbf{x}^{\nu^{(\omega)}}$ divides $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_j^r}$.

"(ii) ⇒ (iii)".
Now remember that $\mathcal{J} : x_{n-j}^\infty$ is a monomial ideal (see e.g [**HH11**, Prop. 1.2.2/1.2.3]) and let B' be its monomial basis. Let statement (ii) hold and let t be an element of B', i.e. $x_{n-j}^r t \in \mathcal{J}$ for some integer r. Since $\mathrm{m}(x_{n-j}^r t) \geq n-j \geq n-\ell$ assertion (ii) entails that for all $i < n-j \leq \mathrm{m}(x_{n-j}^r t)$ there is an integer s_i such that the term $x_i^{s_i} \frac{x_{n-j}^r t}{x_{n-j}^r} = x_i^{s_i} t$ lies in $\mathcal{J}$. Hence $t\langle x_1, \ldots, x_{n-j}\rangle^{(s_1+\cdots+s_{n-j-1}+r)(n-j)} \subseteq \mathcal{J}$ and so $t \in \mathcal{J} : \langle x_1, \ldots, x_{n-j}\rangle^\infty$ which shows (iii).

"(iii) ⇒ (i)".
Finally, assume (5.3) holds and consider a term[1] $\mathbf{x}^\mu \in \mathcal{J}$ such that $\mathrm{m}(\mathbf{x}^\mu) = n-j$ with $j \leq \ell$. Because of (5.3), we have $\frac{\mathbf{x}^\mu}{x_{n-j}^{\mu_{n-j}}} \in \mathcal{J} : x_{n-j}^\infty = \mathcal{J} : \langle x_1, \ldots, x_{n-j}\rangle^\infty$. Hence there is an integer s such that $\frac{\mathbf{x}^\mu}{x_{n-j}^{\mu_{n-j}}}\langle x_1, \ldots, x_{n-j}\rangle^s \subseteq \mathcal{J}$, but this means that $x_i^s \frac{\mathbf{x}^\mu}{x_{n-j}^{\mu_{n-j}}} \in \mathcal{J}$ for all $i < n-j$. Hence there is a minimal generator $\mathbf{x}^\nu$ of $\mathcal{J}$ that divides $x_i^s \frac{\mathbf{x}^\mu}{x_{n-j}^{\mu_{n-j}}}$. Because of $\nu_i \leq \deg B$ it is clear that we may choose $s \leq \deg B$ which finally shows that $\mathcal{J}$ is ℓ-quasi-stable and finishes our proof. □

[1] If such a term does not exist, then any term of $\mathcal{J}$ has an index less than $n-\ell$. Hence $\mathcal{J}$ is obviously ℓ-quasi-stable.

COROLLARY 5.1.5.
Let $\mathcal{J} \lhd \mathcal{P}$ *be a monomial and* ℓ*-quasi-stable ideal with* $\dim(\mathcal{P}/\mathcal{J}) = D$*. If* $\ell \geq D-1$ *then* $\mathcal{J}$ *is quasi-stable.*

PROOF. The assertion follows from Proposition 5.1.4 and Proposition 5.1.3, since for any integer j the equation $\mathcal{J} : x_{n-j}^{\infty} = \mathcal{J} : \langle x_1, \ldots, x_{n-j}\rangle^{\infty}$ implies the equation $\langle \mathcal{J}, x_n, \ldots, x_{n-j+1}\rangle : x_{n-j}^{\infty} = \langle \mathcal{J}, x_n, \ldots, x_{n-j+1}\rangle : \mathfrak{m}^{\infty}$ (compare proof of Proposition 5.1.3). $\square$

PROPOSITION 5.1.6.
Let $\mathcal{J} \lhd \mathcal{P}$ *be a monomial ideal,* B *its minimal basis and* ℓ *an integer. Then the following statements are equivalent:*

(i) $\mathcal{J}$ *is weakly* ℓ*-quasi-stable.*
(ii) Let $\mathbf{x}^{\mu} \in \mathcal{J}$ *with* $\mathrm{m}(\mathbf{x}^{\mu}) \geq n-\ell$ *and* $\mu_j > 0$ *for some* $n-\ell \leq j \leq n$*, then for each* $r \leq \mu_j$ *and* $i \leq n-\ell$ *an integer* $s \geq 0$ *exists such that* $x_i^s \frac{\mathbf{x}^{\mu}}{x_j^r}$ *lies in* $\mathcal{J}$*.*
(iii) For all $0 \leq j \leq \ell$ *holds:*

$$\mathcal{J} : x_{n-j}^{\infty} \subseteq \mathcal{J} : \langle x_1, \ldots, x_{n-\ell}\rangle^{\infty} \tag{5.6}$$

The following proof of Proposition 5.1.6 is essentially equal to the one of Proposition 5.1.4. We only have to do some minor adaptations, which are based on the difference between the definitions of ℓ-quasi-stable and weakly ℓ-quasi-stable concerning the index i.

PROOF.
"$(i) \Rightarrow (ii)$".
Assume first that $\mathcal{J}$ is weakly ℓ-quasi-stable. Let $\mathbf{x}^{\mu} \in \mathcal{J}$ be a term with $\mu_j > 0$ for some $n-\ell \leq j \leq n$, hence $k = \mathrm{m}(\mathbf{x}^{\mu}) \geq j$. Further, let r be an integer with $r \leq \mu_j$. We want to prove (ii) by showing that for all integers $i \leq n-\ell$ there is a term $\mathbf{x}^{\nu} \in \mathcal{J}$ which divides $x_i^{\deg B} \frac{\mathbf{x}^{\mu}}{x_j^r}$. By the definition of weak ℓ-quasi-stability $x_i^{\deg B} \frac{\mathbf{x}^{\mu}}{x_k^{\mu_k}} \in \mathcal{J}$ for $i \leq n-\ell$. Therefore there is a term $\mathbf{x}^{\nu^{(1)}} \in B$ with

$$\mathbf{x}^{\nu^{(1)}} \mid x_i^{\deg B} \frac{\mathbf{x}^{\mu}}{x_k^{\mu_k}} \tag{5.7}$$

and so $k_1 = \mathrm{m}(\mathbf{x}^{\nu^{(1)}}) \leq \mathrm{m}(x_i^{\deg B} \frac{\mathbf{x}^{\mu}}{x_k^{\mu_k}}) < k$. Obviously, $\nu_{\alpha}^{(1)} \leq \mu_{\alpha}$ for all $i \neq \alpha < k$ and $\nu_i^{(1)} \leq \mu_i + \deg B$. If $k_1 \leq i$, then $\mathbf{x}^{\nu^{(1)}}$ divides $x_i^{\deg B} \frac{\mathbf{x}^{\mu}}{x_{i+1}^{\mu_{i+1}} \cdots x_k^{\mu_k}}$, but since the latter monomial is a divisor of $x_i^{\deg B} \frac{\mathbf{x}^{\mu}}{x_j^r}$, we are already done in this case. Otherwise, we know by weak ℓ-quasi-stability, that $x_i^{\deg B} \frac{\mathbf{x}^{\nu^{(1)}}}{x_{k_1}^{\nu_{k_1}^{(1)}}} \in \mathcal{J}$ and so there is a term $\mathbf{x}^{\nu^{(2)}} \in B$ with $\mathbf{x}^{\nu^{(2)}} \mid x_i^{\deg B} \frac{\mathbf{x}^{\nu^{(1)}}}{x_{k_1}^{\nu_{k_1}^{(1)}}}$ and $\mathrm{m}(\mathbf{x}^{\nu^{(2)}}) = k_2 < k_1$. Therefore

$\mathbf{x}^{\nu^{(2)}} \mid x_i^{2\cdot\deg B}\frac{\mathbf{x}^{\mu}}{x_{k_1}^{\nu_{k_1}^{(1)}} x_k^{\mu_k}}$ since by (5.7):

$$\mathbf{x}^{\nu^{(2)}} \mid \left(\frac{x_i^{\deg B}}{x_{k_1}^{\nu_{k_1}^{(1)}}}\right)\mathbf{x}^{\nu^{(1)}} \mid \left(\frac{x_i^{\deg B}}{x_{k_1}^{\nu_{k_1}^{(1)}}}\right) x_i^{\deg B}\frac{\mathbf{x}^{\mu}}{x_k^{\mu_k}} = x_i^{2\cdot\deg B}\frac{\mathbf{x}^{\mu}}{x_{k_1}^{\nu_{k_1}^{(1)}} x_k^{\mu_k}}$$

This entails that

$$\mathbf{x}^{\nu^{(2)}} \mid x_i^{\deg B}\frac{\mathbf{x}^{\mu}}{x_{k_1}^{\mu_{k_1}}\cdots x_k^{\mu_k}} \tag{5.8}$$

because $\deg\mathbf{x}^{\nu^{(2)}} \le \deg B$ and $\mathrm{m}(\mathbf{x}^{\nu^{(2)}}) = k_2$. We go on like this until we end up at a term $\mathbf{x}^{\nu^{(\omega)}} \in B$ with $\mathbf{x}^{\nu^{(\omega)}} \mid x_i^{\deg B}\frac{\mathbf{x}^{\nu^{(\omega-1)}}}{x_{k_{\omega-1}}^{\nu_{k_{\omega-1}}^{(\omega-1)}}}$ and $\mathrm{m}(\mathbf{x}^{\nu^{(\omega)}}) = k_\omega < \cdots < k_1 < k$ such that $k_\omega < j$. Then the following holds:

- $\nu_\alpha^{(\omega)} = 0$ for all $\alpha \ge j > k_\omega$
- $\nu_\alpha^{(\omega)} \le \nu_\alpha^{(\omega-1)} \le \cdots \le \nu_\alpha^{(1)} \le \mu_\alpha$ for all $i \ne \alpha < k_\omega < j$
- $\nu_i^{(\omega)} \le \nu_i^{(\omega-1)} \le \cdots \le \nu_i^{(1)} \le \mu_i + \deg B$

Again with same arguments like above we are immediately done if $k_\omega \le i$, so we may assume $k_\omega > i$ and analogous to (5.7) and (5.8) we get:

$$\mathbf{x}^{\nu^{(\omega)}} \mid x_i^{\deg B}\frac{\mathbf{x}^{\mu}}{x_{k_\omega+1}^{\mu_{k_\omega+1}}\cdots x_k^{\mu_k}}$$

Since $k_\omega < j \le k$ this entails that $x_i^{\deg B}\frac{\mathbf{x}^{\mu}}{x_{k_\omega+1}^{\mu_{k_\omega+1}}\cdots x_k^{\mu_k}}$ divides $x_i^{\deg B}\frac{\mathbf{x}^{\mu}}{x_j^{r}}$ and so in particular $\mathbf{x}^{\nu^{(\omega)}}$ divides $x_i^{\deg B}\frac{\mathbf{x}^{\mu}}{x_j^{r}}$.

"*(ii)* ⇒ *(iii)*".
Now remember that $\mathcal{J} : x_{n-j}^{\infty}$ is a monomial ideal (see e.g. [**HH11**, Prop. 1.2.2/1.2.3]) and let B' be its monomial basis. Let statement (*ii*) hold and let t be an element of B', i.e. $x_{n-j}^{r}t \in \mathcal{J}$ for some integer r. Since $\mathrm{m}(x_{n-j}^{r}t) \ge n-j \ge n-\ell$ assertion (*ii*) entails that for all $i \le n-\ell \le \mathrm{m}(x_{n-j}^{r}t)$ there is an integer s_i such that the term $x_i^{s_i}\frac{x_{n-j}^{r}t}{x_{n-j}^{r}} = x_i^{s_i}t$ lies in $\mathcal{J}$. Hence $t\langle x_1,\dots,x_{n-\ell}\rangle^{(s_1+\cdots+s_{n-\ell-1}+r)(n-\ell)} \subseteq \mathcal{J}$ and so $t \in \mathcal{J} : \langle x_1,\dots,x_{n-\ell}\rangle^{\infty}$ which shows (*iii*).

"*(iii)* ⇒ *(i)*".
Finally, assume (5.6) holds and consider a term[2] $\mathbf{x}^{\mu} \in \mathcal{J}$ such that $\mathrm{m}(\mathbf{x}^{\mu}) = n-j$ with $j \le \ell$. Because of (5.6), we have $\frac{\mathbf{x}^{\mu}}{x_{n-j}^{\mu_{n-j}}} \in \mathcal{J} : x_{n-j}^{\infty} \subseteq \mathcal{J} : \langle x_1,\dots,x_{n-\ell}\rangle^{\infty}$. Hence there is an integer s such that $\frac{\mathbf{x}^{\mu}}{x_{n-j}^{\mu_{n-j}}}\langle x_1,\dots,x_{n-\ell}\rangle^{s} \subseteq \mathcal{J}$, but this means that $x_i^{s}\frac{\mathbf{x}^{\mu}}{x_{n-j}^{\mu_{n-j}}} \in \mathcal{J}$ for all $i \le n-\ell$. Hence there is a minimal generator $\mathbf{x}^{\nu}$ of $\mathcal{J}$ that

[2] If such a term does not exist, then any term of $\mathcal{J}$ has an index less than $n-\ell$. Hence $\mathcal{J}$ is obviously weakly ℓ-quasi-stable.

divides $x_i^s \frac{\mathbf{x}^\mu}{x_{n-j}^{\mu_{n-j}}}$. Because of $\nu_i \le \deg B$ it is clear that we may choose $s \le \deg B$ which finally shows that $\mathcal{J}$ is weakly ℓ-quasi-stable and finishes our proof. $\square$

DEFINITION 5.1.7.
An ideal $\mathcal{I} \lhd \mathcal{P}$ with $\dim(\mathcal{P}/\mathcal{I}) = D$ is in *Noether position* if $\Bbbk[x_{n-D+1}, \dots, x_n]$ is a *Noether normalization* of $\mathcal{P}/\mathcal{I}$, i.e. $\mathcal{P}/\mathcal{I}$ is a finitely generated $\Bbbk[x_{n-D+1}, \dots, x_n]$-module.

LEMMA 5.1.8 ([**BG01**, Lem. 4.1]).
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal with $\dim(\mathcal{P}/\mathcal{I}) = D$. Then the following statements are equivalent:

(i) $\mathcal{I}$ is in Noether position.
(ii) There are integers s_i such that $x_i^{s_i} \in \operatorname{lt}\mathcal{I}$ for all $i \le n - D$.
(iii) $\dim(\mathcal{P}/(\langle \mathcal{I}, x_n, \dots, x_{n-D+1}\rangle)) = 0$.
(iv) $\dim(\mathcal{P}/(\langle \operatorname{lt}\mathcal{I}, x_n, \dots, x_{n-D+1}\rangle)) = 0$.

REMARK 5.1.9.
Assume that the monomial ideal $\mathcal{J}$ with minimal basis B is weakly ℓ-quasi-stable for some ℓ and that $\mathbf{x}^\mu \in \mathcal{J}$. It follows immediately from Definition 5.1.1 that any term of the form $x_1^{\mu_1+\nu_1} \cdots x_{n-\ell}^{\mu_{n-\ell}+\nu_{n-\ell}}$ is also contained in $\mathcal{J}$, where every ν_i is either zero or a multiple of $\deg B$ and $\nu_1 + \dots + \nu_{n-\ell} = k \cdot \deg B$, where $k = \#\{\mu_j \mid j > n - \ell \text{ and } \mu_j > 0\}$.

PROPOSITION 5.1.10.
The monomial ideal $\mathcal{J} \lhd \mathcal{P}$ is weakly D-quasi-stable, if and only if it is in Noether position.

PROOF. Let $\mathcal{J}$ be weakly D-quasi-stable and assume that there exists a term $\mathbf{x}^\mu \in \mathcal{J} \cap \Bbbk[x_{n-D+1}, \dots, x_n]$, i.e. in particular $\mu_1 = \dots = \mu_{n-D} = 0$. Then Remark 5.1.9 immediately implies that there is an integer s such that $x_i^s \in \mathcal{J}$ for all $i \le n - D$ and we are done by Lemma 5.1.8.
If $\mathcal{J} \cap \Bbbk[x_{n-D+1}, \dots, x_n] = \emptyset$, then the D-dimensional cone $\Bbbk[x_{n-D+1}, \dots, x_n] \cdot 1$ lies completely in $\mathcal{P}/\mathcal{J}$. Assume that for some $i \le n - D$ no power of x_i was contained in $\mathcal{J}$. It is not possible that the $(D+1)$-dimensional cone $\Bbbk[x_i, x_{n-D+1}, \dots, x_n] \cdot 1$ lies completely in $\mathcal{P}/\mathcal{J}$, since then obviously $\dim(\mathcal{P}/\mathcal{J}) > D$. Thus we must have $\mathcal{J} \cap \Bbbk[x_i, x_{n-D+1}, \dots, x_n] \ne \emptyset$. But if a term $\mathbf{x}^\mu$ lies in this intersection, then again by Remark 5.1.9 there is an integer s such that $x_i^s \in \mathcal{J}$ in contradiction to our assumption.
Now let $\mathcal{J}$ be in Noether position. By Lemma 5.1.8 there are integers s_i such that $x_i^{s_i}$ is in $\mathcal{J}$ for all $i \le n - D$. Hence $\langle x_1, \dots, x_{n-D}\rangle^{s(n-D)} \subseteq \mathcal{J}$ with $s = \max_i s_i$ and therefore $\mathcal{J} : \langle x_1, \dots, x_{n-D}\rangle^\infty = \mathcal{P}$. This implies of course, that $\mathcal{J} : x_{n-j}^\infty \subseteq \mathcal{J} : \langle x_1, \dots, x_{n-D}\rangle^\infty$ for any j and we are done by Proposition 5.1.6. $\square$

Remark 5.1.11.
With an appropriate adaptation of line 2 it is easy to provide a weakly D-stable version of Algorithm 1. Hence using Proposition 5.1.10 we are able to deliver a deterministic method to put any ideal directly in Noether position, which represents an alternative to the approach described in [**Rob09**].

Furthermore, it is another notable consequence of Proposition 5.1.10 that now we are able to give a combinatorial characterization of the notion Noether position.

The following Algorithm 5 verifies whether a given monomial ideal is D-quasi-stable without a priori knowledge of the dimension D of $\mathcal{P}/\mathcal{J}$.

Algorithm 5 DQS-Test: Test for D-quasi-stability

Input: minimal basis $B = \{t_1, \ldots, t_r\}$ of monomial ideal $\mathcal{J} \lhd \mathcal{P}$
Output: The answer to: is $\mathcal{J}$ D-quasi-stable?
1: $\ell :=$ smallest j such that $x_\alpha^{\deg B} \in \mathcal{J}$ for $\alpha = 1, \ldots, n-j$
2: **for all** $\mathbf{x}^\mu \in B$ with $k := \mathrm{m}(\mathbf{x}^\mu) \geq n - \ell$ **do**
3:    **for** $i = 1, \ldots, k-1$ **do**
4:       **if** $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}} \notin \langle B \rangle$ **then**
5:          **return** false
6:       **end if**
7:    **end for**
8: **end for**
9: **return** true

For showing its correctness, we want to distinguish between the following three cases:

Case I: $\mathcal{J}$ is D-quasi-stable
Case II: $\mathcal{J}$ is not D-quasi-stable but in Noether position
Case III: $\mathcal{J}$ is neither D-quasi-stable nor in Noether position

In Case I we know by Proposition 5.1.10 that $\mathcal{J}$ is in Noether position. Hence the number ℓ computed in Line 1 equals D by Lemma 5.1.8. So by Definition 5.1.1 of D-quasi-stability we never get to Line 5.

If the second case is true, we get again $\ell = D$ by the same argument. But as $\mathcal{J}$ is not D-quasi-stable there must be an obstruction that leads us correctly to Line 5.

In the last case ℓ is greater[3] than D. Since $\mathcal{J}$ is not D-quasi-stable there exists a term $\mathbf{x}^\mu \in B$ with $k = \mathrm{m}(\mathbf{x}^\mu) \geq n - D > n - \ell$ such that $x_i^{\deg B} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}} \notin \mathcal{J}$ for some $i < k$. Our algorithm will detect this obstruction and thus gives the right answer.

[3] We know that $\ell \neq D$ since $\mathcal{J}$ is not in Noether position. The assumption $\ell < D$ leads to a contradiction because then $\dim(\mathcal{P}/\mathcal{J}) \leq n - (n - \ell) = \ell < D$.

This algorithm can be used to optimize the while loop in line 2 of the quasi-stable version of Algorithm 1. We know from Corollary 5.1.5 that instead of transforming to quasi-stable position, it is enough to achieve a D-quasi-stable position. So a combination of the quasi-stable version of Algorithm 1 and the DQS-Test, for which an a priori determination of D is not necessary, leads to a further ansatz to put an ideal into quasi-stable position.

5.2. Associating weakly D-stable Ideals with the Reduction Number

In this section we will now present some generalizations of the stable position and discuss their properties which correspond to the results of the previous section. Subsequently we will introduce the notion *weakly D-minimal stable* that leads us to a generalized version of Theorem 4.3.4.

Definition 5.2.1.
Let $\mathcal{J}$ be a monomial ideal and $0 \leq \ell < n$ an integer.

- $\mathcal{J}$ is *ℓ-stable* if for every term $\mathbf{x}^\mu \in \mathcal{J}$ with $\mathrm{m}(\mathbf{x}^\mu) = k \geq n - \ell$ and every $i < k$ the term $x_i \frac{\mathbf{x}^\mu}{x_k}$ also lies in $\mathcal{J}$.
- $\mathcal{J}$ is *weakly ℓ-stable* if for every term $\mathbf{x}^\mu \in \mathcal{J}$ with $\mathrm{m}(\mathbf{x}^\mu) = k \geq n - \ell$ and every $i \leq n - \ell$ the term $x_i \frac{\mathbf{x}^\mu}{x_k}$ also lies in $\mathcal{J}$.

Further, we call $\mathcal{J}$ *(weakly) D-stable* if $\mathcal{J}$ is (weakly) $\dim(\mathcal{P}/\mathcal{J})$-stable. Finally, a polynomial ideal is in *(weakly) ℓ-stable position* if its leading ideal is (weakly) ℓ-stable.

Remark 5.2.2.
It is easy to derive the following hierarchy immediately from the definition:

$$\mathcal{J} \text{ stable } \Rightarrow \mathcal{J} \ \ell\text{-stable } \Rightarrow \mathcal{J} \text{ weakly } \ell\text{-stable}$$

Compared with Definition 5.1.1 it is also clear that any (weakly) ℓ-stable ideal is (weakly) ℓ-quasi-stable. Again we can argue analogous to Lemma 2.2.4 to see that it is enough to verify the definition on the elements of the minimal basis of the ideal.

Example 5.2.3.
We consider the ideal $\mathcal{J}_1 \lhd \Bbbk[x_1, \ldots, x_6]$ with

$$\begin{aligned} \mathcal{J}_1 = \langle \quad & x_1,\ x_2^2,\ x_2x_3,\ x_2x_4,\ x_3x_4,\ x_4^2,\ x_3^3,\ x_3^2x_5,\ x_2x_5^2,\ x_3x_5^2,\ x_4x_5^2,\ x_5^3, \\ & x_3^2x_6^2,\ x_2x_5x_6^2,\ x_3x_5x_6^2,\ x_4x_5x_6^2,\ x_5^2x_6^2,\ x_2x_6^4,\ x_3x_6^4,\ x_4x_6^4,\ x_5x_6^4,\ x_6^6 \quad \rangle, \end{aligned}$$

which is the leading ideal of the fifth Katsura ideal. One can easily see that here $\dim(\mathcal{P}/\mathcal{J}_1) = 0$ and that $\mathcal{J}_1$ is D-stable since the defining property holds for the generators containing x_6. However, $\mathcal{J}_1$ is not stable, as for example $x_3x_4 \in \mathcal{J}_1$ but $x_3\frac{x_3x_4}{x_4} = x_3^2 \notin \mathcal{J}_1$.

Consider now the monomial ideal $\mathcal{J}_2 \lhd \Bbbk[x_1, \dots, x_5]$ with

$$\begin{aligned}\mathcal{J}_2 = \langle\ & x_1^2,\ x_1x_2^2,\ x_2^3,\ x_1x_2x_3^2,\ x_3^2x_2^2,\ x_1x_3^4,\ x_2x_3^4,\ x_3^5,\ x_1x_3^3x_4^2,\ x_2x_3^3x_4^2,\ x_3^4x_4^2,\\ & x_1x_2x_3x_4^4,\ x_2^2x_3x_4^4,\ x_1x_3^2x_4^4,\ x_2x_3^2x_4^4,\ x_3^3x_4^4,\ x_1x_2x_4^6,\ x_2^2x_4^6,\ x_1x_3x_4^6,\\ & x_1x_2x_3x_4^3x_5^2,\ x_1x_2x_4^5x_5^2,\ x_2^2x_4^5x_5^2,\ x_1x_3x_4^5x_5^3,\ x_2^2x_3x_4^3x_5^4,\\ & x_1x_3^2x_4^3x_5^5,\ x_2x_3^2x_4^3x_5^5,\ x_1x_2x_4^4x_5^6,\ x_1x_4^6x_5^5,\ x_2^2x_4^4x_5^7,\ x_2x_3x_4^5x_5^7 \quad \rangle\,.\end{aligned}$$

Since here $\dim(\mathcal{P}/\mathcal{J}_2) = 2$, we must check the defining property of a weakly D-stable ideal only for the terms containing x_3, x_4, x_5 and one readily verifies that $\mathcal{J}_2$ is weakly D-stable. However, it is not D-stable because $x_1x_4^6x_5^5 \in \mathcal{J}_2$ but $x_4\frac{x_1x_4^6x_5^5}{x_5} = x_1x_4^7x_5^4 \notin \mathcal{J}_2$.

Proposition 5.2.4 ([**HSS14**, Prop. 3.5]).
The monomial ideal $\mathcal{J} \lhd \mathcal{P}$ is ℓ-stable, if and only if it satisfies $\mathcal{J} : x_n = \mathcal{J} : \mathfrak{m}$ and for all $0 < j \leq \ell$

$$\langle \mathcal{J}, x_n, \dots, x_{n-j+1}\rangle : x_{n-j} = \langle \mathcal{J}, x_n, \dots, x_{n-j+1}\rangle : \mathfrak{m}\,. \tag{5.9}$$

Proof. Assume first that $\mathcal{J}$ is ℓ-stable and let t be a term such that $x_nt \in \mathcal{J}$. Then by the definition of ℓ-stable $x_i\frac{x_nt}{x_n} = x_it \in \mathcal{J}$ for all $i < n$ and therefore $t\mathfrak{m} \subseteq \mathcal{J}$, hence $\mathcal{J} : x_n = \mathcal{J} : \mathfrak{m}$. Further, let now t be a term such that

$$x_{n-j}t \in \langle \mathcal{J}, x_n, \dots, x_{n-j+1}\rangle \tag{5.10}$$

for some $j \leq \ell$. If $\mathrm{m}(t) > n-j$, then $t \in \langle x_n, \dots, x_{n-j+1}\rangle \subseteq \langle \mathcal{J}, x_n, \dots, x_{n-j+1}\rangle : \mathfrak{m}$ and we are done. Otherwise, $\mathrm{m}(x_{n-j}t) = n - j \geq n - \ell$ and then (5.10) entails $x_{n-j}t \in \mathcal{J}$. Because of the ℓ-stability, we have $x_i\frac{x_{n-j}t}{x_{n-j}} = x_it \in \mathcal{J}$ for all $i < n - j$. Hence $t\langle x_1, \dots, x_{n-j}\rangle \subseteq \mathcal{J}$ implying $t\mathfrak{m} \subseteq \langle \mathcal{J}, x_n, \dots, x_{n-j+1}\rangle$, which shows (5.9).

For the converse consider a term $t \in \mathcal{J}$ with $\mathrm{m}(t) = n - j \geq n - \ell$. So if $j = 0$ it follows from our assumption that $\frac{t}{x_n} \in \mathcal{J} : x_n = \mathcal{J} : \mathfrak{m}$. Hence $x_i\frac{t}{x_n} \in \mathcal{J}$ for all $i < n$. Otherwise, $0 < j \leq \ell$ and because of (5.9), we have:

$$\frac{t}{x_{n-j}} \in \mathcal{J} : x_{n-j} \subseteq \langle \mathcal{J}, x_n, \dots, x_{n-j+1}\rangle : x_{n-j} = \langle \mathcal{J}, x_n, \dots, x_{n-j+1}\rangle : \mathfrak{m}$$

Hence $x_i\frac{t}{x_{n-j}} \in \langle \mathcal{J}, x_n, \dots, x_{n-j+1}\rangle$ for all $i \leq n$. If $i < n - j$, then $\mathrm{m}(x_i\frac{t}{x_{n-j}}) \leq n - j$ and thus we must have $x_i\frac{t}{x_{n-j}} \in \mathcal{J}$ showing that $\mathcal{J}$ is ℓ-stable. $\square$

Corollary 5.2.5 ([**HSS14**, Cor. 3.6]).
Let $\mathcal{J} \lhd \mathcal{P}$ be a monomial ideal. Then $\mathcal{J}$ is quasi-stable, if it is D-stable.

Proof. According to the previous proposition, we have $\mathcal{J} : x_n = \mathcal{J} : \mathfrak{m}$ and (5.9) holds for all $0 < j \leq D$. By Proposition 5.1.3 it is enough to show that for all $0 < j < D$ the following equation hold:

$$\langle \mathcal{J}, x_n, \dots, x_{n-j+1}\rangle : x_{n-j}^\infty = \langle \mathcal{J}, x_n, \dots, x_{n-j+1}\rangle : \mathfrak{m}^\infty\,. \tag{5.11}$$

Indeed, if a term t lies in the ideal on the left hand side, then an integer s exists such that $x_{n-j}^s t \in \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle$ and therefore by (5.9):

$$x_{n-j}^{s-1} t \in \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle : x_{n-j} = \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle : \mathfrak{m}$$

Applying this argument a second time yields:

$$\begin{aligned} x_{n-j}^{s-2} t &\in \big(\langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle : \mathfrak{m} \big) : x_{n-j} \\ &= \big(\langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle : x_{n-j} \big) : \mathfrak{m} \\ &= \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle : \mathfrak{m}^2 \end{aligned}$$

Thus we find by iteration that $t \in \langle \mathcal{J}, x_n, \dots, x_{n-j+1} \rangle : \mathfrak{m}^s$ proving (5.11).

□

EXAMPLE 5.2.6.
Weak D-stability is not sufficient for quasi-stability, as one can see from the ideal $\mathcal{J}_1 = \langle x_1^2,\ x_1x_3 \rangle \lhd \Bbbk[x_1, x_2, x_3]$ where $D = \dim(\mathcal{P}/\mathcal{J}_1) = 2$. One easily verifies that it is weakly D-stable but not quasi-stable since $x_2^2 \frac{x_1x_3}{x_3} = x_1x_2^2 \notin \mathcal{J}_1$.

And the ideal $\mathcal{J}_2 = \langle x_1x_2,\ x_1^3 \rangle \lhd \Bbbk[x_1, x_2, x_3]$ where again $D = \dim(\mathcal{P}/\mathcal{J}_2) = 2$ shows that the converse of Corollary 5.2.5 does not hold, as it is quasi-stable but not (weakly) D-stable since $x_1 \frac{x_1x_2}{x_2} = x_1^2 \notin \mathcal{J}_2$.

PROPOSITION 5.2.7.
Let $\mathcal{J} \lhd \mathcal{P}$ be a monomial ideal. If $\mathcal{J}$ is weakly ℓ-stable, then it satisfies:

$$\langle \mathcal{J}, x_n, \dots, x_{n-\ell+1} \rangle : x_{n-\ell} = \langle \mathcal{J}, x_n, \dots, x_{n-\ell+1} \rangle : \mathfrak{m} \tag{5.12}$$

Conversely, $\mathcal{J}$ is weakly ℓ-stable if it satisfies for all $0 \le j \le \ell$:

$$\mathcal{J} : x_{n-j} \subseteq \mathcal{J} : \langle x_1, \dots, x_{n-\ell} \rangle \tag{5.13}$$

PROOF. Assume first that $\mathcal{J}$ is weakly ℓ-stable and let t be a term such that $x_{n-\ell} t \in \langle \mathcal{J}, x_n, \dots, x_{n-\ell+1} \rangle$. If $\mathrm{m}(t) > n - \ell$, then $t \in \langle x_n, \dots, x_{n-\ell+1} \rangle \subseteq \langle \mathcal{J}, x_n, \dots, x_{n-\ell+1} \rangle : \mathfrak{m}$ and we are done. Otherwise, $\mathrm{m}(x_{n-\ell} t) = n - \ell$ and we have $x_{n-\ell} t \in \mathcal{J}$. Because of the weak ℓ-stability, this entails that $x_i \frac{x_{n-\ell} t}{x_{n-\ell}} = x_i t \in \mathcal{J}$ for all $i \le n - \ell$. Hence $t \langle x_1, \dots, x_{n-\ell} \rangle \subseteq \mathcal{J}$ implying $t\mathfrak{m} \subseteq \langle \mathcal{J}, x_n, \dots, x_{n-\ell+1} \rangle$, which shows (5.12).

For the converse consider a term $t \in \mathcal{J}$ with $\mathrm{m}(t) = n - j \ge n - \ell$. Because of (5.13), we have $\frac{t}{x_{n-j}} \in \mathcal{J} : x_{n-j} \subseteq \mathcal{J} : \langle x_1, \dots, x_{n-\ell} \rangle$. Hence $x_i \frac{t}{x_{n-j}} \in \mathcal{J}$ for all $i \le n - \ell$ so that $\mathcal{J}$ is weakly ℓ-stable. □

The following example shows that the converse of the second part of Proposition 5.2.7 is not true:

EXAMPLE 5.2.8.
Let us consider the ideal $\mathcal{J} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3,\ x_1x_2x_3 \rangle \lhd \Bbbk[x_1, x_2, x_3]$. $\mathcal{J}$ is weakly 1-stable, but equation (5.13) does not hold for $j = 1$ since:

$$\mathcal{J} : x_2 = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3 \rangle \nsubseteq \langle x_1^2,\ x_1x_2,\ x_2^2 \rangle = \mathcal{J} : \langle x_1, x_2 \rangle$$

However, equation (5.12) does hold, since:

$$\langle \mathcal{J}, x_3 \rangle : x_2 = \langle x_3,\ x_1^2,\ x_1x_2,\ x_2^2 \rangle = \langle \mathcal{J}, x_3 \rangle : \mathfrak{m}$$

Corollary 5.2.9 ([**HSS14**, Prop. 3.9]).
Any weakly D-stable ideal is in Noether position.

Proof. Since a weakly D-stable ideal is always weakly D-quasi-stable, the assertion follows from Proposition 5.1.10. □

The following Algorithm 6 verifies, whether a given monomial ideal is weakly D-stable without a priori knowledge of the dimension D of $\mathcal{P}/\mathcal{I}$.

Algorithm 6 WDS-Test: Test for weak D-stability

Input: minimal basis $B = \{t_1, \dots, t_r\}$ of monomial ideal $\mathcal{J} \lhd \mathcal{P}$
Output: The answer to: is $\mathcal{J}$ weakly D-stable?
1: $\ell :=$ smallest j such that $x_\alpha^{\deg B} \in \mathcal{J}$ for $\alpha = 1, \dots, n-j$
2: **for all** $\mathbf{x}^\mu \in B$ with $k := \mathrm{m}(\mathbf{x}^\mu) \geq n - \ell$ **do**
3:     **for** $i = 1, \dots, n - \ell$ **do**
4:         **if** $x_i \frac{\mathbf{x}^\mu}{x_k} \notin \langle B \rangle$ **then**
5:             **return** false
6:         **end if**
7:     **end for**
8: **end for**
9: **return** true

For showing its correctness, we want to distinguish between the following three cases:

Case I: $\mathcal{J}$ is weakly D-stable
Case II: $\mathcal{J}$ is not weakly D-stable but in Noether position
Case III: $\mathcal{J}$ is neither weakly D-stable nor in Noether position

In Case I we know by Corollary 5.2.9 that $\mathcal{J}$ is in Noether position. Hence the number ℓ computed in Line 1 equals D by Lemma 5.1.8. So by Definition 5.2.1 of weak D-stability we never get to Line 5.

If the second case is true, we get again $\ell = D$ by the same argument. But as $\mathcal{J}$ is not weakly D-stable there must be an obstruction that leads us correctly to Line 5.

In the last case ℓ is greater[4] than D. Since $\mathcal{J}$ is not weakly D-stable there exists a term $\mathbf{x}^\mu \in B$ with $k = \mathrm{m}(\mathbf{x}^\mu) \geq n - D > n - \ell$ such that $x_i \frac{\mathbf{x}^\mu}{x_k} \notin \mathcal{J}$ for some $i \leq n - \ell$. Our algorithm will detect this obstruction and thus gives the right answer.

[4] We know that $\ell \neq D$ since $\mathcal{J}$ is not in Noether position. The assumption $\ell < D$ leads to a contradiction because then $\dim(\mathcal{P}/\mathcal{J}) \leq n - (n - \ell) = \ell < D$.

REMARK 5.2.10.
Assume that the monomial ideal $\mathcal{J}$ is weakly ℓ-stable for some ℓ and that $\mathbf{x}^\mu \in \mathcal{J}$. It follows immediately from Definition 5.2.1 that any term of the form $x_1^{\mu_1+\nu_1}\cdots x_{n-\ell}^{\mu_{n-\ell}+\nu_{n-\ell}}$ with $\nu_1+\cdots+\nu_{n-\ell} = \mu_{n-\ell+1}+\cdots+\mu_n$ is also contained in $\mathcal{J}$. If we introduce for $1 \le j \le \ell$ the homogeneous polynomials

$$g_j = \sum_{\nu_1^{(j)}+\cdots+\nu_{n-\ell}^{(j)}=\mu_{n-\ell+j}} a^{(j)}_{\nu_1^{(j)},\ldots,\nu_{n-\ell}^{(j)}} x_1^{\nu_1^{(j)}}\cdots x_{n-\ell}^{\nu_{n-\ell}^{(j)}}$$

with arbitrary coefficients $a^{(j)}_{\nu_1^{(j)},\ldots,\nu_{n-\ell}^{(j)}} \in \Bbbk$, then it follows from the observation above that the polynomial

$$f_\mu = x_1^{\mu_1}\cdots x_{n-\ell}^{\mu_{n-\ell}} g_1\cdots g_\ell$$

also lies in $\mathcal{J}$. Since each term in its support is of the form $x_1^{\mu_1+\nu_1}\cdots x_{n-\ell}^{\mu_{n-\ell}+\nu_{n-\ell}}$ with $\nu_1+\cdots+\nu_{n-\ell} = \deg(g_1\cdots g_\ell) = \deg g_1+\cdots+\deg g_\ell = \mu_{n-\ell+1}+\cdots+\mu_n$.

THEOREM 5.2.11 ([**HSS14**, Thm. 5.1]).
Let $\mathcal{J} \lhd \mathcal{P}$ be a weakly D-stable monomial ideal. Then $\mathcal{J}$ has a minimal generator x_{n-D}^s and $r(\mathcal{P}/\mathcal{J}) = r_{\mathcal{R}}(\mathcal{P}/\mathcal{J}) = s-1$ for any minimal reduction $\mathcal{R}$ of $\mathcal{J}$.

PROOF. Since $\mathcal{J}$ is assumed to be weakly D-stable, it follows from Corollary 5.2.9 and Lemma 5.1.8 that there are integers s_i such that:

$$x_i^{s_i} \in \mathcal{J} \text{ for all } i \le n-D \tag{5.14}$$

Hence $x_{n-D+1},\ldots,x_n$ induce a minimal reduction[5] of $\mathcal{J}$. We can apply Lemma 4.3.3 which shows that any minimal reduction is induced by D linear forms $z_i = x_{n-D+i} + \sum_{j=1}^{n-D} b_{ij}x_j$ with $1 \le i \le D$ and arbitrary coefficients $b_{ij} \in \Bbbk$. Let $\mathcal{R}_1 = \mathcal{J} + \langle z_1,\ldots,z_D\rangle$ and $\mathcal{R}_2 = \mathcal{J} + \langle x_{n-D+1},\ldots,x_n\rangle$. We want to prove that any minimal reduction leads to the same reduction number.

Claim 1: $r_{\mathcal{R}_1}(\mathcal{P}/\mathcal{J}) = r_{\mathcal{R}_2}(\mathcal{P}/\mathcal{J})$**.**
It is enough to show the identity $\mathcal{J}_1 = \mathcal{J}_2$ where $\mathcal{P}/\mathcal{R}_1 \cong \Bbbk[x_1,\ldots,x_{n-D}]/\mathcal{J}_1$ and $\mathcal{P}/\mathcal{R}_2 \cong \Bbbk[x_1,\ldots,x_{n-D}]/\mathcal{J}_2$. Thereby we interpret $\mathcal{J}_1 \lhd \Bbbk[x_1,\ldots,x_{n-D}]$ as the ideal that arises by replacing x_{n-D+i} with $-\sum_{j=1}^{n-D} b_{ij}x_j$ for all $1 \le i \le D$ in any element of $\mathcal{J}$ and – as one easily sees – we have $\mathcal{J}_2 = \mathcal{J} \cap \Bbbk[x_1,\ldots,x_{n-D}]$. Thus trivially $\mathcal{J}_2 \subseteq \mathcal{J}_1$, since the elements of $\mathcal{J}$, which do not involve the variables $x_{n-D+1},\ldots,x_n$ remain in $\mathcal{J}_1$.

For the converse inclusion $\mathcal{J}_1 \subseteq \mathcal{J}_2$ we should note that every element $f_\mu \in \mathcal{J}_1$ is induced by a term $\mathbf{x}^\mu \in \mathcal{J}$ such that f_μ is of the form $f_\mu = x_1^{\mu_1}\cdots x_{n-D}^{\mu_{n-D}} g_1\cdots g_D$ with $g_j = (-b_{j1}x_1 - \cdots - b_{j(n-D)}x_{n-D})^{\mu_{n-D+j}}$. It follows from Remark 5.2.10 that f_μ lies in $\mathcal{J}$ and – as obviously $f_\mu \in \Bbbk[x_1,\ldots,x_{n-D}]$ – we thus have $f_\mu \in \mathcal{J}_2$ proving Claim 1.

[5] Since $\dim(\mathcal{P}/\langle\mathcal{J}, x_{n-D+1},\ldots,x_n\rangle) = 0$ by [**KR00**, Prop. 3.7.1, Def. 3.7.2].

By (5.14) we know that $\mathcal{J}$ contains a power of x_{n-D}. Let s be minimal such that $x_{n-D}^s \in \mathcal{J}$.

Claim 2: $r_{\mathcal{R}_2}(\mathcal{P}/\mathcal{J}) = s-1$.
Since x_{n-D}^s is a minimal generator of $\mathcal{J}$, we have $x_{n-D}^{s-1} \notin \mathcal{R}_2$ and so in particular $(\mathcal{P}/\mathcal{R}_2)_{s-1} \neq 0$ showing that $r_{\mathcal{R}_2}(\mathcal{P}/\mathcal{J}) \geq s-1$.

On the other hand $x_{n-D}^s \in \mathcal{J}$ implies – because of the weak D-stability of $\mathcal{J}$ – that any term $x_1^{\mu_1}\cdots x_{n-D}^{\mu_{n-D}}$ with $\mu_1 + \cdots + \mu_{n-D} = s$ also belongs to $\mathcal{J}$. Thus $(\mathcal{P}/\mathcal{R}_2)_s = 0$ since $x_{n-D+1}, \ldots, x_n \in \mathcal{R}_2$ by construction. Therefore $r_{\mathcal{R}_2}(\mathcal{P}/\mathcal{J}) \leq s-1$ which finally entails that $r_{\mathcal{R}_2}(\mathcal{P}/\mathcal{J}) = s-1$. □

We have thus identified a class of monomial ideals, the weakly D-stable ideals, for which it is particularly simple to determine their reduction number. Given a polynomial ideal $\mathcal{I}$, we may use the weakly D-stable version of Algorithm 1 to render it and obtain then immediately the reduction number of its leading ideal $\operatorname{lt}\mathcal{I}$. According to Theorem 4.3.1, this number gives us an upper bound for $r(\mathcal{P}/\mathcal{I})$. We introduce now a more specialized class of ideals for which we can guarantee that $\mathcal{I}$ and $\operatorname{lt}\mathcal{I}$ have the same reduction number.

Notation 5.2.12.
We denote in the following for a monomial ideal $\mathcal{J}$ by

$$\deg_{x_k} \mathcal{J} = \max\{s \mid x_k^s \text{ divides a minimal generator of } \mathcal{J}\}$$

the maximal x_k-degree of a minimal generator of $\mathcal{J}$.

Definition 5.2.13.
Let $0 \leq \ell < n$ be an integer. The homogeneous ideal $\mathcal{I} \lhd \mathcal{P}$ is *weakly ℓ-minimal stable* if its leading ideal $\operatorname{lt}\mathcal{I}$ is weakly ℓ-stable and if for any linear change of coordinates $A \in \mathrm{GL}(n, \Bbbk)$ such that $\operatorname{lt}(A\cdot\mathcal{I})$ is still weakly ℓ-stable, we have $\deg_{x_{n-\ell}} \operatorname{lt}\mathcal{I} \leq \deg_{x_{n-\ell}} \operatorname{lt}(A\cdot\mathcal{I})$.

Proposition 5.2.14.
Let $\operatorname{char}\Bbbk = 0$*, then an ideal* $\mathcal{I}$ *is weakly D-minimal stable if it is in* gin*-position.*

Proof. $\operatorname{gin}\mathcal{I}$ is weakly D-stable by Theorem 2.4.4 and Proposition 2.4.5, since we assumed $\operatorname{char}\Bbbk = 0$. Further, let $A \in \mathrm{GL}(n, \Bbbk)$ be a linear change of coordinates such that $\operatorname{lt}(A\cdot\mathcal{I})$ is also weakly D-stable. Hence it follows from Theorem 5.2.11, Theorem 4.3.2 and Theorem 4.3.1 that:

$$\deg_{x_{n-D}} \operatorname{gin}\mathcal{I} = r(\mathcal{P}/\operatorname{gin}\mathcal{I}) = r(\mathcal{P}/\mathcal{I}) \leq r(\mathcal{P}/\operatorname{lt}(A\cdot\mathcal{I})) = \deg_{x_{n-D}} \operatorname{lt}(A\cdot\mathcal{I})$$

□

We will see in Example 5.2.16 that the converse of Proposition 5.2.14 is not true. But before we want to clarify the importance of the assumption $\operatorname{char}\Bbbk = 0$ in this context. Therefore the next example will show us that in positive characteristic even gin-position does not imply weak D-stability.

EXAMPLE 5.2.15.
Let $\mathcal{J} = \langle x_1^2,\ x_2^2\rangle \lhd \Bbbk[x_1, x_2]$ and $\operatorname{char}\Bbbk = 2$. Then $\mathcal{J}$ is Borel-fixed (see Example 2.4.7) and therefore $\mathcal{J} = \operatorname{gin}\mathcal{J}$ by Theorem 2.4.4. However, $\mathcal{J}$ is not weakly D-stable since $x_1\frac{x_2^2}{x_2} = x_1x_2 \notin \mathcal{J}$ and so in particular not weakly D-minimal stable.

EXAMPLE 5.2.16.
We consider for $n = 4$ the ideal

$$\mathcal{I} = \langle x_1x_4 - x_2x_3,\ x_2^3 - x_1x_3^2,\ x_2^2x_4 - x_1^3\rangle$$

which represents the special case $a = 2$, $b = 3$ of [**BH99**, Example 15]. Here $D = 2$ and the ideal $\mathcal{I}$ is not in weakly D-stable position since:

$$x_1\frac{x_2x_3}{x_3} = x_1x_2 \notin \operatorname{lt}\mathcal{I} = \langle x_2x_3,\ x_1^3,\ x_2^3,\ x_1x_3^3\rangle$$

The linear change of coordinates $\Psi : (x_2 \mapsto x_1 + x_2, x_3 \mapsto x_1 + x_3)$ transforms $\mathcal{I}$ into a weakly D-stable (in fact, even strongly stable) ideal $\tilde{\mathcal{I}} = \Psi(\mathcal{I})$ with leading ideal

$$\operatorname{lt}\tilde{\mathcal{I}} = \langle x_1^2,\ x_1x_2^2,\ x_2^3,\ x_1x_2x_3^2,\ x_1x_3^3,\ x_2^2x_3^3,\ x_2x_3^4\rangle\,.$$

Note that although this leading ideal is different from

$$\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2^2,\ x_2^3,\ x_1x_2x_3^2,\ x_2^2x_3^2,\ x_1x_3^4,\ x_2x_3^4\rangle\,,$$

both ideals have the same minimal generator x_2^3. Thus $\tilde{\mathcal{I}}$ is weakly D-minimal stable and we see that in this example the set of transformations leading to weakly D-minimal position is strictly larger than the one leading to the generic initial ideal.

THEOREM 5.2.17 ([**HSS14**, Thm. 5.5]).
Let $\operatorname{char}\Bbbk = 0$ *and* $\mathcal{I} \lhd \mathcal{P}$ *be a weakly D-minimal stable homogeneous ideal. Then* $\operatorname{lt}\mathcal{I}$ *has a minimal generator* x_{n-D}^s *and*

$$r(\mathcal{P}/\mathcal{I}) = r(\mathcal{P}/\operatorname{lt}\mathcal{I}) = \deg_{x_{n-D}}\operatorname{lt}\mathcal{I} - 1 = s - 1.$$

PROOF. Since $\operatorname{lt}\mathcal{I}$ is weakly D-stable, it possesses a minimal generator x_{n-D}^s and $r(\mathcal{P}/\operatorname{lt}\mathcal{I}) = s-1$ by Proposition 5.2.11. As $\operatorname{char}\Bbbk = 0$ $\operatorname{gin}\mathcal{I}$ is also weakly D-stable and thus has a minimal generator $x_{n-D}^{s'}$. As $\mathcal{I}$ and $\operatorname{gin}\mathcal{I}$ are both weakly D-minimal stable we must have $s = s'$. Hence $r(\mathcal{P}/\mathcal{I}) = r(\mathcal{P}/\operatorname{gin}\mathcal{I}) = s-1$ by Theorem 4.3.2 and Proposition 5.2.11. $\square$

REMARK 5.2.18.
Unfortunately, Theorem 5.2.17 is mainly of theoretical interest, as we are not able to provide a simple deterministic algorithm for the construction of a change of coordinates leading to a weakly D-minimal stable position.

If we consider again Example 4.2.3, we see that the leading ideal $\operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_1x_3,\ x_2^3,\ x_2^2x_3\rangle$ is even strongly stable and thus of course weakly D-stable (with $D = 1$ here). However, $\mathcal{I}$ is not weakly D-minimal stable, as $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3^2\rangle$ and thus $\deg_{x_2}\operatorname{gin}\mathcal{I} = 2 < 3 = \deg_{x_2}\operatorname{lt}\mathcal{I}$. Hence analogous to Remark 4.3.7 it is not enough to analyze the leading ideal in order to verify whether $\mathcal{I}$ is weakly D-minimal stable or not.

Remark 5.2.19.
Analogous to Remark 2.2.8 it is clear that any ideal can be put in (weakly) ℓ-quasi-stable or (weakly) ℓ-stable position by using the corresponding version of Algorithm 1. However, we should note that this algorithm only works if $\operatorname{char}\Bbbk = 0$.

5.3. Generalization of Borel-fixed Position

As a final step of our "generalization procedure", we will now have a look at the Borel-fixed position, which we introduced in Section 2.4. Especially if the considered field has positive characteristic this position becomes more important.

With the same method of the preceding sections we are now able to connect the newly defined (weakly) ℓ-quasi-stable position with a position we call *(weakly) ℓ-Borel-fixed*. Therefore we have to refine the Borel group $\mathfrak{B}$ of Definition 2.4.1. Furthermore, using another new stable position allows us the formulation of a correspondingly adapted version of certain results we presented in Section 2.4.

The weakly ℓ-Borel-fixed Position.

Definition 5.3.1.
The subgroup $\mathfrak{B}_\ell^w = \{A = (a_{ij}) \in \mathfrak{B} \mid a_{ij} = 0 \text{ if } j > n-\ell \text{ or } i < n-\ell\} \subseteq \operatorname{Gl}(n,\Bbbk)$ is called *weak ℓ-Borel group*.

So a matrix $A \in \mathfrak{B}_\ell^w$ of the weak ℓ-Borel group is of the form

$$A = \left(\begin{array}{ccc|c|c} & \mathbb{1}_1 & & 0 & 0 \\ \hline a_{n-\ell,1} & \cdots & a_{n-\ell,n-\ell-1} & 1 & 0\cdots 0 \\ \hline & & \mathfrak{a} & & \mathbb{1}_2 \end{array}\right),$$

where $\mathbb{1}_1 \in \operatorname{Gl}(n-\ell-1,\Bbbk)$, $\mathbb{1}_2 \in \operatorname{Gl}(\ell,\Bbbk)$ are identity matrices and $\mathfrak{a}$ is a $\ell \times n-\ell$ matrix of the form:

$$\mathfrak{a} = \begin{pmatrix} a_{n-\ell+1,1} & \cdots & a_{n-\ell+1,n-\ell} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n-\ell} \end{pmatrix}$$

Definition 5.3.2.
A monomial ideal $\mathcal{J}$ is *weakly ℓ-Borel-fixed* if $\mathcal{J} = A \cdot \mathcal{J}$ for all $A \in \mathfrak{B}_\ell^w$. We say that a polynomial ideal $\mathcal{I}$ is in *weakly Borel-fixed position* if $\operatorname{lt}\mathcal{I}$ is weakly ℓ-Borel-fixed.

Analogous to Proposition 2.4.6 we receive the following proposition considering the notion weakly ℓ-Borel-fixed.

Proposition 5.3.3.
Let $\mathcal{J}$ a monomial ideal and B its monomial basis then the following statements are equivalent:

(i) *$\mathcal{J}$ is weakly ℓ-Borel-fixed.*
(ii) *For every $\mathbf{x}^\mu \in B$ with $\mathrm{m}(\mathbf{x}^\mu) \geq n-\ell$, every $j \geq n-\ell$ with $\mu_j > 0$ and every $i \leq n-\ell$ the term $x_i^{\mu_j-u}\frac{\mathbf{x}^\mu}{x_j^{\mu_j-u}}$ also lies in $\mathcal{J}$ for all integers u that satisfy:*

$$\begin{array}{ll} \binom{\mu_j}{u} \neq 0, & \text{if } \operatorname{char} \Bbbk = 0 \\ \binom{\mu_j}{u} \not\equiv 0 \mod p, & \text{if } \operatorname{char} \Bbbk = p > 0 \end{array} \tag{5.15}$$

The following proof of this proposition is inspired by the proof of [**Eis95**, Thm. 15.23].

Proof. Let $A = (a_{kl})$ be a $n \times n$ matrix with $a_{kk} = 1$ for all k and $a_{ji} = a \neq 0$ for some integers $i < j$ such that $i \leq n-\ell \leq j$. Further, let $a_{kl} = 0$ for all $k \neq l$ and $(k,l) \neq (j,i)$. Then we have:

$$A \cdot \mathbf{x}^\mu = (x_j + a x_i)^{\mu_j} \frac{\mathbf{x}^\mu}{x_j^{\mu_j}} = \sum_{u=0}^{\mu_j} \binom{\mu_j}{u} a^{\mu_j-u} x_i^{\mu_j-u} \frac{\mathbf{x}^\mu}{x_j^{\mu_j-u}} \tag{5.16}$$

The weak ℓ-Borel group is generated by nonsingular diagonal matrices and matrices of the form of A. Since any monomial ideal is invariant under transformations induced by diagonal matrices (see [**Eis95**, Thm. 15.23]), it is enough to show that $\mathcal{J}$ is weakly ℓ-Borel-fixed, if and only if it is invariant under A. Now let $\mathbf{x}^\mu \in B$ be a term that is affected by the transformation A, i.e. $\mu_j > 0$ (in particular we have in this case $\mathrm{m}(\mathbf{x}^\mu) \geq j \geq n-\ell$). Since a polynomial lies in a monomial ideal, if and only if every term of its support lies in the ideal, we can derive by (5.16):

$$A \cdot \mathbf{x}^\mu \in \mathcal{J} \quad \Leftrightarrow \quad x_i^{\mu_j-u} \frac{\mathbf{x}^\mu}{x_j^{\mu_j-u}} \in \mathcal{J} \text{ for all integer } u \text{ that satisfy (5.15)}$$

□

Introducing another stable position, we can formulate a generalized version of Proposition 2.4.5 and Theorem 2.4.4.

Definition 5.3.4.
Let $\mathcal{J}$ be a monomial ideal and $0 \leq \ell < n$ an integer. $\mathcal{J}$ is *weakly ℓ-strongly stable*, if for every term $\mathbf{x}^\mu \in \mathcal{J}$ with $\mathrm{m}(\mathbf{x}^\mu) \geq n-\ell$, every $j \geq n-\ell$ with $\mu_j > 0$ and every $i \leq n-\ell$ the term $x_i \frac{\mathbf{x}^\mu}{x_j}$ also lies in $\mathcal{J}$.

Corollary 5.3.5.
Let $\mathcal{J} \lhd \mathcal{P}$ be a monomial ideal. Then $\mathcal{J}$ is weakly ℓ-Borel-fixed if it is weakly ℓ-strongly stable.

Conversely, if $\mathcal{J}$ is weakly ℓ-Borel-fixed and $\operatorname{char} \Bbbk = 0$*, then $\mathcal{J}$ is weakly ℓ-strongly stable.*

Proof. This is a direct consequence of Definition 5.3.4 and Proposition 5.3.3. □

Example 5.3.6.
Let $\operatorname{char} \Bbbk = 2$ and $\mathcal{J} = \langle x_1^2,\ x_2^2,\ x_4^2 \rangle \lhd \Bbbk[x_1, x_2, x_3, x_4]$ be a monomial ideal. We now want to verify that $\mathcal{J}$ is weakly 2-Borel-fixed. According to Proposition 5.3.3 it is enough to check the generators x_2^2 and x_4^2. Further, since $\binom{2}{u} \not\equiv 0 \mod 2$ holds for $u = 0$ or $u = 2$, we only consider the case $u = 0$ as the other case is "uninteresting". Hence $\mathcal{J}$ is weakly 2-Borel-fixed because of:

$$x_1^2 \frac{x_2^2}{x_2^2} = x_1^2 \in \mathcal{J}, \quad x_1^2 \frac{x_4^2}{x_4^2} = x_1^2 \in \mathcal{J}, \quad x_2^2 \frac{x_4^2}{x_4^2} = x_2^2 \in \mathcal{J}$$

However, $\mathcal{J}$ is not weakly 2-strongly stable since $x_1 \frac{x_4^2}{x_4} = x_1 x_4 \notin \mathcal{J}$.

Corollary 5.3.7.
Any weakly ℓ-Borel-fixed ideal is weakly ℓ-quasi-stable.

Proof. Let $\mathcal{J}$ be a monomial weakly ℓ-Borel-fixed ideal and B its monomial basis. Further, let $\mathbf{x}^\mu \in B$ with $\mathrm{m}(\mathbf{x}^\mu) = k \geq n - \ell$. Since the binomial coefficient $\binom{\mu_k}{0}$ is equal to 1, it is nonzero in any characteristic. Therefore by Proposition 5.3.3 the term $x_i^{\mu_k} \frac{\mathbf{x}^\mu}{x_k^{\mu_k}}$ lies in $\mathcal{J}$ for every $i \leq n - \ell$. Hence any element of B and so by Remark 5.1.2 any element of $\mathcal{J}$ fulfills the condition for weak ℓ-quasi-stability. □

Obviously, it follows from Proposition 5.1.10 that any weakly D-Borel-fixed ideal is in Noether position. Hence we can generalize the first part of Theorem 4.3.4 from Bresinsky and Hoa, so that we only have to assume an ideal to be in weakly D-Borel-fixed position in order to follow that it contains a power of x_{n-D}.

The ℓ-Borel-fixed Position.

Definition 5.3.8.
The subgroup $\mathfrak{B}_\ell = \{A = (a_{ij}) \in \mathfrak{B} \mid a_{ij} = 0 \text{ if } j < n - \ell \text{ and } i \neq j\} \subseteq \mathrm{Gl}(n, \Bbbk)$ is called *ℓ-Borel group.*

So a matrix $A \in \mathfrak{B}_\ell$ of the ℓ-Borel group is of the form $A = \left(\begin{array}{c|c} \mathfrak{a}_1 & 0 \\ \hline \mathfrak{a}_2 & \mathbb{1} \end{array}\right)$, where $\mathbb{1} \in \mathrm{Gl}(\ell, \Bbbk)$ is the identity matrices and $\mathfrak{a}_1$ respectively $\mathfrak{a}_2$ are $n-\ell \times n-\ell$ respectively $\ell \times n-\ell$ matrices of the form:

$$\mathfrak{a}_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_{2,1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ & & & 0 \\ a_{n-\ell,1} & \cdots & a_{n-\ell,n-\ell-1} & 1 \end{pmatrix}, \; \mathfrak{a}_2 = \begin{pmatrix} a_{n-\ell+1,1} & \cdots & a_{n-\ell+1,n-\ell} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n-\ell} \end{pmatrix}$$

Definition 5.3.9.
A monomial ideal $\mathcal{J}$ is *ℓ-Borel-fixed* if $\mathcal{J} = A \cdot \mathcal{J}$ for all $A \in \mathfrak{B}_\ell$. We say that a polynomial ideal $\mathcal{I}$ is in *Borel-fixed position* if $\operatorname{lt} \mathcal{I}$ is ℓ-Borel-fixed.

Proposition 5.3.10.
Let $\mathcal{J}$ a monomial ideal and B its monomial basis then the following statements are equivalent:

(i) $\mathcal{J}$ is ℓ-Borel-fixed.

(ii) For every $\mathbf{x}^\mu \in B$ with $\mathrm{m}(\mathbf{x}^\mu) \geq n-\ell$, every $j \geq n-\ell$ with $\mu_j > 0$ and every $i < j$ the term $x_i^{\mu_j - u}\frac{\mathbf{x}^\mu}{x_j^{\mu_j-u}}$ also lies in $\mathcal{J}$ for all integers u that satisfy:

$$\begin{array}{ll} \binom{\mu_j}{u} \neq 0, & \text{if } \operatorname{char} \Bbbk = 0 \\ \binom{\mu_j}{u} \not\equiv 0 \mod p, & \text{if } \operatorname{char} \Bbbk = p > 0 \end{array} \tag{5.17}$$

We leave out the proof for this proposition since it is essentially equal to the one of Proposition 5.3.3.

Again we can introduce another stable position analogous to Definition 5.3.4 that allows us to formulate a generalized version of Proposition 2.4.5 and Theorem 2.4.4 concerning the notion ℓ-Borel-fixed.

Definition 5.3.11.
Let $\mathcal{J}$ be a monomial ideal and $0 \leq \ell < n$ an integer. $\mathcal{J}$ is *ℓ-strongly stable*, if for every term $\mathbf{x}^\mu \in \mathcal{J}$ with $\mathrm{m}(\mathbf{x}^\mu) \geq n-\ell$, every $j \geq n-\ell$ with $\mu_j > 0$ and every $i < j$ the term $x_i \frac{\mathbf{x}^\mu}{x_j}$ also lies in $\mathcal{J}$.

Corollary 5.3.12 ([**HSS14**, Prop. 3.4]).
Let $\mathcal{J} \lhd \mathcal{P}$ be a monomial ideal. Then $\mathcal{J}$ is ℓ-Borel-fixed, if it is ℓ-strongly stable. Conversely, if $\mathcal{J}$ is ℓ-Borel-fixed and $\operatorname{char} \Bbbk = 0$*, then $\mathcal{J}$ is ℓ-strongly stable.*

Proof. This is a direct consequence of Definition 5.3.11 and Proposition 5.3.10. $\square$

In summary, it is easy to derive the following hierarchy immediately from the Propositions 2.4.6, 5.3.3 and 5.3.10, the Definitions 2.2.1, 5.3.4 and 5.3.11 as well as Proposition 2.4.5 and the Corollaries 5.3.5 and 5.3.12:

$$\begin{array}{ccccc}
\mathcal{J} \text{ Borel-fixed} & \Rightarrow & \mathcal{J}\ \ell\text{-Borel-fixed} & \Rightarrow & \mathcal{J} \text{ weakly } \ell\text{-Borel-fixed} \\
\Uparrow \quad (\Downarrow)^6 & & \Uparrow \quad (\Downarrow)^6 & & \Uparrow \quad (\Downarrow)^6 \\
\mathcal{J} \text{ strongly stable} & \Rightarrow & \mathcal{J}\ \ell\text{-strongly stable} & \Rightarrow & \mathcal{J} \text{ weakly } \ell\text{-strongly stable}
\end{array}$$

Example 5.3.13.
Let $\operatorname{char} \Bbbk = 3$ and $\mathcal{J} = \langle x_1x_2,\ x_1^3,\ x_2^3,\ x_3^3\rangle \lhd \Bbbk[x_1, x_2, x_3]$ be a monomial ideal. We now want to verify that $\mathcal{J}$ is 0-Borel-fixed. According to Proposition 5.3.10 it is enough to check the generator x_3^3. Further, since $\binom{3}{u} \not\equiv 0 \mod 3$ holds for $u = 0$ or $u = 3$, we only consider the case $u = 0$ as the other case is "uninteresting". Hence $\mathcal{J}$ is 0-Borel-fixed because of:

$$x_1^3\frac{x_3^3}{x_3^3} = x_1^3 \in \mathcal{J}, \quad x_2^3\frac{x_3^3}{x_3^3} = x_2^3 \in \mathcal{J}$$

However, $\mathcal{J}$ is not 0-strongly stable since $x_1\frac{x_3^3}{x_3} = x_1x_3^2 \notin \mathcal{J}$. Moreover, $x_1\frac{x_1x_2}{x_2} = x_1^2 \notin \mathcal{J}$ causing that $\mathcal{J}$ is also not Borel-fixed by Proposition 2.4.6.

Example 5.3.14.
In Example 5.3.6 we saw that the ideal $\mathcal{J} = \langle x_1^2,\ x_2^2,\ x_4^2\rangle \lhd \Bbbk[x_1, x_2, x_3, x_4]$ is weakly 2-Borel-fixed under the assumption that $\operatorname{char} \Bbbk = 2$. However, it is easy to verify that $\mathcal{J}$ is not 2-Borel-fixed since $x_3^2\frac{x_4^2}{x_4^2} = x_3^2 \notin \mathcal{J}$.

Analogous to Corollary 5.3.7 the next corollary is a consequence of the Propositions 5.3.10 and the fact that the binomial coefficient $\binom{\mu_k}{0}$ is nonzero in any characteristic.

Corollary 5.3.15.
Any ℓ-Borel-fixed ideal is ℓ-quasi-stable.

Finally, we note that with an appropriate[7] adaptation of the condition of the while loop in line 2 resulting from Proposition 5.3.3 respectively 5.3.10, we can use Algorithm 2 to put any ideal in weakly ℓ-Borel-fixed respectively ℓ-Borel-fixed position. Analogously, a suitably[8] modified version of Algorithm 1 let us transform a given ideal into (weakly) ℓ-strongly stable position.

[6]This implication only holds if $\operatorname{char} \Bbbk = 0$.
[7]Compare with Theorem 2.4.11.
[8]Recall the discussion in Remark 2.2.17.

CHAPTER 6

β-maximal Ideals

In the context of computing the reduction number we saw in Remark 4.3.5 how to obtain this invariant of ideals that are in gin-position by using results of Trung and Bresinsky/Hoa. As a next step we defined the notion weakly D-minimal stable in Section 5.2 that induced a generalization of this method.

Looking for further alternatives we will introduce the concept of *β-maximal* ideals in this chapter. Unfortunately, we will see that β-maximality does not deliver any improvements concerning the determination of the reduction number, as this property combined with weak D-stability delivers only a sufficient criterion for weak D-minimal stability (see Proposition 6.3.1). However, β-maximal ideals possess several remarkable algebraic properties that motivate a deeper investigation, especially in relation to Pommaret bases.

6.1. Connection to Pommaret Basis

To figure out the relation between β-maximality and quasi-stability, we make use of the *Hilbert function* of Pommaret cones that we associate with the corresponding *β-vector*. Afterwards we describe a technique to verify, whether a given ideal is in β-maximal position or not.

Definition 6.1.1.
Let $\mathcal{I}$ be an ideal, then we set $B_q(\mathcal{I}) = (\operatorname{lt}\mathcal{I})_q \cap \mathbb{T}$. Further, we define the *β-vector of $\mathcal{I}$ in degree q* by

$$\beta_q(\mathcal{I}) = (\beta_q^{(1)}(\mathcal{I}), \dots, \beta_q^{(n)}(\mathcal{I})),$$

where $\beta_q^{(k)}(\mathcal{I}) = \#\{t \in B_q(\mathcal{I}) \mid \mathrm{m}(t) = k\}$.

Remark 6.1.2.
In the following we list some simple facts concerning the notations from the above definition:

- $B_q(\mathcal{I})$ is the monomial basis of the ideal $\langle B_q(\mathcal{I})\rangle = \langle(\operatorname{lt}\mathcal{I})_q\rangle$
- $\displaystyle\sum_{k=1}^{n} \beta_q^{(k)}(\mathcal{I}) = \#B_q(\mathcal{I}) = \dim_{\mathbb{k}}(\langle B_q(\mathcal{I})\rangle)_q$
- $\#B_q(\mathcal{I}) = \#B_q(A \cdot \mathcal{I})$ for any $A \in \mathrm{Gl}(n, \mathbb{k})$

Definition 6.1.3.
We call an ideal $\mathcal{I}$ in *β-maximal position* if for every $A \in \mathrm{Gl}(n, \Bbbk)$ and all $q > 0$ holds:

$$\beta_q(\mathcal{I}) \succeq_{\mathrm{lex}} \beta_q(A \cdot \mathcal{I})$$

Thereby $\prec_{\mathrm{lex}}$ denotes the *lexicographical*[1] term ordering.

Notation 6.1.4.
An important and well-known function in algebraic geometry is the *Hilbert function*. For a given ideal $\langle F \rangle = \mathcal{I} \lhd \mathcal{P}$ it is defined by:

$$h_{\mathcal{I}} : \mathbb{N} \to \mathbb{N}, \quad s \mapsto \dim_{\Bbbk}(\mathcal{I}_s)$$

Analogous to [**Sei10**, 4.3] we use for $s \geq 0$ the notation:

$$h_{F,\mathscr{P}}(s) = \dim_{\Bbbk}(\langle F \rangle_{\mathscr{P}})_s$$

Remark 6.1.5.
Let $\langle F \rangle = \mathcal{I} \lhd \mathcal{P}$ denote a polynomial ideal where F is autoreduced. Some properties of the Hilbert function are presented below:

First we should note that the Hilbert function is invariant under coordinate transformation, i.e. $h_{\mathcal{I}}(s) = h_{A \cdot \mathcal{I}}(s)$ for any $A \in \mathrm{Gl}(n, \Bbbk)$ since $\dim_{\Bbbk}(\mathcal{I}_s) = \dim_{\Bbbk}(A \cdot \mathcal{I})_s$. Also there is no difference between the Hilbert function of $\mathcal{I}$ and the one of its leading ideal $\operatorname{lt} \mathcal{I}$.

Especially if one considers degrees $s \leq \deg F$, it is worth to mention that $\dim_{\Bbbk}(\mathcal{I}_s) = \dim_{\Bbbk}(\langle \mathcal{I}_{s-1} \rangle_s) + \# F_s$, where F_s denotes the set of those elements contained in F that are of degree s.

Further, by the definition of the Pommaret span (see Definition 3.1.1), we know that $\langle F \rangle_{\mathscr{P}} \subseteq \langle F \rangle$ implying $h_{F,\mathscr{P}}(s) \leq h_{\mathcal{I}}(s)$. Obviously, $h_{F,\mathscr{P}}(s) = h_{\mathcal{I}}(s)$ if F is a Pommaret basis of $\mathcal{I}$ since then $\langle F \rangle_{\mathscr{P}} = \mathcal{I}$. Moreover, if $h_{F,\mathscr{P}}(s) = h_{\mathcal{I}}(s)$ we have $(\langle F \rangle_{\mathscr{P}})_s = (\langle F \rangle)_s$.

Finally, let $q \geq \operatorname{reg}(\operatorname{lt} \mathcal{I})$, $r \geq 0$ and $\beta^{(k)}_{F_q} = \#\{f \in F_q \mid \mathrm{m}(f) = k\}$. Then following [**Sei10**, Rem. 4.3.7] there are coefficients $\mathtt{C}_{ik} \in \Bbbk$ such that:

$$h_{F,\mathscr{P}}(q + r) = \sum_{i=0}^{n-1} \left(\sum_{k=i}^{n-1} \mathtt{C}_{ik} \beta^{(n-k)}_{F_q} \right) r^i \tag{6.1}$$

The explicit formula for these coefficients is $\mathtt{C}_{ik} = \frac{s^{(k)}_{k-i}(0)}{k!}$, where $s^{(k)}_{k-i}(0)$ denotes the modified Stirling numbers (see [**Sei10**, A.4] for more details). Therefore the coefficients $\mathtt{C}_{ik}$ are positive and invariant under coordinate transformations. Of course $h_{F,\mathscr{P}}$ itself does change under coordinate transformation, as the following example shows.

[1] Following [**LA94**, Def. 1.4.2] we set $\mathbf{x}^{\mu} \prec_{\mathrm{lex}} \mathbf{x}^{\nu}$, if and only if $\mu_m < \nu_m$ with $m = \min\{i \mid \mu_i \neq \nu_i\}$.

Example 6.1.6.
Let $\mathcal{J} = \langle x_1x_2\rangle \lhd \Bbbk[x_1, x_2]$, then $\operatorname{reg}(\mathcal{J}) = 2$ and $h_{\mathcal{J}}(3) = 2$. As $\mathcal{J}$ is not quasi-stable, it is clear that $F = \{x_1x_2\}$ is not a Pommaret basis of $\mathcal{J}$. Hence we must have $h_{F,\mathscr{P}}(3) < h_{\mathcal{J}}(3)$. Indeed, formula (6.1) delivers:

$$h_{F,\mathscr{P}}(2+1) = \mathtt{C}_{00}\beta^{(2)}_{F_2} + \mathtt{C}_{01}\beta^{(1)}_{F_2} + \mathtt{C}_{11}\beta^{(1)}_{F_2} = \frac{s^{(0)}_0(0)}{0!}\cdot 1 + \frac{s^{(1)}_1(0)}{1!}\cdot 0 + \frac{s^{(1)}_0(0)}{1!}\cdot 0 \overset{2}{=} 1$$

Performing the coordinate transformation $\Psi : (x_2 \mapsto x_2 + x_1)$ on $\mathcal{J}$, yields to the ideal $\tilde{\mathcal{J}} = \Psi(\mathcal{J}) = \langle x_1^2 + x_1x_2\rangle$, which is in quasi-stable position. Further, $\tilde{F} = \{x_1^2 + x_1x_2\}$ is a Pommaret basis of $\tilde{\mathcal{J}}$ (compare Example 3.1.9), so that now $h_{\tilde{F},\mathscr{P}}(3) = h_{\tilde{\mathcal{J}}}(3) = h_{\mathcal{J}}(3)$ since

$$h_{\tilde{F},\mathscr{P}}(2+1) = \mathtt{C}_{00}\beta^{(2)}_{\tilde{F}_2} + \mathtt{C}_{01}\beta^{(1)}_{\tilde{F}_2} + \mathtt{C}_{11}\beta^{(1)}_{\tilde{F}_2} = \frac{s^{(0)}_0(0)}{0!}\cdot 0 + \frac{s^{(1)}_1(0)}{1!}\cdot 1 + \frac{s^{(1)}_0(0)}{1!}\cdot 1 \overset{2}{=} 2.$$

In the following we investigate how the maximality of the β-vector $\beta_q(\mathcal{I})$ with respect to $\prec_{\text{lex}}$ correlates with $h_{F,\mathscr{P}}(q+r)$ being maximal under coordinate change. Therefore we want to ensure the appearance of $\beta^{(k)}_q(\mathcal{I})$ in formula (6.1). Since obviously $\beta^{(k)}_{F_q} = \beta^{(k)}_q(\mathcal{I})$ if $F = B_q(\mathcal{I})$, we now turn our attention to the ideal $\langle B_q(\mathcal{I})\rangle$ and summarize some facts referring to this ideal in the next remark.

Remark 6.1.7.
The first thing we should note is that $h_{\mathcal{I}}(s) = \dim_{\Bbbk}(\mathcal{I}_s) = h_{\langle B_q(\mathcal{I})\rangle}(s)$ for all $s \geq q$. So in terms of the Hilbert function it does not make a difference whether we consider $\mathcal{I}$ or $\langle B_q(\mathcal{I})\rangle$ as long as we look at degrees higher than $q-1$.

Analogous to (6.1) we can formulate the following equation containing the entries of the β-vector $\beta_q(\mathcal{I})$:

$$h_{B_q(\mathcal{I}),\mathscr{P}}(q+r) = \sum_{i=0}^{n-1}\left(\sum_{k=i}^{n-1}\mathtt{C}_{ik}\beta^{(n-k)}_q(\mathcal{I})\right) r^i \tag{6.2}$$

Further, if $q_1 \leq q_2$ and $s \geq q_2$ we have:

$$(\langle B_{q_1}(\mathcal{I})\rangle_{\mathscr{P}})_s \subseteq (\langle B_{q_2}(\mathcal{I})\rangle_{\mathscr{P}})_s \tag{6.3}$$

To prove (6.3), let us consider a term $t_s \in (\langle B_{q_1}(\mathcal{I})\rangle_{\mathscr{P}})_s$. There is a term $t_{q_1} \in B_{q_1}(\mathcal{I})$ and $\mathbf{x}^\nu \in \Bbbk[x_{\mathrm{m}(t_{q_1})}, \ldots, x_n]$ such that $\mathbf{x}^\nu t_{q_1} = t_s$ and $\deg \mathbf{x}^\nu = s - q_1$. But t_{q_1} also lies in $(\operatorname{lt}\mathcal{I})_{q_1}$ and hence:

$$\mathbf{x}^\mu t_{q_1} \in (\operatorname{lt}\mathcal{I})_{q_2} \text{ for all } \mu \text{ with } \deg \mathbf{x}^\mu = q_2 - q_1 \tag{6.4}$$

As $s \geq q_2$, we can choose an exponent $\hat{\mu}$ that fulfills (6.4) and $\mathbf{x}^{\hat{\mu}} |_{\mathscr{P}} \mathbf{x}^\nu$. This finally entails that $\mathbf{x}^{\hat{\mu}} t_{q_1} \in B_{q_2}(\mathcal{I})$ is a Pommaret divisor of $\mathbf{x}^\nu t_{q_1} = t_s$ and so in particular $t_s \in \langle B_{q_2}(\mathcal{I})\rangle_{\mathscr{P}}$.

---

[2] It follows from (A.40) and (A.42b) of [**Sei10**, A.4] that $s^{(0)}_0(0) = s^{(1)}_1(0) = s^{(1)}_0(0) = 1$.

Before we are able to present an important result of this section we need to formulate a slightly modified version of the Lemmas 3.2.19 and 3.2.24 from [**Sei09b**]. Thereby we should recall that by Remark 3.2.18:

$$(\operatorname{lt}\mathcal{I})_{\geq q} = (\operatorname{lt}\mathcal{I})_{\langle q\rangle} = \langle(\operatorname{lt}\mathcal{I})_q\rangle = \langle B_q(\mathcal{I})\rangle, \quad \text{if } q \geq \operatorname{reg}(\operatorname{lt}\mathcal{I})$$

Proposition 6.1.8 ([**Sei09b**, Lem. 2.2,Prop. 9.6]).
If the ideal $\mathcal{I}$ is in quasi-stable position, then for any degree $q \geq \operatorname{reg}(\operatorname{lt}\mathcal{I})$ the ideal $\langle B_q(\mathcal{I})\rangle$ is stable.

Conversely, if for any degree $q \geq \operatorname{reg}(\operatorname{lt}\mathcal{I})$ the ideal $\langle B_q(\mathcal{I})\rangle$ is quasi-stable, then so is $\operatorname{lt}\mathcal{I}$.

Corollary 6.1.9.
Let $\mathcal{I} \lhd \mathcal{P}$ be an ideal and $q \geq \operatorname{reg}(\operatorname{lt}\mathcal{I})$ then the following statements are equivalent:

(i) $\operatorname{lt}\mathcal{I}$ *is quasi-stable*
(ii) For all integers $s > q$ holds: $h_{B_q(\mathcal{I}),\mathscr{P}}(s) = h_{\mathcal{I}}(s)$
(iii) There is an integer $\hat{s} > q$ such that: $h_{B_q(\mathcal{I}),\mathscr{P}}(\hat{s}) = h_{\mathcal{I}}(\hat{s})$

Proof.
"*(i)* ⇒ *(ii)*".
Let us first assume that $\operatorname{lt}\mathcal{I}$ is quasi-stable then the ideal $\langle B_q(\mathcal{I})\rangle$ is stable by Proposition 6.1.8 and it follows from Lemma 3.2.3 that $B_q(\mathcal{I})$ is its Pommaret basis. But this means that $\langle B_q(\mathcal{I})\rangle_{\mathscr{P}} = \langle B_q(\mathcal{I})\rangle$ and therefore $h_{B_q(\mathcal{I}),\mathscr{P}}(s) = h_{\mathcal{I}}(s)$ for all $s > q$.

Since *(ii)* trivially implies *(iii)*, we assume now that *(iii)* holds.
"*(iii)* ⇒ *(i)*".
So if *(iii)* holds, we can derive:

$$(\langle B_q(\mathcal{I})\rangle)_{\hat{s}} \underset{\text{Rem. 6.1.5}}{\overset{(iii)}{=}} (\langle B_q(\mathcal{I})\rangle_{\mathscr{P}})_{\hat{s}} \overset{(6.3)}{\subseteq} (\langle B_{\hat{s}-1}(\mathcal{I})\rangle_{\mathscr{P}})_{\hat{s}} \subseteq (\langle B_{\hat{s}-1}(\mathcal{I})\rangle)_{\hat{s}} = (\langle B_q(\mathcal{I})\rangle)_{\hat{s}}$$

This entails that $(\langle B_{\hat{s}-1}(\mathcal{I})\rangle_{\mathscr{P}})_{\hat{s}} = (\langle B_{\hat{s}-1}(\mathcal{I})\rangle)_{\hat{s}}$ and so $x_j t \in \langle B_{\hat{s}-1}(\mathcal{I})\rangle_{\mathscr{P}}$ for all indices j and all $t \in B_{\hat{s}-1}(\mathcal{I})$. Hence $B_{\hat{s}-1}(\mathcal{I})$ is a Pommaret basis of $\langle B_{\hat{s}-1}(\mathcal{I})\rangle$ by Proposition 3.1.7. Finally, it follows from Theorem 3.1.3 and Proposition 6.1.8 that $\operatorname{lt}\mathcal{I}$ is quasi-stable since $\hat{s} - 1 \geq q \geq \operatorname{reg}(\mathcal{I})$, which finishes our proof. □

Proposition 6.1.10.
An ideal $\mathcal{I}$ is in quasi-stable position, if and only if for every $A \in \operatorname{Gl}(n, \Bbbk)$ and all $q \geq \operatorname{reg}(\operatorname{lt}\mathcal{I})$ holds:

$$\beta_q(\mathcal{I}) \succeq_{\text{lex}} \beta_q(A \cdot \mathcal{I})$$

Proof. Let us assume there is an integer $q \geq \operatorname{reg}(\operatorname{lt}\mathcal{I})$ and a matrix $A \in \operatorname{Gl}(n, \Bbbk)$ such that $\beta_q(\mathcal{I}) \prec_{\text{lex}} \beta_q(A \cdot \mathcal{I})$. Then by the definition of the lexicographic term order there is an integer $\hat{\ell}$ such that $\beta_q^{(n-\hat{\ell})}(\mathcal{I}) < \beta_q^{(n-\hat{\ell})}(A \cdot \mathcal{I})$ and $\beta_q^{(n-\ell)}(\mathcal{I}) = \beta_q^{(n-\ell)}(A \cdot \mathcal{I})$ for all $\ell > \hat{\ell}$. Let us use the notation

$$b_i^{\mathcal{I}} = \sum_{k=i}^{n-1} \mathtt{C}_{ik} \beta_q^{(n-k)}(\mathcal{I})$$

such that – analogous to equation (6.2) – $h_{B_q(\mathcal{I}),\mathscr{P}}(q+r) = \sum_{i=0}^{n-1} b_i^{\mathcal{I}} r^i$. The coefficients $\mathtt{C}_{ik}$ are positive and invariant under coordinate transformation – as we already mentioned in Remark 6.1.5 – so that $b_{\hat{\ell}}^{\mathcal{I}} < b_{\hat{\ell}}^{A \cdot \mathcal{I}}$ and $b_\ell^{\mathcal{I}} = b_\ell^{A\cdot\mathcal{I}}$ for all $\ell > \hat{\ell}$. In particular, this entails $(b_0^{\mathcal{I}}, \ldots, b_{n-1}^{\mathcal{I}}) \prec_{\text{lex}} (b_0^{A\cdot\mathcal{I}}, \ldots, b_{n-1}^{A\cdot\mathcal{I}})$. If $r \gg 0$, then $\sum_{i=0}^{n-1} b_i^{\mathcal{I}} r^i < \sum_{i=0}^{n-1} b_i^{A\cdot\mathcal{I}} r^i$, if and only if $(b_0^{\mathcal{I}}, \ldots, b_{n-1}^{\mathcal{I}}) \prec_{\text{lex}} (b_0^{A\cdot\mathcal{I}}, \ldots, b_{n-1}^{A\cdot\mathcal{I}})$. Hence for $r \gg 0$ we can follow – using again Remark 6.1.5 – that

$$h_{B_q(\mathcal{I}),\mathscr{P}}(q+r) < h_{B_q(A\cdot\mathcal{I}),\mathscr{P}}(q+r) \leq h_{A\cdot\mathcal{I}}(q+r) = h_{\mathcal{I}}(q+r)$$

and so by Corollary 6.1.9 ($\neg(ii) \Rightarrow \neg(i)$) $\operatorname{lt}\mathcal{I}$ is not quasi-stable.

For the converse let us assume that $\operatorname{lt}\mathcal{I}$ is not quasi-stable. Let $q \geq \operatorname{reg}(\operatorname{lt}\mathcal{I})$ then by Corollary 6.1.9 ($\neg(i) \Rightarrow \neg(iii)$) $h_{B_q(\mathcal{I}),\mathscr{P}}(s) < h_{\mathcal{I}}(s)$ for all $s > q$. Let $A \in \operatorname{Gl}(n, \Bbbk)$ be a transformation matrix such that $A \cdot \mathcal{I}$ is in quasi-stable position. We can use Corollary 6.1.9 ($(i) \Rightarrow (ii)$) again to see that

$$h_{B_q(A\cdot\mathcal{I}),\mathscr{P}}(s) = h_{A\cdot\mathcal{I}}(s) \overset{\text{Rem.6.1.5}}{=} h_{\mathcal{I}}(s) > h_{B_q(\mathcal{I}),\mathscr{P}}(s)$$

for all $s > q$. If $s \gg q$ it follows analogous to the above argumentation that there must be an integer $\hat{\ell}$ such that $\beta_q^{(n-\hat{\ell})}(\mathcal{I}) < \beta_q^{(n-\hat{\ell})}(A\cdot\mathcal{I})$ and $\beta_q^{(n-\ell)}(\mathcal{I}) = \beta_q^{(n-\ell)}(A\cdot\mathcal{I})$ for all $\ell > \hat{\ell}$. Since this is equivalent to $\beta_q(\mathcal{I}) \prec_{\text{lex}} \beta_q(A \cdot \mathcal{I})$ we are done. □

Corollary 6.1.11.
Any ideal that is in β-maximal position, is in quasi-stable position.

The converse of the corollary above is not true, as we can see in the next example.

Example 6.1.12.
Let $\mathcal{J} = \langle x_1^2,\ x_2^2,\ x_3^2\rangle \lhd \Bbbk[x_1, x_2, x_3]$, then $\mathcal{J}$ is quasi-stable since it is zero-dimensional (see Lemma 3.2.1 and Theorem 3.1.3). But $\mathcal{J}$ is not in β-maximal position since $\operatorname{gin}\mathcal{J} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3^2,\ x_2x_3^2,\ x_3^4\rangle$ and therefore:

$$\beta_2(\mathcal{J}) = (1, 1, 1) \prec_{\text{lex}} (1, 2, 0) = \beta_2(\operatorname{gin}\mathcal{J})$$

PROPOSITION 6.1.13.
An ideal $\mathcal{I}$ is in β-maximal position if it is in gin*-position. In particular, if $\mathcal{I}$ is in β-maximal position then $\beta_q(\operatorname{gin}\mathcal{I}) = \beta_q(\mathcal{I})$ for all $q \geq 0$.*

PROOF. To proof this assertion we should recall Theorem 1.3.4 and the fact that the term order we use is index respecting by Remark 1.3.8.

Indeed, in the proof of Theorem 1.3.4 we construct a Zariski open set $\mathcal{U}$ such that $\operatorname{lt}(A\cdot\mathcal{I}) = \operatorname{gin}\mathcal{I}$ for all $A \in \mathcal{U}$. Thereby we choose $\mathcal{U}$ such that for any $A \in \mathcal{U}$

$$\mathcal{V}_{q,k}(A) = \dim_{\Bbbk}(\langle(\operatorname{lt}(A\cdot\mathcal{I}))_q\rangle_{\Bbbk} \cap \langle t_1,\dots,t_k\rangle_{\Bbbk}) \tag{6.5}$$

is maximal[3] in every degree q and all $k \leq s = \dim_{\Bbbk}(\mathcal{P}_q)$, where $\{t_1,\dots,t_s\}$ is a $\Bbbk$-basis of $\mathcal{P}_q$ with $t_1 \succ \cdots \succ t_s$. Now let $\mathscr{L}(B_q(\operatorname{gin}\mathcal{I})) = (\tilde{t}_1,\dots,\tilde{t}_\ell)$ and $\mathscr{L}(B_q(A'\cdot\mathcal{I})) = (\hat{t}_1,\dots,\hat{t}_\ell)$ for some arbitrary $A' \in \operatorname{Gl}(n,\Bbbk)$ Thereby we should note that it follows from Remark 6.1.2 that $\ell = \#B_q(\operatorname{gin}\mathcal{I}) = \#B_q(A'\cdot\mathcal{I})$. Then the maximality[4] of (6.5) entails that:

$$\tilde{t}_i \succeq \hat{t}_i \quad \text{for all } i \tag{6.6}$$

Since our term order $\prec$ is index respecting it follows from (6.6) that:

$$\operatorname{m}(\tilde{t}_i) \leq \operatorname{m}(\hat{t}_i) \quad \text{for all } i \tag{6.7}$$

If $\operatorname{m}(\tilde{t}_i) = \operatorname{m}(\hat{t}_i)$ for all i, then of course $\beta_q(\operatorname{gin}\mathcal{I}) = \beta_q(A'\cdot\mathcal{I})$. Otherwise, we find an index j such that $\tilde{k} = \operatorname{m}(\tilde{t}_j) < \operatorname{m}(\hat{t}_j)$. Choosing j minimal leads together with (6.7) to the situation that $\beta_q^{(\tilde{k})}(\operatorname{gin}\mathcal{I}) > \beta_q^{(\tilde{k})}(A'\cdot\mathcal{I})$ and $\beta_q^{(i)}(\operatorname{gin}\mathcal{I}) = \beta_q^{(i)}(A'\cdot\mathcal{I})$ for all $i < \tilde{k}$. Thus it follows by the definition of the lexicographical term order that $\beta_q(\operatorname{gin}\mathcal{I}) \succ_{\text{lex}} \beta_q(A'\cdot\mathcal{I})$. Hence we finally showed that $\beta_q(\operatorname{gin}\mathcal{I}) \succeq_{\text{lex}} \beta_q(A'\cdot\mathcal{I})$ for all $q \geq 0$ and all $A' \in \operatorname{Gl}(n,\Bbbk)$. □

REMARK 6.1.14.
Combining some results of this section, we can now formulate a procedure to verify whether a given ideal $\mathcal{I}$ is in β-maximal position or not.

At first we check if $\mathcal{I}$ is in quasi-stable position, since if this is not the case we know by Corollary 6.1.11 that $\mathcal{I}$ can not be in β-maximal position. Otherwise, if the ideal is in quasi-stable position, then it follows from Proposition 6.1.10 and Proposition 6.1.13 that:

$$\beta_q(\operatorname{gin}\mathcal{I}) = \beta_q(\mathcal{I}), \quad \text{for all } q \geq \operatorname{reg}(\mathcal{I}) \tag{6.8}$$

As a further consequence of Proposition 6.1.13 we know that $\mathcal{I}$ is in β-maximal position, if and only if the β-vectors of $\mathcal{I}$ coincide with the one of $\operatorname{gin}\mathcal{I}$. So by (6.8) we only have to compare the finitely many β-vectors of $\mathcal{I}$ and $\operatorname{gin}\mathcal{I}$ for all degrees lower than $\operatorname{reg}(\mathcal{I})$.

[3] Recall Remark 1.3.5 and note that if $B \notin \mathcal{U}$, then for at least some values of q and k we have $\dim_{\Bbbk}\mathcal{V}_{q,k}(B) < \dim_{\Bbbk}\mathcal{V}_{q,k}(A)$ for any $A \in \mathcal{U}$.

[4] Note that the maximality holds for any choice of k.

Using the method described in Remark 6.1.14, we can now present an example which shows that β-maximal position does not imply gin-position. Hence the converse of Proposition 6.1.13 is not true.

Example 6.1.15.
Let us consider the ideal $\mathcal{J} = \langle x_1^2,\ x_2^2\rangle \lhd \Bbbk[x_1, x_2]$. $\mathcal{J}$ is quasi-stable and as $\operatorname{gin}\mathcal{J} = \langle x_1^2,\ x_1x_2,\ x_2^3\rangle$, we know that $\operatorname{reg}(\mathcal{J}) = 3$ (see Theorem 1.3.9). Hence we only have to compare the β-vectors of degree 2 in order to check whether $\mathcal{J}$ is in β-maximal position or not. As $\beta_2(\mathcal{J}) = (1, 1) = \beta_2(\operatorname{gin}\mathcal{J})$, we can immediately follow that $\mathcal{J}$ is in β-maximal position.

6.2. Criterion for minimal Length of Pommaret bases in three Variables

We know from Proposition 3.1.6 that the number of elements a Pommaret basis of a given ideal has is unique. This number – which we call *length of a Pommaret basis* – will be investigated in the following section.

Considering polynomial rings with at most three variables, we will prove in this section a one-to-one correspondence between β-maximal position and Pommaret bases of minimal length.

Proposition 6.2.1.
Let $\mathcal{I} \lhd \Bbbk[x_1, \ldots, x_n]$ an ideal and $n \leq 3$. Further, let $\mathcal{I}$ be in quasi-stable position and $A \in \operatorname{Gl}(n, \Bbbk)$ such that $\tilde{\mathcal{I}} = A \cdot \mathcal{I}$ is also in quasi-stable position. Let H be a Pommaret basis of $\mathcal{I}$, $\tilde{H}$ a Pommaret basis of $\tilde{\mathcal{I}}$ and $q < \operatorname{reg}(\mathcal{I})$ be an integer, then the following statements are equivalent:

(i) $h_{B_q(\mathcal{I}),\mathscr{P}}(q+1) < h_{B_q(\tilde{\mathcal{I}}),\mathscr{P}}(q+1)$

(ii) $\beta_q(\mathcal{I}) \prec_{\text{lex}} \beta_q(\tilde{\mathcal{I}})$

(iii) $\#H_{q+1} > \#\tilde{H}_{q+1}$, *using the notation* $H_i = \{h \in H \mid \deg h = i\}$

Further, the equivalence is preserved if one replaces the inequality signs of (i)−(iii) by equality signs.

Proof.
"(i) ⇔ (ii)".
We only consider the case $n = 3$ since the proof for $n = 1, 2$ is similar.

Since $\operatorname{lt}\mathcal{I}$ and $\operatorname{lt}\tilde{\mathcal{I}}$ are quasi-stable, we must have $\beta_q^{(1)}(\mathcal{I}) = \beta_q^{(1)}(\tilde{\mathcal{I}}) = 1$ by Lemma 3.2.4. So let $\beta_q(\mathcal{I}) = (1, a, b)$ and $\beta_q(\tilde{\mathcal{I}}) = (1, c, d)$ for some $a, b, c, d \in \mathbb{N}$. By Remark 6.1.2 $\sum_{k=1}^{3} \beta_q^{(k)}(\mathcal{I}) = \#B_q(\mathcal{I}) = \#B_q(\tilde{\mathcal{I}}) = \sum_{k=1}^{3} \beta_q^{(k)}(\tilde{\mathcal{I}})$ and therefore:

$$a + b = c + d \tag{6.9}$$

Now we consider $h_{B_q(\mathcal{I}),\mathscr{P}}(q+1)$, which presents the number of elements of $B_{q+1}(\mathcal{I})$ that have a Pommaret divisor in $B_q(\mathcal{I})$. Those elements are of the form $x_j t$, where $t \in B_q(\mathcal{I})$ and $j \geq \mathrm{m}(t)$. Hence we get:

$$\begin{aligned} h_{B_q(\mathcal{I}),\mathscr{P}}(q+1) &= \#\{x_j t \mid t \in B_q(\mathcal{I}),\ \mathrm{m}(t) \leq j \leq n\} \\ &= \sum_{k=1}^{n} \#\{j \mid k \leq j \leq n\} \cdot \#\{t \in B_q(\mathcal{I}) \mid \mathrm{m}(t) = k\} \\ &= \sum_{k=1}^{n} (n-k+1) \cdot \beta_q^{(k)}(\mathcal{I}) \end{aligned}$$

This entails for our situation:

$$h_{B_q(\mathcal{I}),\mathscr{P}}(q+1) = 3 + 2a + b, \qquad h_{B_q(\tilde{\mathcal{I}}),\mathscr{P}}(q+1) = 3 + 2c + d \tag{6.10}$$

Now we can derive that statement (i) is by (6.10) equivalent to

$$\begin{aligned} & 3 + 2a + b < 3 + 2c + d \\ \Leftrightarrow\ & a < c + (c+d) - (a+b) \\ \overset{(6.9)}{\Leftrightarrow}\ & a < c \end{aligned}$$

On the other side statement (ii)

$$\beta_q(\mathcal{I}) = (1, a, b) \prec_{\mathrm{lex}} (1, c, d) = \beta_q(\tilde{\mathcal{I}})$$

is equivalent to either $a < c$ or $a = c$ and $b < d$. As the second case is impossible because of (6.9), we see that both statements are equivalent to $a < c$, which proves the assertion. Analogously, it is easy to see that both of the statements $h_{B_q(\mathcal{I}),\mathscr{P}}(q+1) = h_{B_q(\tilde{\mathcal{I}}),\mathscr{P}}(q+1)$ and $\beta_q(\mathcal{I}) = \beta_q(\tilde{\mathcal{I}})$ are equivalent to $a = c$.

"$(i) \Leftrightarrow (iii)$".
Since we assume that $\langle \mathrm{lt}\, H \rangle_{\mathscr{P}} = \mathrm{lt}\, \mathcal{I}$ and $\langle \mathrm{lt}\, \tilde{H} \rangle_{\mathscr{P}} = \mathrm{lt}\, \tilde{\mathcal{I}}$ Remark 6.1.5 implies on the one hand that $h_{H,\mathscr{P}} = h_{\mathcal{I}} = h_{\tilde{\mathcal{I}}} = h_{\tilde{H},\mathscr{P}}$ and on the other hand:

$$\begin{aligned} h_{H,\mathscr{P}}(q+1) &= \dim_{\Bbbk}(\langle H \rangle_{\mathscr{P}})_{q+1} \\ &= \dim_{\Bbbk}(\langle (\langle H \rangle_{\mathscr{P}})_q \rangle_{\mathscr{P}})_{q+1} + \#H_{q+1} \\ &= \dim_{\Bbbk}(\langle (\langle \mathrm{lt}\, H \rangle_{\mathscr{P}})_q \cap \mathbb{T} \rangle_{\mathscr{P}})_{q+1} + \#H_{q+1} \\ &= \dim_{\Bbbk}(\langle (\mathrm{lt}\, \mathcal{I})_q \cap \mathbb{T} \rangle_{\mathscr{P}})_{q+1} + \#H_{q+1} \\ &= h_{B_q(\mathcal{I}),\mathscr{P}}(q+1) + \#H_{q+1} \end{aligned}$$

Combining those two results delivers the following decisive equation:

$$h_{B_q(\mathcal{I}),\mathscr{P}}(q+1) + \#H_{q+1} = h_{B_q(\tilde{\mathcal{I}}),\mathscr{P}}(q+1) + \#\tilde{H}_{q+1}$$

Consequently, $\#H_{q+1} > \#\tilde{H}_{q+1}$ is equivalent to $h_{B_q(\mathcal{I}),\mathscr{P}}(q+1) < h_{B_q(\tilde{\mathcal{I}}),\mathscr{P}}(q+1)$ as well as $\#H_{q+1} = \#\tilde{H}_{q+1}$ is equivalent to $h_{B_q(\mathcal{I}),\mathscr{P}}(q+1) = h_{B_q(\tilde{\mathcal{I}}),\mathscr{P}}(q+1)$. $\square$

In the following we present an example which shows that it is not possible to extend Proposition 6.2.1 to the case $n = 4$.

EXAMPLE 6.2.2.
Let $\mathcal{I} = \langle x_1^4,\ x_1^3x_2 - x_1^3x_3,\ x_2^4,\ x_3^4,\ x_1^2x_2x_4,\ x_1x_2^2x_4\rangle + \langle x_1, x_2, x_3, x_4\rangle^6 \lhd \Bbbk[x_1, x_2, x_3, x_4]$ and $\Psi_1 : (x_3 \mapsto x_3 + x_2)$, $\Psi_2 : (x_4 \mapsto x_4 + x_3)$ coordinate transformations. Then the leading ideal of $\mathcal{I}$ is

$$\operatorname{lt}\mathcal{I} = \langle x_1^4,\ x_1^3x_2,\ x_2^4,\ x_3^4,\ x_1^2x_2x_4,\ x_1x_2^2x_4,\ x_1^3x_3x_4\rangle + \langle x_1, x_2, x_3, x_4\rangle^6$$

and if we set $\tilde{\mathcal{I}} = \Psi_2(\Psi_1(\mathcal{I}))$, then its leading ideal is

$$\operatorname{lt}\tilde{\mathcal{I}} = \langle x_1^4,\ x_2^4,\ x_1^3x_3,\ x_1^2x_2x_3,\ x_1x_2^2x_3,\ x_2^3x_3,\ x_1x_2x_3^3,\ x_2^2x_3^3,\ x_1^3x_2x_4\rangle + \langle x_1, x_2, x_3, x_4\rangle^6.$$

Obviously, $\operatorname{reg}(\mathcal{I}) = 6$ and for the β-vectors in degree 4 we have the following relationship:

$$\beta_4(\tilde{\mathcal{I}}) = (1, 1, 4, 0) \prec_{\mathrm{lex}} (1, 2, 1, 2) = \beta_4(\mathcal{I}) \tag{6.11}$$

Since $\dim(\mathcal{P}/\mathcal{I}) = \dim(\mathcal{P}/\tilde{\mathcal{I}}) = 0$ we know by Lemma 3.2.1 that $\mathcal{I}$ and $\tilde{\mathcal{I}}$ are quasi-stable and hence they have a finite Pommaret basis. So let H denote the Pommaret basis of $\mathcal{I}$. As we are only interested in the number of elements of H_i it is enough to consider the elements of $(\operatorname{lt} H)_i$:

i	$(\operatorname{lt} H)_i$
4	$x_1^4,\ x_1^3x_2,\ x_2^4,\ x_3^4,\ x_1^2x_2x_4,\ x_1x_2^2x_4$
5	$\boldsymbol{x_1x_2^4},\ \boldsymbol{x_1x_3^4},\ \boldsymbol{x_2x_3^4},\ \boldsymbol{x_1^2x_2^2x_4},\ \boldsymbol{x_1x_2^3x_4},\ x_1^3x_3x_4,\ \boldsymbol{x_1^2x_2x_3x_4},\ \boldsymbol{x_1x_2^2x_3x_4}$
6	$\boldsymbol{x_1^2x_2^4},\ x_1^2x_2^3x_3,\ x_1^2x_2^2x_3^2,\ x_1x_2^3x_3^2,\ x_1^3x_3^3,\ x_1^2x_2x_3^3,\ x_1x_2^2x_3^3,\ x_2^3x_3^3,\ \boldsymbol{x_1^2x_3^4},\ \boldsymbol{x_1x_2x_3^4},$ $\boldsymbol{x_2^2x_3^4},\ \boldsymbol{x_1^2x_2^3x_4},\ \boldsymbol{x_1^2x_2^2x_3x_4},\ \boldsymbol{x_1x_2^3x_3x_4},\ \boldsymbol{x_1^3x_3^2x_4},\ \boldsymbol{x_1^2x_2x_3^2x_4},\ \boldsymbol{x_1x_2^2x_3^2x_4},$ $x_2^3x_3^2x_4,\ x_1^2x_3^3x_4,\ x_1x_2x_3^3x_4,\ x_2^2x_3^3x_4,\ x_2^3x_3x_4^2,\ x_1^2x_3^2x_4^2,\ x_1x_2x_3^2x_4^2,\ x_2^2x_3^2x_4^2,$ $x_1x_3^3x_4^2,\ x_2x_3^3x_4^2,\ x_1^3x_4^3,\ x_2^3x_4^3,\ x_1^2x_3x_4^3,\ x_1x_2x_3x_4^3,\ x_2^2x_3x_4^3,\ x_1x_3^2x_4^3,\ x_2x_3^2x_4^3,$ $x_3^3x_4^3,\ x_1^2x_4^4,\ x_1x_2x_4^4,\ x_2^2x_4^4,\ x_1x_3x_4^4,\ x_2x_3x_4^4,\ x_3^2x_4^4,\ x_1x_4^5,\ x_2x_4^5,\ x_3x_4^5,\ x_4^6$

The elements that are not minimal generators are marked bold.

Analogously, let $\tilde{H}$ be the Pommaret basis of $\tilde{\mathcal{I}}$ then:

i	$(\operatorname{lt}\tilde{H})_i$
4	$x_1^4,\ x_2^4,\ x_1^3x_3,\ x_1^2x_2x_3,\ x_1x_2^2x_3,\ x_2^3x_3$
5	$\boldsymbol{x_1x_2^4},\ \boldsymbol{x_1^3x_2x_3},\ \boldsymbol{x_1^2x_2^2x_3},\ \boldsymbol{x_1x_2^3x_3},\ x_1x_2x_3^3,\ x_2^2x_3^3,\ x_1^3x_2x_4$
6	$x_1^3x_2^3,\ \boldsymbol{x_1^2x_2^4},\ \boldsymbol{x_1^3x_2^2x_3},\ \boldsymbol{x_1^2x_2^3x_3},\ x_1^2x_3^4,\ x_1x_3^5,\ x_2x_3^5,\ x_3^6,\ \boldsymbol{x_1^3x_2^2x_4},\ x_1^2x_2^3x_4,$ $x_1^2x_3^3x_4,\ x_1x_3^4x_4,\ x_2x_3^4x_4,\ x_3^5x_4,\ x_1^2x_2^2x_4^2,\ x_1x_2^3x_4^2,\ x_1^2x_3^2x_4^2,\ x_1x_2x_3^2x_4^2,$ $x_2^2x_3^2x_4^2,\ x_1x_3^3x_4^2,\ x_2x_3^3x_4^2,\ x_3^4x_4^2,\ x_1^3x_4^3,\ x_1^2x_2x_4^3,\ x_1x_2^2x_4^3,\ x_2^3x_4^3,\ x_1^2x_3x_4^3,$ $x_1x_2x_3x_4^3,\ x_2^2x_3x_4^3,\ x_1x_3^2x_4^3,\ x_2x_3^2x_4^3,\ x_3^3x_4^3,\ x_1^2x_4^4,\ x_1x_2x_4^4,\ x_2^2x_4^4,\ x_1x_3x_4^4,$ $x_2x_3x_4^4,\ x_3^2x_4^4,\ x_1x_4^5,\ x_2x_4^5,\ x_3x_4^5,\ x_4^6$

The elements that are not minimal generators are marked bold.

Therefore $\#H_5 = 8 > 7 = \#\tilde{H}_5$, which shows together with (6.11) that Proposition 6.2.1 does not hold for $n = 4$.

Corollary 6.2.3.
Let $\mathcal{I} \lhd \Bbbk[x_1, \ldots, x_n]$ an ideal and $n \leq 3$. Then $\mathcal{I}$ is in β-maximal position, if and only if its Pommaret basis is of minimal length, i.e. if H is a Pommaret basis of $\mathcal{I}$ then:

$$\#H = \min\{\#\tilde{H} \mid \tilde{H} \text{ Pommaret basis of } A \cdot \mathcal{I} \text{ for some } A \in \mathrm{Gl}(n, \Bbbk)\}$$

(Recall that by Proposition 3.1.6 any Pommaret basis of $\mathcal{I}$ has the same number of elements.)

Proof. Let us assume that $\mathcal{I}$ is not in β-maximal position. If $\mathcal{I}$ is not even in quasi-stable position, it does not possess a finite Pommaret basis by Theorem 3.1.3 and therefore its Pommaret basis is obviously not of minimal length. Otherwise, it follows from Proposition 6.1.10 and Proposition 6.1.13 that there is an integer $\hat{q} < \operatorname{reg}(\mathcal{I})$ such that $\beta_{\hat{q}}(\mathcal{I}) \prec_{\text{lex}} \beta_{\hat{q}}(\operatorname{gin}\mathcal{I})$ and $\beta_q(\mathcal{I}) \preceq_{\text{lex}} \beta_q(\operatorname{gin}\mathcal{I})$ for all $q \neq \hat{q}$. Hence by Proposition 6.2.1 this is equivalent to $\#H_{\hat{q}+1} > \#\tilde{H}_{\hat{q}+1}$ and $\#H_{q+1} \geq \#\tilde{H}_{q+1}$ for all $q \neq \hat{q}$, where H is a Pommaret basis of $\mathcal{I}$ and $\tilde{H}$ one of $\operatorname{gin}\mathcal{I}$. But this already shows that the Pommaret basis H is not of minimal length since obviously

$$\#H = \sum_{q \leq \operatorname{reg}(\mathcal{I})} \#H_q > \sum_{q \leq \operatorname{reg}(\mathcal{I})} \#\tilde{H}_q = \#\tilde{H}$$

(remember that $\deg H = \deg \tilde{H} = \operatorname{reg}(\mathcal{I})$ by Theorem 3.1.10).

Conversely, suppose that the Pommaret basis H of $\mathcal{I}$ is not of minimal length. If H is of infinite length, then it follows again from Theorem 3.1.3 that $\mathcal{I}$ is not in quasi-stable position. But then $\mathcal{I}$ is not in β-maximal position by Corollary 6.1.11. Otherwise, the Pommaret basis H of $\mathcal{I}$ is finite – but still not of minimal length by assumption – so that $\mathcal{I}$ is in quasi-stable position. Let $A \in \mathrm{Gl}(n, \Bbbk)$ be a matrix such that the Pommaret basis $\tilde{H}$ of the ideal $\tilde{\mathcal{I}} = A \cdot \mathcal{I}$ is of minimal length[5]. Hence $\#\tilde{H} < \#H$ and in particular there must be an integer $\hat{q} \leq \deg H = \deg \tilde{H} = \mathrm{reg}(\mathcal{I})$ with $\#\tilde{H}_{\hat{q}} < \#H_{\hat{q}}$. But this already entails by Proposition 6.2.1 that $\beta_{\hat{q}-1}(\mathcal{I}) \prec_{\mathrm{lex}} \beta_{\hat{q}-1}(\tilde{\mathcal{I}})$ showing that also in this case $\mathcal{I}$ is not in β-maximal position. $\square$

Example 6.2.4.
Resuming Example 6.2.2 we first want to note that:

$$\operatorname{gin}\mathcal{I} = \langle x_1^4,\ x_1^3x_2,\ x_1^2x_2^2,\ x_1x_2^3,\ x_2^4,\ x_1^3x_3,\ x_1^2x_2x_3^2,\ x_1x_2^2x_3^2,\ x_2^3x_3^2,\ x_1^2x_3^3\rangle + \langle x_1, x_2, x_3, x_4\rangle^6$$

The minimal basis B of $\operatorname{gin}\mathcal{I}$ – which is simultaneously its Pommaret basis by Lemma 3.2.3 – is presented in the following:

i	B_i
4	$x_1^4,\ x_1^3x_2,\ x_1^2x_2^2,\ x_1x_2^3,\ x_2^4,\ x_1^3x_3$
5	$x_1^2x_2x_3^2,\ x_1x_2^2x_3^2,\ x_2^3x_3^2,\ x_1^2x_3^3$
6	$x_1x_2x_3^4,\ x_2^2x_3^4,\ x_1x_3^5,\ x_2x_3^5,\ x_3^6,\ x_1x_2x_3^3x_4,\ x_2^2x_3^3x_4,\ x_1x_3^4x_4,\ x_2x_3^4x_4,\ x_3^5x_4,$ $x_1^2x_2x_3x_4^2,\ x_1x_2^2x_3x_4^2,\ x_2^3x_3x_4^2,\ x_1^2x_3^2x_4^2,\ x_1x_2x_3^2x_4^2,\ x_2^2x_3^2x_4^2,\ x_1x_3^3x_4^2,\ x_2x_3^3x_4^2,$ $x_3^4x_4^2,\ x_1^3x_4^3,\ x_1^2x_2x_4^3,\ x_1x_2^2x_4^3,\ x_2^3x_4^3,\ x_1^2x_3x_4^3,\ x_1x_2x_3x_4^3,\ x_2^2x_3x_4^3,\ x_1x_3^2x_4^3,\ x_2x_3^2x_4^3,$ $x_3^3x_4^3,\ x_1^2x_4^4,\ x_1x_2x_4^4,\ x_2^2x_4^4,\ x_1x_3x_4^4,\ x_2x_3x_4^4,\ x_3^2x_4^4,\ x_1x_4^5,\ x_2x_4^5,\ x_3x_4^5,\ x_4^6$

Therefore we can conclude that even in this example the length of the Pommaret basis of $\operatorname{gin}\mathcal{I}$ is minimal compared to those of $\mathcal{I}$ and $\tilde{\mathcal{I}}$ since:

$$\begin{array}{ccccc} \#H & > & \#\tilde{H} & > & \#B \\ \shortparallel & & \shortparallel & & \shortparallel \\ \#H_4 + \#H_5 + \#H_6 & & \#\tilde{H}_4 + \#\tilde{H}_5 + \#\tilde{H}_6 & & \#B_4 + \#B_5 + \#B_6 \\ \shortparallel & & \shortparallel & & \shortparallel \\ 6+8+45=59 & & 6+7+42=55 & & 6+4+39=49 \end{array}$$

We even assume that this holds in general (see Conjecture 4 of the Outlook).

[5] In particular, $\tilde{H}$ is finite and so $\tilde{\mathcal{I}}$ in quasi-stable position.

6.3. Connection to Reduction Number

Finally, we now present the already mentioned relationship between β-maximality and weak D-minimality. As a consequence we are able to connect the reduction number with the notion of β-maximal ideals.

Proposition 6.3.1.
Let $\operatorname{char} \Bbbk = 0$. *If the ideal* $\mathcal{I} \lhd \mathcal{P}$ *is in* β*-maximal and in weakly* D*-stable position, then* $\mathcal{I}$ *is weakly* D*-minimal stable. In particular, we have*

$$r(\mathcal{P}/\mathcal{I}) = r(\mathcal{P}/\operatorname{lt}\mathcal{I}) = \deg_{x_{n-D}} \operatorname{lt}\mathcal{I} - 1$$

in this case.

Proof. Since $\operatorname{lt}\mathcal{I}$ is weakly D-stable Theorem 5.2.11 implies that there is an integer s such that x_{n-D}^s is a minimal generator of $\operatorname{lt}\mathcal{I}$.

Assume now that $\mathcal{I}$ is not weakly D-minimal stable. As $\operatorname{char} \Bbbk = 0$ we know that there is a transformation matrix A such that $A \cdot \mathcal{I}$ is weakly D-minimal stable (see Proposition 5.2.14). Therefore we have:

$$t = \deg_{x_{n-D}} \operatorname{lt}(A \cdot \mathcal{I}) < \deg_{x_{n-D}} \operatorname{lt}\mathcal{I} = s$$

In particular, $\operatorname{lt}(A \cdot \mathcal{I})$ is weakly D-stable so that x_{n-D}^t is a minimal generator of $\operatorname{lt}(A \cdot \mathcal{I})$ by Theorem 5.2.11. Moreover, it follows from Remark 5.2.10 that

$$x_1^{\nu_1} \cdots x_{n-D}^{\nu_{n-D}} \in \operatorname{lt}(A \cdot \mathcal{I}),$$

for all integers ν_i with $\nu_1 + \cdots + \nu_{n-D} = t$. Therefore $\beta_t^{(k)}(A \cdot \mathcal{I}) \geq \beta_t^{(k)}(\mathcal{I})$ for all $k < n - D$ and $\beta_t^{(n-D)}(A \cdot \mathcal{I}) > \beta_t^{(n-D)}(\mathcal{I})$ since $x_{n-D}^t \notin \operatorname{lt}\mathcal{I}$. This entails that $\beta_t(A \cdot \mathcal{I}) \succ_{\text{lex}} \beta_t(\mathcal{I})$, which is a contradiction to the β-maximality of $\mathcal{I}$.

The second assertion follows immediately from Theorem 5.2.17. $\square$

The assumption in the previous proposition that $\mathcal{I}$ is in weakly D-stable position, is necessary since this is not implied by β-maximality[6].

Further, the following example shows that the converse of Proposition 6.3.1 is not true, while Example 6.3.3 symbolizes the meaning of $\operatorname{char} \Bbbk$ in this context.

Example 6.3.2.
Let $\mathcal{J} = \langle x_1^2,\ x_1x_3 \rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then $\mathcal{J}$ is weakly D-stable and $D = \dim(\mathcal{P}/\mathcal{J}) = 2$. Further, $\deg_{x_1} \mathcal{J} = 2 = \deg_{x_1} \operatorname{gin}\mathcal{J}$ since $\operatorname{gin}\mathcal{J} = \langle x_1^2,\ x_1x_2 \rangle$ and so $\mathcal{J}$ is even weakly D-minimal stable. But $\mathcal{J}$ is not in β-maximal position since:

$$\beta_2(\mathcal{J}) = (1, 0, 1) \prec_{\text{lex}} (1, 1, 0) = \beta_2(\operatorname{gin}\mathcal{J})$$

[6] See Example 6.1.15 and Example 5.2.15.

Example 6.3.3.
Let $\mathcal{I} = \langle x_1^2,\ x_2^2 + x_1x_3,\ x_3^2\rangle \lhd \Bbbk[x_1, x_2, x_3]$ and $\operatorname{char}\Bbbk = 2$. At first we want to verify that $\operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_2^2,\ x_3^2\rangle$ is Borel-fixed using Proposition 2.4.6. Thereby, since $\binom{2}{u} \not\equiv 0 \mod 2$ holds for $u = 0$ or $u = 2$, we only consider the case $u = 0$ as the other case is "uninteresting". Hence $\operatorname{lt}\mathcal{I}$ is Borel-fixed because of:

$$x_1^2\frac{x_2^2}{x_2^2} = x_1^2 \in \operatorname{lt}\mathcal{I}, \quad x_1^2\frac{x_3^2}{x_3^2} = x_1^2 \in \operatorname{lt}\mathcal{I}, \quad x_2^2\frac{x_3^2}{x_3^2} = x_2^2 \in \operatorname{lt}\mathcal{I}$$

As $D = \dim(\mathcal{P}/\mathcal{I}) = 0$ and $x_1\frac{x_3^2}{x_3} = x_1x_3 \notin \operatorname{lt}\mathcal{I}$, we see that $\operatorname{lt}\mathcal{I}$ is not (weakly) D-stable. Now we want to check whether $\mathcal{I}$ is in β-maximal position. Therefore we transform the ideal $\mathcal{I}$ by $\Psi : (x_3 \mapsto x_3 + x_2)$. This leads to the ideal

$$\Psi(\mathcal{I}) = \langle x_1^2,\ x_2^2 + x_1(x_3 + x_2),\ (x_3 + x_2)^2\rangle = \langle x_1^2,\ x_1x_2 + x_2^2 + x_1x_3,\ x_2^2 + x_3^2\rangle.$$

As the Gröbner basis of $\Psi(\mathcal{I})$ is $\{x_1^2,\ x_1x_2,\ x_2^2 + x_3^2,\ x_1x_3^2 + x_3^3,\ x_2x_3^2,\ x_3^4\}$, its leading ideal is given by $\operatorname{lt}\Psi(\mathcal{I}) = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3^2,\ x_2x_3^2,\ x_3^4\rangle$ so that:

$$\beta_2(\mathcal{I}) = (1, 1, 1) \prec_{\text{lex}} (1, 2, 0) = \beta_2(\Psi(\mathcal{I}))$$

Hence $\mathcal{I}$ is not in β-maximal position, although its leading ideal is Borel-fixed.

Remark 6.3.4.
Similar to the weakly D-minimal stable position (see Remark 5.2.18), we can also not provide a deterministic algorithm that puts a given ideal in β-maximal position. With arguments analogous to Remark 4.3.7 we can conclude that a simple algorithm, i.e. one that is restricted to the analysis of the corresponding leading ideal does not exist. Therefore we again use Example 4.2.3 and denote $\mathcal{I} = \langle x_1^2,\ x_1x_2 + x_2^2,\ x_1x_3\rangle$ and $\mathcal{I}' = \operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_1x_3,\ x_2^3,\ x_2^2x_3\rangle$. As the ideal $\mathcal{I}'$ is monomial and strongly stable, it is in gin-position by Theorem 2.4.4 and so in particular in β-maximal position by Proposition 6.1.13. But $\mathcal{I}$ is not in β-maximal position since $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3^2\rangle$ and therefore:

$$\beta_2(\mathcal{I}) = (1, 1, 1) \prec_{\text{lex}} (1, 2, 0) = \beta_2(\operatorname{gin}\mathcal{I})$$

Hence we have two distinct ideals $\mathcal{I}$, $\mathcal{I}'$ with identical leading ideals but $\mathcal{I}'$ is in β-maximal position while $\mathcal{I}$ is not.

CHAPTER 7

The Map of Positions

Our final chapter will present 24 examples from which a clear delimitation of the several positions that we discussed throughout this thesis can be derived. At the end of this chapter we sum up all the obtained relationships between the considered positions resulting from the examples by drawing the *the map of positions*. Thereby we restrict us to the case of $\operatorname{char} \Bbbk = 0$ in the following.

EXAMPLE 1.
Let $\mathcal{I} = \langle x_1^2,\ x_2^2,\ x_1x_4\rangle \lhd \Bbbk[x_1,x_2,x_3,x_4]$. Then $\mathcal{I}$ is not quasi-stable since $x_3^2\frac{x_1x_4}{x_4} = x_1x_3^2 \notin \mathcal{I}$. Further, $D = \dim(\mathcal{P}/\mathcal{I}) = 2$ and $\mathcal{I}$ is not weakly D-stable since $x_2\frac{x_1x_4}{x_4} = x_1x_2 \notin \mathcal{I}$. Because $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3^2\rangle$ we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_2(\mathcal{I}) = (1,1,0,1) \prec_{\mathrm{lex}} (1,2,0,0) = \beta_2(\operatorname{gin}\mathcal{I})$$

EXAMPLE 2.
Let $\mathcal{I} = \langle x_1x_2,\ x_1^3\rangle \lhd \Bbbk[x_1,x_2]$. Then $\mathcal{I}$ is quasi-stable but not componentwise quasi-stable since $\mathcal{I}_{\langle 2\rangle} = \langle x_1x_2\rangle$ is not quasi-stable. Further, $D = \dim(\mathcal{P}/\mathcal{I}) = 1$ and $\mathcal{I}$ is not weakly D-stable since $x_1\frac{x_1x_2}{x_2} = x_1^2 \notin \mathcal{I}$. Because $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2^2\rangle$ we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_2(\mathcal{I}) = (0,1) \prec_{\mathrm{lex}} (1,0) = \beta_2(\operatorname{gin}\mathcal{I})$$

EXAMPLE 3.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_3\rangle \lhd \Bbbk[x_1,x_2,x_3]$. Then $\mathcal{I}$ is not quasi-stable since $x_2^2\frac{x_1x_3}{x_3} = x_1x_2^2 \notin \mathcal{I}$. Further, $D = \dim(\mathcal{P}/\mathcal{I}) = 2$ and $\mathcal{I}$ is not D-stable since $x_2\frac{x_1x_3}{x_3} = x_1x_2 \notin \mathcal{I}$. Because $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2\rangle$ we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_2(\mathcal{I}) = (1,0,1) \prec_{\mathrm{lex}} (1,1,0) = \beta_2(\operatorname{gin}\mathcal{I})$$

EXAMPLE 4.
Let $\mathcal{I} = \langle x_1^2,\ x_2^2,\ x_3^2\rangle \lhd \Bbbk[x_1,x_2,x_3]$. Then $D = \dim(\mathcal{P}/\mathcal{I}) = 0$ and $\mathcal{I}$ is not weakly D-stable since $x_1\frac{x_3^2}{x_3} = x_1x_3 \notin \mathcal{I}$. Because $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3^2,\ x_2x_3^2,\ x_3^4\rangle$ we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_2(\mathcal{I}) = (1,1,1) \prec_{\mathrm{lex}} (1,2,0) = \beta_2(\operatorname{gin}\mathcal{I})$$

Example 5.
Let $\mathcal{I} = \langle x_1^3,\ x_1x_2^2 + x_2^2x_3,\ x_2^4\rangle \lhd \Bbbk[x_1,x_2,x_3]$. Then $\operatorname{lt}\mathcal{I} = \langle x_1^3,\ x_1x_2^2,\ x_2^4,\ x_2^2x_3^3\rangle$ and $D = \dim(\mathcal{P}/\mathcal{I}) = 1$. $\operatorname{lt}\mathcal{I}$ is not weakly D-stable since $x_1\frac{x_1x_2^2}{x_2} = x_1^2x_2 \notin \operatorname{lt}\mathcal{I}$. Further, $\operatorname{lt}\mathcal{I}_{\langle 3\rangle} = \langle x_1^3,\ x_1x_2^2,\ x_2^2x_3^3\rangle$ which is not quasi-stable, hence $\mathcal{I}$ is not in componentwise quasi-stable position. Because $\operatorname{gin}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^3,\ x_2^4,\ x_1x_2^2x_3^2,\ x_1^2x_3^4\rangle$ we see that $\mathcal{I}$ is in β-maximal position since[1]

$$\begin{aligned} \beta_3(\mathcal{I}) &= (1,1,0) &= \beta_3(\operatorname{gin}\mathcal{I})\\ \beta_4(\mathcal{I}) &= (1,4,2) &= \beta_4(\operatorname{gin}\mathcal{I})\\ \beta_5(\mathcal{I}) &= (1,5,8) &= \beta_5(\operatorname{gin}\mathcal{I}) \end{aligned}$$

Example 6.
Let $\mathcal{I} = \langle x_1^2,\ x_2^2\rangle \lhd \Bbbk[x_1,x_2]$. Then $D = \dim(\mathcal{P}/\mathcal{I}) = 0$ and $\mathcal{I}$ is not weakly D-stable since $x_1\frac{x_2^2}{x_2} = x_1x_2 \notin \mathcal{I}$. Because $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^3\rangle$ we see that $\mathcal{I}$ is in β-maximal position since:

$$\beta_2(\mathcal{I}) = (1,1) = \beta_2(\operatorname{gin}\mathcal{I})$$

Example 7.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2 + x_3^2,\ x_1x_4\rangle \lhd \Bbbk[x_1,x_2,x_3,x_4]$. Then $\operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_4,\ x_1x_3^2\rangle$ and $D = \dim(\mathcal{P}/\mathcal{I}) = 2$. Further, $\operatorname{lt}\mathcal{I}$ is not D-stable since $x_3\frac{x_1x_4}{x_4} = x_1x_3 \notin \mathcal{I}$. Because $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3,\ x_1x_4^2\rangle$ we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_2(\mathcal{I}) = (1,2,0,1) \prec_{\text{lex}} (1,2,1,0) = \beta_2(\operatorname{gin}\mathcal{I})$$

Example 8.
Let $\mathcal{I} = \langle x_1^3 + x_1x_3^2,\ x_1^2x_2 + x_2x_4^2,\ x_1x_2^2,\ x_2^3,\ x_2^2x_3^2,\ x_2x_3^3\rangle \lhd \Bbbk[x_1,x_2,x_3,x_4]$. Then

$$\begin{aligned} \operatorname{lt}\mathcal{I} = \langle\ & x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3,\ x_1x_2x_3^2,\ x_2^2x_3^2,\ x_2x_3^3,\ x_2^2x_4^2,\\ & x_1x_2x_3x_4^2,\ x_2x_3^2x_4^2,\ x_1x_2x_4^4,\ x_2x_3x_4^4,\ x_2x_4^6 \quad \rangle \end{aligned}$$

and $D = \dim(\mathcal{P}/\mathcal{I}) = 2$. $\operatorname{lt}\mathcal{I}$ is not D-stable since $x_3\frac{x_2^2x_4^2}{x_4} = x_2^2x_3x_4 \notin \operatorname{lt}\mathcal{I}$. Further, $\operatorname{lt}\mathcal{I}_{\langle 3\rangle} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3,\ x_1x_2x_3^2,\ x_2^2x_4^2,\ x_2x_3^2x_4^2\rangle$ which is not quasi-stable, hence $\mathcal{I}$ is not in componentwise quasi-stable position. Because

$$\begin{aligned} \operatorname{gin}\mathcal{I} = \langle\ & x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3,\ x_1^2x_3^2,\ x_1x_2x_3^2,\ x_1x_3^3,\ x_1^2x_3x_4,\\ & x_1x_2x_3x_4^2,\ x_1x_3^2x_4^2,\ x_1^2x_4^3,\ x_1x_2x_4^4,\ x_1x_3x_4^4,\ x_1x_4^6 \quad \rangle \end{aligned}$$

we see that $\mathcal{I}$ is in β-maximal position since:

$$\begin{aligned} \beta_3(\mathcal{I}) &= (1,3,0,0) &= \beta_3(\operatorname{gin}\mathcal{I})\\ \beta_4(\mathcal{I}) &= (1,4,7,5) &= \beta_4(\operatorname{gin}\mathcal{I})\\ \beta_5(\mathcal{I}) &= (1,5,12,20) &= \beta_5(\operatorname{gin}\mathcal{I})\\ \beta_6(\mathcal{I}) &= (1,6,18,40) &= \beta_6(\operatorname{gin}\mathcal{I}) \end{aligned}$$

[1] Recall that by Remark 6.1.14 we only have to compare the β-vectors of degrees lower than $\operatorname{reg}(\mathcal{I})$.

EXAMPLE 9.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_4,\ x_1x_3^2\rangle \lhd \Bbbk[x_1,x_2,x_3,x_4]$. Then $\mathcal{I}_{\langle 2\rangle} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_4\rangle$ is not quasi-stable, hence $\mathcal{I}$ is not in componentwise quasi-stable position. Further, $D = \dim(\mathcal{P}/\mathcal{I}) = 2$ and $\operatorname{lt}\mathcal{I}$ is not D-stable since $x_3\frac{x_1x_4}{x_4} = x_1x_3 \notin \mathcal{I}$. Because $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3,\ x_1x_4^2\rangle$ we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_2(\mathcal{I}) = (1,2,0,1) \prec_{\text{lex}} (1,2,1,0) = \beta_2(\operatorname{gin}\mathcal{I})$$

EXAMPLE 10.
Let $\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3,\ x_2^2x_3^2,\ x_2^2x_4^2\rangle \lhd \Bbbk[x_1,x_2,x_3,x_4]$. Then $D = \dim(\mathcal{P}/\mathcal{I}) = 2$ and $\mathcal{I}$ is not D-stable since $x_3\frac{x_2^2x_4^2}{x_4} = x_2^2x_3x_4 \notin \mathcal{I}$. Because

$$\operatorname{gin}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3,\ x_1^2x_3^2,\ x_1^2x_3x_4,\ x_1^2x_4^3\rangle$$

we see that $\mathcal{I}$ is in β-maximal position since:

$$\begin{array}{rcl} \beta_3(\mathcal{I}) = & (1,3,0,0) & = \beta_3(\operatorname{gin}\mathcal{I}) \\ \beta_4(\mathcal{I}) = & (1,4,5,5) & = \beta_4(\operatorname{gin}\mathcal{I}) \end{array}$$

EXAMPLE 11.
Let $\mathcal{I} = \langle x_2^2,\ x_1x_3,\ x_2x_3,\ x_3^2,\ x_1^3\rangle \lhd \Bbbk[x_1,x_2,x_3]$. Then $\mathcal{I}_{\langle 2\rangle} = \langle x_2^2,\ x_1x_3,\ x_2x_3,\ x_3^2\rangle$ is not quasi-stable, hence $\mathcal{I}$ is not in componentwise quasi-stable position. Further, $\mathcal{I}$ is not stable since $x_1\frac{x_2^2}{x_2} = x_1x_2 \notin \mathcal{I}$. Because $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3,\ x_2x_3^2,\ x_3^4\rangle$ we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_2(\mathcal{I}) = (0,1,3) \prec_{\text{lex}} (1,2,1) = \beta_2(\operatorname{gin}\mathcal{I})$$

EXAMPLE 12.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_2^2 + x_2x_3^2,\ x_2^5,\ x_2^4x_3,\ x_2^3x_3^2,\ x_2^2x_3^3\rangle \lhd \Bbbk[x_1,x_2,x_3]$. Then

$$\operatorname{lt}\mathcal{I}_{\langle 3\rangle} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_1^2x_3,\ x_1x_2x_3^2,\ x_2x_3^4\rangle$$

is not quasi-stable, hence $\mathcal{I}$ is not in componentwise quasi-stable position. Further,

$$\operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_1x_2^2,\ x_1x_2x_3^2,\ x_2^5,\ x_2^4x_3,\ x_2^3x_3^2,\ x_2^2x_3^3,\ x_2x_3^4\rangle$$

is not strongly stable since $x_1\frac{x_1x_2x_3^2}{x_2} = x_1^2x_3^2 \notin \mathcal{I}$. Because

$$\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2^2,\ x_2^4,\ x_2^3x_3^2,\ x_1x_2x_3^3,\ x_2^2x_3^3,\ x_1x_3^4\rangle$$

we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_4(\mathcal{I}) = (1,3,5) \prec_{\text{lex}} (1,4,4) = \beta_4(\operatorname{gin}\mathcal{I})$$

Example 13.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_2 + x_2x_3,\ x_1x_3,\ x_2^3,\ x_2^2x_3\rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then $\operatorname{lt}\mathcal{I}_{\langle 2\rangle} = \langle x_1^2,\ x_1x_2,\ x_1x_3,\ x_2x_3^2\rangle$ is not quasi-stable, hence $\mathcal{I}$ is not in componentwise quasi-stable position. Further,

$$\operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_1x_3,\ x_2^3,\ x_2^2x_3,\ x_2x_3^2\rangle \neq \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3^2,\ x_2x_3^2\rangle = \operatorname{gin}\mathcal{I}$$

and so we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_2(\mathcal{I}) = (1, 1, 1) \prec_{\text{lex}} (1, 2, 0) = \beta_2(\operatorname{gin}\mathcal{I})$$

Example 14.
Let $\mathcal{I} = \langle x_1^3,\ x_2^3,\ x_1x_3^2,\ x_2x_3^2,\ x_3^3\rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then $\mathcal{I}$ is not stable since $x_1\frac{x_2^3}{x_2} = x_1x_2^2 \notin \mathcal{I}$. Because

$$\operatorname{gin}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3,\ x_1^2x_3,\ x_1x_2x_3^2,\ x_2^2x_3^2,\ x_1x_3^4,\ x_2x_3^4,\ x_3^6\rangle$$

we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_3(\mathcal{I}) = (1, 1, 3) \prec_{\text{lex}} (1, 3, 1) = \beta_3(\operatorname{gin}\mathcal{I})$$

Example 15.
Let $\mathcal{I} = \langle x_1^3,\ x_1x_2^2 + x_2^2x_3,\ x_2^4,\ x_1x_3^3,\ x_2x_3^3,\ x_3^4\rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then $\operatorname{lt}\mathcal{I}_{\langle 3\rangle} = \langle x_1x_2^2,\ x_1^3,\ x_2^2x_3^3\rangle$ is not quasi-stable, hence $\mathcal{I}$ is not in componentwise quasi-stable position. Further, $\operatorname{lt}\mathcal{I} = \langle x_1^3,\ x_1x_2^2,\ x_2^4,\ x_1x_3^3,\ x_2x_3^3,\ x_3^4\rangle$ is not stable since $x_1\frac{x_1x_2^2}{x_2} = x_1^2x_2 \notin \operatorname{lt}\mathcal{I}$. Because

$$\operatorname{gin}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^3,\ x_2^4,\ x_1x_2^2x_3,\ x_2^3x_3,\ x_1^2x_3^2,\ x_1x_2x_3^3,\ x_2^2x_3^3,\ x_1x_3^4,\ x_2x_3^5,\ x_3^6\rangle$$

we see that $\mathcal{I}$ is in β-maximal position since:

$$\begin{aligned} \beta_3(\mathcal{I}) &= (1, 1, 0) &&= \beta_3(\operatorname{gin}\mathcal{I}) \\ \beta_4(\mathcal{I}) &= (1, 4, 5) &&= \beta_4(\operatorname{gin}\mathcal{I}) \\ \beta_5(\mathcal{I}) &= (1, 5, 13) &&= \beta_5(\operatorname{gin}\mathcal{I}) \end{aligned}$$

Example 16.
Let $\mathcal{I} = \langle x_1^3,\ x_1x_2^2,\ x_2^3,\ x_1^2x_2x_3,\ x_1^2x_3^2,\ x_1x_2x_3^2,\ x_2^2x_3^2,\ x_1x_3^3,\ x_2x_3^3,\ x_3^4\rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then $\mathcal{I}$ is not stable since $x_1\frac{x_1x_2^2}{x_2} = x_1^2x_2 \notin \mathcal{I}$. Because

$$\operatorname{gin}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^4,\ x_2^3x_3,\ x_1^2x_3^2,\ x_1x_2x_3^2,\ x_2^2x_3^2,\ x_1x_3^3,\ x_2x_3^3,\ x_3^4\rangle$$

we see that $\mathcal{I}$ is in β-maximal position since:

$$\beta_3(\mathcal{I}) = (1, 2, 0) = \beta_3(\operatorname{gin}\mathcal{I})$$

Example 17.
Let $\mathcal{I} = \langle x_1^3,\ x_1^2x_2 + x_2^3,\ x_1^2x_3,\ x_2^4\rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then

$$\operatorname{lt}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1^2x_3,\ x_1x_2^3,\ x_2^4,\ x_2^3x_3\rangle$$

is not strongly stable since $x_1\frac{x_2^3x_3}{x_2} = x_1x_2^2x_3 \notin \operatorname{lt}\mathcal{I}$. Because

$$\operatorname{gin}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^4,\ x_1^2x_3^2\rangle$$

we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_3(\mathcal{I}) = (1,1,1) \prec_{\mathrm{lex}} (1,2,0) = \beta_3(\operatorname{gin}\mathcal{I})$$

Example 18.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_2 + x_2x_3,\ x_2^3,\ x_2^2x_3\rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then $\operatorname{lt}\mathcal{I}_{\langle 2\rangle} = \langle x_1^2,\ x_1x_2,\ x_2x_3^2\rangle$ is not quasi-stable, hence $\mathcal{I}$ is not in componentwise quasi-stable position. Further, $\operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^3,\ x_2^2x_3,\ x_2x_3^2\rangle$ is not strongly stable since $x_1\frac{x_2x_3^2}{x_2} = x_1x_3^2 \notin \operatorname{lt}\mathcal{I}$. Because $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_2^3,\ x_2^2x_3,\ x_1x_3^2\rangle$ we see that $\mathcal{I}$ is in β-maximal position since:

$$\beta_2(\mathcal{I}) = (1,3,4) = \beta_2(\operatorname{gin}\mathcal{I})$$

Example 19.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_2^2,\ x_2^3,\ x_2^2x_3^2\rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then $\mathcal{I}$ is not strongly stable since $x_1\frac{x_2^2x_3^2}{x_2} = x_1x_2x_3^2 \notin \mathcal{I}$. Because $\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2^2,\ x_2^3,\ x_1x_2x_3^2\rangle$ we see that $\mathcal{I}$ is in β-maximal position since:

$$\begin{aligned}\beta_2(\mathcal{I}) &= (1,0,0) = \beta_2(\operatorname{gin}\mathcal{I})\\ \beta_3(\mathcal{I}) &= (1,3,1) = \beta_3(\operatorname{gin}\mathcal{I})\end{aligned}$$

Example 20.
Let $\mathcal{I} = \langle x_1^2,\ x_1x_2 + x_2x_3,\ x_2^3,\ x_2^2x_3\rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then

$$\operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_1x_2,\ x_1x_3,\ x_2^3,\ x_2^2x_3\rangle \neq \langle x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3^2\rangle = \operatorname{gin}\mathcal{I}$$

and so we see that $\mathcal{I}$ is not in β-maximal position since:

$$\beta_2(\mathcal{I}) = (1,1,1) \prec_{\mathrm{lex}} (1,2,0) = \beta_2(\operatorname{gin}\mathcal{I})$$

Example 21.
Let $\mathcal{I} = \langle x_1^3,\ x_1^2x_2 + x_2x_3^2,\ x_1x_2^3,\ x_2^4,\ x_1x_2^2x_3,\ x_1^2x_3^2,\ x_1x_3^4\rangle \lhd \Bbbk[x_1, x_2x_3]$. Then $\operatorname{lt}\mathcal{I}_{\langle 3\rangle} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2x_3^2,\ x_2x_3^4\rangle$ is not quasi-stable, hence $\mathcal{I}$ is not in componentwise quasi-stable position. Further, the leading ideal

$$\operatorname{lt}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^3,\ x_2^4,\ x_1x_2^2x_3,\ x_1^2x_3^2,\ x_1x_2x_3^2,\ x_2^3x_3^2,\ x_2^2x_3^3,\ x_1x_3^4,\ x_2x_3^4\rangle$$

is not equal to

$$\operatorname{gin}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^3,\ x_2^4,\ x_1x_2^2x_3,\ x_2^3x_3,\ x_1^2x_3^2,\ x_1x_2x_3^3,\ x_2^2x_3^3,\ x_1x_3^4,\ x_2x_3^4\rangle$$

and we see that $\mathcal{I}$ is in β-maximal position since:

$$\begin{aligned}\beta_3(\mathcal{I}) &= (1,1,0) = \beta_3(\operatorname{gin}\mathcal{I})\\ \beta_4(\mathcal{I}) &= (1,4,5) = \beta_4(\operatorname{gin}\mathcal{I})\end{aligned}$$

Example 22.
Let $\mathcal{K} = \langle x_2x_3 - x_1x_4,\ x_1^3 - x_2^2x_4,\ x_2^3 - x_1x_3^2\rangle \lhd \Bbbk[x_1, x_2, x_3, x_4]$ and $\mathcal{I} = \Psi_2\Psi_1(\mathcal{K})$ with $\Psi_1 : (x_3 \mapsto x_3 + x_1)$ and $\Psi_2 : (x_2 \mapsto x_2 + x_1)$. Then the leading ideal

$$\operatorname{lt}\mathcal{I} = \langle x_1^2,\ x_1x_2^2,\ x_2^3,\ x_1x_2x_3^2,\ x_1x_3^3,\ x_2^2x_3^3,\ x_2x_3^4\rangle$$

is not equal to

$$\operatorname{gin}\mathcal{I} = \langle x_1^2,\ x_1x_2^2,\ x_2^3,\ x_1x_2x_3^2,\ x_2^2x_3^2,\ x_1x_3^4,\ x_2x_3^4\rangle$$

and we see that $\mathcal{I}$ is in β-maximal position since:

$$\begin{aligned} \beta_2(\mathcal{I}) = &\quad (1,0,0,0) &= \beta_2(\operatorname{gin}\mathcal{I}) \\ \beta_3(\mathcal{I}) = &\quad (1,3,1,1) &= \beta_3(\operatorname{gin}\mathcal{I}) \\ \beta_4(\mathcal{I}) = &\quad (1,4,7,6) &= \beta_4(\operatorname{gin}\mathcal{I}) \end{aligned}$$

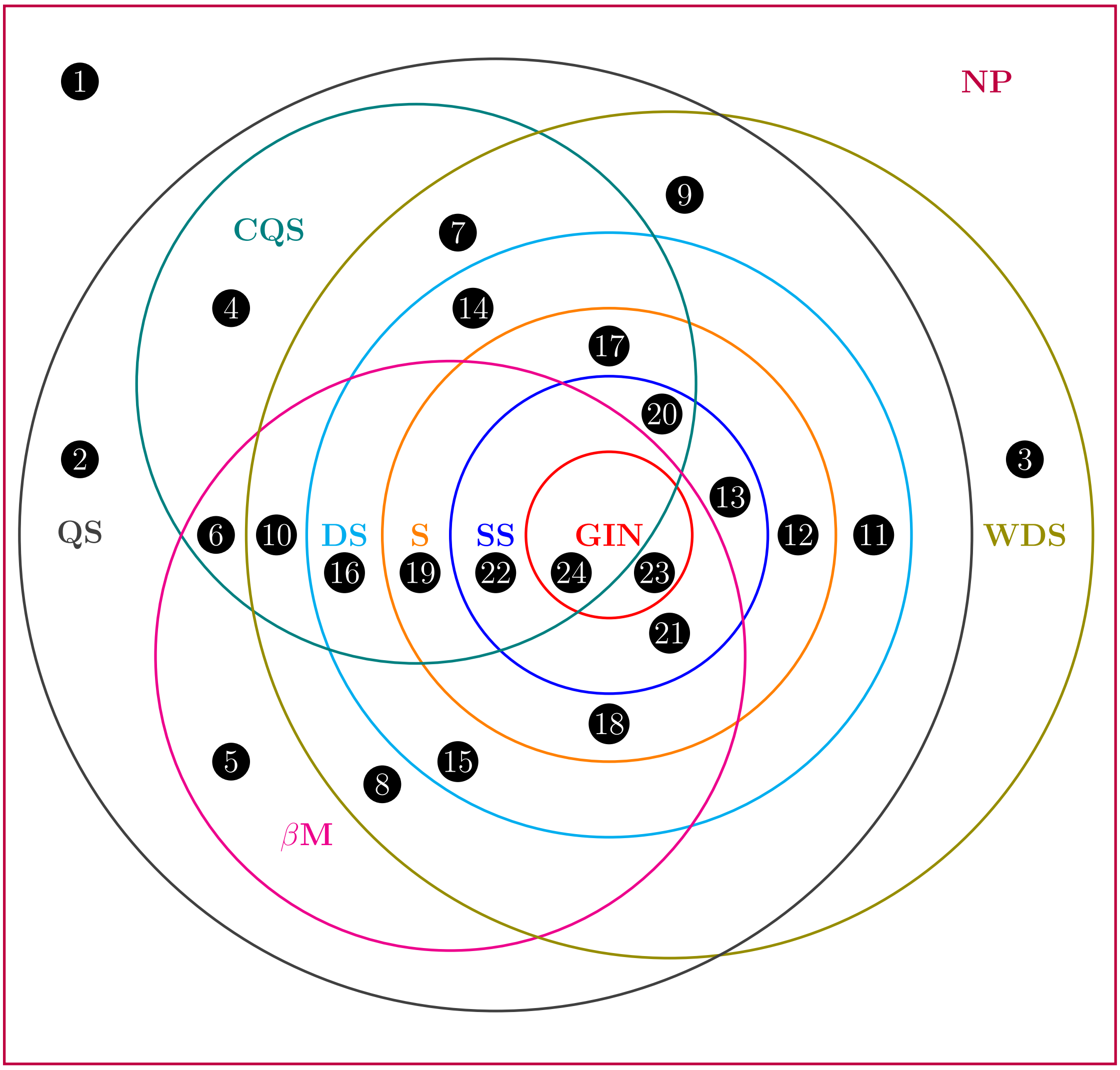
Example 23.
Let $\mathcal{I} = \langle x_1^3,\ x_1^2x_2 + x_1x_2x_3,\ x_1x_2^3,\ x_1x_2^2x_3,\ x_1^2x_3^2\rangle \lhd \Bbbk[x_1, x_2, x_3]$. Then $\operatorname{lt}\mathcal{I}_{\langle 3\rangle} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2x_3^2\rangle$ is not quasi-stable, hence $\mathcal{I}$ is not in componentwise quasi-stable position. Further,

$$\operatorname{lt}\mathcal{I} = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^3,\ x_1x_2^2x_3,\ x_1^2x_3^2,\ x_1x_2x_3^2\rangle = \operatorname{gin}\mathcal{I}$$

and so we see that $\mathcal{I}$ is in β-maximal position.

Example 24.
The final example that is in any position is simply $\langle x_1\rangle \lhd \Bbbk[x_1]$.

With the help of these examples we can now illustrate the relations between the considered positions in the “map” below:



NP=Noether Position, QS=Quasi-Stable, CQS=Componentwise Quasi-Stable, βM=β-Maximal, WDS=Weakly D-Stable, DS=D-Stable, S=Stable, SS=Strongly Stable, GIN=gin-Position

In order to provide a better overview of which properties each of the presented examples satisfies we summarized them in the following tabular:

Example	NP	QS	WDS	CQS	βM	DS	S	SS	GIN
1	✔	✖	✖	✖	✖	✖	✖	✖	✖
2	✔	✔	✖	✖	✖	✖	✖	✖	✖
3	✔	✖	✔	✖	✖	✖	✖	✖	✖
4	✔	✔	✖	✔	✖	✖	✖	✖	✖
5	✔	✔	✖	✖	✔	✖	✖	✖	✖
6	✔	✔	✖	✔	✔	✖	✖	✖	✖
7	✔	✔	✔	✔	✖	✖	✖	✖	✖
8	✔	✔	✔	✖	✔	✖	✖	✖	✖
9	✔	✔	✔	✖	✖	✖	✖	✖	✖
10	✔	✔	✔	✔	✔	✖	✖	✖	✖
11	✔	✔	✔	✖	✖	✔	✖	✖	✖
12	✔	✔	✔	✖	✖	✔	✔	✖	✖
13	✔	✔	✔	✖	✖	✔	✔	✔	✖
14	✔	✔	✔	✔	✖	✔	✖	✖	✖
15	✔	✔	✔	✖	✔	✔	✖	✖	✖
16	✔	✔	✔	✔	✔	✔	✖	✖	✖
17	✔	✔	✔	✔	✖	✔	✔	✖	✖
18	✔	✔	✔	✖	✔	✔	✔	✖	✖
19	✔	✔	✔	✔	✔	✔	✔	✖	✖
20	✔	✔	✔	✔	✖	✔	✔	✔	✖
21	✔	✔	✔	✖	✔	✔	✔	✔	✖
22	✔	✔	✔	✔	✔	✔	✔	✔	✖
23	✔	✔	✔	✖	✔	✔	✔	✔	✔
24	✔	✔	✔	✔	✔	✔	✔	✔	✔

Outlook

In this outlook we want to identify areas of the considered research field that provide a canonical starting point for a further investigation. Thereby we first start with general arising questions and tasks, before we formulate some concrete conjectures afterwards.

Implementation of Algorithm 1. In Section 2.3, we gave a short overview of possible strategies for an implementation of the while loop in line 2 of Algorithm 1. Providing a concrete implementation of this algorithm as well as discussing efficiency aspects in more detail are one of the ideas for an additional study of this topic.

Grothendieck's vanishing and nonvanishing Theorem. We presented in Proposition 3.2.41 an alternative proof for one part of Grothendieck's vanishing and nonvanishing theorem. So it is natural to ask whether it is possible to find an alternative proof using the theory of Pommaret basis for the other assertions.

Combination of weakly D-stable and β-maximal. Ideals that are in weakly D-stable as well as β-maximal position represent a very promising class of ideals. For example – as we have seen in Proposition 6.3.1 – we can directly read off the reduction number of those ideals. Further, such ideals are also in quasi-stable position by Corollary 6.1.11 and so possess a finite Pommaret basis. Hence these ideals can be interpreted as a generalization of the generic initial ideal $\operatorname{gin}\mathcal{I}$.

So on the one hand it would be interesting to discover further properties that the mentioned ideal class share with the gin-position. On the other hand developing an algorithm that directly[2] transforms a given ideal into weakly D-stable and β-maximal position is also a suggestion we recommend to figure out.

[2] Obviously, transforming a given ideal into gin-position is one possibility to receive an ideal, which is in weakly D-stable and β-maximal position. But – as we have seen in Example 8 – weakly D-stability and β-maximality are not sufficient for gin-position. So "directly" means here an algorithm that transforms into weakly D-stable and β-maximal position but not necessarily into gin-position.

Conjecture 1.
Let $\mathfrak{L} = \{\mathcal{L}_i\}_{i=1}^{\ell}$ *be a set where the* $\mathcal{L}_i \lhd \mathcal{P}$ *are the monomial ideals resulting from the Gröbner system of a given ideal* $\mathcal{I}$ *as described in Remark 2.1.2. Without loss of generality we assume that* $\mathcal{L}_1 \prec_{\mathscr{L}} \cdots \prec_{\mathscr{L}} \mathcal{L}_\ell$. *Further, let* α *be an index such that* $\mathcal{L}_\alpha$ *is quasi-stable*[3]. *Then we assume that the following monomial ideals* $\mathcal{L}_{\alpha+1}, \ldots, \mathcal{L}_\ell$ *are also quasi-stable*[4].

Conjecture 2.
Again we take a look at the set $\mathfrak{L} = \{\mathcal{L}_i\}_{i=1}^{\ell}$ *we already mentioned in the previous conjecture and recall that* $\mathcal{L}_m = \max_{\prec_{\mathscr{L}}}\{\mathcal{L}_1, \ldots, \mathcal{L}_\ell\}$ *is Borel-fixed by Corollary 2.4.9. We intuitively expect that in this case we even do have* $\mathcal{L}_m = \operatorname{gin}\mathcal{I}$.

Conjecture 3.
The characteristic of our field $\Bbbk$ *is decisive for the question whether* $\operatorname{gin}\mathcal{I}$ *is weakly D-stable or not (see Example 5.2.15). Therefore we need the assumption* $\operatorname{char}\Bbbk = 0$ *in Proposition 6.3.1 but we suppose that this is not necessary. We conjecture that* $\operatorname{gin}\mathcal{I}$ *is already weakly D-stable if* $\mathcal{I}$ *is weakly D-stable.*

Conjecture 4.
As already indicated in Example 6.2.4, we assume that it is possible to extend the assertion from Corollary 6.2.3 by the following statement:

The length of the Pommaret basis of $\operatorname{gin}\mathcal{I}$ *is minimal for any* n.

---

[3] As $\operatorname{gin}\mathcal{I}$ is Borel-fixed and therefore quasi-stable (see Theorem 2.4.4), such an index must exist since $\operatorname{gin}\mathcal{I} \in \mathfrak{L}$.

[4] Although we have seen in Example 2.2.20 that this does not hold for strong stability, we were not able to construct a corresponding example for the case of quasi-stability.

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