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FREE RESOLUTIONS FROM INVOLUTIVE BASES

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Contents

1 Introduction

In the field of commutative algebra, minimal free resolutions are one of the basic invariants associated to a module $\mathcal{M}$ over a commutative ring R. Several other invariants of a graded module $\mathcal{M}$, for example the Betti numbers or the Hilbert Function, can be defined via, or easily obtained from, this resolution of $\mathcal{M}$. If the module $\mathcal{M}$ comes from a geometrical setting, many of these invariants reflect geometrical properties. From a computational point of view, the question of how to efficiently compute the minimal free resolution and related invariants is an interesting topic in itself.

In the case of a module over a polynomial ring $\mathcal{P} = \Bbbk[x_0, ..., x_n]$, Buchberger introduced Gröbner bases (see [Buc06] for an english version of his PhD thesis): Gröbner bases provide a first idea for a systematic construction of a resolution, which can be minimized to obtain a minimal free resolution. However, arbitrary Gröbner bases have the downside that, in general, they are a primarily computational tool and might not contain much information about the algebraic structure of $\mathcal{M}$. However, the subclass of involutive bases provides a relatively new approach, introduced in [GB98], which reflects interesting algebraic properties of $\mathcal{M}$ in a way that is useful from both a computational-algorithmic perspective as well as from a theoretical perspective. New types of involutive bases have been introduced very recently [GB11].

To obtain an involutive basis, one first needs an involutive division. There are many different classes of involutive divisions, each with advantages and downsides. The Pommaret division is typically the most useful involutive division when one intends to make purely theoretical arguments. Many well-known results such as the Auslander-Buchsbaum formula can also be proved with Pommaret bases, see [Sei10]. While Pommaret bases are helpful for most explicit calculations, their use is hampered by the fact that in some non-generic cases they do not exist. Here, the Janet division can be considered the closest substitute which avoids this pitfall.

Either of these involutive divisions induces a free resolution, which is generally not minimal. Coming from a different direction, discrete algebraic Morse theory (see [For98], [JW09] or [Skö06]) also offers a technique to construct resolutions. While this theory has its origins in the field of cellular complexes, Sköldberg has in [Skö11] applied this theory to modules with initially linear syzygies, a situation which typically arises in the context of involutive bases. The main idea of the present work is to further investigate this construction and see what results it brings.

Our first main result, Theorem 4.2.3, states that the combination of these approaches yields a new construction for the iterative construction of involutive bases of syzygy modules. Additionally, this approach enables us to directly compute some parts of the differential anywhere in the resolution without having to compute other parts of the resolution. Consequently, we can compute Betti numbers without having to do the “unnecessary” computations for lower homological degrees. This algorithm has been implemented in CoCoALib [AB] by Mario Albert, and it appears to be very efficient at computing Betti numbers.

Next, we are able establish that our resolution is highly structured. It is then natural to ask if this structure enables us to make purely theoretical statements, for example which properties can be deduced for the minimal resolution, or more precisely, what part of the structure is retained during a minimising process. It turns out that the rules for minimising a resolution make it possible to give, based on our resolution, some non-vanishing statements about certain Betti numbers of Veronese subrings.

The Veronese subrings $S^{(d)} = \Bbbk[x^\mu \mid \deg(x^\mu) = d] \subseteq \Bbbk[x_0, ..., x_n]$ are among the most studied examples of graded modules. In [EL12], Ein and Lazarsfeld showed that for every $q \geq d + 1$, the (shifted) Betti numbers $\beta'_{p,q}$ of $S^{(d)}$ are nonzero if

$$\binom{d+q}{q} - \binom{d-1}{q} - q \ \leq \ p \ \leq \ \binom{d+n}{n} - \binom{d+n-q}{n-q} + \binom{n}{n-1} - q - 1.$$

Using our theory established earlier, one of our main results, Theorem 6.2.6, is a generalization of the bound on the left to

$$\sum_{i=0}^{\mathbf{d}(q)_{s_q}} \binom{d-i+r_q-1}{d-i} - r_q$$

with certain integers $s_q, \mathbf{d}(q)_{s_q}, r_q$ as in Definition 6.1.14. This bound then holds without any restriction on q, apart from the obvious $1 \leq q \leq \operatorname{reg} S^{(d)}$.

We conclude with a topic that shows a different application of involutive bases: The study of the behavior of the Hilbert function under some ideal-theoretic operations has brought some interesting results such as the Theorem of Macaulay, the Persistence and the Regularity Theorem of Gotzmann or the Hyperplane Restriction Theorem, see [Gre98]. Using the technique of Pommaret bases, we are able to give a new proof of the Hyperplane Restriction Theorem, though we are limited to sufficiently large degrees.

This thesis is structured as follows:

Chapter 2 presents basic definitions and results that are needed throughout the remaining chapters. In particular, we will briefly outline the concept of the minimal free resolution of a graded module, and the invariants defined via this resolution. While we suppose that the readers have encountered these constructions before, the same cannot be said about the other major topic of this chapter, involutive divisions and involutive bases. This field is much more specialized, and therefore we devote the larger part of the chapter to introducing and illustrating the basic ideas behind involutive divisions, with the goal of explaining every aspect of involutive divisions necessary for comprehending the remaining chapters. We close this chapter with some remarks about homological algebra, which build the foundation for the next chapter.

Chapter 3 introduces algebraic discrete Morse Theory. This chapter is heavily based on two papers by Emil Sköldberg, [Skö06] and [Skö11]. We repeat several constructions and theorems from both papers. We have included most of the proofs given in the references. Occasionally, we have revised some of these proofs and added some arguments, hoping to improve accessibility.

Our first original results appear in chapter 4: We see that certain classes of involutive bases can be combined with algebraic discrete Morse theory, yielding a free resolution. We will see that this is essentially the same resolution as the resolution induced by the involutive basis. This approach leads to new possibilities to calculate syzygies, which in particular includes the possibility to compute single Betti numbers without having to compute the entire resolution. The theory of this chapter has been implemented in CoCoALib by M. Albert. We will give a short introduction to this implementation and see that it often favorably compares to other computer algebra systems.

Earlier versions of the content of this chapter, joint work with M. Albert and W.M.Seiler, have been published in [AFSS15] and [AFS15]. In the first paper, the respective results have been given for the special case of Pommaret bases, and in the second paper they were extended to Janet bases.

The next two chapters share the results of chapter 4 as a common basis, but they lead in overall rather different directions.

In chapter 5, we further analyze the newly constructed resolution of chapter 4, which is, in general, not the minimal free resolution. It is interesting to see where in this resolution the differentials contain constants. We will show that in this resolution, the appearance of some constants is interlinked, or more precisely, some sets of constant share a common origin. We caution that stating these theorems in a precise manner makes them look rather technical, and also that their proofs require several technical results to be established in advance. We discuss how these results can be used to further improve future implementations in CoCoALib.

In chapter 6, we apply the results of chapter 4 to the Veronese subrings generated by the monomials of degree d in $n+1$ variables. Here, we restrict to Pommaret division. As first important step, we will construct a Pommaret basis for the ideal arising from the Veronese subrings. As a corollary, we will obtain new proofs for some well-known properties of the Veronese subrings. Additionally, we will prove that some Betti numbers of the Veronese subrings do not vanish. In [EL12], the respective result was proved for $d \geq q+1$, where q is a fixed degree of the "shifted" Betti number in question, but our result covers any value of d. While the main focus of this chapter is different from chapter 5, we do occasionally require some of the lemmata of said chapter. Our result regarding these Betti numbers can be understood via some purely combinatorial conditions regarding multiindices.

In chapter 7, we show another new application of Pommaret bases: We derive a formula that connects a Pommaret basis of an ideal and the unique saturated lex segment ideal with the same Hilbert polynomial. We will use this link to give a new proof for a part of Green's Hyperplane Restriction Theorem. This chapter does not require Morse theory and is independent of Chapters 3-6.

2 Basic definitions and theorems

Definition 2.0.1. A *(directed) graph* $\Gamma = (V, E)$ consists of an arbitrary set V, called *vertices of* Γ and a set $E \subseteq V \times V$, called *edges of* Γ. For an edge $e = (a, b) \in E$, we will use the notation $a \to b$, and say that a, b *are incident to* e. We call a the *source* and b the *target of* e.

Definition 2.0.2. Let $\Gamma = (V, E)$ be a directed graph. A *path in* Γ is a finite (ordered) subset $a_0, ..., a_m \in V$ such that $a_i \to a_{i+1} \in E$ for all $0 \leq i \leq m - 1$. Equivalently, we will view any (ordered) set $\{(a_0 \to a_1), ..., (a_{m-1} \to a_m)\} \subseteq E$ as a path. We will usually write

$$a_0 \to ... \to a_m$$

for such a path.

A *cycle of* Γ is a path $a_0 \to ... \to a_m$ in Γ such that $a_0 = a_m$. We say that Γ is *acyclic* if there are no cycles in Γ.

For two paths $p_1 = (a \to ... \to b)$, $p_2 = (b \to ... \to c)$, we write $p_2 \circ p_1$ for the concatenation of the paths, i.e. the path $a \to ... \to b \to ... \to c$.

Even though the definition of the concatenation possibly looks reversed to what one might initially expect, we prefer this order, for we will later associate maps ρ_p to paths p in certain graphs. For these maps, we will have

$$\rho_{p_2 \circ p_1} = \rho_{p_2} \circ \rho_{p_1},$$

and here our notation is indeed natural.

2.1 Free resolutions

We will start by presenting some basic definitions and results on free resolutions. The presentation is heavily inspired by [CLO98, Chapter 6].

Definition 2.1.1. A chain complex $\mathcal{F}$ over a commutative ring R consists of R-modules F_l for $l \in \mathbb{Z}$ and homomorphisms

$$... \to F_{l+1} \xrightarrow{\varphi_{l+1}} F_l \xrightarrow{\varphi_l} F_{l-1} \to ..$$

such that $\varphi_l \circ \varphi_{l+1} = 0$ for all $l \in \mathbb{Z}$. We call

$$H_i(\mathcal{F}) = \ker(\varphi_i) / \operatorname{im}(\varphi_{i+1})$$

the *i-th homology module of* $\mathcal{F}$. The maps φ_l are called the *differential of* $\mathcal{F}$. We call a chain complex $\mathcal{F}$ an *exact sequence* if $H_i(\mathcal{F}) = 0$ for all $i \in \mathbb{Z}$.

Let $\mathcal{M}$ be an R-module. A *free resolution of* $\mathcal{M}$ is an exact sequence

$$... \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} \mathcal{M} \to 0,$$

where $F_i \cong R^{r_i}$ is a free R-module for all i. We say that the elements of F_i are of *homological degree* i.

Equivalently, we say that a chain complex[1] $\mathcal{F}$ with

$$\dots \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0$$

is a free resolution of $\mathcal{M}$ if and only if every F_l is a free R-module and

$$H_0(\mathcal{F}) = F_0/\operatorname{im}(\varphi_1) \cong \mathcal{M} \quad \text{and} \quad H_i(\mathcal{F}) = 0 \text{ for all } i \geq 1.$$

We say that the (free) resolution is *finite of length* l, if there is an l such that $F_{l+i} = 0$ for $i \geq 1$ and $F_l \neq 0$. In this case, we usually write

$$0 \to F_l \to F_{l-1} \to \dots \to F_1 \to F_0 \to \mathcal{M} \to 0.$$

For our purposes, we usually take R to be the ring $\mathcal{P} = \Bbbk[x_0, \dots, x_n]$ of polynomials in $n+1$ variables over a field $\Bbbk$ of arbitrary characteristic, and $\mathcal{M}$ to be a finitely generated $\mathcal{P}$-module. While in more general situations, finite free resolution do not necessarily exist, the next theorem guarantees the existence of finite resolutions over $\mathcal{P}$.

Theorem 2.1.2 (Hilbert's Syzygy Theorem)**.** *Let $\mathcal{P} = \Bbbk[x_0, \dots, x_n]$. Then for every finitely generated $\mathcal{P}$-module, there is a free resolution of length at most $n+1$.*

Proof. A standard proof is given in [CLO98, Chapter 6, Theorem 2.1]. The statement also follows from Theorem 2.3.59, which gives an alternative proof, using involutive bases. □

Definition 2.1.3. Let $s \geq 0$ be an integer. Then we define $\mathcal{P}_s$ to be the set of all polynomials of total degree s, together with 0. Obviously, we now have

$$\mathcal{P} = \bigoplus_{s \geq 0} \mathcal{P}_s$$

as $\Bbbk$-vector spaces. We will call this decomposition the *(standard) grading on* $\mathcal{P}$. Using this decomposition of $\mathcal{P}$, a *(standard) graded module* over $\mathcal{P}$ is a $\mathcal{P}$-module $\mathcal{M}$ with a family of subgroups $\{\mathcal{M}_t | t \in \mathbb{Z}\} \subseteq \mathcal{M}$ of the additive group of $\mathcal{M}$ such that

- $\mathcal{M} \cong \bigoplus_{t \in \mathbb{Z}} \mathcal{M}_t$ as additive groups,
- $\mathcal{P}_s \mathcal{M}_t \subseteq \mathcal{M}_{s+t}$ for all $s \geq 0$ and $t \in \mathbb{Z}$.

The elements of M_t are called *(homogeneous) elements of degree* t.

This notation is consistent with the standard grading on $\mathcal{P}$. More generally, any decomposition of $\mathcal{P}$ as a direct sum

$$\mathcal{P} = \bigoplus_{s \geq 0} \mathcal{P}'_s$$

[1] To formally match the definition of a chain complex, we extend $\mathcal{F}$ to the right with a chain of trivial modules 0 and trivial homomorphisms.

of finite-dimensional $\Bbbk$-vector spaces $\mathcal{P}'_s$ such that $\mathcal{P}'_s\mathcal{P}'_t \subseteq \mathcal{P}'_{t+s}$ for all $s, t \geq 0$ allows us to define a grading by taking as elements of degree s the nonzero elements of $\mathcal{P}'_s$. In the same manner, the definition can be extended to graded modules with respect to the new grading. Examples of gradings that are different from the standard grading can be found, for example, in [KR05, Section 4.1.].

We will work exclusively with the standard grading on $\mathcal{P}$ and graded modules with respect to the standard grading on polynomial rings:

Assumption 2.1.4. From now on, unless stated otherwise, any graded module $\mathcal{M}$ is a standard graded $\mathcal{P}$-module for a polynomial ring $\mathcal{P}$. With the exception of Chapter 6, we take $\mathcal{P} = \Bbbk[x_0, ..., x_n]$ to be the polynomial ring in $n+1$ variables over a field $\Bbbk$.

Assumption 2.1.5. We write $\mathbb{N}$ for the set of non-negative integers. In particular, $\mathbb{N}$ contains 0.

Definition 2.1.6. A *monomial in* $\mathcal{P}$ is a term of shape $x_0^{\mu_0} \cdots x_n^{\mu_n}$ where we have $\mu_0, ..., \mu_n \in \mathbb{N}$. We write $\mathbb{T}$ for the set of all monomials in $\mathcal{P}$.

To be consistent with this general assumption, we define multiindices to also have $n+1$ entries:

Definition 2.1.7. We call a vector $\mu = (\mu_0, ..., \mu_n) \in \mathbb{Z}^{n+1}$ a *multiindex*. For a multiindex μ such that $\mu_i \geq 0$ for all $0 \leq i \leq n$, let x^μ be the monomial $x^\mu = x_0^{\mu_0} \cdots x_n^{\mu_n} \in \Bbbk[x_0, ..., x_n]$. We also say that μ is the *exponent vector of* x^μ. For a monomial x^μ, we define

$$\operatorname{supp}(x^\mu) = \operatorname{supp}(\mu) = \{i \mid \mu_i > 0\}.$$

For $0 \leq i \leq n$, we define $\mathbf{1}_i$ the multiindex whose i-th entry is 1 and whose remaining entries are 0, i.e.

- $(\mathbf{1}_i)_i = 1$ and
- $(\mathbf{1}_i)_j = 0$ for $j \neq 1$.

Thanks to the identification of multiindices and monomials as in this definition, many objects and definitions related to multiindices can just as well be understood via the corresponding monomials. Occasionally, when the context ensures that there cannot be confusion, we will use this fact for a minor abuse of notation and write μ instead of x^μ and vice versa, depending on what is better suited for the respective context.

Theorem 2.1.8. *Let $d \in \mathbb{Z}$. Let $\mathcal{M}(d)$ be the direct sum*

$$\mathcal{M}(d) = \bigoplus_{t \in \mathbb{Z}} \mathcal{M}(d)_t,$$

where $\mathcal{M}(d)_t = \mathcal{M}_{d+t}$. Then $\mathcal{M}(d)$ is a graded $\mathcal{P}$-module.

Proof. Obvious. □

We call such an $\mathcal{M}(d)$ a shifted module. Of special interest are the shifted modules $\mathcal{P}(d)$.

Definition 2.1.9. Let $\mathcal{M}, \mathcal{N}$ be graded $\mathcal{P}$-modules. We say that a homomorphism $\varphi\colon \mathcal{M} \to \mathcal{N}$ is a *graded homomorphism of degree d* if $\varphi(\mathcal{M}_t) \subseteq \mathcal{N}_{t+d}$ for all $t \in \mathbb{Z}$. A graded resolution of $\mathcal{M}$ is a free resolution

$$... \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} \mathcal{M} \to 0$$

such that each F_l is a shifted graded free module $\mathcal{P}(-d_1) \oplus ... \oplus \mathcal{P}(-d_p)$ and each φ_l is a graded homomorphism of degree 0.

Assumption 2.1.10. Unless stated otherwise, we will from now on take any graded homomorphism to be of degree 0. In particular, if we speak of isomorphisms of graded modules, we will always assume that the modules are isomorphic via a graded isomorphism of degree 0, unless stated otherwise.

Theorem 2.1.11 (Graded Hilbert Syzygy Theorem)**.** *Let $\mathcal{P} = \Bbbk[x_0, ..., x_n]$. Then any finitely generated graded $\mathcal{P}$-module has a finite graded resolution of length at most $n+1$.*

Proof. See for example [CLO98, Chapter 6, Theorem 3.8]. □

Definition 2.1.12. Let $\mathcal{M}$ be a finitely generated graded $\mathcal{P}$-module and

$$... \to F_l \to F_{l-1} \to ... \to F_0 \to \mathcal{M} \to 0$$

a graded resolution of $\mathcal{M}$. The resolution is called *minimal* if for all $l \geq 1$, the non-vanishing entries of the matrix of φ_l (which represents the map $F_l \to F_{l-1}$) are of positive degree. If one of these matrices contains has an entry contained in $\mathcal{P}_0 = \Bbbk$, we call this entry a *constant of $\mathcal{F}$* and we say that $\mathcal{F}$ *contains a constant.*

We will briefly explain the use of the word "minimal" in this context: In Section 2.2, we will see that from any resolution for which a matrix of some φ_l contains a non-vanishing entry of degree 0 (i.e. a constant), we can construct another free resolution of $\mathcal{M}$ whose modules have smaller ranks. Thus, it makes sense to say that any resolution which contains a constant is not minimal, justifying this definition of minimality for resolutions.

Definition 2.1.13. Two chain complexes

$$... \to F_{l+1} \xrightarrow{\varphi_{l+1}} F_l \to ... \quad \text{and} \quad ... \to G_{l+1} \xrightarrow{\psi_{l+1}} G_l \to ...$$

are called *isomorphic* if there are graded isomorphisms $\alpha_l\colon F_l \to G_l$ (of degree 0) for $l \geq 1$ such that $\alpha_l \circ \varphi_{l+1} = \psi_l \circ \alpha_{l+1}$ for all $l \geq 0$ holds.

Theorem 2.1.14. *Any two minimal resolutions of $\mathcal{M}$ are isomorphic.*

Proof. [CLO98, Chapter 6, Theorem 3.13] □

Remark 2.1.15. From definition 2.1.1, we now immediately see that for any isomorphism $\mathcal{M} \cong \mathcal{M}'$, the minimal free resolutions of $\mathcal{M}$ and $\mathcal{M}'$ are the same, up to isomorphism. In particular, this is the case if $\mathcal{M}'$ arises from $\mathcal{M}$ via a change of coordinates, i.e. an automorphism of $\mathcal{P}$ given by $x_i' = \sum_{j=0}^{n} A_{ij}x_j$ for an invertible matrix $A = (A_{ij}) \in \Bbbk^{(n+1)\times(n+1)}$.

Additionally, by Theorem 2.1.14, the minimal free resolution is unique up to isomorphism. For each F_l, there are numbers $d_1, ..., d_p \in \mathbb{Z}$ such that there exists a graded isomorphism $F_l \cong R(-d_1) \oplus ... \oplus R(-d_p)$ of degree 0. Thus the invariants of the definitions below are indeed well-defined:

Definition 2.1.16. Let $\mathcal{M}$ be a graded $\mathcal{P}$-module and

$$0 \to \bigoplus_{d\in\mathbb{Z}} \mathcal{P}(-d)^{\beta_{p,d}} \to \bigoplus_{d\in\mathbb{Z}} \mathcal{P}(-d)^{\beta_{p-1,d}} \to ... \to \bigoplus_{d\in\mathbb{Z}} \mathcal{P}(-d)^{\beta_{0,d}} \to \mathcal{M} \to 0 \tag{2.1.1}$$

a minimal free resolution of $\mathcal{M}$.

- The numbers $\beta_{i,j} = \beta_{i,j}(\mathcal{M})$ are called the *graded Betti numbers* of $\mathcal{M}$. $\beta_i(\mathcal{M}) = \sum_{j\in\mathbb{Z}} \beta_{i,j}(\mathcal{M})$ is called the *j-th total Betti number* of $\mathcal{M}$.
- We call the numbers $\beta'_{i,j} = \beta_{i,i+j}$ the *shifted graded Betti numbers (of $\mathcal{M}$).*
- $\mathrm{pd}(\mathcal{M}) = \max\{i \in \mathbb{Z} | \exists i : \beta_{i,j}(\mathcal{M}) \neq 0\}$ is called the *projective dimension* of $\mathcal{M}$.
- $\mathrm{reg}(\mathcal{M}) = \max\{j \in \mathbb{Z} | \exists j : \beta'_{i,j}(\mathcal{M}) \neq 0\}$ is called the *Castelnuovo-Mumford-regularity* of $\mathcal{M}$ (or simply the regularity of $\mathcal{M}$).

In particular, any two minimal generating sets of $\mathcal{M}$ have the same number of generators in every degree.

Remark 2.1.17. The graded Betti numbers of a module $\mathcal{M}$ can be represented in a compact way by a (finite) matrix, which is called the *(graded) Betti table of* $\mathcal{M}$. The graded Betti table of $\mathcal{M}$ contains the numbers $\beta'_{i,j}$. For example, suppose $\mathcal{M}$ has a minimal free resolution of shape

$$\begin{aligned} 0 \to \mathcal{P}(-7) \to \mathcal{P}(-5) \oplus \mathcal{P}^3(-6) \to \\ \mathcal{P}(-3) \oplus \mathcal{P}^5(-4) \oplus P(-5) \to \\ \mathcal{P}^3(-2) \oplus \mathcal{P}^2(-3) \to \mathcal{P}(0) \to \mathcal{M} \to 0, \end{aligned}$$

then the graded Betti table of $\mathcal{M}$ is

	0	1	2	3	4
0	1	0	0	0	0
1	0	3	1	0	0
2	0	2	5	1	0
3	0	0	1	3	1.

In fact, in example 2.3.24 we will introduce an ideal I for which the module $\mathcal{M} = \mathcal{P}/I$ has exactly this Betti table (see also example 2.3.66).

Definition 2.1.18. The *Hilbert function of* $\mathcal{M}$ is defined as

$$\mathrm{HF}_{\mathcal{M}} : \mathbb{Z} \to \mathbb{Z}, \quad t \mapsto \dim_K(\mathcal{M}_t).$$

In the context of an ideal $I \trianglelefteq \mathcal{P}$ and the module $\mathcal{P}/I$, we call HF_I the *volume function of* I, while the *Hilbert function of* I is given by $\mathrm{HF}_{P/I}$.

It is a well-known fact that for large values of t, the Hilbert function becomes a polynomial function; this fact also follows as a corollary from the ideas of Remark 2.1.20 Lemma 2.3.64. Again, we see ensures that the next definition is well-defined:

Definition 2.1.19. Let $\mathcal{M}$ be a $\mathcal{P}$-module. The *Hilbert polynomial* $\mathrm{HP}_{\mathcal{M}}$ of $\mathcal{M}$ is the unique polynomial such that for $t \gg 0$, we have

$$\mathrm{HP}_{\mathcal{M}}(t) = \mathrm{HF}_{\mathcal{M}}(t).$$

As with the Hilbert function, for an ideal $I \trianglelefteq \mathcal{P}$, we will refer to $\mathrm{HP}_{\mathcal{P}/I}$ as the Hilbert polynomial of I. Whenever it is necessary to make a distinction between HP_I and $\mathrm{HP}_{\mathcal{P}/I}$, we will explicitly state the polynomial in question.

Remark 2.1.20. It is possible to calculate the Hilbert function and the Hilbert polynomial from the ranks of the modules in a free resolution, i.e. the Betti number, via an inclusion-exclusion-principle. We note that here we are implicitly using Assumption 2.1.10, thanks to which the formulas here remain as compact as possible. Let

$$... \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} \mathcal{M} \to 0$$

be a (finite) free resolution of M. Then we have[1]

$$\mathrm{HF}_{\mathcal{M}}(t) = \dim \mathcal{M}_t = \sum_{i \geq 0} (-1)^i \dim_{\Bbbk} F_{i,t}.$$

Using the Betti numbers as in Definition 2.1.16, i.e. for the minimal resolution given by

$$F_i = \bigoplus_{d \in \mathbb{Z}} \mathcal{P}(-d)^{\beta_{i,d}},$$

we obtain

$$\mathrm{HP}_{\mathcal{M}}(t) = \sum_{i \geq 0} \sum_{d \in \mathbb{Z}} (-1)^i \beta_{i,d} \binom{n+t-d}{t-d}$$

as soon as t is large enough. In particular, we see that Betti numbers contain more information than the Hilbert function or the Hilbert polynomial. Additionally, we remark that we could even drop the assumption of the resolution being minimal, as the analogous sum formula holds for any free resolution, not just the minimal one.

[1] We denote by $F_{i,t}$ the component of degree t of the module F_i.

2.2 Minimising free resolutions

Lemma 2.2.2 below explains how an exact sequence of $\mathcal{P}$-modules can be replaced by a smaller exact sequence. We will use this lemma in multiple instances since it gives us a way to algorithmically "minimise" a given resolution step-by-step by iterating the construction given in the lemma. If the original resolution was finite, we will obtain a minimal free resolution after a finite number of steps. The idea is as follows: For a nonzero constant in the differential (i.e. a constant entry in a matrix representation of one of the φ_l, see definition 2.1.1), we eliminate the generators that belong to this constant and slightly change the differential to obtain a "smaller" exact sequence, i.e. an exact sequence where the ranks of most modules remain unchanged, but some ranks are indeed smaller than in the original exact sequence. The sum of all ranks of the free modules in the new exact sequence is smaller than the sum for the exact sequence we started with. Hence at some point we obtain a resolution without any constants, i.e. a minimal free resolution. Therefore, if we started with an arbitrary finite free resolution, we can reduce it until we obtain a minimal free resolution. The lemma is implicitly stated and proved in [CLO98, Chapter 6, Theorem 3.15]; we formulate it here in an explicit manner, using mostly the same notation. In [CLO98], only graded modules are considered, yet the generalisation to any free module holds trivially. Since we will often remove elements from (index) sets, it is useful to introduce the following notation of the $\hat{}$-symbol:

Definition 2.2.1. For any finite set $\mathbf{a} = \{a_1, ..., a_m\}$, let

$$\mathbf{a}_r = \{a_1, ..., \hat{a}_r, ..., a_m\} = \{a_1, ..., a_m\} \setminus \{a_r\}.$$

In the same way, we extend this notation to any ordered set.

Lemma 2.2.2. *Let $F_{l+1}, ..., F_{l-2}$ be free $\mathcal{P}$-modules. Let*

$$... \to F_{l+1} \xrightarrow{\varphi_{l+1}} F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} F_{l-2} \to ...$$

be an exact sequence. Let $\{e_1, ..., e_m\}$ be a basis of F_l and $\{u_1, ..., u_t\}$ a basis of F_{l-1}. Let $(A_l)_{r,s}$ be a non-vanishing constant entry of the matrix A_l of φ_l. Let $G_l \subseteq F_l$ be the module with basis $\{e_1, ..., \hat{e}_s, ..., e_m\}$ and $G_{l-1} \subseteq F_{l-1}$ the module with basis $\{u_1, ..., \hat{u}_r, ..., u_t\}$. Then there is an exact sequence

$$... \to F_{l+1} \xrightarrow{\psi_{l+1}} G_l \xrightarrow{\psi_l} G_{l-1} \xrightarrow{\psi_{l-1}} F_{l-2} \to ...,$$

where the differentials are given by $\psi_{l+1} = \pi_{G_l}(\varphi_{l+1})$ with the canonical projection $\pi_{G_l} \colon F_l \to G_l$, the map ψ_l given by

$$\psi_l(e_i) = \varphi_l \left(e_i - \frac{(A_l)_{r,i}}{(A_l)_{r,s}} e_s \right)$$

for $i \in \{1, ..., \hat{r}, ..., m\}$, and $\psi_{l-1} = \varphi_{l-1}|_{G_{l-1}}$. The remaining differentials are unchanged.

Proof. See proof of [CLO98, Chapter 6, Theorem 3.15]. □

Definition 2.2.3. For a given exact sequence

$$\mathcal{F} = \quad F_{l+1} \xrightarrow{\varphi_{l+1}} F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} F_{l-2}$$

and $(A_l)_{r,s}$ a non-vanishing constant entry of the matrix A_l of φ_l, we use the notation $\mathcal{F}_{r,s}$ for the the exact sequence

$$\mathcal{F}_{r,s} = \quad (F_{l+1} \xrightarrow{\psi_{l+1}} G_l \xrightarrow{\psi_l} G_{l-1} \xrightarrow{\psi_{l-1}} F_{l-2}),$$

as constructed in Lemma 2.2.2 and call this exact sequence the $\mathcal{F}_{r,s}$*-sequence of* $\mathcal{F}$.

Additionally, as a consequence of this lemma, we see that any free resolution of $\mathcal{M}$ gives upper bounds for the Betti numbers and the invariants of definition 2.1.16.

In particular, applying Lemma 2.2.2 to the resolution whose existence is guaranteed by Theorem 2.1.11, we immediately obtain:

Corollary 2.2.4. *We have* $\mathrm{pd}(\mathcal{M}) \leq n + 1$.

Now if we are minimising a given (finite) resolution step-by-step via Lemma 2.2.2, from Theorem 2.1.14 we know that, after a finite number of steps, we always obtain the unique (up to isomorphism) minimal free resolution. However, we can pick the minimisations (or equivalently, the constants) in any order, and therefore depending on the chosen order of minimisations, the resolutions obtained during this process may vary. The next lemma aims to show that the appearance of constants in the process is not completely arbitrary. Later, we will use this lemma as a criterion to see that certain generators cannot vanish throughout any minimisation process:

Lemma 2.2.5. *Let* $F_{l+1}, ..., F_{l-2}$ *be free* $\mathcal{P}$*-modules of finite rank. Let* $\mathcal{F}$ *be an exact sequence*

$$F_{l+1} \xrightarrow{\varphi_{l+1}} F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} F_{l-2}.$$

Let $(A_l)_{r,s}$ *be a non-vanishing constant entry of the matrix* A_l *of* φ_l*. If we have that* $(A_l)_{t,u} = 0$*, but* $(B_l)_{t,u}$ *is a non-vanishing constant entry of the matrix* B_l *of* ψ_l *in the resolution* $\mathcal{F}_{r,s}$*, then the entries* $(A_l)_{r,u}$ *and* $(A_l)_{t,s}$ *are non-vanishing constants, the entry* $(D_l)_{t,s}$ *of the matrix of the differential in* $\mathcal{F}_{r,u}$ *is nonzero and we have* $(\mathcal{F}_{r,s})_{t,u} \cong (\mathcal{F}_{r,u})_{t,s}$*.*

Proof. If $(A_l)_{t,u} = 0$, we have

$$(B_l)_{t,u} = (A_l)_{t,u} - (A_l)_{t,s}\frac{(A_l)_{r,u}}{(A_l)_{r,s}} = -(A_l)_{t,s}\frac{(A_l)_{r,u}}{(A_l)_{r,s}},$$

so if the left side is nonzero, then so are all terms on the right side.

Since the assumption $(D_l)_{t,s} = (A_l)_{t,s} - (A_l)_{t,u}\frac{(A_l)_{r,s}}{(A_l)_{r,u}} = 0$ is equivalent to $(B_l)_{t,u} = (A_l)_{t,u} - (A_l)_{t,s}\frac{(A_l)_{r,u}}{(A_l)_{r,s}} = 0$, we see that $(D_l)_{t,s}$ must be nonzero.

Writing $A_{t,u}$ instead of $(A_l)_{t,u}$, we obtain that an entry $C_{v,w}$ in the matrix of the differential in the exact sequence $(\mathcal{F}_{r,s})_{t,u}$ is given by

$$\begin{aligned}
&C_{v,w}\\
=&B_{v,w} - \frac{B_{v,u}B_{t,w}}{B_{t,u}}\\
=&A_{v,w} - \frac{A_{v,s}A_{r,w}}{A_{r,s}} - \frac{(A_{v,u} - \frac{A_{v,s}A_{r,u}}{A_{r,s}})(A_{t,w} - \frac{A_{t,s}A_{r,w}}{A_{r,s}})}{A_{t,u} - \frac{A_{t,s}A_{r,u}}{A_{r,s}}}\\
=&\frac{A_{v,w}A_{t,u}A_{r,s} - A_{v,w}A_{t,s}A_{r,u} - A_{v,s}A_{r,w}A_{t,u}A_{r,s}A_{r,s}^{-1} + A_{v,s}A_{r,w}A_{t,s}A_{r,u}A_{r,s}^{-1}}{A_{t,u}A_{r,s} - A_{t,s}A_{r,u}}\\
&+ \frac{-A_{v,u}A_{t,w}A_{r,s} + A_{v,u}A_{t,s}A_{r,w} + A_{t,w}A_{v,s}A_{r,u} + A_{v,s}A_{r,u}A_{t,s}A_{r,w}A_{r,s}^{-1}}{A_{t,u}A_{r,s} - A_{t,s}A_{r,u}}\\
=&\frac{A_{v,w}A_{t,u}A_{r,s} - A_{v,w}A_{t,s}A_{r,u} - A_{v,s}A_{r,w}A_{t,u}}{A_{t,u}A_{r,s} - A_{t,s}A_{r,u}}\\
&+ \frac{-A_{v,u}A_{t,w}A_{r,s} + A_{v,u}A_{t,s}A_{r,w} + A_{t,w}A_{v,s}A_{r,u}}{A_{t,u}A_{r,s} - A_{t,s}A_{r,u}}
\end{aligned}$$

We see that the last term is invariant under permutation of s and u (the sign changes of the numerator and denominator cancel each other out). □

Lemma 2.2.6. *Let $F_{l+1}, ..., F_{l-2}$ be free $\mathcal{P}$-modules of finite rank. Let $\mathcal{F}$ be an exact sequence*

$$F_{l+1} \xrightarrow{\varphi_{l+1}} F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} F_{l-2}.$$

Let $(A_l)_{r,s}$ be a non-vanishing constant entry of the matrix A_l of φ_l. Let $(A_l)_{r,s}$ and $(A_l)_{t,u}$ be non-vanishing constant entries of the matrix A_l of φ_l.

- *If $(B_l)_{t,u}$ is a non-vanishing constant entry of the matrix B_l of ψ_l, then we have $(\mathcal{F}_{r,s})_{t,u} = (\mathcal{F}_{t,u})_{r,s}$.*
- *If $(B_l)_{t,u} = 0$, then also the entry $(C_l)_{r,s}$ in the matrix of the differential of $\mathcal{F}_{t,u}$ is zero.*

Proof. For the first point, we see that the formula for the differential of the complex $(\mathcal{F}_{r,s})_{t,u}$ in the proof of Lemma 2.2.5 remains invariant if we exchange the roles of r and t, and s and u.

For the second point, we note that $(C_l)_{r,s} = (A_l)_{r,s} - (A_l)_{r,u}\frac{(A_l)_{t,s}}{(A_l)_{t,u}} = 0$ is equivalent to $(B_l)_{t,u} = (A_l)_{t,u} - (A_l)_{t,s}\frac{(A_l)_{r,u}}{(A_l)_{r,s}} = 0$, since by assumption $(A_l)_{r,s}$ and $(A_l)_{t,u}$ are nonzero. □

2.3 Involutive bases

Involutive bases are special classes of Gröbner bases. Compared to Gröbner bases, involutive bases tend to contain a larger amount of combinatorial properties than minimal Gröbner bases; this information is often contained in additional (compared to a minimal Gröbner bases) elements of involutive basis.

Involutive bases and were introduced by Gerdt and Blinkov (see [GB98]) who combined ideas from the Janet-Riquier theory of differential equations with the theory of Gröbner bases. Some of the results on Pommaret bases presented in this chapter were obtained earlier by Amasaki [Ama90], who was using the term *Weierstraß bases*.

2.3.1 Monomial orders

Definition 2.3.1. Let $\mathcal{M} = \mathcal{P}^m$. Let $\{e_1, ..., e_m\}$ be the standard basis of $\mathcal{M}$. A *monomial in* $\mathcal{M}$ is an element $x^\alpha e_i$, where x^α is a monomial in the polynomial ring $\mathcal{P}$ in the sense of Definition 2.1.6. A *monomial order* on $\mathcal{P}^m$ is a relation $\prec$ on the set of monomials in $\mathcal{P}^m$ such that

- $\prec$ is a well-ordering and
- if $\mathbf{m}_\alpha, \mathbf{m}_\beta \in \mathcal{P}^m$ and $x^\gamma \in \mathcal{P}$ are monomials in $\mathcal{M}$ respectively $\mathcal{P}$ with $\mathbf{m}_\alpha \prec \mathbf{m}_\beta$, then also $x^\gamma \mathbf{m}_\alpha \prec x^\gamma \mathbf{m}_\beta$.

Given a term $f = \sum_\alpha c_\alpha \mathbf{m}_\alpha$, where the $\mathbf{m}_\alpha \in \mathcal{M}$ are monomials and $c_\alpha \neq 0$ for all α, the largest monomial appearing in f with respect to a given monomial order $\prec$ is called the *leading monomial of* f, which we will denote by $\mathrm{lt}_\prec(f)$, or just $\mathrm{lt}(f)$ if the monomial order is clear from the context. Given a set B of terms in $\mathcal{P}$ (or $\mathcal{M}$), we will write $\mathrm{lt}_\prec(B) = \{x^\alpha | \exists f \in B : \mathrm{lt}_\prec(f) = x^\alpha\} \subseteq \mathcal{P}$ or $\mathrm{lt}_\prec(B) = \{x^\alpha e_i | \exists f \in B : \mathrm{lt}_\prec(f) = x^\alpha e_i\} \subseteq \mathcal{M}$ respectively for the set of leading monomials of B. In cases where B itself is an ideal or a submodule, we will also use the notation $\mathrm{lt}_\prec(B)$ for the ideal generated by these terms, which is called the *leading ideal of* B.

It appears to us that this ambiguous use of the notation $\mathrm{lt}_\prec(B)$ is common practice. To avoid confusion, whenever necessary we will explicitly state when we want $\mathrm{lt}_\prec(B)$ to be (just) a set of monomials, or the leading ideal of B.

Definition 2.3.2. Let $\mathcal{M} \subseteq \mathcal{P}^m$ be a nonzero polynomial module. Let $\prec$ be a monomial order order on $\mathcal{P}^m$. A finite set $\mathcal{G} \subseteq \mathcal{M}$ is a *Gröbner basis (with respect to* $\prec$ if $\mathrm{lt}_\prec(\mathcal{M}) = \mathrm{lt}_\prec(\mathcal{G})$.

Depending on the given problem, there are various practically relevant monomial orders on a given polynomial ring $\mathcal{P}$. For a reference, see [CLO98, Chapter 1, Section 2] or other books about Gröbner bases. For our purposes, we will mention some monomial orders which are particularly interesting:

Definition 2.3.3. Let $x^\alpha, x^\beta \in \mathcal{P}$ be two monomials, where $\alpha = (\alpha_0, ..., \alpha_n)$ and $\beta = (\beta_0, ..., \beta_n)$ are the exponent vectors.

- The *degree reverse lexicographic order* $\prec_{\text{degrevlex}}$ (also called the *graded reverse lexicographic order*) on $\mathcal{P}$ is given by $x^\alpha \succ_{\text{degrevlex}} x^\beta$ if

$$\deg \alpha = \sum_{i=0}^{n} \alpha_i > \sum_{i=0}^{n} \beta_i = \deg \beta$$

  or if $\sum_{i=0}^{n} \alpha_i = \sum_{i=0}^{n} \beta_i$ and in the difference $\alpha - \beta \in \mathbb{Z}^{n+1}$ the first nonzero entry is negative.

- The *lexicographic order* $\prec_{\text{lex}}$ on $\mathcal{P}$ is given by $x^\alpha \succ_{\text{lex}} x^\beta$ if in the difference $\alpha - \beta \in \mathbb{Z}^{n+1}$, the last nonzero entry is positive.

- The *degree lexicographic order* $\prec_{\text{deglex}}$ on $\mathcal{P}$ is given by $x^\alpha \succ_{\text{deglex}} x^\beta$ if

$$\deg \alpha = \sum_{i=0}^{n} \alpha_i > \sum_{i=0}^{n} \beta_i = \deg \beta,$$

  or if $\sum_{i=0}^{n} \alpha_i = \sum_{i=0}^{n} \beta_i$ and in the difference $\alpha - \beta \in \mathbb{Z}^{n+1}$, the last nonzero entry is positive.

In our context, we will usually only we compare multiindices of the same degree, and for such multiindices, the lex and the deglex order obviously coincide. We note that by our definitions, we have $x_0 \prec x_1 \prec ... \prec x_n$ for either of these three orders. We encountered both this and the reverse convention in literature, so we had to make choice which we will stick to for the remainder of this work.

The next two lemmata explain how via Gröbner bases, we obtain induced monomial orders on syzygy modules.

Lemma 2.3.4. *Let $\mathcal{G} = \{\mathbf{g}_1, ..., \mathbf{g}_s\} \subseteq \mathcal{P}^m \setminus \{0\}$ be a finite set. Let $\prec$ be a monomial order on $\mathcal{P}^m$. Then there is a monomial order $\prec_{\mathcal{G}}$ on $\mathcal{P}^s$ defined as follows:*

$$x^\alpha e_i \prec_{\mathcal{G}} x^\beta e_j$$

if

$$\text{lt}_\prec(x^\alpha \mathbf{g}_i) \prec \text{lt}_\prec(x^\beta \mathbf{g}_j)$$

or if

$$\text{lt}_\prec(x^\alpha \mathbf{g}_i) = \text{lt}_\prec(x^\beta \mathbf{g}_j) \quad \textit{and} \quad i > j.$$

Proof. This lemma is a slightly more general version of one part of [CLO98, Chapter 5, Theorem 3.3], where its proof is left to the reader as an exercise. It is obvious that $\prec_{\mathcal{G}}$ is a linear ordering and that it is preserved under multiplication with monomials $x^\alpha \in \mathcal{P}$. To show that it is a well-ordering, we remark that $x^\alpha e_i \succ_{\mathcal{G}} x^\beta e_j$ implies $\text{lt}_\prec(x^\alpha \mathbf{g}_i) \geq \text{lt}_\prec(x^\beta \mathbf{g}_j)$. But since for a given monomial $\mathbf{m} \in \mathcal{P}^m$, we can have at most $\leq s$ monomials $x^\beta e_j$ such that $\text{lt}_\prec(x^\beta \mathbf{g}_j) = \mathbf{m}$. So the existence of a strictly descending infinite chain

$$x^{\alpha_1} e_{i_1} \succ_{\mathcal{G}} x^{\alpha_2} e_2 \succ_{\mathcal{G}} ...$$

with respect to $\prec_{\mathcal{G}}$ gives an infinite chain

$$x^{\alpha_1}\mathbf{g}_{i_1} \geq x^{\alpha_2}\mathbf{g}_2 \geq ...$$

where we have a strong inequality at at least every s-th step. So we have a strictly descending infinite descending chain with respect to $\prec$. But then since $\prec$ is a well-ordering, this is impossible. □

Definition 2.3.5. In the situation of Lemma 2.3.4, the monomial order $\prec_{\mathcal{G}}$ is called the *Schreyer order* on $\mathcal{P}^s$ (with respect to $\mathcal{G}$ and $\prec$).

Definition 2.3.6. For a finite set $G = \{t_1, ..., t_s\} \subseteq \mathcal{P}^m$, we call

$$\operatorname{Syz}(G) = \left\{ \sum_{j=1}^{s} \mathcal{P}_j e_j \in \mathcal{P}^s \mid \sum_{j=1}^{s} \mathcal{P}_j t_j = 0 \right\}$$

the *(first) syzygy module of* T. An element of $\operatorname{Syz}(T)$ is called a *syzygy of* T. We recursively define $\operatorname{Syz}^1(G) = \operatorname{Syz}(G)$ and $\operatorname{Syz}^i(G) = \operatorname{Syz}\big(\operatorname{Syz}^i(G)\big)$ for $i \geq 2$.

Definition 2.3.7. Let $\mathcal{G} = \{\mathbf{g}_1, ..., \mathbf{g}_s\} \subseteq \mathcal{P}^m$ be a Gröbner basis with respect to a monomial oder $\prec$ on $\mathcal{P}^m$. For two monomials $x^\mu e_i, x^\nu e_j \in \mathcal{P}^m$, we define the least common multiple to be

$$\operatorname{lcm}(x^\mu e_i, x^\nu e_j) = \begin{cases} \operatorname{lcm}(x^\mu, x^\nu) e_i & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

where $\operatorname{lcm}(x^\mu, x^\nu)$ is the usual least common multiple in $\mathcal{P}$. Since $\mathcal{G}$ is a Gröbner bases, for any $\mathbf{g}_i, \mathbf{g}_j \in \mathcal{G}$ with $\operatorname{lcm}\big(\operatorname{lt}(\mathbf{g}_i), \operatorname{lt}(\mathbf{g}_j)\big) \neq 0$, there are polynomials $P_k^{i,j}$ such that

$$\frac{\operatorname{lcm}\big(\operatorname{lt}(\mathbf{g}_i), \operatorname{lt}(\mathbf{g}_j)\big)}{\operatorname{lt}(\mathbf{g}_i)}\mathbf{g}_i - \frac{\operatorname{lcm}\big(\operatorname{lt}(\mathbf{g}_i), \operatorname{lt}(\mathbf{g}_j)\big)}{\operatorname{lt}(\mathbf{g}_j)}\mathbf{g}_j = \sum_{k=1}^{s} P_k^{i,j}\mathbf{g}_k \in \mathcal{P}^s,$$

where the sum on the right is a standard representation (which is not necessarily unique) with respect to $\mathcal{G}$, i. e. we have

$$\operatorname{lt}\left(\frac{\operatorname{lcm}\big(\operatorname{lt}(\mathbf{g}_i), \operatorname{lt}(\mathbf{g}_j)\big)}{\operatorname{lt}(\mathbf{g}_i)}\mathbf{g}_i - \frac{\operatorname{lcm}\big(\operatorname{lt}(\mathbf{g}_i), \operatorname{lt}(\mathbf{g}_j)\big)}{\operatorname{lt}(\mathbf{g}_j)}\mathbf{g}_j\right) \succeq \operatorname{lt}(P_k^{i,j}\mathbf{g}_k)$$

for all $P_k^{i,j}\mathbf{g}_k$. For such a standard representation, we have that

$$\mathbf{s}_{i,j} = \frac{\operatorname{lcm}\big(\operatorname{lt}(\mathbf{g}_i), \operatorname{lt}(\mathbf{g}_j)\big)}{\operatorname{lt}(\mathbf{g}_i)}e_i - \frac{\operatorname{lcm}\big(\operatorname{lt}(\mathbf{g}_i), \operatorname{lt}(\mathbf{g}_j)\big)}{\operatorname{lt}(\mathbf{g}_j)}e_j - \sum_{k=1}^{s} P_k^{i,j} e_k \in \mathcal{P}^s$$

is an element of $\operatorname{Syz}(\mathcal{G})$. For a given pair (i,j), there may be multiple syzygies of shape $\mathbf{s}_{i,j}$, as the choice of the $P_k^{i,j}e_k$ is not necessarily unique.

Theorem 2.3.8 (Schreyer's Theorem for Gröbner bases)**.** *Let $\mathcal{G} = \{\mathbf{g}_1, ..., \mathbf{g}_s\}$ be a Gröbner basis for $\langle \mathcal{G} \rangle \subseteq \mathcal{P}^m$ with respect to the monomial order $\prec_{\mathcal{G}}$. Then any set $S \subseteq P^s$ which contains exactly one $\mathbf{s}_{i,j}$ for each pair (i, j) with $i \neq j$ and $\operatorname{lcm}\big(\operatorname{lt}(\mathbf{g}_i), \operatorname{lt}(\mathbf{g}_j)\big) \neq 0$, is a Gröbner basis for the syzygy module* $\operatorname{Syz}(\mathbf{g}_1, ..., \mathbf{g}_s)$ *with respect to the Schreyer order $\prec_{\mathcal{G}}$.*

Proof. This is the other part of [CLO98, Chapter 5, Theorem 3.3], which is proved in the reference: There, a fixed division algorithm is considered, and as a consequence, the $P_k^{i,j} e_k$ are uniquely defined. But the proof remains unchanged for any other valid choice of $P_k^{i,j} e_k$. □

Remark 2.3.9. There are two aspects of the Schreyer order and the Schreyer Theorem which we emphasize:

- The Schreyer order depends on how the set $\mathcal{G}$ is ordered. We will later see in Section 2.3.5 that some involutive bases admit certain "intrinsic" orderings. This property of involutive bases is typically not present for arbitrary Gröbner basis.

- Given a chain complex, and in particular a free resolution

$$... \to F_l \to F_{l-1} \to ... \to F_0 \to \mathcal{M} \to 0,$$

  we can construct Schreyer orders for every module F_l. The idea is as follows: Let $e_1, ..., e_s$ be the standard basis on $\mathcal{P}^s$. For any given homomorphism $\varphi \colon \mathcal{P}^s \to \mathcal{P}^m$, with the properties that

$$\varphi(e_\alpha) \neq 0 \text{ for all } \alpha$$

  any monomial order on $\mathcal{P}^m$ induces a Schreyer order on $\mathcal{P}^s$ (after ordering the elements of the set $\{e_1, ..., e_s\}$, or their images under φ). Now we can obtain Schreyer orders on the F_l by iterating this principle, provided no generator of an F_l is mapped to 0. Explicitly, this means that from a monomial order $\prec_{\mathcal{P}^m}$ on $\mathcal{P}^m$, we obtain a (Schreyer) order $\prec_{\mathcal{P}^s}$ on $\mathcal{P}^s$ by defining

$$x^\mu e_\alpha \prec_{\mathcal{P}^s} x^\nu e_\beta$$

  if and only if

$$\operatorname{lt}\left(\varphi(x^\mu e_\alpha)\right) \prec_{\mathcal{P}^m} \operatorname{lt}\left(\varphi(x^\nu e_\beta)\right)$$

  or

$$\operatorname{lt}\left(\varphi(x^\mu e_\alpha)\right) \prec_{\mathcal{P}^m} \operatorname{lt}\left(\varphi(x^\nu e_\beta)\right) \text{ and } \alpha < \beta.$$

  Later on, we will consider the situation where the images (under the differential) the basis elements of F_{l+1} are an involutive basis of the image of F_l, so here indeed no generator is mapped to 0 and this idea can be applied to such a situation.

2.3.2 General involutive divisions

We will state some definitions about involutive bases and their properties. First, we will shortly sketch the general idea: Given any finite set $\mathcal{H} \subseteq \mathcal{P}$, we want to assign to each $\mathbf{h} \in \mathcal{H}$ a subset of the variables $\{x_0, ..., x_n\}$, which we will call the *multiplicative variables of* $\mathbf{h}$. Now an involutive division can simply be seen as a set of rules which tell us how to find these multiplicative variables. If these rules depend on $\mathbf{h}$, but not on $\mathcal{H}$, we will call the involutive division a global (involutive) division. Additionally, we will also assign to $\mathbf{h}$ the set of *non-multiplicative variables*; this set simply contains all variables that are not multiplicative for $\mathbf{h}$. The presentation in this chapter is based on [Sei10, Chapter 3].

Definition 2.3.10. An *involutive division* L is defined on the monoid $(\mathbb{T}, \cdot)$, if for any finite subset $B \subseteq \mathbb{T}$ and every $x^\nu \in \mathbb{T}$, there is a subset

$$N_{L,B}(x^\nu) = N_{L,B}(\nu) \subseteq \{0, ..., n\},$$

called the *multiplicative variables*, and a submonoid

$$L(\nu, B) = L(x^\nu, B) = \{x^\mu \in \mathbb{T} \mid \forall j \notin N_{L,B}(x^\nu)\colon \mu_j = 0\},$$

such that for the *involutive cones* $C_{L,B}(x^\nu) = x^\nu \cdot L(x^\nu, B) \subseteq \mathbb{T}$, the following conditions hold:

- If $x^\nu, x^\mu \in B$ and $C_{L,B}(x^\nu) = C_{L,B}(x^\mu) \neq \emptyset$, then $C_{L,B}(x^\nu) \subseteq C_{L,B}(x^\mu)$ or $C_{L,B}(x^\mu) \subseteq C_{L,B}(x^\nu)$ holds.
- If $B' \subseteq B$, then $N_{L,B}(x^\nu) \subseteq N_{L,B'}(x^\nu)$ for all $x^\nu \in B'$.

We call $\overline{N}_{L,B}(x^\nu) = \{0, ..., n\} \setminus N_{L,B}(x^\nu)$ the *non-multiplicative* variables of x^ν. An involutive division is *global*, if for any given x^ν, the set $N_{L,B}(x^\nu)$ is independent of the choice of B. We say that x^μ is an *involutive divisor of* x^ν, or that x^ν *is involutively divisible by* x^μ if $x^\nu \in C_{L,B}(x^\mu)$.

Definition 2.3.11. For an involutive division L and a finite set $B \subseteq \mathbb{T}$, the *involutive span of* B is given by

$$\langle B \rangle_L = \bigcup_{x^\nu \in B} C_{L,B}(x^\nu). \tag{2.3.1}$$

The set B is called *(weakly) involutive* if

$$\langle B \rangle_L = \langle B \rangle \cap \mathbb{T},$$

and a *weak involutive basis* of a monomial ideal I, if we have

$$\langle B \rangle_L = I \cap \mathbb{T}.$$

We say that B is *(strong) involutive basis* or an *L-basis* of I if B is a weak involutive basis of I and additionally, the union in Equation (2.3.1) is disjoint.

Let $I \trianglelefteq \mathcal{P} = \Bbbk[x_0, ..., x_n]$ be an ideal, $\prec$ a monomial order and $\mathcal{H} \subseteq I$ a finite subset with $\mathrm{lt}(\mathbf{h}_1) \neq \mathrm{lt}(\mathbf{h}_2)$ for $\mathbf{h}_1, \mathbf{h}_2 \in \mathcal{H}$ with $\mathbf{h}_1 \neq \mathbf{h}_2$. We say that $\mathcal{H}$ is an *involutive basis of* I (with respect to the monomial order $\prec$), if we have that the finite set $\{x^\nu \mid \exists \mathbf{h} \in \mathcal{H} \colon \mathrm{lt}_\prec(\mathbf{h}) = x^\nu\}$ is a strong involutive basis of the leading ideal $\mathrm{lt}_\prec(I)$; i.e. the leading monomials of $\mathcal{H}$ are a strong involutive basis of the leading ideal $\mathrm{lt}_\prec(I)$.

Given a polynomial submodule $\mathcal{M} \subseteq \mathcal{P}^m$, we say that a finite subset $\mathcal{H} \subseteq \mathcal{M}$ (again with $\mathrm{lt}(\mathbf{h}_1) \neq \mathrm{lt}(\mathbf{h}_2)$ for $\mathbf{h}_1, \mathbf{h}_2 \in \mathcal{H}$ with $\mathbf{h}_1 \neq \mathbf{h}_2$) is an *involutive basis of* $\mathcal{M}$ (for the monomial order $\prec$), if for any $1 \leq j \leq m$, the set

$$\mathcal{H}_j = \{x^\nu \mid \exists \mathbf{h} \in \mathcal{H} \colon \mathrm{lt}_\prec(\mathbf{h}) = x^\nu e_j\},$$

where $\{e_1, ..., e_m\}$ is the standard basis of $\mathcal{P}^m$ is an involutive basis of the ideal

$$\langle \{x^\nu \mid \exists f \in \mathcal{M} \colon \mathrm{lt}_\prec(f) = x^\nu e_j\} \rangle \subseteq \mathcal{P}.$$

For ideals and modules, we will use the notations $\mathcal{X}_{L,\mathcal{H},\prec}(\mathbf{h})$ for the multiplicative variables of $\mathbf{h}$ and $\overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h})$ for the non-multiplicative variables.

A first important fact about involutive bases is the Theorem below, which follows immediately from the the definition of involutive bases.

Theorem 2.3.12. *Any involutive basis is a Gröbner basis.*

Obviously, any statement concerning involutive bases can be given as a statement about ideals in $\mathcal{P}$; we will usually prefer this approach. As Definition 2.3.11 above is key to this work, we reformulate it once more for ideals of $\mathcal{P}$.

Remark 2.3.13. Let $\prec$ be a monomial order on $\mathcal{P}$, $\mathcal{H} \subseteq \mathcal{P}$ a finite set such that $\mathrm{lt}(\mathbf{h}_1) \neq \mathrm{lt}(\mathbf{h}_2)$ for $\mathbf{h}_1, \mathbf{h}_2 \in \mathcal{H}$ with $\mathbf{h}_1 \neq \mathbf{h}_2$, and L an involutive division. The set $\mathcal{H}$ is a weak L-basis for the ideal $I = \langle \mathcal{H} \rangle \subseteq \mathcal{P}$, if we have [1]

$$I = \sum_{\mathbf{h} \in \mathcal{H}} \Bbbk[\mathcal{X}_{L,\mathcal{H},\prec}(\mathbf{h})]\mathbf{h}. \tag{2.3.2}$$

The set $\mathcal{H}$ is a strong L-basis for the ideal $I \trianglelefteq \mathcal{P}$, if we have

$$I = \bigoplus_{\mathbf{h} \in \mathcal{H}} \Bbbk[\mathcal{X}_{L,\mathcal{H},\prec}(\mathbf{h})]\mathbf{h}.$$

In both cases, we take (direct) sums of $\Bbbk$-vector spaces.

In general, the involutive span of $\mathcal{H}$ (with respect to L and $\prec$) is given by

$$\langle \mathcal{H} \rangle_{L,\prec} = \sum_{\mathbf{h} \in \mathcal{H}} \Bbbk[\mathcal{X}_{L,\mathcal{H},\prec}(\mathbf{h})]\mathbf{h}.$$

For involutive bases for polynomial modules $\mathcal{M} \subseteq \mathcal{P}^m$, these notions are extended in analogy to ideals.

[1] For any set X of variables, we understand $\Bbbk[X] = \Bbbk[x_i \mid i \in X]$.

Definition 2.3.14. Let L be an involutive division with respect to a monomial order $\prec$ on $\mathcal{P}^m$. A finite set $\mathcal{G} \subseteq \mathcal{P}^m$ is *(head) auto reduced*[1] if no leading term of an element of $\mathcal{G}$ is an involutive divisor of another leading term, i.e. for all $\mathbf{g}_1, \mathbf{g}_2 \in G$ with $\mathbf{g}_1 \neq \mathbf{g}_2$, we have $\mathrm{lt}_\prec(\mathbf{g}_1) \notin \mathcal{X}_{L,\mathcal{G}}(\mathbf{g}_2)$.

Combining this concept with Definition 2.3.10 of involutive divisions, we see that for an auto reduced set $\mathcal{G}$, the involutive cones of the elements of $\mathcal{G}$ are pairwise disjoint. Hence we have an alternative definition of involutive bases:

Theorem 2.3.15. *Let L be an involutive basis and $\prec$ a monomial order on $\mathcal{P}^m$. Let $\mathcal{H}$ be a finite subset of the polynomial module $\mathcal{M} \trianglelefteq \mathcal{P}^m$. $\mathcal{H}$ is an involutive basis of $\mathcal{M}$ (with respect to L and $\prec$) if and only if the following conditions are satisfied:*

- $\mathrm{lt}_\prec\big(I\big) = \mathrm{lt}_\prec(\langle\mathcal{H}\rangle_{L,\prec})$.
- *$\mathcal{H}$ is auto reduced.*

Assumption 2.3.16. For simplicity of notation, from now on we will refer to strong involutive bases simply as involutive bases.

For later use, we note two more properties related to involutive divisions:

Definition 2.3.17. Let L be an involutive division. The division L is *continuous*, if it satisfies the following condition: For any finite set $B \subseteq \mathbb{N}_0^{n+1}$ and any finite sequence $b^1, ..., b^m \in B$ where for each $p < m$ there is a non-multiplicative variable $i \in \overline{N}_{L,B}(b^p)$ with $b^p + 1_i \in C_{L,B}(b^{p+1})$, we have that all elements of this sequence are pairwise distinct.

Definition 2.3.18. An involutive division L is of *Schreyer type* for the monomial order $\prec$, if for any involutive set $\mathcal{H} \subseteq \mathcal{P}$ and any $\mathbf{h} \in \mathcal{H}$, the set $\overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h})$ is again (weakly) involutive.

In order to check if a given set is an involutive basis, at least in the case of a continuous division, it is often easier to take the next lemma as a criterion:

Lemma 2.3.19. *Let $\prec$ be a monomial order on $\mathcal{P}$, $\mathcal{H} \subseteq \mathcal{P}$ a finite set and L an involutive division. The set $\mathcal{H}$ is an L-basis for the ideal $I = \langle\mathcal{H}\rangle \subseteq \mathcal{P}$ if and only if for each $\mathbf{h} \in H$ and each $x_i \in \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h})$, we have $x_i\mathbf{h} \in \langle\mathcal{H}\rangle_{L,\prec}$.*

Proof. Obviously, if $\mathcal{H}$ is an involutive basis, then $\langle\mathcal{H}\rangle = \langle\mathcal{H}\rangle_{L,\prec}$ and any product of $x_i\mathbf{k}$ and an $\mathbf{h} \in \mathcal{H}$ and a variable $x_i \in \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h})$ is contained in $\langle\mathcal{H}\rangle_{L,\prec}$.

For the other direction, it suffices to check that any product $x^\mu\mathbf{h}$ is contained in $\langle\mathcal{H}\rangle_{L,\prec}$. The proof is a variation of the proof of Lemma 4.1.6, which is based on the proof of [AFS15, Lemma 13]:

Consider the following algorithm:

[1] One could also define $\mathcal{G}$ to be auto reduced if no arbitrary monomial summand of $\mathbf{g}_1$ is contained in a $\mathcal{X}_{L,\mathcal{G}}(\mathbf{g}_2)$. Then, the term "head auto reduced" would be used to imply that one is only concerned about the leading term of $\mathbf{g}_1$, as is the case in the definition presented here. However, this more general definition is of no real further importance for this work.

Given a product $x^\mu \mathbf{h}$, we check if x^μ contains any variables which are non-multiplicative for $\mathbf{h}$. If this is not the case, then the algorithm terminates, giving $x^\mu \mathbf{h}$ as the output. Otherwise, we pick one such non-multiplicative variable x_i. Let $\mathbf{g}$ be the representation of $x_i \mathbf{h}$ as an element of $\langle \mathcal{H} \rangle_{L,\prec}$, which exists by assumption, so $\mathbf{g} = \sum_{\mathbf{h}_\alpha \in \mathcal{H}} P_\alpha \mathbf{h}_\alpha$ with $P_\alpha \in \Bbbk[\mathcal{X}_{L,\prec}(\mathbf{h}_\alpha)]$ for all α. Then we have $\frac{x^\mu}{x_i} \operatorname{lt}(\mathbf{g}) = x^\mu \operatorname{lt}(\mathbf{h})$

Let $\mathbf{h}_\beta$ be such that $\operatorname{lt}_\prec(P_\beta \mathbf{h}_\beta) = \operatorname{lt}_\prec \mathbf{g}$. Now we iterate our algorithm, by looking at the leading monomial of $\frac{x^\mu}{x_i} \operatorname{lt}_\prec(P_\beta \mathbf{h}_\beta)$, which, as we have just seen, has the same leading monomial as the input.

Our claim is that this algorithm

- terminates after a finite number of steps.
- gives as output a term $x^\nu \mathbf{h}'$ whose leading monomial is a product of an element $\mathbf{h}' \in \mathcal{H}$ with a monomial $x^\nu \in \Bbbk[\mathcal{X}_{L,\prec}(\mathbf{h}')]$, i.e. x^ν is multiplicative for $\mathbf{h}'$.

Indeed, the last part is obvious by construction of the algorithm, if it does terminate.

Now assume the algorithm does not terminate: Then we obtain an infinite chain

$$\mathbf{h}_{\gamma_1} \to \mathbf{h}_{\gamma_2} \to \dots$$

of elements of $\mathcal{H}$ by picking as $\mathbf{h}_{\gamma_i}$ the $\mathbf{h}_\beta$ calculated in the i-th iteration of the algorithm. Since $\mathcal{H}$ is finite, this means that at least one element appears more than once in this chain. But since L is continuous, this is impossible (see definition 2.3.17).

Now since the difference of $x^\mu \operatorname{lt}(\mathbf{h}_\alpha)$ and the output of the algorithm is smaller than $x^\mu \operatorname{lt}(\mathbf{h}_\alpha)$ with respect to the monomial order $\prec$, we can use this algorithm to construct the involutive standard representation of $x^\mu \operatorname{lt}(\mathbf{h}_\alpha)$. □

Remark 2.3.20. We note that this proof does in fact leads to an idea how one could try to find an involutive basis: Given a finite set $\mathcal{H}$, one can calculate the non-multiplicative variables, take the involutive product and then iteratively reduce the leading monomial like in the algorithm in the proof above. If not this process terminates with a nonzero element, we add this element to the set $\mathcal{H}$ and continue. While this algorithm might terminate in a finite number of steps, with an involutive basis as the output, there are nevertheless some problems, for example situations for which this algorithm does not terminate even in cases where an involutive basis exists. See [Sei10, Section 4.1] for a more detailed treatment of these topics.

For our purposes, this lemma is usually a sufficient criterion to check if a given set is an involutive basis, which is exactly what we will need at several points later in this work. Nevertheless, we point out that from our experience, the task of finding an involutive basis for a given ideal tends to be more challenging than verifying that the candidate set, once it is found, is indeed an involutive basis.

2.3.3 Examples of involutive bases

Now we will introduce some examples of involutive divisions: The Pommaret division, the Janet division and the class of pairwise divisions, which was recently introduced by Gerdt and Blinkov [GB11]. We will see that both the Pommaret and the Janet are continuous and of Schreyer type.

Definition 2.3.21 (The Pommaret division)**.** Let $\mathcal{P} = \Bbbk[x_0, ..., x_n]$ and $\prec$ be a monomial order on $\mathcal{P}$. For an exponent vector $\mu = (\mu_0, ..., \mu_n)$, and the monomial $x^\mu \neq 0$, we set

$$\mathrm{cls}(x^\mu) = \min\{i \mid \mu_i \neq 0\},$$

or $\mathrm{cls}(x^\mu) = n$ if x^μ is a nonzero constant. We call $\mathrm{cls}(x^\mu)$ the *class of* x^μ (or the *class of* μ). Then we define $\mathcal{X}_P(x^\mu) = \{x_0, ..., x_{\mathrm{cls}(x^\mu)}\}$. For a term $f \in \mathcal{P}$, we define the (Pommaret) multiplicative variables to be

$$\mathcal{X}_{P,\prec}(f) = \mathcal{X}_P(\mathrm{lt}_\prec(f)) = \left\{x_0, .., x_{\mathrm{cls}\left(\mathrm{lt}_\prec(f)\right)}\right\}.$$

Since this assignment is independent of $\mathcal{H}$, we will omit the index $\mathcal{H}$ when talking about (non)-multiplicative variables with respect to the Pommaret division. For the non-multiplicative variables, we have

$$\overline{\mathcal{X}}_{P,\prec}(f) = \left\{x_{\mathrm{cls}\left(\mathrm{lt}_\prec(f)\right)+1}, ..., x_n\right\}.$$

For a module $\mathcal{M} = \mathcal{P}^m$, a monomial order $\prec$ on $\mathcal{M}$ and an element $f \in \mathcal{M}$ with leading monomial $\mathrm{lt}_\prec(f) = x^\mu e_i$, we define $\mathrm{cls}(f) = \mathrm{cls}(x^\mu)$, implying

$$\mathcal{X}_{P,\prec}(f) = \mathcal{X}_P(\mathrm{lt}_\prec(f)) = \left\{x_0, .., x_{\mathrm{cls}(x^\alpha)}\right\}.$$

Theorem 2.3.22. [Sei10, Theorem 3.1.8] *The Pommaret division is an involutive division.*

We point out a special relationship between the degree reverse lexicographic order and the Pommaret division (or more precisely, the concept of the class $\mathrm{cls}(f)$ of f as in definition 2.3.21). The degrevlex-order trivially satisfies the definition below.

Definition 2.3.23. We say that a monomial order $\succ$ on $\mathcal{P}$ is a *class-respecting* order, if it satisfies

$$\mathrm{cls}(f) < \mathrm{cls}(g) \Longrightarrow f \prec_{\mathrm{degrevlex}} g$$

of all polynomials $f, g \in \mathcal{P}$.

Note that even in the case of the degrevlex-order, the inverse of this definition does not hold.

We will now give an example of a Pommaret basis. Throughout this work, we will often come back to this example, using it to illustrate newly introduced constructions.

Example 2.3.24. Let $\Bbbk$ be of characteristic $\neq 2$. Let $I \trianglelefteq \Bbbk[x_0, x_1, x_2, x_3]$ be the ideal

$$I = \langle x_1^3 + 2x_0^2 x_1, \quad x_1^2 x_3 + 2x_0^2 x_3, \quad x_1 x_2, \quad x_2^2, \quad x_3^2 \rangle.$$

Obviously, this generating set is minimal. If we use in advance the fact that the Pommaret division is continuous by Lemma 2.3.36, and the alternative definition for involutive basis of Lemma 2.3.19 in this special case, one can check that a Pommaret basis of I (with respect to the degrevlex order $\prec_{\text{degrevlex}}$) is given by

$$\mathcal{H} = \{x_0^2 x_2 x_3,\, x_1^3 + 2x_0^2 x_1,\, x_1^2 x_3 + 2x_0^2 x_3,\, x_1 x_2 x_3,\, x_2^2 x_3,\, x_1 x_2,\, x_2^2,\, x_3^2\}.$$

The multiplicative variables are given by

$\mathbf{h}$	$x_0^2 x_2 x_3$	$x_1^3 + 2x_0^2 x_1$	$x_1^2 x_3 + 2x_0^2 x_3$	$x_1 x_2 x_3$	$x_2^2 x_3$	$x_1 x_2$	x_2^2	x_3^2
$\mathcal{X}_{P,\prec}(\mathbf{h})$	x_0	x_0 x_1	x_0 x_1	x_0 x_1	x_0 x_1 x_2	x_0 x_1	x_0 x_1 x_2	x_0 x_1 x_2 x_3

Definition 2.3.25 (The Janet division)**.** Let $\mathcal{P} = \Bbbk[x_0, ..., x_n]$ and $\prec$ be a monomial order on $\mathcal{P}$. For a finite set $\mathcal{H} \subseteq \mathcal{P}$ of monomials, we define the sets

$$(d_k, ..., d_n) = \{x^\alpha \in \mathcal{H} | \alpha_i = d_i \,\forall k \leq i \leq n\}.$$

The variable x_n is (Janet) multiplicative for x^α, if $\alpha_n = \max_{x^\beta \in \mathcal{H}}\{\beta_n\}$ and x_k with $k < n$ is multiplicative for x^α if $\alpha_k = \max_{x^\beta \in (\alpha_{k+1}, ..., \alpha_n)}\{\beta_k\}$. For a finite set $\mathcal{H}$ of terms, we then define

$$\mathcal{X}_{J,\prec,\mathcal{H}}(f) = \mathcal{X}_{J,\prec,\text{lt}_\prec(\mathcal{H})}(\text{lt}_\prec(f)).$$

For a module $\mathcal{M} = \mathcal{P}^m$, a monomial order $\prec$ on $\mathcal{M}$ and a finite set $\mathcal{H} \subseteq \mathcal{M}$, we define the sets

$$\mathcal{H}_j = \{x^\beta \in \mathcal{P} | \exists g \in \mathcal{H} : \text{lt}_\prec(g) = x^\beta e_j\}$$

for all $1 \leq j \leq m$. Now for a term $f \in \mathcal{H}$ with leading monomial $\text{lt}_\prec(f) = x^\alpha e_j$, this means we have

$$\mathcal{X}_{J,\prec,\mathcal{H}}(f) = \mathcal{X}_{J,\prec,\text{lt}_\prec(\mathcal{H}_j)}(x^\alpha).$$

Theorem 2.3.26. [Sei10, Theorem 3.1.5] *The Janet division is an involutive division.*

Note that the Pommaret division is a global division, while the Janet division is not. We will reflect this fact in the notation for the multiplicative variables, as for a finite set $\mathcal{H}$ and $f \in \mathcal{H}$, we will simply write $\mathcal{X}_{P,\prec}(f)$ instead of $\mathcal{X}_{P,\mathcal{H},\prec}(f)$ when we restrict to the Pommaret division. With this in mind, we will keep $\mathcal{H}$ as an index when making statements about involutive divisions, in order to cover the general case.

Now, another obvious question is the (non-)existence of (strong) involutive bases. This time, the answer actually depends on the involutive division:

Theorem 2.3.27. *Let $\prec$ be a monomial order on $\mathcal{P}$. Then every ideal $I \trianglelefteq \mathcal{P}$ has a strong Janet basis with respect to $\prec$.*

Proof. This theorem is a special case of [Sei10, Theorem 4.5.13]. □

Remark 2.3.28. Unfortunately, there are modules for which Pommaret bases do not exist. However, we will see that this is only a question of "choosing the right coordinate system", at least if $\Bbbk$ is an infinite field. It even suffices to require the field $\Bbbk$ to be a sufficiently large field (with respect to the number of elements in a minimal generating system and their degrees). We will not treat this question in full detail, but just sketch the ideas and give the main results. For a more detailed view on this subject, see [Sei10, Section 4.3].

For the ring $\mathcal{P} = \Bbbk[x_0, ..., x_n]$, an invertible matrix $A \in \Bbbk^{(n+1)\times(n+1)}$ defines an automorphism of $\mathcal{P}$ by setting $x'_i = \sum_{j=0}^{n} A_{ij}x_j$, called a change of coordinates. In this way, we can view $\mathbf{x} = (x'_0, ..., x'_n)$ (or respectively, the invertible matrix $A \in \Bbbk^{(n+1)\times(n+1)}$) as a coordinate system. Of course, the initial coordinate system corresponds the identity matrix. Now if we have an ideal $I \trianglelefteq \mathcal{P}$ for which no Pommaret basis exists with respect to a given monomial order $\prec$, we can apply a change of coordinates and look at the image I' of I in the ring $\mathcal{P}' = \Bbbk[x'_0, ..., x'_n]$. Next, we ask the question whether I' has a Pommaret basis for the monomial order $\prec$ on $\mathcal{P}'$, which we want to be the "same" monomial order as it was on $\mathcal{P}$, in the sense that $x'^{\alpha} \prec x'^{\beta} \Leftrightarrow x^{\alpha} \prec x^{\beta}$. We will see that, at least for infinite fields, it is always possible for a given ideal to find a suitable coordinate system (or equivalently, a change of coordinates) for which the ideal has a Pommaret basis.

While we consider this question of coordinate systems, we seize this opportunity to define the notion of "genericity":

Definition 2.3.29. We say that a statement holds *generically* or *in generic coordinates*, if there is an open subset (with respect to the Zariski topology on $\Bbbk^{(n+1)\times(n+1)}$) of coordinate systems for which the statement holds.

Definition 2.3.30. Let $I \trianglelefteq \mathcal{P}$ be an ideal and $\prec$ a monomial order on $\mathcal{P}$. The variables $\mathbf{x} = (x_0, ..., x_n)$ are called *δ-regular* for I and $\prec$, if there is a Pommaret basis for I with respect to $\prec$. Otherwise, the variables $\mathbf{x}$ are called *δ-singular* for I (and $\prec$).

We will also say that an ideal $I \trianglelefteq \mathcal{P}$ is δ-regular if I has a Pommaret basis. If I is additionally a monomial ideal, we say that I is *quasi-stable*.

Definition 2.3.31. Let $I \trianglelefteq \mathcal{P}$ be a monomial ideal. I is *stable* if for any $x^{\mu} \in I$ with $\mu_i > 0$ and any $j < i$, we have that $x^{\mu - 1_i + 1_j} \in I$.

Theorem 2.3.32. [Sei10, Theorem 4.3.15] *Let $\Bbbk$ be infinite and let $\prec$ be a monomial order on $\mathcal{P}$. Then every ideal $I \trianglelefteq \mathcal{P}$ has a Pommaret basis in suitably chosen coordinates.*

Recall that by our definition, any involutive basis is finite. We note that in the theorem it would be enough to have a sufficiently large field, if the number and the degree of the generators of I are fixed.

In fact, "almost every" coordinate system is δ-regular:

Corollary 2.3.33. [Sei10, Theorem 4.3.16] *The set of coordinate systems* $\mathbf{x}$ *(seen as matrices, see Remark 2.3.28) which are* δ*-singular, for a given ideal* $I \trianglelefteq \mathcal{P}$ *and a monomial order* $\prec$*, form a Zariski closed proper subset of the affine space* $\mathbb{A}_{\Bbbk}^{(n+1)\times(n+1)}$*.*

Obviously, I is δ-regular if and only if $\operatorname{lt}(I)$ is quasi-stable. Additionally, δ-regular coordinates are generic coordinates in the sense that if a statement is true in δ-regular coordinates, then it holds generically.

Example 2.3.34. The "most basic" example of an ideal which does not have a Pommaret basis is the ideal

$$\langle x_0 \rangle \trianglelefteq \Bbbk[x_0, x_1],$$

for any monomial order $\prec$ satisfying $x_0 \prec x_1$. However, after any change of coordinates which maps x_0 to $ax_1' + bx_0'$ with $a \neq 0$, we obtain the ideal

$$\langle ax_1' + bx_0' \rangle \trianglelefteq \Bbbk[x_0', x_1'],$$

whose leading term is x_1'. Hence this ideal is δ-regular for any monomial order satisfying $x_0' \prec x_1'$, in particular for the degrevlex order.

In characteristic 2, an example of an ideal which is neither δ-regular in the given coordinates nor after any change of coordinates is the ideal

$$\langle x_0^2 x_1 + x_0 x_1^2 \rangle \trianglelefteq \mathbb{F}_2[x_0, x_1].$$

This behavior is independent of the given coordinate system, as the ideal is in fact invariant under any change of coordinates. Therefore, with respect to any monomial order satisfying $x_0 \prec x_1$, the leading monomial of the generator is of class 0 in any coordinate system.

The following theorem states that the Janet and the Pommaret division are closely related:

Theorem 2.3.35. *Let* $I \trianglelefteq \mathcal{P}$ *be an ideal and* $\prec$ *a monomial order on* $\mathcal{P}$*. If* I *has a Pommaret basis* $\mathcal{H}$ *with respect to* $\prec$*, then* $\mathcal{H}$ *is also a Janet basis with respect to* $\prec$*, and for each* $\mathbf{h} \in \mathcal{H}$*, the Pommaret and Janet multiplicative variables coincide, i.e.* $\mathcal{X}_{P,\prec}(\mathbf{h}) = \mathcal{X}_{J,\mathcal{H},\prec}(\mathbf{h})$*.*

So example 2.3.24 is also an example of a Janet basis.

Proof. [Sei10, Corollary 4.3.9] states that any Pommaret multiplicative variable is also Janet multiplicative. But now if $\mathcal{H}$ is a strong Pommaret basis and if there were more Janet multiplicative variables, the involutive cone of $\mathcal{H}$ with respect to the Janet division would not be a direct sum, contradicting [Sei10, Corollary 4.3.11]. □

For later use, we state:

Lemma 2.3.36. [Sei10, Lemma 4.1.5] *The Pommaret and the Janet division are continuous.*

and

Lemma 2.3.37. [Sei10, Lemma 5.4.9] *The Pommaret and the Janet division are of Schreyer type for any monomial order* $\prec$*.*

One major advantage of Pommaret bases is that several properties of an ideal $I \trianglelefteq \mathcal{P}$ and the quotient ring $\mathcal{P}/I$ (or a submodule $\mathcal{M} \subseteq \mathcal{P}^s$ and $\mathcal{P}^s/\mathcal{M}$) can be read off from a Pommaret basis of I in very simple ways. We use a different language than in given reference, to avoid the introduction of more technical terms which would be of no further use for the purposes of this work.

Definition 2.3.38. Let $\mathcal{M}$ be a $\mathcal{P}$-module. A finite sequence $r_1, ..., r_k \in \mathcal{P}$ is called a *regular sequence for* $\mathcal{M}$, if r_1 is not a zero divisor for $\mathcal{M}$ and for $1 < i \leq k$, r_i is not a zero divisor for $\mathcal{M}/\langle r_1, ..., r_{i-1}\rangle\mathcal{M}$. A regular sequence is *maximal*, if it cannot be extended to a longer regular sequence. It is a well-known fact that all maximal regular sequences of $\mathcal{M}$ have the same length. This length is called the *depth of* $\mathcal{M}$.

Theorem 2.3.39. [Sei10, Theorem 5.2.7] *Let* $\mathcal{H}$ *be a Pommaret basis of the ideal* $I \trianglelefteq \mathcal{P}$ *with respect to the degrevlex order. Let* $d = \min_{\mathbf{h}\in\mathcal{H}} \operatorname{cls} \mathbf{h}$*. Then the variables* $x_0, ..., x_d$ *are a maximal regular sequence of* I *and we have* $\operatorname{depth} I = d + 1$ *and* $\operatorname{depth}(\mathcal{P}/I) = d$*.*

Definition 2.3.40. Let $\mathcal{M}$ be a $\mathcal{P}$-module. The *dimension of* $\mathcal{M}$ is given by $\dim(\mathcal{M}) = 1 + \deg \mathrm{HP}_{\mathcal{M}}$. The module $\mathcal{M}$ is called a *Cohen-Macaulay module* if $\dim \mathcal{M} = \operatorname{depth} \mathcal{M}$.

Theorem 2.3.41. [Sei10, Theorem 5.2.1] *Let* $\mathcal{H}$ *be a Pommaret basis of the ideal* $I \trianglelefteq \mathcal{P}$ *with respect to a monomial order. Let* $q = \max\limits_{\mathbf{h}\in\mathcal{H}} \deg \mathbf{h}$*. Then we have*

$$\dim(P/I) = \min\{i \mid \langle \mathcal{H}, x_0, ..., x_{i-1}\rangle_q = \mathcal{P}_q\}.$$

Theorem 2.3.42. [Sei10, Theorem 5.2.9] *Let* $\mathcal{H}$ *be a Pommaret basis of the ideal* $I \trianglelefteq \mathcal{P}$ *with respect to the degrevlex order.* P/I *is a Cohen-Macaulay module, if and only* $\langle \mathcal{H}, x_0, ..., x_{d-1}\rangle_q = \mathcal{P}_q$*, where* $d = \operatorname{depth}(\mathcal{P}/I) = \min_{\mathbf{h}\in\mathcal{H}} \operatorname{cls} \mathbf{h}$ *and* $q = \max_{\mathbf{h}\in\mathcal{H}} \deg \mathbf{h}$*.*

Theorem 2.3.43. [Sei10, Theorem 5.5.11] *Let* $\mathcal{H}$ *be a Pommaret basis of the polynomial module* $\mathcal{M} \subseteq \mathcal{P}^s$ *with respect to class-respecting monomial order* $\prec$*. Let* $d = \operatorname{depth}(\mathcal{P}/I) = \min_{\mathbf{h}\in\mathcal{H}} \operatorname{cls} \mathbf{h}$*. Then the projective dimension of* $\mathcal{M}$ *is* $\operatorname{pd} \mathcal{M} = n - d$*. Equivalently, we have* $\operatorname{pd}(\mathcal{P}^s/\mathcal{M}) = n - d + 1$*.*

Corollary 2.3.44 (The Auslander-Buchsbaum formula)**.** [Sei10, Corollary 5.5.12] *Let* $\mathcal{M}$ *be a polynomial* $\mathcal{P}$*-module. Then we have*

$$\operatorname{depth} \mathcal{M} + \operatorname{pd} \mathcal{M} = n + 1.$$

Theorem 2.3.45. [Sei10, Theorem 5.5.15] *Let* $\mathcal{H}$ *be a Pommaret basis of the module* $\mathcal{M} \trianglelefteq \mathcal{P}^s$ *with respect to the degrevlex order. Let* $q = \max_{\mathbf{h}\in\mathcal{H}} \deg \mathbf{h}$*. Then the Castelnuovo-Mumford regularities of* I *and* $\mathcal{P}/I$ *are given by* $\operatorname{reg} I = q$ *and* $\operatorname{reg}(\mathcal{P}/I) = q - 1$*.*

Definition 2.3.46 (Pairwise divisions)**.** An involutive division L is *pairwise* if for any finite set $U \subseteq \mathbb{T}$, and any $\mu \in U$, we have

$$\overline{\mathcal{X}}_{L,U}(\mu) = \bigcup_{\nu \in U} \overline{\mathcal{X}}_{L,\{\mu,\nu\}}(\mu). \tag{2.3.3}$$

Theorem 2.3.47. [GB11, Theorem 1] *Let $\sqsubset$ be a total order on $\mathbb{T}$. Let $\mu \in \mathbb{T}$ be a multiindex and $\sigma \in S_{n+1}$ be a permutation of the variables. For any $\nu \in \mathbb{T}$, let an assignment for non-multiplicative variables be given by*

$$\overline{\mathcal{X}}_{L,\{\mu,\nu\}}(\mu) = \begin{cases} \emptyset & \text{if } \nu \sqsubset \mu \text{ or } (\mu \sqsubset \nu \wedge \nu | \mu). \\ x_{\sigma(i)} & \text{where } i = \max\{j \mid \mu_j < \nu_j\} \text{ otherwise.}^1 \end{cases}$$

For any finite set $U \subseteq \mathbb{T}$, let $\overline{\mathcal{X}}_{L,U}(\mu)$ be given by Equation (2.3.3)*. Then this assignment defines an involutive division L.*

Remark 2.3.48. An involutive division given by Theorem 2.3.47 is called a $\sqsubset$-division[2]. There are some orders $\sqsubset$ which are particularly interesting:

- If $\sqsubset$ is the lex-ordering, the $\prec_{\text{lex}}$-division is in fact the Janet division (for $\sigma = \text{id}$), see also [Sem06, page 266].
- In [GB11] the authors introduce the $\sqsubset_{\text{alex}}$-division for the $\sqsubset_{\text{alex}}$-order given by

  $$f \prec_{\text{alex}} g \Leftrightarrow \big(\deg(f) > \deg(g)\big) \vee \big(\deg(f) = \deg(g) \wedge f \prec_{\text{lex}} g\big).$$

  The authors argue that the $\sqsubset_{\text{alex}}$-division is, from a computational point of view, heuristically better than the Janet division, which until then has been considered the computationally best involutive division. The main reason behind is that compared to the Janet division, there tend to be more $\sqsubset_{\text{alex}}$-multiplicative than Janet-multiplicative variables. Consequentially, one can expect $\sqsubset_{\text{alex}}$-bases to usually contain fewer elements than Janet- or Pommaret bases.

Nevertheless, if one looks the example given in Example 2.3.24, we see that occasionally, the inverse behavior can occur: Again, consider the ideal given by

$$I = \langle x_1^3 + 2x_0^2x_1, \quad x_1^2x_3 + 2x_0^2x_3, \quad x_1x_2, \quad x_2^2, \quad x_3^2\rangle.$$

One can check, again using Lemma 2.3.19, that a $\sqsubset_{\text{alex}}$-basis of I with respect tot the degrevlex-order and for the permutation $\sigma = \text{id}$ is given by the set $\mathcal{H}$ whose elements are given in the chart below, together with the respective multiplicative variables:

---

[1] more precisely, we want $\overline{\mathcal{X}}_{L,\{\mu,\nu\}}(\mu)$ to be the set containing this element

[2] In the literature about such $\sqsubset$-divisions, it is more common to use the term $\prec$-division. However, in our context, $\prec$ commonly denotes the monomial order which determines the leading terms, while the total order $\sqsubset$ is in general different from $\prec$.

$\mathbf{h}$	$\mathcal{X}_{\sqsubset_{\text{alex}},\mathcal{H},\prec}(\mathbf{h})$
x_3^2	x_0, x_1, x_2, x_3
x_2^2	x_0, x_1, x_2
x_2x_1	x_0, x_1
$x_3x_2^2$	x_0, x_2
$x_3x_2x_1$	x_0, x_1
$x_3x_1^2 + 2x_3x_0^2$	x_0, x_1
$x_1^3 + 2x_1x_0^2$	x_0, x_1
$x_3x_2^2x_1$	x_0, x_2
$x_3x_2x_0^2$	x_0
$x_3x_2^2x_1^2$	x_2
$x_3x_2^2x_1^3$	x_1, x_2
$x_3x_2^2x_1^2x_0$	x_2
$x_3x_2^2x_1^3x_0$	x_1, x_2
$x_3x_2^2x_1^2x_0^2$	x_0, x_2
$x_3x_2^2x_1^3x_0^2$	x_0, x_1, x_2

As one immediately sees, this involutive basis is much larger than the Pommaret basis given in Example 2.3.24; additionally the appearing sets of multiplicative variables are more diverse than in the case of Pommaret division.

2.3.4 Combinatorial decompositions

Involutive bases induce a decomposition of an ideal $I \trianglelefteq \mathcal{P}$ as in Equation (2.3.2). Analogously, often one is also interested in an analogous decomposition of the module $\mathcal{P}/I$.

Definition 2.3.49. A *Stanley decomposition of* $\mathcal{P}/I$ consists of a homomorphism as $\Bbbk$-linear spaces

$$\mathcal{P}/I \cong \bigoplus_{\mathbf{g}\in\mathcal{G}} \Bbbk[X_{\mathbf{g}}] \cdot \mathbf{g},$$

where $\mathcal{G} \subseteq \mathbb{T}$ is a finite set of monomials, $X_{\mathbf{g}} \subseteq \{0, ..., n\}$ is a set of variables for each $\mathbf{g} \in G$. The elements of $X_{\mathbf{g}}$ are called the *multiplicative variables of* $\mathbf{g}$.

From this definition it immediately follows that a Stanley decomposition of $\mathcal{P}/I$ is also a Stanley decomposition of $\mathcal{P}/\operatorname{lt}(I)$.

Remark 2.3.50. From a decomposition of $I \neq \mathcal{P}$ given by an involutive bases $\mathcal{H}$ of I, it is possible to directly construct a Stanley decomposition of $\mathcal{P}/I$. We will present an algorithm for Pommaret bases, as this decomposition is, directly or implicitly, used at several points later in this work. This will also serve as motivation why we speak of multiplicative variables both in the context of involutive bases and Stanley decompositions.

Let $\mathcal{H}$ be a Pommaret basis of an ideal $I \trianglelefteq \mathcal{P}$. Let $r = \max\{\deg \mathbf{h} \mid \mathbf{h} \in \mathcal{H}\}$. We set

$$\mathcal{G} = \bigcup_{t=0}^{r} \{x^\nu \mid \deg x^\nu = t, x^\nu \notin \mathrm{lt}(I)\}.$$

The multiplicative variables are given by

- $X_{\mathbf{g}} = \emptyset$ if $\deg \mathbf{g} < r$.
- $X_{\mathbf{g}} = \{0, ..., \mathrm{cls}\, \mathbf{g}\}$ if $\deg \mathbf{g} = r$.

Using the decomposition of I from Equation (2.3.2), we see that any monomial is contained either in $\mathrm{lt}(I)$ or in some $\mathbb{k}[X_{\mathbf{g}}]\mathbf{g}$, from which the correctness of this definition follows immediately.

However, this decomposition is not optimal, as we can remove some redundant elements: If for an $\mathbf{g}$ in $\mathcal{G}$ with $X_{\mathbf{g}} = \emptyset$, we have that

- $x_i\mathbf{g} \in \mathcal{G}$ for all $i \leq \mathrm{cls}\, \mathbf{g}$ and
- $X_{x_i\mathbf{g}} = \{0, ..., \mathrm{cls}(x_i\mathbf{g})\}$

then we can remove all $x_i\mathbf{g}$ from $\mathcal{G}$ and redefine $X_{\mathbf{g}} = \{0, ..., \mathrm{cls}\, \mathbf{g}\}$, since

$$\mathbb{k}[x_0, ..., x_{\mathbf{g}}\}]\mathbf{g} = \mathbb{k}\mathbf{g} \oplus \bigoplus_{i=0}^{\mathrm{cls}\, \mathbf{g}} \mathbb{k}[x_0, .., x_i]x_i\mathbf{g}.$$

A class examples of such Stanley decompositions which is minimal (i.e. contains no redundant elements) will be constructed in the proof of Corollary 7.2.6.

2.3.5 Syzygies of involutive bases

Lemma 2.3.51. [Sei10, Theorem 3.4.4] *Let $M \subseteq \mathcal{P}^m$ be a polynomial module, $\mathcal{H} = \{\mathbf{h}_1, ..., \mathbf{h}_s\}$ an involutive basis for $\mathcal{M}$ with respect to an involutive division L and $\prec$ a monomial order on $\mathcal{P}^m$. By definition 2.3.11, we have*

$$M = \bigoplus_{\alpha=1}^{s} \mathbb{k}[\mathcal{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)]\mathbf{h}_\alpha.$$

For any $f \in \mathcal{M}$, there are unique polynomials $P_\alpha^f \in \mathbb{k}[\mathcal{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)]$ such that

$$f = \sum_{\alpha=1}^{s} P_\alpha^f \mathbf{h}_\alpha$$

and

$$\mathrm{lt}_\prec(P_\alpha^f \mathbf{h}_\alpha) \preceq \mathrm{lt}_\prec(f) \text{ for all } \alpha.$$

Definition 2.3.52. In the situation of Lemma 2.3.51, given the representation

$$f = \sum_{\alpha=1}^{s} P_\alpha^f \mathbf{h}_\alpha,$$

we call the sum on the right the *involutive standard representation of* f (with respect to $\mathcal{H}, L, \prec$).

Now for a polynomial module $\mathcal{M} \subseteq \mathcal{P}^m$, an involutive basis $\mathcal{H}$ of $\mathcal{M}$, $\mathbf{h}_\alpha \in \mathcal{H}$ and any variable $x_k \in \mathcal{P}$, we may look at the involutive standard representation of $x_k \mathbf{h}_\alpha$. There are two cases which can occur:

- $x_k \in \mathcal{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)$: Here $x_k \mathbf{h}_\alpha$ is its own involutive standard representation.
- $x_k \in \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)$: Here by definition, $x_k \mathbf{h}_\alpha$ is not an involutive standard representation, so the involutive standard representation can be written in a unique way as

  $$x_k \mathbf{h}_\alpha = \sum_{\beta=1}^{s} P_\beta^{(\alpha,k)} \mathbf{h}_\beta, \tag{2.3.4}$$

  which serves as the defining equation of the polynomials $P_\beta^{(\alpha,k)}$.

Therefore, taking the involutive standard representation of $x_k \mathbf{h}_\alpha$, we see that we obtain a syzygy

$$\vec{S}_{\alpha;k} = x_k e_\alpha - \sum_{\beta=1}^{s} P_\beta^{(\alpha,k)} e_\beta \in \operatorname{Syz}(\mathcal{H}). \tag{2.3.5}$$

So for each product $xk\mathbf{h}_\alpha$ of an element $\mathbf{h}_\alpha$ of an involutive basis $\mathcal{H}$ with a non-multiplicative variable $x_k \in \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)$, we obtain an element of the syzygy module $\operatorname{Syz}(\mathcal{H})$ of $\mathcal{H}$. This corresponds to taking $\mathbf{s}_{i,j}$ in the theory of Gröbner basis, see definition 2.3.7 and Theorem 2.3.8.

For a Gröbner basis $\mathcal{G}$, we obtain a Gröbner basis of $\operatorname{Syz}(\mathcal{G})$ with the help of the S-polynomials, i.e. by calculating standard representations. We will now work towards the corresponding result for involutive bases. In order to obtain such a result, we first need to introduce some definitions.

Definition 2.3.53. Let $\mathcal{H} \subseteq \mathcal{P}^m$ be an involutive basis for an involutive division L. Any partial ordering $\sqsubset$ on $\mathcal{H}$ satisfying

$$\exists x_k \in \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha) : \operatorname{lt}_\prec(\mathbf{h}_\beta) | x_k \operatorname{lt}_\prec(\mathbf{h}_\alpha) \Longrightarrow \mathbf{h}_\alpha \sqsubset \mathbf{h}_\beta$$

for all $\mathbf{h}_\alpha, \mathbf{h}_\beta \in \mathcal{H}$ is called an *L-ordering on $\mathcal{H}$.*

If there is an L-ordering $\sqsubset$ on $\mathcal{H}$ such that $\mathbf{h}_\alpha \sqsubset \mathbf{h}_\beta \Longrightarrow \alpha < \beta$, we say that *$\mathcal{H}$ is ordered according to an L-ordering*, or simply that *$\mathcal{H}$ is L-ordered.*

Now, an obvious question is whether L-orderings exist.

Definition 2.3.54. Let $\mathcal{H} \subseteq P^m$ be an involutive basis for an involutive division L. To $\mathcal{H}$, we associate a directed graph, called the *L-graph of $\mathcal{H}$*, defined as follows:

- The set of vertices is $\mathcal{H}$.
- For any two vertices $\mathbf{h}_\alpha, \mathbf{h}_\beta \in \mathcal{H}$, the graph contains the directed edge $\mathbf{h}_\alpha \to \mathbf{h}_\beta$ if and only if there is a $x_k \in \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)$ such that $\mathrm{lt}_\prec(\mathbf{h}_\beta)$ is involutively divisible by $x_k \, \mathrm{lt}_\prec(\mathbf{h}_\alpha)$.

With this definition, we see that the question of the existence of L-orders can be reformulated in terms of the L-graph: An L-ordering exists, if and only if there are no elements $\mathbf{h}_{\alpha_1}, ..., \mathbf{h}_{\alpha_t} \in \mathcal{H}$ such that there is a cycle

$$\mathbf{h}_{\alpha_1} \to ... \to \mathbf{h}_{\alpha_t} \to \mathbf{h}_{\alpha_1},$$

in the L-graph, i.e. the L-graph of $\mathcal{H}$ is acyclic.

Lemma 2.3.55. *Let $\mathcal{H} \subseteq \mathcal{P}^m$ be an involutive basis for continuous involutive division L. Then the L-graph of $\mathcal{H}$ is acyclic. In particular, for any Pommaret or Janet basis, P-orders (or J-orders) exist.*

Proof. In [Sei10, Lemma 5.4.5], the lemma is proven for ideals of $\mathcal{P}$ and involutive bases with respect to any continuous involutive division L. But since any L-graph of $\mathcal{H}$ consists of (at most) m disjoint graphs

$$\{x^\mu \mid \exists f \in \langle \mathcal{H} \rangle \colon \mathrm{lt}(f) = x^\mu e_i\},$$

one for each each $1 \leq i \leq m$, the lemma also holds for modules. From Lemma 2.3.36, we see that both the Pommaret division and the Janet division are continuous, so the claim of the lemma follows for either type of involutive bases. □

Lemma 2.3.56. *Let $\mathcal{H} = \{\mathbf{h}_1, ..., \mathbf{h}_s\} \subseteq \mathcal{P}^m$ be an involutive basis for a continuous involutive division L with respect to a monomial order $\prec$ on $\mathcal{P}^m$. Let $\mathbf{h}_\alpha \in \mathcal{H}$. Let $\mathcal{H}$ be ordered according to an L-ordering. Let $x_k \in \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)$. Then we have*

$$\mathrm{lt}_{\prec_\mathcal{H}}(\vec{S}_{\alpha;k}) = x_k e_\alpha.$$

Proof. [Sei10, Lemma 5.4.7] establishes the lemma for involutive bases of $\mathcal{P}$, but the proof remains the same for $\mathcal{P}^m$. □

Definition 2.3.57. Let $\mathcal{H} \subseteq \mathcal{P}^m$ be involutive basis for a continuous involutive division L, ordered according to an L-ordering. Then we define

$$\mathcal{H}_{\mathrm{Syz}} = \left\{\vec{S}_{\alpha;k} \mid \mathbf{h}_\alpha \in \mathcal{H}, x_k \in \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha)\right\}.$$

Now we have finished the preparations to state the involutive version of the Schreyer Theorem 2.3.8.

Theorem 2.3.58. [Sei10, Theorem 5.4.10] *Let L be a continuous involutive division of Schreyer type. Let $\mathcal{H} \subseteq \mathcal{P}^m$ be an L-ordered involutive L-basis with respect to a monomial order $\prec$. Then the set $\mathcal{H}_{\mathrm{Syz}}$ is an L-basis for the module* $\mathrm{Syz}(\mathcal{H})$ *with respect to the Schreyer order $\prec_{\mathcal{H}}$.*

In particular, combining Lemma 2.3.36 and Lemma 2.3.37, we see that this theorem holds for the Pommaret and the Janet division.

By iterating the construction of $\mathcal{H}_{\mathrm{Syz}}$, we obtain:

Theorem 2.3.59. *Let $\mathcal{H} \subseteq \mathcal{P}^m$ be an involutive basis for the polynomial module $M = \langle \mathcal{H} \rangle \subseteq \mathcal{P}^m$, with respect to a continuous involutive division L of Schreyer type. We define $\beta_0^{(k)}$ to be the number of elements of $\mathcal{H}$ with exactly k non-multiplicative variables and $d = \min\{k | \beta_0^{(k)} > 0\}$. Then $\mathcal{M}$ has a free resolution*

$$0 \to \mathcal{P}^{t_{n-d}} \to \ldots \to \mathcal{P}^{t_1} \to \mathcal{P}^{t_0} \to \mathcal{M} \to 0$$

where the ranks of the free modules are given by

$$t_i = \sum_{k=0}^{n-i} \binom{n-k}{i} \beta_0^{(k)}.$$

Proof. See [Sei10, Theorem 5.4.12] for Pommaret bases, and [Sei10, Remark 5.4.13] for the more general case. □

Lemma 2.3.60. *The resolution introduced in Theorem 2.3.59 is minimal if and only if all first syzygies $\vec{S}_{\alpha;k}$ do not contain any constants.*

Proof. In [Sei09, Theorem 8.1], a proof for Pommaret bases is given. However, the proof remains unchanged for any other involutive basis satisfying the assumptions of Theorem 2.3.59. □

Remark 2.3.61. We note that this lemma can also be adapted for the resolution

$$0 \to \mathcal{P}^{t_{n-d}} \to \ldots \to \mathcal{P}^{t_1} \to \mathcal{P}^{t_0} \to \mathcal{P}^m \to \mathcal{P}^m/\mathcal{M} \to 0$$

to get an equivalent statement for $\mathcal{P}^m/\mathcal{M}$.

Remark 2.3.62. In [Sei10, page 200], it is explained how the resolution introduced in Theorem 2.3.59 can be presented in an explicit manner as a complex, using the exterior algebra. While in the reference, the construction is established only for Pommaret bases, again by [Sei10, Remark 5.4.13], it can be extended to any involutive basis with respect to a continuous involutive division L. This idea is central for the statement of one of our main results, Theorem 4.2.3. We will make some minor adaptations in our notations, but the idea of the construction remains unchanged. The correctness of this approach is treated en detail in the given references, so we will skip it here.

Let $\mathcal{H}$ be an involutive basis of $\mathcal{M}$ with $|\mathcal{H}| = s$ elements. Let $\mathcal{W}$ be a free $\Bbbk$-module with s elements whose elements will be denoted by $\{\mathbf{v}_\emptyset \otimes_\Bbbk \mathbf{h}_\alpha | \mathbf{h}_\alpha \in \mathcal{H}\}$. Let $\mathcal{V}$ be a free $\Bbbk$-module of rank $n+1$ with basis $\mathbf{v}_0, \ldots, \mathbf{v}_n$, i.e. we have one generator $\mathbf{v}_i$ for each variable $x_i \in \mathcal{P}$. Let $\Lambda\mathcal{V}$ be the exterior algebra over $\mathcal{V}$. We set $C_i = \mathcal{P} \otimes_{\Bbbk} (\Lambda^i \mathcal{V} \otimes_{\Bbbk} \mathcal{W}) \cong (\mathcal{P} \otimes_{\Bbbk} \Lambda^i \mathcal{V}) \otimes_{\mathcal{P}} (\mathcal{P} \otimes_{\Bbbk} \mathcal{W})$. A basis of $\Lambda^i \mathcal{V}$ is given by the elements of shape $\mathbf{v}_{\mathbf{k}} = \mathbf{v}_{k_i} \wedge ... \wedge \mathbf{v}_{k_i}$, where $\mathbf{k} = (k_1, ..., k_i)$ is an ordered sequence of length i, i.e. $0 \le k_1 < ... < k_i \le n$. Now a basis for $\Lambda^i \mathcal{V} \otimes_{\Bbbk} \mathcal{W}$ is given by the elements of shape $\mathbf{v}_{\mathbf{k}} \otimes_{\Bbbk} \mathbf{v}_{\emptyset} \otimes_{\Bbbk} \mathbf{h}_\alpha$. In order to shorten our notation, for these elements, we will just write $\mathbf{v}_{\mathbf{k}} \otimes_{\Bbbk} \mathbf{h}_\alpha$ for such an element. Now consider the submodule $G_i \subseteq C_i$ which is generated by the basis elements $\mathbf{v}_{\mathbf{k}} \otimes_{\Bbbk} \mathbf{h}_\alpha$ for which $\mathbf{k} \subseteq \overline{\mathcal{X}}_{L,\prec}(\mathbf{h}_\alpha)$ holds.

For each pair $\mathbf{k}, \mathbf{h}_\alpha$ with $|\mathbf{k}| = i \ge 1$ which gives a generator of a G_i, we have an element $\vec{S}_{(\alpha,\mathbf{k})} \in \mathrm{Syz}^i(\mathcal{H}) \subseteq G_{i-1}$; and for each $k_{i+1} > k_i$, we have a unique involutive standard representation

$$x_{k_{i+1}} \vec{S}_{(\alpha,\mathbf{k})} = \sum_{\beta=1}^{s} \sum_{\boldsymbol{\ell}} P_{\beta,\boldsymbol{\ell}}^{(\alpha,\mathbf{k},k_{i+1})} \vec{S}_{\beta;\boldsymbol{\ell}},$$

where the second sum ranges over all ordered sequences $\boldsymbol{\ell} = (l_1, ..., l_i)$ of length i such that $0 \le l_1 < ... < l_i \le n$. We define homomorphisms $d_0 \colon G_0 \to \mathcal{M}$ by

$$d_0(\mathbf{v}_{\emptyset} \otimes_{\Bbbk} \mathbf{h}_\alpha) = \mathbf{h}_\alpha \in \mathcal{M}$$

and $d_i \colon G_{i+1} \to G_i$ by

$$d_i(\mathbf{v}_{\mathbf{k}} \otimes_{\Bbbk \cup \{k_{i+1}\}} \mathbf{h}_\alpha) = \vec{S}_{(\alpha,\Bbbk \cup \{k_{i+1}\})} = x_{k_{i+1}} \mathbf{v}_{\mathbf{k}} \otimes_{\Bbbk} \mathbf{h}_\alpha - \sum_{\beta=1}^{s} \sum_{\boldsymbol{\ell}} P_{\beta,\boldsymbol{\ell}}^{(\alpha,\mathbf{k},k_{i+1})} \mathbf{v}_{\boldsymbol{\ell}} \otimes_{\Bbbk} \mathbf{h}_\beta.$$

Now with these homomorphisms as differential, $(\mathcal{G}, d)$ is a free resolution of $\mathcal{M}$.

If we further take the degrees of the elements of the involutive basis $\mathcal{H}$ into consideration, and recall the definition 2.1.16 of the bigraded Betti numbers, we have the motivation for

Definition 2.3.63. In the situation of Theorem 2.3.59, let $\beta_{0,f}^{(k)}$ be the number of elements of $\mathcal{H}$ of degree f and with exactly k non-multiplicative variables, i.e.

$$\beta_{0,f}^{(k)} = \left|\left\{\mathbf{h} \in \mathcal{H} \,\middle|\, \deg \mathbf{h} = f, |\overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h})| = k\right\}\right|.$$

Then we say that the numbers

$$\beta_{i,f}^{*} = \sum_{k=0}^{n-i} \binom{n-k}{i} \beta_{0,f}^{(k)}$$

are the (bigraded) *pseudo Betti numbers of $\mathcal{M}$ for the involutive basis $\mathcal{H}$*. In analogy to the Betti table, we define the *pseudo Betti table of $\mathcal{M}$*.

With equation (2.3.2), which can be easily generalised to be applicable to modules, we see how to obtain the Hilbert function and the Hilbert polynomial of a $\mathcal{P}$-module $\mathcal{M}$ from an involutive basis of $\mathcal{M}$ with a simple combinatorial argument:

Lemma 2.3.64. [Sei10, Equation (4.6)] *Let $\mathcal{H} = \{\mathbf{h}_1, ..., \mathbf{h}_t\}$ be an involutive basis of the module $\mathcal{M} = \langle \mathcal{H} \rangle \subseteq \mathcal{P}^m$. For every $\mathbf{h}_\alpha$, let k_α be the number of multiplicative variables of $\mathbf{h}_\alpha$. Then we have*

$$\mathrm{HF}_{\mathcal{M}}(t) = \sum_{\alpha=1}^{t} \binom{t - \deg(\mathbf{h}_\alpha) + k_\alpha - 1}{t - \deg(\mathbf{h}_\alpha)} = \sum_{f \geq 0} \sum_{k=0}^{n} \binom{t - f + k - 1}{t - f} \beta_{0,f}^{(k)}. \tag{2.3.6}$$

Remark 2.3.65. Note that only finitely many $\beta_{0,f}^{(k)}$ are nonzero, so the second sum is indeed a finite sum. We understand the binomial coefficient $\binom{a}{b}$ to be 0 if $a < b$ (or equivalently $b < 0$).

Example 2.3.66. Obviously, the pseudo Betti numbers are upper bounds for the Betti numbers.

If we look at how the set $\mathcal{H}_{\mathrm{Syz}}$ is defined, we see that (for a graded module $\mathcal{M}$) for any $\mathbf{h}_\alpha \in \mathcal{H}$, the degree of $\vec{S}_{\alpha;k}$ is the degree of $\mathbf{h}_\alpha$, plus 1. Hence for graded modules, this statement can be reformulated to a more precise version including statements about the degrees of the generators of the shifted graded free modules in the graded resolution of $\mathcal{M}$.

For Pommaret bases, any set of non-multiplicative variables is of the shape $\{x_{\mathrm{cls}(\mathbf{h}_\alpha)+1}, ..., x_n\}$. Thus we see that for Pommaret bases, we have

$$\beta_0^{(k)} = \#\{\mathbf{h}_\alpha \in \mathcal{H} | \operatorname{cls}(\mathbf{h}_\alpha) = n - k\}.$$

Going back to the ideal I given in example 2.3.24, the pseudo Betti table of I for the Pommaret basis given there is

	0	1	2	3
2	3	3	1	0
3	4	7	3	0
4	1	3	3	1,

where all other numbers are understood to be 0. By remark 2.3.61, this gives the pseudo Betti table

0	1	2	3	4	5
0	1	0	0	0	0
1	0	3	3	1	0
2	0	4	7	3	0
3	0	1	3	3	1.

of $\mathcal{M} = \mathcal{P}/I$. However, computing the Betti table of I (for example, with CoCoALib), one sees that the Betti table of I is given by

	0	1	2	3
2	3	1	0	0
3	2	5	1	0
4	0	1	3	1,

which translates to the Betti table of $\mathcal{M} = \mathcal{P}/I$ given in remark 2.1.17.

2.4 Homological algebra

The presentation in this chapter is based on [Wei95].

Definition 2.4.1. Let $\mathcal{F}$ and $\mathcal{G}$ be two chain complexes

$$\mathcal{F} = ... \to F_{l+1} \xrightarrow{\varphi_{l+1}} F_l \to ... \quad \text{and} \quad \mathcal{G} = ... \to G_{l+1} \xrightarrow{\psi_{l+1}} G_l \to ....$$

- A *chain map* $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a set of homomorphisms $\alpha_l\colon F_l \to \mathcal{G}_l$ (of degree 0) such that $\alpha_l \circ \varphi_{l+1} = \psi_l \circ \alpha_{l+1}$ for all l.
- It is a simple fact that any chain map $\alpha\colon \mathcal{F} \to \mathcal{G}$ satisfies $\alpha(\operatorname{im}\varphi_l) \subseteq \operatorname{im}\psi_l$ and $\alpha(\ker\varphi) \subseteq \ker\psi_l$. Hence α induces a well-defined map $\alpha^*\colon H_l(\mathcal{F}) \to H_l(\mathcal{G})$ on the homology groups for all l by setting $\alpha^*(\overline{g}) = \overline{\alpha(g)}$, where $\bar{\cdot}$ denotes the respective equivalence classes.
- A chain map $\alpha\colon \mathcal{F} \to \mathcal{G}$ is called *null homotopic* if there is a set s of maps $s_l\colon F_l \to G_{l+1}$ such that[1]
$$\alpha = \psi s + s\varphi.$$
s is called a *chain contraction of* α.
- Two chain maps $\alpha, \beta\colon \mathcal{F} \to \mathcal{G}$ are called *chain homotopic* if the difference $\alpha - \beta$ is null homotopic.
- A chain map $\alpha\colon \mathcal{F} \to \mathcal{G}$ is called *a chain homotopy equivalence* if there is a chain map $\beta\colon \mathcal{G} \to \mathcal{F}$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are chain homotopic to the respective identity maps of $\mathcal{F}$ and $\mathcal{G}$. In this situation, we say that $\mathcal{F}$ and $\mathcal{G}$ are *homotopy equivalent.*

Lemma 2.4.2. [Wei95, Lemma 1.4.5] *If $\alpha, \beta\colon \mathcal{F} \to \mathcal{G}$ are chain homotopic, then they induce the same maps $H_l(\mathcal{F}) \to H_l(\mathcal{G})$.*

Corollary 2.4.3. *If $\mathcal{F}$ and G are homotopy equivalent, then their homology modules are identical. In particular, if $\mathcal{F}$ is a free resolution of a module $\mathcal{M}$ and $\mathcal{G}$ is homotopic to $\mathcal{F}$, then $\mathcal{G}$ is also a free resolution of $\mathcal{M}$.*

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are homotopy equivalent, then the concatenated induced maps $H_l(\mathcal{F}) \to H_l(\mathcal{G}) \to H_l(\mathcal{F})$ and $H_l(\mathcal{G}) \to H_l(\mathcal{F}) \to H_l(\mathcal{G})$ are the identity maps on $H_l(\mathcal{F})$ and $H_l(\mathcal{G})$. □

We will now give an alternative definition for the Betti numbers, using the a special case of the Tor-functor. We abstain from giving a more general definition of this functor, as it would serve no further purpose for this work.

Definition 2.4.4. Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{P}$-modules. Let

$$... \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} \mathcal{M} \to 0$$

[1] For such sets of maps it is common practice to omit the indices of the maps ϕ_l, i.e. the given equations are supposed to hold for all eligible values of l.

be a free resolution of $\mathcal{M}$ and $\mathcal{F} \otimes_{\mathcal{P}} \mathcal{N}$ be the chain complex

$$... \to F_2 \otimes_{\mathcal{P}} \mathcal{N} \to F_1 \otimes_{\mathcal{P}} \mathcal{N} \to F_0 \otimes_{\mathcal{P}} \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{P}} \mathcal{N} \to 0.$$

Then we define

$$\operatorname{Tor}_i^{\mathcal{P}}(\mathcal{M}, \mathcal{N}) = H_i(\mathcal{F} \otimes_P \mathcal{N}).$$

One can show that this definition is indeed independent of the choice of $\mathcal{F}$. Of course, this definition can be easily extended to keep track of gradings. With this in mind, we obtain another interpretation of the graded Betti numbers. Note that we have $\Bbbk \cong \mathcal{P}/\langle x_0, ..., x_n\rangle$, thus the action of $\mathcal{P}$-module on $\Bbbk$ is given by, for a polynomial $f \in \mathcal{P}$ acting on $\Bbbk$, multiplication with the constant term of f. This gives a well-known alternative approach to calculate the Betti numbers, presented in the lemma below.

Lemma 2.4.5. *Let $\mathcal{M}$ be a $\mathcal{P}$-module. Then we have*

$$\operatorname{Tor}_i^{\mathcal{P}}(M, \Bbbk) = \bigoplus_{j \in \mathbb{Z}} \Bbbk\big(-(j)\big)^{\beta_{i,j}}.$$

2.5 Splitting homotopies and strong deformation retracts

In this section, we explain another idea which allows us reduce a given chain complex to a smaller complex while preserving homology. The concept of splitting homotopies was introduced by Barnes and Lambe in [BL91]. The advantage of this idea, when compared to the step-by-step approach of Section 2.2, is that, given suitable circumstances, it enables us to "minimise" infinite resolutions.

Assumption 2.5.1. For the remainder of Section 2.5, let $\mathcal{F}$ be a chain complex

$$... \to F_{l+1} \xrightarrow{d_{l+1}} F_l \to ...$$

with differential d.

Definition 2.5.2. A *splitting homotopy* is a set $\phi = \{\phi_l\}$ of homomorphisms $\phi_l \colon F_l \to F_{l+1}$ such that

- $\phi^2 = 0$.
- $\phi d \phi = \phi$.

Lemma 2.5.3. *Let ϕ be a splitting homotopy. Let the set π of maps $\pi_l \colon F_l \to F_l$ be given by*

$$\pi = \operatorname{id}_{\mathcal{F}} - \phi d - d\phi$$

and let $\iota \colon \pi(\mathcal{F}) \to \mathcal{F}$ be the inclusion. Then π satisfies

- $\pi^2 = \pi$.
- $\pi\iota = \operatorname{id}_\pi \mathcal{F}$.

- $\iota\pi = \pi$.
- $d\pi = \pi d$.

Proof. Regarding the first point, we have

$$\begin{aligned}\pi^2 &= \mathrm{id} - \phi d - d\phi - \phi d + \phi d\phi d + \phi dd\phi - d\phi + d\phi\phi d + d\phi d\phi \\ &= \mathrm{id} - \phi d - d\phi - \phi d + \phi d - d\phi + d\phi \\ &= \mathrm{id} - \phi d - d\phi \\ &= \pi.\end{aligned}$$

From this equation, the second point follows at once, while the third point is obvious anyway. Regarding the last point, we note the equations

$$d\pi = d - d\phi d - dd\phi = d - d\phi d$$

and

$$\pi d = d - \phi dd - d\phi d = d - d\phi d.$$

□

We would like to turn $\pi(\mathcal{F})$ into a chain complex. The lemma below ensures we can indeed define a differential on $\pi(\mathcal{F})$.

Lemma 2.5.4. *Let the maps ϕ, π, ι be given as in Lemma 2.5.3. For the maps $\delta\colon \pi(\mathcal{F}_{l+1}) \to \pi(\mathcal{F}_l)$ defined by*

$$\delta = \pi d\iota,$$

we have $\delta\delta = 0$, i.e. $\pi(\mathcal{F})$ together with the differential δ is a chain complex.

Proof. Using Lemma 2.5.3, we obtain

$$\delta^2 = \pi d\iota\pi d\iota = \pi d\pi d\iota = \pi\pi dd\iota = 0.$$

□

Now what are the properties of this complex? The most important fact for our purposes is that the homology remains unchanged.

Theorem 2.5.5. *The chain complexes $\mathcal{F}$ and $\pi(\mathcal{F})$ as in Lemma 2.5.4 are homotopy equivalent. A chain homotopy equivalence is given by the maps*[1] *π and ι.*

Proof. Checking definition 2.4.1 for homotopy equivalence. Again using Lemma 2.5.3, we have

$$\iota\pi - \mathrm{id}_{\mathcal{F}} = \pi - \mathrm{id}_{\mathcal{F}} = -(\phi d + d\phi)$$

and

$$\pi\iota - \mathrm{id}_{\pi(\mathcal{F})} = \mathrm{id}_{\pi(\mathcal{F})} - \mathrm{id}_{\pi(\mathcal{F})} = 0.$$

□

[1] Strictly speaking, we are working with the map obtained from π by restricting its codomain to $\pi(\mathcal{F})$.

Remark 2.5.6. We can also give a formula for δ in terms of d and ϕ instead of d and π. Depending on the context, either formula may be useful. We have

$$\delta = \pi d\iota = (\mathrm{id}_{\mathcal{F}} - \phi d - d\phi)d\iota = d\iota - \phi dd - d\phi d\iota = d\iota - d\phi d\iota.$$

Usually we will omit ι in there formulas and simply write

$$\delta = d - d\phi d.$$

3 Algebraic discrete Morse theory

In this chapter, we will discuss some elements of algebraic discrete Morse theory. Our goal is to explain how algebraic discrete Morse theory allows us to construct a free resolution of some modules. We will see that this concept can be applied to a class of involutive bases with respect to a continuous involutive division of Schreyer type, which in particular includes Pommaret and Janet bases.

Discrete Morse theory was developed by Forman, see [For98] and [For02], allowing a CW-complex to be reduced to a smaller, homotopy-equivalent CW-complex. An algebraic version of this theory was established by Sköldberg in [Skö06] and by Jöllenbeck and Welker in [JW09].

Our basic notation is the same as in the paper [AFSS15]; the presentation there is in turn based on the papers [Skö06] and [Skö11] by Sköldberg. Unless stated otherwise, any proof of a theorem, lemma or corollary which is cited from the papers [Skö06] and [Skö11] is mathematically the same proof as in those papers; with some occasional minor changes to the notation. However, in a few cases, some proofs in this chapter have been partially rewritten, or contain some additional arguments; if this is the case, it is explicitly mentioned at the beginning of the respective proof.

In order to make the definitions apply to a more general situation than in [AFSS15], we will make some minor adaptations.

3.1 Basics of discrete Morse theory

Definition 3.1.1. Let R be a commutative ring and $\mathcal{C}$ a finite chain complex of R-modules

$$0 \longrightarrow C_p \longrightarrow C_{p-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0 \tag{3.1.1}$$

where each module $C_m = \bigoplus_{a \in I_m} K_a$ is written as a direct sum of R-modules with disjoint index sets I_m (Sköldberg call auch a complex a *based complex*). Note that we require $\mathcal{C}$ to be of finite length, however the index sets I_m may be infinite. To such a complex, we associate a directed graph $\Gamma_{\mathcal{C}}$: The set of vertices is the disjoint union $\sqcup_m I_m$ and the graph contains the edge $a \to b$ if and only if $a \in I_{m+1}$, $b \in I_m$ for some $m \geq 0$ and $d_{b,a} = \pi_b(d_{\mathcal{C}}|_{K_a}) \neq 0$. Here, $d_{\mathcal{C}}$ is the differential in $\mathcal{C}$ and for $C_m = \bigoplus_{a \in I_m} K_a$, π_b is the canonical projection $\pi_b \colon C_m \to K_b$ for $b \in I_m$ while $\cdot|_{K_a}$ denotes the restriction to K_a.

A *partial matching* on a directed graph $D = (V, E)$ with vertices V and edges E is a subset $A \subseteq E$ of edges such that no vertex is incident to 2 or more edges in A. Given a partial matching A, we define a new directed graph $D^A = (V, E^A)$ which is obtained from D by reversing all the arrows contained in A: Thus the graph D^A has the same vertices as D and contains the edge $a \to b$ if and only if

$$(b \to a) \in A \vee \big((a \to b) \in E \setminus A\big).$$

We define $A^+ \subseteq V$ to be the (sub-)set of vertices of D that are sources of the arrows in A, i.e.

$$A^+ = \{a \in V \mid \exists b \in V \colon (a \to b) \in A\},$$

and $A^- \subseteq V$ as the sources of said arrows, i.e.

$$A^- = \{a \in V \mid \exists b \in V \colon (b \to a) \in A\};$$

finally $A^0 \subseteq V$ contains all vertices which are not incident to any arrow contained in A, i.e.

$$A^0 = \{a \in V \mid \nexists b \in V \colon (a \to b) \in A \vee (b \to a) \in A\} = V \setminus (A^+ \cup A^-).$$

Vertices contained in A^0 are called *A-critical*. Furthermore, we define

$$A^-_m = A^- \cap I_m, \quad A^+_m = A^+ \cap I_m, \quad A^0_m = A^0 \cap I_m$$

for each $m \in \mathbb{N}$.

For our applications, starting with chapter 4, we will work exclusively in a situation where every $C_m = \bigoplus_{a \in I_m} K_a$ is a free module and every K_a is a shifted module $\mathcal{P}(d)$.

Definition 3.1.2. A *Morse matching* on the directed graph $\Gamma_{\mathcal{C}}$ is a partial matching A satisfying the following conditions:

- For every edge $(a \to b) \in A$, the map $d_{b,a}$ is an isomorphism.
- For every index set I_m, there is a well-founded partial order $\sqsubset$ on I_m such that for any $a, c \in I_m$ with $a \neq c$, we have $c \sqsubset a$ if and only if there is a path $a \to b \to c$ in $\Gamma^A_{\mathcal{C}}$. We say that such an order $\sqsubset$ *respects the Morse matching A*.

If the modules of the complex $\mathcal{C}$ are finitely generated, the differential can be represented by matrices. So in this case, the graph $\Gamma_{\mathcal{C}}$ contains a directed edge for every nonzero entry of these matrices. For an edge to be contained in a Morse matching, it is necessary that the edge corresponds to a constant entry in the matrix representing the differential.

However, we already know a way how to minimise a finite resolution in this case, see Lemma 2.2.2. Thus, for our purposes, Morse theory will be needed in particular when (some of) the modules C_m are not finitely generated.

First, we give an alternative definition of Morse matchings under the condition that the graph $\Gamma_{\mathcal{C}}$ is finite. While we have just mentioned that we do not require Morse theory for finite graphs, we will later use this Lemma in situations where we can represent an infinite graph by (infinitely many) finite equivalence classes.

Lemma 3.1.3. [Skö06, Lemma 1] *Let $\mathcal{C}$ be a based complex such that the graph $\Gamma_{\mathcal{C}}$ is finite. Let A be a partial matching on $\Gamma_{\mathcal{C}}$ such that $d_{b,a}$ is an isomorphism whenever the edge $a \to b$ is contained in A. Then A is a Morse matching if and only if $\Gamma^A_{\mathcal{C}}$ has no directed cycles.*

Proof. $\Longleftarrow$: Let $u \in I_m$ and define

$$l(u) = \max\{t \mid \exists u_1, ..., u_t, v_1, ..., v_t \in I_m : u_t \to v_t \to u_{t-1} \to ... \to u_1 \to v_1 \to u \in \Gamma_{\mathcal{C}}\}.$$

Since $\Gamma_{\mathcal{C}}$ is finite and there are no directed cycles in $\Gamma_{\mathcal{C}}$, $l(u)$ is finite. We set $u \sqsubset v$ if $l(u) < l(v)$. Obviously, this is a well founded partial order which respects the Morse matching.

$\Longrightarrow$: Let $\prec$ be an order that respects the Morse matching. We immediately see that the existence of a directed cycle

$$u \to v_1 \to u_1 \to ... \to u_{s-1} \to v_{s-1} \to u_s = u \in \Gamma_{\mathcal{C}}$$

would imply $u \sqsupset u_1 \sqsupset ... \sqsupset u$, contradicting the fact that $\sqsubset$ is a well founded. $\square$

Even when a based complex is not finitely generated, Lemma 3.1.3 can still be useful thanks to the next lemma:

Lemma 3.1.4. [Skö06, Lemma 7] *Let $\mathcal{C}$ be a based complex. Let $\sim$ be an equivalence relation on the set V of vertices of $\Gamma_{\mathcal{C}}$. Let $\blacktriangleleft$ be a partial order on the set of equivalence classes satisfying $[b] \blacktriangleleft [a]$ whenever there is an edge $a \to b$ in $\Gamma_{\mathcal{C}}$. If there is a Morse matching $A_{[a]}$ on $\Gamma_{\mathcal{C}} \cap [a]$ for each $[a] \in V/\sim$, then $\bigcup_{[a]\in V/\sim} A_{[a]}$ is a Morse matching on $\Gamma_{\mathcal{C}}$.*

Proof. If $A_{[a]}$ is a Morse matching on $[a]$, then there is a well-founded partial order $\sqsubset_{[a]}$ on $[a]$ such that $\sqsubset_{[a]}$ respects the Morse matching $A_{[a]}$. Let $\sqsubset$ be a partial order on V defined by $a \sqsubset b$ if $[a] \blacktriangleleft [b]$ or $[a] = [b]$ and $a \sqsubset_{[a]} b$. Let $(a_i)_{i\in\mathbb{N}}$ be a decreasing sequence with respect to $\sqsubset$, so $a_i \sqsupseteq a_{i+1}$ for all i. Since $\blacktriangleleft$ is well-founded on $V/\sim$, there is an $N \in \mathbb{N}$ such that $[a_i] = [a_N]$ for all $i \geq N$. Since $\sqsubset_{[a_i]}$ is well-founded on $[a_i]$, there is a $M \geq N$ such that $a_j = a_M$ for all $j \geq M$. So $\sqsubset$ is well-founded. $\square$

Later, we will apply the idea of this Lemma to the following situation: Given a based complex of certain $\mathcal{P}$-modules which are not necessarily finitely generated, but still only have a finite number of generators in every degree and no generators in degree $\leq N$, we take two generators to be equivalent if they are of the same (total) degree.

Our next goal is to use a Morse matching on $\Gamma_{\mathcal{C}}$ to construct a complex that is "smaller" than $\mathcal{C}$, but with the same homology.

Remark 3.1.5. Given a based complex $\mathcal{C}$ and a Morse matching A on $\Gamma_{\mathcal{C}}$, we recursively define a $\mathcal{P}$-linear map ϕ as follows:

For an a that is minimal with respect to $\sqsubset$ and $x \in K_a$, let

$$\phi(x) = \begin{cases} d_{a,b}^{-1}(x) & \text{if } b \to a \in A \text{ for some } b \\ 0 & \text{otherwise.} \end{cases}$$

If a is not minimal with respect to $\sqsubset$ and $x \in K_a$, let

$$\phi(x) = \begin{cases} d_{a,b}^{-1}(x) - \sum_{\substack{b\to c\\ a\neq c}} \big(\phi \circ d_{c,b} \circ d_{a,b}^{-1}\big)(x) & \text{if } b \to a \in A \text{ for some } b \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.1.6. *Let $x \in K_a$. If $\pi_{K_b}\big(\phi(x)\big) \neq 0$, then the graph Γ_C^A contains a path of shape $a \to \ldots \to b$.*

Proof. By the recursive definition of ϕ, this is obvious. □

For simplicity of notation, from now on we will often omit the $\circ$-symbol for concatenations of maps.

Lemma 3.1.7. [Skö06, Lemma 2] *Let A be a Morse matching on a based complex $\mathcal{C}$. Then the map ϕ satisfies:*

$$\phi^2 = 0,\ \phi d \phi = \phi.$$

In other words, ϕ is a splitting homotopy, and therefore we can later on make use of the theory presented in Section 2.5.

Proof. The first equation follows from the fact that A is a partial matching, since there are no a, b, c such that both $a \to b$ and $b \to c$ are contained in A. In fact, we see that for $\pi_b\big(\phi(x)\big)$ to be nonzero, it is necessary that $b \in A^-$ and $x \in K_a$ for some $a \in A^+$.

We will prove $\phi d \phi = \phi$ by induction over $\sqsubset$. Let a be minimal with respect to $\sqsubset$ and $x \in K_a$. If $a \notin A^-$, we have just seen that

$$\phi d \phi(x) = 0 = \phi(x).$$

Now assume that $a \in A^-$ and $b \to a \in A$. Since a is minimal, we see that $d|_{K_b} = d_{a,b}$, as otherwise $\Gamma_{\mathcal{C}}$ would contain the edge $b \to c$ for some c, and so $a \to b \to c \in \Gamma_{\mathcal{C}}^A$, contradicting the minimality of a. So we have:

$$\phi d \phi(x) = \phi d d_{a,b}^{-1}(x) = \phi d_{a,b} d_{a,b}^{-1}(x) = \phi(x).$$

Now suppose a is not minimal with respect to $\sqsubset$. If $a \notin A^-$, the argument remains unchanged. So we assume that $a \in A^-$ with $b \to a \in A$ and $x \in K_a$. Now we obtain

$$\begin{aligned} \phi d \phi(x) &= \phi d\big(d_{a,b}^{-1}(x) - \sum_{\substack{b\to c\\ c\neq a}} \phi d_{c,b} d_{a,b}^{-1}(x)\big) \\ &= \phi d d_{a,b}^{-1}(x) - \sum_{\substack{b\to c\\ c\neq a}} \phi d \phi d_{c,b} d_{a,b}^{-1}(x) \\ &= \phi d_{a,b} d_{a,b}^{-1}(x) + \sum_{\substack{b\to c\\ c\neq a}} \phi d \phi d_{c,b} d_{a,b}^{-1}(x) - \sum_{\substack{b\to c\\ c\neq a}} \phi d \phi d_{c,b} d_{a,b}^{-1}(x) \\ &= \phi(x). \end{aligned}$$

Regarding the second-to-last equality, we note that for any c appearing in the sum, we have $a \to b \to c \in \Gamma_{\mathcal{C}}^{A}$ and therefore $c \sqsubset a$, so by induction $\phi d\phi(y) = \phi(y)$ for all $y \in K_c$. □

Lemma 3.1.8. [Skö06, Lemma 3] *Let $\mathcal{C}$ be a based complex, A a Morse matching on $\Gamma_{\mathcal{C}}$ and the map ϕ as defined in remark 3.1.5. For $x \in K_a$, we have*

$$d\phi(x) = \begin{cases} x + \sum_{b \sqsubset \alpha} y_b & \text{if } a \in A^- \\ 0 & \text{otherwise.} \end{cases}$$

Here, with the notation $\sum_{b \sqsubset \alpha} y_b$ we mean that the sum is taken over some elements y_b where $y_b \in K_b$ for all b.

Proof. By induction over $\sqsubset$. Let a be minimal with respect to $\sqsubset$. If $a \notin A^-$, we again have $d\phi(x) = 0$. So let $a \in A^-$ with $b \to a \in A$. As in the proof of Lemma 3.1.3, we have $d|_{K_b} = d_{a,b}$ and so:

$$d\phi(x) = dd_{a,b}^{-1}(x) = d_{a,b}d_{a,b}^{-1}(x) = x.$$

Finally, let a be non-minimal with respect to $\sqsubset$. Again, if $a \notin A^-$, then we have $d\phi(x) = 0$. So if $a \in A^-$ and $b \to a \in A$, then:

$$\begin{aligned} d\phi(x) &= d\big(d_{a,b}^{-1}(x) - \sum_{\substack{b \to c \\ c \neq a}} \phi d_{c,b}, d_{a,b}^{-1}(x)\big) \\ &= x + \sum_{\substack{b \to c \\ c \neq a}} d_{c,b}d_{a,b}^{-1}(x) - d\phi \sum_{\substack{b \to c \\ c \neq a}} d_{c,b}d_{a,b}^{-1}(x). \end{aligned}$$

For each term appearing in these sums, we have $c \sqsubset a$, since $(a \to b \to c) \in \Gamma_{\mathcal{C}}^{A}$. □

Lemma 3.1.9. [Skö06, Lemma 4] *Let $\mathcal{C}$ be a based complex, A a Morse matching on $\Gamma_{\mathcal{C}}$ and the map ϕ as defined in remark 3.1.5. For $x \in K_a$, we have*

$$\phi d(x) = \begin{cases} x & \text{if } a \in A^+ \\ \sum_{b \sqsubset a} y_b & \text{otherwise,} \end{cases}$$

where again the sum is taken over some elements $y_b \in K_b$ for all b, as in Lemma 3.1.8.

Proof. While as mentioned before, this proof presented here is mathematically the same as the one in [Skö06], here we have also included some arguments added for clarification.

Induction over $\sqsubset$. Let a be minimal with respect to $\sqsubset$. If $a \in A^+$ with $(a \to b) \in A$, then for any $c \neq b$ with $a \to c$ we have $c \notin A^-$, for otherwise there would be some d such that $a \to c \to d$ is a path in $\Gamma_{\mathcal{C}}^{a}$ and hence $d \sqsubset a$, contradicting the minimality of a. So $\phi|_{K_c} = 0$ and we get

$$\phi d(x) = \phi d_{b,a}(x) + \sum_{\substack{a \to c \\ c \neq b}} \phi d_{c,a}(x) = d_{b,a}^{-1} d_{b,a}(x) = x.$$

If $a \notin A^+$, we get

$$\phi d(x) = \phi \sum_{a \to b} d_{b,a}(x) = 0$$

since again $b \notin A^-$ for all b appearing in the sum, for otherwise a would not be minimal.

Now let a be non-minimal with respect to $\sqsubset$ and $x \in K_a$. If $a \in A^+$ with $(a \to b) \in A$, then

$$\begin{aligned}
\phi d(x) &= \phi d_{b,a}(x) + \sum_{\substack{a \to c \\ c \neq b}} \phi d_{c,a}(x) \\
&= d_{b,a}^{-1} d_{b,a}(x) - \sum_{\substack{a \to c \\ c \neq b}} \phi d_{c,a}(x) + \sum_{\substack{a \to c \\ c \neq b}} \phi d_{c,a}(x) \\
&= x.
\end{aligned}$$

If $a \notin A^+$, then

$$\begin{aligned}
\phi d(x) &= \phi \sum_{a \to b} d_{b,a}(x) \\
&= \sum_{\substack{a \to b \\ c \to b \in A}} d_{b,c}^{-1} d_{b,a}(x) - \sum_{\substack{a \to b \\ c \to b \in A}} \sum_{\substack{c \to d \\ d \neq b}} \phi d_{d,c} d_{b,c}^{-1} d_{b,a}(x).
\end{aligned}$$

For the summands in the first sum, we immediately see that any summand is contained in some K_c for an index c such that $a \to b \to c$ is a path in $\Gamma_{\mathcal{C}}^{A}$ and hence $c \sqsubset a$. So these summands are of the shape given in the lemma. In the same manner, for the summands in the second sum, using Lemma 3.1.6, we see that whenever the term $\pi_{K_e}\big(\phi d_{d,c} d_{b,c}^{-1} d_{b,a}(x)\big)$ is nonzero for some e, there are paths

$$a \to b \to c \to d \to \ldots \to e$$

in $\Gamma_{\mathcal{C}}^{A}$. This concludes the proof. □

Theorem 3.1.10. [Skö06, Theorem 1] *Let A be a Morse matching on $\Gamma_{\mathcal{C}}$. Let $\pi \colon \mathcal{C} \to \mathcal{C}$ be defined by $\pi = \mathrm{id} - (\phi d + d\phi)$. Then the complexes $\mathcal{C}$ and $\pi(\mathcal{C})$ are homotopy equivalent. Additionally, for each m there is an isomorphism of modules*

$$\pi(\mathcal{C}_m) \cong \bigoplus_{a \in A_m^0} K_a.$$

Proof. The homotopy equivalence follows directly from Theorem 2.5.5.

All that is left to prove is

$$\pi(\mathcal{C}) = \pi\left(\bigoplus_{c\in A^0} K_c\right).$$

We will do so by showing that $\pi(K_a) \subseteq \pi(\bigoplus_{c\in A^0} K_c)$ for all a by induction over $\sqsubset$.

Let $x \in K_a$ where a is minimal with respect to $\sqsubset$. If $a \in A^0$, the statement is obvious. If $a \notin A^0$, then $\pi(x) = 0$ by Lemma 3.1.8 and Lemma 3.1.9.

Now let $x \in K_a$ and a be not minimal with respect to $\sqsubset$. Again if $a \in A^0$, there is nothing to prove. Using Lemma 3.1.8 and Lemma 3.1.9, we see that there is a set K_J with $c \sqsubset a$ for all $c \in J$ and some $y_c \in K_J$ such that

$$\pi(x) = \pi^2(x) = \pi\big(\sum_{c\in J} y_c\big) = \sum_{\substack{c\in J\\ c\in A^0}} \pi(y_c) + \sum_{\substack{c\in J\\ c\notin A^0}} \pi(y_c).$$

From this equation the statement follows by induction.

To conclude the proof, we will prove that the homomorphism

$$\pi\colon \bigoplus_{a\in A^0_m} K_a = \pi(C_m) \to C_m,$$

which is obtained from the homomorphisms of π by restricting the domain, is injective. Again, from Lemma 3.1.8 and Lemma 3.1.9, we see that for $a \in A^0$ and $x \in K_a$ there is once again a set K_J, with $c \sqsubset a$ for all $c \in J$ and and some $y_c \in K_J$, such that

$$\pi(x) = x + \sum_{c\in J} y_c.$$

So the restricted map is injective. □

Now we define a new chain complex $(\mathcal{D}, \tilde{d})$ with modules

$$\mathcal{D}_m = \pi(C_m) = \bigoplus_{a\in A^0} K_a.$$

For the differential $\tilde{d}$ on $\mathcal{D}$, we know from Remark 2.5.6 that is given by

$$\tilde{d} = (d - d\phi d). \tag{3.1.2}$$

Altogether, we once again state that we have found the theorem below. In the given reference requires a proof, for there the complexes which take the role of $\mathcal{D}$ and $\pi(\mathcal{C})$ are not a priori identical.

Theorem 3.1.11. [Skö06, Theorem 2] $(\mathcal{D}, \tilde{d})$ *is a chain complex which is homotopy equivalent to the complex* $\mathcal{C}$*.*

3.2 Constructing the differential in the complex $\mathcal{D}$

Now while we know how to construct the complex $\mathcal{D}$, for the differential $\tilde{d}$ we only have the recursive definition involving the splitting homotopy ϕ. In this chapter, we will see that is possible to give an alternative definition of $\tilde{d}$ which is based on paths in the Morse graph. This apporach allows us to give a non-recursive description of ϕ and $\tilde{d}$, at the cost of an increase in technical language.

As a first step, the construction below and Lemma 3.2.1 establish how we can use paths in the Graph $\Gamma_{\mathcal{C}}^{A}$ to construct the splitting homotopy ϕ and the "reduced differential" $\tilde{d}$.

If we have $a \in I_m$ and $b \in I_{m+1}$ (or $b \in I_{m-1}$ resp.), let $\Gamma_{b,a}$ be the set of directed paths p in the Graph $\Gamma_{\mathcal{C}}^{A}$ of shape

$$p = \quad c_1 \to c_2 \to ... \to c_{2k-1} \to c_{2k},$$

where $c_1 = a, c_{2k} = b$ and all c_j are contained in I_m if j is odd, or in I_{m+1} (or I_{m-1} resp.) if j is even.

If $b \in I_{m+1}$ and $p \in \Gamma_{b,a}$, let

$$\varrho_p = (-1)^{k-1} d_{c_{2k},c_{2k-1}} d^{-1}_{c_{2k-2},c_{2k-1}} \cdots d^{-1}_{c_2,c_3} d_{c_2,c_1},$$

and if $b \in I_{m-1}$ and $p \in \Gamma_{b,a}$, let

$$\varrho_p = (-1)^{k-1} d^{-1}_{c_{2k},c_{2k-1}} d_{c_{2k-2},c_{2k-1}} \cdots d_{c_2,c_3} d^{-1}_{c_2,c_1}.$$

Using this notation, we have

Lemma 3.2.1. [Skö06, Lemma 5] *Let $a \in I_m$ and $x \in K_a$. Then we have*

$$\phi(x) = \sum_{b \in I_{m+1}} \sum_{p \in \Gamma_{b,a}} \varrho_p(x).$$

Proof. We use induction with respect to $\sqsubset$. If a is minimal with respect to $\sqsubset$ and $a \in A^-$, then $\phi(x) = d^{-1}_{a,b}(x)$, for there is exactly one b with $b \to a \in A$, and for this b there is no c with $b \to c \in \Gamma_{\mathcal{C}}$. If $a \notin A^-$, then there is no b with $b \to a \in A$ and we have $\phi(x) = 0$.

Now let a be non-minimal with respect to $\sqsubset$. If $a \notin A^-$, we still have $\phi(x) = 0$ by the same argument as before. So let $a \in A^-$. Then we get

$$\begin{aligned}
\phi(x) &= d^{-1}_{a,b} - \sum_{\substack{b \to e \\ e \neq a}} \phi d_{e,b} d^{-1}_{a,b}(x) \\
&= d^{-1}_{a,b} - \sum_{\substack{b \to e \\ e \neq a}} \sum_{c \in I_{m+1}} \sum_{p \in \Gamma_{c,e}} \varrho_p d_{c,b} d^{-1}_{a,b}(x) \\
&= \sum_{b \in I_{m+1}} \sum_{p \in \Gamma_{b,a}} \varrho_p(x),
\end{aligned}$$

where to obtain the last equation, we have used the fact that we obtain all paths $a \to ... \to e$ (of length > 1) by taking the disjoint union (over b) of all paths of the shape $a \to b \to c \to ... \to e$. □

Corollary 3.2.2. [Skö06, Corollary 3] *For $a \in I_m$ and $x \in K_a$, we have*

$$\tilde{d}(x) = \sum_{b \in I_{m-1}} \sum_{p \in \Gamma_{b,a}} \varrho_p(x).$$

Proof. By Lemma 3.2.1, we have

$$(d - d\phi d)(x) = \sum_{b \in I_{m-1}} \sum_{p \in \Gamma_{b,a}} \varrho_p(x),$$

where we again use the same disjoint union as in the proof of Lemma 3.2.1. □

3.3 Morse theory for modules with initially linear syzygies

In his paper [Skö11], Sköldberg applied algebraic discrete Morse theory, as introduced in Sections 3.1 and 3.2, to modules with "initially linear syzygies". One of his goals was to directly construct the minimal free resolution of a module. However, his construction is still useful for constructing other non-minimal finite resolutions, which might not be minimal, but can be minimized using lemma 2.2.2.

From a technical point of view, an important feature of this chapter will be the introduction of reduction paths. This concept provides a way to represent and calculate the differential obtained from initially linear syzygies (see below) with the help of a Morse matching. While this construction is rather technical, it cannot be avoided, as it is used later in many theoretical proofs while also serving as foundation of the implementation of the theory in the computer algebra system CoCoALib.

Definition 3.3.1. A polynomial module $\mathcal{M}$ has *initially linear syzygies* if $\mathcal{M}$ possesses a finite presentation

$$0 \longrightarrow \ker \eta \longrightarrow \mathcal{W} = \mathcal{P}^s = \bigoplus_{\alpha=1}^{s} \mathcal{P}\mathbf{w}_\alpha \xrightarrow{\eta} \mathcal{M} \longrightarrow 0 \tag{3.3.1}$$

such that with respect to some monomial order $\prec$ on the free module $\mathcal{W}$, the leading module $\operatorname{lt} \ker \eta$ of the kernel of η is generated by terms of the form $x_j \mathbf{w}_\alpha$. We say that $\mathcal{M}$ has *initially linear minimal syzygies*, if the presentation is minimal in the sense that $\ker \eta \subseteq \mathfrak{m}^s$, where by $\mathfrak{m}$ we denote the homogeneous maximal ideal $\mathfrak{m} = \bigoplus_{i \geq 1} \mathcal{P}_i$.

These notions go back to [Skö11] who, however, does not consider the non-minimal case. In his work, the term "initially linear syzygies" always refers to initially linear *minimal* syzygies. His construction begins with the following two-sided Koszul complex $(\mathcal{F}, d_\mathcal{F})$, defining a free resolution of the module $\mathcal{M}$ (see Lemma 3.3.4), which we will take as a fixed notation for the remainder of this work.

Definition 3.3.2. For an (ordered) sequence $\mathbf{k} = (k_1, ..., k_j)$ and $1 \le i \le j$, we denote by $\mathbf{k}_i$ the (ordered) sequence $\mathbf{k}$ without the element k_i, i.e. $\mathbf{k}_i = \mathbf{k} \setminus \{k_i\}$.

Assumption 3.3.3. We will fix a complex $\mathcal{F}$ with modules F_j and differential $d_{\mathcal{F}}$ as follows: Let $\mathcal{V}$ be a $\Bbbk$-linear space with basis $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ (recall that $n+1$ is the number of variables) and let

$$F_j = \mathcal{P} \otimes_{\Bbbk} \Lambda_j \mathcal{V} \otimes_{\Bbbk} \mathcal{M}, \tag{3.3.2}$$

which obviously yields a free $\mathcal{P}$-module. Choosing a $\Bbbk$-linear basis $\{m_a \mid a \in B\}$ of $\mathcal{M}$, a $\mathcal{P}$-linear basis of $\mathcal{F}_j$ is given by the elements of shape $1 \otimes \mathbf{v}_{\mathbf{k}} \otimes m_a$ with ordered sequences $\mathbf{k}$ of length j. The differential is now defined by

$$d_{\mathcal{F}}(1 \otimes_{\Bbbk} \mathbf{v}_{\mathbf{k}} \otimes_{\Bbbk} m_a) = \sum_{i=1}^{j} (-1)^{i+1} \big(x_{k_i} \otimes_{\Bbbk} \mathbf{v}_{\mathbf{k}_i} \otimes_{\Bbbk} m_a - 1 \otimes_{\Bbbk} \mathbf{v}_{\mathbf{k}_i} \otimes_{\Bbbk} x_{k_i} m_a \big). \tag{3.3.3}$$

It should be noted that in general, the second term of the summands on the right hand side is not yet expressed in the chosen $\Bbbk$-linear basis of $\mathcal{M}$. For notational simplicity, we will drop in the sequel the index of $\otimes_{\Bbbk}$, or even the tensor signs $\otimes_{\Bbbk}$ entirely. We will also omit the leading factor if it is equal to 1, i.e. we will use the notation

$$\mathbf{v}_{\mathbf{k}} \mathbf{h}_{\alpha} = 1 \otimes_{\Bbbk} \mathbf{v}_{\mathbf{k}} \otimes_{\Bbbk} \mathbf{h}_{\alpha}.$$

We remark that, unless $\mathcal{M}$ is an artinian module, the modules of this two-sided Koszul complex are not finitely generated. Thus, if we want to minimise this complex later to the point where the reduced complex is finite, we cannot rely on the step-by-step approach from Lemma 2.2.2. However, using Morse theory, such a "minimisation" becomes possible.

We also point out that this complex also on the surface looks similar to the free resolution induced by an involutive basis, understood as Remark 2.3.62. A large part of Chapter 4 is dedicated to formally establishing links between the behavior of these complexes.

Lemma 3.3.4. [Skö11, Lemma 1] *The complex $\mathcal{F}$ is a free resolution of $\mathcal{M}$ (in the sense that we have a complex as in definition 2.1.1, but its modules are not necessarily finitely generated).*

Proof. While this proof is based on the proof given in [Skö11], it also includes some additional aspects which have been added for further clarification.

Since $d_{\mathcal{F}}(F_1)$ is generated by all relations $x_i \mathbf{v}_{\emptyset} m - \mathbf{v}_{\emptyset} x_i m$, we immediately see that $H_0(\mathcal{F}) \cong \mathcal{M}$.

Now we have to prove that $H_i(\mathcal{F}) = 0$ for all $i \ge 1$.

We consider $\mathcal{F}$ to be a based complex of $\Bbbk$-vector spaces via the natural decomposition

$$\mathcal{P} \otimes_{\Bbbk} \Lambda \otimes_{\Bbbk} \mathcal{M} \cong \bigoplus \Bbbk \cdot x^{\mu} \mathbf{v}_I m_a,$$

where the direct sum is taken over all μ, I, m_a such that $\mu \in \mathbb{N}^n, I \subseteq \{0, .., n\}$ and $m_a \in B$ where B is a basis of $\mathcal{M}$ as a $\Bbbk$-vector space.

For each $i \geq 0$ and $m_a \in B$, let

$$V_{i,m_a} = \{x^\mu \mathbf{v}_I m_a \mid \deg x^\mu + |I| = i\}$$

be a subset of the vertices of $\Gamma_\mathcal{F}$. We define a partial matching E_{i,m_a} on $\Gamma_\mathcal{F}|_{V_{i,m_a}}$ by

$$(x^\mu \mathbf{v}_I m_a \to x^\mu x_j \mathbf{v}_{I\setminus j} m_a) \in E_{i,m_a} \text{ if and only if } j = \min(\operatorname{supp} \mu \cup I) \wedge j \in I.$$

It is clear that E_{i,m_a} is a partial matching. Since any V_{i,m_a} is finite, by Lemma 3.1.3, E_{i,m_a} is a Morse matching on $\Gamma_\mathcal{F}|_{V_{i,m_a}}$ if there are no directed cycles on $\left(\Gamma_\mathcal{F}|_{V_{i,m_a}}\right)^{E_{i,m_a}}$, where by this notation meant to describe the graph that is given by taking the graph $\Gamma_\mathcal{F}|_{V_{i,m_a}}$ and reversing the arrows contained in the Morse matching E_{i,m_a} (see Definition 3.1.1). So let

$$x^{\mu_0}\mathbf{v}_{I_0}m_a \to x^{\mu_1}\mathbf{v}_{I_1}m_a \to ... \to x^{\mu_{2k-1}}\mathbf{v}_{I_{2k-1}}m_a \to x^{\mu_0}\mathbf{v}_{I_0}m_a = x^{\mu_k}\mathbf{v}_{I_k}m_a$$

be a directed cycle in $\left(\Gamma_\mathcal{F}|_{V_{i,m_a}}\right)^{E_{i,m_a}}$. Since E_{i,m_a} is a partial matching on a based complex, for such a cycle we can assume that $x^{\mu_l}\mathbf{v}_{I_l}m_a \in (E_{i,m_a})^+$ if l is even and $x^{\mu_l}\mathbf{v}_{I_l}m_a \in (E_{i,m_a})^-$ if l is odd. From the definition of E_{i,m_a}, we then have

$$\mu_0 \prec_{\text{lex}} \mu_2 \prec_{\text{lex}} .. \prec_{\text{lex}} \mu_{2k} = \mu_0,$$

which is impossible (recall that according to our conventions of Definition 2.3.3, we have $x_0 \prec x_1 \prec ... \prec x_n$ for the lex order). So $E_i = \bigcup_{m_a \in B} E_{i,m_a}$ is a Morse matching on the graph $\Gamma_\mathcal{F}|_{V_i}$, where $V_i = \bigcup_{m_a \in B} V_{i,m_a}$.

Now for any $(x^{\mu_0}\mathbf{v}_{I_0}m_a \to x^{\mu_1}\mathbf{v}_{I_1}m_a) \in \Gamma_\mathcal{F}$, we must have

$$\deg x^{\mu_0} + |I_0| \geq \deg x^{\mu_1} + |I_1|,$$

and so by Lemma 3.1.4, $E = \bigcup_i E_i$ is a Morse matching on $\Gamma_\mathcal{F}$. But then, there are no E-critical vertices in (homological) degree ≥ 1. By Theorem 3.1.10, this means that all homology modules $H_i(\mathcal{F})$ for $i \geq 0$ vanish, if we view $\mathcal{F}$ as a complex over $\Bbbk$. But then of course, they also vanish over $\mathcal{P}$. □

Remark 3.3.5. Under the assumption that the module $\mathcal{M}$ has initially linear syzygies via the presentation of equation (3.3.1), [Skö11] constructs a Morse matching leading to a smaller resolution $(\mathcal{G}, d_\mathcal{G})$. He calls the variables

$$\operatorname{crit}(\mathbf{w}_\alpha) = \{x_j \mid x_j \mathbf{w}_\alpha \in \operatorname{lt} \ker \eta\} \tag{3.3.4}$$

critical for the generator $\mathbf{w}_\alpha$; the remaining *non-critical* ones are contained in the set $\operatorname{ncrit}(\mathbf{w}_\alpha)$. A $\Bbbk$-linear basis of $\mathcal{M}$ is then given by all elements $x^\mu \mathbf{h}_\alpha$ with $\mathbf{h}_\alpha = \eta(\mathbf{w}_\alpha)$ and monomials $x^\mu \in \Bbbk[\operatorname{ncrit}(\mathbf{w}_\alpha)]$.

For each $m \in \mathcal{M}$, consider the following set of vertices in the graph $\Gamma_\mathcal{F}$:

$$V_m = \{\mathbf{v}_I x^\mu \mathbf{h}_\alpha \mid x^I x^\mu \mathbf{h}_\alpha = m\} \tag{3.3.5}$$

Then V_m is not empty if and only if m is the product of some generator $\mathbf{h}_\alpha$ with a monomial. Furthermore, we define[1]

$$A_m = \left\{ \mathbf{v}_I x^\mu \mathbf{h}_\alpha \to \mathbf{v}_{I\setminus i} x_i x^\mu \mathbf{h}_\alpha \in \Gamma_\mathcal{F}|_{V_m} \mid \right. \\ \left. i = \min \left\{ I \cap \mathrm{ncrit}\left(\mathbf{w}_\alpha\right) \right\} \wedge i \leq \mathrm{cls}\left(x^\mu\right) \right\}. \quad (3.3.6)$$

Now we see that we have constructed a Morse matching:

Lemma 3.3.6. [Sköl1, Lemma 2] *The union $A = \bigcup_{m\in\mathcal{M}} A_m$ is a Morse matching on $\Gamma_\mathcal{F}$. The set of unmatched vertices consists of all $\mathbf{v_k h}_\alpha$ with $\mathbf{k} \subseteq \mathrm{crit}(\mathbf{w}_\alpha)$.*

Proof. It is clear that A is a partial matching, since any set I of indices contains every index at most once. In analogy to the proof of Lemma 3.3.4, we see that the existence of an oriented cycle

$$\mathbf{v}_{I_0} x^{\mu_0} \mathbf{h}_{a_0} \to \mathbf{v}_{I_1} x^{\mu_1} \mathbf{h}_{a_1} \to ... \\ ... \to \mathbf{v}_{I_{2k-1}} x^{\mu_{2k-1}} \mathbf{h}_{a_{2k-1}} \to \mathbf{v}_{I_{2k}} x^{\mu_{2k}} \mathbf{h}_{a_{2k}} = \mathbf{v}_{I_0} x^{\mu_0} \mathbf{h}_{a_0} \quad (3.3.7)$$

in $\Gamma_\mathcal{F}|_{V_m}{}^{A_m}$ implies $I_0 >_{\mathrm{lex}} I_2 >_{\mathrm{lex}} ... >_{\mathrm{lex}} I_{2k} = I_0$, which is again impossible, as in the proof of Lemma 3.3.4. Using Lemma 3.1.3 and Lemma 3.1.4 again, we see that A is a Morse matching on $\Gamma_\mathcal{F}$.

The statement about the unmatched vertices is clear by definition (3.3.6) of the matching A in remark 3.3.5. ☐

A vertex $\mathbf{v_k h}_\alpha$ is not contained in A if and only if $\mathbf{k} \subseteq \mathrm{crit}\left(\mathbf{w}_\alpha\right)$; additionally, all vertices of the form $\mathbf{v}_I x^\mu \mathbf{h}_\alpha$ with $\mu \neq 0$ appear in this Morse matching. We now define $G_j \subseteq F_j$ as the free submodule generated by those vertices $\mathbf{v_k h}_\alpha$ where the ordered sequences $\mathbf{k}$ are of length j and such that every entry $\mathbf{k}_i$ of $\mathbf{k}$ is critical for $\mathbf{w}_\alpha$. In particular, $\mathcal{W} \cong G_0$ with an isomorphism induced by $\mathbf{w}_\alpha \mapsto \mathbf{v}_\emptyset \mathbf{h}_\alpha$.

Combining Lemma 3.1.7, Theorem 3.1.10, Lemma 3.3.4 and Lemma 3.3.6, we get the following result:

Theorem 3.3.7. [Sköl1, Theorem 1] *This complex $\mathcal{G}$, with modules G_j is a finite free resolution of $\mathcal{M}$. The differential $d_\mathcal{G}$ is induced by the Morse matching A (see Section 3.2).*

This theorem is one of the main results of [Sköl1], where it is stated in the form that $(\mathcal{G}, d_\mathcal{G})$ is the *minimal* free resolution of $\mathcal{M}$ if one starts with initially linear *minimal* syzygies. However, independently of the minimality assumption, his construction always yields some free resolution. The minimality condition holds in the case of initially linear minimal syzygies since the condition $\ker \eta \subseteq \mathfrak{m}^s$ is equivalent to the fact that if we write a product $x_i m_a$ with $i \in \mathrm{crit}(m_a)$ as a linear combination over $\mathcal{P}$, then no constant coefficients appear in this sum. But then there are also no constant coefficients in any of the $d_\mathcal{F}(\mathbf{v_k} m_a)$ and by Equation 3.1.2, this property translates to $d_\mathcal{G}$, hence $\mathcal{G}$ is minimal. Notably, for Pommaret bases, the analogous property holds:

[1] In [Sköl1], a slightly different definition for the sets A_m is given.

Lemma 3.3.8. [Sei10, Lemma 5.5.1] *The resolution induced by a Pommaret basis as in Theorem 2.3.59 is minimal if and only if all first syzygies $\vec{S}_{\alpha;k}$ are free of constant terms.*

Remark 3.3.9. If we reduce a general complex $\mathcal{C}$ to a smaller complex with the same homology by taking advantage of a Morse matching A, it is natural to ask how we can calculate the differential map in the smaller complex. In particular, we are interested in the differential of the complex $\mathcal{G}$, which is obtained by reducing the two-sided Koszul complex $\mathcal{F}$ via the Morse matching A of equation (3.3.6). For the definition of this differential, we will use *reduction paths* in $\Gamma^A_{\mathcal{C}_\bullet}$. An *elementary reduction path* is a "zig-zag" path $\alpha_0 \to \beta \to \alpha_1$ of length[1] two in $\Gamma^A_{C_\bullet}$ with $\alpha_0, \alpha_1 \in I_m$ that also satisfies

$$\beta \in I_{m-1} \Longleftrightarrow \alpha_0 \in A^0 \cup A^+ \quad \text{and} \quad \beta \in I_{m+1} \Longleftrightarrow \alpha_0 \in A^- \,. \tag{3.3.8}$$

Note that there are also zig-zag-paths $\alpha_0 \to \beta \to \alpha_1$ of length two in the graph $\Gamma^A_{C_\bullet}$ with $\alpha_0, \alpha_1 \in I_m$ which are *not* elementary reduction paths: a path with $\beta \in I_{m-1}$ and $\alpha_0 \in A^-$ is not considered to be an elementary reduction path; we will see later in Lemma 4.1.5 that, that for our goal of giving a formula for the differential in $\mathcal{G}$, these paths are irrelevant anyway. Note that since A is a Morse matching, the existence of a path $\alpha_0 \to \beta$ is equivalent to $\alpha_0 \in A^-$, so if $\alpha_0 \in A^0 \cup A^+$, there cannot be any path $\alpha_0 \to \beta \to \alpha_1$ where $\beta \in I_{m+1}$. For the elementary reduction path $\alpha_0 \to \beta \to \alpha_1$, we define the corresponding *elementary reduction* as the map

$$\rho_{\alpha_1,\alpha_0} = \begin{cases} -d^{-1}_{\beta,\alpha_1} \circ d_{\beta,\alpha_0} & \text{if } \beta \in I_{m-1}\,, \\ -d_{\alpha_1,\beta} \circ d^{-1}_{\alpha_0,\beta} & \text{if } \beta \in I_{m+1}\,. \end{cases} \tag{3.3.9}$$

A *(general) reduction path* p is a composition of elementary reduction paths

$$p = (\alpha_0 \to \beta_1 \to \alpha_1 \to \cdots \to \beta_q \to \alpha_q)$$

where $q \geq 0$.

For two indices $\alpha, \alpha^* \in I_m$, several reduction paths from α to α^* may exist; we write $[\alpha \rightsquigarrow \alpha^*]$ for the set of all such paths. For a general reduction path p, the (associated) *reduction* ρ_p is given by

$$\rho_p = \rho_{\alpha_q,\alpha_{q-1}} \circ \rho_{\alpha_{q-1},\alpha_{q-2}} \circ \cdots \circ \rho_{\alpha_1,\alpha_0}\,.$$

[Skö11] gives two descriptions of the differential $d_{\mathcal{G}}$ in the reduced complex, the recursive one of equation (3.1.2), which is based on the splitting homotopy ϕ of remark 3.1.5; and another one that makes use of the concept of reduction paths. The latter one is better suited for our purposes. It is based on reduction paths in the associated Morse graph and expresses the differential as a triple

[1] Length as defined in Definition 2.0.2.

sum. If we assume that after expanding the right hand side of (3.3.3) in the chosen $\Bbbk$-linear basis of $\mathcal{M}$, the differential of the complex $\mathcal{F}$ is expressed as

$$d_{\mathcal{F}}(\mathbf{v}_{\mathbf{k}}\mathbf{h}_{\alpha}) = \sum_{\mathbf{m},\mu,\gamma} Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}\mathbf{v}_{\mathbf{m}}(x^{\mu}\mathbf{h}_{\gamma})\,, \tag{3.3.10}$$

then by Corollary 3.2.2, $d_{\mathcal{G}}$ is given by

$$d_{\mathcal{G}}(\mathbf{v}_{\mathbf{k}}\mathbf{h}_{\alpha}) = \sum_{\boldsymbol{\ell},\beta}\sum_{\mathbf{m},\mu,\gamma}\sum_{p}\rho_p\big(Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}\mathbf{v}_{\mathbf{m}}(x^{\mu}\mathbf{h}_{\gamma})\big), \tag{3.3.11}$$

where the first sum ranges over all ordered sequences $\boldsymbol{\ell}$ which consist entirely of critical indices for $\mathbf{w}_{\beta}$ and the second sum may be restricted to all values such that a polynomial multiple of $\mathbf{v}_{\mathbf{m}}(x^{\mu}\mathbf{h}_{\gamma})$ effectively appears in $d_{\mathcal{F}}(\mathbf{v}_{\mathbf{k}}\mathbf{h}_{\alpha})$, and the third sum ranges over all reduction paths p going from $\mathbf{v}_{\mathbf{m}}(x^{\mu}\mathbf{h}_{\gamma})$ to $\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_{\beta}$, see [Skö11, Equation (2)] Finally, ρ_p is the reduction associated with the reduction path p satisfying $\rho_p\big(\mathbf{v}_{\mathbf{m}}(x^{\mu}\mathbf{h}_{\gamma})\big) = q_p\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_{\beta}$ for some polynomial $q_p \in \mathcal{P}$.

4 Combining Morse theory and involutive bases

Definition 4.0.1. If $p = p_m \circ \cdots \circ p_1$ is a reduction path where all p_i are elementary reduction paths, then from now on, we will call m the *length of* p, written $l(p) = m$. In the classical sense of graphs, see definition 2.0.2, such a path would be of length $2m$, for it contains $2m$ arrows. However, as for the purposes of this chapter, elementary reduction paths are the minimal building blocks of (general) reduction paths, this definition of the length is better suited.

Now we combine Sköldberg's construction with involutive bases for continuous involutive divisions of Schreyer type, which by lemmata 2.3.36 and 2.3.37 include the Pommaret and Janet divisions. In the paper [AFSS15], this idea has been established for the case of Pommaret division, and in [AFS15], for the Janet division. In this chapter, we will take another step and generalize these results to any continuous involutive division of Schreyer type. Assume that the considered graded module $\mathcal{M}$ in the definition of initially linear syzygies in Equation (3.3.1) is presented by a quotient $\mathcal{P}^m/\mathcal{M}$ for a "different" graded submodule $\mathcal{M} \subseteq \mathcal{P}^m$. Obviously, a free resolution of $\mathcal{M}$ immediately yields one of $\mathcal{P}^m/\mathcal{M}$ and vice versa. Therefore we will restrict to the construction of resolutions for polynomial submodules given by an involutive basis $\mathcal{H} = \{\mathbf{h}_1, \ldots, \mathbf{h}_s\}$ for a continuous involutive division L.

Assumption 4.0.2. Unless stated otherwise, we will always assume that any involutive basis is enumerated according to a L-ordering.

As an immediate consequence of Lemma 2.3.56, with respect to the presentation given in equation (3.3.1), we obtain the following trivial assertion:

Lemma 4.0.3. *In the situation of Definition 3.3.1, let the map η be given by $\eta(\mathbf{w}_\alpha) = \mathbf{h}_\alpha$. Then the submodule $\mathcal{M} \subseteq \mathcal{P}^m$ has initially linear syzygies with respect to the Schreyer order $\prec_{\mathcal{H}}$ and* $\operatorname{crit}(\mathbf{w}_\alpha) = \overline{\mathcal{X}}_L(\mathbf{h}_\alpha)$*, i. e. the critical variables of the generator $\mathbf{w}_\alpha$ are the non-multiplicative variables of $\mathbf{h}_\alpha = \eta(\mathbf{w}_\alpha)$.*

From now on, we will consider exclusively such initially linear syzygies originating from an involutive division:

Assumption 4.0.4. From now, let $\mathcal{H}$ be an involutive basis for the module $\mathcal{M} \trianglelefteq \mathcal{P}^m$ with respect to a continuous involutive division L and a monomial order $\prec$. The presentation Equation (3.3.1) comes from the homomorphism defined by $\eta(\mathbf{w}_\alpha) = \mathbf{h}_\alpha$. For the $\Bbbk$-basis of $\mathcal{M}$ as in Assumption 3.3.3, we take the basis induced by the direct sum decomposition defined by $\mathcal{H}$ as in Equation (2.3.2), i.e. the basis is given by all terms $x^\mu \mathbf{h}_\alpha$ where $\mathbf{h}_\alpha \in \mathcal{H}$ and

$$\operatorname{supp}(\mu) \subseteq \mathcal{X}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha) = \operatorname{ncrit}(\mathbf{w}_\alpha).$$

In particular, whenever an element of the complex $\mathcal{F}$ appears, for example $x^\kappa \mathbf{v}_\mathbf{k} x^\mu \mathbf{h}_\alpha$, we always implicitly assume $\operatorname{supp}(\mu) \subseteq \operatorname{ncrit}(\mathbf{w}_\alpha)$.

We will also occasionally use the notation $\operatorname{ncrit}(\mathbf{h}_\alpha) = \operatorname{ncrit}(\mathbf{w}_\alpha)$.

Definition 4.0.5. Let $\mathcal{M} \subseteq \mathcal{P}^m$ be a graded polynomial module. $\mathcal{M}$ is *componentwise linear* if for each $e \geq 0$, the module $\langle \mathcal{M}_e \rangle$ (i.e. the module generated by $\mathcal{M}_e$, the elements of degree e in $\mathcal{M}$) has a linear resolution, i.e. the only non-vanishing Betti numbers of $\langle \mathcal{M}_e \rangle$ are the $\beta_{i,i+e}(\langle \mathcal{M}_e \rangle)$ for $i \geq 0$.

Theorem 4.0.6. [Sei09, Theorem 9.12.] *Let $\mathcal{M}$ be componentwise linear. Then for the Pommaret division, generically the resolution introduced in Theorem 2.3.59 is minimal.*

[Skö11, Corollary 4] shows that a module with initially linear minimal syzygies is always componentwise linear. Now, it follows from the combination of Lemma 2.3.60, Theorem 4.0.6 and Lemma 4.0.3 that the converse is also true: modulo a coordinate transformation, any componentwise linear module has initially linear minimal syzygies:

Corollary 4.0.7. *If the polynomial module $\mathcal{M} \subseteq \mathcal{P}^m$ is componentwise linear, then $\mathcal{M}$ has initially linear minimal syzygies in generic coordinates.*

Of course, the question of how to find a generic coordinate system is not that interesting from a theoretical point of view, but nevertheless relevant when one wants to do actual computations. A discussion regarding this question the situation can be found in the preprint [HSS16, in particular Remark 6.5.].

4.1 Classification of reduction paths

For later use, we will now classify elementary reduction paths p in the graph $\Gamma_{\mathcal{F}}^{\mathcal{A}}$ into three different types. Our classifications follows the presentation in [Skö11], which was in turn using [JW09]. This classification covers all elementary reduction path.

Definition 4.1.1. Let $\mathbf{k} \subseteq \mathbb{N}$ be a finite set. Then for any $i \in \mathbb{N}$, let

$$\varepsilon(i; \mathbf{k}) = (-1)^{|\{j \in \mathbf{k} | j < i\}|}.$$

This generalises the corresponding notation in [Skö06, Skö11] in the sense that we do not require $i \in \mathbf{k}$.

Type 0: In this case p is a path $\alpha_0 \to \beta \to \alpha_1$ with $\alpha_0, \alpha_1 \in I_m$ and $\beta \in I_{m-1}$. We will later see that these elementary reduction paths are irrelevant for the construction of the differential $d_{\mathcal{G}}$ of the reduced complex $\mathcal{G}$.

All other elementary reduction paths are of the form

$$\mathbf{v}_{\mathbf{k}}(x^\mu \mathbf{h}_\alpha) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}\left(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\right) \longrightarrow \mathbf{v}_{\boldsymbol{\ell}}(x^\nu \mathbf{h}_\beta)\,.$$

Here $\mathbf{k} \cup i$ is the ordered sequence which arises when i is inserted into $\mathbf{k}$; likewise $\mathbf{k} \setminus i$ stands for the ordered sequence given by the removal of an index $i \in \mathbf{k}$, in analogy to definition 2.2.1.

Type 1: Here we have $\boldsymbol{\ell} = (\mathbf{k} \cup i)\backslash j$, $x^\nu = \frac{x^\mu}{x_i}$ and $\beta = \alpha$. Note that $i = j$ is allowed. The associated reduction is

$$\rho(\mathbf{v_k} x^\mu \mathbf{h}_\alpha) = \varepsilon(i; \mathbf{k} \cup i)\varepsilon(j; \mathbf{k} \cup i) x_j \mathbf{v}_{(\mathbf{k}\cup i)\backslash j}\big(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\big)\,.$$

Type 2: Now $\boldsymbol{\ell} = (\mathbf{k} \cup i) \setminus j$ and $x^\nu \mathbf{h}_\beta$ appears in the involutive standard representation of $\frac{x^\mu x_j}{x_i}\mathbf{h}_\alpha$ with a coefficient $\lambda_{j,i,\alpha,\mu,\beta,\nu} \in \mathbb{k}$. In this case, by construction of the Morse matching, (see also remark 3.3.5 and Lemma 3.3.6), we have $i \neq j$. The reduction is

$$\rho(\mathbf{v_k} x^\mu \mathbf{h}_\alpha) = -\varepsilon(i; \mathbf{k} \cup i)\varepsilon(j; \mathbf{k} \cup i)\lambda_{j,i,\alpha,\mu,\beta,\nu} \mathbf{v}_{(\mathbf{k}\cup i)\backslash j}(x^\nu \mathbf{h}_\beta)\,.$$

Note that by definition 4.1.1, we obviously have $\varepsilon(i, \mathbf{k} \cup i) = \varepsilon(i, \mathbf{k})$, which we could use to slightly shorten the representation of the reduction maps, at the expense of less "symmetrically" looking formulas.

These reductions come from the differential Equation (3.3.3)

$$d_\mathcal{F}(1 \otimes_\mathbb{k} \mathbf{v_k} \otimes_\mathbb{k} m_a) = \sum_{i=1}^{j} (-1)^{i+1}\big(x_{k_i} \otimes_\mathbb{k} \mathbf{v}_{\mathbf{k}_i} \otimes_\mathbb{k} m_a - 1 \otimes_\mathbb{k} \mathbf{v}_{\mathbf{k}_i} \otimes_\mathbb{k} x_{k_i} m_a\big).$$

The summands appearing there are either of the form $x_{k_i}\mathbf{v}_{\mathbf{k}_i} m_a$ or of the form $\mathbf{v}_{\mathbf{k}_i}(x_{k_i} m_a)$. Recall Definition 3.3.2 for the notation $\mathbf{k}_i$. For each of these summands, we have a directed edge in the graph $\Gamma^\mathcal{A}_\mathcal{F}$; or in cases where the involutive standard representation of x_{k_i} consists of more than one summand, multiple edges. Thus for an elementary reduction path

$$\mathbf{v_k}(x^\mu \mathbf{h}_\alpha) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}\left(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\right) \longrightarrow \mathbf{v}_{\boldsymbol{\ell}}\left(x^\nu \mathbf{h}_\beta\right)\,,$$

the second edge can originate from summands of either form. For the first form we then have an elementary reduction path of type 1 and for the second form we have type 2.

For completeness, we note the following simple result which just shows that the free resolution $\mathcal{G}$ indeed extends the presentation (3.3.1) and hence yields essentially the same first syzygies as the involutive basis, see Equation (2.3.5).

Lemma 4.1.2. *Let*[1] $i \in \operatorname{crit}(\mathbf{h}_\alpha)$ *and* $x_i \mathbf{h}_\alpha = \sum_{\beta=1}^{s} P_\beta^{(\alpha;i)}\mathbf{h}_\beta$ *be the involutive standard representation. Then we have* $d_\mathcal{G}(\mathbf{v}_i \mathbf{h}_\alpha) = x_i \mathbf{v}_\emptyset \mathbf{h}_\alpha - \sum_{\beta=1}^{s} P_\beta^{(\alpha;i)}\mathbf{v}_\emptyset \mathbf{h}_\beta$ *and* $\operatorname{lt}_\mathcal{H}\big(d_\mathcal{G}(\mathbf{v}_i \mathbf{h}_\alpha)\big) = x_i \mathbf{v}_\emptyset \mathbf{h}_\alpha$.

Proof. Looking at the different types of reduction paths, we immediately see that in the differential (3.3.11), we can only have concatenations of elementary reduction paths of type 1 which are of the form

$$\mathbf{v}_\emptyset(x^\mu \mathbf{h}_\alpha) \longrightarrow \mathbf{v}_i\left(\frac{x^\mu}{x_i}\mathbf{h}_\alpha\right) \longrightarrow \mathbf{v}_\emptyset\left(\frac{x^\mu}{x_i}\mathbf{h}_\beta\right)\,.$$

[1] Recall that for notational simplicity, we have identified sets X of variables with sets of the corresponding indices, allowing us to simply write $i \in X$ instead of $x_i \in X$.

The corresponding reduction is $\rho(\mathbf{v}_\emptyset x^\mu \mathbf{h}_\alpha) = x_i \mathbf{v}_\emptyset(\frac{x^\mu}{x_i}\mathbf{h}_\alpha)$. As

$$d_\mathcal{F}(\mathbf{v}_i\mathbf{h}_\alpha) = x_i\mathbf{v}_\emptyset\mathbf{h}_\alpha - \sum_{\beta=1}^{s} \mathbf{v}_\emptyset P_\beta^{(\alpha;i)}\mathbf{h}_\beta,$$

the reduction paths move the variables in a way that gives us the correct reduced differential $d_\mathcal{G}$.

The statement about the leading monomial follows directly from the definition of the Schreyer order. □

Example 4.1.3. We will go back to the ideal I from example 2.3.24, to see an explicit example of how to construct the differential $d_\mathcal{G}$, and how a part of the graph $\Gamma_\mathcal{C}$ looks like. The ideal is given by the Pommaret basis

$$\mathcal{H} = \{x_0^2x_2x_3,\ x_1^3 + 2x_0^2x_1,\ x_1^2x_3 + 2x_0^2x_3,\ x_1x_2,\ x_1x_2x_3,\ x_2^2,\ x_2^2x_3,\ x_3^2\}.$$

We want to find the differential $d_\mathcal{G}$ of $\mathbf{v}_{2,3}(x_1^3+2x_0^2x_1)$. We start by constructing the part of the Morse graph which contains all paths originating in the vertex $\mathbf{v}_{2,3}(x_1^3+2x_0^2x_1)$. We note the coefficient for each differential $d_{\beta,\alpha}$ or $d_{\beta,\alpha}^{-1}$ along the respective edge. Every blue path has 1 as the coefficient for the associated reduction map. We have omitted these coefficients in order to nor overload the graphs. A more detailed discussion of these coefficients follows below.

We also use Lemma 4.1.4 and Lemma 4.1.5 in advance, which allows us to immediately ignore all edges that are not incident to vertices in $\mathcal{F}_1$ and $\mathcal{F}_2$. This gives us a graph, which is displayed in figures 4.1.1 and 4.1.2. In this graph, we have marked vertices of $\mathcal{F}_1$

in green, if they are also generators of modules in $\mathcal{G}$. By the definition of the Morse matching in equation (3.3.6), no elementary reductions paths of type 1 or 2 originate in these vertices, i.e. the paths ending there do contribute to the differential $d_\mathcal{G}$.

in red, if we know from the definition of the Morse matching in equation (3.3.6) that there are no elementary reduction paths of type 1 or 2 originating in this vertex, but the vertex is not a generator of a module of $\mathcal{G}$; i.e. the paths ending there do not contribute to the differential $d_\mathcal{G}$.

in purple, if we know from the not yet proven Lemma 4.1.4 that no path originating in these vertices leads to a generator of a module of $\mathcal{G}$; i.e. these paths do not contribute to $d_\mathcal{G}$

in light blue, if we could use the not yet proven Theorem 5.1.4 as a shortcut to obtain the differential, instead of calculating reduction paths originating in this vertex: In this example, this applies to the vertices of shape $x_0^{\mu_0}\mathbf{h}_\alpha$, and since x_0 is of minimal class, this variable is of minimal class and therefore multiplicative for any element of $\mathcal{H}$. In this case Theorem 5.1.4 gives a formula that instantly gives the reduction map associated to any path originating in this vertex.

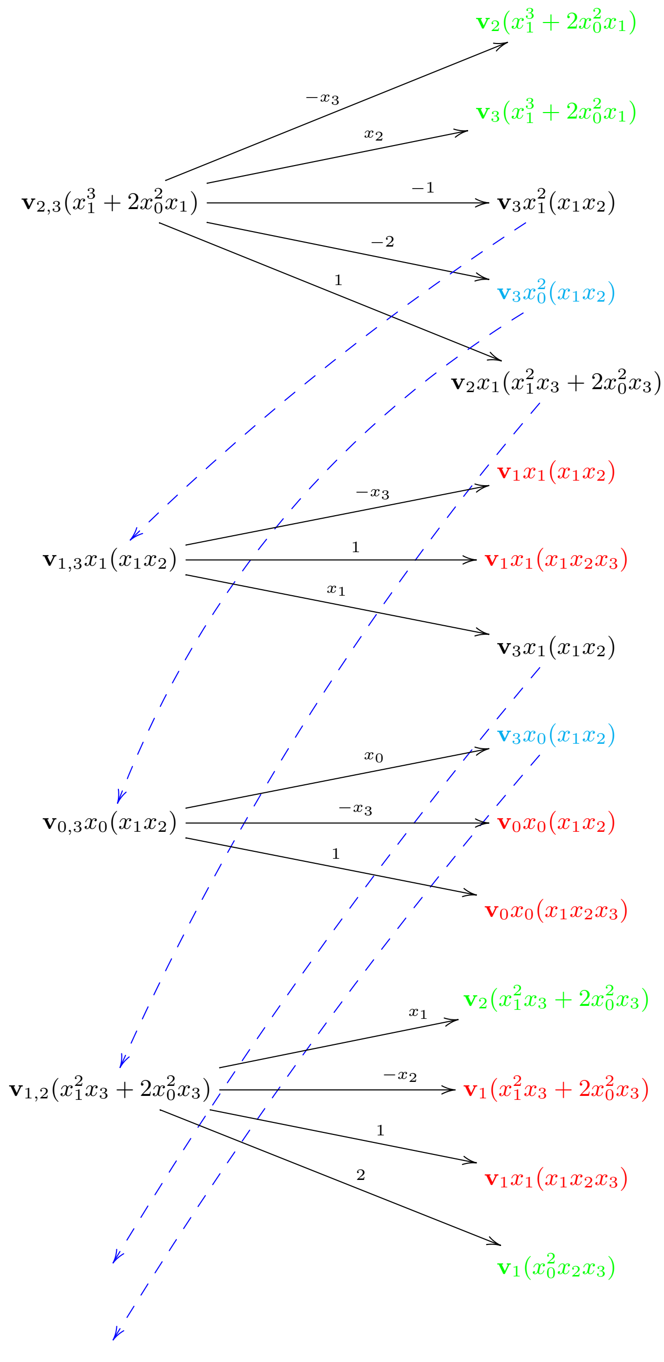

Figure 4.1.1: Top part of the graph of example 4.1.3

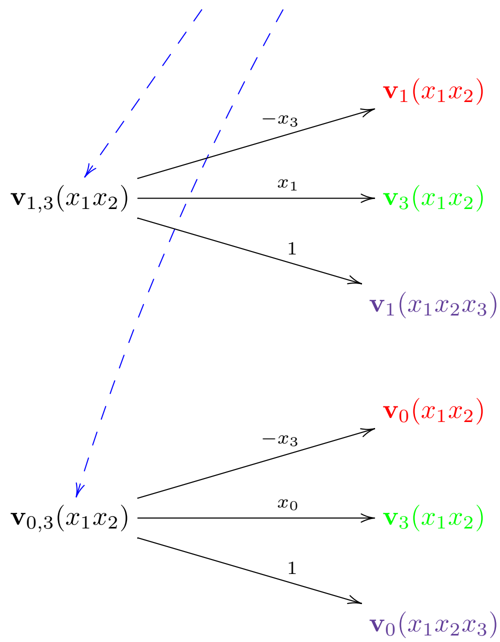

Figure 4.1.2: Bottom part of the graph of example 4.1.3

Note that the in general, these categories might not be mutually exclusive; in fact even any purple vertex could also be marked in red.

From this graph, we obtain the differential $d_{\mathcal{G}}\big(\mathbf{v}_{2,3}(x_1^3 + x_0^2 x_1)\big)$ by taking the sum over all reduction maps ending in a green vertex; the coefficient of such a summand (for one path) is given by multiplying all coefficients noted along a path. Note that while in the example each blue path, which comes from an edge in the Morse matching that has been reversed, has 1 as associated coefficient, this is not the **not** the case in more general situations: We might also have -1 as coefficient for such paths. If we look at the reduction maps, as displayed at the beginning of Section 4.1, the coefficients of the blue paths are the $\varepsilon(i; \mathbf{k} \cup i)$, while the coefficients of the **black paths** are the $\varepsilon(j; \mathbf{k} \cup i)x_j$ (in case of type 1) or the $-\varepsilon(j; \mathbf{k} \cup i)\lambda_{j,i,\alpha,\mu,\nu,\beta}$ (in case of type 2).

Of course, the result does not depend on the way chosen to calculate the differential. We obtain:

$$\begin{aligned}
&d_{\mathcal{G}}\big(\mathbf{v}_{2,3}(x_1^3 + 2x_0^2 x_1)\big) \\
&= -x_3\mathbf{v}_2(x_1^3 + 2x_0^2x_1) + x_2\mathbf{v}_3(x_1^3 + 2x_0^2x_1) + x_1\mathbf{v}_2(x_1^2x_3 + 2x_0^2x_3) \\
&\quad + 2\mathbf{v}_1(x_0^2x_2x_3) - x_1^2\mathbf{v}_3(x_1x_2) - 2x_0^2\mathbf{v}_3(x_1x_2).
\end{aligned}$$

Our next result formalizes the following idea: If one starts at a vertex

$\mathbf{v}_i(x^\mu \mathbf{h}_\alpha)$ with $i \in \operatorname{ncrit}(\mathbf{h}_\alpha)$ and follows through all possible reduction paths in the graph, one will never get to a vertex where one must calculate an involutive standard representation. If there are no critical (i. e. non-multiplicative) variables present at the source of a reduction path, then this property will also hold for any vertex incident to a reduction path originating in this source. In order to generalize this Lemma to higher homological degrees, one must simply replace the conditions $i \in \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $j \in \operatorname{ncrit}(\mathbf{h}_\beta)$ by ordered sequences $\mathbf{k}, \boldsymbol{\ell}$ with $\mathbf{k} \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $\boldsymbol{\ell} \subseteq \operatorname{ncrit}(\mathbf{h}_\beta)$.

Lemma 4.1.4. *Let L be of Schreyer type. Let $i \cup \operatorname{supp}(\mu) \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$. Then for any reduction path $p = \big(\mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_j(x^\nu \mathbf{h}_\beta)\big)$ we have $j \in \operatorname{ncrit}(\mathbf{h}_\beta)$. In particular, in this situation there is no reduction path $p = \mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_k \mathbf{h}_\beta$ with $k \in \operatorname{crit}(\mathbf{h}_\beta)$.*

Proof. Assume first that p is an elementary reduction path. We separately consider two cases, depending on the source of p.

Case 1 $\mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \in A^0 \cup A^+$. Then the elementary reduction path must be of type 0 and p is either of the form

$$\mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \mathbf{v}_\emptyset(x_i x^\mu \mathbf{h}_\alpha) \to \mathbf{v}_{\operatorname{cls}(x_i x^\mu)}\left(\frac{x_i x^\mu}{\operatorname{cls}(x_i x^\mu)} \mathbf{h}_\alpha\right)$$

or

$$\mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to x_i \mathbf{v}_\emptyset(x^\mu \mathbf{h}_\alpha) \to x_i \mathbf{v}_{\operatorname{cls}(x^\mu)}\left(\frac{x^\mu}{\operatorname{cls}(x^\mu)} \mathbf{h}_\alpha\right).$$

The assumption, $i \cup \operatorname{supp}(\mu) \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$ assures $\operatorname{cls}(x_i x^\mu) \in \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $\operatorname{cls}(x^\mu) \in \operatorname{ncrit}(\mathbf{h}_\alpha)$, resp., as claimed.

Case 2 $\mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \in A^-$. Then p can be either of type 1 or type 2.

Type 1 If p is of the form

$$\mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \mathbf{v}_{i,\operatorname{cls}(x^\mu)}\left(\frac{x^\mu}{\operatorname{cls}(x^\mu)} \mathbf{h}_\alpha\right) \to \mathbf{v}_i\left(\frac{x^\mu}{\operatorname{cls}(x^\mu)} \mathbf{h}_\alpha\right),$$

then the statement is obvious. If, however, p is of the form

$$\mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \mathbf{v}_{i,\operatorname{cls}(x^\mu)}\left(\frac{x^\mu}{\operatorname{cls}(x^\mu)} \mathbf{h}_\alpha\right) \to \mathbf{v}_{\operatorname{cls}(x^\mu)}\left(\frac{x^\mu}{\operatorname{cls}(x^\mu)} \mathbf{h}_\alpha\right),$$

then Assumption 4.0.4 entails that $\operatorname{cls}(x^\mu) \in \operatorname{ncrit}(\mathbf{h}_\alpha)$.

Type 2 Here the path p is of the form

$$\mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \mathbf{v}_{i,\operatorname{cls}(x^\mu)}\left(\frac{x^\mu}{\operatorname{cls}(x^\mu)} \mathbf{h}_\alpha\right) \to \mathbf{v}_{\operatorname{cls}(x^\mu)}\left(\frac{x_i x^\mu}{\operatorname{cls}(x^\mu)} \mathbf{h}_\alpha\right).$$

As above, $\operatorname{cls}(x^\mu) \in \operatorname{ncrit}(\mathbf{h}_\alpha)$ and by assumption $i \in \operatorname{ncrit}(\mathbf{h}_\alpha)$. Thus $\frac{x_i x^\mu}{\operatorname{cls}(x^\mu)} \mathbf{h}_\alpha$ is already an involutive standard representation.

For arbitrary reduction paths p, the claim now follows by an induction over the length of p. $\square$

Now we can show, as we have claimed earlier, that reduction paths of type 0 are irrelevant for the differential $d_{\mathcal{G}}$. Implicitly, this statement is already contained in [Skö06, Lemma 5], in a more general setting.

Lemma 4.1.5. *Let L be of Schreyer type. In the differential* (3.3.11)*, no reduction path appearing in the third sum contains an elementary reduction path of type 0; i. e. all reduction paths appearing in the third sum are concatenations of elementary reduction paths of type 1 or 2.*

Proof. Let $p = p_m \circ ... \circ p_1$ be a reduction path appearing in the sum in equation (3.3.11) with elementary reduction paths p_i. Suppose p ends at the vertex $\mathbf{v_k h}_\gamma$, and let

$$p_r = \quad \left(\mathbf{v}_{\boldsymbol{\ell}}(x^\mu \mathbf{h}_\alpha) \to \mathbf{v}_{\boldsymbol{\ell}_i}(x^\nu \mathbf{h}_\beta) \to \mathbf{v}_{\boldsymbol{\ell}_i \cup \operatorname{cls}(x^\nu)} \left(\frac{x^\nu}{\operatorname{cls}(x^\nu)} \mathbf{h}_\beta\right)\right)$$

be an elementary reduction path of type 0 appearing in p. Now we have that $\mathbf{v}_{\boldsymbol{\ell}_i \cup \operatorname{cls}(x^\nu)}(\frac{x^\nu}{\operatorname{cls}(x^\nu)}\mathbf{h}_\beta) \in A^+$. From the discussion of elementary reduction paths in remark 3.3.9, in particular equations (3.3.8) and (3.3.9), we learn that p_{r+1} is of type 0, and then so is p_m by induction. But then we see that $\mathbf{k}$ must contain an index that is multiplicative for $\mathbf{h}_\gamma$, so $\mathbf{v_k h}_\gamma$ is not contained in $\Gamma_{\mathcal{G}}$, i.e. it is not a generator of a module of $\mathcal{G}$. $\square$

The lemma below is a generalized version of [AFS15, Lemma 13]. The proof here is a generalized version of the proof given in the reference.

Lemma 4.1.6. *Let L be of Schreyer type. If $\operatorname{lt}(\mathbf{h}_\beta)$ is an involutive divisor of $x^\mu \operatorname{lt}(\mathbf{h}_\alpha)$ for some x^μ, then $\mathbf{h}_\beta$ is greater or equal than $\mathbf{h}_\alpha$ according to the L-ordering. In particular, we have $\alpha < \beta$.*

Proof. Since $\mathcal{H}$ is an involutive basis and therefore auto reduced, $x^\mu \in \Bbbk$ occurs only if $\mathbf{h}_\alpha = \mathbf{h}_\beta$. So now let x^μ be non-constant.

Consider the following algorithm:

Given a product $x^\mu \operatorname{lt}(\mathbf{h}_\alpha)$, we check if x^μ contains any variables which are non-multiplicative for $\mathbf{h}_\alpha$. If this is not the case the algorithm terminates, giving $x^\mu \mathbf{h}_\alpha$ as the output. Otherwise, we pick one such non-multiplicative variable x_i and find the involutive standard representation of $x_i \mathbf{h}_\alpha$. Let $x^\nu \operatorname{lt}(\mathbf{h}_\gamma)$ be the leading monomial of this representation, i.e. $x_i \operatorname{lt}(\mathbf{h}_\alpha) = x^\nu \operatorname{lt}(\mathbf{h}_\gamma)$.

Now we iterate our algorithm, by looking at $(\frac{x^\mu}{x_i} x^\nu)\mathbf{h}_\gamma$.

Our claim is that this algorithm terminates after a finite number of steps, with the output being the leading monomial of the involutive standard representation of $x^\mu \mathbf{h}_\alpha$:

Indeed, as the leading monomials remain unchanged during the algorithm, the last part is obvious if the algorithm terminates.

Now assume the algorithm does not terminate: Then we obtain an infinite chain

$$\mathbf{h}_\alpha \to \mathbf{h}_{\gamma_1} \to \mathbf{h}_{\gamma_2} \to ...$$

of elements of the involutive basis. By construction of these elements, this chain corresponds to a path in the L-graph. As our basis is finite, this means we obtain a cycle in the L-graph. But since L is continuous, this is impossible (see definition 2.3.17).

□

Lemma 4.1.7. *Let L be of Schreyer type, $|\mathcal{H}| = s$ and let $\mathcal{H}_1$ be an involutive basis of the syzygy module* $\mathrm{Syz}(\mathcal{H}) \subseteq \mathcal{P}^s$ *with $|\mathcal{H}_1| = t$. Let $p = \mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_j(x^\nu \mathbf{h}_\beta)$ be a reduction path that appears in the differential* (3.3.11) *(potentially as part of a longer path). If $\rho_p\big(\mathbf{v}_i(x^\mu \mathbf{h}_\alpha)\big) = x^\kappa \mathbf{v}_j(x^\nu \mathbf{h}_\beta)$, then*

$$\mathrm{lt}_{\prec_{\mathcal{H}_1}}(x^{\kappa+\nu}\mathbf{v}_j \mathbf{h}_\beta) \preceq_{\mathcal{H}_1} \mathrm{lt}_{\prec_{\mathcal{H}_1}}(x^\mu \mathbf{v}_i \mathbf{h}_\alpha).$$

Here, $\prec_{\mathcal{H}_1}$ denotes the Schreyer order on $\mathcal{P}^t$ which is induced by $\mathcal{H}_1$, see Definition 2.3.5.

Proof. We prove the assertion only for an elementary reduction path p and the general case follows by induction over the length of the path. If p is of type 1, we can easily prove the assertion by using the same arguments as for the corresponding Lemma in the Pommaret case, see [AFSS15, Lemma 4.6]: We either have $\rho_p(\mathbf{v}_i x^\mu \mathbf{h}_\alpha) = x_k \mathbf{v}_i\big(\frac{x^\mu}{x_k}\mathbf{h}_\alpha\big)$, where the claim is obvious, or $\rho_p(\mathbf{v}_i x^\mu \mathbf{h}_\alpha) = x_i \mathbf{v}_k\big(\frac{x^\mu}{x_k}\mathbf{h}_\alpha\big)$ for an index $k \in \mathrm{supp}(x^\mu)$, so $k \in \mathrm{ncrit}(\mathbf{h}_\alpha)$. But by Lemma 4.1.5, the last case cannot occur, for this would imply $j \in \mathrm{ncrit}(\mathbf{h}_\beta)$.

If p is of type 2, there exists an index $j \in \mathrm{supp}(\mu)$ $\big($which in particular implies $j \in \mathrm{ncrit}(\mathbf{h}_\alpha)\big)$ and thus $j \in \mathcal{X}_{J,\mathcal{H},\prec}(\mathbf{h}_\alpha)$, a multiindex ν and a scalar $\lambda \in \Bbbk$ such that $\rho_p(\mathbf{v}_i(x^\mu \mathbf{h}_\alpha)) = \lambda \mathbf{v}_j(x^\nu \mathbf{h}_\gamma)$ where $x^\nu \mathbf{h}_\gamma$ appears in the involutive standard representation of $\frac{x^\mu x_i}{x_j}\mathbf{h}_\alpha$ with a non-vanishing coefficient. Lemma 4.1.4 now implies $j \in \mathrm{crit}(\mathbf{h}_\gamma)$. By construction, $\mathrm{lt}_\prec(\frac{x_i x^\mu}{x_j}\mathbf{h}_\alpha) \succeq \mathrm{lt}_\prec(x^\nu \mathbf{h}_\gamma)$.

Here, we separately consider equality and strict inequality. If strict inequality holds, then also $\mathrm{lt}_\prec(x_i x^\mu \mathbf{h}_\alpha) \succ \mathrm{lt}_\prec(x_j x^\nu \mathbf{h}_\gamma)$. Hence by definition of the Schreyer order, we get $\mathrm{lt}_{\prec_{\mathcal{H}_1}} x^\mu \mathbf{v}_i \mathbf{h}_\alpha \succ_{\mathcal{H}_1} \mathrm{lt}_{\prec_{\mathcal{H}_1}}(x^\nu \mathbf{v}_j \mathbf{h}_\beta)$. In the case of equality, we note that $x^\nu\, \mathrm{lt}_\prec(\mathbf{h}_\gamma)$ must be the involutive divisor of $\frac{x_i x^\mu}{x_j}\, \mathrm{lt}_\prec(\mathbf{h}_\alpha)$. Hence Lemma 4.1.6 guarantees that $\mathbf{h}_\alpha$ is smaller than $\mathbf{h}_\gamma$ according to the L-ordering and hence the claim follows for this special case directly from the definition of the Schreyer order, see definition 2.3.5. □

4.2 Involutive bases via Morse theory

Before we proceed, we recall the resolution induced by an involutive basis (with respect to a continuous involutive division L) of Theorem 2.3.59 and its representation as a complex via the exterior algebra, see Remark 2.3.62: Using Lemma 4.0.3, we immediately see that the two a priori different complexes $\mathcal{G}$ of Remark

2.3.62 and Theorem 3.3.7 do in fact have the same generators $\mathbf{v_k h}_\alpha \in G_i$. It should be noted that this fact, combined with the knowledge that either complex is a free resolution of $\mathcal{M}$, is enough to know that both resolutions are isomorphic by [Eis95, Theorem 20.2]. However, we do not stop here and ask if these similarities go even further.

For notational simplicity, we formulate the two decisive corollaries only for the special case of second syzygies, but they remain valid in any homological degree: In Corollary 4.2.1 below, one replaces[1] $\mathbf{v}_{i,j}$ with $\mathbf{v_k}$ and $x_j\mathbf{v}_i\mathbf{h}_\alpha$ with $x_{\max \mathbf{k}}\mathbf{v}_{\mathbf{k}\setminus(\max \mathbf{k})}\mathbf{h}_\alpha$, while for Corollary 4.2.2 the analogous statement is true if one replaces 2 with any integer ≥ 2. Note that the special case of $|\mathbf{k}| = 1$ has been covered in Lemma 4.1.2. Corollary 4.2.1 already indicates the great similarity between Sköldbergs resolution and the one induced by an involutive basis with respect to a continuous involutive division of Schreyer type, as a comparison with Lemma 2.3.56 shows that there is a one-to-one correspondence between the leading monomials of the syzygies contained in the two resolutions.

Corollary 4.2.1. *Let L be an involutive division of Schreyer type, $|\mathcal{H}| = s$ and let $\mathcal{H}_1$ be an involutive basis of the syzygy module* $\operatorname{Syz}(\mathcal{H}) \subseteq \mathcal{P}^s$ *with $|\mathcal{H}_1| = t$. If $i < j$, then*

$$\operatorname{lt}_{\prec_{\mathcal{H}_1}}\big(d_{\mathcal{G}}(\mathbf{v}_{i,j}\mathbf{h}_\alpha)\big) = x_j\mathbf{v}_i\mathbf{h}_\alpha.$$

Here, $\prec_{\mathcal{H}_1}$ is again the Schreyer order as in Lemma 4.1.7.

Proof. As described in Section 2.3.5, we assume that the elements of the given involutive basis are numbered according to an L-ordering. Consider now the differential $d_{\mathcal{G}}$. We first compare the terms $x_i\mathbf{v}_j\mathbf{h}_\alpha$ and $x_j\mathbf{v}_i\mathbf{h}_\alpha$. The minimality of these terms with respect to any order respecting the Morse matching entails that there are no reduction paths $[\mathbf{v}_j\mathbf{h}_\alpha \rightsquigarrow \mathbf{v}_k\mathbf{h}_\delta]$ with $k \in \operatorname{crit}(\mathbf{h}_\delta)$ (except trivial reduction paths of length 0), since $\mathbf{v}_j\mathbf{h}_\alpha \in A^0$; the same argument applies to $\mathbf{v}_i\mathbf{h}_\alpha$. By Definition 2.3.5 of the Schreyer order, we have $x_i\mathbf{v}_j\mathbf{h}_\alpha \prec_{\mathcal{H}_1} x_j\mathbf{v}_i\mathbf{h}_\alpha$.

Now consider any other term in this sum. We will prove $x_j\mathbf{v}_i\mathbf{h}_\alpha \succ_{\mathcal{H}_1} x^\kappa\mathbf{v}_i\mathbf{h}_\beta$, where $x^\kappa\mathbf{h}_\beta$ effectively appears in the involutive standard representation of $x_j\mathbf{h}_\alpha$. The claim then follows from applying Lemma 4.1.7 with

$$x_j\mathbf{v}_i\mathbf{h}_\alpha \succ_{\mathcal{H}_1} x^\kappa\mathbf{v}_i\mathbf{h}_\beta \succeq_{\mathcal{H}_1} \operatorname{lt}_{\prec_{\mathcal{H}_1}}\big(\rho_p(\mathbf{v}_i x^\kappa\mathbf{h}_\beta)\big).$$

We always have $\operatorname{lt}_\prec(x_jx_i\mathbf{h}_\alpha) \succeq \operatorname{lt}_\prec(x^\kappa x_i\mathbf{h}_\beta)$.

If this is a strict inequality, then $x_j\mathbf{v}_i\mathbf{h}_\alpha \succ_{\mathcal{H}_1} x^\kappa\mathbf{v}_i\mathbf{h}_\beta$ follows at once by definition of the Schreyer order.

So now assume $\operatorname{lt}_\prec(x_jx_i\mathbf{h}_\alpha) = \operatorname{lt}_\prec(x^\kappa x_i\mathbf{h}_\beta)$. By construction, we have $x^\kappa \in \Bbbk[x_0,\dots,x_{\operatorname{cls}(\mathbf{h}_\beta)}]$. Again by definition of the Schreyer order, the claim follows, if we can prove $\operatorname{lt}_{\prec_{\mathcal{H}_0}}(x_jx_i\mathbf{v}_\emptyset\mathbf{h}_\alpha) \succ_{\mathcal{H}_0} \operatorname{lt}_{\prec_{\mathcal{H}_0}}(x^\kappa x_i\mathbf{v}_\emptyset\mathbf{h}_\beta)$. Since $j \in \operatorname{crit}(\mathbf{h}_\alpha)$ and $\operatorname{lt}(x_j\mathbf{h}_\alpha)$ is involutively divisible by $\operatorname{lt}(\mathbf{h}_\beta)$, we have $\alpha < \beta$, by definition of the L-ordering. As we have $\operatorname{lt}_\prec(x_j\mathbf{h}_\alpha) = \operatorname{lt}_\prec(x^\kappa\mathbf{h}_\beta)$, we also obtain

$$\operatorname{lt}_{\prec_{\mathcal{H}_0}}(x_jx_i\mathbf{v}_\emptyset\mathbf{h}_\alpha) \succ_{\mathcal{H}_0} \operatorname{lt}_{\prec_{\mathcal{H}_0}}(x^\kappa x_i\mathbf{v}_\emptyset\mathbf{h}_\beta)$$

[1] Formally, we should write $\mathbf{v}_{(i,j)}$ instead of $\mathbf{v}_{i,j}$. But since the meaning is clear from the context, we omit unnecessary brackets.

and therefore

$$\operatorname{lt}_{\prec_{\mathcal{H}_1}}(x_j \mathbf{v}_i \mathbf{h}_\alpha) \succ_{\mathcal{H}_1} \operatorname{lt}_{\prec_{\mathcal{H}_1}}(x^\kappa \mathbf{v}_i \mathbf{h}_\beta).$$

□

The leading monomials are therefore the same as the leading monomials given by Theorem 2.3.58, so we have:

Corollary 4.2.2. *Let L be of Schreyer type. The set*

$$\bigl\{d_{\mathcal{G}}(v_{\mathbf{k}} \otimes \mathbf{h}_\alpha) \mid |\mathbf{k}| = 2; \mathbf{k} \subseteq \operatorname{crit}(\mathbf{w}_\alpha)\bigr\}$$

is an involutive basis for the involutive division L with respect to the term order $\prec_{\mathcal{H}_0}$.

Based on these two corollaries, it is now comparatively straightforward to prove our main theoretical result of this chapter, by explicitly constructing an isomorphism between the two resolutions we consider.

Theorem 4.2.3. *Let $\mathcal{H}$ be an involutive basis with respect to a continuous involutive division L of Schreyer type. Assume the situation of Lemma 4.0.3, i. e. we consider a submodule $\mathcal{M} \subseteq \mathcal{P}^m$ and the presentation* (3.3.1) *comes from an involutive basis $\mathcal{H}$ that is ordered according to an L-ordering. Then the resolution $(\mathcal{G}, d_{\mathcal{G}})$ is isomorphic to the resolution induced by $\mathcal{H}$ as in Theorem 2.3.59 and Remark 2.3.62, via a family of automorphisms $\varphi_i \colon G_i \to G_i$ satisfying $\operatorname{lt}\bigl(\varphi_i(f)\bigr) = \operatorname{lt}(f)$ for the leading monomials with respect to $\prec_{\mathcal{H}_{i-1}}$ of any $f \in G_i$. In particular, if we denote $d_{\mathcal{G},i} \colon G_{i+1} \to G_i$, then the images of the basis elements of G_{i+1} are an involutive basis of $\ker d_{\mathcal{G},i}$ (for the involutive division L with respect to $\prec_{\mathcal{H}_{i-1}}$), and both the image and the preimage of any involutive basis under φ_i is again an involutive basis.*

Proof. We write the two resolutions as rows in a diagram denoting the components of the resolution induced by an involutive basis (see Remark 2.3.62) by d_i and those of $d_{\mathcal{G}}$ by d_i^*:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & G_2 & \xrightarrow{d_2} & G_1 & \xrightarrow{d_1} & G_0 & \xrightarrow{d_0} & \mathcal{M} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0 = \mathrm{id}} & & \| & & \\
\cdots & \longrightarrow & G_2 & \xrightarrow{d_2^*} & G_1 & \xrightarrow{d_1^*} & G_0 & \xrightarrow{d_0^* = d_0} & \mathcal{M} & \longrightarrow & 0
\end{array} \quad . \tag{4.2.1}$$

Let $\{\mathbf{v}_{\mathbf{k}} \mathbf{h}_\alpha \mid \mathbf{k} \subseteq \overline{\mathcal{X}}_{L,\mathcal{H}\prec}(\mathbf{h}_\alpha), |\mathbf{k}| = i\}$ be the basis of the free module G_i. By Remark 2.3.62, the vectors $\mathbf{h}_{i,\mathbf{k},\alpha} = d_{i+1}(\mathbf{v}_{\mathbf{k}} \mathbf{h}_\alpha)$ define[1] an involutive basis $\mathcal{H}_i$ of $\operatorname{im} d_i$. Analogously, we obtain an involutive basis $\mathcal{H}_i^*$ of $\operatorname{im} d_i^*$. Here we set $\mathcal{H}_{-1} = \mathcal{H}_{-1}^* = \mathcal{H}$, the given involutive basis of $\mathcal{M}$, and define the term orders $\prec_i$ on G_i recursively as the Schreyer orders $\prec_i = \prec_{\mathcal{H}_{i-1}}$. Because of Corollary 4.2.1,

[1] Recall that these vectors are the same as the $\vec{S}_{(\alpha,\mathbf{k})}$ from Remark 2.3.62; but since here our point of view is to see them as elements of an involutive basis, we use the notation $\mathbf{h}_{i,\mathbf{k},\alpha}$ associated with involutive bases, instead of the notation $\vec{S}_{(\alpha,\mathbf{k})}$ associated with syzygies.

we always have $\operatorname{lt} \mathbf{h}_{i,\mathbf{k},\alpha} = \operatorname{lt} \mathbf{h}^*_{i,\mathbf{k},\alpha}$ and hence also $\prec_i = \prec_{\mathcal{H}^*_{i-1}}$, so the Schreyer order is the same for either involutive basis.

Assume now that an automorphism $\varphi_0 : G_0 \to G_0$ is given which satisfies

$$\varphi_0\bigl(\operatorname{im}(d_1)\bigr) = \operatorname{im}(d_1^*)$$

and which preserves the term order $\prec_0$ in the sense that $\operatorname{lt}_{\prec_0}\bigl(\varphi_0(\mathbf{f})\bigr) = \operatorname{lt}_{\prec_0}(\mathbf{f})$ holds for all vectors $0 \neq \mathbf{f} \in G_0$. Obviously, the identity is such an automorphism (since by construction the maps $d_0, d_0^* \colon G_0 \to \mathcal{M}$ are the same). We now show that φ_0 can be lifted to automorphisms $\varphi_i : G_i \to G_i$ preserving the term orders $\prec_i$ such that the diagram (4.2.1) commutes.

If

$$\varphi_i(\mathbf{h}_{i,\mathbf{k},\alpha}) = \sum_{\beta=1}^{s} \sum_{\boldsymbol{\ell} \subseteq \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\beta)} P^{\boldsymbol{\ell},\beta}_{\mathbf{k},\alpha} \mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}^*_{i,\boldsymbol{\ell},\beta}$$

is an involutive standard representation with respect to the involutive basis $\mathcal{H}^*_i$, then we set

$$\varphi_{i+1}(\mathbf{v}_{\mathbf{k}} \mathbf{h}_\alpha) = \sum_{\beta=1}^{s} \sum_{\boldsymbol{\ell} \subseteq \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\beta)} P^{\boldsymbol{\ell},\beta}_{\mathbf{k},\alpha} \mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}_\beta$$

and extend $\mathcal{P}$-linearly. It is trivial that for this iterative choice of φ_{i+1}, the diagram (4.2.1) becomes commutative.

We temporarily renumber the elements of the involutive bases $\mathcal{H}_i$ and $\mathcal{H}^*_i$ according to an ordering $\sqsubset$ such that

$$\mathbf{v}_{\mathbf{k}} \mathbf{h}_\alpha \sqsubset \mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}_\beta \quad \text{if and only if} \quad \operatorname{lt} \mathbf{h}_{i,\mathbf{k},\alpha} \prec_i \operatorname{lt} \mathbf{h}_{i,\boldsymbol{\ell},\beta}.$$

This is possible since by construction, the images of the $\mathbf{v}_{\mathbf{k}} \mathbf{h}_\alpha$ are an involutive basis and therefore have pairwise distinct leading terms. By definition of an involutive standard representation, the matrix $(P^{\boldsymbol{\ell},\beta}_{\mathbf{k},\alpha})$, whose entries are defined by the involutive standard representations

$$\varphi_i(\mathbf{h}_{i,\mathbf{k},\alpha}) = \sum_{\beta=1}^{s} \sum_{l \in \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\beta)} P^{\boldsymbol{\ell},\beta}_{\mathbf{k},\alpha} \mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}^*_{i,\boldsymbol{\ell},\beta}$$

is then an upper triangular matrix for this ordering $\sqsubset$. Since φ_i preserves the term order $\prec_i$, the elements $(P^{\mathbf{k},\alpha}_{\mathbf{k},\alpha})$ on the diagonal of the matrix are non vanishing constants. This fact trivially implies that φ_{i+1} is an automorphism.

Finally, we must show that φ_{i+1} preserves the term order $\prec_{i+1}$. Obviously, it suffices to check this for terms. By definition of φ_{i+1}, we have

$$\varphi_{i+1}(\mathbf{v}_{\mathbf{k}} \mathbf{h}_\alpha) = \sum_{\beta=1}^{s} \sum_{\boldsymbol{\ell} \subseteq \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\beta)} P^{\boldsymbol{\ell},\beta}_{\mathbf{k},\alpha} \mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}_\beta.$$

Using the definition of the Schreyer order and the fact that the coefficients $P^{\boldsymbol{\ell},\beta}_{\mathbf{k},\alpha}$ come from involutive standard representations, we find

$$
\begin{aligned}
& x^\nu \mathbf{v}_\mathbf{m} \mathbf{h}_\gamma = \operatorname{lt}_{\prec_{i+1}} \varphi_{i+1}(x^\kappa \mathbf{v}_\mathbf{k} \mathbf{h}_\alpha) \\
\Leftrightarrow \quad & x^\nu \mathbf{v}_\mathbf{m} \mathbf{h}_\gamma = \max_{\prec_{i+1}} \left\{ x^\kappa \operatorname{lt}_{\prec_i} P^{\boldsymbol{\ell},\beta}_{\mathbf{k},\alpha} \mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}_\beta \mid \beta = 1,\dots,s; \kappa \in \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha) \right\} \\
\Leftrightarrow \quad & x^\nu \mathbf{h}^*_{i,\mathbf{m},\gamma} = \max_{\prec_i} \left\{ x^\kappa \operatorname{lt}_{\prec_i} (P^{\boldsymbol{\ell},\beta}_{\mathbf{k},\alpha} \mathbf{h}^*_{i,\boldsymbol{\ell},\beta}) \mid \beta = 1,\dots,s; \kappa \in \overline{\mathcal{X}}_{L,\mathcal{H},\prec}(\mathbf{h}_\alpha) \right\} \\
\Leftrightarrow \quad & x^\nu \mathbf{h}^*_{i,\mathbf{m},\gamma} = x^\kappa \operatorname{lt}_{\prec_i} (\mathbf{h}^*_{i,\mathbf{k},\alpha}) \\
\Leftrightarrow \quad & x^\nu \mathbf{h}_{i,\mathbf{m},\gamma} = x^\kappa \operatorname{lt}_{\prec_i} (\mathbf{h}_{i,\mathbf{k},\alpha}) \\
\Leftrightarrow \quad & x^\nu \mathbf{v}_\mathbf{m} \mathbf{h}_\gamma = \operatorname{lt}_{\prec_{i+1}} (x^\kappa \mathbf{v}_\mathbf{k} \mathbf{h}_\alpha)
\end{aligned}
$$

as required. Note again that here we have used the fact that both $\mathcal{H}_i$ and $\mathcal{H}_i^*$ are involutive bases, and therefore the leading monomials of $\mathcal{H}_i$ (or $\mathcal{H}_i^*$ resp.) are pairwise distinct, which simplifies the comparison of terms elements with respect to the Schreyer order.

□

4.3 Calculating individual Betti numbers

Another application of the combination of involutive bases and Morse theory is as follows:

Given a $\mathcal{P}$-module $\mathcal{M}$, suppose we are interested in only one of the bigraded Betti numbers $\beta_{i,j}(\mathcal{M})$. Betti numbers are defined via ranks of the graded modules appearing in the minimal free resolution of $\mathcal{M}$, one could be tempted to calculate the minimal free resolution of $\mathcal{M}$ (at least for up to homological degree j and total degree i) and then read off the rank corresponding to the Betti number in question. But this approach usually involves calculations that are irrelevant to this special problem.

Our approach now allows us to directly calculate the differential of the free (non-minimal) resolution $\mathcal{G}$ of $\mathcal{M}$ in any degree, total and homological, without having to compute the differential for smaller (total and homological) degrees:

We obtain the modules (and the differential) of the minimal free resolution by minimising the free resolution $\mathcal{G}$. From Lemma 2.2.2 we see that the only minimisations that involve the module[1] $G_{j,i}$ are those that come from the maps $G_{j+1,i-1} \to G_{j,i}$ and $G_{j,i} \to G_{j-1,i+1}$. Even better, for the purpose of obtaining the modules of $\mathcal{G}$ it is sufficient to know all constants in the differential $d_\mathcal{G}$, for after performing a minimisation, the formula from Lemma 2.2.2 ensures that the constants in the new smaller reduced complex arise via a formula determined by the constants in the original complex.

It should be noted that this process of partially[2] minimising $\mathcal{G}$ involves only linear algebra over $\Bbbk$, as the necessary operations are matrix operations over

[1]In these notations, $G_{j,i}$ denotes the component of degree i of the module G_j of homological degree j in the chain complex $\mathcal{G}$.

[2]I.e. calculating the modules in the minimal resolution, but not the differential

$\Bbbk$. Following the approach of first constructing the minimal free resolution, one would have, in order to obtain the differential of this resolution, perform matrix operations over $\mathcal{P}$, which can usually be expected to be much more computationally challenging.

So our algorithm to compute a single Betti number $\beta_{i,j}(\mathcal{M})$ looks as follows:

- Construct an involutive basis of $\mathcal{M}$, for a suitable involutive division, i.e. a continuous involutive division of Schreyer type.
- Construct the modules $F_{j+1,i-1}, F_{j,i}$ and $F_{j-1,i+1}$ and the differential between these modules. In Chapter 5 below, we explain how with our approach, it is even possible to restrict the calculation of the differential in a way that only computes the constants, i.e. we skip the (here) unnecessary part of calculation the non-constant part of the differential.
- Construct the subgraph of the Morse graph of $\mathcal{F}$ which contains only the generators of these three modules. So if we use this idea, we can skip the next two steps in this algorithm.
- From this graph, construct the constant parts of the chain complex

  $$G_{j+1,i-1} \to G_{j,i} \to G_{j-1,i+1}.$$

- Compute the homology module at the middle of the complex

  $$\Bbbk \otimes G_{j+1,i-1} \to \Bbbk \otimes G_{j,i} \to \Bbbk \otimes G_{j-1,i+1}.$$

  In particular, this computation only involves $\Bbbk$-linear spaces, i.e. linear algebra over $\Bbbk$ is sufficient to compute this homology.
- $\beta_{i,j}(\mathcal{M})$ is the rank of this homology module, i.e. its $\Bbbk$-dimension.

Of course, we can iterate this algorithm to compute the entire Betti diagram; or even, as some computations for different Betti numbers are the same, combine several calculations. In the next section we compare see how efficient this algorithm is in comparison to other approaches.

4.3.1 Discussion of possible applications

What are the applications for this algorithm? Apart from the efficiency of the implementation of this algorithm as explained in Section 4.4 below, it very much depends on the given setting.

Usually, given a module, one would like to know the complete Betti diagram, which can of course be computed with our algorithm. However, are there situations where it might suffice to compute only a small number of Betti numbers?

In some situations, often some which are described in a geometrical way, the shape of the Betti diagram can only be one of a limited number of well-known tables[1]. For example, from [Eis05, Section 2C], we know that the Betti table of (the module corresponding to) seven points in linearly general position in $\mathbb{P}^2_{\Bbbk}$ is either

	0	1	2	3
0	1	–	–	–
1	–	3	**0**	–
2	–	**1**	6	3

or

	0	1	2	3
0	1	–	–	–
1	–	3	**2**	–
2	–	**3**	6	3

,

depending on whether or not the seven points lie on a curve of degree 3 (second table) or not (first table). So if we know that only one of either cases is possible, the knowledge of just one of the Betti numbers marked in red is sufficient to deduce the entire betti diagram.

In the same manner, there are results classifying Betti tables of canonical curves up to genus 8, see [Sch]; in [Sag06], a complete classification of the Betti tables of (smooth, irreducible) canonical curves of genus 9 was given. The possible Betti tables, classified as in the given reference, are:

	general								$\exists g^1_5$								$\exists$ two g^1_5							
	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-
1	-	21	64	**70**	-	-	-	-	-	21	64	**70**	4	-	-	-	-	21	64	**70**	8	-	-	-
2	-	-	-	**0**	70	64	21	-	-	-	-	**4**	70	64	21	-	-	-	-	**8**	70	64	21	-
3	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1
	$\exists$ three g^1_5								$\exists g^2_7$								$\exists g^1_4$							
	0	1	2	3	4	5	6	7		1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-
1	-	21	64	**70**	12	-	-	-	-	21	64	**70**	24	-	-	-	-	21	64	**75**	24	5	-	-
2	-	-	-	**12**	70	64	21	-	-	-	-	**24**	70	64	21	-	-	-	5	**24**	75	64	21	-
3	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1
	$\exists g^1_4 \times g^1_5$								$\exists g^2_6$								$\exists g^1_3$							
	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
0	1	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-
1	-	21	64	**75**	44	5	-	-	-	21	64	**90**	64	20	-	-	-	21	70	**105**	84	35	6	-
2	-	-	5	**44**	75	64	21	-	-	-	20	**64**	90	64	21	-	-	6	35	**84**	105	70	21	-
3	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1	-	-	-	-	-	-	-	1

Hence it suffices to compute the red Betti numbers, and in the cases where this number is 24, to additionally compute the blue Betti numbers. Of course, these are not the only possible choices.

A few words of caution are in order though: In neither of the quoted references, the results obtained were by working with explicit generating systems, whose existence is a necessity to use our approach. Thus the actual relevance of these results for the fields of origin of these examples might be limited. Additionally, we expect the knowledge that the assumptions of either example (linearly general position, canonical curve of certain genus) are given, to not come easily. They should require some computations or theoretical arguments on their own.

[1] The examples given here are based on talk by M. Albert at CASC 2015.

In particular, we note that to our knowledge, there is no argument linking involutive bases and the genus of a variety. We expect that if one compares the task of verifying the assumptions to the task of computing of a single Betti number, the former tends to be much more difficult than the latter.

4.4 Implementation in CoCoALib

The theory of this chapter, creating a free resolution via involutive bases and algebraic discrete Morse theory, and in particular the algorithm of Section 4.3 to compute individual Betti numbers has been implemented by M. Albert in the computer algebra system CoCoALib[AB]. From our experience, it appears that for most examples, this implementation outperforms Macaulay2 [GS15] and Singular[DGPS15]. While there are some examples where these computer algebra systems are faster, most of the time CoCoALib is faster than either, often by orders of magnitude. Notably, this is before some relatively obvious, but yet to be implemented, optimisations of the algorithm in CoCoALib, which we will mention at the end of this section.

In this section, we shortly sketch some aspects of this implementation and compare it to the two computer algebra systems mentioned above. Most of the contents of this chapter, including the tables containing the benchmarks, have already been presented in [AFS15, Section 4]. We refer to this paper for a more detailed outline.

The calculations for the tables were performed by M. Albert using an Intel i5-4570 processor with 8GB DDR3 main memory, the operating system Fedora 20 and CoCoALib. CoCoALib was compiled by gcc 4.8.3. The running times are given in seconds. The ground field for each example was $\mathbb{F}_{101} \cong \mathbb{Z}/101\mathbb{Z}$. This field was chosen to keep the coefficients within a manageable limit for either system; the choice was made in order to limit effects that the unrelated aspect of how either system is able to deal with rational numbers (or large finite fields) might have on our computations. The maximal time usage was limited to 2 hours and and the maximal memory usage to 7.5 GB. In the tables, a * marks when the computation was exceeding the time limit, while ** marks when it was running out of memory.

The examples considered were taken from the website [GBY], where one can also find more information about these examples, in particular the defining equations. This website is an ongoing project by Gerdt, Blinkov and Yanukovich, documenting their results regarding computation of involutive bases and Gröbner bases. A large number of these examples also features in articles published by these authors, for example [GBY01].

The following list describes the columns of the tables:

Example: name of the example

#JB: number of elements in the minimal Janet basis

#GB: number of elements in the reduced Gröbner basis

$\frac{\#\mathbf{JB}}{\#\mathbf{GB}}$**:** the quotient of #JB and #GB

Example	Time Macaulay2	Time Singular	Time CoCoALib
butcher8	**126.25**	**19.92**	**1.20**
camera1s	**0.09**	**6.00**	**0.13**
chandra6	0.64	8.00	0.13
cohn2	**0.03**	**1.00**	**0.03**
cohn3	**1.47**	**5.90**	**0.32**
cpdm5	14.71	5.05	0.64
cyclic6	0.99	1.26	0.37
cyclic7	1 093.66	*	37.42
cyclic8	*****	*****	**1 663.00**
des18_3	433.45	20.84	3.15
des22_24	*	**	52.19
dessin1	428.13	20.89	3.10
dessin2	*	*	32.90
f633	**591.08**	**7.70**	**49.06**
hcyclic5	**0.03**	**2.00**	**0.09**
hcyclic6	**11.00**	**47.12**	**7.41**
hcyclic7	*****	*****	**3 688.01**
hemmecke	**0.00**	**0.00**	**2.69**
hietarinta1	**443.15**	**170.29**	**4.12**
katsura6	51.41	13.90	1.22
katsura7	**	1 373.70	15.87
katsura8	*	**	412.90
kotsireas	**51.89**	**17.84**	**0.83**
mckay	**0.84**	**3.20**	**0.38**
noon5	0.13	6.00	0.27
noon6	15.14	5.07	5.25
noon7	6 979.40	821.64	122.61
rbpl	**58.81**	**22.69**	**57.91**
redcyc5	**0.02**	**2.00**	**0.01**
redcyc6	**6.79**	**1.95**	**0.13**
redcyc7	*****	*****	**8.26**
redcyc8	*****	******	**207.02**
redeco7	2.72	2.20	0.42
redeco8	355.30	11.83	5.01
redeco9	**	312.49	84.89
redeco10	**	**	2 694.05
reimer4	0.01	1.00	0.01
reimer5	1.39	5.00	0.35
reimer6	1 025.89	176.08	19.01
speer	**0.20**	**3.00**	**0.13**

Table 1: Various examples for computing Betti diagrams

Example	**#JB**	**#GB**	$\frac{\#\mathbf{JB}}{\#\mathbf{GB}}$	**ppd**	**pd**	**preg**	**reg**	**bprk**	**brk**	$\frac{\mathbf{bprk}}{\mathbf{brk}}$
butcher8	**64**	**54**	**1.19**	**8**	**8**	**3**	**3**	**3 732**	**2 631**	**1.42**
camera1s	**59**	**29**	**2.03**	**6**	**6**	**4**	**4**	**863**	**337**	**2.56**
chandra6	32	32	1.00	6	6	5	5	684	64	10.69
cohn2	**33**	**23**	**1.43**	**4**	**4**	**7**	**7**	**179**	**67**	**2.67**
cohn3	**106**	**92**	**1.15**	**4**	**4**	**7**	**7**	**696**	**370**	**1.88**
cpdm5	83	77	1.08	5	5	9	9	1 020	100	10.20
cyclic6	46	45	1.02	6	6	9	9	1 060	320	3.31
cyclic7	210	209	1.00	7	7	11	11	10 356	1 688	6.14
cyclic8	**384**	**372**	**1.03**	**8**	**8**	**12**	**12**	**34 136**	**6 400**	**5.33**
des18_3	104	39	2.67	8	8	4	4	8 132	2 048	3.97
des22_24	129	45	2.87	10	10	4	4	32 632	6 192	5.27
dessin1	104	39	2.67	8	8	4	4	8 132	2 048	3.97
dessin2	122	46	2.65	10	10	4	4	22 760	6 192	3.68
f633	**153**	**47**	**3.26**	**10**	**10**	**3**	**3**	**17 390**	**4 987**	**3.49**
hcyclic5	**52**	**38**	**1.37**	**6**	**5**	**11**	**10**	**932**	**32**	**29.13**
hcyclic6	**221**	**99**	**2.23**	**7**	**7**	**14**	**14**	**9 834**	**146**	**67.36**
hcyclic7	**1 182**	**443**	**2.67**	**8**	**8**	**17**	**17**	**105 957**	**1 271**	**83.37**
hemmecke	**983**	**9**	**109.22**	**4**	**4**	**61**	**61**	**6 242**	**38**	**164.26**
hietarinta1	**52**	**51**	**1.02**	**10**	**10**	**2**	**2**	**6 402**	**3 615**	**1.77**
katsura6	43	41	1.05	7	7	6	6	1 812	128	14.16
katsura7	79	74	1.07	8	8	7	7	6 900	256	26.95
katsura8	151	143	1.06	9	9	8	8	27 252	512	53.23
kotsireas	**78**	**70**	**1.11**	**6**	**6**	**5**	**5**	**1 810**	**1 022**	**1.77**
mckay	**126**	**51**	**2.47**	**4**	**4**	**15**	**9**	**840**	**248**	**3.39**
noon5	137	72	1.90	5	5	8	8	1 618	130	12.45
noon6	399	187	2.13	6	6	10	10	9 558	322	29.68
noon7	1 157	495	2.34	7	7	12	12	56 666	770	73.59
rbpl	**309**	**126**	**2.45**	**7**	**7**	**14**	**14**	**13 834**	**1 341**	**10.32**
redcyc5	**23**	**10**	**2.30**	**5**	**5**	**7**	**7**	**276**	**88**	**3.14**
redcyc6	**46**	**21**	**2.19**	**6**	**6**	**9**	**9**	**1 060**	**320**	**3.31**
redcyc7	**210**	**78**	**2.69**	**7**	**7**	**11**	**11**	**10 356**	**1 688**	**6.14**
redcyc8	**371**	**193**	**1.92**	**8**	**8**	**12**	**12**	**32 459**	**6 973**	**4.65**
redeco7	48	33	1.45	7	7	5	5	1 708	128	13.34
redeco8	96	65	1.48	8	8	6	6	6 828	256	26.67
redeco9	192	129	1.49	9	9	7	7	27 308	512	53.34
redeco10	384	257	1.49	10	10	8	8	109 228	1 024	106.67
reimer4	19	17	1.12	4	4	6	6	118	16	7.38
reimer5	55	38	1.45	5	5	9	9	694	32	21.69
reimer6	199	95	2.09	6	6	12	12	5 302	64	82.84
speer	**49**	**44**	**1.11**	**5**	**5**	**7**	**7**	**359**	**133**	**2.70**

Table 2: Statistics for examples from Table 1

ppd: the projective pseudo-dimension, i.e. the length of the pseudo Betti diagram

pd: the projective dimension

preg: the pseudo-regularity, i.e. the maximal total degree appearing in the pseudo Betti table

reg: the Castelnuovo-Mumford-regularity

bprk: the Betti pseudo-rank, i.e. the total rank of the pseudo Betti table

brk: the Betti rank, i.e. the total rank of the Betti table

$\frac{\textbf{bprk}}{\textbf{brk}}$**:** the quotient of bprk and brk.

In the rows, **boldface** marks examples which are not δ-regular. As the computations were done for the Janet division, we avoided the unpleasant topic of coordinate transformations which would have been an issue for the Pommaret division.

Comparing the time of CoCoALib with the various numbers contained in table 2, we see that as one would intuitively expect, there seems to be a loose, but nevertheless noticeable correlation between the time of the computation and **#JB**, the size of the Janet basis; but it appears the correlation of time and **bprk**, the size of the pseudo Betti table, is stronger. For Macaulay2 and Singular, the size of the Gröbner bases are a better indicator. And consequentially, we indeed see that the relative performance of these system seems to be linked to the quotients $\frac{\#\textbf{JB}}{\#\textbf{GB}}$ and $\frac{\textbf{bprk}}{\textbf{brk}}$, again with the latter more often being the better indicator. Large values of these quotient tend to be bad for CoCoALib, as in these cases, there are many more additional elements in the Janet basis compared to the Gröbner basis. Nevertheless, these correlations suggest that other factors are still of importance; for example a natural factor in the computation would be the sparseness of the polynomials involved.

Additionally, it appears that most of the time, the resolution constructed by CoCoALib already gives a correct bounding box for the actual Betti table, i.e. the values of **ppd** and **pd** and of **preg** and **reg** coincide.

Analyzing these tables more closely, we see that as mentioned before, our algorithm in CoCoALib is faster than both Macaulay2 and Singular for most examples. Often, our algorithm is faster by orders of magnitude. When just one of these systems is faster, such as in the noon5, noon6 or **rbpl** examples, our algorithm usually performs within the same orders of magnitude. With a small stretch, one can still say the same about the less favorable **f633** example. The only clear exception where our algorithm is considerably slower, is the **hemmecke** example.

Regarding the example **hemmecke**, we note the this is a an example of an ideal which has very few generators, which however contain high powers of different variables. The defining equations of the **hemmecke** example are

$$x_0^{18}x_2^2 - x_0^{10}x_4^{10} - x_4^{20}, \quad x_0^{26}x_2x_3^3 - x_0^{20}x_4^{10} - x_4^{30}, \quad -x_1^{40}x_2^4 + x_0^{38}x_3^6.$$

From the definition of the Janet division, we quickly see that this combination of a small number of generators with high powers of different variables most likely leads to very large Janet basis: When trying to construct a Janet bases with the help of the algorithm sketched in Remark 2.3.20 in the **hemmecke** example, we have that x_4 is non-multiplicative for the second generator; and since the first generator has leading monomial x_4^{20}, we probably have to multiply the second generator (or some of its involutive standard representations) 20 times with x_4 to obtain a Janet basis, giving up to 19 new elements for the Janet basis. Such bases are not good for our algorithm, as the Gröbner bases in these situation tend to be much smaller than the Janet bases.

We also point out that our algorithm in CoCoALib is still far from optimal, and possible optimisations could come both from theoretical and practical backgrounds:

- The algorithm itself has a huge potential for parallelization: Once the (constant part of) the complex is calculated, we know from Section 4.3 that single Betti numbers can be computed independently of the others. So in principle, the associated operations can be parallelized.

- The implementations of the algorithms which are used to minimise the resolution are still in a basic state. In principle it follows the ideas given in Lemma 2.2.2. As it suffices to consider linear algebra over $\Bbbk$, we expect that we are not the first to encounter this problem. It stands to check if there are more advanced algorithms for the minimisation process, which could further increase the speed of this implementation.

- Another idea might be to take other involutive bases than Janet bases. As stated in Remark 2.3.48, the $\sqsubset_{\text{alex}}$-division might be more favorable than the Janet-division from a combinatorial point of view.

- In Chapter 5, we explain an idea how some constants are essentially the same. This idea is not yet implemented in CoCoALib and likely has potential to further improve the algorithm. A more detailed explanation of the possible improvements originating from this idea can be found in the corresponding chapter, see in particular Remarks 5.1.5 and 5.4.2.

4.5 Theoretical limits for algorithms with involutive bases

As we have seen in Section 4.4, a large value of the quotient $\frac{\#\mathbf{JB}}{\#\mathbf{GB}}$ (i.e. the involutive basis contains a sizable proportion of elements which do not feature in a minimal Gröbner basis) tends be an indicator or a close-to-necessary condition for our algorithm to perform relatively unfavorable. We shortly explain while in some situations, these elements sometimes cannot be avoided when using involutive bases:

Remark 4.5.1. Looking at the formula of the resolution induced by an involutive basis as in Theorem 2.3.59, we see that in the pseudo Betti diagram, the entries in each row are given by (sums of) binomial coefficients, whose exact values are determined by the distribution of the elements of the involutive basis $\mathcal{H}$ over the various degrees and occurring sets of non-multiplicative variables. Now since the Betti diagram is obtained from the pseudo Betti diagram via minimisations, this restriction to the shape of the pseudo Betti diagrams can have some rather unpleasant consequences. We will restrict ourselves here to explain these effects for some examples, as a precise mathematical expression would both require highly technical language, if at all possible, and possibly hide the actually relevant consequence.

For the sake of simplicity, we will suppose that in the remainder of this remark, we consider polynomial modules $\mathcal{M} \trianglelefteq \mathcal{P}^m$ instead of quotients $\mathcal{P}^m/\mathcal{M}$.. Suppose for example that the Betti diagram of $\mathcal{M}$ is of shape

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline * & * & * & * & * & * \\ 5 & 0 & 0 & 0 & 0 & 1 \end{array},$$

with all other entries further right or further below being 0. The only way this is possible if $\mathcal{H}$ contains exactly one element of degree 5 with exactly 4 non-multiplicative variables. Assuming that the pseudo Betti table has the same bounding box[1] implies that in the pseudo Betti table, the entries of the last row are at least given by

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline * & * & * & * & * & * \\ 5 & 1 & 4 & 6 & 4 & 1. \end{array}$$

Recall that the entries of the pseudo Betti diagram correspond to generators in the complex $\mathcal{G}$, and we immediately see that $\mathcal{G}$ contains $1 + 4 + 6 + 4 = 15$ redundant generators. In fact it is even worse: Every minimisation removes two generators in a way that can be expressed as subtracting a diagram of shape

$$\begin{array}{cccc} ... & 0 & 1 & 0 \\ 0 & 1 & 0 & ... \end{array}$$

from the pseudo Betti diagram. This means that in order to remove the entries of the last row (except for the bottom right entry 1), the entries of the second-to-last row must be at least

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline * & * & * & * & * & * \\ 4 & * & 1 & 4 & 6 & 4 \\ 5 & 1 & 4 & 6 & 4 & 1. \end{array}$$

Now for the rightmost entry in the second-to last row implies to be at least 4, we have that $\mathcal{H}$ must contain at least 4 elements of degree 4 with exactly

[1] This assumption does not hold in general. However, for most of the examples given in Section 4.4, this behavior is indeed satisfied, so we consider this assumption to be reasonable. Considering the general case where the pseudo Betti table has a potentially larger bounding box would complicate the situation even further.

4 non-multiplicative variables, which in fact implies that entries in the pseudo Betti diagram are at least

	0	1	2	3	4
$*$	$*$	$*$	$*$	$*$	$*$
4	4	16	24	16	4
5	1	4	6	4	1.

Now performing the minimisations that lead to a last row as in the actual Betti diagram, we have that after these minimisations a lower bound for the entries of the (partially) reduced diagram is

	0	1	2	3	4
$*$	$*$	$*$	$*$	$*$	$*$
4	4	15	20	10	0
5	0	0	0	0	1.

Now depending on how the second-to-last row in the Betti diagram looks, this gives further restrictions. Note that now we even need to consider that the remaining entries might get canceled both by entries in the row above, one column to the right, or by an entries in the row below, one column to the left.

A (likely) worst case scenario would be that the second-to-last row in the Betti diagram vanishes entirely. Since then again by iterating our argument, if the rightmost entry in the third-to-last row is least a, then be third-from-the-right entry in the bottom row must be at least $10 - a$. These conditions can then be transferred back to restrictions on $\mathcal{H}$. Remember that all entries that are removed by minimisations correspond to redundant basis elements in the complex $\mathcal{G}$, which means that in such a situation, our algorithm to compute Betti numbers (or a free resolution) works with a relatively large number of redundant elements.

It is worth pointing out that by the results of [EFW11], pure resolutions exist: These are resolutions of shape

$$0 \to \mathcal{P}(-d_m)^{\beta_{m,d_m}} \to \mathcal{P}(-d_{m-1})^{\beta_{m-1,d_{m-1}}} \to ... \to \mathcal{P}(-d_0)^{\beta_{0,d_0}}.$$

In the cited article, it is established that for every sequence $d_m > d_{m-1} > ... > d_0$, such a pure resolution does indeed exist. The $\beta_{i,j}$ are uniquely determined by the choice of the d_i. Now for these resolutions, the Betti diagram contains exactly one non-vanishing entry per column. Conversely, for every such choice of non-vanishing entries, one has a corresponding pure resolution. Continuing our example, we now know that there is indeed a module with a Betti diagram of shape

	0	1	2	3	4
3	$*$	$*$	$*$	$*$	0
4	0	0	0	0	0
5	0	0	0	0	1,

where the entries marked with $*$ are the only non-vanishing entries except for the bottom right 1.

In particular, this implies that there are arbitrarily "sparse" Betti diagrams with "many" zero rows. The consideration presented earlier suggests that for modules with such Betti diagrams, the involutive bases contain a high proportion of redundant elements.

5 Structural analysis of the induced resolution

The resolution $\mathcal{G}$ constructed in Theorem 2.3.59 from an involutive basis is in general not minimal. For a minimisation of this chain complex $\mathcal{G}$, it is decisive where constants appear in the differential. Furthermore, knowledge of all constants is sufficient for the computation of the Betti numbers; no further information about the differential is necessary. In this chapter, we will specify to the case of a Pommaret basis and show that in the resolution obtained by the combination of this Pommaret basis and algebraic discrete Morse theory as in Chapters 3 and 4, there exist relations between constants of $\mathcal{G}$ in different homological degrees.

Assumption 5.0.1. For the remainder of chapter 5, $\mathcal{H} = \{\mathbf{h}_1 \dots, \mathbf{h}_s\}$ will always be the degrevlex Pommaret basis of a $\mathcal{P}$-module $\mathcal{M} \subseteq \mathcal{P}^m$ and

$$d = \min\{k \in \mathbb{N} \mid \exists \mathbf{h} \in \mathcal{H}\colon \operatorname{cls}(\mathbf{h}) = k\}$$

the minimal class of the elements of $\mathcal{H}$. Throughout this chapter, ordered sets of pairwise different indices will appear; for these, we will employ the notations $\mathbf{k}, \boldsymbol{\ell}, \mathbf{m}$ or $(j_1, \dots, j_m)$ $(l_1, \dots, l_f)$, etc. We consider these sets to be ordered, i. e. the notation $(j_1, ..., j_m)$ implies $j_1 < \dots < j_m$.

Unless stated otherwise, the notation $x^\mu \mathbf{h}_\alpha$ implies

$$x^\mu \in \operatorname{ncrit}(\mathbf{h}_\alpha) = \mathcal{X}_{P,\prec_{\text{degrevlex}}}(\mathbf{h}_\alpha)\,.$$

However, if we multiply an element of the Pommaret basis $\mathcal{H}$ explicitly with a single variable, for example in a product $x_i \mathbf{h}_\alpha$, we also allow $x_i \in \operatorname{crit}(\mathbf{h}_\alpha)$; in fact, in cases where such a kind of multiplication arises, this will usually be the more interesting case. We will make use of reduction paths, their decomposition into elementary paths and the classification of these into different types (see Section 4.1 or [JW09, Skö06, AFSS15] for details).

We will fix the resolution $\mathcal{G}$ as constructed in Theorem 2.3.59.

Remark 5.0.2. We note one particular argument for Pommaret bases, which will appear in a shortened version on several occasions in the proofs of this chapter. Let $\mathbf{h}_\alpha$ be an element of our Pommaret basis and $x_i \in \operatorname{crit}(\mathbf{h}_\alpha) = \overline{\mathcal{X}}_{\mathcal{P},\prec}(\mathbf{h}_\alpha)$. Let $x_i \mathbf{h}_\alpha = \sum_{\beta=1}^{s} P_\beta^{(\alpha,i)} \mathbf{h}_\beta$ be the involutive standard representation. By definition, we have $\operatorname{lt}(x_i \mathbf{h}_\alpha) \succeq \operatorname{lt}(P_\beta^{(\alpha,i)} \mathbf{h}_\beta)$ for any term appearing in the involutive standard representation. This implies

$$\operatorname{cls}(x_i \mathbf{h}_\alpha) = \operatorname{cls}(\mathbf{h}_\alpha) \geq \operatorname{cls}\bigl(P_\beta^{(\alpha,i)} \mathbf{h}_\beta\bigr),$$

since $\mathcal{H}$ is a Pommaret basis for the degrevlex order. Now, we consider two cases:

$\operatorname{cls}(\mathbf{h}_\beta) > \operatorname{cls}(\mathbf{h}_\alpha)$**:** Since $\operatorname{cls}(\mathbf{h}_\alpha) \leq \operatorname{cls}\bigl(P_\beta^{(\alpha,i)} \mathbf{h}_\beta\bigr)$, we must have that the polynomial $P_\beta^{(\alpha,i)}$ is non-constant, for otherwise

$$\operatorname{cls}(\mathbf{h}_\alpha) \geq \operatorname{cls}\bigl(P_\beta^{(\alpha,i)} \mathbf{h}_\beta\bigr) = \operatorname{cls}(\mathbf{h}_\beta) > \operatorname{cls}(\mathbf{h}_\alpha).$$

So we have $\operatorname{cls}(\mathbf{h}_\alpha) \geq \operatorname{cls}\bigl(P_\beta^{(\alpha,i)}\mathbf{h}_\beta\bigr) = \operatorname{cls}\bigl(P_\beta^{(\alpha,i)}\bigr)$, and then there is an index $j \leq \operatorname{cls}(\mathbf{h}_\alpha)$ such that x_j divides $\operatorname{lt}\bigl(P_\beta^{(\alpha,i)}\bigr)$.

$\operatorname{cls}(\mathbf{h}_\beta) \leq \operatorname{cls}(\mathbf{h}_\alpha)$: Since $P_\beta^{(\alpha,i)} \in \Bbbk[x_0, \dots, x_{\operatorname{cls}(\mathbf{h}_\beta)}]$, if the polynomial $P_\beta^{(\alpha,i)}$ is not a constant, again there is an index $j \leq \operatorname{cls}(\mathbf{h}_\beta) \leq \operatorname{cls}(\mathbf{h}_\alpha)$ such that x_j divides $\operatorname{lt}\bigl(P_\beta^{(\alpha,i)}\bigr)$.

So whenever $P_\beta^{(\alpha,i)}$ is not a constant, there is an index $j \leq \operatorname{cls}(\mathbf{h}_\alpha)$ such that x_j divides $\operatorname{lt}\bigl(P_\beta^{(\alpha,i)}\bigr)$ (and, as a consequence, the same holds for any other monomial summand of $P_\beta^{(\alpha,i)}$).

This argument does not necessarily hold for other involutive divisions: Consider the ideal $\langle \mathbf{h}_1 = x_0x_1, \mathbf{h}_2 = x_0x_2\rangle \subseteq \Bbbk[x_0, x_1, x_2]$. One can check that the given generators are a Janet basis $\mathcal{H}$ for the ideal, where

$$\mathcal{X}_{J,\mathcal{H},\prec}(\mathbf{h}_1) = \{x_0, x_1\} \text{ and } \mathcal{X}_{J,\mathcal{H},\prec}(\mathbf{h}_2) = \{x_0, x_1, x_2\}.$$

However, we have $x_2\mathbf{h}_1 = x_1\mathbf{h}_2$, and therefore $P_2^{(1,1)} = x_1 > \operatorname{cls}(\mathbf{h}_1)$.

As this property of the Pommaret division is crucial in the proofs of lemmata 5.1.8 and 5.1.13, a generalisation of our results to arbitrary initially linear syzygies (for example coming from a Janet basis) is at least not straightforward, if at all possible.

Remark 5.0.3. Let $\mathbf{k} \cap \operatorname{ncrit}(\mathbf{h}_\alpha) \neq \emptyset$. If $\operatorname{ncrit}(\mathbf{h}_\alpha)$ contains an index e, it also contains all indices smaller than e, which means that in this situation, we have $\min\bigl(\mathbf{k} \cap \operatorname{ncrit}(\mathbf{h}_\alpha)\bigr) = \min \mathbf{k}$. This follows from Lemma 4.0.3 and the fact that we are considering initially linear syzygies originating from a Pommaret Basis. Again, in a more general situation, for example for Janet bases, the non-critical variables might not show such a nice behaviour. As we will take advantage of this property of the Pommaret division, we have another possible obstruction regarding the generalisation of our results of this chapter to other involutive divisions.

5.1 Some technical lemmata

Another key observation which we will implicitly use in the remainder of this chapter is:

Lemma 5.1.1. *Let $p = \mathbf{v_m}x^\mu\mathbf{h}_\gamma \to \dots \to \mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta$ be a reduction path appearing in the differential*

$$d_{\mathcal{G}}(\mathbf{v_k}\mathbf{h}_\alpha) = \sum_{\boldsymbol{\ell},\beta}\sum_{\mathbf{m},\mu,\gamma}\sum_p \rho_p\left(Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}\mathbf{v_m}(x^\mu\mathbf{h}_\gamma)\right)$$

from equation (3.3.11)[1]*. Then for any reduction path p, the reduction map ρ_p contributes a non-vanishing constant in this sum if and only if p consists entirely*

[1] see the comments following said equation for a discussion for the restrictions to the summation indices

of elementary reduction paths of type 2 (including trivial elementary reduction paths of type 0).

Proof. By Lemma 4.1.5, the path is made up from elementary reduction paths of type 1 or 2. From the description of reduction paths of type 1 in Section 4.1, we see that for any path containing, the degree of the coefficient of the reduction map is given by the numbers of elementary reduction paths of type 1 contained in the entire path. In particular, the coefficient is a constant if and only if the path consists exclusively of elementary reduction paths of type 2. □

We continue with two lemmata which we will use in proofs later in this chapter.

Lemma 5.1.2. *Let*

$$\mathbf{v}_{\mathbf{k}}(x^{\mu}\mathbf{h}_{\alpha}) \longrightarrow ... \longrightarrow \mathbf{v}_{\boldsymbol{\ell}}(x^{\nu}\mathbf{h}_{\beta})$$

be a reduction path that consists of a concatenation of elementary reduction paths of type 1 or 2, but not type 0.

Let $\mathbf{k} \cap \{0, ..., e\} \neq \emptyset$ *for some* $e \geq 0$. *Then we also have* $\boldsymbol{\ell} \cap \{0, ..., e\} \neq \emptyset$.

Proof. It suffices to prove the lemma for elementary reduction paths. Let

$$\mathbf{v}_{\mathbf{k}}(x^{\mu}\mathbf{h}_{\alpha}) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}\left(\frac{x^{\mu}}{x_i}\mathbf{h}_{\alpha}\right) \longrightarrow \mathbf{v}_{\boldsymbol{\ell}}(x^{\nu}\mathbf{h}_{\beta})$$

be an elementary reduction path of type 1 or 2. Looking at equation (3.3.6) which defines the Morse matching, we see that we must have

$$i < \min\left\{\mathbf{k} \cap \operatorname{ncrit}(\mathbf{w}_{\alpha})\right\} \text{ and } i = \operatorname{cls}(x^{\mu}).$$

Let $f \in \mathbf{k} \cap \{0, ..., e\}$.

Suppose $f \leq i$: Then because of $i \in \operatorname{ncrit}(\mathbf{h}_{\alpha})$, we also have $f \in \operatorname{ncrit}(\mathbf{h}_{\alpha})$. But then $f \leq i < \min\left\{\mathbf{k} \cap \operatorname{ncrit}(\mathbf{w}_{\alpha})\right\} \leq f$.

So $e \geq f > i$ must hold. By definition of elementary reduction paths, we have $\boldsymbol{\ell} = \{\mathbf{k} \cup i\} \setminus j$ for some $j \in \{\mathbf{k} \cup i\}$.

If $i = j$, then $\boldsymbol{\ell} = \mathbf{k}$. In this case, we have $f \in \mathbf{k} \cap \{0, ..., e\} = \boldsymbol{\ell} \cap \{0, ..., e\}$, and now $\boldsymbol{\ell} \cap \{0, ..., e\} \neq \emptyset$.

If $i \neq j$, then $i \in \boldsymbol{\ell}$. But then because of $i \leq e$, we have $i \in \boldsymbol{\ell} \cap \{0, ..., e\}$. □

Lemma 5.1.3. *Let*

$$\mathbf{v}_{\mathbf{k}}(x^{\mu}\mathbf{h}_{\alpha}) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}\left(\frac{x^{\mu}}{x_i}\mathbf{h}_{\alpha}\right) \longrightarrow \mathbf{v}_{\boldsymbol{\ell}}\left(x^{\nu}\mathbf{h}_{\beta}\right)$$

be an elementary reduction path of type 1 or 2, which additionally is **not** *of form*

$$\mathbf{v}_{\mathbf{k}}(x^{\mu}\mathbf{h}_{\alpha}) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}\left(\frac{x^{\mu}}{x_i}\mathbf{h}_{\alpha}\right) \longrightarrow \mathbf{v}_{\mathbf{k}}\left(\frac{x^{\mu}}{x_i}\mathbf{h}_{\beta}\right).$$

Let $\operatorname{cls}(x^{\mu}) \leq e$. *Then also* $\boldsymbol{\ell} \cap \{0, ..., e\} \neq \emptyset$.

Proof. For all such reduction paths, we have $\boldsymbol{\ell} = \big(\mathbf{k} \cup \mathrm{cls}(x^\mu)\big) \setminus j$ for some $j \in \mathbf{k}$. The claim now follows from $\mathrm{cls}(x^\mu) \leq e$. $\square$

Theorem 5.1.4. *Let $p = \mathbf{v_m} x^\mu \mathbf{h}_\gamma \to \ldots \to \mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}_\beta$ be a reduction path appearing in the differential*

$$d_{\mathcal{G}}(\mathbf{v_k h}_\alpha) = \sum_{\boldsymbol{\ell},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p} \rho_p \left(Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma} \mathbf{v_m} (x^\mu \mathbf{h}_\gamma) \right)$$

from equation (3.3.11) *and $x^\nu \in \Bbbk[x_0, ..., x_d]$ be a term with $x^\nu | x^\mu$ (recall that by Assumption 5.0.1, d is the minimal class of an element of $\mathcal{H}$). Then we have*

$$\rho_{\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta, \mathbf{v_m} x^\mu \mathbf{h}_\gamma}(\mathbf{v_m} x^\mu \mathbf{h}_\gamma) = x^\nu \rho_{\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta, \mathbf{v_m} \frac{x^\mu}{x^\nu} \mathbf{h}_\gamma} \left(\mathbf{v_m} \frac{x^\mu}{x^\nu} \mathbf{h}_\gamma \right).$$

Remark 5.1.5. In a more compact manner, the lemma states that there is a reduction path $p' = \mathbf{v_m} \frac{x^\mu}{x^\nu} \mathbf{h}_\gamma \to \cdots \to \mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}_\beta$ such that

$$\rho_p(\mathbf{v_m} x^\mu \mathbf{h}_\gamma) = x^\nu \rho_{p'} \left(\mathbf{v_m} \frac{x^\mu}{x^\nu} \mathbf{h}_\gamma \right).$$

Given p, the path p' can be effectively constructed. While by construction, any reduction map $\rho_* \colon F_j = \mathcal{P} \otimes \Lambda_j \mathcal{V} \otimes \mathcal{M} \to F_j$ is a $\mathcal{P}$-homomorphism with respect to $\mathcal{P}$ acting on the first component of F_j, this lemma essentially states that any ρ_p is also a "pseudo-homomorphism" for polynomials of class $\leq d$ acting on $\mathcal{M}$, i.e. the third component of F_j written as a tensor product.

Additionally, this lemma has the potential to further improve the implementations of our algorithms in, as have hinted in Section 4.4. Whenever such a variable appears, we could use the "pseudo-homomorphism" property and replace the original reduction path with a shorter one, allowing us to skip the calculation of some reduction paths and maps in the process; see also the comments in example 4.1.3 considering the light blue vertices of said example. Hence it might be possible to avoid some computations which are essentially the same. The discussion in Remark 5.4.2 suggests that it should even be possible to just compute one particular differential per element $\mathbf{h}_\alpha$ of $\mathcal{H}$ from which it is possible to obtain all constants for any generator of shape $\mathbf{v}_* \mathbf{h}_\alpha$. But most importantly for our theoretical analysis of the constant parts of the resolution $\mathcal{G}$, we now already see that, whenever a variable of class $i \leq d$ is present at some point in the path p in an $\mathcal{M}$-part of a vertex, then p cannot contribute a constant to the differential.

Additionally, the idea presented in Lemma 5.1.1 implies that the degree of a coefficient contributed by a reduction path is equal to number of elementary reduction paths of type 1 that appear in this path. Of course, after minimising this resolution, we still have that coefficients of degree f originate from the coefficients of degree f in the original unminimised resolution. So if one is interested differential of the minimal free resolution up to a certain degree f, one can also use this idea and restrict to the calculation of reduction paths which contain at most f elementary reduction paths of type 1.

Proof. (of Theorem 5.1.4) Without loss of generality assume that x^ν is not constant, so $\operatorname{cls}(x^\nu) \le d$.

Let $p = \quad \mathbf{v_m}(x^\mu \mathbf{h}_\gamma) \to ... \to \mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}_\beta$.

Consider the partition of p into elementary reduction paths $p_1, ..., p_m$, i.e. $p = p_m \circ p_2 \circ ... \circ p_1$. By Lemma 4.1.5, or [Skö11, Equation (2)] or [Skö06, lemma 5], these elementary reduction paths are of type 1 or 2.

Now we claim

$$p_1 = \quad \mathbf{v_m} x^\mu \mathbf{h}_\gamma \to \mathbf{v}_{\mathbf{m} \cup \operatorname{cls}(x^\mu)} \left(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}} \mathbf{h}_\gamma \right) \to \mathbf{v_m} \left(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}} \mathbf{h}_\gamma \right).$$

If not, then from Lemma 5.1.3 follows that p_1 is of shape

$$\mathbf{v_m} x^\mu \mathbf{h}_\gamma \to \mathbf{v}_{\mathbf{m} \cup \operatorname{cls}(x^\mu)} \left(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}} \mathbf{h}_\gamma \right) \to \mathbf{v}_{(\mathbf{m} \cup \operatorname{cls}(x^\mu)) \setminus j} \left(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}} \mathbf{h}_\gamma \right)$$

for some $j \in \mathbf{m} \setminus \operatorname{cls} x^\mu$. Because of $\operatorname{cls}(x^\mu) \le \operatorname{cls}(x^\nu) \le d$, we then have $\Big(\big(\mathbf{m} \cup \operatorname{cls}(x^\mu) \big) \setminus j \Big) \cap \{0, ..., d\} \ne \emptyset$. Via induction, using Lemma 5.1.2, this implies $\boldsymbol{\ell} \cap \{0, ..., d\} \ne \emptyset$. But since $\{0, .., d\} \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$ for all $1 \le \alpha \le s$, we also get $\boldsymbol{\ell} \not\subseteq \operatorname{crit}(\mathbf{h}_\beta)$. Such a reduction path does not appear in the formula (3.3.11) for $d_{\mathcal{G}}$.

So we have $p_1 = \mathbf{v_m} x^\mu \mathbf{h}_\gamma \to \mathbf{v}_{\mathbf{m} \cup \operatorname{cls}(x^\mu)} \left(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}} \mathbf{h}_\gamma \right) \to \mathbf{v_m} \left(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}} \mathbf{h}_\gamma \right)$ and we see

$$\begin{aligned}
\rho_p(\mathbf{v_m} x^\mu \mathbf{h}_\gamma) &= \rho_{p_m} \circ .... \circ \rho_{p_2} \circ \rho_{p_1}(\mathbf{v_m} x^\mu \mathbf{h}_\gamma) \\
&= \rho_{p_m} \circ .... \circ \rho_{p_2} \circ \rho_{\mathbf{v_m} \frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}} \mathbf{h}_\gamma, \mathbf{v_m} x^\mu \mathbf{h}_\gamma}(\mathbf{v_m} x^\mu \mathbf{h}_\gamma) \\
&= \rho_{p_m} \circ .... \circ \rho_{p_2} x_{\operatorname{cls}(x^\mu)} \mathbf{v_m} \left(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}} \mathbf{h}_\gamma \right) \\
&= x_{\operatorname{cls}(x^\mu)} \rho_{p_m} \circ .... \circ \rho_{p_2} \mathbf{v_m} \left(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}} \mathbf{h}_\gamma \right).
\end{aligned}$$

Now let $x^\xi \in \Bbbk[x_0, ..., x_d]$ satisfying $\frac{x^\mu}{x^\xi} \in \Bbbk[x_{d+1}, ..., x_n]$ (x^ξ is uniquely determined by these properties). Iterating this argument, we obtain

$$\begin{aligned}
\rho_p(\mathbf{v_m} x^\mu \mathbf{h}_\gamma) &= x_{\operatorname{cls}(x^\mu)} \rho_{p_m} \circ .... \circ \rho_{p_2} \mathbf{v_m} \left(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}} \mathbf{h}_\gamma \right) \\
&= ... \\
&= x^\xi \rho_{p_m} \circ ... \circ \rho_{p_r} \mathbf{v_m} \left(\frac{x^\mu}{x^\xi} \mathbf{h}_\delta \right).
\end{aligned}$$

Additionally, for each x^ν with $x^\nu | x^\xi$, if we do follow the ideas of these computations in the reversed order, we get

$$\rho_p(\mathbf{v_m} x^\mu \mathbf{h}_\gamma) = x^\xi \rho_{p_m} \circ \ldots \circ \rho_{p_r} \mathbf{v_m} \left(\frac{x^\mu}{x^\xi} \mathbf{h}_\delta \right)$$
$$= \frac{x^\xi}{x_{\mathrm{cls}\left(\frac{x^\xi}{x^\nu}\right)}} \rho_{p'_m}$$
$$circ \ldots \circ \rho_{p'_r} \mathbf{v_m} \left(\frac{x^\mu x_{\mathrm{cls}\left(\frac{x^\xi}{x^\nu}\right)}}{x^\xi} \mathbf{h}_\delta \right)$$
$$= \ldots$$
$$= x^\nu \rho_{p_*}$$
$$circ \ldots \circ \rho_{p_*} \left(\frac{x^\mu}{x^\nu} \mathbf{h}_\delta \right).$$

□

Remark 5.1.6. For later use, we introduce some more notations. We write $[a \rightsquigarrow b]_2$ for the set containing all reduction paths $a \to \cdots \to b$ which consist exclusively of concatenations of elementary reduction paths of type 2. Recall that by Lemma 5.1.1, these are exactly the paths that give the constants in $d_{\mathcal{G}}$. This notation is in analogy to a notation in [Skö11], where $[a \rightsquigarrow b]$ denotes the set of reduction paths without any further restrictions. For $\mu = 0, \gamma = \alpha$, let $Q^{j,\alpha}_{\mu,\gamma} = -x_j$. Otherwise, let $Q^{j,\alpha}_{\mu,\gamma}$ be the coefficient of $x^\mu \mathbf{h}_\gamma$ in the involutive standard representation of $x_j \mathbf{h}_\alpha$. This means that these polynomials are defined by the equations

$$d_{\mathcal{F}}(\mathbf{v_k h}_\alpha) = \sum_{j \in \mathbf{k}, \mu, \gamma} \varepsilon(j, \mathbf{k}) Q^{j,\alpha}_{\mu,\gamma} \mathbf{v}_{\mathbf{k} \setminus j}(x^\mu \mathbf{h}_\gamma). \tag{5.1.1}$$

We will see in the lemma below that these polynomials are essentially the polynomials $Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}$ from Equation (3.3.10), but since this new notation removes one index, we tend to prefer it. Even more so, the lemma will tell us that some of these coefficients $Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}$ are actually the same, a fact we will use in some proofs later on. Finally, we introduce an "extended" ε-symbol, generalizing the respective notation of definition 4.1.1:

$$\varepsilon\big((j_1, \ldots, j_f); (l_1, \ldots, l_q)\big) = \prod_{g=1}^{f} \varepsilon\big(j_g; (l_1, \ldots, l_q)\big).$$

We state some elementary properties of the extended ε-symbol which we will use later:

Lemma 5.1.7. *1. If $\mathbf{m} \subseteq \boldsymbol{\ell}$, then $\varepsilon(\mathbf{k}; \boldsymbol{\ell} \setminus \mathbf{m}) \varepsilon(\mathbf{k}; \mathbf{m}) = \varepsilon(\mathbf{k}; \boldsymbol{\ell})$.*

2. $\varepsilon(\mathbf{k}; \boldsymbol{\ell})^2 = 1$.

3. $\varepsilon\big((j);\boldsymbol{\ell}\big) = \varepsilon(j;\boldsymbol{\ell})$.

4. $\varepsilon\big((j_1,\dots,j_f);(l_1,\dots,l_q)\big) = (-1)^{\sum_{g=1}^{f}|\{l\in(l_1,\dots,l_q)|l<j_g\}|}$.

5. Let $\mathbf{m}\subseteq\mathbf{k}$*. For the coefficients* $Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}$*, as defined in Equation* (3.3.10) *or in [AFSS15, Equation (15)], we have*

$$Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma} = \varepsilon(\mathbf{k}\setminus\mathbf{m};\mathbf{k})Q^{\mathbf{k}\setminus\mathbf{m},\alpha}_{\mu,\gamma},$$

where we set $Q^{(j),\alpha}_{\mu,\gamma} = Q^{j,\alpha}_{\mu,\gamma}$.

Proof. The first four points are obvious. Regarding the last point, note that the $Q^{\mathbf{k}\setminus\mathbf{m},\alpha}_{\mu,\gamma}$ are defined by the equation

$$d_{\mathcal{F}}(\mathbf{v}_{\mathbf{k}}\mathbf{h}_\alpha) = \sum_{j,\mu,\gamma}\varepsilon(j;\mathbf{k})Q^{j,\alpha}_{\mu,\gamma}\mathbf{v}_{\mathbf{k}\setminus j}(x^\mu\mathbf{h}_\gamma).$$

In the construction of the differential in the complex $\mathcal{F}$ as in Equation (3.3.10)

$$d_{\mathcal{F}}(\mathbf{v}_{\mathbf{k}}\mathbf{h}_\alpha) = \sum_{\mathbf{m},\mu,\gamma}Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}\mathbf{v}_{\mathbf{m}}(x^\mu\mathbf{h}_\gamma),$$

we must have $\mathbf{m}=\mathbf{k}\cup\{j\}$ for some $j\in\mathbf{k}$, which in combination with Equation (3.3.3)

$$\begin{aligned}d_{\mathcal{F}}(1\otimes_{\Bbbk}\mathbf{v}_{\mathbf{k}}\otimes_{\Bbbk}m_a) &= \sum_{i=1}^{|\mathbf{k}|}(-1)^{i+1}\big(x_{k_i}\otimes_{\Bbbk}\mathbf{v}_{\mathbf{k}_i}\otimes_{\Bbbk}m_a - 1\otimes_{\Bbbk}\mathbf{v}_{\mathbf{k}_i}\otimes_{\Bbbk}x_{k_i}m_a\big)\\ &= \sum_{j\in\mathbf{k}}\varepsilon(j;\mathbf{k})\big(x_j\otimes_{\Bbbk}\mathbf{v}_{\mathbf{k}\setminus\{j\}}\otimes_{\Bbbk}m_a - 1\otimes_{\Bbbk}\mathbf{v}_{\mathbf{k}\setminus\{j\}}\otimes_{\Bbbk}x_jm_a\big)\end{aligned}$$

yields the given formula. □

Lemma 5.1.8. *Let* $\operatorname{cls}(x^\mu\mathbf{h}_\gamma) = d+e-1$ *and* $p\in[\mathbf{v}_{\mathbf{m}}(x^\mu\mathbf{h}_\gamma)\rightsquigarrow\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta]_2$.

- *We have* $l(p)\le e-1$ *(with the definition for the length of reduction paths as in Assumption 4.0.1).*
- *We have* $\boldsymbol{\ell}\setminus\mathbf{m}\in\{0,\dots,d+e-1\}$.
- *Let* $p=p_m\circ\cdots\circ p_1$*, where the* p_i *are elementary reduction paths and let* p_i *be of shape* $p_i=(\cdots\to\mathbf{v}_{\mathbf{n}_i}\mathbf{h}_*)$*. Then for* $i<j$*, we have* $\mathbf{m}\cap\mathbf{n}_j\subseteq\mathbf{m}\cap\mathbf{n}_i$.

Remark 5.1.9. Assume that $\operatorname{cls}(\mathbf{h}_\alpha)\le d+e-1$ and let $d_{\mathcal{G}}(\mathbf{v}_{\mathbf{k}}\mathbf{h}_\alpha)$ be given by equation (3.3.11). Then for any reduction path $p\in[\mathbf{v}_{\mathbf{m}}(x^\mu\mathbf{h}_\gamma)\rightsquigarrow\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta]_2$ which appears in this sum, we have $\operatorname{cls}(x^\mu\mathbf{h}_\gamma)\le d+e-1$, as the path originates from repeatedly computing involutive standard representations. So the lemma can be applied to such reduction paths. In other words, the lemma says that, if you keep track of the index set of $\mathbf{v}_*$ along such a reduction path, indices of class $\geq d+e$ can only disappear, or equivalently, all newly introduced indices are of class $\leq d+e-1$ (though those might disappear again). Any elementary reduction path of type 2 replaces indices in the $\mathbf{v}_*$ by smaller ones.

Proof. (of Lemma 5.1.8) If $x^\mu \in \Bbbk$, the claims of the lemma hold trivially.

So let $x^\mu \notin \Bbbk$. We consider two cases:

- $\operatorname{cls}(\mathbf{h}_\gamma) > d+e-1$: Then $\operatorname{cls}(x^\mu) \leq d+e-1$.
- $\operatorname{cls}(\mathbf{h}_\gamma) \leq d+e-1$: Here we have $x^\mu \in \Bbbk[x_0, ..., x_{d+e-1}]$.

Therefore $\operatorname{cls}(x^\mu) \leq d+e-1$ holds in either case (this is the argument from remark 5.0.2).

Now let $p \in [\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma) \rightsquigarrow \mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta]_2$ with elementary reduction paths $p_1, ..., p_m$ of type 2, such that $p = p_m \circ ... \circ p_1$ $\big($so $l(p) = m\big)$.

Let

$$p_1 = \quad \mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma) \to \mathbf{v}_{\mathbf{m}\cup\operatorname{cls}(x^\mu)}\left(\frac{x^\mu}{x_{\operatorname{cls}(x^\mu)}}\mathbf{h}_\gamma\right) \to \mathbf{v}_{\left(\mathbf{m}\cup\operatorname{cls}(x^\mu)\right)\setminus i}\left(x^\xi \mathbf{h}_\delta\right)$$

for some $i \in \mathbf{m}$ $\big($By the definition of elementary reduction paths of type 2, we have $i \neq \operatorname{cls}(x^\mu)\big)$.

Further, let p_2 be the reduction path

$$\begin{aligned} p_2 = \quad & \mathbf{v}_{\left(\mathbf{m}\cup\operatorname{cls}(x^\mu)\right)\setminus i}\left(x^\xi \mathbf{h}_\delta\right) \to \\ & \qquad \mathbf{v}_{\left(\left(\mathbf{m}\cup\operatorname{cls}(x^\mu)\right)\setminus i\right)\cup\operatorname{cls}(x^\xi)}\left(x^\xi \mathbf{h}_\delta\right) \to \\ & \qquad\qquad \mathbf{v}_{\left(\left(\left(\mathbf{m}\cup\operatorname{cls}(x^\mu)\right)\setminus i\right)\cup\operatorname{cls}(x^\xi)\right)\setminus j}\left(x^\pi \mathbf{h}_\eta\right) \end{aligned} \tag{5.1.2}$$

for some $j \in (\mathbf{m} \cup \operatorname{cls}(x^\mu)) \setminus i$. The Morse matching condition requires

$$\operatorname{cls}(x^\xi) < \Big(\big(\mathbf{m} \cup \operatorname{cls}(x^\mu)\big) \setminus i\Big) \cap \operatorname{ncrit}(\mathbf{h}_\delta).$$

Again, a priori two cases can occur:

$\operatorname{cls}(\mathbf{h}_\delta) \geq d+e-1$: Because of $\operatorname{cls}(x^\mu) \leq d+e-1$, we have $\operatorname{cls}(x^\mu) \in \operatorname{ncrit}(\mathbf{h}_\delta)$ and therefore

$$\operatorname{cls}(x^\xi) < \Big(\big(\mathbf{m} \cup \operatorname{cls}(x^\mu)\big) \setminus i\Big) \cap \operatorname{ncrit}(\mathbf{h}_\delta) \leq \operatorname{cls}(x^\mu) = d+e-1.$$

$\operatorname{cls}(\mathbf{h}_\delta) < d+e-1$: Here we directly have $\operatorname{cls}(x^\xi) \leq \operatorname{cls}(\mathbf{h}_\delta) < d+e-1$.

So $\operatorname{cls}(x^\xi) < d+e-1$ holds, and then $\operatorname{cls}(x^\xi) \leq d+e-2$.

By iterating this argument, we see that $\min(\boldsymbol{\ell}) \leq d+e-m$. Since for $d_0 \leq d$, we always have $d_0 \in \operatorname{ncrit}(\mathbf{h}_\beta)$ (see proof of Lemma 5.1.2), $m < e$ holds in any case, and so $l(p) = m \leq e-1$.

The other statements of the lemma follow from the same considerations, using induction in the process. □

Corollary 5.1.10. *Assume that* $\operatorname{cls}(\mathbf{h}_\alpha) = d+e-1$ *and let* $d_{\mathcal{G}}(\mathbf{v_k h}_\alpha)$ *be given by equation* (3.3.11)*. Then for any reduction path* $p \in [\mathbf{v_m}(x^\mu \mathbf{h}_\gamma) \rightsquigarrow \mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta]_2$ *which appears in the sums in equation* (3.3.11)*, we have* $|\mathbf{m} \setminus \boldsymbol{\ell}| \leq e-1$ *and* $|\mathbf{k} \setminus \boldsymbol{\ell}| \leq e$*.*

Proof. By Lemma 5.1.8, $l(p) \leq e-1$. Since any elementary reduction path of type 2 replaces exactly one index in the index set of $\mathbf{v}_*$ with a different index, p replaces in total at most $e-1$ indices. Furthermore, we have $\mathbf{m} = \mathbf{k} \setminus \{i\}$ for an $i \in \mathbf{k}$. □

Now we would like to proceed to our main theorems, which establish some relations between the constants of the differentials $d_{\mathcal{G}}$ in different homological degrees. First however, in order to formulate the results in a compact way, we introduce an abbreviation:

Definition 5.1.11. For $\operatorname{cls}(\mathbf{h}_\alpha) = d+e-1$ and $0 \leq f \leq e$, we set

$$P^{(j_1,\dots,j_f),\alpha}_{(k_1,\dots,k_{f-1}),\beta} = \sum_{\gamma,\mu} \sum_{j_r \in (j_1,\dots,j_f)} \sum_{\substack{p \in [\mathbf{v}_{(j_1,\dots,\hat{j}_r,\dots,j_{f-1})} x^\mu \mathbf{h}_\gamma \\ \rightsquigarrow \mathbf{v}_{(k_1,\dots,k_{f-1})}\mathbf{h}_\beta]}} \varepsilon\big(j_r;(j_1,\dots,j_f)\big) Q^{j_r;\alpha}_{\mu,\gamma} q_p\,,$$

where q_p is the polynomial coefficient of the map p (see also equation (5.2.1)). Note that by remark 5.1.6, only finitely many of the $Q^{j_r;\alpha}_{\mu,\gamma}$ are nonzero.

With this definition, we obtain the following chain of equations, which serves as the motivation of using a notation similar to the polynomials in an involutive standard representation:

$$\begin{aligned}
&d_{\mathcal{G}}(\mathbf{v}_{(j_1,\dots,j_f)}\mathbf{h}_\alpha) \\
&= \sum_{\boldsymbol{\ell},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p} \rho_p\big(Q^{(j_1,\dots,j_f),\alpha}_{\mathbf{m},\mu,\gamma} \mathbf{v_m}(x^\mu \mathbf{h}_\gamma)\big) \\
&= \sum_{\boldsymbol{\ell},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p} Q^{(j_1,\dots,j_f),\alpha}_{\mathbf{m},\mu,\gamma} q_p \mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta \\
&= \sum_{\boldsymbol{\ell},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p} \varepsilon\big((j_1,\dots,j_f)\setminus\mathbf{m};(j_1,\dots,j_f)\big) Q^{(j_1,\dots,j_f)\setminus\mathbf{m},\alpha}_{\mu,\gamma} q_p \mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\beta \\
&= \sum_{(k_1,\dots,k_{f-1}),\beta} \sum_{\substack{j_r,\mu,\gamma \\ j_r \in (j_1,\dots,j_f)}} \sum_{p} \varepsilon\big(j_r;(j_1,\dots,j_f)\big) Q^{j_r,\alpha}_{\mu,\gamma} q_p \mathbf{v}_{(k_1,\dots,k_{f-1})}\mathbf{h}_\beta \\
&= \sum_{(k_1,\dots,k_{f-1}),\beta} \sum_{\substack{j_r,\mu,\gamma \\ j_r \in (j_1,\dots,j_f)}} \\
&\qquad \sum_{\substack{p \in [\mathbf{v}_{(j_1,\dots,\hat{j}_r,\dots,j_{f-1})} x^\mu \mathbf{h}_\gamma \\ \rightsquigarrow \mathbf{v}_{(k_1,\dots,k_{f-1})}\mathbf{h}_\beta]}} \varepsilon\big(j_r;(j_1,\dots,j_f)\big) Q^{j_r,\alpha}_{\mu,\gamma} q_p \mathbf{v}_{(k_1,\dots,k_{f-1})}\mathbf{h}_\beta
\end{aligned}$$

$$= \sum_{k_1,\dots,k_{f-1}} \sum_{\beta=1}^{s} P^{(j_1,\dots,j_f),\alpha}_{(k_1,\dots,k_{f-1}),\beta} \mathbf{v}_{(k_1,\dots,k_{f-1})} \mathbf{h}_\beta;$$

Thus, the polynomials $P^{(j_1,\dots,j_f),\alpha}_{(k_1,\dots,k_{f-1}),\beta}$ appear as entries of the differential, in a manner similar to the way the polynomials of the involutive standard representation did in Lemma 4.1.2. This justifies our notation.

Remark 5.1.12. We are mainly interested in the constants. If we consider the isomorphisms

$$\Bbbk \otimes_{\mathcal{P}} \mathcal{F} = \Bbbk \otimes_{\mathcal{P}} (\mathcal{P} \otimes_{\Bbbk} \Lambda\mathcal{V} \otimes_{\Bbbk} \mathcal{M}) \cong \Bbbk \otimes_{\Bbbk} \Lambda\mathcal{V} \otimes_{\Bbbk} \mathcal{M} \cong \Lambda\mathcal{V} \otimes_{\Bbbk} \mathcal{M},$$

the constants of $\mathcal{F}$ are exactly those entries of the differential in $\mathcal{F}$ which "survive" after taking the tensor product with $\Bbbk$. Of course, the isomorphisms also hold when one replaces $\mathcal{F}$ by $\mathcal{G}$. It is obvious that the order in which we take the tensor product and perform a minimisation does matter. We recall that the ranks of the homology modules of the complex $\Bbbk \otimes \mathcal{G}$ are the Betti numbers of $\mathcal{M}$, see Lemma 2.4.5. Our main results are statements about the constants of $\mathcal{G}$ and thus can also be stated in the complex $\Bbbk \otimes \mathcal{G}$. In the sequel, the notation $1 \otimes \dots$ is meant to imply that we are talking about elements of $\Bbbk \otimes \mathcal{G}$ or $\Bbbk \otimes \mathcal{P} \cong \Bbbk$, depending on the context.

As a first example, we prove Lemma 5.1.13 below, which will later follow as a corollary to Theorem 5.3.1. However, we can prove it right now, avoiding the use of more technical notations for now.

Lemma 5.1.13. *Let* $\operatorname{cls}(\mathbf{h}_\alpha) = d = \min \operatorname{cls}(\mathcal{H})$*, let* $x_i \mathbf{h}_\alpha = \sum_{\beta=1}^{s} P^{(\alpha,i)}_\beta \mathbf{h}_\beta$ *be the involutive standard representation and* $\mathbf{k} = (k_1, \dots, k_j)$*. Then*

$$1 \otimes d_{\mathcal{G}}(\mathbf{v}_{\mathbf{k}} \mathbf{h}_\alpha) = \sum_{i=1}^{j} (-1)^i \sum_{\beta=1}^{s} 1 \otimes P^{(\alpha,k_i)}_\beta \mathbf{v}_{(k_1,\dots \hat{k}_i,\dots,k_j)} \mathbf{h}_\beta \,.$$

The sum over β *may be restricted to those* β *for which* $\operatorname{cls}(\mathbf{h}_\beta) = d$.

Proof. A priori, three cases are possible.

$\operatorname{cls}(\mathbf{h}_\beta) > d$**:** As $\operatorname{cls}(\mathbf{h}_\alpha) = d$, for any monomial x^μ which appears in the involutive standard representation of a $P^{(\alpha,k_i)}_\beta$, we must have that if $\operatorname{cls}(\mathbf{h}_\beta) > d$, then there is a $d_0 \le d$ with $x_{d_0} | x^\mu$. Theorem 5.1.4 guarantees that all respective reduction paths do not add constants to the differential.

$\operatorname{cls}(\mathbf{h}_\beta) = d$**:** Any monomial x^μ that appears in one of the $P^{(\alpha,i)}_\beta$ as a summand must be multiplicative for $\mathbf{h}_\beta$. If $x^\mu \notin \Bbbk$, then again there is a $d_0 \le \operatorname{cls}(\mathbf{h}_\beta) = d$ with $x_{d_0} | x^\mu$. Here, at most the $x^\mu \in \Bbbk$ add a constant to the differential. In fact, the trivial reduction paths of length 0 do indeed add exactly these as constants to the differential. The sign is therefore determined by Equation 3.3.3.

$\mathrm{cls}(\mathbf{h}_\beta) < d$**:** This case cannot occur, for

$$d = \min\{m \in \mathbb{N} \mid \exists \mathbf{h} \in \mathcal{H}\colon\ \mathrm{cls}(\mathbf{h}) = m\}\,;$$

see Assumption 5.0.1.

□

5.2 Related reduction paths and constants

Lemma 5.2.1. *Let*

$$\boldsymbol{\ell} \subseteq \mathbf{m} \subseteq \mathbf{n} \subseteq \{0, ..., n\}$$

be (ordered) sets of indices. Let $\mathrm{cls}(x^\mu) < \min \mathbf{n}$. *Then for any index set* $\mathbf{c}$ *such that* $|\mathbf{c}| = |\boldsymbol{\ell}|$ *and* $\max \mathbf{c} < \min \mathbf{n}$ *, there is a bijection*

$$\Psi\colon [\mathbf{v_n} x^\mu \mathbf{h}_\gamma \rightsquigarrow \mathbf{v}_{\mathbf{c}\cup(\mathbf{n}\setminus\boldsymbol{\ell})} x^\nu \mathbf{h}_\delta]_2 \to [\mathbf{v_m} x^\mu \mathbf{h}_\gamma \rightsquigarrow \mathbf{v}_{\mathbf{c}\cup(\mathbf{m}\setminus\boldsymbol{\ell})} x^\nu \mathbf{h}_\delta]_2\,.$$

given by

$$\Psi\left(\mathbf{v_n} x^\mu \mathbf{h}_\gamma \to \cdots \to \mathbf{v}_{\mathbf{c}\cup(\mathbf{n}\setminus\boldsymbol{\ell})} x^\nu \mathbf{h}_\delta\right) = \mathbf{v_m} x^\mu \mathbf{h}_\gamma \to \cdots \to \mathbf{v}_{\mathbf{c}\cup(\mathbf{m}\setminus\boldsymbol{\ell})} x^\nu \mathbf{h}_\delta$$

i. e. Ψ *is given by removing (from the index set of any* $\mathbf{v}_*$ *appearing in a given path) the indices contained in* $\mathbf{n} \setminus \mathbf{m}$. *Furthermore, assume that*

$$p \in [\mathbf{v_n} x^\mu \mathbf{h}_\gamma \rightsquigarrow \mathbf{v}_{\mathbf{c}\cup(\mathbf{n}\setminus\boldsymbol{\ell})} x^\nu \mathbf{h}_\delta]_2$$

and

$$\rho_p\left(\mathbf{v_n} x^\mu \mathbf{h}_\gamma\right) = q_p \mathbf{v}_{\mathbf{c}\cup(\mathbf{n}\setminus\boldsymbol{\ell})} x^\nu \mathbf{h}_\delta. \tag{5.2.1}$$

Then we have

$$\begin{aligned}\rho_{\Psi(p)}\left(\mathbf{v_m} x^\mu \mathbf{h}_\gamma\right) &= \prod_{i\in\boldsymbol{\ell}}\left(\varepsilon\big(i; \mathbf{n}\setminus\mathbf{m}\big)\right) q_p \mathbf{v}_{\mathbf{c}\cup(\mathbf{m}\setminus\boldsymbol{\ell})} x^\nu \mathbf{h}_\delta \\ &= \varepsilon\big(\boldsymbol{\ell}; \mathbf{n}\setminus\mathbf{m}\big) q_p \mathbf{v}_{\mathbf{c}\cup(\mathbf{m}\setminus\boldsymbol{\ell})} x^\nu \mathbf{h}_\delta.\end{aligned}$$

In particular, up to sign, we have q_p *equals* $q_{\Psi(p)}$.

We point out that the condition $p \in [\mathbf{v_n} x^\mu \mathbf{h}_\gamma \rightsquigarrow \mathbf{v}_{\mathbf{c}\cup(\mathbf{n}\setminus\boldsymbol{\ell})} x^\nu \mathbf{h}_\delta]_2$ implies that this lemma holds only for those reduction paths which are a concatenation of elementary reduction path of type 2.

Proof. First we check that the image of Ψ is indeed contained in the given codomain: Lemma 5.1.8 in combination with the assumption $\mathrm{cls}(x^\mu) < \min \mathbf{n}$ assures that, in the index set of the $\mathbf{v}_*$, indices larger than $\min \mathbf{n}$ can only disappear in each elementary reduction (sub-)path. For any given elementary reduction path, its image under Ψ is indeed also an elementary reduction path, as both arise from taking the same involutive standard representations, which naturally appear in the differential $d_{\mathcal{F}}$ of both generators. In particular, each elementary reduction path contained in an image is again of type 2.

For Ψ, the inverse map is given by, for a reduction path

$$\mathbf{v_m} x^{\mu} \mathbf{h}_{\gamma} \to \cdots \to \mathbf{v}_{\mathbf{c}\cup(\mathbf{m}\setminus\boldsymbol{\ell})} x^{\nu} \mathbf{h}_{\delta},$$

inserting in every index set of the $\mathbf{v}_*$ in the path, the indices contained in $\mathbf{n}\setminus\mathbf{m}$, so Ψ is indeed a bijection.

All that is left to consider are the signs:

If $p = p_m \circ ... \circ p_1$, then obviously $\Psi(p) = \Psi(p_m) \circ ... \circ \Psi(p_1)$ (All necessary assumptions to inductively apply the lemma to parts of a longer path are satisfied, which the reader can make sure of for himself, if he feels that this chapter is short on technical lemmata and juggling with sets of various types). Hence it suffices to consider elementary reduction paths.

So now let $\mathbf{c}, \mathbf{k}$ be ordered sets such that $\mathbf{k} \subseteq \mathbf{m}$, $|\mathbf{c}| = |\mathbf{k}|$ and $\max \mathbf{c} < \min \mathbf{n}$. Let

$$\begin{aligned} p_* = \quad & \mathbf{v}_{\mathbf{c}\cup(\mathbf{n}\setminus\mathbf{k})}(x^{\xi}\mathbf{h}_{\alpha}) \to \\ & \qquad \mathbf{v}_{\{\mathrm{cls}(x^{\xi})\}\cup\mathbf{c}\cup(\mathbf{n}\setminus\mathbf{k})} \left(\frac{x^{\xi}}{x_{\mathrm{cls}(x^{\xi})}} \mathbf{h}_{\alpha}\mathbf{h}_{\alpha} \right) \to \\ & \qquad\qquad \mathbf{v}_{\{\mathrm{cls}(x^{\xi})\}\cup\big((\mathbf{c}\setminus\{i\})\cup(\mathbf{n}\setminus\mathbf{k})\big)}(x^{\pi}\mathbf{h}_{\beta}) \end{aligned}$$

or

$$\begin{aligned} p_* = \quad & \mathbf{v}_{\mathbf{c}\cup(\mathbf{n}\setminus\mathbf{k})}(x^{\xi}\mathbf{h}_{\alpha}) \to \\ & \qquad \mathbf{v}_{\mathrm{cls}(x^{\xi})\cup\mathbf{c}\cup(\mathbf{n}\setminus\mathbf{k})} \left(\frac{x^{\xi}}{x_{\mathrm{cls}(x^{\xi})}} \mathbf{h}_{\alpha} \right) \to \\ & \qquad\qquad \mathbf{v}_{\{\mathrm{cls}(x^{\xi})\}\cup\mathbf{c}\cup\big(\mathbf{n}\setminus(\mathbf{k}\cup\{i\})\big)}(x^{\pi}\mathbf{h}_{\beta}) \end{aligned}$$

respectively be elementary reduction paths of type 2. Note that by Equation 3.3.6, we must have $\mathrm{cls}(x^{\xi}) < \min \mathbf{c}$. The defining difference between these two cases is that in the first case, we want the index i to be contained in $\mathbf{c}$, while in the second case, we have $i \in \mathbf{n} \setminus \mathbf{k}$.

For the reduction maps as introduced in Section 4.1, we then have

$$\begin{aligned} & \rho_{p_*}\left(\mathbf{v}_{\mathbf{c}\cup(\mathbf{n}\setminus\mathbf{k})}(x^{\xi}\mathbf{h}_{\alpha})\right) \\ = {} & \varepsilon\big(\mathrm{cls}(x^{\xi}); \{\mathrm{cls}(x^{\xi})\} \cup \mathbf{c}(\cup\mathbf{n}\setminus\mathbf{k})\big)\varepsilon\big(i; \{\mathrm{cls}(x^{\xi})\} \cup \mathbf{c} \cup (\mathbf{n}\setminus\mathbf{k})\big) \\ & \quad \cdot q_{p_*}\mathbf{v}_{\{\mathrm{cls}(x^{\xi})\}\cup\big((\mathbf{c}\setminus\{i\})\cup(\mathbf{n}\setminus\mathbf{k})\big)}(x^{\pi}\mathbf{h}_{\beta}) \\ = {} & \varepsilon(i; \{\mathrm{cls}(x^{\xi})\} \cup \mathbf{c})q_{p_*}... \end{aligned}$$

or

$$\begin{aligned} & \rho_{p_*}(...) \\ = {} & \varepsilon\big(\mathrm{cls}(x^{\xi}); \{\mathrm{cls}(x^{\xi})\} \cup \mathbf{c} \cup (\mathbf{n}\setminus\mathbf{k})\big)\varepsilon\big(i; \{\mathrm{cls}(x^{\xi})\} \cup \mathbf{c}(\cup\mathbf{n}\setminus\mathbf{k})\big)q_{p_*}(...) \\ = {} & (-1)^{|\mathbf{c}|+1}\varepsilon(i; \mathbf{n}\setminus\mathbf{k})q_{p_*}(...) \end{aligned}$$

respectively. In the same manner, we have

$$
\begin{aligned}
&\rho_{\Psi_{(p_*)}}\left(\mathbf{v}_{\mathbf{c}\cup(\mathbf{m}\setminus\mathbf{k})}(x^\xi\mathbf{h}_\alpha)\right)\\
&=\varepsilon\big(\operatorname{cls}(x^\xi);\{\operatorname{cls}(x^\xi)\}\cup\mathbf{c}\cup\mathbf{m}(\setminus\mathbf{k})\big)\varepsilon\big(i;\{\operatorname{cls}(x^\xi)\}\cup\mathbf{c}(\cup\mathbf{m}\setminus\mathbf{k})\big)\\
&\quad\cdot q_{p_*}\mathbf{v}_{\{\operatorname{cls}(x^\xi)\}\cup\left((\mathbf{c}\setminus\{i\})\cup(\mathbf{m}\setminus\mathbf{k})\right)}(x^\pi\mathbf{h}_\beta)\\
&=\varepsilon(i;\{\operatorname{cls}(x^\xi)\}\cup\mathbf{c})q_{p_*}(...)
\end{aligned}
$$

or

$$
\begin{aligned}
&\rho_{\Psi_{(p_*)}}(...)\\
&=\varepsilon\big(\operatorname{cls}(x^\xi);\{\operatorname{cls}(x^\xi)\}\cup\mathbf{c}\cup(\mathbf{m}\setminus\mathbf{k})\big)\varepsilon\big(i;\{\operatorname{cls}(x^\xi)\}\cup\mathbf{c}(\cup\mathbf{m}\setminus\mathbf{k})\big)q_{p_*}(...)\\
&=(-1)^{|\mathbf{c}|+1}\varepsilon(i;\mathbf{m}\setminus\mathbf{k})q_{p_*}(...)
\end{aligned}
$$

respectively. We were using $\operatorname{cls}(x^\xi)<\min\mathbf{c}<\max\mathbf{n}$ in both cases.

In the first case ("removing" an index which was not present at the origin), the sign remains unchanged. If however an index which was present at the origin of the path, is "removed", the coefficients change by the sign

$$
\begin{aligned}
&\varepsilon\big(i;\mathbf{n}\setminus\mathbf{k}\big)\varepsilon\big(i;\mathbf{m}\setminus\mathbf{k}\big)\\
&=\varepsilon(i;\mathbf{n})\varepsilon(i;\mathbf{k})\varepsilon(i;\mathbf{m})\varepsilon(i;\mathbf{k})\\
&=\varepsilon(i;\mathbf{n})\varepsilon(i;\mathbf{m})\\
&=\varepsilon(i;\mathbf{n}\setminus\mathbf{m})
\end{aligned}
$$

In order to obtain the correcting factor for the entire path, we see that we need a correcting factor for each index which both was present at the start of the path and was removed along the path. By multiplying all these factors, we obtain the correcting factor for the entire path. □

Remark 5.2.2. The necessity to precisely determine the signs of the constants makes this lemma look very technical. Our bijection essentially expresses the following idea, of which we already have seen a hint in the proof of corollary 5.1.10: Those reduction paths which give us the constants can only be of limited length. Thus, if the index set of $\mathbf{v}_*$ is larger than this length, some indices remain the same along the entire path p. For shorter paths, this behavior may or may not occur. But those unchanged indices are then almost irrelevant for the coefficient q_p of p; they can at most change the sign. The bijection in the lemma expresses this fact in the manner that if we fix two sets $\mathbf{n}\setminus\mathbf{m}$ of indices which are not in any way moved by any of the elementary reduction paths of which the entire path consists, we can identify those reduction paths that change the same sets of (not-fixed) variables. For two paths identified in this way, the key observation is stated in the last sentence of the lemma.

We note that it is possible to replace the condition $\mathbf{m}\subseteq\mathbf{n}$ by a more general one, at the cost of having to introduce more combinatorial notations: For example, one could only require $\boldsymbol{\ell}\subseteq(\mathbf{m}\cap\mathbf{n})$, and still to get a similar result, yet this would further complicate the formulae for the signs and is not necessary for the proof of our main theorems.

5.3 Calculating constants in higher homological degree via lower degrees

Recall our notation of Remark 5.1.12.

Theorem 5.3.1. *Let* $\operatorname{cls}(\mathbf{h}_\alpha) = d + e - 1$. *Let* $\boldsymbol{\ell}$ *be an ordered sequence such that*[1] $\min \boldsymbol{\ell} > \operatorname{cls} \mathbf{h}_\alpha$ *and* $e < |\boldsymbol{\ell}| \leq n - e$. *Then we have*

$$1 \otimes d_{\mathcal{G}}(\mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}_\alpha) = \sum_{f=1}^{e} \sum_{\substack{\mathbf{j} \subseteq \boldsymbol{\ell} \\ |\mathbf{j}|=f}} \varepsilon(\mathbf{j}; \boldsymbol{\ell} \setminus \mathbf{j}) \sum_{\substack{|\mathbf{k}|=f-1 \\ \min \mathbf{k} \leq \operatorname{cls}(\mathbf{h}_\alpha)}} \sum_{\beta=1}^{s} \left(1 \otimes P^{\mathbf{j},\alpha}_{\mathbf{k},\beta}\right) \mathbf{v}_{\mathbf{k} \cup (\boldsymbol{\ell} \setminus \mathbf{j})} \mathbf{h}_\beta.$$

Remark 5.3.2. One can express this theorem as follows: Consider a fixed element $\mathbf{h}_\alpha$ of the Pommaret basis $\mathcal{H}$. Then in our complex $\mathcal{G}$, there are some basis elements of shape $\mathbf{v}_* \mathbf{h}_\alpha$. The theorem establishes relations between the constants in the differentials of the basis elements of these $\mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}_\alpha$ in the following way: First, we need to take into account how much larger the class of $\mathbf{h}_\alpha$ is than the minimal class d appearing in our Pommaret basis. This difference is $e-1$. Then we need to know all constants in the differentials for the $\mathbf{v}_* \mathbf{h}_\alpha$ up to homological degree $f \leq e$. These differential give us some coefficients P^*_*. Now the theorem says that for all $\mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}_\alpha$ of homological degrees $|\boldsymbol{\ell}| > e$, the constants of the differentials of these $\mathbf{v}_* \mathbf{h}_\alpha$ are essentially some constants which already appeared for the smaller homological degrees $f \leq e$; we only need to reassemble them (and sometimes change the sign) in some ways which depend only on the index sets of the $\mathbf{v}_*$ involved.

We also remark that, if we combine this theorem with Theorem 5.4.1 below, it is in fact enough to know the constants exactly in homological degree e to deduce all constants (note that e depends on the given generator $\mathbf{h}_\alpha$).

The proof of the theorem consists in principle of a single chain of equations, yet it is a rather technical and lengthy computation, involving sums over up to six different (multi-)indices. In the process, we are using Lemma 5.1.8, corollary 5.1.10, Lemma 5.2.1 and some properties of our generalised ε-symbol.

Proof. For simplicity of notation, we will write j instead $\{j\}$ or (j) when considering ordered sets with a single element.

$$\begin{aligned} & d_{\mathcal{G}}(\mathbf{v}_{\boldsymbol{\ell}} \mathbf{h}_\alpha) \\ = {} & 1 \otimes \sum_{\mathbf{a},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p \in [\ldots]_2} \rho_p \left(Q^{\boldsymbol{\ell},\alpha}_{\mathbf{m},\mu,\gamma} \mathbf{v}_{\mathbf{m}} (x^\mu \mathbf{h}_\gamma)\right) \end{aligned}$$

[1] Note that $\min \boldsymbol{\ell} > \operatorname{cls} \mathbf{h}_\alpha$ implies $\boldsymbol{\ell} \subseteq \operatorname{crit}(\mathbf{h}_\alpha)$.

See the discussion following Equation (3.3.11) for the restrictions to the indices of either sum. As we are interested only in the constants, by Lemma 5.1.1, we can restrict the sum over p to all reduction paths which consist exclusively of elementary reduction paths of type 2, which we indicate by the notation $p \in [...]_2$. Using the isomorphism $\Bbbk^m \cong \Bbbk \otimes_{\mathcal{P}} \mathcal{P}^m$ given by $(v_1, ..., v_m) \mapsto 1 \otimes (v_1, .., v_m)$, we also make slight abuse of notation and temporarily drop $1 \otimes ...$, as we have ensured via $p \in [...]_2$ that any summand is a constant, which temporarily makes the necessity to the tensor product redundant. Now we are using Lemma 5.1.7 to obtain

$$
\begin{aligned}
&= \sum_{\mathbf{a},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p\in[...]_2} \rho_p\Big(\varepsilon\big(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}\big) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} \mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\Big) \\
&= \sum_{\mathbf{a},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p\in[...]_2} \varepsilon\big(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}\big) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} \rho_p\big(\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big) \\
&= \sum_{\mathbf{j}} \sum_{\substack{\mathbf{a},\beta \\ \boldsymbol{\ell}\setminus\mathbf{a}=\mathbf{j}}} \sum_{\mathbf{m},\mu,\gamma} \sum_{p\in[...]_2} \varepsilon\big(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}\big) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} \rho_p\big(\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big)
\end{aligned}
$$

Now we make use of Lemma 5.1.8 and corollary 5.1.10, which tell us which shapes for $\mathbf{j}$ are possible: In particular, $\mathbf{j}$ contains at most e elements.

$$
= \sum_{f=1}^{e} \sum_{\substack{\mathbf{j}\subseteq\boldsymbol{\ell} \\ |\mathbf{j}|=f}} \sum_{\substack{\mathbf{a},\beta \\ \boldsymbol{\ell}\setminus\mathbf{a}=\mathbf{j}}} \sum_{\mathbf{m},\mu,\gamma} \sum_{p\in[...]_2} \varepsilon\big(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}\big)\big) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} \rho_p\big(\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big)
$$

If $\boldsymbol{\ell} \setminus \mathbf{a} = \mathbf{j}$, then we know by Lemma 5.1.8 that $\mathbf{a} = \mathbf{k} \cup (\boldsymbol{\ell} \setminus \mathbf{j})$ with some $\mathbf{k} \subseteq \{0, ..., \mathrm{cls}(\mathbf{h}_\alpha)$.

$$
\begin{aligned}
&= \sum_{f=1}^{e} \sum_{\substack{\mathbf{j}\subseteq\boldsymbol{\ell} \\ |\mathbf{j}|=f}} \sum_{\substack{|\mathbf{k}|=f-1 \\ \min \mathbf{k}\leq \mathrm{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\mathbf{m},\mu,\gamma} \\
&\qquad \sum_{p\in\left[\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{j})}\mathbf{h}_\beta\right]_2} \varepsilon\big(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}\big) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} \rho_p\big(\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big) \\
&= \sum_{f=1}^{e} \sum_{\substack{\mathbf{j}\subseteq\boldsymbol{\ell} \\ |\mathbf{j}|=f}} \sum_{\substack{|\mathbf{k}|=f-1 \\ \min \mathbf{k}\leq \mathrm{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\mathbf{m},\mu,\gamma} \\
&\qquad \sum_{p\in\left[\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{j})}\mathbf{h}_\beta\right]_2} \varepsilon\big(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}\big) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} q_p \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{j})}\mathbf{h}_\beta
\end{aligned}
$$

Using Lemma 5.2.1 in order to replace the condition in the last sum. Additionally, since again by Lemma 5.2.1, indices greater than $\mathrm{cls}(\mathbf{h}_\alpha)$ can only disappear along reduction paths, we need that $\mathbf{m} = \boldsymbol{\ell} \setminus j$ with some $j \in \mathbf{j}$.

$$
= \sum_{f=1}^{e} \sum_{\substack{\mathbf{j}\subseteq\boldsymbol{\ell} \\ |\mathbf{j}|=f}} \sum_{\substack{|\mathbf{k}|=f-1 \\ \min \mathbf{k}\leq \mathrm{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\substack{j,\mu,\gamma \\ j\in\mathbf{j}}}
$$

$$\sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus j}(x^{\mu}\mathbf{h}_{\gamma})\rightsquigarrow\mathbf{v}_{\mathbf{k}}\mathbf{h}_{\beta}\right]_2} \varepsilon\big(\boldsymbol{\ell}\setminus\mathbf{m};\boldsymbol{\ell}\big)\varepsilon\big(\mathbf{j}\setminus j;\boldsymbol{\ell}\setminus\mathbf{j}\big)Q^{j,\alpha}_{\mu,\gamma}q_p\mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{j})}\mathbf{h}_{\beta}$$

$$=\sum_{f=1}^{e}\sum_{\substack{\mathbf{j}\subseteq\boldsymbol{\ell}\\|\mathbf{j}|=f}}\sum_{\substack{|\mathbf{k}|=f-1\\\min\mathbf{k}\leq\operatorname{cls}(\mathbf{h}_\alpha)}}\sum_{\beta}\sum_{\substack{j,\mu,\gamma\\j\in\mathbf{j}}}$$
$$\sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus j}(x^{\mu}\mathbf{h}_{\gamma})\rightsquigarrow\mathbf{v}_{\mathbf{k}}\mathbf{h}_{\beta}\right]_2} \varepsilon\big(j;\boldsymbol{\ell}\big)\varepsilon\big(\mathbf{j}\setminus j;\boldsymbol{\ell}\setminus\mathbf{j}\big)Q^{j,\alpha}_{\mu,\gamma}q_p\mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{j})}\mathbf{h}_{\beta}$$

$$=\sum_{f=1}^{e}\sum_{\substack{\mathbf{j}\subseteq\boldsymbol{\ell}\\|\mathbf{j}|=f}}\sum_{\substack{|\mathbf{k}|=f-1\\\min\mathbf{k}\leq\operatorname{cls}(\mathbf{h}_\alpha)}}\sum_{\beta}\sum_{\substack{j,\mu,\gamma\\j\in\mathbf{j}}}$$
$$\sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus j}(x^{\mu}\mathbf{h}_{\gamma})\rightsquigarrow\mathbf{v}_{\mathbf{k}}\mathbf{h}_{\beta}\right]_2} \varepsilon\big(j;\boldsymbol{\ell}\big)\varepsilon\big(j;\boldsymbol{\ell}\setminus\mathbf{j}\big)\varepsilon\big(j;\boldsymbol{\ell}\setminus\mathbf{j}\big)\varepsilon\big(\mathbf{j}\setminus j;\boldsymbol{\ell}\setminus\mathbf{j}\big)Q^{j,\alpha}_{\mu,\gamma}q_p\mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{j})}\mathbf{h}_{\beta}$$

$$=\sum_{f=1}^{e}\sum_{\substack{\mathbf{j}\subseteq\boldsymbol{\ell}\\|\mathbf{j}|=f}}\sum_{\substack{|\mathbf{k}|=f-1\\\min\mathbf{k}\leq\operatorname{cls}(\mathbf{h}_\alpha)}}\sum_{\beta}\sum_{\substack{j,\mu,\gamma\\j\in\mathbf{j}}}$$
$$\sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus j}(x^{\mu}\mathbf{h}_{\gamma})\rightsquigarrow\mathbf{v}_{\mathbf{k}}\mathbf{h}_{\beta}\right]_2} \varepsilon\big(j;\boldsymbol{\ell}\big)\varepsilon\big(j;\boldsymbol{\ell}\big)\varepsilon(j;\mathbf{j})\varepsilon\big(j;\boldsymbol{\ell}\setminus\mathbf{j}\big)\varepsilon\big(\mathbf{j}\setminus j;\boldsymbol{\ell}\setminus\mathbf{j}\big)Q^{j,\alpha}_{\mu,\gamma}q_p\mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{j})}\mathbf{h}_{\beta}$$

$$=\sum_{f=1}^{e}\sum_{\substack{\mathbf{j}\subseteq\boldsymbol{\ell}\\|\mathbf{j}|=f}}\sum_{\substack{|\mathbf{k}|=f-1\\\min\mathbf{k}\leq\operatorname{cls}(\mathbf{h}_\alpha)}}\sum_{\beta}\sum_{\substack{j,\mu,\gamma\\j\in\mathbf{j}}}$$
$$\sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus j}(x^{\mu}\mathbf{h}_{\gamma})\rightsquigarrow\mathbf{v}_{\mathbf{k}}\mathbf{h}_{\beta}\right]_2} \varepsilon\big(j;\mathbf{j}\big)\varepsilon\big(\mathbf{j};\boldsymbol{\ell}\setminus\mathbf{j}\big)Q^{j,\alpha}_{\mu,\gamma}q_p\mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{j})}\mathbf{h}_{\beta}$$

$$=\sum_{f=1}^{e}\sum_{\substack{\mathbf{j}\subseteq\boldsymbol{\ell}\\|\mathbf{j}|=f}}\varepsilon\big(\mathbf{j};\boldsymbol{\ell}\setminus\mathbf{j}\big)\sum_{\substack{|\mathbf{k}|=f-1\\\min\mathbf{k}\leq\operatorname{cls}(\mathbf{h}_\alpha)}}\sum_{\beta}\sum_{\substack{j,\mu,\gamma\\j\in\mathbf{j}}}$$
$$\sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus j}(x^{\mu}\mathbf{h}_{\gamma})\rightsquigarrow\mathbf{v}_{\mathbf{k}}\mathbf{h}_{\beta}\right]_2} \varepsilon\big(j;\mathbf{j}\big)Q^{j,\alpha}_{\mu,\gamma}q_p\mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{j})}\mathbf{h}_{\beta}$$

Finally, we reintroduce the tensor product notation which we had omitted earlier.

$$=\sum_{f=1}^{e}\sum_{\substack{\mathbf{j}\subseteq\boldsymbol{\ell}\\|\mathbf{j}|=f}}\varepsilon\big(\mathbf{j};\boldsymbol{\ell}\setminus\mathbf{j}\big)\sum_{\substack{|\mathbf{k}|=f-1\\\min\mathbf{k}\leq\operatorname{cls}(\mathbf{h}_\alpha)}}\sum_{\beta}\left(1\otimes P^{\mathbf{j},\alpha}_{\mathbf{k},\beta}\right)\mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{j})}\mathbf{h}_{\beta}$$

□

5.4 Calculating constants in lower homological degree via higher degrees

Theorem 5.4.1. *Let* $\operatorname{cls}(\mathbf{h}_\alpha)=d+e-1$*,* $\boldsymbol{\ell}\subseteq\mathbf{j}$ *and* $|\boldsymbol{\ell}|\leq|\mathbf{j}|\leq e$*. Then we have:*

$$1 \otimes d_{\mathcal{G}}(\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\alpha) = \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k}\leq \operatorname{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \varepsilon(\mathbf{i};\mathbf{j}\setminus\boldsymbol{\ell})\big(1\otimes P^{\mathbf{j},\alpha}_{\mathbf{k}\cup(\mathbf{j}\setminus\mathbf{i}),\beta}\big)\mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta.$$

Remark 5.4.2. The proof is very similar to the proof of Theorem 5.3.1. This time, we take the inverse point of view: Theorem 5.3.1 tells us how to construct constants in higher (homological) degree from constants in lower degrees. Here, we construct constants in lower degrees from constants in higher degrees. Again, suppose that we have a generator $\mathbf{h}_\alpha \in \mathcal{H}$ and that we know the constants in the differential for a $\mathbf{v}_{\mathbf{j}}\mathbf{h}_\alpha$ in some homological degree f. Then we can give the constants for all $\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\alpha$ where $\boldsymbol{\ell} \subseteq \mathbf{j}$ (again, with some sign- and settheoretic computations).

Most importantly, we state the following consequence: If for a fixed $\mathbf{h}_\alpha$ we know the constants in the differential of $\mathbf{v}_{\left(\operatorname{cls}(\mathbf{h}_\alpha)+1,\dots,n\right)}\mathbf{h}_\alpha$, i.e. in the highest homological degree in which a generator of shape $\mathbf{v}_*\mathbf{h}_\alpha$ is present in the resolution, we can from this differential deduce **all** constants in the differential of all other elements of shape $\mathbf{v}_*\mathbf{h}_\alpha$. This if of course possible because the index set of the $\mathbf{v}_*$ of this last generator contains exactly all non-multiplicative variables. From a computational point of view, this means that for our Pommaret basis $\mathcal{H}$, we need to compute only one differential for each element of $\mathcal{H}$, and additionally do some purely sign- and settheoretic computations, to obtain all constants in the complex $\mathcal{G}$. This gives a lot of potential to further increase the speed of the algorithm presented in [AFSS15].

Proof. Again because of Lemma 5.2.1, we have a bijection of those reduction paths which consist entirely of elementary reduction paths of type 2: To any reduction path originating in $\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\alpha$, we assign the reduction path which originates in $\mathbf{v}_{\mathbf{j}}\mathbf{h}_\alpha$ and leaves the indices contained in $\mathbf{j}\setminus\boldsymbol{\ell}$ unchanged within each of its elementary reduction paths. One again, when talking about sets with a single element, we write i instead of $\{i\}$. Basically, this proof repeats the arguments of the proof of Theorem 5.3.1.

$$\begin{aligned} &1 \otimes d_{\mathcal{G}}(\mathbf{v}_{\boldsymbol{\ell}}\mathbf{h}_\alpha) \\ =& \sum_{\mathbf{a},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p} \rho_p\big(Q^{\boldsymbol{\ell},\alpha}_{\mathbf{m},\mu,\gamma}\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big) \end{aligned}$$

Essentially, throughout this proof, we repeat our remarks from the previous proof: See the discussion following Equation (3.3.11) for the restrictions to the indices of either sum. As we are interested only in the constants, by Lemma 5.1.1, we can restrict the sum over p to all reduction paths which consist exclusively of elementary reduction paths of type 2, which we indicate by the notation $p \in [...]_2$. Using the isomorphism $\Bbbk^m \cong \Bbbk \otimes_{\mathcal{P}} \mathcal{P}^m$ given by $(v_1, ..., v_m) \mapsto 1 \otimes (v_1, .., v_m)$, we also make slight abuse of notation and temporarily drop $1 \otimes ...$, as we have ensured via $p \in [...]_2$ that we any summand is a constant, which temporarily makes the necessity to the tensor product redundant. Now we are using Lemma 5.1.7 to obtain

$$= \sum_{\mathbf{a},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p\in[...]_2} \rho_p\big(\varepsilon(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} \mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big)$$

$$= \sum_{\mathbf{a},\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p\in[...]_2} \varepsilon\big(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}\big) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} \rho_p\big(\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big)$$

$$= \sum_{\mathbf{i}} \sum_{\substack{\mathbf{a},\beta \\ \boldsymbol{\ell}\setminus\mathbf{a}=\mathbf{i}}} \sum_{\mathbf{m},\mu,\gamma} \sum_{p\in[...]_2} \varepsilon\big(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}\big) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} \rho_p\big(\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big)$$

Again using Lemma 5.1.8 and corollary 5.1.10, which tell us which shapes for $\mathbf{a}$ are possible:

$$= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{\mathbf{a},\beta \\ \boldsymbol{\ell}\setminus\mathbf{a}=\mathbf{i}}} \sum_{\mathbf{m},\mu,\gamma} \sum_{p\in[...]_2} \varepsilon\big(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}\big) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} \rho_p\big(\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big)$$

If $\boldsymbol{\ell} \setminus \mathbf{a} = \mathbf{i}$, then we know by Lemma 5.1.8 that $\mathbf{a} = \mathbf{k} \cup (\boldsymbol{\ell} \setminus \mathbf{i})$ with some $\mathbf{k} \subseteq \{0, ..., \mathrm{cls}(\mathbf{h}_\alpha)$.

$$= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k}\leq \mathrm{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p\in\left[\mathbf{v}_{\mathbf{m}}(x^\mu\mathbf{h}_\gamma)\rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta\right]_2} \varepsilon\big(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}\big) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} \rho_p\big(\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big)$$

$$= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k}\leq \mathrm{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\mathbf{m},\mu,\gamma} \sum_{p\in\left[\mathbf{v}_{\mathbf{m}}(x^\mu\mathbf{h}_\gamma)\rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta\right]_2} \varepsilon\big(\boldsymbol{\ell} \setminus \mathbf{m}; \boldsymbol{\ell}\big) Q^{\boldsymbol{\ell}\setminus\mathbf{m},\alpha}_{\mu,\gamma} q_p \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta$$

$$= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k}\leq \mathrm{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\substack{i,\mu,\gamma \\ i\in\mathbf{i}}} \sum_{p\in\left[\mathbf{v}_{\boldsymbol{\ell}\setminus i}(x^\mu\mathbf{h}_\gamma)\rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta\right]_2} \varepsilon\big(i; \boldsymbol{\ell}\big) Q^{i,\alpha}_{\mu,\gamma} q_p \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta$$

Using Lemma 5.2.1 in order to replace the condition in the last sum. Additionally, since again by Lemma 5.2.1, indices greater than $\mathrm{cls}(\mathbf{h}_\alpha)$ can only disappear along reduction paths, we need that $\mathbf{m} = \boldsymbol{\ell} \setminus i$ with some $i \in \mathbf{i}$.

$$= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k}\leq \mathrm{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\substack{i,\mu,\gamma \\ i\in\mathbf{i}}}$$
$$\sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus i}(x^\mu\mathbf{h}_\gamma)\rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\mathbf{j}\setminus\mathbf{i})}\mathbf{h}_\beta\right]_2} \varepsilon(i; \boldsymbol{\ell})\varepsilon\big(\mathbf{i} \setminus i; (\mathbf{j} \setminus i) \setminus (\boldsymbol{\ell} \setminus i)\big) Q^{i,\alpha}_{\mu,\gamma} q_p \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta$$

$$
\begin{aligned}
&= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k} \leq \operatorname{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\substack{i,\mu,\gamma \\ i\in\mathbf{i}}} \\
&\qquad \sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus i}(x^\mu \mathbf{h}_\gamma) \rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\mathbf{j}\setminus\mathbf{i})}\mathbf{h}_\beta\right]_2} \varepsilon(i;\boldsymbol{\ell})\varepsilon\big(\mathbf{i}\setminus i;(\mathbf{j}\setminus i)\big)\varepsilon\big(\mathbf{i}\setminus i;(\boldsymbol{\ell}\setminus i)\big) Q^{i,\alpha}_{\mu,\gamma} q_p \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta \\
&= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k} \leq \operatorname{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\substack{i,\mu,\gamma \\ i\in\mathbf{i}}} \\
&\qquad \sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus i}(x^\mu \mathbf{h}_\gamma) \rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\mathbf{j}\setminus\mathbf{i})}\mathbf{h}_\beta\right]_2} \varepsilon(i;\boldsymbol{\ell})\varepsilon\big(\mathbf{i}\setminus i;\mathbf{j})\big)\varepsilon\big(\mathbf{i}\setminus i;i\big)\varepsilon\big(\mathbf{i}\setminus i;\boldsymbol{\ell}\big)\varepsilon\big(\mathbf{i}\setminus i;i)\big) Q^{i,\alpha}_{\mu,\gamma} q_p \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta \\
&= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k} \leq \operatorname{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\substack{i,\mu,\gamma \\ i\in\mathbf{i}}} \\
&\qquad \sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus i}(x^\mu \mathbf{h}_\gamma) \rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\mathbf{j}\setminus\mathbf{i})}\mathbf{h}_\beta\right]_2} \varepsilon(i;\boldsymbol{\ell})\varepsilon\big(\mathbf{i}\setminus i;\mathbf{j})\big)\varepsilon\big(\mathbf{i}\setminus i;\boldsymbol{\ell}\big) Q^{i,\alpha}_{\mu,\gamma} q_p \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta \\
&= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k} \leq \operatorname{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\substack{i,\mu,\gamma \\ i\in\mathbf{i}}} \\
&\qquad \sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus i}(x^\mu \mathbf{h}_\gamma) \rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\mathbf{j}\setminus\mathbf{i})}\mathbf{h}_\beta\right]_2} \varepsilon(i;\boldsymbol{\ell})\varepsilon\big(\mathbf{i}\setminus i;\mathbf{j}\setminus\boldsymbol{\ell}\big) Q^{i,\alpha}_{\mu,\gamma} q_p \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta \\
&= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k} \leq \operatorname{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\substack{i,\mu,\gamma \\ i\in\mathbf{i}}} \\
&\qquad \sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus i}(x^\mu \mathbf{h}_\gamma) \rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\mathbf{j}\setminus\mathbf{i})}\mathbf{h}_\beta\right]_2} \varepsilon(i;\boldsymbol{\ell})\varepsilon\big(\mathbf{i};\mathbf{j}\setminus\boldsymbol{\ell})\big)\varepsilon\big(i;\mathbf{j}\setminus\boldsymbol{\ell}\big) Q^{i,\alpha}_{\mu,\gamma} q_p \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta \\
&= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k} \leq \operatorname{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \sum_{\substack{i,\mu,\gamma \\ i\in\mathbf{i}}} \\
&\qquad \sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus i}(x^\mu \mathbf{h}_\gamma) \rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\mathbf{j}\setminus\mathbf{i})}\mathbf{h}_\beta\right]_2} \varepsilon\big(\mathbf{i};\mathbf{j}\setminus\boldsymbol{\ell}\big)\varepsilon(i;\mathbf{j}) Q^{i,\alpha}_{\mu,\gamma} q_p \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta \\
&= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq\boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k} \leq \operatorname{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \varepsilon\big(\mathbf{i};\mathbf{j}\setminus\boldsymbol{\ell})\big) \\
&\qquad \Big(\sum_{\substack{i,\mu,\gamma \\ i\in\mathbf{i}}} \sum_{p\in\left[\mathbf{v}_{\mathbf{j}\setminus i}(x^\mu \mathbf{h}_\gamma) \rightsquigarrow \mathbf{v}_{\mathbf{k}\cup(\mathbf{j}\setminus\mathbf{i})}\mathbf{h}_\beta\right]_2} \varepsilon(i;\mathbf{j}) Q^{i,\alpha}_{\mu,\gamma} q_p \Big) \mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta
\end{aligned}
$$

Finally, we reintroduce the tensor product notation which we had omitted earlier.

$$= \sum_{h=1}^{e} \sum_{\substack{\mathbf{i}\subseteq \boldsymbol{\ell} \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k}\leq \operatorname{cls}(\mathbf{h}_\alpha)}} \sum_{\beta} \varepsilon\big(\mathbf{i};\mathbf{j}\setminus\boldsymbol{\ell})\big)\big(1\otimes P^{\mathbf{j},\alpha}_{\mathbf{k}\cup(\mathbf{j}\setminus\mathbf{i}),\beta}\big)\mathbf{v}_{\mathbf{k}\cup(\boldsymbol{\ell}\setminus\mathbf{i})}\mathbf{h}_\beta$$

□

Example 5.4.3. To have a more concrete look at the previous two theorems, we again turn back to the ideal from example 2.3.24. Suppose we visualize the complex $\mathcal{G}$ with a graph $\Gamma_{\mathcal{G}}$, see definition 3.1.1. If we omit the edges belonging to non-constant parts of the differential, we obtain the graph given in figure 5.4.3, plotting the constants of the resolution. The arrows/constants of same color in the graph, i.e. red or blue, are the ones which are linked by Theorem 5.4.1. We note that formally, Theorem 5.3.1 is a priori not helpful for the calculations of the red or blue arrows, since either such arrow has as source an element of shape $\mathbf{v}_*\mathbf{h}_\alpha$ with $\operatorname{cls}(\mathbf{h}_\alpha) = 2$, which means that in Theorem 5.3.1 we have $e = 2$, and hence we could only use it to obtain arrows with sources in homological degree greater or equal to $e = 2$; but obviously there are no such possible sources. The underlying reason is that when going from sources in F_1 to sources in F_2, it is possible to miss out on some constants such as the **black** arrow. A more detailed discussion of this idea follows below. The sign change of the red constants come from the $(-1)^*$-terms and the ε-symbols in these theorems, which in turn were needed to be introduced because of the possible additional indices in the index sets of the $\mathbf{v}_*$. For the blue constants, these terms leave the sign unchanged.

Nevertheless, we can use 5.4.1 for the red and blue arrows. Looking at the right red arrows, we see that we have

$$P^{(2,3),x_1x_2}_{(2),x_1x_2x_3} = 1, \tag{5.4.1}$$

and this is the only constant contained in the differential of this source, i.e. for all $m \neq 2$ or $\beta \neq x_1x_2x_3$, we have

$$P^{(2,3),x_1x_2}_{(m),\beta} = 0. \tag{5.4.2}$$

Now we look at the formula 5.4.1: We have

$$\operatorname{cls}(\mathbf{h}_\alpha) = \operatorname{cls}(x_1x_2) = 1, \quad d = \min_{\mathbf{h}\in\mathcal{H}} \operatorname{cls}(\mathbf{h}) = 0$$

and hence

$$e = \operatorname{cls}(\mathbf{h}_\alpha) - d + 1 = 2, \quad \boldsymbol{\ell} = (3) \quad \text{and} \quad \mathbf{j} = (2,3).$$

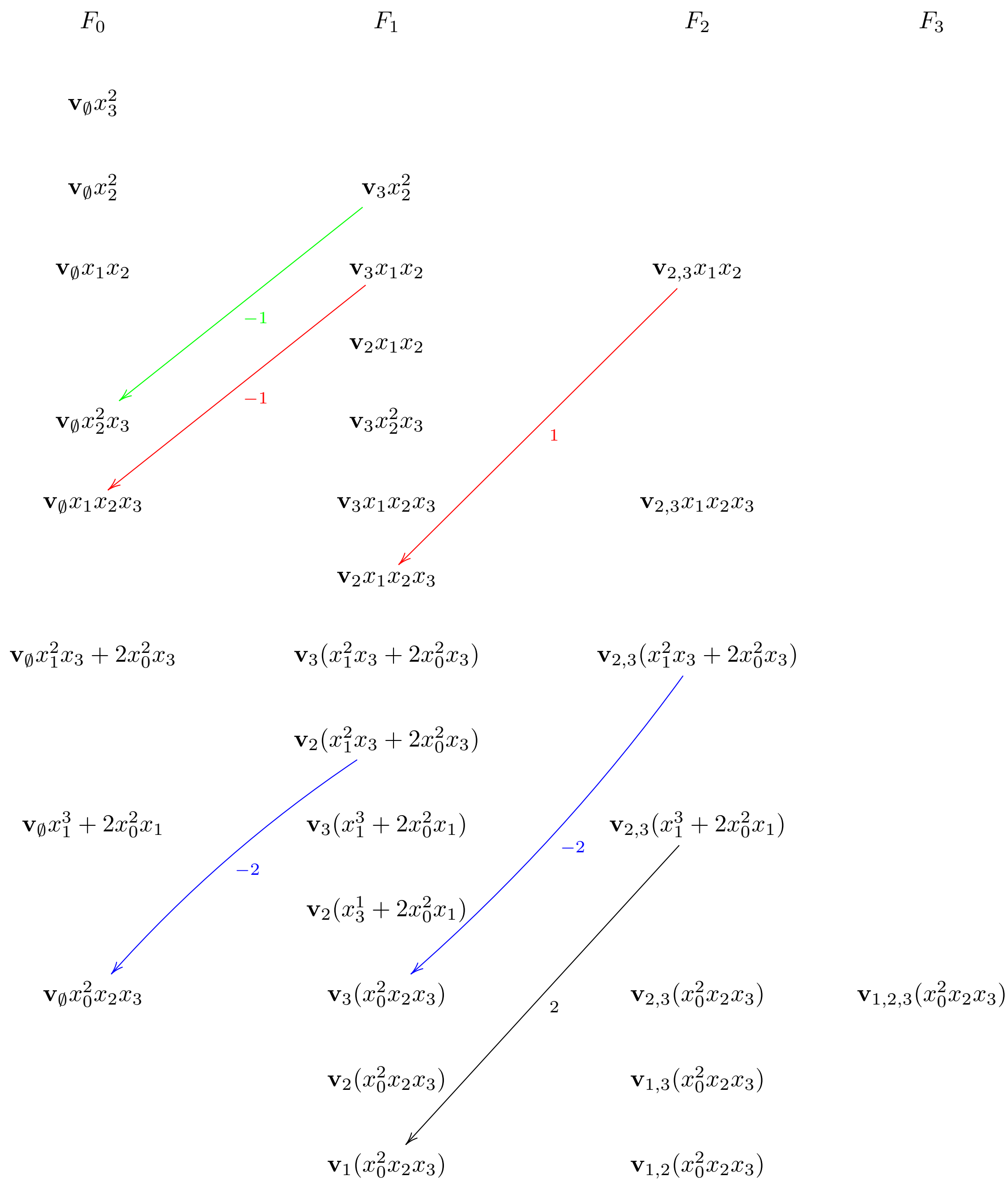

Figure 5.4.1: The graph of constants for example 5.4.3

This gives

$$\begin{aligned}
&1 \otimes d_{\mathcal{G}}(\mathbf{v}_3 x_1 x_2) \\
&= \sum_{h=1}^{2} \sum_{\substack{\mathbf{i} \subseteq (3) \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k} \leq 1}} \sum_{\beta} \varepsilon\big(\mathbf{i}; (2)\big) \Big(1 \otimes P^{(2,3),x_1x_2}_{\mathbf{k} \cup \big((2,3)\setminus \mathbf{i}\big),\beta}\Big) \mathbf{v}_{\mathbf{k} \cup \big((3)\setminus \mathbf{i}\big)} \mathbf{h}_\beta
\end{aligned}$$

Looking at the first two sums, we immediately see that only the cases $h = 1$, $\mathbf{i} = (3)$ give any summands, implying $\mathbf{k} = \emptyset$.

$$= \sum_{\beta} \varepsilon\big((3); (2)\big) \Big(1 \otimes P^{(2,3),x_1x_2}_{(2),\beta}\Big) \mathbf{v}_\emptyset \mathbf{h}_\beta$$

Now we know from Equation (5.4.1) and (5.4.2) that there is only one β for which this summon is nonzero, so

$$\begin{aligned}
&= -1 \cdot 1 \otimes P^{(2,3),x_1x_2}_{(2),x_1x_2x_3} \mathbf{v}_\emptyset x_1 x_2 x_3 \\
&= -\mathbf{v}_\emptyset x_1 x_2 x_3.
\end{aligned}$$

In the same manner, for the blue arrows, we see that we have

$$P^{(2,3),x_1^2x_3+2x_0^2x_3}_{(3),x_0^2x_2x_3} = -2,$$

and for all $m \neq 3$ or $\beta \neq x_0^2 x_2 x_3$, we have

$$P^{(2,3),x_1^2x_3+2x_0^2x_3}_{(3),\beta} = 0.$$

Again, we have

$$\operatorname{cls}(\mathbf{h}_\alpha) = \operatorname{cls}(x_1^2 x_3 + 2x_0^2 x_3) = 1, \quad d = \min_{\mathbf{h} \in \mathcal{H}} \operatorname{cls}(\mathbf{h}) = 0$$

and this time

$$e = \operatorname{cls}(\mathbf{h}_\alpha) - d + 1 = 2, \quad \boldsymbol{\ell} = (2) \quad \text{and} \quad \mathbf{j} = (2,3).$$

This gives

$$\begin{aligned}
&1 \otimes d_{\mathcal{G}}(\mathbf{v}_2 x_1^2 x_3 + 2x_0^2 x_3) \\
&= \sum_{h=1}^{2} \sum_{\substack{\mathbf{i} \subseteq (2) \\ |\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1 \\ \min \mathbf{k} \leq 1}} \sum_{\beta} \varepsilon\big(\mathbf{i}; (3)\big) \Big(1 \otimes P^{(2,3),x_1^2x_3+2x_0^2x_3}_{\mathbf{k} \cup \big((2,3)\setminus \mathbf{i}\big),\beta}\Big) \mathbf{v}_{\mathbf{k} \cup \big((2)\setminus \mathbf{i}\big)} \mathbf{h}_\beta \\
&= \sum_{\beta} \varepsilon\big((2); (3)\big) \Big(1 \otimes P^{(2,3),x_1^2x_3+2x_0^2x_3}_{(3),\beta}\Big) \mathbf{v}_\emptyset \mathbf{h}_\beta \\
&= 1 \cdot 1 \otimes P^{(2,3),x_1^2x_3+2x_0^2x_3}_{(3),x_0^2x_2x_3} \mathbf{v}_\emptyset x_1 x_2 x_3 \\
&= -2\mathbf{v}_\emptyset x_1 x_2 x_3.
\end{aligned}$$

We note that the **black** arrow has no corresponding arrow in lower homological degrees. Along all other arrows, the index set $\mathbf{v}_*$ of the target is just the index set of the source, with one element removed. However, for the **black** arrow, the index set of the target actually contains a new index, 1, which is not contained in the index set of the source, while both indices, 2 and 3, of the source are no longer present. But for arrows going from first to 0-th homological degree, this behavior cannot be matched: If there is just one index in the source, it is impossible to remove two indices; and in the same way, if the target has no index, it impossible to insert one. So this example also shows that Theorem 5.3.1 is not enough to construct all constants in homological degrees lesser or equal to e, for otherwise we would miss the **black** arrow, as it has no corresponding arrow in lower homological degree.

However, Theorem 5.4.1 does not have this problem of missing some constants in lower homological degrees. It is in fact possible to get all constants by calculating $d_{\mathcal{G}}(\mathbf{v}_{(\operatorname{cls}\mathbf{h}_\alpha)+1,\dots,n}\mathbf{h}_\alpha)$ for all $\mathbf{h}_\alpha \in \mathcal{H}$ and then applying Theorem 5.4.1 to obtain the constants in the other differentials. To finish this example, we explain how Theorem 5.4.1 indeed formally assures that the **black** does not give a constant in lower homological degree. Consider the vertex $\mathbf{v}_2(x_1^3+2x_0^2x_1)$: We have

$$P^{(2,3),x_1^3+2x_0^2x_1}_{(1),x_0^2x_2x_3} = 2,$$

and again for all $m \neq 1$ or $\beta \neq x_0^2x_2x_3$, we have

$$P^{(2,3),x_1^3+2x_0^2x_1}_{(m),\beta} = 0. \tag{5.4.3}$$

Once more, we have

$$\operatorname{cls}(\mathbf{h}_\alpha) = \operatorname{cls}(x_1^3 + 2x_0^2x_1) = 1, \quad d = \min_{\mathbf{h}\in\mathcal{H}} \operatorname{cls}(\mathbf{h}) = 0$$

and

$$e = \operatorname{cls}(\mathbf{h}_\alpha) - d + 1 = 2, \quad \boldsymbol{\ell} = (2) \text{ and } \mathbf{j} = (2,3).$$

So we have

$$\begin{aligned} &1 \otimes d_{\mathcal{G}}\big(\mathbf{v}_2(x_1^3 + 2x_0^2x_1)\big) \\ &= \sum_{h=1}^{2} \sum_{\substack{\mathbf{i}\subseteq(2)\\|\mathbf{i}|=h}} \sum_{\substack{|\mathbf{k}|=h-1\\ \min\mathbf{k}\leq 1}} \sum_{\beta} \varepsilon\big(\mathbf{i};(3)\big)\Big(1\otimes P^{(2,3),x_1^3+2x_0^2x_1}_{\mathbf{k}\cup\big((2,3)\setminus\mathbf{i}\big),\beta}\Big)\mathbf{v}_{\mathbf{k}\cup\big((2)\setminus\mathbf{i}\big)}\mathbf{h}_\beta \end{aligned}$$

But now from Equation (5.4.3), we see that this time for $h = 1$, any choice of $\mathbf{i}$ with $|\mathbf{i}| = h = 1$ gives a zero summand.

$$= \sum_{\substack{\mathbf{i}\subseteq(2)\\|\mathbf{i}|=2}} \sum_{\substack{|\mathbf{k}|=1\\ \min\mathbf{k}\leq 1}} \sum_{\beta} \varepsilon\big(\mathbf{i};(3)\big)\Big(1\otimes P^{(2,3),x_1^3+2x_0^2x_1}_{\mathbf{k}\cup\big((2,3)\setminus\mathbf{i}\big),\beta}\Big)\mathbf{v}_{\mathbf{k}\cup\big((2)\setminus\mathbf{i}\big)}\mathbf{h}_\beta$$

And now the condition $\mathbf{i} \subseteq (2), |\mathbf{i}| = 2$ of the first sum cannot be satisfied either, so there are no summands left.

$$= 0.$$

6 Syzygies of Veronese subrings

Through the paper [PS09], which, as its title states, presents "Open problems on syzygies and Hilbert functions", we have first encountered the following questions: Given a $\mathcal{P}$-module $\mathcal{M} \cong \mathcal{P}/I$ for an ideal $I \trianglelefteq \mathcal{P}$ generated in degree greater or equal to 2, and its minimal free resolution

$$\ldots \to F_l \xrightarrow{\varphi_l} F_{l-1} \xrightarrow{\varphi_{l-1}} \ldots \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} \mathcal{M} \to 0,$$

what can be said about the subcomplex

$$\ldots \to F_{l,l+1} \to F_{l-1,l} \to \ldots \to F_{1,2},$$

where by $F_{l,l+1}$ we denote the denote the submodule of F_l of total degree $l+1$. So if[1] $F_l = \bigoplus_{i \geq 1} \mathcal{P}\big(-(i)\big)^{\beta_{l,i}}$, then $F_{l,l+1} = \mathcal{P}\big(-(l+1)\big)^{\beta_{l,l+1}}$. This subcomplex is called the *2-linear strand of* $\mathcal{M}$. Note that as by assumption, I is generated in degree greater or equal to 2, so for $l > 0$, all Betti numbers $\beta_{l,l}$ are 0. The length of the 2-linear strand is $\max\{l \mid \beta_{l,l+1} \neq 0\}$. Now one can ask several questions related to the 2-linear strand (for classes of ideals):

- Up to which homological degree does the 2-linear strand coincide with the minimal free resolution? If we express this number in the language of Betti numbers, it is equivalent to finding the maximal number p such that $\beta_{q,q+i} = 0$ for $q \geq 2$ and $i \leq p$. In this case, we say that $\mathcal{M}$ *satisfies the* N_p*-property*. In terms of the shifted graded Betti numbers and the Betti table, we want to know the integer p such that the Betti table is of shape

	0	1	2	...	$p-1$	p	...
0	1	$-$	$-$	...	$-$	$-$	...
1	$-$	$\beta'_{1,1}$	$\beta'_{2,1}$	...	$\beta'_{p-1,1}$	$\beta'_{p,1}$	...
2	$-$	$-$	$-$	...	$-$	$*$	...
$\vdots$							
r	$-$	$-$	$-$	...	$-$	$*$	...

  where one of the Betti numbers $*$ is nonzero.

- What is the length of the 2-linear strand?

- If these questions cannot be answered, what can be said about lower or upper bounds? Obviously, finding a non-vanishing Betti number other than $\beta_{l,l+1}$ immediately gives an upper bound for the p in the N_p-property. In fact, one of our main result of this chapter is of this shape.

- What can be said about similar subcomplexes of the minimal free resolution? For example the subcomplex generated by all $F_l = \bigoplus_{i=1}^{t} \mathcal{P}\big(-(i)\big)^{\beta_{l,i}}$ for a fixed $t \geq 1$. Note that for $t = 1$, we have the 2-linear strand, while $t = \operatorname{reg} \mathcal{M}$ corresponds to the entire minimal free resolution.

[1] see also definition 2.1.16

One particular class of ideals, for which there are some results and conjectures regarding these questions, are ideals originating from Veronese subrings. In this present chapter, we try to work towards the questions mentioned above with respect to this class of modules. We have used the articles [EL12] and [OP01] as inspiration.

We will apply the theory presented in previous chapters to Veronese subrings, but we will make some minor changes to our notations and conventions, adapting to the special situation in question:

Assumption 6.0.1. For this chapter, we will work with the conventions below:

- The field $\Bbbk$ is algebraically closed of arbitrary characteristic. While for the area of involutive bases, this is rather unusual, in the context of algebraic geometry, this is typically a basic assumption, as some tools such as Hilbert's Nullstellensatz require $\Bbbk$ to be algebraically closed. We will later see in Lemma 6.1.7 below, that for our methods, it would be enough to assume that $\Bbbk$ contains an element of large enough order over the minimal subfield of $\Bbbk$.
- As polynomial rings with different numbers of variables will appear, in order to avoid confusion, we will use the identification

  $$\mathcal{P} \cong \Bbbk[x_0, ..., x_m],$$

  with m as in the definition below. This is a change from previous chapters, where $\mathcal{P}$ was denoting the polynomial ring in $n+1$ variables. We make this change in order to be more consistent with the literature about Veronese subrings mentioned above, where in the situation of definition 6.0.3 below, the domain of the map ν_d is usually denoted with $\mathbb{P}^n_{\Bbbk}$.
- Unless stated otherwise, we will assume if we work with a fixed monomial order $\prec$ in the polynomial rings $\Bbbk[x_0, ..., x_n]$ or $\Bbbk[x_0, ..., x_m]$, that for any product $x_{i_1} \cdots x_{i_l}$ of variables we have $x_{i_1} \preceq ... \preceq x_{i_l}$, i.e. any product of variables is ordered. In particular, with respect to Section 6.1.1 below, we will use this implicit ordering for the renamed variables in the polynomial ring $\mathcal{P}$ and the degrevlex order on $\mathcal{P}$, which both will be introduced in Section 6.1.1.
- We will fix the number d as in Definitions 6.0.2 and 6.0.3 below. This is a change from Chapter 6, where d was the minimal class of elements in $\mathcal{H}$.

Definition 6.0.2. The *(d-th) Veronese subring* $S^{(d)}$ of $\Bbbk[x_0, ..., x_n]$ is given by $S^{(d)} = \Bbbk[x^\mu \mid \deg x^\mu = d] \subseteq \Bbbk[x_0, ..., x_n]$. In particular, we have[1]

$$S^{(d)} = \bigoplus_{i \geq 0} \Bbbk[x_0, ..., x_n]_{i \cdot d}$$

[1]Recall that $\Bbbk[x_0, ..., x_n]_{id}$ contains the elements of degree id.

Definition 6.0.3. The Veronese embedding of degree d in $n+1$ variables is given by the map

$$\nu_d \colon \mathbb{P}^n_{\Bbbk} \to \mathbb{P}^m_{\Bbbk},$$

where $m = \binom{n+d}{d} - 1$ and ν_d maps any point $[\tau_0 \colon \ldots \colon \tau_n]$ to the point whose entries are given by all monomials of degree d in the variables $\tau_0, \ldots, \tau_n$. The image Y of this map is an irreducible variety. This fact easily follows from the remark below, but it does not bear any further relevance for our work.

Remark 6.0.4. Combining Definitions 6.0.2 and 6.0.3, we see that a Veronese subring can be presented as a $\mathcal{P} \cong \Bbbk[x_0, \ldots, x_m]$-algebra via the homomorphism

$$\phi\colon \quad \mathcal{P} \quad \to \quad S^{(d)}$$

which maps each variable $x_0, \ldots, x_m$ to a different monomial $x^\mu \in \Bbbk[x_0, \ldots, x_n]$ of degree d.

Of course, now the vanishing ideal of Y is given by $\ker \phi$.

6.1 A Pommaret basis for Veronese subrings

Our goal is to apply the theory presented in previous chapters, in particular chapter 4, to gain information about the vanishing ideal $I(Y) \cong \ker \phi$.

For this approach, it is necessary to construct an involutive basis of $I(Y)$. We choose to construct a Pommaret basis with respect to the degrevlex order. As we have seen in example 2.3.34, Pommaret bases might not exist in a given coordinate system. So our first task will be to construct δ-regular coordinates.

6.1.1 Constructing δ-regular coordinates for $I(Y)$

First we note that we can choose any order in which the monomials in the image of ν_d appear. Any change of this order induces an isomorphism of varieties: Permuting the order of monomials is equivalent to a permutation of the variables $x_0, \ldots, x_m$. Before we describe one particular ordering which we use for the remainder of this chapter, we will rename the variables $x_0, \ldots, x_m \in \mathcal{P}$ in a way that is better suited for Veronese subrings. We take inspiration from Definition 6.0.3 and Remark 6.0.4:

Since we have one variable x_i for every monomial τ^μ of degree $\deg \mu = d$ in the variables $\tau_0, \ldots, \tau_n$, we will use multi-indices $\mathbf{d}$ of length $n+1$ and total degree d to enumerate the variables $x_0, \ldots, x_m$. We extend the notion of a class in definition 2.3.21 to these new indices; i.e. the class of a variable is now a multiindex.

Now we will look at the monomial order. We already mentioned that it is degrevlex order, but having introduced new indices to the variables, we need to explain how these new indices interact with the monomial order; i.e. we need to define the degrevlex order for our "new" variables. While the order we will define now might look unusual, its usefulness is given in Theorem 6.1.9 below, where we see that, for this monomial order, a Pommaret basis does indeed exist:

Assumption 6.1.1. For the remainder of this chapter, we define $\prec$ to be the degrevlex order on $\mathcal{P}$ induced by the following ordering of the variables $x_{\mathbf{d}}$ where $\mathbf{d} \in \mathbb{N}^{n+1}$ is a multiindex of length d: For two multiindices $\mathbf{d}, \mathbf{e} \in \mathbb{N}^{n+1}$ of degree d, we have $x_{\mathbf{d}} \prec x_{\mathbf{e}}$, if and only if

- $\mathbf{d}$ and $\mathbf{e}$ both are of the form $(0, ..., d, ..., 0)$, and $\mathbf{d} \prec_{\text{degrevlex}} \mathbf{e}$.
- $\mathbf{d}$ is of the form $(0, ..., d, ..., 0)$, but $\mathbf{e}$ is not.
- $\mathbf{d}$ and $\mathbf{e}$ both are **not** of the form $(0, ..., d, ..., 0)$, and $\mathbf{d} \prec_{\text{degrevlex}} \mathbf{e}$.

Note that the $\prec_{\text{degrevlex}}$ order appearing in this definition is the degrevlex order for multiindices of length $n+1$, with our usual convention that $\mathbf{d} \prec_{\text{degrevlex}} \mathbf{e}$ if and only if the leftmost entry of $\mathbf{e} - \mathbf{d}$ is negative, see Definition 2.3.3.

There is one minor downside to this approach: In a context where we have a multiindex $\mathbf{d}$, according to our notation, $\mathbf{d}_i$ denotes the i-th entry of $\mathbf{d}$. Now occasionally, it will be necessary to refer to entries of multiindices in a product of variables, for example, if we have a product $\prod_{i=1}^{t} x_{\mathbf{d}_i}$, we denote by $(\mathbf{d}_i)_j$ the j-th entry of the i-th multiindex. Here $\mathbf{d}_i$ now has two possible meanings. We try to avoid this ambiguity as much as possible; in any case where the meaning of the notation might not be obvious from the context, we explicitly mention which meaning we are referring to.

Example 6.1.2. If $n = 2$ and $d = 3$, this ordering is given by

$$x_{(3,0,0)} \prec x_{(0,3,0)} \prec x_{(0,0,3)} \prec x_{(2,1,0)} \prec x_{(2,0,1)} \prec \\ x_{(1,2,0)} \prec x_{(1,1,1)} \prec x_{(1,0,2)} \prec x_{(0,2,1)} \prec x_{(0,1,2)}.$$

With respect to the existence of a Pommaret basis, we will see that we only need variables of shape $x_{(0,...,d,...,0)}$ to be the smallest variables with respect to $\prec$. The order of those monomials which are not pure powers of variables does not matter. Nevertheless, it is advantageous to work with this fixed monomial order, since we will see that for this particular monomial order, it is possible to obtain statements for Betti numbers of $I(Y)$. We think that one possible idea for further research might be to change the monomial order and see if it is still possible to get results similar to those presented later in this chapter.

Definition 6.1.3. For any multiindex $\mathbf{d}$ with $\deg \mathbf{d} = i \cdot d$ for some integer $i \geq 1$, we define the *minimal monomial of* $\mathbf{d}$ to be the monomial $x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_i}$ with $\sum_{j=1}^{i} \mathbf{d}_j = \mathbf{d}$ which is minimal with respect to $\prec$. We use the notation $\operatorname{MinMon}(\mathbf{d})$.

Example 6.1.4. Let $n = 2$ and $d = 3$. For $\mathbf{d} = (2, 4, 3)$, we have

$$\operatorname{MinMon}(\mathbf{d}) = x_{(0,3,0)} x_{(0,0,3)} x_{(2,1,0)}.$$

We state three simple properties of minimal monomials, which we will later use repeatedly in our proofs:

Lemma 6.1.5. *Let* $\mathbf{d}$ *be a multiindex with* $\deg \mathbf{d} = t \cdot d$ *for some integer* $t \geq 1$.

- *If there is an index* j *such that* $\mathbf{d}_j \geq d$*, then we have*[1]

$$\mathrm{MinMon}(\mathbf{d}) = x_{d\cdot \mathbf{1}_j} \,\mathrm{MinMon}(\mathbf{d} - d \cdot \mathbf{1}_j).$$

- *If* $\mathrm{MinMon}(\mathbf{d}) = \prod_{i=1}^{t} x_{\mathbf{d}_i}$*, then* $\mathrm{MinMon}(\mathbf{d} - \mathbf{d}_1) = \prod_{i=2}^{t} x_{\mathbf{d}_i}$.

- *Let* $\mathbf{e}$ *be the minimal multiindex of degree* d *such that* $\mathbf{e}_j \leq \mathbf{d}_j$ *for all* $0 \leq j \leq n$*. Then*

$$\mathrm{MinMon}(\mathbf{d}) = x_{\mathbf{e}} \cdot \mathrm{MinMon}(\mathbf{d} - \mathbf{e}).$$

Proof. (Recall that by our Assumption 6.0.1, we have $\mathbf{d}_1 \preceq ... \preceq \mathbf{d}_t$)

- Regarding the first point, without loss of generality, let j be the minimal index such that $\mathbf{d}_j \geq d$, i.e. the class (in the renumbered variables) of $\mathrm{MinMon}(\mathbf{d})$ is at least $d\cdot\mathbf{1}_j$. But since the class of $x_{d\cdot\mathbf{1}_j}\,\mathrm{MinMon}(\mathbf{d}-d\cdot\mathbf{1}_j)$ is $d\cdot\mathbf{1}_j$, we see that the class of $\mathrm{MinMon}(\mathbf{d})$ is also at most $d\cdot\mathbf{1}_j$. So we have $\mathrm{cls}\big(\mathrm{MinMon}(\mathbf{d})\big) = d\cdot\mathbf{1}_j$, and therefore $\mathrm{MinMon}(\mathbf{d})$ is divisible by $x_{d\cdot\mathbf{1}_j}$.

- Regarding the second point, we note that if there was a monomial $\prod_{i=2}^{t} x_{\mathbf{e}_i}$ with $\sum_{i=2}^{t} \mathbf{e}_i = \sum_{i=2}^{t} \mathbf{e}_i$ and $\prod_{i=2}^{t} x_{\mathbf{e}_i} \prec \prod_{i=2}^{t} x_{\mathbf{d}_i}$, then also

$$x_{\mathbf{d}_1} \prod_{i=2}^{t} x_{\mathbf{e}_i} \prec x_{\mathbf{d}_1} \prod_{i=2}^{t} x_{\mathbf{d}_i},$$

  contradicting the fact that $\mathrm{MinMon}(\mathbf{d}) = \prod_{i=1}^{t} x_{\mathbf{d}_i}$.

- The third point is obvious, as we are considering the minimal monomial with respect to the degrevlex order.

□

Example 6.1.6. With $\mathbf{d} = (2,4,3)$ as in example 6.1.4, we do have

$$\mathbf{d}_0 = 2, \mathbf{d}_1 = 4, \mathbf{d}_2 = 3$$

and so the minimal j as in the proof of lemma 6.1.5 is $j = 1$. Indeed, we have

$$\mathrm{MinMon}(\mathbf{d}) = x_{(0,3,0)}x_{(0,0,3)}x_{(2,1,0)} = x_{3\cdot\mathbf{1}_1}x_{3\cdot\mathbf{1}_2}x_{(2,1,0)} = x_{3\cdot\mathbf{1}_1}\,\mathrm{MinMon}(2,1,3),$$

which illustrates the statements of lemma 6.1.5.

[1] Recall that by Definition 2.1.7, $\mathbf{1}_j = (0,...0,1,0,..,0)$ is the multiindex for which the j-th entry is 1 while all other indices are 0. As the set of multiindices has a natural $\mathbb{Z}$-module structure, we have $d\cdot\mathbf{1}_j = (0,...,0,d,0,...,0)$.

6.1.2 The Pommaret basis

Before we write down our Pommaret basis, we note the following lemma. Probably it is clear to anyone more familiar with algebraic geometry. Nevertheless, since this lemma will be essential in one of our proofs later, we find that it is helpful to explicitly formulate and prove it right now, for it is here that we make use of one of the special assumptions 6.0.1 that only hold for this chapter, namely that $\Bbbk$ is an algebraically closed field.

Lemma 6.1.7. *Let $\mathbf{g} \in I(Y)$. Let $x_{\mathbf{d}_1}^{\mu_1} \cdots x_{\mathbf{d}_t}^{\mu_t} = x^\mu = \mathrm{lt}(\mathbf{g})$. Then $\mathbf{g}$ contains another monomial $x_{\mathbf{e}_1}^{\pi_1} \cdots x_{\mathbf{e}_t}^{\pi_t}$ such that*

$$\sum_{i=1}^{t} (\mu_i \cdot \mathbf{d}_i) = \sum_{i=1}^{t} (\pi_i \cdot \mathbf{e}_i) \in \mathbb{N}^{n+1}$$

as a summand.

Proof. Let k be the prime field of $\Bbbk$, i.e. the minimal subfield of $\Bbbk$ containing 1. k is not algebraically closed, since obviously $k \cong \mathbb{Q}$ or $k = \mathbb{F}_p$. We note that k is a perfect field in either case. Let $\phi \in \Bbbk$ be an element such that $\big[k(\phi) : k\big]$ is sufficiently large. We shortly explain why we can always pick such an element ϕ: Obviously, if there is a $\phi \in \Bbbk$ that is transcendent over k, we are done. Otherwise, $\big[k(\phi) : k\big]$ is finite for any $\phi \in \Bbbk$. Since $\big[\Bbbk : k\big] = \infty$, we can pick an index $i \in \mathbb{N}$ and a set of elements $\phi_i \in \Bbbk \setminus k(\phi_1, ..., \phi_{i-1})$ such that $\big[k(\phi_1, ..., \phi_i) : k\big]$ is arbitrary large, yet finite. Since k is a perfect field and therefore separable, this extension is separable, and by the Primitive Element Theorem (see for example [Lan05, V,§4, Theorem 4.5]) there is a ϕ such that $k(\phi) = k(\phi_1, ..., \phi_i)$.

Now consider the image of $\Phi = [\phi^{(d \cdot \deg \mu + 1)^0} : \phi^{(d \cdot \deg \mu + 1)^1} : ... : \phi^{(d \cdot \deg \mu + 1)^n}]$ under ν_d. We have $\mathbf{g}\big(\nu_d(\Phi)\big) = 0$, since $\mathbf{g} \in I(Y)$. If we evaluate $\mathrm{lt}(\mathbf{g})$ at $\nu_d(\Phi)$, we obtain

$$\mathrm{lt}(\mathbf{g})\big(\nu_d(\Phi)\big) = \prod_{i=1}^{t} \prod_{j=0}^{n} \Big(\phi^{(d \cdot \deg \mu + 1)^j}\Big)^{\mu_i \cdot (\mathbf{d}_i)_j} = \phi^{\sum_{j=0}^{n} \sum_{i=1}^{t} \mu_i \cdot (\mathbf{d}_i)_j \cdot \left((d \cdot \deg \mu + 1)^j\right)} \in \Bbbk, \tag{6.1.1}$$

where $(\mathbf{d}_i)_j$ denotes the j-th entry of the multiindex $\mathbf{d}_i$. By construction, we have $(\mathbf{d}_i)_j \leq d$, and therefore

$$\mu_i \cdot (\mathbf{d}_i)_j \leq (\deg \mu) \cdot d \leq d \deg \mu + 1.$$

So if we choose ϕ such that $\big[k(\phi) : k\big] \geq (d \cdot \deg \mu + 1)(n + 1)$, the numbers

$$\sum_{i=1}^{t} \mu_i \cdot (\mathbf{d}_i)_j$$

are uniquely determined by equation (6.1.1). But then since $\mathbf{g}\big(\nu_d(\Phi)\big) = 0$, $\mathbf{g}\big(\nu_d(\Phi)\big)$ must contain another summand whose evaluation at $\nu_d(\Phi)$ is the element of $\Bbbk$ given in (6.1.1). But this is only possible if $\mathbf{g}$ contains a summand of the shape given in the lemma. $\square$

Example 6.1.8. Lemma 6.1.7 can indeed fail if $\Bbbk$ is not algebraically closed: Consider

$$\nu_2 \colon \mathbb{P}^1_{\mathbb{F}_2} \to \mathbb{P}^2_{\mathbb{F}_2},$$

which maps $[\tau_0 : \tau_1]$ to $[\tau_0^2 : \tau_1^2 : \tau_0\tau_1]$, so

$$\mathrm{im}(\nu_d) = \{[1:0:0], [0:1:0], [1:1:1]\}.$$

Hence $I(Y)$ contains the element $\mathbf{h} = x_{(0,2)}x_{(1,1)} - x_{(2,0)}x_{(1,1)}$, and for this $\mathbf{h}$, we have

$$1 \cdot (0,2) + 1 \cdot (1,1) = (1,3) \neq (3,1) = 1 \cdot (2,0) + 1 \cdot (1,1).$$

So indeed lemma 6.1.7 can fail if $\Bbbk$ is not algebraically closed.

Now we continue the process of finding a Pommaret basis for $I(Y)$.

Theorem 6.1.9. *For $t \geq 2$, let*

$$\mathcal{H}_t = \Big\{ x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t} - \mathrm{MinMon}\Big(\sum_{j=1}^{t} \mathbf{d}_j\Big) \mid x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t} \neq \mathrm{MinMon}\Big(\sum_{j=1}^{t} \mathbf{d}_j\Big),$$
$$x_{\mathbf{d}_2} \cdots x_{\mathbf{d}_t} = \mathrm{MinMon}\Big(\sum_{j=2}^{t} \mathbf{d}_j\Big), \mathbf{d}_1 \notin \{(d,0,...,0), ..., (0,...,0,d)\} \Big\}.$$

Then $\mathcal{H} = \bigcup_{t \geq 2} \mathcal{H}_t$ is a Pommaret basis of $I(Y)$ (Recall that by assumption 6.0.1 we assume $x_{\mathbf{d}_1} \preceq ... \preceq x_{\mathbf{d}_t}$).

In particular this theorem implies that $I(Y)$ is a toric ideal, i.e. generated by binomials $x^{\mu} - x^{\nu}$.

Remark 6.1.10. Note that the statement of this theorem can be split into two separate aspects: First, that $\mathcal{H}$ is a Pommaret basis (for the ideal generated by $\mathcal{H}$), and second, that its involutive span $\langle \mathcal{H} \rangle_P$ is equal to $I(Y)$. Looking at the proof, we see that the first aspect does not require the fact that $\Bbbk$ is algebraically closed; so any statement made later in this chapter still holds over any field if one replaces $I(Y)$ by $\langle \mathcal{H} \rangle_P$. For the second aspect however, if we omitted the assumption that $\Bbbk$ is algebraically closed, the ideal $I(Y)$ might contain additional elements that are not contained in the involutive span $\langle \mathcal{H} \rangle_P$, so $\langle \mathcal{H} \rangle_P \subsetneqq I(Y)$ is possible: Indeed, for the generator $\mathbf{h}$ of example 6.1.8, one sees that $\mathbf{h}$ has leading monomial $x_{(0,2)}x_{(1,1)}$, which is not the leading monomial of any element of $\mathcal{H}$, as it contains the variable $x_{(0,2)}$. On the other hand $\mathrm{lt}(\mathbf{h})$ cannot be involutively divisible by any element of $\mathcal{H}$, for any involutive divisor would be of degree 1. But obviously, $I(Y)$ is generated in degree greater or equal to 2. So we have indeed $\mathbf{h} \in I(Y) \setminus \langle \mathcal{H} \rangle_P$.

Proof. (of Theorem 6.1.9) Now from definition 2.3.13, we see that we need to show that we have $\langle \mathcal{H} \rangle_P = \langle \mathcal{H} \rangle_{P,\prec} = I(Y)$ for the involutive span of $\mathcal{H}$ with respect to the Pommaret division P. Obviously, we have $\mathcal{H} \subseteq I(Y)$. By Theorem 2.3.15, using $\mathcal{H} \subseteq I(Y)$, it is equivalent to show that

- $\mathrm{lt}_\prec\big(I(Y)\big) = \mathrm{lt}_\prec(\langle\mathcal{H}\rangle_P)$,
- $\mathcal{H}$ is Pommaret auto reduced and
- $\mathcal{H}$ is finite.

Since any element of $\langle\mathcal{H}\rangle_P$ is obviously contained in $I(Y)$, we only need to show that $\mathrm{lt}_\prec\big(I(Y)\big) \subseteq \mathrm{lt}_\prec(\langle\mathcal{H}\rangle_P)$ for the first point.

We start by proving $\mathrm{lt}_\prec\big(I(Y)\big) = \mathrm{lt}_\prec(\langle\mathcal{H}\rangle_P)$. We do so by using induction over the degree t of the elements in $\mathcal{H}$.

$t = 2$: If $x_{\mathbf{d}_1}x_{\mathbf{d}_2}$ is the leading monomial of some element of $I(Y)$, then by Lemma 6.1.7, this element of $I(Y)$ contains a term $x_{\mathbf{c}_1}x_{\mathbf{c}_2}$ with

$$\mathbf{d}_1 + \mathbf{d}_2 = \mathbf{c}_1 + \mathbf{c}_2$$

and $x_{\mathbf{d}_1}x_{\mathbf{d}_2} \succ x_{\mathbf{c}_1}x_{\mathbf{c}_2}$, so $x_{\mathbf{d}_1}x_{\mathbf{d}_2} \succ \mathrm{MinMon}\,(\mathbf{d}_1 + \mathbf{d}_2)$. But now since $\mathcal{H}_2$ contains $x_{\mathbf{d}_1}x_{\mathbf{d}_2} - \mathrm{MinMon}\,(\mathbf{d}_1 + \mathbf{d}_2)$, we have $\mathrm{lt}\big(\langle\mathcal{H}\rangle_P\big)_2 = \mathrm{lt}\big(I(Y)\big)_2$.

$t > 2$: Our goal is to prove $\mathrm{lt}\big(I(Y)\big)_t \subseteq \mathrm{lt}(\langle\mathcal{H}\rangle_P)_t$. Let $x_{\mathbf{d}_1}\cdots x_{\mathbf{d}_t}$ be the leading monomial of some $r \in I(Y)_t$. We separately consider two sub-cases for $x_{\mathbf{d}_1}$ (Recall that by our conventions, we have $x_{\mathbf{d}_1} \preceq x_{\mathbf{d}_i}$ for all $1 \leq i \leq t$):

Case 1: $x_{\mathbf{d}_2}\cdots x_{\mathbf{d}_t} = \mathrm{MinMon}\big(\sum_{j=2}^t \mathbf{d}_j\big)$: If $\mathbf{d}_1 \neq (0, ..., d, ..., 0)$, then by definition, $\mathcal{H}$ contains $x_{\mathbf{d}_1}\cdots x_{\mathbf{d}_t} - \mathrm{MinMon}\big(\sum_{j=1}^t \mathbf{d}_j\big)$, so there is nothing to prove. So let $\mathbf{d}_1 = (0, ..., d, ..., 0)$. We claim that there is some g such that $x_{\mathbf{d}_g}\cdots x_{\mathbf{d}_t} \neq \mathrm{MinMon}\big(\sum_{j=g}^t \mathbf{d}_j\big)$. Then H_{t-g+1} contains $x_{\mathbf{d}_g}\cdots x_{\mathbf{d}_t} - \mathrm{MinMon}\big(\sum_{j=g}^t \mathbf{d}_j\big)$, for which the variables $x_{\mathbf{d}_1}, ..., x_{\mathbf{d}_{g-1}}$ are multiplicative, implying $x_{\mathbf{d}_1}\cdots x_{\mathbf{d}_t} \in \mathrm{lt}(\langle\mathcal{H}\rangle_P)$. So assume that there is no such g. But then by lemma 6.1.5, we have

$$x_{\mathbf{d}_1}\cdots x_{\mathbf{d}_t} = \mathrm{MinMon}\Big(\sum_{j=1}^t \mathbf{d}_j\Big).$$

Now by lemma 6.1.7, r contains another term $x_{\mathbf{e}_1}\cdots x_{\mathbf{e}_t}$ such that

$$\sum_{j=1}^t \mathbf{e}_j = \sum_{j=1}^t \mathbf{d}_j.$$

But then we have

$$x_{\mathbf{e}_1}\cdots x_{\mathbf{e}_t} \succ \mathrm{MinMon}\Big(\sum_{j=1}^t \mathbf{d}_j\Big) = x_{\mathbf{d}_1}\cdots x_{\mathbf{d}_t},$$

contradicting $\mathrm{lt}\, r = x_{\mathbf{d}_1}\cdots x_{\mathbf{d}_t}$.

Case 2: $x_{\mathbf{d}_2} \cdots x_{\mathbf{d}_t} \neq \mathrm{MinMon}(\sum_{j=2}^{t} \mathbf{d}_j)$. So we have

$$x_{\mathbf{d}_2} \cdots x_{\mathbf{d}_t} - \mathrm{MinMon}\big(\sum_{j=2}^{t} \mathbf{d}_j\big) \in I(Y).$$

However, by assumption, we have $\mathrm{lt}(\langle \mathcal{H} \rangle_P)_{t-1} = \mathrm{lt}\big(I(Y)\big)_{t-1}$, so $\langle \mathcal{H} \rangle_{P,t-1}$ (the set of elements of degree $t-1$ in $\langle \mathcal{H} \rangle_P$) contains some $\mathbf{h}$ with leading monomial $x_{\mathbf{d}_2} \cdots x_{\mathbf{d}_t}$ and then of course $x_{\mathbf{d}_1}$ is multiplicative for $\mathbf{h}$, so $\langle \mathcal{H} \rangle_{P,t}$ contains the element $x_{\mathbf{d}_1}\mathbf{h}$ with leading monomial $x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t}$.

Now we show that $\mathcal{H}$ is Pommaret auto reduced: If $x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t}$ is the leading monomial of some element of $\mathcal{H}$, then $x_{\mathbf{d}_2} \cdots x_{\mathbf{d}_t}$ is a minimal monomial. Due to the second statement of lemma 6.1.5, then also $x_{\mathbf{d}_i} \cdots x_{\mathbf{d}_t}$ for $2 \leq i \leq t$ is a minimal monomials, and so neither of these monomials is the leading monomial of another element of $\mathcal{H}$. This is exactly what is needed to see that $\mathcal{H}$ is Pommaret auto reduced (Again recall assumption 6.0.1, which says that we always have $x_{\mathbf{d}_1} \preceq \ldots \preceq x_{\mathbf{d}_t}$).

Now all that is left to show is that $\mathcal{H}$ is finite. Since any $\mathcal{H}_t$ is obviously finite, it suffices to show that only finitely many $\mathcal{H}_t$ are nonempty.

Let $t > \frac{(d-1)(n+1)}{d} + 1$. Assume $\mathcal{H}_t \neq \emptyset$: Let $x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t}$ be the leading monomial of an element of $\mathcal{H}_t$. Since $\deg(\sum_{i=2}^{t} \mathbf{d}_i) = (t-1)d > (d-1)(n+1)$, there is some index $j \in \{0, \ldots, n\}$ for which $(\sum_{i=2}^{t} \mathbf{d}_i)_j \geq d$. As $x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t}$ is the leading monomial of an element $\mathcal{H}_t$, we have $x_{\mathbf{d}_2} \cdots x_{\mathbf{d}_t} = \mathrm{MinMon}(\sum_{i=2}^{t} \mathbf{d}_i)$. But now from lemma 6.1.5, we see that this implies $x_{(0,\ldots,0,d,0,\ldots,0)} = x_{d \cdot \mathbf{1}_j}$ (where the entry corresponding to j is the only nonzero entry) for some $k \in \{2, \ldots, t\}$. Since $x_{\mathbf{d}_1} \preceq x_{\mathbf{d}_2}$, we have that $\mathbf{d}_1$ is also of shape $d \cdot \mathbf{1}_k$ for some $k \leq j$. But this is a contradiction, since by definition of the $\mathcal{H}_t$, no leading monomial of an element of $\mathcal{H}_t$ can be of this shape. ☐

Example 6.1.11. We have a closer look at this Pommaret basis: One naive, but far from optimal, algorithm to construct the elements of $\mathcal{H}$ would be as follows: Starting with degree $t = 2$, we iteratively construct $\mathcal{H}_t$. In order to do so, for any multiindex $\mathbf{e}$ of degree $2d$, we need to find all decompositions of $\mathbf{e}$ into sums of two multiindices $\mathbf{e}_1 \preceq \mathbf{e}_2$ of degree d. In the process, we can discard any sum containing a multiindex of shape $d \cdot \mathbf{1}_j$ (except when this sum is $\mathrm{MinMon}(\mathbf{e})$), for a product of shape $x_{d \cdot \mathbf{1}_j} x_{\mathbf{e}_2}$ cannot be the leading term of an element of $\mathcal{H}$. For every other sum $\mathbf{e}_1 + \mathbf{e}_2 = \mathbf{e}$, we obtain the element

$$x_{\mathbf{e}_1} x_{\mathbf{e}_2} - \mathrm{MinMon}(\mathbf{e}) \in \mathcal{H}_2,$$

(except when $x_{\mathbf{e}_1} x_{\mathbf{e}_2} = \mathrm{MinMon}(\mathbf{e})$). Proceeding to higher degrees $t > 2$, we also need to take into account that for any $\sum_{i=1}^{t} \mathbf{e}_t = \mathbf{e}$ with $\mathbf{e}_1 \preceq \ldots \preceq \mathbf{e}_t$, the condition $x_{\mathbf{e}_2} \cdots x_{\mathbf{e}_t} = \mathrm{MinMon}(\sum_{i=2}^{t} \mathbf{e}_t)$ is satisfied.

So now let $n = 1, d = 2$. Let $t = 2$, i.e. we consider the degree $2d = 4$. The multiindices $\mathbf{e}$ in question are

$$(4,0), \quad (3,1), \quad (2,2), \quad (1,3), \quad (0,4).$$

Of these multiindices, only $(2,2)$ can be written as a sum of two multiindices $\mathbf{e}_1 = \mathbf{e}_2 = (1,1)$ such that $x_{\mathbf{e}_1}x_{\mathbf{e}_2} \neq \operatorname{MinMon}\big((2,2)\big) = x_{(2,0)}x_{(0,2)}$, and we obtain

$$x^2_{(1,1)} - x_{(2,0)}x_{(0,2)} \in \mathcal{H}_2.$$

We know that $\mathcal{H}_t = \emptyset$ as soon as $t > \frac{(d-1)(n+1)}{d} + 1 = 2$, so here indeed the Pommaret basis contains a single element. Equivalently, one could have shown that this element generates $I(Y)$.

A more interesting example arises in the case[1] $n = 3, d = 4$. Even in this relatively small case, we have $m = \binom{7}{4} - 1 = 34$, i.e. the ring $\mathcal{P}$ has 35 variables. Going to degree $t = 4$, one can check

$$x^2_{(3,1,0,0)}x_{(0,2,2,0)}x_{(0,0,1,3)} - \operatorname{MinMon}\big((6,4,3,3)\big) \in \mathcal{H}_4.$$

This element is contained in $I(Y)$. The condition

$$x_{\mathbf{e}_2} \cdots x_{\mathbf{e}_t} = \operatorname{MinMon}\big(\sum_{i=2}^{t} \mathbf{e}_t\big)$$

now translates to

$$x_{(3,1,0,0)}x_{(0,2,2,0)}x_{(0,0,1,3)} = \operatorname{MinMon}\big((3,3,3,3)\big),$$

which is indeed satisfied, as one can easily check with help of Lemma 6.1.5. In fact, we claim that $\mathcal{H}_4$ contains only this single element. While this could be proven by the naive "brute-force"-algorithm outlined at the beginning of the current remark, in Example 6.1.18 below, we introduce a more systematic notation, which allows us to prove a generalized version of this claim by introducing a better algorithm.

In fact, it was this example $n = 3, d = 4$ that served as a primary inspiration for the general construction of Pommaret bases for arbitrary values of n, d.

Remark 6.1.12. It should be noted that this Pommaret basis is in general larger than the minimal Gröbner basis, which is given by $\mathcal{H}_2$, as we will prove in Theorem 6.1.13 below. The fact that the ideal $I(Y)$ defining the Veronese subring has a quadratic Gröbner bases is well known, see for example [Stu, Theorem 14.2], [ERT94, Theorem 6] or [PM15, page 246], who attribute this statement to [BM81].

From Section 4.4 we recall that the quotient of the number of elements in $\mathcal{H}$ and a minimal Gröbner basis, i.e. $\mathcal{H}_2$ is of interest for performing calculations with computers. Since a pseudo Betti table gives upper bounds for the Betti numbers, we have another reason to be interested in how much larger $\mathcal{H}$ is than $\mathcal{H}_2$. Unfortunately, the definition of $\mathcal{H}$ would quickly turn the precise

[1] One can check that for $\mathcal{H}_4$ be a nonempty set, one could also have chosen $d = 2, n \geq 6$ or $d = 3, n \geq 4$. However, we have already seen an example for $d = 2$; and while for $d = 3, n = 4$ we also have $m = 34$, in this example, $\mathcal{H}_4$ then contains more than just a single element. Hence, the given example also illustrates Theorem 6.1.19 below.

determination of $|\mathcal{H}|$ into a combinatorial nightmare, which would also serve no further purpose for the theoretical results of this work.

However, we conjecture that in general, the quotient $\frac{|\mathcal{H}|}{|\mathcal{H}_2|}$ is given by a polynomial function involving n and d and hence can be arbitrarily large. Recall that according to Section 4.4, this suggests that, apart from the obvious issue of the large number m of variables which we expect to be a problem for any computer algebra system, our algorithm to actually compute Betti numbers with a computer might not be optimal here.

Nevertheless, we have constructed our Pommaret basis with the goal of obtaining theoretical results, and for this fact it is indeed very useful. As a first application, we will in Corollary 6.1.17 see that this Pommaret basis immediately yields formulas for the projective dimension and the regularity of $S^{(d)}$.

Theorem 6.1.13. *$\mathcal{H}_2$ is a quadratic minimal Gröbner basis for $I(Y)$ (with respect to the degrevlex order as defined above).*

Proof. We need to show that $\operatorname{lt}(\mathcal{H}_t) \subseteq \operatorname{lt}(\mathcal{H}_2)$ for $t \geq 3$. So let $t \geq 3$ and

$$\mathbf{h}_t = x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t} - \operatorname{MinMon}\big(\sum_{j=1}^{t} \mathbf{d}_j\big) \in \mathcal{H}_t.$$

By definition, we have

$$x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t} \neq \operatorname{MinMon}\big(\sum_{j=1}^{t} \mathbf{d}_j\big),$$

but

$$x_{\mathbf{d}_2} \cdots x_{\mathbf{d}_t} = \operatorname{MinMon}\big(\sum_{j=2}^{t} \mathbf{d}_j\big).$$

If we can show that there is an $i \geq 2$ with

$$x_{\mathbf{d}_1} x_{\mathbf{d}_i} \neq \operatorname{MinMon}(\mathbf{d}_1 + \mathbf{d}_i),$$

the statement follows from recursion with respect to the monomial order, since then the leading term of $\mathbf{h}_t$ is divisible (but not necessarily involutively divisible) by the leading term of

$$\mathbf{h}_2 = x_{\mathbf{d}_1} x_{\mathbf{d}_i} \neq \operatorname{MinMon}(\mathbf{d}_1 + \mathbf{d}_i).$$

So what is a suitable choice of i? From the definition of the monomial order on $\mathcal{P}$, we see that there has to be an index $1 \leq k \leq n$ with $(\mathbf{d}_1)_k \geq 1$ (i.e. the k-th entry of $\mathbf{d}_1$ is greater than 0) and an index $0 \leq l < k$ with $(\sum_{j=2}^{t} \mathbf{d}_j)_l > 0$: For otherwise $\mathbf{d}_1$ is the unique minimal multiindex which divides $\sum_{j=1}^{t} \mathbf{d}_j$. Then Lemma 6.1.5 implies

$$\operatorname{MinMon}\big(\sum_{j=1}^{t} \mathbf{d}_j\big) = x_{\mathbf{d}_1}\big(\sum_{j=2}^{t} \mathbf{d}_j\big) = x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t},$$

and hence $\mathbf{h}_t = 0 \notin \mathcal{H}_t$, a contradiction. So we can pick a $\mathbf{d}_i$ with $2 \le i \le t$ with $(\mathbf{d}_i)_l \ge 1$ and for this choice of i, we have

$$x_{\mathbf{d}_1} x_{\mathbf{d}_i} \succ x_{\mathbf{d}_1+\mathbf{1}_l-\mathbf{1}_k} x_{\mathbf{d}_i-\mathbf{1}_l+\mathbf{1}_k} \succeq \mathrm{MinMon}(\mathbf{d}_1+\mathbf{d}_i),$$

so indeed

$$x_{\mathbf{d}_1} x_{\mathbf{d}_i} \ne \mathrm{MinMon}(\mathbf{d}_1+\mathbf{d}_i).$$

□

Definition 6.1.14. For any integer q with $1 \le q \le \frac{(d-1)(n+1)}{d}$, let $\mathbf{d}(q)$ be the unique multiindex of degree $q \cdot d$ and of shape

$$\mathbf{d}(q) = (0, ..., 0, \mathbf{d}(q)_{s_q}, d-1, ..., d-1)$$

such that $1 \le \mathbf{d}(q)_{s_q} \le d-1$. We define s_q and $\mathbf{d}(q)_{s_q}$ to be the integers that are uniquely determined by this multiindex $\mathbf{d}(q)$. We denote by r_q the maximal integer that is strictly smaller than $\frac{qd}{d-1}$, i.e.[1]

$$r_q = \left\lfloor \frac{qd-1}{d-1} \right\rfloor.$$

For any $q \le \frac{(d-1)(n+1)}{d}$, we define multiindices $\mathbf{d}^1, ..., \mathbf{d}^q$ by

$$\mathbf{d}^1 = (0, ..., 1, d-1)$$

and

$$\mathbf{d}^i = \mathbf{d}(i) - \sum_{j=1}^{i-1} \mathbf{d}^j.$$

For $q \ge 2$, we define $\mathbf{d}^{q,+}$ to be the multiindex of degree d such that $x_{\mathbf{d}^{q,+}}$ is the successor of $x_{\mathbf{d}^q}$ (with respect to the degrevlex order of Assumption 6.1.1).

Remark 6.1.15. By construction of $\mathbf{d}^q$, for any $q \le \frac{(d-1)(n+1)}{d}$, we immediately have

$$\mathbf{d}^q = \mathbf{d}(q) - \mathbf{d}(q-1).$$

Any $\mathbf{d}^q$ is a multiindex in $\mathbb{N}^{n+1}$ of degree d. Obviously, we have

$$n = s_q + r_q$$

and

$$\mathbf{d}(q)_{s_q} = qd - r_q(d-1).$$

Additionally, we have

$$\mathrm{MinMon}\big(\mathbf{d}(q)\big) = x_{\mathbf{d}_q} \cdots x_{\mathbf{d}_1}$$

and

$$\mathbf{d}^q = \mathrm{cls}\Big(\mathrm{MinMon}\big(\mathbf{d}(q)\big)\Big).$$

[1]For a real number $z \in \mathbb{R}$, we denote by $\lfloor z \rfloor = \max\{a \in \mathbb{Z} \mid a \le z\}$ the lower Gauss bracket of z.

Example 6.1.16. For $n = 5, d = 4, q = 4$, we have

$$\mathbf{d}(q) = (1, 3, 3, 3, 3, 3), \quad s_q = 0, \quad \mathbf{d}(q)_{s_q} = 1$$

and

$$\mathbf{d}^1 = (0, 0, 0, 0, 1, 3), \quad \mathbf{d}^2 = (0, 0, 0, 2, 2, 0),$$
$$\mathbf{d}^3 = (0, 0, 3, 1, 0, 0), \quad \mathbf{d}^4 = (1, 3, 0, 0, 0, 0).$$

For the multiindices $\mathbf{d}^{q,+}$, we indeed see that $\mathbf{d}^1$ has no successor, so $\mathbf{d}^{1,+}$ cannot be defined in the same manner as for $q \geq 2$. For $2 \leq q \leq 4$, we obtain

$$\mathbf{d}^{2,+} = (0, 0, 0, 2, 1, 1), \quad \mathbf{d}^{3,+} = (0, 0, 3, 0, 1, 0), \quad \mathbf{d}^{4,+} = (1, 2, 1, 0, 0, 0).$$

Now we can give another proof for the well-known property that Veronese subrings are Cohen-Macaulay, cf. [GM14, Theorem 3.5.] or [Pau13, Prop. 9]. We also have a formula for the regularity.

Corollary 6.1.17. *The ring $S^{(d)} \cong \mathcal{P}/I(Y)$ is a Cohen-Macaulay ring. Its regularity is*

$$\operatorname{reg}\big(P/I(Y)\big) = \left\lfloor \frac{(d-1)(n+1)}{d} \right\rfloor = n + 1 + \left\lfloor \frac{-n-1}{d} \right\rfloor$$

and its projective dimension is $m - n = \binom{n+d}{d} - n$.

Proof. From the proof of Theorem 6.1.9, we know that $\mathcal{H}_t = \emptyset$ as soon as $t > r$. However, $\mathcal{H}_r$ contains $x^2_{\mathbf{d}^{r-1}} x_{\mathbf{d}^r} \cdots x_{\mathbf{d}^1} - \operatorname{MinMon}\big(\mathbf{d}^{r-1} + \mathbf{d}(r-1)\big)$. Using Theorem 2.3.45, we see that $\operatorname{reg}\big(\mathcal{P}/I(Y)\big) = r$.

Additionally, $\mathcal{H}$ contains

$$x^2_{(d-1,1,0,\ldots,0)} - x_{(d,0,\ldots,0)} x_{(d-2,2,0,\ldots,0)},$$

but no elements of smaller class, implying $\operatorname{depth}(\mathcal{P}/I) = n$ by Theorem 2.3.39. But then since $(\langle\mathcal{H}\rangle)_r$ is obviously δ-regular, $\operatorname{lt}\big((\langle\mathcal{H}\rangle)_r\big)$ is stable, implying $\big(\mathbb{k}[\{x_{\mathbf{d}} \mid \mathbf{d} \succ_{\text{lex}} (d_1, 1, 0, \ldots, 0)\}]\big)_r = \operatorname{lt}\langle\mathcal{H}\rangle_r$. However, by construction $\mathcal{H}_r$ obviously contains no leading term of shape $x^a_{(0,\ldots,0,d)}$, and therefore $\dim\big(\mathcal{P}/I(Y)\big) = n$ by Theorem 2.3.41. Now we immediately see from Theorem 2.3.41 that

$$n = \dim\big(\mathcal{P}/I(Y)\big) = \operatorname{depth}\big(\mathcal{P}/I(Y)\big),$$

so $\mathcal{P}/I(Y)$ is indeed Cohen-Macaulay.

The statement about projective dimension now follows from Theorem 2.3.43, or the Auslander-Buchsbaum formula, see Corollary 2.3.44. □

Example 6.1.18. Now we give a more refined algorithm for the construction of $\mathcal{H}$. Using this algorithm, we can show that for $n = 3, d = 4$, the only element of $\mathcal{H}_4$ is given by

$$x^2_{(3,1,0,0)} x_{(0,2,2,0)} x_{(0,0,1,3)} - \operatorname{MinMon}\big((6, 4, 3, 3)\big) \in \mathcal{H}_4,$$

as suggested in Example 6.1.11.

The idea is as follows: Looking at the conditions defining the sets

$$\mathcal{H}_t = \Big\{ x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t} - \mathrm{MinMon}\big(\sum_{j=1}^{t} \mathbf{d}_j\big) \mid x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t} \neq \mathrm{MinMon}\,\big(\sum_{j=1}^{t} \mathbf{d}_j\big),$$
$$x_{\mathbf{d}_2} \cdots x_{\mathbf{d}_t} = \mathrm{MinMon}\,\big(\sum_{j=2}^{t} \mathbf{d}_j\big), \mathbf{d}_1 \notin \{(d,0,...,0),...,(0,...,0,d)\} \Big\},$$

we see that for an element of $\mathcal{H}_t$, the condition $\mathbf{d}_1 \notin \{(d,0,...,0),...,(0,...,0,d)\}$ is satisfied. Recalling our general assumption $x_{\mathbf{d}_1} \preceq ... \preceq x_{\mathbf{d}_t}$, we see that implies that the multiindex $\sum_{j=2}^{t} \mathbf{d}_j$ cannot have an entry $\geq d$. Contrary, it is not difficult to see that for any multiindex $\mathbf{e}$ of degree $d \cdot (t-1)$ with $x_{\mathbf{e}_2} \cdots x_{\mathbf{e}_t} = \mathrm{MinMon}(\mathbf{e})$, and any $\mathbf{d}$ such that $x_{(0,...,d)} \prec x_{(d-1,1,0,...,0)} \preceq x_{\mathbf{d}} \preceq x_{\mathbf{e}_2}$, we have

$$x_{\mathbf{d}} x_{\mathbf{e}_2} \cdots x_{\mathbf{e}_t} - \mathrm{MinMon}(\mathbf{d}+\mathbf{e}) \in \mathcal{H}_t.$$

So another idea to construct $\mathcal{H}_t$ is to look at all such multiindices $\mathbf{e}$ of degree $d \cdot (t-1)$ whose entries are less or equal to $d-1$, and construct the generators in the manner described above. This algorithm is more efficient than the one given in Example 6.1.11, as here the number of multiindices that we actually need to do calculations for can generally be expected to be much smaller: In the first naive algorithm, we had to consider all multiindices of degree $d \cdot t$ without any restriction to the indices.

Going back to the special case $n = 3, d = 4$, for $t = 4$ there is only one multiindex of degree $d \cdot (t-1) = 4 \cdot 3 = 12$ for which no entry is larger than $d = 4$, namely $\mathbf{d}(3) = (3,3,3,3)$. Its minimal monomial can now be described as

$$x_{\mathbf{d}^3} x_{\mathbf{d}^2} x_{\mathbf{d}^1} = x_{(3,1,0,0)} x_{(0,2,2,0)} x_{(0,0,1,3)} = \mathrm{MinMon}\big((3,3,3,3)\big) = \mathrm{MinMon}\big(\mathbf{d}(3)\big).$$

Now we need to find all $\mathbf{d}$ which satisfy

$$x_{(0,0,0,4)} \prec x_{(3,1,0,0)} \preceq x_{\mathbf{d}} \preceq x_{(3,1,0,0)}.$$

So $\mathbf{d} = (3,1,0,0)$ is the only eligible monomial, and we indeed obtain that

$$x_{(3,1,0,0)}^2 x_{(0,2,2,0)} x_{(0,0,1,3)} - \mathrm{MinMon}\big((6,4,3,3)\big)$$

is the only element of $\mathcal{H}_4$.

Generalizing this idea, whenever $n+1 = a \cdot d$ for some $a \in \mathbb{N}$, we have for $t = \frac{(d-1)(n+1)}{d} = a(d-1) = \mathrm{reg}(\mathcal{P}/I)$, the set $\mathcal{H}_t$ contains only a single element,

$$x_{\mathbf{d}^{t-1}}^2 x_{\mathbf{d}^{t-2}} \cdots x_{\mathbf{d}^1} - \mathrm{MinMon}\big((2d-2, d, d-1, ..., d-1)\big).$$

As for $t > \frac{(d-1)(n+1)}{d} = a(d-1)$, we know that $\mathcal{H}_t = \emptyset$, we also obtain that this generator the only element of $\mathcal{H}$ of both maximal degree and minimal class among all elements of $\mathcal{H}$, and hence defines a non-vanishing generator. All in all, we can now proof the theorem below

Theorem 6.1.19. *Let $n+1 = a \cdot d$ for some integer $a \geq 1$. Then*

$$\beta_{\mathrm{pd}(\mathcal{P}/I),\mathrm{reg}(\mathcal{P}/I)}\big(I(Y)\big) = 1,$$

i.e. the bottom right entry of the Betti Diagram of $\mathcal{P}/I$ is 1.

Proof. Essentially, we could repeat the arguments of the proof of Theorem 2.3.45 in [Sei10, Theorem 5.5.15]. A slightly different is to use the idea of the minimisation product translates to the Betti diagram as explained in Section 4.5:

We have just explained in Remark 6.1.18 that $\mathcal{H}_{\mathrm{reg}(\mathcal{P}/I)}$ contains only a single element, for which we also know by Assumption 6.0.1 and Corollary 6.1.17 that is of minimal class among all elements of $\mathcal{H}$. This means that the pseudo Betti table of $\mathcal{P}/I$ is of shape

	0	1	...	reg($\mathcal{P}/I$)
$*$	$*$	$*$	...	$*$
pd($\mathcal{P}/I$)	0	$*$	...	1

with all other entries further right or further below being 0. As we have explained in Section 4.5, the Betti diagram arises from the Pseudo Betti diagram by subtracting some diagrams of shape

...	0	1	0
0	1	0	...

while have to keep in mind that all entries have to be at least 0. But this trivially implies that the 1 in the bottom right corner of the diagram remains unchanged by any such subtraction. □

6.2 Some non-vanishing syzygies of Veronese subrings

Remark 6.2.1. It is obvious that $x_{\mathbf{d}^q} \cdots x_{\mathbf{d}^1} = \mathrm{MinMon}\big(\mathbf{d}(q)\big)$. So our Pommaret basis $\mathcal{H}$ contains elements of form

$$x^2_{\mathbf{d}^q} x_{\mathbf{d}^{q-1}} \cdots x_{\mathbf{d}^1} - \mathrm{MinMon}\left(\mathbf{d}^q + \mathbf{d}(q)\right).$$

Now, for a fixed $q \leq \frac{(d-1)(n+1)}{d}$, we look at the basis element

$$\mathbf{h} = \mathbf{v}_{\mathbf{d}^q,+,\ldots,(0,\ldots,1,q-1)}\left(x^2_{\mathbf{d}^q} x_{\mathbf{d}^{q-1}} \cdots x_{\mathbf{d}^1} - \mathrm{MinMon}\left(\mathbf{d}^q + \mathbf{d}(q)\right)\right).$$

Our goal is to show that these generators survive all minimisations that occur in the process of minimising the chain complex $\mathcal{G}$. Recall lemma 2.2.5, which states that, if after performing a minimisation, $\mathbf{h}$ appears with constant coefficient in the differential of some $\mathbf{g}$, then this must have been the case even before the minimisation (though not necessarily for the same $\mathbf{g}$); and in the same way, if the differential of $\mathbf{h}$ contains some constant after a minimisation, then this differential also was containing a constant before the minimisation. So in order to make sure that $\mathbf{h}$ is not removed during the entire minimisation process, it is sufficient to show that for the differential $d_{\mathcal{G}}$ at the start of the minimisation process, the following two conditions hold:

- The $\mathcal{G}$-differential of $\mathbf{h}$ does not contain any constant.
- $\mathbf{h}$ does not appear with a constant coefficient in the $\mathcal{G}$-differential of some other generator.

We will begin with the second point, as it is is easier to prove:

Lemma 6.2.2.

$$\mathbf{h} = \mathbf{v}_{\mathbf{d}^{q,+},...,(0,....,1,q-1)}\Big(x_{\mathbf{d}^q}^2 x_{\mathbf{d}^{q-1}} \cdots x_{\mathbf{d}^1} - \mathrm{MinMon}\big(\mathbf{d}^q + \mathbf{d}(q)\big)\Big)$$

does not appear with a constant coefficient in the $\mathcal{G}$-differential of any other generator.

Proof. The statement follows from an argument involving classes: We need to check that $\mathbf{h}$ does not appear in the differential of some $\mathbf{g}$ with a constant coefficient, with $\mathbf{g}$ being a generator whose homological degree is 1 larger than that of $\mathbf{h}$. As this means that for $\mathbf{g} = \mathbf{v_k h}_\alpha$, the index set $\mathbf{k}$ needs to contain one more element than $(\mathbf{d}^{q,+}, ..., (0, ..., 1, q-1))$, the index set belonging to $\mathbf{h}$. Any element of $\mathbf{k}$ is non-multiplicative for $\mathbf{h}_\alpha$ by construction of the complex $\mathcal{G}$. This is only possible if $\mathrm{cls}(\mathbf{h}_\alpha) < \mathbf{d}^q$, for otherwise there would not even be $|\mathbf{k}|$ non-multiplicative variables for $\mathbf{h}_\alpha$. By lemma 5.1.8 (see also the remarks following said lemma), $\mathbf{h}$ can appear with a constant coefficient in the differential of $\mathbf{g}$ only if $x_{\mathbf{d}^q}^2 x_{\mathbf{d}^{q-1}} \cdots x_{\mathbf{d}^1} - \mathrm{MinMon}\big(\mathbf{d}^q + \mathbf{d}(q)\big)$ appears as a summand (with a constant coefficient) in the involutive standard representation of some $x_{\mathbf{d}_s}\mathbf{h}_\alpha$, where $\mathrm{cls}(\mathbf{h}_\alpha) < \mathbf{d}_s \leq \mathbf{d}^q$. But then the inequality

$$\mathrm{cls}(x_{\mathbf{d}_s}\mathbf{h}_\alpha) = \mathrm{cls}(\mathbf{h}_\alpha) < \mathbf{d}^q = \mathrm{cls}\Big(x_{\mathbf{d}^q}^2 x_{\mathbf{d}^{q-1}} \cdots x_{\mathbf{d}^1} - \mathrm{MinMon}\big(\mathbf{d}^q + \mathbf{d}(q)\big)\Big) = \mathrm{cls}\,\mathbf{h}$$

makes this is impossible. □

Now we consider the first point given in remark 6.2.1:

Lemma 6.2.3. *The $\mathcal{G}$-differential of*

$$\mathbf{h} = \mathbf{v}_{\mathbf{d}^{q,+},...,(0,....,1,q-1)}\Big(x_{\mathbf{d}^q}^2 x_{\mathbf{d}^{q-1}} \cdots x_{\mathbf{d}^1} - \mathrm{MinMon}\big(\mathbf{d}^q + \mathbf{d}(q)\big)\Big)$$

does not contain a constant.

Remark 6.2.4. Before we begin with the proof, we note in advance an idea which we will use in the proofs of both Lemma 6.2.3 and Lemma 6.2.5:

- To calculate the $\mathcal{G}$-differential of $\mathbf{h}$, by equations (3.3.10) and (3.3.11), we first need to know the involutive standard representations of $x_{\mathbf{e}}\mathbf{h}$ for all $\mathbf{e} > \mathrm{cls}(\mathbf{h})$.
- From the construction of the Pommaret basis $\mathcal{H}$ in Theorem 6.1.9 and the monomial order given in Assumption 6.1.1, we immediately see that any variable of shape $x_{(0,...,d,...,0)}$ is multiplicative for any element of the Pommaret basis $\mathcal{H}$.

- From Theorem 5.1.4, we know that any summand of shape

$$\mathbf{v}_{\boldsymbol{\ell}}\left(x_{(0,\ldots,d,\ldots 0)}x^{\nu}\mathbf{h}'\right)$$

with $\mathbf{h}' \in \mathcal{H}$ does not contribute a constant to the differential.

- Now if at any point in the computation of the involutive standard representation of $x_{\mathbf{e}}\mathbf{h}$, we have a summand with a leading term of shape $x_{(0,\ldots,d,\ldots,0)}x^{\nu}\mathbf{h}$, we can ignore this term, for none of the basis elements of $\mathcal{G}$ arising from this generator contributes a constant to the differential. Additionally, remaining summands have a leading term smaller than $x_{(0,\ldots,d,\ldots,0)}x^{\nu}\mathbf{h}$. As we are using the monomial order defined in Assumption 6.1.1, this implies that any such leading term also has a factor of shape $x_{(0,\ldots,d,\ldots,0)}$. So we can ignore all of these summands thanks to same argument.

Using this idea, we will be left only with summands which share one special shape. For terms of these shape, Lemma 6.2.5 will ensure that, by recursion, we cannot obtain a constant from these summands either.

Proof. of lemma 6.2.3. We start by having a closer look at the involutive standard representations in question:

Let $\mathbf{e} = (0,\ldots.,0,\mathbf{e}_{s_q},\ldots,\mathbf{e}_j,0,\ldots,0) \succeq \mathbf{d}^{q,+} \succ \mathbf{d}^q$ be a multiindex with $\deg \mathbf{e} = d$ such that $\mathbf{e}_j > 0$. In particular, since $\mathbf{e} \succ \mathbf{d}^q$, $\mathbf{e}$ is not of shape $d \cdot \mathbf{1}_k$, and hence $j > s_q$. We claim

$$\big(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i\big)_j \geq d.$$

Separately, consider the two cases of equality and strong inequality:

$j = s_q + 1$: We have $\mathbf{e} > \mathbf{d}_q = (0,\ldots 0,\mathbf{d}(q)_{s_q}, d - \mathbf{d}(q)_{s_q}, 0,..,0)$. This implies $\mathbf{e}_{s_q} < \mathbf{d}(q)_{s_q}$ and $\mathbf{e}_j = \mathbf{e}_{s_q+1} > d - \mathbf{d}(q)_{s_q}$. We also have

$$\big(\sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i\big)_j = \big(\sum_{\mathbf{d}^i \succ \mathbf{d}^q} \mathbf{d}^i\big)_j = \big(\sum_{\mathbf{d}^i \succ \mathbf{d}^q} \mathbf{d}^i\big)_{s_q+1} = \mathbf{d}(q)_{s_q} - 1$$

and hence

$$\big(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i\big)_j = \mathbf{e}_{s_q+1} + \big(\sum_{\mathbf{d}^i \succ \mathbf{d}^q} \mathbf{d}^i\big)_{s_q+1} > d - \mathbf{d}(q)_{s_q} + \mathbf{d}(q)_{s_q} - 1 = d - 1.$$

$j > s_q + 1$: Since $\big(\sum_{\mathbf{d}^i \succ \mathbf{d}^q} \mathbf{d}^i\big)_{s_q+1} = \mathbf{d}(q)_{s_q} - 1$, we have $\big(\sum_{\mathbf{d}^i \succ \mathbf{d}^q} \mathbf{d}^i\big)_j = d - 1$ and hence

$$\big(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i\big)_j = \mathbf{e}_j + \big(\sum_{\mathbf{d}^i \succ \mathbf{d}^q} \mathbf{d}^i\big)_j \geq 1 + d - 1 = d.$$

So in either case $\left(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i\right)_j \geq d$ holds, which entails

$$\operatorname{MinMon}(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i) = x_{d\cdot\mathbf{1}_j}\operatorname{MinMon}\bigl(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i - d\cdot\mathbf{1}_j\bigr)$$

by lemma 6.1.5. Further, by definition 6.1.14 of $\mathbf{d}^q$ and s_q, we have $\mathbf{d}^q_{s_q+1}\geq 1$ and $\bigl(\mathbf{d}(q)\bigr)_{s_q+1} = d-1$. This implies

$$\bigl(\mathbf{d}^q+\mathbf{d}(q)\bigr)_{s_q+1} \geq 1+d-1 = d,$$

which again by lemma 6.1.5 gives

$$\operatorname{MinMon}\bigl(\mathbf{d}^q+\mathbf{d}(q)\bigr) = x_{d\cdot\mathbf{1}_{s_q+1}}\operatorname{MinMon}\bigl(\mathbf{d}^q+\mathbf{d}(q)-d\cdot\mathbf{1}_{s_q+1}\bigr).$$

Using these equations, we obtain

$$\begin{aligned}
&x_{\mathbf{e}}\Bigl(x^2_{\mathbf{d}^q}x_{\mathbf{d}^{q-1}}\cdots x_{\mathbf{d}^1} - \operatorname{MinMon}\left(\mathbf{d}^q+\mathbf{d}(q)\right)\Bigr)\\
&= x_{\mathbf{d}^q}\prod_{\mathbf{d}^i\prec\mathbf{e}} x_{\mathbf{d}^i}\bigl(x_{\mathbf{e}}\prod_{\mathbf{d}^i\succeq\mathbf{e}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i)\bigr)\\
&\quad + \bigl(x_{\mathbf{d}^q}\prod_{\mathbf{d}^i\prec\mathbf{e}} x_{\mathbf{d}^i}\operatorname{MinMon}(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i)\bigr) - \Bigl(x_{\mathbf{e}}\operatorname{MinMon}\left(\mathbf{d}^q+\mathbf{d}(q)\right)\Bigr)\\
&= x_{\mathbf{d}^q}\prod_{\mathbf{d}^i\prec\mathbf{e}} x_{\mathbf{d}^i}\Bigl(x_{\mathbf{e}}\prod_{\mathbf{d}^i\succeq\mathbf{e}} - x_{d\cdot\mathbf{1}_j}\operatorname{MinMon}\bigl(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i-d\cdot\mathbf{1}_j\bigr)\Bigr)\\
&\quad + x_{d\cdot\mathbf{1}_j}x_{\mathbf{d}^q}\prod_{\mathbf{d}^i\prec\mathbf{e}} x_{\mathbf{d}^i}\operatorname{MinMon}\bigl(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i-d\cdot\mathbf{1}_j\bigr)\\
&\quad - x_{d\cdot\mathbf{1}_{s_q+1}}x_{\mathbf{e}}\operatorname{MinMon}\bigl(\mathbf{d}^q+\mathbf{d}(q)-d\cdot\mathbf{1}_{s_q+1}\bigr)\\
&= x_{\mathbf{d}^q}\prod_{\mathbf{d}^i\prec\mathbf{e}} x_{\mathbf{d}^i}\bigl(x_{\mathbf{e}}\prod_{\mathbf{d}^i\succeq\mathbf{e}} x_{\mathbf{d}^i} - x_{d\cdot\mathbf{1}_j}\operatorname{MinMon}(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i-d\cdot\mathbf{1}_j)\bigr)\\
&\quad + x_{d\cdot\mathbf{1}_j}x_{\mathbf{d}^q}\prod_{\mathbf{d}^i\prec\mathbf{e}} x_{\mathbf{d}^i}\operatorname{MinMon}\bigl(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i-d\cdot\mathbf{1}_j\bigr)\\
&\quad - x_{d\cdot\mathbf{1}_{s_q+1}}x_{\mathbf{e}}\operatorname{MinMon}\bigl(\mathbf{d}^q+\mathbf{d}(q)-d\cdot\mathbf{1}_{s_q+1}\bigr).
\end{aligned}$$

Looking at this last sum, the factor

$$x_{\mathbf{e}}\prod_{\mathbf{d}^i\succeq\mathbf{e}} x_{\mathbf{d}^i} - x_{d\cdot\mathbf{1}_j}\operatorname{MinMon}\bigl(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i-d\cdot\mathbf{1}_j\bigr)$$

of the first summand is an element of $\mathcal{H}$, for by construction, $\prod_{\mathbf{d}^i\succeq\mathbf{e}} x_{\mathbf{d}^i}$ is indeed a minimal monomial (in fact, the first summand even is its own involutive standard representation). As an immediate consequence, the first summand is an element of $I(Y)$, and then so is the remainder of the last term in this chain of equations, i.e. the sum of the last and the second-to-last line.

But the leading monomial of the sum of these two lines (as long as this sum is not 0, in which case we do not need to consider it at all) has a factor of shape $x_{(0,\ldots,d,\ldots,0)}$; namely $x_{d\cdot\mathbf{1}_j}$ or $x_{d\cdot\mathbf{1}_{s_q}}$. So using remark 6.2.4, we cannot get a constant in the differential of this summand.

So we now ask the question if the summand

$$\mathbf{h}^{\mathbf{e}} = x_{\mathbf{d}^q} \prod_{\mathbf{d}^i \prec \mathbf{e}} x_{\mathbf{d}^i} \Big(x_{\mathbf{e}} \prod_{\mathbf{d}^i \succeq \mathbf{e}} x_{\mathbf{d}^i} - x_{d\cdot\mathbf{1}_j} \operatorname{MinMon}\big(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i - d\cdot\mathbf{1}_j\big)\Big)$$

can contribute constants to the differential. The technical lemma 6.2.5 below ensures that no constants can come from vertices of shape $\mathbf{v}_*\mathbf{h}^{\mathbf{e}}$ belonging to this generator, which will finish this proof. We state this lemma on its own, as this allows us to simplify the language and hence the more general statement, which includes all that is necessary to conclude this current proof, can be expressed in a more compact manner. □

Lemma 6.2.5. *Let* $\mathbf{e}$ *be a multiindex of degree* d *in* $\mathbb{N}^{n+1}$ *that is not of shape* $d\cdot\mathbf{1}_j$ *for all* $0 \le j \le n$. *Let*

$$\mathbf{v}_{\boldsymbol{\ell}}\Big(x^{\nu}\big(x_{\mathbf{e}} \prod_{\mathbf{d}^i \succeq \mathbf{e}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}_i)\big)\Big)$$

be a vertex, where x^{ν} *is a non-constant monomial*[1]. *If this vertex appears at some point in a reduction path* p *in the sum* (3.3.11)*, then* p *contains at least one elementary reduction path of type 1, i.e. the reduction map* ρ_p *is not constant.*

Proof. We are using induction with respect to the class of $x_{\mathbf{e}}$, starting with maximal class.

The assumption $\mathbf{e} \neq d\cdot\mathbf{1}_j$ assures that $x_{\mathbf{e}} \prod_{\mathbf{d}^i \succeq \mathbf{e}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}_i)$ is indeed an element of the Pommaret basis. Recall that by our conventions, the notation in the lemma implies that x^{ν} is multiplicative for

$$x^{\nu}\big(x_{\mathbf{e}} \prod_{\mathbf{d}^i \succeq \mathbf{e}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}_i)\big) \in \mathcal{H}.$$

We use induction over the class of $x_{\mathbf{e}}$:

If $x_{\mathbf{e}}$ has maximal class, we have $\mathbf{e} = \mathbf{d}^1 = (0, ..., 0, 1, d-1)$. In this case, the form of the element given in the lemma is necessarily

$$\mathbf{v}_{\boldsymbol{\ell}} x^{\nu}\big(x_{\mathbf{d}^1}^2 - \operatorname{MinMon}(2\mathbf{d}^1)\big),$$

as no other shapes are possible. Now the element $x_{\mathbf{d}^1}^2 - \operatorname{MinMon}(2\mathbf{d}^1)$ is of maximal class, and therefore there are no reduction paths of type 2 originating in this vertex by lemma 4.1.4 (see also the discussion preceding said lemma), because in this case any variable is multiplicative for the leading monomial $x_{\mathbf{d}^1}^2$.

[1] Recall that by our assumptions, the notation $x^{\nu}\big(x_{\mathbf{e}} \prod_{\mathbf{d}^i \succeq \mathbf{e}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}_i)\big)$ implies $\nu \in \operatorname{ncrit}\big(x_{\mathbf{e}} \prod_{\mathbf{d}^i \succeq \mathbf{e}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}_i)\big)$.

Now let $x_{\mathbf{e}}$ be not of maximal class. Let t be the number of monomials of type $\mathbf{d}^i$ (see definition 6.1.14) for which $\mathbf{d}^i \succeq \mathbf{e}$ holds. This means $\mathbf{d}^i \succeq \mathbf{e}$ if and only if $i \leq t$. By construction, $\mathbf{d}^t$ is of shape

$$\mathbf{d}^t = \big(0, ..., 0, \mathbf{d}(t)_{s_t}, d - \mathbf{d}(t)_{s_t}, 0, ..., 0\big)$$

with $1 \leq d - \mathbf{d}(t)_{s_t} \leq d-1$. Since $x_{\mathbf{d}^t} \succeq x_{\mathbf{e}}$ holds, for entries of $\mathbf{e}$, we have $\mathbf{e}_{j'} = 0$ for $j' < s_t$ and $\mathbf{d}(t)_{s_t} \leq \mathbf{e}_{s_t} \leq d-1$. Since $\mathbf{e}$ is a multiindex of total degree d, this implies that there is a $j_{\mathbf{e}} > s_t$ such that $\mathbf{e}_{j_{\mathbf{e}}} \geq 1$. We fix this index $j_{\mathbf{e}}$. Now consider the multiindex $\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i$. By remark 6.1.15, we have

$$\sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i = \sum_{i=1}^{t} \mathbf{d}^i = \mathbf{d}(t) = (0, ..., 0, \mathbf{d}(t)_{s_t}, d-1, ..., d-1).$$

But this implies that for the multiindex $\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i$, the $j_{\mathbf{e}}$-th entry must be greater or equal to d, and therefore by lemma 6.1.5, we have

$$\operatorname{MinMon}(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i) = x_{d \cdot \mathbf{1}_{j_{\mathbf{e}}}} \operatorname{MinMon}(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i - d \cdot \mathbf{1}_{j_{\mathbf{e}}}). \qquad (6.2.1)$$

By lemma 5.1.1, it suffices to consider elementary reduction paths of type 2 originating in the vertex given in the current lemma. Any such path is of shape

$$\mathbf{v}_{\boldsymbol{\ell}}\Big(x^{\nu}\big(x_{\mathbf{e}} \prod_{\mathbf{d}^i \succeq \mathbf{e}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i)\big)\Big)$$

$$\downarrow$$

$$\mathbf{v}_{\boldsymbol{\ell} \cup \mathbf{n}}\Big(\tfrac{x^{\nu}}{x_{\mathbf{n}}}\big(x_{\mathbf{e}} \prod_{\mathbf{d}^i \succeq \mathbf{e}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i)\big)\Big)$$

$$\downarrow$$

$$\ldots$$

where the omitted target comes from the involutive standard representation (of the $\mathcal{M}$-component) of

$$\mathbf{v}_{\boldsymbol{\ell} \setminus \mathbf{m} \cup \mathbf{n}} x_{\mathbf{m}}\Big(\frac{x^{\nu}}{x_{\mathbf{n}}}\big(x_{\mathbf{e}} \prod_{\mathbf{d}^i \succeq \mathbf{e}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{e} + \sum_{\mathbf{d}^i \succeq \mathbf{e}} \mathbf{d}^i)\big)\Big),$$

where by remark 3.3.5, we have[1] $\mathbf{m} \in \boldsymbol{\ell}$ and $\mathbf{n} = \operatorname{cls}(x^{\nu}) < \operatorname{cls}(\boldsymbol{\ell})$. In fact, this vertex is its own involutive standard representation if and only if $x_{\mathbf{m}}$ is multiplicative for $x_{\mathbf{e}} \prod_{\mathbf{d}^i \succeq \mathbf{e}} x_{\mathbf{d}^i}$. However, as in this case we have

$$\deg(\frac{x^{\nu}}{x_{\mathbf{n}}} x_{\mathbf{m}}) = \deg x^{\nu} \geq 1,$$

[1] Note that here $\boldsymbol{\ell}$ is actually a multiindex of multiindices, as the variables are enumerated multiindices.

the entire path cannot be a concatenation of exclusively such reduction paths where $x_{\mathbf{m}}$ is multiplicative for $x_{\mathbf{e}}\prod_{\mathbf{d}^i\succeq\mathbf{e}} x_{\mathbf{d}^i}$. Additionally, when $x_{\mathbf{m}}$ is multiplicative, if we choose $\boldsymbol{\ell}' = (\boldsymbol{\ell}\setminus\mathbf{m})\cup\mathbf{n}$ and $x^{\nu'} = \frac{x^{\nu}x_{\mathbf{n}}}{x_{\mathbf{m}}}$, the target of the path is again of the same form as the source of the path, yet it is smaller than the source of this elementary reduction path with respect to the ordering induced by the Morse matching. Therefore we can proceed by recursion. Without loss of generality, we can assume that $x_{\mathbf{m}}$ is non-multiplicative for $x_{\mathbf{e}}\prod_{\mathbf{d}^i\succeq\mathbf{e}} x_{\mathbf{d}^i}$.

Now we proceed by calculating the involutive standard representation of

$$x_{\mathbf{m}}\Big(\frac{x^{\nu}}{x_{\mathbf{n}}}\big(x_{\mathbf{e}}\prod_{\mathbf{d}^i\succeq\mathbf{e}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i)\big)\Big).$$

First we need to find the unique involutive divisor of the leading monomial

$$x_{\mathbf{m}}\Big(\frac{x^{\nu}}{x_{\mathbf{n}}}x_{\mathbf{e}}\prod_{\mathbf{d}^i\succeq\mathbf{e}} x_{\mathbf{d}^i}\Big).$$

From the definition of $\mathbf{d}^i$, we immediately see $\prod_{\mathbf{d}^i\succeq\mathbf{m}} x_{\mathbf{d}^i}$ is a minimal monomial, so the Pommaret basis contains the element

$$x_{\mathbf{m}}\prod_{\mathbf{d}^i\succeq\mathbf{m}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{m}+\sum_{\mathbf{d}^i\succeq\mathbf{m}}\mathbf{d}^i),$$

whose leading monomial is the involutive divisor we are looking for (note that we have $x_{\mathbf{m}}\succ x_{\mathbf{e}}$ as we are in the case where $\mathbf{m}$ is non-multiplicative).

So we have

$$\begin{aligned}
&\frac{x^{\nu}}{x_{\mathbf{n}}}x_{\mathbf{m}}\big((x_{\mathbf{e}}\prod_{\mathbf{d}^i\succeq\mathbf{e}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i)\big)\\
=&\frac{x^{\nu}}{x_{\mathbf{n}}}x_{\mathbf{e}}\prod_{\mathbf{e}\preceq\mathbf{d}^i\prec\mathbf{m}} x_{\mathbf{d}^i}\big(x_{\mathbf{m}}\prod_{\mathbf{d}^i\succeq\mathbf{m}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{m}+\sum_{\mathbf{d}^i\succeq\mathbf{m}}\mathbf{d}^i)\big)\\
&+\frac{x^{\nu}}{x_{\mathbf{n}}}x_{\mathbf{e}}\prod_{\mathbf{e}\preceq\mathbf{d}^i\prec\mathbf{m}} x_{\mathbf{d}^i}\operatorname{MinMon}(\mathbf{m}+\sum_{\mathbf{d}^i\succeq\mathbf{m}}\mathbf{d}^i) - \frac{x^{\nu}}{x_{\mathbf{n}}}x_{\mathbf{m}}\operatorname{MinMon}(\mathbf{e}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i)\\
=&\frac{x^{\nu}}{x_{\mathbf{n}}}x_{\mathbf{e}}\prod_{\mathbf{e}\preceq\mathbf{d}^i\prec\mathbf{m}} x_{\mathbf{d}^i}\big(x_{\mathbf{m}}\prod_{\mathbf{d}^i\succeq\mathbf{m}} x_{\mathbf{d}^i} - \operatorname{MinMon}(\mathbf{m}+\sum_{\mathbf{d}^i\succeq\mathbf{m}}\mathbf{d}^i)\big)\\
&+x_{d\cdot\mathbf{1}_{j_{\mathbf{m}}}}\frac{x^{\nu}}{x_{\mathbf{n}}}x_{\mathbf{e}}\prod_{\mathbf{e}\preceq\mathbf{d}^i\prec\mathbf{m}} x_{\mathbf{d}^i}\operatorname{MinMon}(\mathbf{m}-d\cdot\mathbf{1}_{j_{\mathbf{m}}}+\sum_{\mathbf{d}^i\succeq\mathbf{m}}\mathbf{d}^i)\\
&-x_{\mathbf{m}}\frac{x^{\nu}}{x_{\mathbf{n}}}x_{d\cdot\mathbf{1}_{j_{\mathbf{e}}}}\operatorname{MinMon}(\mathbf{e}-d\cdot\mathbf{1}_{j_{\mathbf{e}}}+\sum_{\mathbf{d}^i\succeq\mathbf{e}}\mathbf{d}^i),
\end{aligned}$$

where $j_{\mathbf{e}}$ is the index associated to the multiindex $\mathbf{e}$ earlier in this proof, and $j_{\mathbf{m}}$ is the index that is constructed in the analogous manner from the multiindex

$\mathbf{m}$. Regarding the last sum, we again see, as in the proof of Lemma 6.2.3, that the sum of the last two lines is an element of $I(Y)$, since we know that the first summand of the last sum is an element of $I(Y)$, and from the first line of the equations, so is the entire sum. Now once again, the arguments from remark 6.2.4 tell us that the last two summands cannot contribute any constants, i.e. no reduction path appearing in the sum (3.3.11) may contain a vertex $\mathbf{v}_*\mathbf{h}_*$ where $\mathbf{h}_*$ is either of those two summands.

For the first summand, the leading monomial of

$$x_{\mathbf{m}} \prod_{\mathbf{d}^i \succeq \mathbf{m}} x_{\mathbf{d}^i} - \mathrm{MinMon}(\mathbf{m} + \sum_{\mathbf{d}^i \succeq \mathbf{m}} \mathbf{d}^i)$$

is of class $\mathbf{m} > \mathbf{e}$, i.e. for this summand we can apply induction. This concludes the proof. $\square$

This lemma was all that was needed to finish the proof of Lemma 6.2.3. We have constructed a generator which survives any minimisation, and thus gives us a non-vanishing shifted Betti number $\beta'_{m-\mathrm{cls}(x_{\mathbf{d}^q})+1,q}$ (the $+1$ appears in this sum as we are constructing Betti numbers of $\mathcal{P}/I$, not I). Now we calculate the class of $\mathbf{d}^q$:

Theorem 6.2.6. *Let $S^{(d)} = \mathcal{P}/I(Y)$ and q be an integer with*

$$1 \le q \le \left\lfloor \frac{(d-1)(n+1)}{d} \right\rfloor = \operatorname{reg} S^{(d)}.$$

Then for every q, there is a non-vanishing Betti number

$$\beta'_{m-\mathrm{cls}(x_{\mathbf{d}^q})+1,q}(S^{(d)}) \neq 0, \qquad \text{where} \quad m-\mathrm{cls}(\mathbf{d}^q)+1 = \sum_{i=0}^{\mathbf{d}(q)_{s_q}} \binom{d-i+r_q-1}{d-i} - r_q.$$

Proof. We have

$$\begin{aligned}
& m - \mathrm{cls}(x_{\mathbf{d}^q}) + 1 \\
&= \#\{\mathbf{e} \mid \mathbf{e} \succ \mathbf{d}^q = (0, ..., 0, \mathbf{d}(q)_{s_q}, d - \mathbf{d}(q)_{s_q}, 0, ..., 0)\} + 1 \\
&= \#\Big(\bigcup_{i=0}^{\mathbf{d}(q)_{s_q}} \{\mathbf{e} \mid \mathbf{e} = (0, ..., 0, i, \mathbf{e}_{s_q+1}, ..., \mathbf{e}_n), \sum_{j=s_q+1}^{n} \mathbf{e}_j = d - i\} \\
&\qquad \setminus \{\mathbf{d}^q, d \cdot \mathbf{1}_{s_q+1}, ..., d \cdot \mathbf{1}_n\}\Big) + 1 \\
&= \Big(\sum_{i=0}^{\mathbf{d}(q)_{s_q}} \#\{\mathbf{e} \mid \mathbf{e} = (0, ..., 0, i, \mathbf{e}_{n-r_q+1}, ..., \mathbf{e}_n), \sum_{j=1}^{r_q} \mathbf{e}_{n-r_q+j} = d - i\}\Big) \\
&- \#\{\mathbf{d}^q, d \cdot \mathbf{1}_{s_q+1}, ..., d \cdot \mathbf{1}_n\} + 1 \\
&= \sum_{i=0}^{\mathbf{d}(q)_{s_q}} \binom{d-i+r_q-1}{d-i} - r_q - 1 + 1
\end{aligned}$$

with a simple combinatorial argument. □

Remark 6.2.7. If we have $d \geq q+1$ we immediately see that the numbers from Definition 6.1.14 simplify to $r_q = q$, $s_q = n - q$, $\mathbf{d}(q)_{s_q} = q$. Using basic identities of binomial coefficients, we obtain

$$m - \operatorname{cls}(\mathbf{d}^q) + 1 = \sum_{i=0}^{q} \binom{d-i+q-1}{d-i} - q = \binom{d+q}{q} - \binom{d-1}{q} - q,$$

which is the same bound[1] as in [EL12, Theorem 6.1 and 6.2]. There, the respective result is given for $d \geq q+1$ (see [EL12, Theorem 6.3]), while our results cover all values of d. Finally, we point out that the arguments in this chapter are independent of the characteristic of $\Bbbk$.

If we go back to the beginning of this chapter, we see that in terms of the Betti table, we have found non vanishing entries

	0	1	2	...						
0	1	–	–	...	?	?	...			
1	–	$\beta_{1,2} = \beta'_{1,1}$	$\beta'_{2,2}$	...						
2	–	?	?	...	?	$\beta'_{m-\operatorname{cls}(x_{\mathbf{d}^2})+1,2}$	...			
$\vdots$								$\ddots$		
r	–	?	?	...	?	?	...	?	$*$	...

marked with $*$. To precisely determine the question of the N_p property, we would have to prove that all entries marked with a red ? are in fact 0. But we also see that we have an upper bound for the N_p property, given by the non-vanishing shifted Betti number $\beta'_{m-\operatorname{cls}(x_{\mathbf{d}^2})+1,2}$. This leads to the following corollary:

$$m - \operatorname{cls}(x_{\mathbf{d}^2}) + 1 = \sum_{i=0}^{\mathbf{d}(2)_{s_2}} \binom{d-i+r_2-1}{d-i} - r_2$$

Corollary 6.2.8. *The Veronese subrings satisfy the N_p property for some*

$$p \leq m - \operatorname{cls}(x_{\mathbf{d}^2}) + 1 = \sum_{i=0}^{\mathbf{d}(2)_{s_2}} \binom{d-i+r_2-1}{d-i} - r_2.$$

6.3 Interpretation of the constructions of this chapter

Some of the constructions of this chapter may appear to be somewhat arbitrary at first glance. In this section, we explain the ideas behind the constructions earlier in this chapter and hint at where we see potential for generalizations.

[1] Note that our situation corresponds to $b = 0$ in the referenced paper.

6.3.1 The ideas behind the construction

While in most literature about Veronese subrings, explicit constructions of the ideal $I(Y)$ are usually done with goal of obtaining a generating system with square-free leading terms, such an approach would be counterproductive when one wants to use the Pommaret division as a tool: If we combine Theorem 2.3.42 and Corollary 6.1.17, we see that any Pommaret must contain exactly $m-n$ pure powers of variables as leading terms, which of course then have to be the largest variables for some elements of $\mathcal{H}$ with respect to the degrevlex-order (whose exact definition at this point of the thought process is yet to be made). As another consequence, the $n+1$ smallest variables cannot appear as classes of leading terms of the Pommaret basis. How should we pick these variables? There are $n+1$ special multiindices of degree d which are somewhat natural candidates, namely the multiindices of shape $(0,\dots,d,\dots,0) = d\cdot\mathbf{1}_j$. This motivates the part of definition of the degrevlex order in Assumption 6.1.1, which defines the variables of shape $x_{d\cdot\mathbf{1}_j}$ to be the smallest variables.

Now if we want to find a systematical way to construct all elements of a Pommaret basis, we start with the fact that $I(Y)$ can be generated by elements of shape $x^\nu - x^\pi$, where x^μ and x^π are monomials as in Lemma 6.1.7, i.e. we have

$$\mathbf{e} = \sum_{i=1}^{t}(\mu_i \cdot \mathbf{d}_i) = \sum_{i=1}^{t}(\pi_i \cdot \mathbf{e}_i) \in \mathbb{N}^{n+1}$$

for some multiindex $\mathbf{e}$. As we have explained in Example 6.1.11, we obtain such a generator by decomposing the multiindex $\mathbf{e}$ into different sums of multiindices. There are many such sums and for any choice of $x^\mu \succ_{\text{degrevlex}} x^\pi$, we have a $x^\mu - x^\pi \in I(Y)$. This leads to the question which indices one should chose. Every such x^μ, except for $\text{MinMon}(\mathbf{e})$, must appear as leading term in exactly one element of the Pommaret basis. It appears natural to pick the elements of shape $x^\mu - \text{MinMon}(\mathbf{e})$. But in fact, there is even another heuristic reason for this choice: We guarantee that the elements of the Pommaret basis are auto reduced[1], i.e. no summand in any element of $\mathcal{H}$ is involutively divisible by the leading term of another element of $\mathcal{H}$. So we have two intuitive reasons why elements of $\mathcal{H}$ should be of the shape given in Theorem 6.1.9.

Regarding the defining conditions of the sets $\mathcal{H}_t$ in Theorem 6.1.9, we now have given the reasons behind every condition except

$$x_{\mathbf{d}_2}\cdots x_{\mathbf{d}_t} = \text{MinMon}\Big(\sum_{j=2}^{t}\mathbf{d}_j\Big).$$

This condition does not so much come from the Veronese subrings, but it takes care of a basic property of involutive bases: If this condition is not satisfied, we have that $\mathcal{H}_{t-1}$ contains the element

$$\mathbf{h} = x_{\mathbf{d}_2}\cdots x_{\mathbf{d}_t} - \text{MinMon}\Big(\sum_{j=2}^{t}\mathbf{d}_j\Big).$$

[1]See Definition 2.3.14: This time we mean *auto reduced* as opposed to *head auto reduced*.

Given our general assumption $x_{\mathbf{d}_1} \preceq x_{\mathbf{d}_2}$, the leading term of $\mathbf{h}$ is an involutive divisor of the leading term of

$$x_{\mathbf{d}_1} \cdots x_{\mathbf{d}_t} - \mathrm{MinMon}\big(\sum_{j=1}^{t} \mathbf{d}_j\big)$$

and hence $\mathcal{H}$ would not be involutively (head) auto reduced. Additionally, this condition has another beneficial side effect: In combination with the fact that the variables $x_{d\cdot\mathbf{1}_j}$ are the smallest variables, it ensures that $\mathcal{H}$ is indeed finite: This combination implies that in the multiindex $\sum_{j=2}^{t} \mathbf{d}_j$, no entry can be larger than d, for otherwise one of the $\mathbf{d}_j$ would be of shape $d \cdot \mathbf{1}_i$, which is not allowed, as we have seen above. This idea is directly formalized in the proof of Theorem 6.1.9.

Going back to the degrevlex order defined in Assumption 6.1.1, we have to reason why we would sort variables of shape $x_{\mathbf{d}}$ with $\mathbf{d} \neq d \cdot \mathbf{1}_j$ in the given way. This part of the monomial order does not come into play until much later in the chapter: We need it for the construction of the non-vanishing generator of Remark 6.2.1. We recall that in our proofs, we had to invest quite a lot of computations to show that the differential of the given generator

$$\mathbf{h} = \mathbf{v}_{\mathbf{d}^q,+,\ldots,(0,\ldots,1,q-1)}\Big(x_{\mathbf{d}^q}^2 x_{\mathbf{d}^{q-1}} \cdots x_{\mathbf{d}^1} - \mathrm{MinMon}\big(\mathbf{d}^q + \mathbf{d}(q)\big)\Big)$$

does not contain a constant. The fact that it was not contained (with a constant coefficient) in the differential of another generator follows simply from a class argument, which is based on the fact that the index set of $\mathbf{v}_*$ contains **all** non-multiplicative variables. So regarding the differential of this generator it seems natural to choose a particular useful combination of non-multiplicative variables; since we are considering the Pommaret division, this implies that we want these variables to be the largest with respect to the degrevlex order. Now what is this useful choice?

In the process of calculating the differential of $\mathbf{h}$, we have to take the product of a $\mathbf{v}_*\mathbf{h}_\alpha$, we have to consider the product $x_{\mathbf{d}}\mathbf{h}_\alpha$ for all $\mathbf{d}$ in the index of $\mathbf{v}_*$. How can we guarantee that neither of these summands contributes a constant to the differential? Here we recall Theorem 5.1.4: In the involutive standard representations of $x_{\mathbf{d}}\mathbf{h}_\alpha$, any summand where a variable of shape $x_{d\cdot\mathbf{1}_j}$ appears will not contribute a constant to the differential. So under which circumstances will many, or even all, such involutive standard representations be of this shape? For a variable $x_{d\cdot\mathbf{1}_j}$ to appear in the involutive standard representation of some $x_{\mathbf{d}}(x^\mu - x^\pi)$, we expect it to be necessary that the j-th entry of the multiindex

$$\mathbf{d} + \sum_{i=1}^{t} (\mu_i \cdot \mathbf{d}_i)$$

is at least d. For this to happen, it would be a good start if the entries of

$$\sum_{i=1}^{t} (\mu_i \cdot \mathbf{d}_i)$$

are as large as possible. However, we still want the entries to be smaller than $d-1$, for we would like to be able to construct the Pommaret basis via the algorithm of Example 6.1.18. A candidate for such a multiindex is of shape

$$\mathbf{d}(q) = (0, ..., 0, \mathbf{d}(q)_{s_q}, d-1, ..., d-1),$$

see Definition 6.1.14. It seems also to be the case that this multiindex has the property that, if we count the number of different multiindices of degree d appearing in some decomposition of $\mathbf{d}(q)$ into sums of multiindices, then this number is probably minimal. Whenever we add $\mathbf{d}(q)$ to a multiindex $\mathbf{d}$ with an entry $\mathbf{d}_j \geq 1$ for a $j > s_q$, the j-th entry of the sum $\mathbf{d}(q) + \mathbf{d}$ is at least d. Obviously, for any $\mathbf{d}$ with $\mathbf{d} \succeq_{\text{degrevlex}} \mathbf{d}(q)$, this condition is satisfied. So we we want the conditions

- $x_{\mathbf{d}} \in \overline{\mathcal{X}}_{\prec_{\text{degrevlex}}, P}(\mathbf{h}_\alpha)$ and
- $\mathbf{d} \succeq_{\text{degrevlex}} \mathbf{d}(q)$

to be satisfied simultaneously. This suggests that we should indeed order the variables $x_{\mathbf{d}}$ as in Assumption 6.1.1. This is the final part of the idea behind the definition of our degrevlex-order.

Now we should also explain how we came to choose the generator

$$x_{\mathbf{d}^q}^2 x_{\mathbf{d}^{q-1}} \cdots x_{\mathbf{d}^1} - \text{MinMon}\big(\mathbf{d}^q + \mathbf{d}(q)\big)$$

appearing in $\mathbf{h}$. We have seen that the multiindex

$$\mathbf{d}(q) = \big(0, ..., 0, \mathbf{d}(q)_{s_q}, d-1, ..., d-1\big)$$

was appearing somewhat naturally along our line of thought. For every decomposition of this multiindex into multiindices of degree d, we obtain a candidate for an element of $I(Y)$. But we found that there is no clear answer to the question which one these decompositions is a good candidate to pick; furthermore, some candidates might actually fail to be elements of $\mathcal{H}$. But there is more natural way to construct an element of $\mathcal{H}$ from $\mathbf{d}(q)$: We consider the monomial $\text{MinMon}\,\big(\mathbf{d}(q)\big)$, and multiply it with its class, obtaining the generator

$$x_{\text{cls MinMon}\,(\mathbf{d}(q))} \cdot \text{MinMon}\,\big(\mathbf{d}(q)\big) - \text{MinMon}\,\big(\mathbf{d} + \mathbf{d}(q)\big).$$

A systematical way to describe this construction is by using the numbers of Definition 6.1.14, since

$$\text{MinMon}\,\big(\mathbf{d}(q)\big) = x_{\mathbf{d}_q} \cdots x_{\mathbf{d}_1}$$

and hence

$$\mathbf{d}^q = \text{cls}\,\Big(\,\text{MinMon}\,\big(\mathbf{d}(q)\big)\Big).$$

The generator

$$x_{\text{cls MinMon}\,(\mathbf{d}(q))} \cdot \text{MinMon}\,\big(\mathbf{d}(q)\big) - \text{MinMon}\,\big(\mathbf{d} + \mathbf{d}(q)\big)$$

is in fact nothing else than

$$x_{\mathbf{d}^q}^2 x_{\mathbf{d}^{q-1}} \cdots x_{\mathbf{d}^1} - \operatorname{MinMon}\big(\mathbf{d}^q + \mathbf{d}(q)\big).$$

The remainder of Section 6.2 shows that these heuristically justified constructions indeed have the intended behavior.

6.3.2 Possibilities for generalization

We have explained in Section 6.3.1 that behind the technical language introduced earlier in this chapter, there are combinatorial arguments which can be expressed purely by multiindices. Recall that it was our key idea to consider the multiindex

$$\mathbf{d}(q) = (0, ..., 0, \mathbf{d}(q)_{s_q}, d-1, ..., d-1).$$

Ignoring for the moment some correctional terms in the formulas, we have seen that

- the degree of elements of the Pommaret basis, and hence the total degree of the non-vanishing Betti numbers in Theorem 6.2.6 corresponds to the question if this multiindex can be decomposed into sums of multiindices such that the defining conditions of Theorem 6.1.9 are satisfied.
- the homological degree of the non-vanishing Betti numbers in Theorem 6.2.6 corresponds to the number of multiindices for which every entry is at most as large as the respective entry of $\mathbf{d}(q)$.

Of course, this idea leads to many new questions: If we exchange $\mathbf{d}(q)$ for another multiindex, is it possible to obtain other non-vanishing Betti numbers? Or even lower bounds larger than 1? It is likely that we need some adaptations to the monomial order of Assumption 6.1.1.

We also point out that so far we have been aiming to construct non-vanishing entries which are, as conjectured by [EL12] in their case $d \geq q+1$, possibly the leftmost non-vanishing entries in each row. As a corollary, we had bounds for the N_p-property. However, the question of the linear strand asks us to try to construct non-vanishing entries of the Betti table which are as much to right as possible (and the also to find the rightmost such entry). Thus, it could be beneficial to identify multiindices which can be decomposed into sums where as many multiindices as possible appear. Recall that for $\mathbf{d}(q)$, we suggest that is number might be minimal.

Since we were working with Veronese subrings, any multiindex of degree d needs to be considered. But what if we change the map

$$\nu_d \colon \mathbb{P}^n_{\Bbbk} \to \mathbb{P}^m_{\Bbbk},$$

by omitting some monomials? Of course, this means we are removing some multiindices from our constructions and therefore also possibly some decompositions of $\mathbf{d}(q)$. Do the purely combinatorial arguments remain valid if we simply add the condition that the multiindices in question need to be contained in a subset? On first glance, we consider this to be likely, but a more rigid investigation remains necessary. If the answer is yes, how does it effect our Pommaret basis and the formula for non-vanishing Betti numbers?

There are also other different generalizations of Veronese subrings, such as Veronese modules given by

$$S_{n,d,k} = \bigoplus_{i \geq 0} \mathbb{k}[x_0, ..., x_n]_{k+id}.$$

While our initial impression is that applying our ideas to the modules would require complicating our language, it still looks possible.

7 Green's Hyperplane Restriction Theorem

In this chapter, we aim to prove a part of Green's Hyperplane Restriction Theorem. The contents of this chapter are independent of Chapters 3 to 6. To state the theorem, we need the definition below.

Definition 7.0.1. Let $c, t > 0$ be two integers. By [Gre98, Theorem 3.1], there are unique integers $k_t, ..., t_\delta \geq \delta > 0$, such that

$$c = \binom{k_t}{t} + \binom{k_{t-1}}{t-1} + ... + \binom{k_\delta}{\delta}.$$

In this situation, we define

$$c_{<t>} = \binom{k_t - 1}{t} + \binom{k_{t-1} - 1}{t-1} + ... + \binom{k_\delta - 1}{\delta},$$

where $\binom{a}{b} = 0$ if $a < b$.

Now the full version of the Hyperplane Restriction Theorem can be formulated as follows.

Theorem 7.0.2. [Gre98, Theorem 3.4] *Let $I \trianglelefteq \mathcal{P}$ be an ideal with Hilbert function $\mathrm{HF}_{\mathcal{P}/I}(t)$. Let I_H be the restriction of I to a general hyperplane*[1]*, seen as an ideal in the polynomial ring $\mathcal{P}_H$. Then for any integer $t \geq 1$, we have*

$$\mathrm{HF}_{\mathcal{P}_H/I_H}(t) \leq \left(\mathrm{HF}_{\mathcal{P}/I}(t)\right)_{<t>}.$$

We will prove a weaker statement, which is given in Theorem 7.4.4; essentially we restrict ourselves to sufficiently large values of t.

In [Gre98], the theorem comes alongside a number of other theorems, for example the Persistence Theorem and the Regularity Theorem of Gotzmann, which follow the common question of examining the behavior (in particular the growth) of the Hilbert function and its relation to the Castelnuovo-Mumford-regularity. Green proves the Hyperplane Restriction theorem together with Theorem 7.0.3 below, by doing one large induction which covers both theorems. However, looking at the proof given [BH98, Theorem 4.2.10], Theorem 7.0.3 can also be understood to be a corollary of Theorem 7.0.2: If Theorem 7.0.2 holds for a degree t, then so does Theorem 7.0.3 for the same t. Hence, if we can prove Theorem 7.0.2 for sufficiently large values of t, then we also obtain a proof Theorem 7.0.3.

Theorem 7.0.3 (Macaulay's Estimate on the Growth of Ideals)**.** [Gre98, Proposition 3.5] *Let $I \trianglelefteq \mathcal{P}$ be an ideal. Then for any integer*[2] *$t \geq 1$,*

$$\mathrm{HF}_{\mathcal{P}/I}(t+1) \leq \mathrm{HF}_{\mathcal{P}/I}(t)^{<t>}.$$

[1] More precisely, we mean that there is an open subset of hyperplanes for which the statement holds, see Definition 2.3.29.

[2] Here the $c^{<t>}$ is defined in analogy to the $c_{<t>}$-notation of Definition 7.0.1, by increasing each k_t by one.

Definition 7.0.4. Let $I \trianglelefteq \mathcal{P}$ be an ideal. The *saturation of* I is the ideal

$$I^{\text{sat}} = \{f \in \mathcal{P} \mid \exists k \in \mathbb{N} : f \cdot \mathcal{P}_k \subseteq I\}.$$

I is *saturated* if and only if $I = I^{\text{sat}}$.

We revert back to our conventions stated in Assumption 2.1.4, while in particular discarding both the temporary Assumption 5.0.1 and Assumption 6.0.1 of the two previous chapters. However, we still consider only Pommaret bases.

Definition 7.0.5. Given an ideal $I \trianglelefteq \mathcal{P}$, we define $B_q(I) = \operatorname{lt}(I)_q \cap \mathbb{T}$ and

$$\beta_q^{(k)}(I) = |\{t \in B_q(I) | \operatorname{cls}(t) = k\}|.$$

The *β-vector* of I is

$$\beta_q(I) = \left(\beta_q^{(0)}(I), \ldots, \beta_q^{(n)}(I)\right).$$

Lemma 7.0.6. *Let $\mathcal{H} = \{\mathbf{h}_1, ..., \mathbf{h}_s\}$ be a Pommaret basis of the ideal $I \trianglelefteq \mathcal{P}$. For every $\mathbf{h}_\alpha \in \mathcal{H}$, let k_α be the number of multiplicative variables of $\mathbf{h}_\alpha$, i.e. $k_\alpha = \operatorname{cls}(\mathbf{h}_\alpha) + 1$. Then we have*

$$\begin{aligned}\beta_q^{(c)}(I) &= \sum_{\alpha=1}^{s} \binom{q - \deg(\mathbf{h}_\alpha) + k_\alpha - c - 2}{q - \deg(\mathbf{h}_\alpha) - 1} \\ &= \sum_{f \geq 0} \sum_{k=0}^{n} \binom{q - f + k - c - 2}{q - f - 1} \beta_{0,f}^{(k)},\end{aligned}$$

where the $\beta_{0,f}^{(k)}$ are the numbers from definition 2.3.63.

Proof. Essentially this is nothing else than a refinement of the representation of the Hilbert polynomial given in lemma 2.3.64: Instead of counting all elements of degree q, we count elements of degree q and class c. For any $\mathbf{h}_\alpha \in \mathcal{H}$ and a degree $q \geq \deg(\mathbf{h}_\alpha)$, picking a monomial x^μ with $\deg x^\mu = \deg(\mathbf{h}_\alpha) - q$ gives a basis vector $x^\mu \mathbf{h}_\alpha \in I_q$, and if we additionally require $x^\mu \in \Bbbk[\mathcal{X}_{P,\mathcal{H}}(\mathbf{h}_\alpha)]$, we obtain a basis of I_q by taking the union over all such monomials and all $\mathbf{h}_\alpha$. Now $\beta_q^{(c)}(I)$ is given by the number of basis elements $x^\mu \mathbf{h}_\alpha \in I_q$ of class c. To count those, we have to count the multiindices μ for which $\mu_c \geq 1$, $\mu_k = 0$ for $k < c$ and $k \geq \operatorname{cls}(\mathbf{h}_\alpha)$. This means that in the formulae of equation (2.3.6) in lemma 2.3.64, we have to replace t by $q - 1$, k_α by $k_\alpha - c$ and k by $k - c$. This gives the sums of the lemma. □

7.1 Lex segment ideals

In this section, we will introduce lex segment ideals, which are monomial ideals with additional "nice" properties. Later on in this chapter, we will link (saturated) lex segment ideals with various representations of the Hilbert polynomial. This kind of connection is by no means a new idea, and closely related constructions can be found for example in [Geh03] and its foundation [Ree92].

Definition 7.1.1.

- A set of monomials of degree q is called a *lex segment* (of x^μ) if it is given by
$$\operatorname{Lex}(x^\mu) = \left\{ x^\nu \in \mathbb{T} \mid \deg(x^\nu) = q,\ x^\mu \preceq_{\text{lex}} x^\nu \right\}$$
for some monomial x^μ of degree q. We also understand the empty set to be a lex segment.

- A $\Bbbk$-linear subspace V of $\mathcal{P}$ is called a *lex segment space* if $V \cap \mathbb{T}_d$ is a lex segment for all $d \in \mathbb{N}$.

- A monomial ideal I of $\mathcal{P}$ is called a *lex segment ideal* if I_d is a lex segment space for all $d \in \mathbb{N}$.

These definitions are in analogy to [KR05, Definitions 5.5.12 and 5.5.30]

Example 7.1.2. Let $P = \Bbbk[x_0, x_1, x_2]$. Then the lex segment of x_1^3 is given by

$$\operatorname{Lex}(x^\mu) = \{x_1^3, x_0^2 x_2, x_0 x_1 x_2, x_1^2 x_2, x_0 x_2^2, x_1 x_2^2, x_2^3\}.$$

Lemma 7.1.3. *Let $I \trianglelefteq \mathcal{P}$ be a lex segment ideal. Then I is stable (and therefore also quasi-stable). The unique monomial minimal generating system of I is a Pommaret basis.*

Proof. Let $x^\mu \in I$ with $\operatorname{cls}(x^\mu) = i$. Since I_d is a lex segment, then $x^{\mu - \mathbf{1}_i + \mathbf{1}_j} \in I$ for $j > i$, which is the unique involutive divisor of $x^{\mu + \mathbf{1}_j}$. Using lemma 2.3.19, we see that $\mathbb{T} \cap I_d$ is a Pommaret basis of $\langle I_d \rangle$, and then so is the monomial minimal generating system of I. □

Lemma 7.1.4. *Let $I \trianglelefteq \mathcal{P}$ be a lex segment ideal. If $I_d = \langle \operatorname{Lex}(x^\mu) \rangle$ for some d, we have $\langle \operatorname{Lex}(x_0 x^\mu) \rangle \subseteq I_{d+1}$, and equality holds if and only if I has no minimal generator in degree $d+1$.*

Proof. Let $x^\nu \in \operatorname{Lex}(x_0 x^\mu)$, so $\nu \succeq_{\text{lex}} \mu + \mathbf{1}_0$. We separately consider two cases:

$\operatorname{cls}(x^\nu) = 0$**:** Then we have $\nu_0 \geq 1$, so $\nu - \mathbf{1}_0 \succeq_{\text{lex}} \mu$. Then $x^{\nu - \mathbf{1}_0} \in I_d$ and therefore $x^\nu \in I_{d+1}$.

$\operatorname{cls}(x^\nu) = i > 0$**:** Here we have $\nu \succeq_{\text{lex}} \nu - \mathbf{1}_i + \mathbf{1}_0$. Now if $\nu - \mathbf{1}_i + \mathbf{1}_0 \succeq_{\text{lex}} \mu + \mathbf{1}_0$, we have $\nu - \mathbf{1}_i \succeq_{\text{lex}} \mu$ and therefore $x^\nu \in I_{d+1}$.

Now let I have no minimal generator in degree $d+1$, so $\mathcal{P} \cdot I_d = I_{d+1}$. For an $x^\nu \in I_{d+1}$, we have $x^\nu = x_i x^\kappa$ for some i and x^κ with $\kappa \succeq_{\text{lex}} \mu$. But then

$$\nu = \kappa + \mathbf{1}_i \succeq_{\text{lex}} \kappa + \mathbf{1}_0 \succeq_{\text{lex}} \mu + \mathbf{1}_0,$$

and therefore $x^\nu \in \operatorname{Lex}(x_0 x^\mu)$. □

7.2 Saturated lex segment ideals

Lemma 7.2.1. [Sei09, Corollary 10.2] *Let $\mathcal{H}$ be a Pommaret basis of an ideal $I \trianglelefteq \mathcal{P}$. Then I is saturated if and only if $\mathcal{H}$ contains no elements of class 0.*

Lemma 7.2.2. *A lex segment ideal $I \trianglelefteq \mathcal{P}$ is saturated if and only if*

$$I \cap \mathbb{T} = \{x^\nu \mid x^\nu \succeq_{\text{lex}} x^\mu\}$$

for some $x^\mu \in \mathbb{T}$ with $\text{cls}(x^\mu) > 0$.

Proof. Let $\mathcal{H}$ be a monomial Pommaret basis of a saturated lex segment ideal $I \trianglelefteq \mathcal{P}$. In particular, $\mathcal{H}$ is a minimal generating system of I by lemma 7.1.3. Now let x^μ be the element of $\mathcal{H}$ which is minimal with respect to the lex order among all elements of maximal degree in $\mathcal{H}$. We claim that for this x^μ, we have $I \cap \mathbb{T} = \{x^\nu \mid x^\nu \succeq_{\text{lex}} x^\mu\}$. Since I is saturated, we have $\text{cls}(x^\mu) > 0$ by lemma 7.2.1. Suppose there is an $x^\kappa \in I$ with $x^\kappa \prec_{\text{lex}} x^\mu$. Let $d = \deg x^\kappa$. We separately consider two cases:

$d \leq \deg(x^\mu)$: Let $\mu = \mu^1 + \mu^2$ such that $\deg x^{\mu^2} = d$ and such that there is an index i_μ such that[1] $\mu^1_i = 0$ for $i > i_\mu$ and $\mu^2_i = 0$ for $i < i_\mu$. So we have

$$\mu^1 = (\mu_0, ..., \mu_{i_\mu - 1}, \mu_{i_\mu} - a_\mu, 0, ..., 0)$$

and

$$\mu^2 = (0, ..., 0, a_\mu, \mu_{i_\mu+1}, ..., \mu_n)$$

where i_μ and a_μ are the unique numbers such that $d+1 = a_\mu + \sum_{j=i_\mu+1}^n \mu_j$ and $0 \leq a_\mu < \mu_{i_\mu}$ (if such numbers do not exist, then because of $d \leq \deg \mu$, we must have $\mu = \nu$, contradicting the assumption $x^\kappa \prec_{\text{lex}} x^\mu$). Since additionally we have $\deg x^{\mu^2} = d + 1 > d = \deg x^\kappa$, $x^\kappa \prec_{\text{lex}} x^\mu$ implies $x^\kappa \prec_{\text{lex}} x^{\mu^2}$. We have $x_0^{\deg(x^\mu)-d} x^\kappa \in I$. But now we obtain

$$\kappa + (\deg(x^\mu) - d) \cdot \mathbf{1}_0 \preceq_{\text{lex}} \kappa + \mu^1 \prec_{\text{lex}} \mu^2 + \mu^1 = \mu.$$

This contradicts the choice of x^μ.

$d > \deg(x^\mu)$: In analogy to the case $d \leq \deg(x^\mu)$, let $\kappa = \kappa^1 + \kappa^2$ such that $\deg x^{\kappa^2} = \deg(x^\mu)$ and such that there is an index i_κ such that $\kappa^{\mathbf{1}}_i = 0$ for $i > i_\kappa$ and $\kappa^2_i = 0$ for $i < i_\kappa$. So we have

$$\kappa^1 = (\kappa_0, ..., \kappa_{i_\kappa - 1}, \kappa_{i_\kappa} - a_\kappa, 0, ..., 0)$$

and

$$\kappa^2 = (0, ..., 0, a_\kappa, \kappa_{i_\kappa+1}, ..., \kappa_n)$$

where i_κ and a_κ are the unique numbers such that

$$\deg(x^\mu) = a_\kappa + \sum_{j=i_\kappa+1}^n \kappa_j$$

[1] μ^1_i refers to the i-entry of the multiindex μ^1.

and $0 \leq a_\kappa < \kappa_{i_\kappa}$ (such numbers do exist because of the assumption $d > \deg(x^\mu)$). Now by construction, κ^2 is an involutive divisor of κ, and since I is generated in degree $\deg(x^\mu) = d = \deg(x^{\kappa^2})$, we have $x^\kappa \in I$ if and only if $x^{\kappa^2} \in I$. So $x^{\kappa^2} \in I$. Now if $\kappa^2 \succeq \mu$, then also $\kappa \succ_{\text{lex}} \kappa^2 \succeq_{\text{lex}} \mu$, so we must have have $\kappa^2 \prec_{\text{lex}} \mu$, contradicting the definition of μ.

Now take I to be a monomial ideal with $I \cap \mathbb{T} = \{x^\nu \mid x^\nu \succeq_{\text{lex}} x^\mu\}$ for some $x^\mu \in \mathbb{T}$ with $\text{cls}(x^\mu) > 0$. By lemma 7.2.1, it suffices to show that if $x_0 x^\nu \in I$, we also have $x^\nu \in I$.

So let $x_0 x^\nu \in I$. Then we have $\nu + \mathbf{1}_0 \succeq_{\text{lex}} \mu$, so the last non-vanishing entry of $\mu - \nu - \mathbf{1}_0$ is negative. Since the entries of $\mu - \nu - \mathbf{1}_0$ and $\mu - \nu$ are the same, with the exception of the 0-th entry (belonging to x_0), we easily see that $\nu \succeq_{\text{lex}} \mu$ unless the 0-th entry of $\mu - \nu$ is positive. But because of $\text{cls}(x^\mu) > 0$, we have $\mu_0 = 0$, and so this is impossible and therefore $x^\nu \in I$. □

Now in the special case of an ideal for which $\text{lt}(I) \cap \mathbb{T} = \{x^\nu \mid x^\nu \succeq_{\text{lex}} x^\mu\}$ for an $x^\mu \in \mathbb{T}$ holds, it is possible to directly give a Stanley decomposition for the module $\mathcal{P}/I$, from which we then can read off the Hilbert polynomial of $\mathcal{P}/I$. In fact, we will even know slightly more about the Hilbert polynomial, as this decomposition will even give us the Gotzmann representation (see definition 7.3.1) of the Hilbert polynomial.

Lemma 7.2.3. *Let $I \trianglelefteq \mathcal{P}$ be a monomial ideal with*

$$\text{lt}(I) \cap \mathbb{T} = \{x^\nu \mid x^\nu \succeq_{\text{lex}} x^\mu\}$$

for some $x^\mu \in \mathbb{T}$. Then

$$\mathcal{P}/\text{lt}(I) \cong \bigoplus_{i=0}^{\mu_n - 1} \mathbb{k}[x_0, ... x_{n-1}] x_n^i \oplus \bigoplus_{i=0}^{\mu_{n-1}-1} \mathbb{k}[x_0, ..., x_{n-2}] x_{n-1}^i x_n^{\mu_n}$$
$$\oplus ... \oplus \bigoplus_{i=0}^{\mu_1 - 1} \mathbb{k}[x_0] x_1^i x_2^{\mu_2} \cdots x_n^{\mu_n} \oplus \bigoplus_{i=0}^{\mu_0 - 1} \mathbb{k} x_0^i x_1^{\mu_1} \cdots x_n^{\mu_n}$$

Proof. Note that a $\mathbb{k}$-basis of $\mathcal{P}/\text{lt}(I)$ is given by $\{x^\nu \in \mathbb{T} \mid x^\nu \prec_{\text{lex}} x^\mu\}$. The direct sum decomposition essentially does nothing more than partitioning those generators by the last non-vanishing entries of the difference $\mu - \nu$. It is obvious that no elements of this basis are missing, and that no redundant elements are added either. □

We explicitly state one corollary of this lemma. Due to lemma 7.2.2, the corollary does in particular hold for saturated lex segment ideals.

Corollary 7.2.4. *Let $I \trianglelefteq \mathcal{P}$ be a monomial ideal with*

$$\text{lt}(I) \cap \mathbb{T} = \{x^\nu \mid x^\nu \succeq_{\text{lex}} x^\mu\}$$

for some $x^\mu \in \mathbb{T}$. Then the regularity of $\mathcal{P}/I$ is given by $\text{reg}(P/I) = \deg \mu$.

Proof. From the decomposition of lemma 7.2.3, we immediately see that a Pommaret basis of $\mathrm{lt}(I)$ is given by

$$\langle x_n^{\mu_n+1}, x_n^{\mu_n}x_{n-1}^{\mu_{n-1}+1}, ..., x_n^{\mu_n}\cdots x_1^{\mu_1}x_0^{\mu_0+1}\rangle.$$

We now obtain the formula for the regularity by Theorem 2.3.45. □

Definition 7.2.5. For an integer $a \geq 0$, let

$$\binom{t+a}{a} = \frac{(t+a)(t+(a-1)\cdots(t+1)}{a!} \in \mathbb{Q}[t]$$

be a polynomial function. We understand $\binom{t+0}{0} = 1$.

Corollary 7.2.6. *Let $I \trianglelefteq \mathcal{P}$ be an ideal with $\mathrm{lt}(I)\cap\mathbb{T} = \{x^\nu \mid x^\nu \succeq_{\mathrm{lex}} x^\mu\}$ for some $x^\mu \in \mathbb{T}$. Then the Hilbert polynomial of $\mathcal{P}/I$ is given by*

$$\begin{aligned}\mathrm{HP}_{\mathcal{P}/I}(t) = &\sum_{i=0}^{\mu_n-1}\binom{(t-i)+(n-1)}{n-1} + \sum_{i=\mu_n}^{\mu_n+\mu_{n-1}-1}\binom{(t-i)+(n-2)}{n-2}\\ &+ ... + \sum_{i=\mu_n+...+\mu_1}^{\mu_n+...+\mu_2+\mu_1-1}\binom{(t-i)+1}{1} + \sum_{i=\mu_n+...+\mu_1}^{\mu_n+...+\mu_0-1}\binom{(t-i)+0}{0}\end{aligned}$$

Proof. If we restrict the decomposition of lemma 7.2.3 to terms of degree t with $t \geq \deg(x^\mu)$, we obtain

$$\begin{aligned}\left(\mathcal{P}/\mathrm{lt}(I)\right)_t \cong &\bigoplus_{i=0}^{\mu_n-1}\mathbb{k}[x_0,...x_{n-1}]_{t-i}\cdot x_n^i \oplus \bigoplus_{i=0}^{\mu_{n-1}-1}\mathbb{k}[x_0,...,x_{n-2}]_{t-\mu_n-i}.x_{n-1}^i x_n^{\mu_n}\\ &... \oplus \bigoplus_{i=0}^{\mu_1-1}\mathbb{k}[x_0]_{t-\mu_n-...-\mu_2-i}\cdot x_1^i x_2^{\mu_2}\cdots x_n^{\mu_n} \oplus \bigoplus_{i=0}^{\mu_0-1}\mathbb{k}x_0^i\cdot x_1^{\mu_1}\cdots x_n^{\mu_n}\end{aligned}$$

which gives

$$\begin{aligned}
&\mathrm{HP}_{\mathcal{P}/I}(t)\\
=\ &\mathrm{HP}_{\mathcal{P}/\operatorname{lt}(I)}(t)\\
=\ &\sum_{i=0}^{\mu_n-1}\binom{(t-i)+(n-1)}{n-1}+\sum_{i=0}^{\mu_{n-1}-1}\binom{(t-\mu_n-i)+(n-2)}{n-2}+...\\
&+\sum_{i=0}^{\mu_1-1}\binom{(t-\mu_n-....-\mu_2-i)+1}{1}+\sum_{i=0}^{\mu_0-1}\binom{(t-\mu_n-....-\mu_1-i)+0}{0}\\
=\ &\sum_{i=0}^{\mu_n-1}\binom{(t-i)+(n-1)}{n-1}+\sum_{i=\mu_n}^{\mu_n+\mu_{n-1}-1}\binom{(t-i)+(n-2)}{n-2}+...\\
&+\sum_{i=\mu_n+...+\mu_1}^{\mu_n+...+\mu_2+\mu_1-1}\binom{(t-i)+1}{1}+\sum_{i=\mu_n+...+\mu_1}^{\mu_n+...+\mu_0-1}\binom{(t-i)+0}{0}.
\end{aligned}$$

□

Note that if we understand $\binom{a}{b}=0$ if $a<b$, the proof ensures that the formula from lemma 7.2.6 gives the Hilbert function, and not just the Hilbert polynomial.

7.3 The Hyperplane Restriction Theorem

Definition 7.3.1. Let $f\in\mathbb{Q}[t]$ be a polynomial. If f can be written as

$$f=\binom{t+a_1}{a_1}+\binom{(t-1)+a_2}{a_2}+\cdots+\binom{\big(t-(g-1)\big)+a_g}{a_g}$$

with integers $a_1\geq a_2\geq ...\geq a_g\geq 0$, we say that f has a *Gotzmann representation* and we call

$$(a_1,a_2,...,a_g)\in\mathbb{N}^g$$

the *Gotzmann difference set* of f. For $i\geq 0$, we call $\mathfrak{a}_i=\mathfrak{a}_i(f)=\#\{a_j\mid a_j=i\}$ the *i-th Gotzmann coefficient of* f. The set

$$G(f)=(\mathfrak{a}_0,\mathfrak{a}_1,...,\mathfrak{a}_{a_1-1},\mathfrak{a}_{a_1},0,0,...)\in\mathbb{N}^{\mathbb{N}}$$

is called the *Gotzmann vector of* f. If $f=\mathrm{HP}_{\mathcal{P}/I}$ is the Hilbert polynomial of an ideal, we use the notation

$$G(I)=G\big(\mathrm{HP}_{\mathcal{P}/I}\big)$$

and call $G(I)$ the *Gotzmann vector of* I.

This notation is a minor adaptation of the notation used in [AGS09, Definition 2.10].

Obviously, the Gotzmann vector can be determined from the Gotzmann difference set and vice versa.

Lemma 7.3.2. *Let $\alpha = (a_0, \alpha_1, ..., \alpha_n) \in \mathbb{N}^{n+1}$ be a multiindex. Then the Gotzmann vector of* $\mathrm{HP}_{\langle \mathrm{Lex}(\alpha) \rangle}$ *is given by*

$$(\alpha_1, ..., \alpha_n, 0, ...).$$

In particular, a polynomial $f \in \mathbb{Q}[t]$ has a Gotzmann representation if and only if there is a lex segment ideal with Hilbert polynomial f.

Note that α_0 is irrelevant for the Gotzmann vector.

Proof. See (proof of) lemma 7.2.6. □

Lemma 7.3.3. *The Gotzmann representation is unique, provided it exists.*

Proof. This follows by recursion, using the fact that we must have $a_1 = \deg f$. □

We also state the relation between the Gotzmann vector and the $(\cdot)_{<t>}$-notation of Definition 7.0.1

Lemma 7.3.4. *Let the polynomial $f \in \mathbb{Q}[t]$ be given by the Gotzmann representation*

$$f = \binom{t + a_1}{a_1} + \binom{(t-1) + a_2}{a_2} + \cdots + \binom{\big(t - (g-1)\big) + a_g}{a_g}.$$

Then for any $q \in \mathbb{Z}$ with $q \geq g - 1$, we have

$$f(q)_{<q>} = \binom{t + a_1 - 1}{a_1 - 1} + \binom{(t-1) + a_2 - 1}{a_2 - 1} + \cdots + \binom{\big(t - (g-1)\big) + a_g - 1}{a_g - 1}.$$

In particular, if f is defined by the Gotzmann vector

$$(\mathfrak{a}_0, \mathfrak{a}_1, ..., \mathfrak{a}_{a_1 - 1}, \mathfrak{a}_{a_1}, 0, 0, ...) \in \mathbb{N}^{\mathbb{N}},$$

and $h \in \mathbb{Q}[t]$ is the polynomial defined by the Gotzmann vector

$$(\mathfrak{a}_1, ..., \mathfrak{a}_{a_1 - 1}, \mathfrak{a}_{a_1}, 0, 0, 0, ...) \in \mathbb{N}^{\mathbb{N}},$$

then we have that for any $q \geq g$ that

$$h(q) = f(q)_{<q>}.$$

Proof. Using the identity $\binom{t+a}{a} = \binom{t+a}{t}$ the statements follow immediately from the respective definitions. □

7.4 Connecting Pommaret bases and Lex segment ideals

Remark 7.4.1. Our next goal is to find a relation between a Pommaret basis $\mathcal{H}$ of an ideal $I \neq \{0\}$, and the Gotzmann representation of the Hilbert polynomial $\mathrm{HP}_{\mathcal{P}/I}$. As a byproduct, in the process we will also find a another proof for one part of Lemma 7.3.2; namely that the Hilbert polynomial of an arbitrary graded ideal has a Gotzmann representation. Our idea is to use the Pommaret basis $\mathcal{H}$ to construct a (saturated) lex segment ideal I' with the same Hilbert polynomial as I. For this ideal I', we will then use the decomposition of lemma 7.2.3 and the formula for the Hilbert polynomial $\mathrm{HP}_{P/\operatorname{lt}(I)}$ of corollary 7.2.6.

We deviate from the usual convention of first stating the precise lemma we want to prove, for the following reason: If one starts with a rather intuitive computation of the Hilbert polynomial, after some calculations, one naturally arrives at a point where the suitable (recursive) definition of the lex segment ideal I' is clearly visible. A concrete illustration of the calculation can be found in Example 7.4.3 below.

Without loss of generality, we can assume that the ideal I is δ-regular, i.e. it has a Pommaret basis. I' is quasi-stable by lemma 7.1.3 (provided it exists). Now if we look at the formulae from lemma 2.3.64, we see that the Hilbert polynomials $\mathrm{HP}_{\mathcal{P}/I}$ and $\mathrm{HP}_{\mathcal{P}/I'}$ of I and I' are identic if and only if the beta-vectors of I and I' are identic for large degrees. More specifically, we want the ideal I' to be a lex segment ideal with

$$I' \cap \mathbb{T} = \{x^\nu \mid x^\nu \succeq_{\mathrm{lex}} x^\mu\}$$

for an $x^\mu \in \mathbb{T}$, since for such ideals we can make use of corollary 7.2.6. Since I' is obviously generated by $I' \cap \mathbb{T}$, finding a suitable μ is a sufficient answer for the question of constructing I'. We also restrict ourselves to the case where $\dim(I) \neq 0$, as in this case the Hilbert function of I and I_H is 0 eventually, so the Gotzmann Hyperplane Restriction Theorem trivially holds in sufficiently large degrees.

We set $\beta_d^{(j)}(\mathcal{P}/I) = |\{x^\nu \in \mathbb{T}_d \mid x^\nu \notin \operatorname{lt}(I), \operatorname{cls} x^\nu = j\}|$, in a manner similar to the β-vector, see definition 7.0.5. In particular, we note

$$\sum_{j=0}^{n} \beta_d^{(j)} = \dim(P/I)_d.$$

We will construct the entries of μ recursively (starting with a maximal nonzero index μ_k and then working "backwards"). Assuming that we have have constructed $\mu_{k-(i-1)}, ..., \mu_k$, we will develop a formula for μ_{k-i} depending exclusively on $\mu_{k-(i-1)}, ..., \mu_k$ and the $\beta_d^{(j)}(\mathcal{P}/I)$. As a technical tool, we also define integers t_i depending on μ_{k-i}. The t_i correspond to degrees which we consider in the process.

- To start the recursion, let k be the maximal integer such that $\beta_{\operatorname{reg} I}^{(k)}(\mathcal{P}/I) \neq 0$. This choice is possible, as I is not a zero-dimensional ideal. Then we set $\mu_{k+1} = \beta_{\operatorname{reg} I}^{(k)}(\mathcal{P}/I)$ and $\mu_j = 0$ for $j > k+1$ (note that if $k = n$, we would have $\mathcal{P}/I \cong \mathcal{P}$ and therefore $I = \{0\}$). We also set $t_{-1} = \max\{\operatorname{reg} I, \mu_{k+1}\} + 1$.

With this choice of k, we have

$$\beta_t^{(k)}(\mathcal{P}/I) = \beta_{\operatorname{reg} I}^{(k)}(\mathcal{P}/I) = \mu_{k+1}$$

for $t \geq \operatorname{reg} I = \deg \mathcal{H}$, for the following reason: Considering I', for $t \geq \mu_{k+1}$, the monomials of degree t and class k which are not in I'_t, are

$$x_k^t, \quad x_k^{t-1}x_{k+1}, \quad ...., \quad x_k^{t-(\mu_{k+1}-1)}x_{k+1}^{\mu_{k+1}-1}.$$

Therefore $\beta_t^{(k)}(\mathcal{P}/I') = \mu_{k+1} = \beta_t^{(k)}(\mathcal{P}/I)$ as soon as

$$t \geq t_{-1} = \max\{\operatorname{reg} I, \mu_{k+1}\} + 1.$$

While the additional summand of 1 in the definition of t_{-1} may look unnatural at first glance, it is necessary to make our computations work..

- Now we want to define μ_{k-i} and t_i for $i \leq k-1$ such that for any choice of $\mu_0, ..., \mu_{k-i-1}$, the equality

  $$\beta_t^{(k-i-1)}(\mathcal{P}/I') = \beta_t^{(k-i-1)}(\mathcal{P}/I)$$

  for $t \geq \mu_{k-i-1} + t_i$ holds.

So we fix an integer $0 \leq i \leq k-1$: First, we define

$$t_i = \mu_{k-i} + t_{i-1},$$

so we only have define μ_{k-i} now. We want to ensure that $\beta_t^{(k-l)}$ is identic for I and I', given $t \geq t_{i+1}$, where t_{i+1} is yet to be defined. Using induction for the higher classes, we can assume that

$$\beta_t^{(k-l)}(\mathcal{P}/I') = \beta_t^{(k-l)}(\mathcal{P}/I)$$

holds for all classes $k - l \geq k - i$, all $\mu_0, ..., \mu_{k-i}$ and all $t \geq t_i = \mu_{k-i} + t_{i-1}$.

So for any μ_{k-i-1}, let

$$t_{i+1} = \mu_{k-i-1} + t_i = \sum_{j=k-i-1}^{k} \mu_j + \max\{\operatorname{reg} I, \mu_{k+1}\} + 1,$$

the minimal degree for which we want $\beta_{t_{i+1}}^{(k-i-1)}(\mathcal{P}/I') = \beta_{t_{i+1}}^{(k-i-1)}(\mathcal{P}/I)$ to hold. For any choice of $\mu_0, ..., \mu_{k-i-1}$, we have

$$\begin{aligned} &\beta_{t_{i+1}}^{(k-i-1)}(\mathcal{P}/I') \\ =&\left|\{x^\nu \in \mathbb{T}_{t_{i+1}} \mid \nu \prec_{\text{lex}} (\mu_0, ..., \mu_{k+1}, 0, ..., 0), \operatorname{cls} \nu = k-i-1\}\right| \end{aligned}$$

If we have $\operatorname{cls}\nu = k-i$, then $\nu_0, ..., \nu_{k-i-2} = 0$. Note that we have $i \le k-1$.

$$=\left|\{x^\nu \in \mathbb{T}_{t_{i+1}} \mid \nu \prec_{\text{lex}} (0, ..., 0, \mu_{k-i-1}, ..., \mu_{k+1}, 0, ..., 0), \operatorname{cls}\nu = k-i-1\}|\right|$$
$$=\left|\{x^\nu \in \mathbb{T}_{t_{i+1}} \mid \nu \prec_{\text{lex}} (\mu_{k-i-1}, 0, ..., 0, \mu_{k-i}, ..., \mu_{k+1}, 0, ..., 0)), \operatorname{cls}\nu = k-i-1\}\right|$$
$$+\left|\{x^\nu \in \mathbb{T}_{t_{i+1}} \mid (\mu_{k-i-1}, 0, ..., 0, \mu_{k-i}, ..., \mu_{k+1}, 0, ..., 0) \preceq_{\text{lex}} \nu \prec_{\text{lex}} (0, ..., 0, \mu_{k-i-1}, ..., \mu_{k+1}, 0, ..., 0), \operatorname{cls}\nu = k-i-1\}\right|$$

But since $t_{i+1} = \sum_{j=k-i-1}^{k} \mu_j + \max\{\operatorname{reg} I, \mu_{k+1}\} + 1$, the second set is empty for degree reasons.

$$=\left|\{x^\nu \in \mathbb{T}_{t_{i+1}} \mid \nu \prec_{\text{lex}} (\mu_{k-i-1}, 0, ..., 0, \mu_{k-i}, ..., \mu_{k+1}, 0, ..., 0)), \operatorname{cls}\nu = k-i-1\}\right|$$
$$=\beta_{t_{i+1}}^{(k-i-1)}\big(\mathcal{P}/\langle \operatorname{Lex}(x^{(\mu_{k-i-1},0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)})\rangle\big)\big|$$

Using lemma 7.1.4 and lemma 7.2.3.

$$=\sum_{l=0}^{i+1}\binom{(t_{i+1}-t_i-1)+(i-l+2)-1}{t_{i+1}-t_i-1}\beta_{t_i}^{(k-l)}\big(\mathcal{P}/\langle \operatorname{Lex}(x^{(0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)})\rangle\big)$$
$$=\sum_{l=0}^{i}\binom{(t_{i+1}-t_i-1)+(i-l+2)-1}{t_{i+1}-t_i-1}\beta_{t_i}^{(k-l)}\big(\mathcal{P}/\langle \operatorname{Lex}(x^{(0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)})\rangle\big)$$
$$+\beta_{t_i}^{(k-i-1)}\big(\mathcal{P}/\langle \operatorname{Lex}(x^{(0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)})\rangle\big)$$

Now we use the fact that in order to determine the numbers $\beta_{t_i}^{(k-l)}$, we need to count elements of class $k-l \ge k-i$ (see also the definition earlier in this remark). But for any choice of $\mu_0, ..., \mu_{k-i-1}$, these numbers are obviously the same for ideals $\operatorname{lex}(x^{(0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)})\rangle$ and $\operatorname{lex}(x^{(\mu_0....,\mu_{k+1},0,...,0)})\rangle = I'$.

$$=\sum_{l=0}^{i}\binom{(t_{i+1}-t_i-1)+(i-l+2)-1}{t_{i+1}-t_i-1}\beta_{t_i}^{(k-l)}\big(\mathcal{P}/I'\big)$$
$$+\beta_{t_i}^{(k-i-1)}\big(\mathcal{P}/\langle \operatorname{Lex}(x^{(0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)})\rangle\big)$$

Now we make use of the assumption that we have $\big(\mathcal{P}/I'\big) = \beta_{t_i}^{(k-l)}(\mathcal{P}/I)$ for $0 \le l \le i$.

$$=\sum_{l=0}^{i}\binom{(t_{i+1}-t_i-1)+(i-l+2)-1}{t_i-t_{i-1}-1}\beta_{t_i}^{(k-l)}(\mathcal{P}/I)$$
$$+\beta_{t_i}^{(k-i-1)}\big(\mathcal{P}/\langle \operatorname{Lex}(x^{(0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)})\rangle\big)$$
$$=\sum_{l=0}^{i+1}\binom{(t_{i+1}-t_i-1)+(i-l+2)-1}{t_{i+1}-t_i-1}\beta_{t_i}^{(k-l)}(\mathcal{P}/I)-\beta_{t_i}^{(k-i-1)}(\mathcal{P}/I)$$
$$+\beta_{t_i}^{(k-i-1)}\big(\mathcal{P}/\langle \operatorname{Lex}(x^{(0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)})\rangle\big)$$
$$=\beta_{t_{i+1}}^{(k-i-1)}(\mathcal{P}/I)-\beta_{t_i}^{(k-i-1)}(\mathcal{P}/I)+\beta_{t_i}^{(k-i-1)}\big(\mathcal{P}/\langle \operatorname{Lex}(x^{(0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)})\rangle\big).$$

- So we see that (for any choice of $\mu_0, ..., \mu_{k-i-1}$) we have

$$\beta_{t_{i+1}}^{(k-i-1)}(\mathcal{P}/I') = \beta_{t_{i+1}}^{(k-i-1)}\big(\mathcal{P}/I\big)$$

  if and only if

$$\beta_{t_i}^{(k-i-1)}(\mathcal{P}/I) = \beta_{t_i}^{(k-i-1)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)})\rangle\big). \quad (7.4.1)$$

We continue by studying the terms in this equation (7.4.1), using ideas similar to those used in deriving this equation:

$$\begin{aligned}
&\beta_{t_i}^{(k-i-1)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)})\rangle\big)\\
=&\big|\{x^\nu \in \mathbb{T}_{t_i} \mid \nu \prec_{\mathrm{lex}} (0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0), \operatorname{cls}\nu = k-i-1\}\big|
\end{aligned}$$

Again note that we consider the case $i \le k-1$.

$$\begin{aligned}
=&\big|\{x^\nu \in \mathbb{T}_{t_i} \mid \nu \prec_{\mathrm{lex}} (\mu_{k-i},0,....,0,\mu_{k-i+1},....,\mu_{k+1},0,...,0)), \operatorname{cls}\nu = k-i-1\}\big|\\
&+\big|\{x^\nu \in \mathbb{T}_{t_i} \mid (\mu_{k-i},0,....,0,\mu_{k-i+1},....,\mu_{k+1},0,...,0) \preceq_{\mathrm{lex}} \nu\\
&\qquad \prec_{\mathrm{lex}} (0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0), \operatorname{cls}\nu = k-i-1\}\big|
\end{aligned}$$

Now the elements of the second set can be given in a simple explicit manner.

$$\begin{aligned}
=&\big|\{x^\nu \in \mathbb{T}_{t_i} \mid \nu \prec_{\mathrm{lex}} (\mu_{k-i},0,....,0,\mu_{k-i+1},....,\mu_{k+1},0,...,0)), \operatorname{cls}\nu = k-i-1\}\big|\\
&+\big|\{(0,....,0,\mu_{k-i},0,\mu_{k-i+1}....,\mu_{k+1},0,...,0),\\
&\qquad (0,....,0,\mu_{k-i}-1,1,\mu_{k-i+1}....,\mu_{k+1},0,...,0),...,\\
&\qquad\qquad (0,....,0,1,\mu_{k-i}-1,\mu_{k-i+1}....,\mu_{k+1},0,...,0)\}\big|\\
=&\big|\{x^\nu \in \mathbb{T}_{t_i} \mid \nu \prec_{\mathrm{lex}} (\mu_{k-i},0,....,0,\mu_{k-i},....,\mu_{k+1},0,...,0)), \operatorname{cls}\nu = k-i-1\}\big|\\
&+\mu_{k-i}\\
=&\beta_{t_i}^{(k-i-1)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(\mu_{k-i},0,....,0,\mu_{k-i+1},....,\mu_{k+1},0,...,0)})\rangle\big)+\mu_{k-i}
\end{aligned}$$

Using lemma 7.1.4 and lemma 7.2.3 again.

$$\begin{aligned}
=&\sum_{l=0}^{i+1}\binom{(t_i-t_{i-1}-1)+(i-l+2)-1}{t_i-t_{i-1}-1}\beta_{t_{i-1}}^{(k-l)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(0,....,0,\mu_{k-i+1},....,\mu_{k+1},0,...,0)})\rangle\big)\\
&+\mu_{k-i}\\
=&\sum_{l=0}^{i}\binom{(t_i-t_{i-1}-1)+(i-l+2)-1}{t_i-t_{i-1}-1}\beta_{t_{i-1}}^{(k-l)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(0,....,0,\mu_{k-i+1},....,\mu_{k+1},0,...,0)})\rangle\big)\\
&+\beta_{t_{i-1}}^{(k-i-1)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(0,....,0,\mu_{k-i+1},....,\mu_{k+1},0,...,0)})\rangle\big)+\mu_{k-i}\\
=&\sum_{l=0}^{i}\binom{(t_i-t_{i-1}-1)+(i-l+2)-1}{t_i-t_{i-1}-1}\beta_{t_{i-1}}^{(k-l)}\big(\mathcal{P}/I'\big)\\
&+\beta_{t_{i-1}}^{(k-i-1)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(0,....,0,\mu_{k-i+1},....,\mu_{k+1},0,...,0)})\rangle\big)+\mu_{k-i}
\end{aligned}$$

Now again we make use of the assumption that for $0 \leq l \leq i$, we have the equation $\beta_{t_i}^{(k-l)}(\mathcal{P}/I') = \beta_{t_i}^{(k-l)}(\mathcal{P}/I)$.

$$
\begin{aligned}
=&\sum_{l=0}^{i} \binom{(t_i - t_{i-1} - 1) + (i - l + 2) - 1}{t_i - t_{i-1} - 1} \beta_{t_{i-1}}^{(k-l)}(\mathcal{P}/I) \\
&+ \beta_{t_{i-1}}^{(k-i-1)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(0,\dots,0,\mu_{k-i+1},\dots,\mu_{k+1},0,\dots,0)})\rangle\big) + \mu_{k-i} \\
=&\sum_{l=0}^{i+1} \binom{(t_i - t_{i-1} - 1) + (i - l + 2) - 1}{t_i - t_{i-1} - 1} \beta_{t_i}^{(k-l)}(\mathcal{P}/I) - \beta_{t_{i-1}}^{(k-i-1)}(\mathcal{P}/I) \\
&+ \beta_{t_{i-1}}^{(k-i-1)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(0,\dots,0,\mu_{k-i+1},\dots,\mu_{k+1},0,\dots,0)})\rangle\big) + \mu_{k-i} \\
=&\beta_{t_i}^{(k-i-1)}(\mathcal{P}/I) - \beta_{t_{i-1}}^{(k-i-1)}(\mathcal{P}/I) \\
&+ \beta_{t_{i-1}}^{(k-i-1)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(0,\dots,0,\mu_{k-i+1},\dots,\mu_{k+1},0,\dots,0)})\rangle\big) + \mu_{k-i}
\end{aligned}
$$

Comparing the first and the last line of this chain of equations, we see that the desired equality (7.4.1) holds if and only if

$$
\mu_{k-i} = \beta_{t_{i-1}}^{(k-i-1)}(\mathcal{P}/I) - \beta_{t_{i-1}}^{(k-i-1)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(0,\dots,0,\mu_{k-i+1},\dots,\mu_{k+1},0,\dots,0)})\rangle\big).
$$

However, since $\langle \mathrm{Lex}(x^{(0,\dots,0,\mu_{k-i+1},\dots,\mu_{k+1},0,\dots,0)})\rangle$ is a lex segment ideal, we can use the decomposition from lemma 7.2.3 and corollary 7.2.6 (which can easily be adapted to take into consideration only elements of one fixed class), to explicitly express the second summand on the right side in this equation in terms of the $\mu_{k-i+1}, \dots, \mu_{k+1}$. We obtain:

$$
\begin{aligned}
&\beta_{t_{i-1}}^{(k-i-1)}\big(\mathcal{P}/\langle \mathrm{Lex}(x^{(0,\dots,0,\mu_{k-i+1},\dots,\mu_{k+1},0,\dots,0)})\rangle \\
&= \sum_{j=0}^{i} \sum_{l=0}^{\mu_{k-j+1}-1} \binom{\big(t_{i-1} - l - (\mu_{k-j+2} + \dots + \mu_{k+1}) - 1\big) + (i - j + 2) - 1}{t_{i-1} - l - (\mu_{k-j+2} + \dots + \mu_{k+1}) - 1}.
\end{aligned}
$$

So if we define

$$
\begin{aligned}
&\mu_{k-i} = \\
&\beta_{t_{i-1}}^{(k-i-1)}(\mathcal{P}/I) - \sum_{j=0}^{i} \sum_{l=0}^{\mu_{k-j+1}-1} \binom{\big(t_{i-1} - l - (\mu_{k-j+2} + \dots + \mu_{k+1})\big) + (i - j)}{t_{i-1} - l - (\mu_{k-j+2} + \dots + \mu_{k+1}) - 1},
\end{aligned}
$$

we have the equalities we were looking for. Note that this definition of μ_{k-i} does indeed only depend on $\mu_{k-i+1}, \dots, \mu_{k+1}$ and I, since

$$
t_{i-1} = \sum_{j=k-i+1}^{k} \mu_j + \max\{\mathrm{reg}\, I, \mu_{k+1}\} + 1.
$$

Because of lemma 7.0.6, the β-vectors of I and I' are then also identic, for sufficiently large values of the degree t.

We also point out that this construction is independent of the characteristic of the ground field $\Bbbk$.

If we set $\mu_0 = 0$, the ideal $\text{lex}(x^\mu)$ is saturated by lemma 7.2.2.

Altogether, using Lemma 7.3.2, we have proven:

Theorem 7.4.2. *Let $I \trianglelefteq \mathcal{P}$ be an ideal that is neither the zero ideal nor zero-dimensional. Let k be the maximal integer such that $\beta^{(k)}_{\text{reg}\, I}(\mathcal{P}/I) \neq 0$. Then the unique saturated lex segment ideal I' with the same Hilbert polynomial as I is given by $I' = \langle \text{Lex}(x^\mu) \rangle$, for μ given by*

- $\mu_j = 0$ *for* $j > k+1$,
- $\mu_{k+1} = \beta^{(k)}_{\text{reg}\, I}(\mathcal{P}/I)$,
- $\mu_{k-i} = \beta^{(k-i-1)}_{t_{i-1}}(\mathcal{P}/I) - \sum\limits_{j=0}^{i-1} \sum\limits_{l=0}^{\mu_{k-j+1}-1} \binom{\left(t_{i-1}-l-(\mu_{k-j+2}+\ldots+\mu_{k+1})\right)+(i-j)}{t_{i-1}-l-(\mu_{k-j+2}+\ldots+\mu_{k+1})}$ *for* $0 \leq i \leq k-1$,
- *and* $\mu_0 = 0$,

where the numbers t_i are given by

$$t_{-1} = \max\{\text{reg}\, I, \mu_{k+1}\} + 1 \qquad \textit{and} \qquad t_i = \mu_{k-i} + t_{i-1} = t_{-1} + \sum_{j=0}^{i} \mu_j$$

for $i \geq 0$. The Gotzmann vector of I and I' is given by

$$G(I) = G(I') = (0, \mu_1, \ldots., \mu_{k+1}, 0, \ldots).$$

Example 7.4.3. Via the construction given above, we calculate the saturated lex segment ideal with the same Hilbert polynomial as the ideal

$$I = \langle x_2^2 x_3, x_2 x_3^2, x_3^3 \rangle \trianglelefteq \Bbbk[x_0, x_1, x_2, x_3].$$

Its Hilbert polynomial is given by

$$\text{HP}_{\mathcal{P}/I}(t) = \frac{1}{2}t^2 + \frac{9}{2}t - 1.$$

Obviously, the given set of generators is a Pommaret basis for I (with respect to the degrevlex order). From Theorem 2.3.45, we learn that

$$\text{reg}(I) = \text{reg}(\mathcal{P}/I) + 1 = 3.$$

We see that $\mathbb{T}_{\text{reg}(I)} \cap I = \mathbb{T}_3 \cap \text{lt}(I)$ does not contain the monomials

$$x_0^3, \quad x_0^2 x_1, \quad x_0^2 x_2, \quad x_0^2 x_3, \quad x_0 x_1^2, \quad x_0 x_1 x_2, \quad x_0 x_1 x_3, \quad x_0 x_2^2,$$
$$x_0 x_2 x_3, \quad x_0 x_3^2, \quad x_1^3, \quad x_1^2 x_2, \quad x_1^2 x_3, \quad x_1 x_2^2, \quad x_1 x_2 x_3, \quad x_1 x_3^2, \quad x_2^3.$$

So for the numbers given in the construction above, we obtain

$$\beta_3^{(0)}(\mathcal{P}/I) = 10, \quad \beta_3^{(1)}(\mathcal{P}/I) = 6, \quad \beta_3^{(2)}(\mathcal{P}/I) = 1, \quad \beta_3^{(3)}(\mathcal{P}/I) = 0,$$

and hence $k = 2$ and $\mu_{k+1} = \mu_3 = 1$. To continue, we first need

$$t_{-1} = \max\{\operatorname{reg} I, \mu_{k+1}\} + 1 = \max\{3, 1\} + 1 = 4.$$

Now let $i = 0$, i.e. our goal is to calculate $\mu_k = \mu_2$. The formula

$$\mu_{k-i} =$$
$$\beta_{t_{i-1}}^{(k-i-1)}(\mathcal{P}/I) - \sum_{j=0}^{i} \sum_{l=0}^{\mu_{k-j+1}-1} \binom{\left(t_{i-1} - l - (\mu_{k-j+2} + ... + \mu_{k+1})\right) + (i-j)}{t_{i-1} - l - (\mu_{k-j+2} + ... + \mu_{k+1}) - 1},$$

now simplifies to

$$\mu_2 = \beta_4^{(1)}(\mathcal{P}/I) - \binom{4}{3} = \beta_4^{(1)}(\mathcal{P}/I) - 4.$$

So all this left to do is to calculate $\beta_4^{(1)}(\mathcal{P}/I)$. We can either explicitly write down $\mathbb{T}_4 \cap I = \mathbb{T}_4 \cap \operatorname{lt}(I)$, or use the fact that a Pommaret basis induces a Stanley decomposition of $\mathcal{P}/I$ as explained in Remark 2.3.50, or a formula in analogy to Lemma 7.0.6 to see that

$$\beta_4^{(1)}(\mathcal{P}/I) = \binom{1-1}{0}\beta_3^{(1-1)}(\mathcal{P}/I) + \binom{2-1}{0}\beta_3^{(2)}(\mathcal{P}/I) = 7$$

and hence $\mu_2 = 3$. Now to find μ_1, let $i = 1$. We have

$$t_0 = \mu_k + t_{-1} = \mu_2 + 4 = 5.$$

The formula for μ_{k-i} now simplifies to

$$\begin{aligned}
\mu_1 &= \beta_{t_0}^{(0)}(\mathcal{P}/I) - \sum_{j=0}^{1} \sum_{l=0}^{\mu_{k-j+1}-1} \binom{(t_0 - l - (\mu_{k-j+2} + ... + \mu_{k+1})) + (1-j)}{t_0 - l - (\mu_{k-j+2} + ... + \mu_{k+1}) - 1} \\
&= \beta_{t_0}^{(0)}(\mathcal{P}/I) - \binom{(t_0+1)}{t_0 - 1} - \sum_{l=0}^{2} \binom{(t_0 - l - \mu_{k+1})}{t_0 - l - \mu_{k+1} - 1} \\
&= \beta_5^{(0)}(\mathcal{P}/I) - \binom{6}{4} - \sum_{l=0}^{2} \binom{4-l}{3-l}. \\
&= \beta_5^{(0)}(\mathcal{P}/I) - 15 - 4 - 3 - 2. \\
&= \beta_5^{(0)}(\mathcal{P}/I) - 24.
\end{aligned}$$

Again, we have

$$\begin{aligned}\beta_5^{(0)}(\mathcal{P}/I) &= \binom{1-1}{1}\beta_3^{(0)}(\mathcal{P}/I) + \binom{2-1}{1}\beta_3^{(1)}(\mathcal{P}/I) + \binom{3-1}{1}\beta_3^{(0)}(\mathcal{P}/I)\\ &= 10 + 2\cdot 6 + 3\cdot 1\\ &= 25\end{aligned}$$

and hence $\mu_1 = 1$. As mentioned above, we have $\mu_0 = 0$ for any ideal. So the ideal I' is given by

$$I' = \operatorname{Lex}(x_1x_2^3x_3) = \langle x_1x_2^3x_3, x_2^4x_3, x_3^2\rangle,$$

where this set of generators is also a Pommaret basis for I'.

Now we want to prove a part of the Hyperplane Restriction Theorem of Green (recall in the given reference, this theorem is proven for all values of t):

Theorem 7.4.4. [Gre98, Theorem 3.4] *Let $I \trianglelefteq \mathcal{P}$ be an ideal with Hilbert function $\operatorname{HF}_{\mathcal{P}/I}(t)$. Let I_H be the restriction of I to a general hyperplane, seen as an ideal in the polynomial ring $\mathcal{P}_H$. Then for sufficiently large values of t, we have*

$$\operatorname{HF}_{\mathcal{P}_H/I_H}(t) \leq \big(\operatorname{HF}_{\mathcal{P}/I}(t)\big)_{<t>}.$$

Proof. Working in generic coordinates, we can assume that for $t \geq \operatorname{reg} I$, the ideal $\langle I_t\rangle$ has a Pommaret basis whose elements are of degree t [1]and that the generic hyperplane is given by $H = \langle x_0\rangle$. We will use the identification

$$\mathcal{P}_H \cong \mathcal{P}/\langle x_0\rangle \cong \Bbbk[x_0, ..., x_{n-1}]$$

and view I_H as an ideal in $\Bbbk[x_0, ..., x_{n-1}]$ (even though this means we will identify $x_i \in \mathcal{P}$ with $x_{i-1} \in \mathcal{P}_H$, we choose to work with the notation $\Bbbk[x_0, ..., x_{n-1}]$ over $\Bbbk[x_1, ..., x_n]$ to be consistent with our conventions). Theorem 2.3.45 ensures that we have $\operatorname{reg} I \geq \operatorname{reg} I_H$. By definition of the Hilbert polynomial, we know that $\operatorname{HP}_{\mathcal{P}/I}(t) = \operatorname{HF}_{\mathcal{P}/\operatorname{lt}(I)}(t)$ for $t \geq \operatorname{reg} I$. Furthermore, for the β-vectors of I and I_H, we have the relation

$$\beta_q^{(k)}(I'_H) = \beta_q^{(k)}(I_H) = \beta_q^{(k+1)}(I) = \beta_q^{(k+1)}(I'),$$

for $k \geq 0$ and sufficiently large q and therefore

$$\beta_q(I_H) = \big(\beta_q^{(1)}(I), ..., \beta_q^{(n)}(I)\big).$$

Note that that $\beta_q(I_H)$ is of length n, while $\beta_q(I)$ is of length $n+1$.

Now we look at the saturated lex segment ideals I', I'_H with the same Hilbert polynomials as I, I_H. The construction of I', I'_H and their respective Gotzmann vectors according to Remark 7.4.1 and Theorem 7.4.2 tells us that for sufficiently

[1] This implies that the leading ideal $\operatorname{lt}(I_t)$ of is stable.

large values, we can construct the β-vector of I or I' from its Gotzmann vector, and vice versa. The key observation is now that the formulae given in Theorem 7.4.2 for this process imply that the shifting relation

$$\beta_q^{(k)}(I'_H) = \beta_q^{(k)}(I_H) = \beta_q^{(k+1)}(I) = \beta_q^{(k+1)}(I'),$$

translates to a corresponding shift in the Gotzmann vectors, i.e. we obtain that if the Gotzmann vector of I' is given by

$$(0, \mu_1, ..., \mu_k, \mu_{k+1}, 0, ...),$$

then the Gotzmann vector of I'_H is given by

$$(0, \mu_2, ..., \mu_{k+1}, 0, 0, ...).$$

Here, the entry μ_1 vanishes, which may appear surprising at first glance; however from Theorem 7.4.2 we see that this is indeed the correct formula.

Obviously, for two polynomials $f, g \in \mathbb{Q}[t]$, the equation $G(f) \preceq_{\text{lex}} G(g)$ implies $f(t) \leq g(t)$ for t sufficiently large. Using this property, from the definition of the $(\cdot)_{<t>}$-symbol, we immediately see by Lemma 7.3.4 that $\left(\text{HF}_{\mathcal{P}/I}(t)\right)_{<t>}$ has the Gotzmann vector

$$(\mu_1, \mu_2, ..., \mu_{k+1}, 0, 0, ...).$$

This finishes the proof. $\square$

8 Conclusion

If one asks the question if the ideas and constructions of this work have been explored up to their possible limits, our impression is that this could be answered with a yes at most for Chapter 7: However, while we expect that our main new idea of this chapter, the link between Pommaret basis and the lex segment ideals given in Remark 7.4.1 and Theorem 7.4.2, is not enough to extend Theorem 7.4.4 to cover any value of t and hence prove the entire Hyperplane Restriction Theorem 7.0.2, it stands to check if this link might be of use for other related fields.

However, we believe that further examination of the resolution introduced in Chapter 4 might be much more fruitful, as even the first applications we have presented in Chapter 5, and in particular in Chapter 6, opened up numerous avenues on which one might continue.

In Chapter 5, we investigated relations between constants appearing in the resolution from Chapter 4. While so far, the results are of a rather technical nature, we recall that this resolution has been implemented in CoCoALib, and that this implementation is efficient for computing Betti numbers, as explained in Section 4.4. Right now, CoCoALib calculates every single constant on its own, in the sense that involutive standard representations are calculated only once, but all reduction paths are calculated on their own. We expect that in the case where constants are related as explained in Chapter 5, many of these computations are redundant. Thus an implementation of these technical results might further accelerate the computation of Betti numbers with CoCoALib. From a theoretical perspective, for the goal of computing Betti numbers from a non-minimal resolution, it is sufficient to first compute the constants, and then compute the ranks of the matrices given by these constants. Now, if we know something about the relations between constants in different homological degrees, one can ask if it is also possible to obtain results regarding the ranks of the matrices. We have tried to continue in this direction, but so far without success. Nevertheless, we consider it likely that in a situation where there are overall “relatively few” constants, it should be possible to find at least some results.

Another interesting question bears some similarity to the topic of Chapter 6: Instead of asking where minimisations might happen, i.e. finding constants in the differential of the resolution, one might instead ask where minimisations do **not** occur. So one reverses the question of Chapter 5 in the sense that we no longer want to find relations between constants, but now we want to find relations between places where there are no constants. Of course, it seems likely that the theorem from Chapter 5 could yield some results in this direction. Then by the idea explained in remark 6.2.1, we would obtain some generators that will never vanish during the minimisation process, and therefore give non-vanishing Betti numbers, or maybe even better, lower bounds for certain Betti numbers. The feasibility of this approach might however depend on the given ideals or modules, and a good choice of a Pommaret bases, just like for the Veronese subrings in Chapter 6.

While the constants of our resolution contain sufficient information to compute Betti numbers, the minimal resolution itself still remains interesting, and in some sense the optimal goal. This leads to the question if the results for constants can be generalised to non-constant entries in the differential. We recall that the degree of the entry is given by the number of times an elementary reduction path of type 1 appears in a longer reduction path. As stated in Remark 5.1.5, it might be of particular interest to study the linear entries, i.e. the paths where at most one path of type 1 appears.

Turning to Chapter 6, we first point out that at the heart of this chapter, we constructed a Pommaret basis for the Veronese subrings. In general, finding a Pommaret basis (or some other kind of involutive basis) for a given a ideal over polynomial ring $\mathcal{P} = \Bbbk[x_0, .., x_n]$ in a "concrete" situation such as in example 2.3.24 is a task that can often be solved without much computation, or for example by using the algorithm mentioned in Remark 2.3.20, possibly after a change of coordinates. For an entire class of ideals however, such as the class of ideals given by the Veronese subrings, finding an associated class of Pommaret bases is usually more difficult. This leads to the question if the methods employed in this Chapter can be applied to other classes of ideals: For example, one could consider, as first generalization, any embedding $X \to \mathbb{P}^n_{\Bbbk}$ given by arbitrary sets of monomials, i.e. where some monomials of degree d are missing, as explained in Section 6.3.2. As another step, one can ask to what extent our concept of renumbering variables by multiindices as introduced in Section 6.1.1 remains useful if one replaces monomials by leading monomials of polynomials.

If we stay at Veronese subrings, we note that in the paper [EL12], the statement about non-vanishing Betti numbers covered other Betti numbers we have not considered in this work, see [EL12, Theorem 6.1]. Therefore, a further topic of research could be to check if we could obtain comparable results via Pommaret bases and algebraic discrete Morse theory for these Betti numbers. We suspect that we would have to make some adaptations to the approach presented in Chapter 6: For example, a change of the monomial order given in Assumption 6.1.1 might be prove to be useful. This is another interesting question, as we still believe this order has an aspect of "free parameters", corresponding to the ordering of the monomials in the Veronese embedding. However, a change of the monomial order is likely to entail a new computation for a different Pommaret basis, but we think that the usefulness of an order most likely depends on the precise question one would like to answer (for example, which Betti numbers one is interested in).

Again, in this chapter, one can also adopt an inverse question compared to our results: Instead of asking which Betti numbers do not vanish, one can ask which Betti numbers are 0. In fact, as we explained at the beginning of Chapter 6, this was originally the first question. However, it is assumed (see for example again [EL12, Chapter 7]) that the task of proving which Betti numbers for the Veronese subrings vanish, is harder than finding non-vanishing ones.

Another possible class of modules that might be worth considering from our point of view are Veronese modules $S_{n,d,k}$, given by

$$S_{n,d,k} = \bigoplus_{i \geq 0} \mathbb{k}[x_0, ..., x_n]_{k+id},$$

which are a generalization of Veronese subrings. In the paper [OP01], one can find several results for these modules. The first big step to apply our approach of Chapter 6 to these modules would be to find a Pommaret basis for these modules and then to check if the methods utilized for Veronese subrings still prove to be fruitful. After a first, not yet detailed, glance at this subject, we are rather optimistic that this topic might prove to be a suitable field for generalizing our methods employed for the Veronese subrings.

A Erklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbstständig, ohne unerlaubte Hilfe Dritter angefertigt und andere als die in der Dissertation angegebenen Hilfsmittel nicht benutzt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen sind, habe ich als solche kenntlich gemacht. Dritte waren an der inhaltlich-materiellen Erstellung der Dissertation nicht beteiligt; insbesondere habe ich hierfür nicht die Hilfe eines Promotionsberaters in Anspruch genommen. Kein Teil dieser Arbeit ist in einem anderen Promotions- oder Habilitationsverfahren verwendet worden.

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