# Computing Quot Schemes

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# Contents

# Introduction

In the late 1950s and 1960s Alexander Grothendieck revolutionized algebraic geometry. He introduced an abstract algebraic point of view for algebraic geometry. In particular, he introduced the language of schemes which enabled us to use sheaf theory to study algebraic varieties. Furthermore, he developed the theory of Hilbert and Quot schemes, which is still today an area of active research.

Quot schemes were introduced by Grothendieck [25] as schemes representing the contravariant Quot functor $\mathrm{Quot}_{X,S}^{\mathcal{F},\mathsf{HP}(t)}$ which is a functor from the category of $S$-schemes to the category of sets. The Quot functor parametrizes quotient sheaves of a fixed quasi-coherent sheaf $\mathcal{F}$ on an $S$-scheme $X$ that are flat over the base and have Hilbert polynomial $\mathsf{HP}(t)$ on the fibres. Grothendieck showed the representability of the Quot functor by embedding it into a suitable Grassmann functor $\mathrm{Gr}_{N,\mathsf{HP}(s)}$ where $s$ is a sufficiently large integer and $N$ is the number of module terms in degree $s$.

For the case of Hilbert functors, which are special Quot functors there were several attempts to simplify the proof of representability. The first crucial point was the concept of regularity, introduced by Mumford [35], for the choice of degree $s$. A further simplification was found by Gotzmann [23]: his regularity theorem gives a formula for the minimal $s$ depending only on the polynomial $\mathsf{HP}(t)$. These classical approaches allow us to determine equations for specifying the embedding of the Hilbert scheme into the Grassmann scheme. But the number of equations and their degrees are usually so high, that it is impossible to compute these equations.

Instead of considering global equations to embed the Hilbert scheme into the Grassmann scheme, Bayer [6] suggested a reduction to the local case, that is to consider an open covering of the Hilbert scheme induced by an open covering of the Grassmann scheme. However, even this approach was still far away from practical for computing concrete equations of a Hilbert scheme.

Using the strategy of Bayer [6] Brachat et al. [11] constructed a special open covering of the Grassmann functor which is called the Borel fixed open covering. Using this covering they defined subfunctors of the Grassmann and the Hilbert functor. Then they showed that these subfunctors are representable which induces the representability of the Grassmann and Hilbert functors. In particular, they showed that the subfunctors of the Hilbert functor are represented by marked schemes. The advantage of the approach of Brachat et al. [11] is that we have to consider only a really small open covering which is provided by a much smaller set of equations then the approaches mentioned above. With this approach it was possible for the first time to really compute non trivial Hilbert schemes.

Marked schemes appeared for the first time in [13]. If $\mathcal{J}$ is a strongly stable ideal in the polynomial ring $A[x_0, \ldots, x_n]$ then the family of ideals $\mathcal{I} \subseteq A[x_0, \ldots, x_n]$ such that $A[x_0, \ldots, x_n] = \mathcal{I} \oplus \langle \{x^\alpha \in \mathbb{T} \mid x^\alpha \notin \mathcal{J}\} \rangle$ is naturally an algebraic scheme, called the marked scheme. Using the marked scheme it is possible to embed the Hilbert schemes in affine spaces, which allows computing explicit equations for it. One major restriction of this approach is that this only works over fields with characteristic zero.

In this thesis we extend the ideas developed by Brachat et al. [11] in two directions. First, we show that we can construct a similar covering, called a quasi-stable covering, which let us prove the representability of Hilbert functors over fields of arbitrary characteristic. Secondly, we show that we can extend this approach to Quot functors, which allows us to give a new proof for the representability of the Quot functor. Using this new approach we formulate algorithms to compute explicitly an open covering of a Quot scheme. The new algorithms allow us for the first time to compute concrete Quot schemes.

This thesis starts with a chapter about resolving decompositions. They provide a unifying framework for computing free resolutions. Furthermore, we introduce in this chapter the notion of Pommaret bases. They are an example of resolving decompositions, and they will be the foundation for the theory which is developed in the following chapters. One disadvantage of Pommaret bases is that they exist only in generic coordinates. Therefore, a part of the first chapter is devoted to introducing several generic coordinate positions.

Another example of resolving decompositions are marked bases over modules; a topic we also introduce in the first chapter. They may be considered as a form of Gröbner basis which do not depend on a term order. Instead, one chooses for each generator some term as head module term such that the head module terms generate a prescribed monomial module. We show that the involutive normal form algorithm with respect to Pommaret division will terminate if the prescribed monomial module is quasi-stable.

The second chapter is devoted to the Hilbert function and the Hilbert polynomial. Both objects are defined in this chapter. Furthermore, we analyse the important persistence and regularity theorem for ideals of Gotzmann [23]. We provide for both theorems new alternative proofs which are much simpler to understand than previous proofs. They are based on the theory of Pommaret bases. We stated above that the regularity theorem of Gotzmann is an important step in the proof of the representability of the Hilbert functor. We want to use this step for the Quot functor, too. For this we briefly recall the work of Dellaca [15] who extended the regularity theorem of Gotzmann to modules.

In the third chapter we introduce at first the Hilbert, Quot and Grassmann functors. Then we construct the "quasi-stable covering" of the Grassmann functor and show that we can restrict this covering to the Quot functor. The covering is represented by subfunctors. In an additional step we show that every subfunctor can be represented by a marked scheme. This marked scheme is an extension of the marked scheme defined by Brachat et al. [11] and uses the marked bases for modules that we defined in the first chapter.

The fourth chapter is devoted to the computation of concrete Quot schemes. In the first part we investigate the computation of saturated quasi-stable or $p$-Borel fixed monomial modules. In the second part we present two algorithms for computing marked schemes. Finally, we prove that $\mathbf{Hilb}_4^3$ is a reduced scheme, we compute for the first time a concrete representation for a Quot scheme and give an example for Hilbert schemes over fields of finite characteristic.

## Conventions

Throughout this work we always consider $\mathbb{k}$ to be an algebraically closed field of arbitrary characteristic and $\mathbb{N}$ the set of natural numbers including zero:

- $A$ is a $\mathbb{k}$-algebra.
- $\mathcal{P} = A[\mathbf{x}]$ is a polynomial ring with variables $\mathbf{x} = (x_0, \ldots, x_n)$.
- $\mathcal{P}^m_{\mathbf{d}} = \bigoplus_{k=1}^m \mathcal{P}(-d_k)\mathbf{e}_k$ is a finitely generated free $\mathcal{P}$-module with grading $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{Z}^m$ and free generators $\mathbf{e}_1, \ldots, \mathbf{e}_m$. If $\mathbf{d} = (0, \ldots, 0)$ we write simply $\mathcal{P}^m$ instead of $\mathcal{P}^m_{\mathbf{d}}$.
- We consider only finitely generated graded submodules $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$.
- Let $\mathcal{B} \subseteq \mathcal{P}^m_{\mathbf{d}}$, then $\langle \mathcal{B} \rangle$ is the module generated by $\mathcal{B}$.
- $\mathbb{T}$ is the set of terms in $\mathcal{P}$.
- $x^\alpha \in \mathbb{T}$, then $\deg_i(x^\alpha)$ is the degree of $x_i$ in $x^\alpha$. Sometimes we also write just $\alpha_i$ instead of $\deg_i(x^\alpha)$.
- $x^\alpha \in \mathbb{T}$, then $x^{\alpha_{\mathrm{sat}}} := \frac{x^\alpha}{x_0^{\alpha_0}}$.

- If $M \subseteq \{0, \ldots, n\}$, we define

$$\mathbb{T}_M := \left\{ x^\mu \in \mathbb{T} \mid \deg_i(x^\mu) = 0 \; \forall \; i \in \{0, \ldots, n\} \setminus M \right\}.$$

- $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ is a *monomial module* if it is of the form $\bigoplus_{k=1}^m \mathcal{J}^{(k)} \mathbf{e}_k$ where $\mathcal{J}^{(k)}$ is a monomial ideal in $\mathcal{P}$.
- A *module term (with index k)* is a term of the form $x^\mu \mathbf{e}_k$.
- For a monomial ideal $\mathcal{J} \subseteq \mathcal{P}$ we define $\mathcal{N}(\mathcal{J}) \subseteq \mathbb{T}$ as the set of terms in $\mathbb{T}$ not belonging to $\mathcal{J}$.
- For a monomial module $\mathcal{V} = \bigoplus_{i=1}^m \mathcal{J}^{(k)} \mathbf{e}_k$ we define $\mathcal{N}(\mathcal{V}) = \bigcup_{k=1}^m \mathcal{N}(\mathcal{J}^{(k)}) \mathbf{e}_k$.
- For an element $\mathbf{f} \in \mathcal{P}^m_{\mathbf{d}}$ we define $\operatorname{supp}(\mathbf{f})$ to be the set of module terms appearing in $\mathbf{f}$ with nonzero coefficient: $\mathbf{f} = \sum_{x^\alpha \mathbf{e}_{i_\alpha} \in \operatorname{supp}(\mathbf{f})} c_\alpha x^\alpha \mathbf{e}_{i_\alpha}$.
- If $\mathcal{B}$ is a set of homogeneous elements of degree $s$ in $\mathcal{P}^m_{\mathbf{d}}$ we write $\langle \mathcal{B} \rangle^A$ for the $A$-module space generated by $\mathcal{B}$ in $(\mathcal{P}^m_{\mathbf{d}})_s$.
- For a module $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ we denote by $\operatorname{pd}(\mathcal{U})$ the *projective dimension* of $\mathcal{U}$.
- For a module $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ we denote by $\operatorname{reg}(\mathcal{U})$ the *(Castelnuovo-Mumford) regularity* of $\mathcal{U}$.
- For a module $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ we denote by $\operatorname{depth}(\mathcal{U})$ the *depth* of $\mathcal{U}$.
- The binomial coefficient is defined as usual. For $k > n$ we set $\binom{n}{k} = 0$. Furthermore, $\binom{n}{0} = \binom{n}{n} = 1$.

# 1 Decomposition of Polynomial Modules

The first chapter is about the decomposition of polynomial modules. Decompositions of polynomial modules are an important aspect in computational algebraic geometry and commutative algebra because they allow the structured analysis of polynomial modules.

Now we introduce resolving decompositions, which are decompositions which contain much information about the underlying free resolution of the polynomial module. Furthermore, we introduce involutive bases and marked bases, which are the foundation for the following chapters. In this thesis we often work in generic coordinates due to that we introduce several notions of stability of polynomial modules, too.

## 1.1 Resolving Decompositions

The determination of free resolutions for polynomial modules is a fundamental task in computational commutative algebra and algebraic geometry. Free resolutions are needed for derived functors like Ext and Tor. Also, many important homological invariants like the projective dimension or the Castelnuovo-Mumford regularity are defined via the minimal resolution. Moreover, the Betti numbers contain much geometric and topological information.

Unfortunately, it is rather expensive to compute a resolution. As a rough rule of thumb, one may say that computing a resolution of length $\ell$ corresponds to computing $\ell$ Gröbner bases. In many cases one needs only partial information about the resolution like the Betti numbers simply measuring its size. However, all classical algorithms require always determining a full resolution.

A new approach is provided in [4] by combining the theory of Pommaret bases and algebraic discrete Morse theory (together with an implementation in the CoCoALib). It allows for the first time to determine Betti numbers –even individual ones– without computing a full resolution and thus is for most problems much faster than classical approaches (see [4, 5, 19] for detailed benchmarks). In addition, it scales much better and can be easily parallelised.

By reason of these good properties it is of great interest to generalize this approach to other situations. In [5], the approach is extended to Janet bases and in [19] to arbitrary continuous involutive divisions of Schreyer type. While the proofs remained essentially the same, the use of another involutive division required the adoption of a number of technical points.

The main objective of this section is the development of an axiomatic framework that unifies all the above works. We introduce the novel concept of a *resolving decomposition* which is defined via several direct sum decompositions. It implies in particular the existence of standard representations and normal forms. Then we show that such a decomposition allows for the explicit determination of a free resolution and of Betti numbers.

The point of such a unification is *not* that it leads to any new algorithms. Indeed, we will not present a general algorithm for the construction of resolving decomposition. Instead, one should see the results as a "meta-machinery" which given any concept of a basis that induces a resolving decomposition delivers automatically an effective syzygy theory for this kind of basis. For the concrete case of resolving decompositions induced by Janet or Pommaret bases, an implementation of this effective theory is described (together with benchmarks) in [4, 5]. For other types of underlying bases only fairly trivial modifications of this implementation would be required.

Large parts of the following section have already been published in [2].

### 1.1.1 Definition of Resolving Decompositions

Let $\mathcal{B} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\} \subset \mathcal{P}_{\mathbf{d}}^m$ be a finite homogeneous generating set such that there exists module terms $x^{\mu_i}\mathbf{e}_{k_i} \in \operatorname{supp}(\mathbf{h}_i)$ with coefficient $1_A$ for every $\mathbf{h}_i \in \mathcal{B}$. Among all of these special module terms we choose one for every $\mathbf{h}_i$ and denote it by $\operatorname{hm}(\mathbf{h}_i)$ and call it *head module term*. We define $\mathcal{U} = \langle \mathcal{B} \rangle$ and define the *head module terms of* $\mathcal{B}$, $\operatorname{hm}(\mathcal{B}) := \{\operatorname{hm}(\mathbf{h}) \mid \mathbf{h} \in \mathcal{B}\}$ and the *head module of* $\mathcal{U}$, $\operatorname{hm}(\mathcal{U}) := \langle \operatorname{hm}(\mathcal{B}) \rangle$. Note that $\operatorname{hm}(\mathcal{U})$ depends on the choice of $\mathcal{B}$ and on the choice of the head module terms in $\mathcal{B}$.

**Definition 1.1.** *Choose a submodule $\mathcal{U} \subseteq \mathcal{P}_{\mathbf{d}}^m$ such that there exists a finite generating set $\mathcal{B}$ described like above. Then we define a* resolving decomposition *of the submodule $\mathcal{U}$ as a quadruple $(\mathcal{B}, \operatorname{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$ with the following five properties:*

*(i)* $\mathcal{U} = \langle \mathcal{B} \rangle$.

*(ii)* *Let $\mathbf{h} \in \mathcal{B}$ be an arbitrary generator. Then, for every module term $x^{\mu}\mathbf{e}_k \in \operatorname{supp}(\mathbf{h}) \setminus \{\operatorname{hm}(\mathbf{h})\}$, we have $x^{\mu}\mathbf{e}_k \notin \operatorname{hm}(\mathcal{U})$.*

*(iii) We assign a set of* multiplicative variables $X_{\mathcal{B}}(\mathbf{h}) \subseteq \mathbf{x}$ *to every head module term* $\mathrm{hm}(\mathbf{h})$ *with* $\mathbf{h} \in \mathcal{B}$ *such that we have direct sum decompositions of both the head module*

$$\mathrm{hm}(\mathcal{U}) = \bigoplus_{\mathbf{h}\in\mathcal{B}} A[X_{\mathcal{B}}(\mathbf{h})] \cdot \mathrm{hm}(\mathbf{h}) \tag{1.1}$$

*and of the module itself*

$$\mathcal{U} = \bigoplus_{\mathbf{h}\in\mathcal{B}} A[X_{\mathcal{B}}(\mathbf{h})] \cdot \mathbf{h}\,. \tag{1.2}$$

*(iv)* $(\mathcal{P}^m_{\mathbf{d}})_r = \mathcal{U}_r \oplus \langle \mathcal{N}(\mathrm{hm}(\mathcal{U}))_r \rangle^A$ *for all* $r \geq 0$.

*(v) Let* $\{\mathbf{f}_1, \dots, \mathbf{f}_s\}$ *denote the standard basis of the free module* $\mathcal{P}^s$. *Given an arbitrary term* $x^\delta \in \mathbb{T}$ *and an arbitrary generator* $\mathbf{h}_\alpha \in \mathcal{B}$*, we find for every term* $x^\epsilon \mathbf{e}_i \in \mathrm{supp}(x^\delta \mathbf{h}_\alpha) \cap \mathrm{hm}(\mathcal{U})$ *a unique* $\mathbf{h}_\beta \in \mathrm{hm}(\mathcal{B})$ *such that* $x^\epsilon \mathbf{e}_i = x^{\delta'}\,\mathrm{hm}(\mathbf{h}_\beta)$ *with* $x^{\delta'} \in A[X_{\mathcal{B}}(\mathbf{h}_\beta)]$ *by* (iii). *Then the term order* $\prec_{\mathcal{B}}$ *on* $\mathcal{P}^s$ *must satisfy* $x^\delta \mathbf{f}_\alpha \succeq_{\mathcal{B}} x^{\delta'} \mathbf{f}_\beta$.

In the sequel, we always assume that $(\mathcal{B}, \mathrm{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$ is a resolving decomposition of the finitely generated graded module $\mathcal{U} = \langle \mathcal{B} \rangle \subseteq \mathcal{P}^m_{\mathbf{d}}$. In addition to the multiplicative variables, we define for $\mathbf{h} \in \mathcal{B}$ the *non-multiplicative variables* as $\overline{X}_{\mathcal{B}}(\mathbf{h}) := \{x_0, \dots, x_n\} \setminus X_{\mathcal{B}}(\mathbf{h})$. Note, that for notational simplicity, we will identify sets $X \subseteq \mathbf{x}$ of variables often with sets of the corresponding indices and thus simply write $i \in X$ instead of $x_i \in X$. Let $\mathbf{h} \in \mathcal{B}$, then we call a product $x_i \cdot \mathbf{h}$ *non-multiplicative prolongation* if $x_i \in \overline{X}_{\mathcal{B}}(\mathbf{h})$.

**Remark 1.2.** *Resolving decompositions may be considered as a refinement of Stanley decompositions. A Stanley decomposition of a module* $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ *is a representation of* $\mathcal{U}$ *as an isomorphism of graded A-linear spaces*

$$\mathcal{U} \cong \bigoplus_{\mathbf{f}\in\mathcal{B}} A[X_{\mathbf{f}}] \cdot \mathbf{f}$$

*with a finite set* $\mathcal{B} \subseteq \mathbb{T}^m$ *and sets* $X_{\mathbf{f}} \subseteq \{x_0, \dots, x_n\}$.

*Indeed,* (1.1) *gives us a Stanley decomposition of the head module of* $\mathcal{U}$ *and* (1.2) *of* $\mathcal{U}$ *itself.*

**Remark 1.3.** *The third condition gives us for every* $\mathbf{f} \in \mathcal{U}$ *a unique* standard representation

$$\mathbf{f} = \sum_{\alpha=1}^{s} P_\alpha \mathbf{h}_\alpha$$

*with* $P_\alpha \in A[X_{\mathcal{B}}(\mathbf{h}_\alpha)]$. *Condition* (iv) *implies the existence of unique* normal forms *for all homogeneous elements* $\mathbf{f} \in \mathcal{P}^m$. *Due to this condition, we find unique* $P_\alpha \in A[X_{\mathcal{B}}(\mathbf{h}_\alpha)]$ *for every* $\mathbf{h}_\alpha \in \mathcal{B}$ *such that* $\mathbf{f}' = \mathbf{f} - \sum_{\alpha=1}^{s} P_\alpha \mathbf{h}_\alpha$ *and* $\mathbf{f}' \in \langle \mathcal{N}(\mathrm{hm}(\mathcal{U})) \rangle^A$. *Another important consequence of the definition of a resolving decomposition is that* (1.1) *implies that every generator in* $\mathcal{B}$ *has a different head module term.*

While for the purposes of this work the mere existence of normal forms is sufficient, we note that (v) implies that they can be effectively computed. The choice of head terms and multiplicative variables in a resolving decomposition induces a natural reduction relation. If $\mathbf{f} \in \mathcal{P}^m_{\mathbf{d}}$ contains a module term $x^\epsilon \mathbf{e}_i \in \operatorname{hm}(\mathcal{U})$, then there exists a unique generator $\mathbf{h} \in \operatorname{hm}(\mathcal{B})$ such that $x^\epsilon \mathbf{e}_i = x^\delta \operatorname{hm}(\mathbf{h})$ with $x^\delta \in \mathbb{T}_{X_\mathcal{B}(\mathbf{h})}$ and we have a possible reduction $\mathbf{f} \xrightarrow{\mathcal{B}} \mathbf{f} - cx^\delta \mathbf{h}$ for a suitably chosen coefficient $c \in A$.

**Lemma 1.4.** *For any resolving decomposition $(\mathcal{B}, \operatorname{hm}(\mathcal{B}), X_\mathcal{B}, \prec_\mathcal{B})$ the transitive closure $\xrightarrow{\mathcal{B}}{}^*$ of $\xrightarrow{\mathcal{B}}$ is noetherian and confluent.*

*Proof.* It is sufficient to prove that for every term $x^\gamma \mathbf{e}_k$ in $\operatorname{hm}(\mathcal{U})$, there is a unique $\mathbf{g} \in \mathcal{P}^m_{\mathbf{d}}$ such that $x^\gamma \mathbf{e}_k \xrightarrow{\mathcal{B}}{}^* \mathbf{g}$ and $\mathbf{g} \in \langle \mathcal{N}(\operatorname{hm}(\mathcal{U})) \rangle$.

Since $x^\gamma \mathbf{e}_k \in \operatorname{hm}(\mathcal{U})$, there exists a unique $x^\delta \mathbf{h}_\alpha \in \mathcal{U}$ such that $x^\delta \operatorname{hm}(\mathbf{h}_\alpha) = x^\gamma \mathbf{e}_k$ and $x^\delta \in X_\mathcal{B}(\mathbf{h}_\alpha)$. Hence, $x^\gamma \mathbf{e}_k \xrightarrow{\mathcal{B}} x^\gamma \mathbf{e}_k - cx^\delta \mathbf{h}_\alpha$ for a suitably chosen coefficient $c \in A$. Denoting again the standard basis of $\mathcal{P}^s$ by $\{\mathbf{f}_1, \dots, \mathbf{f}_s\}$, we associated the term $x^\delta \mathbf{f}_\alpha$ with this reduction step. If we could proceed infinitely with further reduction steps, then the reduction process would induce a sequence of terms in $\mathcal{P}^s$ containing an infinite chain which, by condition (v) of Definition 1.1, is strictly descending for $\prec_\mathcal{B}$. But this is impossible, since $\prec_\mathcal{B}$ is a well-ordering. Hence, $\xrightarrow{\mathcal{B}}{}^*$ is noetherian. Confluence is immediate by the uniqueness of the element that is used at each reduction step. □

Furthermore, every resolving decomposition $(\mathcal{B}, \operatorname{hm}(\mathcal{B}), X_\mathcal{B}, \prec_\mathcal{B})$ induces a directed graph naturally. Its vertices are given by the elements in $\mathcal{B}$. If $x_j \in \overline{X}_\mathcal{B}(\mathbf{h})$ for some $\mathbf{h} \in \mathcal{B}$, then, by definition, $\mathcal{B}$ contains a unique generator $\mathbf{h}'$ such that $x_j \operatorname{hm}(\mathbf{h}) = x^\mu \operatorname{hm}(\mathbf{h}')$ with $x^\mu \in \mathbb{T}_{X_\mathcal{B}(\mathbf{h}')}$. In this case we include a directed edge from $\mathbf{h}$ to $\mathbf{h}'$. We call the thus defined graph the $\mathcal{B}$-graph.

**Lemma 1.5.** *The $\mathcal{B}$-graph of a resolving decomposition $(\mathcal{B}, \operatorname{hm}(\mathcal{B}), X_\mathcal{B}, \prec_\mathcal{B})$ is acyclic.*

*Proof.* Assume the $\mathcal{B}$-graph was cyclic. In this case we find generators $\mathbf{h}_{k_1}, \dots, \mathbf{h}_{k_t} \in \mathcal{B}$ which are pairwise distinct variables $x_{i_1}, \dots, x_{i_t}$ such that $x_{i_j} \in \overline{X}_\mathcal{B}(\operatorname{hm}(\mathbf{h}_{k_j}))$ for all $j \in \{1, \dots, t\}$ and terms $x^{\mu_1}, \dots, x^{\mu_t}$ such that $x^{\mu_j} \in \mathbb{T}_{X_\mathcal{B}(\operatorname{hm}(\mathbf{h}_{k_j}))}$ for all $j \in \{1, \dots, t\}$ satisfying:

$$\begin{aligned} x_{i_1} \operatorname{hm}(\mathbf{h}_{k_1}) &= x^{\mu_2} \operatorname{hm}(\mathbf{h}_{k_2}), \\ x_{i_2} \operatorname{hm}(\mathbf{h}_{k_2}) &= x^{\mu_3} \operatorname{hm}(\mathbf{h}_{k_3}), \\ &\vdots \\ x_{i_t} \operatorname{hm}(\mathbf{h}_{k_t}) &= x^{\mu_1} \operatorname{hm}(\mathbf{h}_{k_1}). \end{aligned}$$

Multiplying with some variables, we obtain the following chain of equations:

$$\begin{aligned} x_{i_1}\cdots x_{i_t}\,\mathrm{hm}(\mathbf{h}_{k_1}) &= x_{i_2}\cdots x_{i_t}x^{\mu_2}\,\mathrm{hm}(\mathbf{h}_{k_2})\\ &= x_{i_3}\cdots x_{i_t}x^{\mu_2}x^{\mu_3}\,\mathrm{hm}(\mathbf{h}_{k_3})\\ &\vdots\\ &= x_{i_t}x^{\mu_2}\cdots x^{\mu_t}\,\mathrm{hm}(\mathbf{h}_{k_t})\\ &= x^{\mu_1}\cdots x^{\mu_t}\,\mathrm{hm}(\mathbf{h}_{k_1}) \end{aligned}$$

which implies that $x_{i_1}\cdots x_{i_t} = x^{\mu_1}\cdots x^{\mu_t}$. Furthermore, condition (v) of Definition 1.1 implies in $\mathcal{P}^s$ the following chain:

$$x_{i_1}\cdots x_{i_t}\mathbf{f}_{k_1} \succeq_{\mathcal{B}} x_{i_2}\cdots x_{i_t}x^{\mu_2}\mathbf{f}_{k_2} \succeq_{\mathcal{B}} \cdots \succeq_{\mathcal{B}} x^{\mu_1}\cdots x^{\mu_t}\mathbf{f}_{k_1}\,.$$

Because of $x_{i_1}\cdots x_{i_t} = x^{\mu_1}\cdots x^{\mu_t}$, we must have throughout equality entailing that $k_1 = \cdots = k_t$ which contradicts our assumptions. □

The main obstacle to check if a given quadruple $(\mathcal{B}, \mathrm{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$ is a resolving decomposition is condition (v). In the following we show, that one can easily choose an ordering for a monomial module if the $\mathcal{B}$-graph is acyclic.

**Lemma 1.6.** *Let $\mathcal{B}$ be a generating set with multiplicative variables $X_{\mathcal{B}}$ such that $\mathcal{B} = \mathrm{hm}(\mathcal{B})$, e.g. $\mathcal{B}$ consists only of module terns. If the $\mathcal{B}$-graph is acyclic, then there is no cycle of the form*

$$\begin{aligned} x^{\nu_1}\,\mathrm{hm}(\mathbf{h}_{k_1}) &= x^{\mu_2}\,\mathrm{hm}(\mathbf{h}_{k_2}),\\ x^{\nu_2}\,\mathrm{hm}(\mathbf{h}_{k_2}) &= x^{\mu_3}\,\mathrm{hm}(\mathbf{h}_{k_3}),\\ &\vdots\\ x^{\nu_t}\,\mathrm{hm}(\mathbf{h}_{k_t}) &= x^{\mu_1}\,\mathrm{hm}(\mathbf{h}_{k_1}) \end{aligned}$$

*such that $x^{\mu_i} \in \mathbb{T}_{X_{\mathcal{B}}(\mathbf{h}_{k_i})}$ and $x^{\nu_i} \in \mathbb{T}$.*

*Furthermore, if $x^{\nu}\,\mathrm{hm}(\mathbf{h}_i) = x^{\mu}\,\mathrm{hm}(\mathbf{h}_j)$ such that $x^{\mu} \in \mathbb{T}_{X_{\mathcal{B}}(\mathbf{h}_j)}$ then there is a path from $\mathbf{h}_i$ to $\mathbf{h}_j$ in the $\mathcal{B}$-graph.*

*Proof.* Let us assume that there is such a cycle. In the following we show that this cycle induces a cycle in the $\mathcal{B}$-graph.

Without loss of generality we assume that $\gcd(x^{\nu_i}, x^{\mu_{i+1}}) = 1$ and $x^{\nu_i} \notin \mathbb{T}_{X_{\mathcal{B}}(\mathbf{h}_{k_i})}$.

Take $x_{i_0} \in \overline{X}_{\mathcal{B}}(\mathbf{h}_{k_1})$ such that $x_{i_0}$ divides $x^{\nu_1}$. In addition, we define $x^{\rho_0} = \frac{x^{\nu_1}}{x_{i_0}}$. Let the normal form of $x_{i_0}\,\mathrm{hm}(\mathbf{h}_{k_1})$ be $x^{\tau_1}\,\mathrm{hm}(\mathbf{h}_{l_1})$. Then $x^{\rho_0}x^{\tau_1}\,\mathrm{hm}(\mathbf{h}_{l_1}) = x^{\mu_2}\,\mathrm{hm}(\mathbf{h}_{k_2})$. Now we take $x_{i_1}$ which is non-multiplicative for $\mathbf{h}_{l_1}$ and divides $x^{\rho_0}x^{\tau_1}$. Again we define $x_{\rho_1}$ as $\frac{x^{\rho_0}x^{\tau_1}}{x_{i_1}}$ and then we repeat the procedure. Due to the fact that the $\mathcal{B}$-graph is acyclic and there are only finitely many $x^\rho, x^\tau$ and $\mathbf{h}_l$ such that $x^\rho x^\tau \operatorname{hm}(\mathbf{h}_l) = x^{\mu_2}\operatorname{hm}(\mathbf{h}_{k_2})$ we find after finitely many steps $x_{l_t}$ and $\mathbf{h}_{l_t}$ such that $x_{l_t} \in \overline{X}_\mathcal{B}(\mathbf{h}_{l_t})$, $x_{l_t}|x^{\rho_{t-1}}x^{\tau_t}$ and the normal form of $x_{l_t}\operatorname{hm}(\mathbf{h}_{l_t})$ is $x^{\mu_2}\operatorname{hm}(\mathbf{h}_{k_2})$.

Now we do the same for $\mathbf{h}_{k_2}, \mathbf{h}_{k_3}, \dots$. At the end we reach again $\mathbf{h}_{k_1}$. Hence, we have constructed a cycle in the $\mathcal{B}$-graph, which is a contradiction to our assumption.

The last statement of the lemma follows immediately from the construction above. □

**Lemma 1.7.** *Choose $\mathcal{B} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ and $X_\mathcal{B}$ such that all $\mathbf{h}_i$ are module terms and the conditions* (i) *to* (iv) *of Definition 1.1 are satisfied. Furthermore, the $\mathcal{B}$-graph should be acyclic and the elements of $\mathcal{B}$ are numbered in such a way that for a path from $\mathbf{h}_i$ to $\mathbf{h}_j$ in the $\mathcal{B}$-graph we always have $i < j$.*

*Let $\prec_\mathcal{B}$ be an arbitrary term ordering in $\mathcal{P}^s$ such that $x^\alpha \mathbf{f}_i \succ_\mathcal{B} x^\beta \mathbf{f}_j$ when $i < j$. Then ($\mathcal{B}$, $\operatorname{hm}(\mathcal{B}), X_\mathcal{B}, \prec_\mathcal{B}$) is a resolving decomposition.*

*Proof.* We only have to proof condition (v) of Definition 1.1. Take an arbitrary $\mathbf{h}_i \in \mathcal{B}$ and $x^\delta \in \mathbb{T}$. Then $x^\delta \mathbf{h}_i = x^\alpha \mathbf{h}_j$ for a suitable $x^\alpha \in \mathbb{T}_{X_\mathcal{B}(\mathbf{h}_j)}$. Due to Lemma 1.6 there is a path from $\mathbf{h}_i$ to $\mathbf{h}_j$ in the $\mathcal{B}$-graph and hence $i < j$. But then $x^\delta \mathbf{f}_i \succ_\mathcal{B} x^\alpha \mathbf{f}_j$ which proves the last condition of the definition of a resolving decomposition. □

The last lemma gives us a tool to check easily if a monomial generating set together with the assignment of multiplicative variables is a resolving decomposition. We only have to check if the corresponding $\mathcal{B}$-graph is acyclic. Then we can choose an arbitrary ordering which fulfils the property of Lemma 1.7 to finish the definition of a resolving decomposition. The existence of such an ordering is obvious because every position over term order[1] fulfils this property.

**Example 1.8.** *Let $\mathcal{P} = \Bbbk[x_0, x_1, x_2, x_3]$, $m = 1$ and consider the standard grading. Let $\mathcal{U}$ be the ideal generated by $x_0x_1, x_1^2, x_2x_3, x_3^3$ in $\mathcal{P}$. A Stanley decomposition of $\mathcal{U}$ is then given by the set*

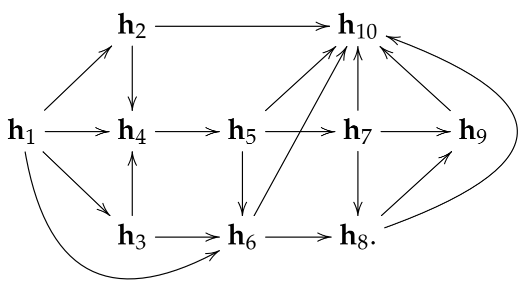
$$\begin{aligned}\mathcal{B} = \{&\mathbf{h}_1 = x_0x_1,\ \mathbf{h}_2 = x_0x_1x_2,\ \mathbf{h}_3 = x_0^2x_1,\ \mathbf{h}_4 = x_1^2,\ \mathbf{h}_5 = x_1^2x_3,\\ &\mathbf{h}_6 = x_0x_1x_3,\ \mathbf{h}_7 = x_1^2x_3^2,\ \mathbf{h}_8 = x_0x_1x_3^2,\ \mathbf{h}_9 = x_3^3,\ \mathbf{h}_{10} = x_2x_3\}\end{aligned}$$

[1] A position over term order $\prec_{POT}$ is an order for $\mathcal{P}^m_\mathbf{d}$ based on a term order $\prec$ of $\mathcal{P}$. Let $x^\mu\mathbf{e}_k, x^\nu\mathbf{e}_l \in \mathcal{P}^m_\mathbf{d}$. Then $x^\mu\mathbf{e}_k \succ_{POT} x^\nu\mathbf{e}_l$ if $k < l$ or if $k = l$ and $x^\mu \succ x^\nu$

*with multiplicative variables*

$$\begin{aligned}
X_{\mathcal{B}}(\mathbf{h}_1) &= \emptyset, & X_{\mathcal{B}}(\mathbf{h}_2) &= \{x_0, x_2\}\\
X_{\mathcal{B}}(\mathbf{h}_3) &= \{x_0, x_2\}, & X_{\mathcal{B}}(\mathbf{h}_4) &= \{x_0, x_1, x_2\}\\
X_{\mathcal{B}}(\mathbf{h}_5) &= \{x_1\}, & X_{\mathcal{B}}(\mathbf{h}_6) &= \{x_0, x_1\}\\
X_{\mathcal{B}}(\mathbf{h}_7) &= \{x_1\}, & X_{\mathcal{B}}(\mathbf{h}_8) &= \{x_0, x_1\}\\
X_{\mathcal{B}}(\mathbf{h}_9) &= \{x_0, x_1, x_3\}, & X_{\mathcal{B}}(\mathbf{h}_{10}) &= \{x_0, x_1, x_2, x_3\}\,.
\end{aligned}$$

*The $\mathcal{B}$-graph of this set is*




*This graph is obviously acyclic and hence we can choose an arbitrary ordering $\prec_{\mathcal{B}}$ like described in Lemma 1.7 which completes the definition of a resolving decomposition $(\mathcal{B}, \operatorname{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$.*

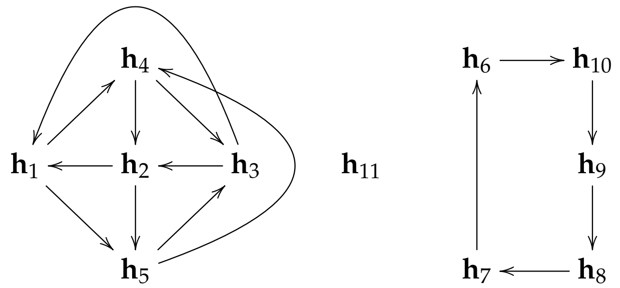
**Example 1.9.** *Even in the case of a monomial module, not every Stanley decomposition can be extended to a resolving decomposition. For the example we use $\mathcal{P} = \Bbbk[x_0, x_1, x_2, x_3, x_4]$, $m = 1$ and the standard grading. Let $\mathcal{U}$ be the homogeneous maximal ideal in $\mathcal{P}$. A Stanley decomposition of $\mathcal{U}$ is then given by the set*

$$\begin{aligned}
\mathcal{B} = \{&\mathbf{h}_1 = x_0, \mathbf{h}_2 = x_1, \mathbf{h}_3 = x_2, \mathbf{h}_4 = x_3, \mathbf{h}_5 = x_4, \mathbf{h}_6 = x_0x_1x_3, \mathbf{h}_7 = x_0x_2x_3,\\
&\mathbf{h}_8 = x_0x_2x_4, \mathbf{h}_9 = x_1x_2x_4, \mathbf{h}_{10} = x_1x_3x_4, \mathbf{h}_{11} = x_0x_1x_2x_3x_4\}
\end{aligned}$$

*with multiplicative variables*

$$\begin{aligned}
X_{\mathcal{B}}(\mathbf{h}_1) &= \{x_0, x_1, x_2\}, & X_{\mathcal{B}}(\mathbf{h}_2) &= \{x_1, x_2, x_3\}\\
X_{\mathcal{B}}(\mathbf{h}_3) &= \{x_2, x_3, x_4\}, & X_{\mathcal{B}}(\mathbf{h}_4) &= \{x_0, x_3, x_4\}\\
X_{\mathcal{B}}(\mathbf{h}_5) &= \{x_0, x_1, x_4\}, & X_{\mathcal{B}}(\mathbf{h}_6) &= \{x_0, x_1, x_2, x_3\}\\
X_{\mathcal{B}}(\mathbf{h}_7) &= \{x_0, x_2, x_3, x_4\}, & X_{\mathcal{B}}(\mathbf{h}_8) &= \{x_0, x_1, x_2, x_4\}\\
X_{\mathcal{B}}(\mathbf{h}_9) &= \{x_1, x_2, x_3, x_4\}, & X_{\mathcal{B}}(\mathbf{h}_{10}) &= \{x_0, x_1, x_3, x_4\}\\
X_{\mathcal{B}}(\mathbf{h}_{11}) &= \{x_0, x_1, x_2, x_3, x_4\}\,.
\end{aligned}$$

*It is not possible to find a term order $\prec_{\mathcal{B}}$ which makes this Stanley decomposition to a resolving one, as the corresponding $\mathcal{B}$-graph contains several cycles (note that here obviously $\operatorname{hm}(\mathbf{h}_i) = \mathbf{h}_i$):*




### 1.1.2 Syzygy Resolutions via Resolving Decompositions

Let $\mathcal{P}^m_{\mathbf{d}_0}$ be a graded free polynomial module with standard basis $\{\mathbf{e}^{(0)}_1, \dots, \mathbf{e}^{(0)}_m\}$ and grading $\mathbf{d}_0 = (d^{(0)}_1, \dots, d^{(0)}_m)$. Furthermore, let $(\mathcal{B}^{(0)}, \mathrm{hm}(\mathcal{B}^{(0)}), X_{\mathcal{B}^{(0)}}, \prec_{\mathcal{B}^{(0)}})$ be a resolving decomposition of a finitely generated graded module $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}_0}$ with $\mathcal{B}^{(0)} = \{\mathbf{h}_1, \dots, \mathbf{h}_{s_1}\}$. Our first goal is now to construct a resolving decomposition of the syzygy module $\mathrm{Syz}(\mathcal{B}^{(0)}) \subseteq \mathcal{P}^{s_1}$ which may be considered as a refined version of the well-known Schreyer theorem for Gröbner bases.

For every non-multiplicative variable $x_k$ of a generator $\mathbf{h}_\alpha$, we have a standard representation $x_k \mathbf{h}_\alpha = \sum_{\beta=1}^{s_1} P^{(\alpha;k)}_\beta \mathbf{h}_\beta$ and thus a syzygy

$$\mathbf{S}_{\alpha;k} := x_k \mathbf{e}^{(1)}_\alpha - \sum_{\beta=1}^{s_1} P^{(\alpha;k)}_\beta \mathbf{e}^{(1)}_\beta \tag{1.3}$$

where $\{\mathbf{e}^{(1)}_1, \dots, \mathbf{e}^{(1)}_{s_1}\}$ denotes the standard basis of the free module $\mathcal{P}^{s_1}_{\mathbf{d}_1}$ with grading $\mathbf{d}_1 = (\deg(\mathbf{h}_1), \dots, \deg(\mathbf{h}_s))$. Let $\mathcal{B}^{(1)}$ be the set of all these syzygies.

**Lemma 1.10.** *Let $S = \sum_{l=1}^{s_1} S_l \mathbf{e}^{(1)}_l$ be an arbitrary syzygy of $\mathcal{B}^{(0)}$ with coefficients $S_l \in \mathcal{P}$. Then $S_l \in A[X_{\mathcal{B}^{(0)}}(\mathbf{h}_l)]$ for all $1 \leq l \leq s_1$ if and only if $S = 0$.*

*Proof.* If $S \in \mathrm{Syz}(\mathcal{B}^{(0)})$, then $\sum_{l=1}^{s_1} S_l \mathbf{h}_l = 0$. Each $\mathbf{f} \in \mathcal{U}$ can be uniquely written in the form $\mathbf{f} = \sum_{l=1}^{s_1} P_l \mathbf{h}_l$ with $\mathbf{h}_l \in \mathcal{B}^{(0)}$ and $P_l \in A[X_{\mathcal{B}^{(0)}}(\mathbf{h}_l)]$. In particular, this holds for $0 \in \mathcal{U}$. Thus, $0 = S_l \in A[X_{\mathcal{B}^{(0)}}(\mathbf{h}_l)]$ for all $l$ and hence $S = 0$. □

For $\mathbf{h}_\alpha \in \mathcal{B}^{(0)}$ we denote the non-multiplicative variables by $\{x_{i^\alpha_1}, \dots, x_{i^\alpha_{r_\alpha}}\}$ with $i^\alpha_1 < \dots < i^\alpha_{r_\alpha}$. Thus, $\mathcal{B}^{(1)} = \bigcup_{j=1}^{s_1} \{\mathbf{S}_{j;i^j_k} \mid 1 \leq k \leq i^j_{r_j}\}$.

**Theorem 1.11.** *For every syzygy* $\mathbf{S}_{\alpha;i_k^\alpha} \in \mathcal{B}^{(1)}$ *we set*

$$\mathrm{hm}(\mathbf{S}_{\alpha;i_k^\alpha}) = x_{i_k^\alpha}\mathbf{e}_\alpha^{(1)}$$

*and*

$$X_{\mathcal{B}^{(1)}}(\mathbf{S}_{\alpha;i_k^\alpha}) = \{x_0,\dots x_n\} \setminus \{x_{i_1^\alpha},\dots,x_{i_{k-1}^\alpha}\}\,.$$

*Furthermore, we define* $\prec_{\mathcal{B}^{(1)}}$ *as the Schreyer order*[2] *associated to* $\mathcal{B}^{(1)}$ *and* $\prec_{\mathcal{B}^{(0)}}$*. Then the quadruple* $(\mathcal{B}^{(1)}, \mathrm{hm}(\mathcal{B}^{(1)}), X_{\mathcal{B}^{(1)}}, \prec_{\mathcal{B}^{(1)}})$ *is a resolving decomposition of the syzygy module* $\mathrm{Syz}(\mathcal{B}^{(0)})$*.*

*Proof.* We first show that $(\mathcal{B}^{(1)}, \mathrm{hm}(\mathcal{B}^{(1)}), X_{\mathcal{B}^{(1)}}, \prec_{\mathcal{B}^{(1)}})$ is a resolving decomposition of $\langle\mathcal{B}^{(1)}\rangle$. In a second step we show that $\langle\mathcal{B}^{(1)}\rangle = \mathrm{Syz}(\mathcal{B}^{(0)})$.

The first condition of Definition 1.1 is trivially satisfied. By construction, it is obvious to see that

$$\mathrm{hm}(\langle\mathcal{B}^{(1)}\rangle) = \bigoplus_{i=1}^{s_1}\langle\overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_i)\rangle\mathbf{e}_i^{(1)}. \tag{1.4}$$

A term $x^\mu\mathbf{e}_l^{(1)} \in \mathrm{supp}(\mathbf{S}_{\alpha;k} - x_k\mathbf{e}_\alpha^{(1)})$ must satisfy by (1.3) that $x^\mu \in \mathbb{T}_{X_{\mathcal{B}^{(0)}}(\mathbf{h}_l)}$ and hence $x^\mu\mathbf{e}_l^{(1)} \notin \mathrm{hm}(\langle\mathcal{B}^{(1)}\rangle)$ which implies the second condition of a resolving decomposition. The first part of third condition is again easy to see. It is obvious that

$$\langle\overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)\rangle\mathbf{e}_\alpha^{(1)} = \bigoplus_{k=1}^{r_\alpha} A[X_{\mathcal{B}^{(1)}}(\mathbf{S}_{\alpha,i_k^\alpha})]x_{i_k^\alpha}\mathbf{e}_\alpha^{(1)}.$$

If we combine this equation with (1.4) the first part of the third condition follows.

The second part of this condition is a bit harder to prove. We take an arbitrary $\mathbf{f} \in \langle\mathcal{B}^{(1)}\rangle$ and construct a standard representation for this module element. We construct this representation according to $\mathrm{hm}(\langle\mathcal{B}^{(1)}\rangle)$. We take the biggest term $x^\mu\mathbf{e}_\alpha^{(1)} \in \mathrm{supp}(\mathbf{f}) \cap \mathrm{hm}(\langle\mathcal{B}^{(1)}\rangle)$ with respect to the order $\prec_{\mathcal{B}^{(0)}}$. There must be a syzygy $\mathbf{S}_{\alpha;i}$ such that $x_i \mid x^\mu$ and $\frac{x^\mu}{x_i} \in \mathbb{T}_{X_{\mathcal{B}^{(1)}}(\mathbf{S}_{\alpha;i})}$. We reduce $\mathbf{f}$ by this element and get

$$\mathbf{f}' = \mathbf{f} - c\frac{x^\mu}{x_i}\mathbf{S}_{\alpha;i}$$

for a suitable constant $c \in A$ such that the term $x^\mu\mathbf{e}_\alpha^{(1)}$ is no longer in the support of $\mathbf{f}'$. Every term $x^\lambda\mathbf{e}_\beta^{(1)}$ newly introduced by $\frac{x^\mu}{x_i}\mathbf{S}_{\alpha;i}$ which also lies in $\mathrm{hm}(\mathcal{B}^{(1)})$ is strictly less

[2] Let $\mathcal{B} = \{\mathbf{h}_1,\dots\mathbf{h}_s\} \subset \mathcal{P}_{\mathbf{d}}^m$ be a finite subset, $\prec$ a term order on $\mathcal{P}_{\mathbf{d}}^m$ and $\mathbf{e}_1^{(1)},\dots\mathbf{e}_s^{(1)}$ the free generators of $\mathcal{P}^s$. The Schreyer order $\prec_{\mathcal{B}}$ is the term order on $\mathcal{P}^s$ defined by $x^\mu\mathbf{e}_\alpha^{(1)} \prec_{\mathcal{B}} x^\nu\mathbf{e}_\beta^{(1)}$, if $\mathrm{lt}_\prec(x^\mu\mathbf{h}_\alpha) \prec \mathrm{lt}_\prec(x^\eta\mathbf{h}_\beta)$ or if these leading terms are equal and $\beta < \alpha$.

than $x^\mu \mathbf{e}_\alpha^{(1)}$ according to the fifth condition of Definition 1.1 and equation (1.3) defining the syzygies $\mathbf{S}_{\alpha;i}$. Now we repeat this procedure until we arrive at an $\mathbf{f}''$ such that $\mathrm{supp}(\mathbf{f}'') \cap \mathrm{hm}\left(\langle \mathcal{B}^{(1)} \rangle\right) = \emptyset$. It is clear that we reach such an $\mathbf{f}''$ in a finite number of steps, since the terms during the reduction decrease with respect to $\prec_{\mathcal{B}^{(0)}}$ which is a well-order. We know that all $x^\epsilon \mathbf{e}_\alpha^{(1)} \in \mathrm{supp}(\mathbf{f}'')$ have the property that $x^\epsilon \in X_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)$. Therefore, we get that $\mathbf{f}'' = 0$ due to Lemma 1.10 which finishes the proof of this condition.

The procedure above provides us with an algorithm to compute arbitrary normal forms and hence the fourth condition of Definition 1.1 follows immediately. For the last condition we note that now each head term $x_i \mathbf{e}_\alpha^{(1)}$ is actually the leading term of $\mathbf{S}_{\alpha;i}$ with respect to the order $\prec_{\mathcal{B}^{(0)}}$. Hence, the corresponding Schreyer order satisfies the last condition of Definition 1.1. □

As with the usual Schreyer theorem, we can iterate this construction and derive this way a free resolution of $\mathcal{U}$. In contrast to the classical situation, it is however now possible to make precise statements about the *shape* of the resolution (even if we do not obtain explicit formulae for the differentials).

**Theorem 1.12.** *Let $\beta_{0,j}^{(k)}$ be the number of generators $\mathbf{h} \in \mathcal{B}^{(0)}$ of degree $j$ having $k$ multiplicative variables and set $d = \min\{k \mid \exists j : \beta_{0,j}^{(k)} > 0\}$. Then $\mathcal{U}$ possesses a finite free resolution*

$$0 \to \bigoplus \mathcal{P}(-j)^{r_{n+1-d,j}} \to \cdots \to \bigoplus \mathcal{P}(-j)^{r_{1,j}} \to \bigoplus \mathcal{P}(-j)^{r_{0,j}} \to U \to 0 \tag{1.5}$$

*of length $n+1-d$ where the ranks of the free modules are given by*

$$r_{i,j} = \sum_{k=1}^{n+1-i} \binom{n+1-k}{i} \beta_{0,j-i}^{(k)}.$$

*Proof.* According to Theorem 1.11, $(\mathcal{B}^{(1)}, \mathrm{hm}(\mathcal{B}^{(1)}), X_{\mathcal{B}^{(1)}}, \prec_{\mathcal{B}^{(1)}})$ is a resolving decomposition for the module $\mathrm{Syz}_1(\mathcal{U})$. Applying the theorem again, we can construct a resolving decomposition of the second syzygy module $\mathrm{Syz}_2(U)$ and so on. Recall that for every index $1 \leq l \leq s_1$ and for every non-multiplicative variable $x_k \in \overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_{\alpha(l)})$ we have $|\overline{X}_{\mathcal{B}^{(1)}}(\mathbf{S}_{l;k})| < |\overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_{\alpha(l)})|$.

If $D$ is the minimal number of multiplicative variables for a head module term in $\mathcal{B}^{(0)}$, then the minimal number of multiplicative variables for a head term in $\mathcal{B}^{(1)}$ is $D+1$. This observation yields the length of the resolution (1.5). Furthermore, $\deg(\mathbf{S}_{k;i}) = \deg(\mathbf{h}_k) + 1$, e. g. from the $j$th to the $(j+1)$th module the degree from the basis element to the corresponding syzygies grows by one.

The formula for the ranks of the modules follows from a rather straightforward combinatorial calculation. Let $\beta_{i,j}^{(k)}$ denote the number of generators of degree $j$ of the $i$th syzygy module $\mathrm{Syz}_i(\mathcal{U})$ with $k$ multiplicative variables according to the head module terms. By definition of the generators, we find

$$\beta_{i,j}^{(k)} = \sum_{t=1}^{k-1} \beta_{i-1,j-1}^{(n+1-t)}$$

as each generator with less multiplicative variables and degree $j-1$ in the resolving decomposition of $\mathrm{Syz}_i(\mathcal{B}^{(0)})$ contributes one generator with $k$ multiplicative variables. A lengthy induction allows us to express $\beta_{i,j}^{(k)}$ in terms of $\beta_{0,j}^{(k)}$:

$$\beta_{i,j}^{(k)} = \sum_{t=1}^{k-i} \binom{k-l-1}{i-1} \beta_{0,j-i}^{(t)}.$$

Now we are able to compute the ranks of the free modules via

$$r_{i,j} = \sum_{k=1}^{n+1} \beta_{i,j}^{(k)} = \sum_{k=1}^{n+1} \sum_{t=1}^{k-i} \binom{k-t-1}{i-1} \beta_{0,j-i}^{(t)} = \sum_{k=1}^{n+1-i} \binom{n-k}{i} \beta_{0,j-i}^{(k)}.$$

The last equality follows from a classical identity for binomial coefficients. □

Theorem 1.12 allows us to construct recursively resolving decompositions for the higher syzygy modules. In the sequel, we denote the corresponding resolving decomposition of the syzygy module $\mathrm{Syz}_j(\mathcal{U})$ by $(\mathcal{B}^{(j)}, \mathrm{hm}(\mathcal{B}^{(j)}), X_{\mathcal{B}^{(j)}}, \prec_{\mathcal{B}^{(j)}})$. To define an element of $\mathcal{B}^{(j)}$, we consider for each generator $\mathbf{h}_\alpha \in \mathcal{B}^{(0)}$ all ordered integer sequences $\mathbf{k} = (k_1, \ldots, k_j)$ with $0 \leq k_1 < \cdots < k_j \leq n$ of length $|\mathbf{k}| = j$ such that $x_{k_i} \in \overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)$ for all $1 \leq i \leq j$. We denote for any $1 \leq i \leq j$ by $\mathbf{k}_i$ the sequence obtained by eliminating $k_i$ from $\mathbf{k}$. Then the generator $\mathbf{S}_{\alpha;\mathbf{k}}$ arises recursively from the standard representation of $x_{k_j}\mathbf{S}_{\alpha;\mathbf{k}_j}$ according to the resolving decomposition $(\mathcal{B}^{(j-1)}, \mathrm{hm}(\mathcal{B}^{(j-1)}), X_{\mathcal{B}^{(j-1)}}, \prec_{\mathcal{B}^{(j-1)}})$:

$$x_{k_j}\mathbf{S}_{\alpha;\mathbf{k}_j} = \sum_{\beta=1}^{s_1} \sum_{\mathbf{l}} P_{\beta;\mathbf{l}}^{(\alpha;\mathbf{k})} \mathbf{S}_{\beta;\mathbf{l}}. \tag{1.6}$$

The second sum is taken over all ordered integer sequences $\mathbf{l}$ of length $j-1$ such that for all entries $\ell_i$ the variables $x_{\ell_i}$ is non-multiplicative for the generator $\mathbf{h}_\beta \in \mathcal{B}^{(0)}$. Denoting the free generators of the free module which contains the $j$th syzygy module by $\mathbf{e}_{\alpha,\mathbf{l}}^{(j)}$ such that $\alpha \in \{1, \ldots, s_1\}$ and $\mathbf{l}$ is an ordered subset of $\overline{X}_{\mathcal{B}^{(0)}}(\mathbf{h}_\alpha)$ of length $j-1$ we get the following representation for $\mathbf{S}_{\alpha,\mathbf{k}}$:

$$\mathbf{S}_{\alpha;\mathbf{k}} = x_{k_j}\mathbf{e}_{\alpha;\mathbf{k}_j}^{(j)} - \sum_{\beta=1}^{s_1} \sum_{\mathbf{l}} P_{\beta;\mathbf{l}}^{(\alpha;\mathbf{k})} \mathbf{e}_{\beta;\mathbf{l}}^{(j)}.$$

**Corollary 1.13.** *In the situation of Theorem 1.12, set* $d = \min\{k \mid \exists j : \beta_{0,j}^{(k)} > 0\}$ *and* $q = \deg(\mathcal{B}^{(0)}) = \max\{\deg(\mathbf{h}) \mid \mathbf{h} \in \mathcal{B}^{(0)}\}$*. Then we obtain the following bounds for the projective dimension, the Castelnuovo-Mumford regularity and the depth, respectively, of the submodule* $\mathcal{U}$*:*

$$\mathrm{pd}(\mathcal{U}) \le n+1-d\,, \qquad \mathrm{reg}(\mathcal{U}) \le q\,, \qquad \mathrm{depth}(\mathcal{U}) \ge d\,.$$

*Proof.* The first estimate follows immediately from the resolution (1.5) induced by the resolving decomposition $(\mathcal{B}^{(0)}, \mathrm{hm}(\mathcal{B}^{(0)}), X_{\mathcal{B}^{(0)}}, \prec_{\mathcal{B}^{(0)}})$ of $\mathcal{U}$. The last estimate is a simple consequence of the first one and the graded form of the Auslander-Buchsbaum formula. Finally, the $i$th module of this resolution is obviously generated by elements of degree less than or equal to $q+i$. This observation implies that $\mathcal{U}$ is $q$-regular and thus the second estimate. □

### 1.1.3 An Explicit Formula for the Differential

In the former sections we used polynomials rings over $\Bbbk$-algebras. For this section this is not applicable. Hence, we restrict us to the easier case that we only have polynomial rings $\mathcal{P}$ over $\Bbbk$. As before let $\mathcal{P}^m_{\mathbf{d}}$ be a graded free module with free generators $\mathbf{e}_1, \dots \mathbf{e}_m$ and grading $\mathbf{d} = (d_1, \dots, d_m)$. We always work with a finitely generated graded module $\mathcal{U} \in \mathcal{P}^m_{\mathbf{d}}$ and a resolving decomposition $(\mathcal{B}, \mathrm{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$ of $\mathcal{U}$ where $\mathcal{B} = \{\mathbf{h}_1, \dots, \mathbf{h}_{s_1}\}$.

First we give an alternative description of the complex underlying the resolution (1.5). Let $\mathcal{W} = \bigoplus_{\alpha=1}^{s_1} \Bbbk \cdot \mathbf{w}_\alpha$ and $\mathcal{V} = \bigoplus_{i=0}^{n} \Bbbk \cdot \mathbf{v}_i$ be two free $\Bbbk$-vector spaces whose dimensions are given by the size of $\mathcal{B}$ and by the number of variables in $\mathcal{P}$, respectively. Then we set $\mathcal{C}_i = \mathcal{W} \otimes_{\mathcal{P}} \Lambda_i \mathcal{V}$ where $\Lambda_\bullet$ denotes the exterior product. A $\mathcal{P}$-linear basis of $\mathcal{C}_i$ is provided by the elements $\mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k}$ where $\mathbf{v}_\mathbf{k} = \mathbf{v}_{k_1} \wedge \cdots \wedge \mathbf{v}_{k_i}$ for an ordered sequence $\mathbf{k} = (k_1, \dots, k_i)$ with $0 \le k_1 < \cdots < k_i \le n$. Then the free subcomplex $\mathcal{S}_\bullet \subset \mathcal{C}_\bullet$ generated by all elements $\mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k}$ with $\mathbf{k} \subseteq \overline{X}_{\mathcal{B}}(\mathbf{h}_\alpha)$ corresponds to (1.5) upon the identification $\mathbf{e}^{(i+1)}_{\alpha;\mathbf{k}} \leftrightarrow \mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k}$. Let $k_{i+1} \in \overline{X}_{\mathcal{B}}(\mathbf{h}_\alpha) \setminus \mathbf{k}$, then the differential comes from (1.6),

$$d_{\mathcal{S}}(\mathbf{w}_\alpha \otimes \mathbf{v}_{\mathbf{k},k_{i+1}}) = x_{k_{i+1}} \mathbf{w}_\alpha \otimes \mathbf{v}_\mathbf{k} - \sum_{\beta,\mathbf{l}} P^{(\alpha;\mathbf{k},k_{i+1})}_{\beta;\mathbf{l}} \mathbf{w}_\beta \otimes \mathbf{v}_\mathbf{l}\,,$$

and thus requires the explicit determination of all the higher syzygies (1.6).

In this section we present a method to directly compute the differential without computing higher syzygies. It is based on ideas of Sköldberg [45, 46] and generalises the theory which we developed in [4, 5] for the special case of a resolution induced by a Pommaret or a Janet basis for a given term order.

**Definition 1.14.** *A graded polynomial module $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ has* head linear syzygies *if it possesses a finite presentation*

$$0 \longrightarrow \ker\eta \longrightarrow \mathcal{W} = \bigoplus_{\alpha=1}^{s} \mathcal{P}\mathbf{w}_\alpha \stackrel{\eta}{\longrightarrow} \mathcal{U} \longrightarrow 0 \tag{1.7}$$

*with a finite generating set $\mathcal{B} = \{\mathbf{h}_1, \dots, \mathbf{h}_t\}$ of $\ker\eta$ where one can choose for each generator $\mathbf{h}_\alpha \in \mathcal{B}$ a head module term $\mathrm{hm}(\mathbf{h}_\alpha)$ of the form $x_i\mathbf{w}_\alpha$.*

Sköldberg's construction begins with the following two-sided Koszul complex $(\mathcal{F}, d_\mathcal{F})$ defining a free resolution of $\mathcal{U}$. Let $\mathcal{V}$ be a $\Bbbk$-linear space with basis $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ (with $n+1$ still the number of variables in $\mathcal{P}$) and set $\mathcal{F}_j = \mathcal{P} \otimes_\Bbbk \Lambda_j\mathcal{V} \otimes_\Bbbk \mathcal{U}$ which obviously yields a free $\mathcal{P}$-module. Let $\{m_a \mid a \in A\}$ be a $\Bbbk$-linear basis of $\mathcal{U}$, where $A$ is a Morse matching of $\mathcal{U}$ (see [4, Ch. 3]). Then a $\mathcal{P}$-linear basis of $\mathcal{F}_j$ is given by the elements $1 \otimes \mathbf{v}_\mathbf{k} \otimes m_a$ with ordered sequences $\mathbf{k}$ of length $j$. The differential is now defined by

$$d_\mathcal{F}(1 \otimes \mathbf{v}_\mathbf{k} \otimes m_a) = \sum_{i=1}^{j} (-1)^{i+1} \big( x_{k_i} \otimes \mathbf{v}_{\mathbf{k}_i} \otimes m_a - 1 \otimes \mathbf{v}_{\mathbf{k}_i} \otimes x_{k_i} m_a \big). \tag{1.8}$$

Here it should be noted that the second term on the right hand side is not yet expressed in the chosen $\Bbbk$-linear basis of $\mathcal{U}$. For notational simplicity, we will drop in the sequel the tensor sign $\otimes$ and leading factors 1 when writing elements of $\mathcal{F}_\bullet$.

Sköldberg uses a specialisation of head linear terms. He requires that for a given term order $\prec$ the leading module of $\ker\eta$ in the presentation (1.7) must be generated by terms of the form $x_i\mathbf{w}_\alpha$. In this case he says that $\mathcal{U}$ has *initially linear syzygies*. Our definition is term order free.

Under the assumption that the module $\mathcal{U}$ has initially linear syzygies via a presentation (1.7), Sköldberg [46] constructs a Morse matching leading to a smaller resolution $(\mathcal{G}, d_\mathcal{G})$. He calls the variables

$$\mathrm{crit}\,(\mathbf{w}_\alpha) = \{x_j \mid x_j\mathbf{w}_\alpha \in \mathrm{lt}_\prec(\ker\eta)\}\,;$$

*critical* for the generator $\mathbf{w}_\alpha$; the remaining *non-critical* ones are contained in the set $\mathrm{ncrit}\,(\mathbf{w}_\alpha)$. Then a $\Bbbk$-linear basis of $\mathcal{U}$ is given by all elements $x^\mu\mathbf{h}_\alpha$ with $\mathbf{h}_\alpha = \eta(\mathbf{w}_\alpha)$ and $x^\mu \in \Bbbk[\mathrm{ncrit}\,(\mathbf{w}_\alpha)]$.

According to [45] we define $\mathcal{G}_j \subseteq \mathcal{F}_j$ as the free submodule generated by those vertices $\mathbf{v}_\mathbf{k}\mathbf{h}_\alpha$ where the ordered sequences $\mathbf{k}$ are of length $j$ and such that every entry $k_i$ is critical for $\mathbf{w}_\alpha$. In particular $\mathcal{W} \cong \mathcal{G}_0$ with an isomorphism induced by $\mathbf{w}_\alpha \mapsto \mathbf{v}_\emptyset\mathbf{h}_\alpha$.

The description of the differential $d_\mathcal{G}$ is based on reduction paths in the associated Morse graph (for a detailed treatment of these notions, see [4, 45] or [30]) and expresses the differential as a triple sum. If we assume that, after expanding the right hand side of (1.8) in the chosen $\Bbbk$-linear basis of $\mathcal{U}$, the differential of the complex $\mathcal{F}_\bullet$ can be expressed as

$$d_{\mathcal{F}}(\mathbf{v}_{\mathbf{k}}\mathbf{h}_\alpha) = \sum_{\mathbf{m},\mu,\gamma} Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\,,$$

then $d_{\mathcal{G}}$ is defined by

$$d_{\mathcal{G}}(\mathbf{v}_{\mathbf{k}}\mathbf{h}_\alpha) = \sum_{\mathbf{l},\beta}\sum_{\mathbf{m},\mu,\gamma}\sum_{p}\rho_p\big(Q^{\mathbf{k},\alpha}_{\mathbf{m},\mu,\gamma}\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big) \tag{1.9}$$

where the first sum ranges over all ordered sequences $\mathbf{l}$ which consists entirely of critical indices for $\mathbf{w}_\beta$. Moreover, the second sum may be restricted to all values such that a polynomial multiple of $\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)$ effectively appears in $d_{\mathcal{F}}(\mathbf{v}_{\mathbf{k}}\mathbf{h}_\alpha)$ and the third sum ranges over all reduction paths $p$ going from $\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)$ to $\mathbf{v}_{\mathbf{l}}\mathbf{h}_\beta$. Finally, $\rho_p$ is the reduction associated to the reduction path $p$ satisfying

$$\rho_p\big(\mathbf{v}_{\mathbf{m}}(x^\mu \mathbf{h}_\gamma)\big) = q_p \mathbf{v}_{\mathbf{l}}\mathbf{h}_\beta$$

for some polynomial $q_p \in \mathcal{P}$.

It turns out that Sköldberg uses the term order $\prec$ only for distinguishing the critical and non-critical variables. Therefore, it is straightforward to see that his construction also works for modules which have head linear syzygies. We simply replace the definition of critical and non-critical variables. We define

$$\operatorname{crit}(\mathbf{w}_\alpha) = \{x_j \mid x_j\mathbf{w}_\alpha \in \operatorname{hm}(\mathcal{H})\}\,,$$

where $\mathcal{H}$ is chosen as in Definition 1.14. Again the remaining variables are contained in the set $\operatorname{ncrit}(\mathbf{w}_\alpha)$.

In the sequel we will show that for a finitely generated graded module $\mathcal{U}$ with a resolving decomposition $(\mathcal{B}, \operatorname{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$ the resolution constructed by Sköldberg's method is isomorphic to the resolution which is induced by the resolving decomposition if we choose the head linear syzygies properly. Firstly we obtain the following trivial assertion.

**Lemma 1.15.** *If the graded submodule $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ possesses a resolving decomposition $(\mathcal{B}, \operatorname{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$, then it has head linear syzygies. More precisely, we can set $\operatorname{crit}(\mathbf{w}_\alpha) = \overline{X}_{\mathcal{B}}(\mathbf{h}_\alpha)$, i.e. the critical variables of the generator $\mathbf{w}_\alpha$ are the non-multiplicative variables of $\mathbf{h}_\alpha = \eta(\mathbf{w}_\alpha)$.*

The lemmata which we subsequently cite from [4] are formulated for a Pommaret basis which is an involutive basis. Nevertheless, we can apply them directly in our setting, if not stated otherwise, because their proofs remain applicable for resolving decompositions. The reason for this is, that they only need the existence of unique standard representations and the division of variables into multiplicative and non-multiplicative ones. Some proofs in [4] explicitly use the class of a generator in $\mathcal{B}$, a notion arising in the context of Pommaret bases. When working with resolving decompositions, one has to replace it by the maximal index of a multiplicative variable.

The reduction paths can be divided into elementary paths of length two. There are essentially three types of reductions paths [4, Section 4]. The elementary reductions of *type 0* are not of interest [4, Lemma 4.5]. All remaining elementary reductions paths are of the form

$$\mathbf{v_k}(x^\mu \mathbf{h}_\alpha) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}(\frac{x^\mu}{x_i}\mathbf{h}_\alpha) \longrightarrow \mathbf{v_l}(x^\nu \mathbf{h}_\beta)\,.$$

Here $\mathbf{k} \cup i$ is the ordered sequence which arises when $i$ is inserted into $\mathbf{k}$; likewise $\mathbf{k} \setminus i$ stands for the removal of an index $i \in \mathbf{k}$.

**Type 1:** Here $\mathbf{l} = (\mathbf{k} \cup i)\backslash j$, $x^\nu = \frac{x^\mu}{x_i}$ and $\beta = \alpha$. Note that $i = j$ is allowed. We define $\epsilon(i;\mathbf{k}) := (-1)^{|\{j\in\mathbf{k}|j>i\}|}$. Then the corresponding reduction is

$$\rho(\mathbf{v_k}x^\mu \mathbf{h}_\alpha) = \epsilon(i;\mathbf{k}\cup i)\epsilon(j;\mathbf{k}\cup i)x_j \mathbf{v}_{(\mathbf{k}\cup i)\backslash j}(\frac{x^\mu}{x_i}\mathbf{h}_\alpha)\,.$$

**Type 2:** Now $\mathbf{l} = (\mathbf{k} \cup i) \setminus j$ and $x^\nu \mathbf{h}_\beta$ appears in the involutive standard representation of $\frac{x^\mu x_j}{x_i}\mathbf{h}_\alpha$ with the coefficient $\lambda_{j,i,\alpha,\mu,\nu,\beta} \in \Bbbk$. In this case, by construction of the Morse matching, we have $i \neq j$. The reduction is

$$\rho(\mathbf{v_k}x^\mu \mathbf{h}_\alpha) = -\epsilon(i;\mathbf{k}\cup i)\epsilon(j;\mathbf{k}\cup i)\lambda_{j,i,\alpha,\mu,\nu,\beta}\mathbf{v}_{(\mathbf{k}\cup i)\backslash j}(x^\nu \mathbf{h}_\beta)\,.$$

These reductions originate from the differential (1.8): The summands appearing there are either of the form $x_{k_i}\mathbf{v}_{\mathbf{k}_i}m_a$ or of the form $\mathbf{v}_{\mathbf{k}_i}(x_{k_i}m_a)$. For each of these summands, we have a directed edge in the Morse graph $\Gamma^A_{\mathcal{F}_\bullet}$. Thus, for an elementary reduction path

$$\mathbf{v_k}(x^\mu \mathbf{h}_\alpha) \longrightarrow \mathbf{v}_{\mathbf{k}\cup i}(\frac{x^\mu}{x_i}\mathbf{h}_\alpha) \longrightarrow \mathbf{v_l}(x^\nu \mathbf{h}_\beta)\,,$$

the second edge can originate from summands of either form. For the first form we then have an elementary reduction path of type 1 and for the second form we have type 2.

To show that the resolution induced by a resolving decomposition is isomorphic to the resolution constructed via Sköldberg's method we need a classical theorem concerning the uniqueness of free resolutions.

**Theorem 1.16.** *[17, Thm. 1.6] Let $\mathcal{U}$ be a finitely generated graded $\mathcal{P}_{\mathbf{d}}^m$-module. If $\mathcal{F}$ is the graded minimal free resolution of $\mathcal{U}$ and $\mathcal{G}$ an arbitrary graded free resolution of $\mathcal{U}$, then $\mathcal{G}$ is isomorphic to the direct sum of $\mathcal{F}$ and a trivial complex.*

Assume that we have two graded free resolutions $\mathcal{F}$, $\mathcal{G}$ of the same module $\mathcal{U}$ with the same shape (which means that the homogeneous components of the free modules in the two resolutions have always the same dimensions: $\dim_{\mathbb{k}}((\mathcal{F}_i)_j) = \dim_{\mathbb{k}}((\mathcal{G}_i)_j)$). Then Theorem 1.16 implies that the two resolutions are isomorphic. For the next theorem, we note the following important observation. The bases of the free modules in the resolution $\mathcal{G}$ of Sköldberg are given by the generators $\mathbf{v}_{\mathbf{k}}\mathbf{h}_\alpha$ with $\mathbf{k} \subseteq \overline{X}_{\mathcal{B}}(\mathbf{h}_\alpha)$.

**Theorem 1.17.** *Let $\mathcal{F}$ be the graded free resolution which is induced by the resolving decomposition $(\mathcal{B}, \mathrm{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$ and $\mathcal{G}$ the graded free resolution which is constructed by the method of Sköldberg when the head linear syzygies are chosen such that $\mathrm{crit}(\mathbf{h}_\alpha) = \overline{X}_{\mathcal{B}}(\mathbf{h}_\alpha)$ for every $\mathbf{h}_\alpha \in \mathcal{B}$. Then the resolutions $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.*

*Proof.* According to the observation made above, it is obvious that the two resolutions $\mathcal{F}$ and $\mathcal{G}$ have the same shape. Together with Theorem 1.16, the claim follows then immediately. □

For completeness, we repeat some simple results from [4]. They will show us, that the differentials of both resolutions are very similar. In fact, we show for the resolution constructed via Sköldberg's method, that we can find head module terms in the higher syzygies which are equal to the head module terms of the resolving decompositions of the higher syzygies of the induced free resolution.

**Lemma 1.18.** *[4, Lem. 4.3] For a non-multiplicative index $i \in \mathrm{crit}\,(\mathbf{h}_\alpha)$ let*

$$x_i \mathbf{h}_\alpha = \sum_{\beta=1}^{s_1} P_\beta^{(\alpha;i)} \mathbf{h}_\beta$$

*be the standard representation. Then we have*

$$d_{\mathcal{G}}(\mathbf{v}_i \mathbf{h}_\alpha) = x_i \mathbf{v}_\emptyset \mathbf{h}_\alpha - \sum_{\beta=1}^{s_1} P_\beta^{(\alpha;i)} \mathbf{v}_\emptyset \mathbf{h}_\beta \,.$$

The next result states that if one starts at a vertex $\mathbf{v}_i(x^\mu \mathbf{h}_\alpha)$ with certain properties and follows through all possible reduction paths in the graph, one will never get to a point where one must calculate a standard representation with respect to the given resolving decomposition. If there are no critical (i. e. non-multiplicative) variables present at the starting point, then this will not change throughout any reduction path. In order to generalise this lemma to higher homological degrees, one must simply replace the conditions $i \in \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $j \in \operatorname{ncrit}(\mathbf{h}_\beta)$ by ordered sequences $\mathbf{k}, \mathbf{l}$ with $\mathbf{k} \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$ and $\mathbf{l} \subseteq \operatorname{ncrit}(\mathbf{h}_\beta)$.

**Lemma 1.19.** *[4, Lem. 4.4] Assume that $i \cup \operatorname{supp}(\mu) \subseteq \operatorname{ncrit}(\mathbf{h}_\alpha)$. Then for any reduction path $p = \mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_j(x^\nu \mathbf{h}_\beta)$ we have $j \in \operatorname{ncrit}(\mathbf{h}_\beta)$. In particular, in this situation there is no reduction path $p = \mathbf{v}_i(x^\mu \mathbf{h}_\alpha) \to \cdots \to \mathbf{v}_k \mathbf{h}_\beta$ with $k \in \operatorname{crit}(\mathbf{h}_\beta)$.*

The next corollary asserts that we can choose in Sköldberg's resolution head module terms in such a way that there is a one-to-one correspondence to the head terms of the syzygies contained in the free resolution induced by the resolving decomposition. This corollary is a direct consequence of Lemma 1.19.

**Corollary 1.20.** *Let $(k_1, \ldots, k_j) = \mathbf{k} \subseteq \operatorname{crit} \mathbf{h}_\alpha$, then*

$$x_{k_l} \mathbf{v}_{\mathbf{k} \setminus k_l} \mathbf{h}_\alpha \in \operatorname{supp}(d_{\mathcal{G}}(\mathbf{v}_{\mathbf{k}} \mathbf{h}_\alpha)).$$

In [4] and [5] we show a method to effectively compute graded Betti numbers via the induced free resolution of Janet and Pommaret bases and the method of Sköldberg. We show that we can compute the graded Betti numbers with computing only the constant part of the resolution. With this method it is also possible to compute only a single Betti number without computing the complete constant part of the free resolution. The reason for that is that Sköldberg's formula allows to compute a differential in the free resolution independently of the rest of the free resolution. Furthermore, Theorem 1.12 gives us a formula to compute the ranks of this resolution. These methods are also applicable for an arbitrary resolving decomposition due to the fact that we proved Theorem 1.12 and the form of the differential (1.9).

## 1.2 Pommaret Bases

The computation of "good" generating systems of polynomial modules is an important task in commutative algebra and algebraic geometry. It is one of the starting points for further studies of polynomial modules. Gröbner bases are well-known generating systems of polynomial modules. A main advantage of them is that they are relative easy to compute.

Using Gröbner bases to study different kind of algebraic invariants turns out to be laborious. Usually the invariants could only be computed by costly operations on Gröbner bases. Involutive bases are a special kind of Gröbner bases. On the one hand they are as easy to compute as Gröbner bases but on the other hand they provide more easy accessible algebraic invariants of the represented polynomial modules. In fact, we show that they are a special kind of resolving decomposition.

Among all involutive bases it turns out that the Pommaret bases are special. At first Pommaret bases seem to be useless because a Pommaret basis does not always exist. By performing suitable linear coordinate changes it is possible to solve this obstacle. Assuming that we are acting in suitable coordinates they provide directly many algebraic invariants like the regularity or the depth. Hence, Pommaret bases are a good point to start a further analysis of polynomial modules.

### 1.2.1 Involutive Bases

In the following we use the abelian monoid $(\mathbb{N}^{n+1}, +)$ with componentwise addition. We call an element $\nu \in \mathbb{N}^{n+1}$ a *multi index* where we count the indices from $0, \dots, n$, that is $\nu = (\nu_0, \dots, \nu_n)$. Let $\nu \in \mathbb{N}^{n+1}$, then we define the *cone of* $\nu$ as $\mathcal{C}(\nu) := \nu + \mathbb{N}^{n+1}$. We say that $\nu$ *divides* $\mu$ $(\nu \mid \mu)$ if $\mu \in \mathcal{C}(\nu)$. If $\mathcal{V} \subseteq \mathbb{N}^{n+1}$ then the *span of* $\mathcal{V}$ is the monoid ideal

$$\langle \mathcal{V} \rangle = \bigcup_{\nu \in \mathbb{N}^{n+1}} \mathcal{C}(\nu)\,.$$

The idea of the involutive division is to restrict the cones in such a way that the union above is a disjoint one. We restrict the cone by restricting the addition of the multi indices. That means we only allow the addition in the cone for certain multi indices. Instead of speaking about the restriction of the cones we also speak about a restriction of the divisibility relation.

Let $N \subseteq \{0, \dots, n\}$ be an arbitrary subset; then we write $\mathbb{N}^{n+1}_N = \{\nu \in \mathbb{N}^{n+1} \mid \forall j \notin N : \nu_j = 0\}$ for the set of all multi indices where the only nonzero entries have an index which is contained in $N$.

**Definition 1.21.** *An* involutive division $L$ *is defined on the abelian monoid* $(\mathbb{N}^{n+1}, +)$ *if for any finite set* $\mathcal{V} \subset \mathbb{N}^{n+1}$ *a subset* $X_{L,\mathcal{V}}(\nu) \subseteq \{0, \dots, n\}$ *of* multiplicative indices *is associated to every multi index* $\nu \in \mathcal{V}$ *such that the following two conditions on the* involutive cones $\mathcal{C}_{L,\mathcal{V}}(\nu) := \nu + \mathbb{N}^n_{X_{L,\mathcal{V}}(\nu)}$ *are satisfied.*

*(i) It either holds* $\mathcal{C}_{L,\mathcal{V}}(\mu) \subseteq \mathcal{C}_{L,\mathcal{V}}(\nu)$ *or* $\mathcal{C}_{L,\mathcal{V}}(\nu) \subseteq \mathcal{C}_{L,\mathcal{V}}(\mu)$ *if there exist two elements* $\mu, \nu \in \mathcal{V}$ *with* $\mathcal{C}_{L,\mathcal{V}}(\mu) \cap \mathcal{C}_{L,\mathcal{V}}(\nu) \neq \emptyset$.

*(ii) If* $\mathcal{V}' \subset \mathcal{V}$, *then* $X_{L,\mathcal{V}}(\nu) \subseteq X_{L,\mathcal{V}'}(\nu)$ *for all* $\nu \in \mathcal{V}'$.

*An arbitrary multi index* $\mu \in \mathbb{N}^{n+1}$ *is* involutively divisible *by* $\nu \in \mathcal{V}$, *written* $\nu \mid_{L,\mathcal{V}} \mu$ *if* $\mu \in \mathcal{C}_{L,\mathcal{V}}(\nu)$.

Involutive divisibility is defined with respect to the involutive division $L$ *and* a fixed set $\mathcal{V} \subset \mathbb{N}^{n+1}$. We note that the only possible divisors must be in $\mathcal{V}$ and that involutive divisibility implies ordinary divisibility. In the definition we introduced multiplicative indices $X_{L,\mathcal{V}}(\nu)$. As for the definition of multiplicative variables for a resolving decomposition, we call the counterpart $\overline{X}_{L,\mathcal{V}}(\nu) := \{0,\dots,n\} \setminus X_{L,\mathcal{V}}(\nu)$ *non-multiplicative indices*.

Now we will see a first example of an involutive division

**Example 1.22.** *First we introduce certain subsets of the given set $\mathcal{V} \subset \mathbb{N}^{n+1}$ for $0 \leq k \leq n$:*

$$(d_k,\dots,d_n) = \{\nu \in \mathcal{V} \mid v_i = d_i,\ k \leq i \leq n\}.$$

*The* Janet division *$J$ is defined as the following assignment of multiplicative variables for the elements in $\mathcal{V}$: the index $n$ is multiplicative for $\nu$, if $\nu_n = \max_{\mu \in \mathcal{V}}\{\mu_n\}$, and $0 \leq k < n$ is multiplicative for $\nu \in (d_{k+1},\dots,d_n)$, if $\nu_k = \max_{\mu \in (d_{k+1},\dots,d_n)}\{\mu_k\}$.*

Removing an element $\nu$ of a given set $\mathcal{V} \subset \mathbb{N}^{n+1}$ and determining the multiplicative indices of the remaining elements with respect to $\mathcal{V}' = \mathcal{V} \setminus \{\nu\}$ again, gives in general a different result than before: the second condition of the definition of involutive division implies that only a non-multiplicative index can become multiplicative for some $\mu \in \mathcal{V}'$, but the converse cannot happen. The next definition treats involutive divisions where removing elements do not change the multiplicative indices of the remaining elements e.g. the set $\mathcal{V}$ does not play a role in computing multiplicative indices.

**Definition 1.23.** *The division $L$ is* globally defined, *if the assignment of the multiplicative indices is independent of the set $\mathcal{V}$. In this case we write $X_L(\nu)$.*

For an element $\nu \in \mathbb{N}^n$ we define the *class of $\nu$* as $\operatorname{cls}(\nu) := \min(\{i \mid \nu_i \neq 0\})$ when $\nu \neq [0,\dots,0]$ and $\operatorname{cls}([0,\dots,0]) := n$.

**Example 1.24.** *An important globally defined division is the* Pommaret division *$P$. It assigns the multiplicative indices according the following rule: Let $\nu \in \mathbb{N}^{n+1}$ and $k = \operatorname{cls}(\nu)$, then we set $X_P(\nu) = \{0,\dots,k\}$. Finally, we define $X_P([0,\dots,0]) = \{0,\dots,n\}$.*

Now we define the involutive span and an involutive basis, which is a disjoint union of involutive cones.

**Definition 1.25.** *The* involutive span *of a finite set $\mathcal{V} \subset \mathbb{N}^{n+1}$ is*

$$\langle \mathcal{V} \rangle_L = \bigcup_{\nu \in \mathcal{V}} \mathcal{C}_{L,\mathcal{V}}(\nu). \tag{1.10}$$

*The set $\mathcal{V}$ is* weakly involutive *for the division $L$ or a* weak involutive basis *of the monoid ideal $\langle\mathcal{V}\rangle$, if $\langle\mathcal{V}\rangle_L = \langle\mathcal{V}\rangle$. A weak involutive basis is a* (strong) involutive basis*, if the union on the right hand side of* (1.10) *is disjoint. That is, the intersections of the involutive cones are empty. Any finite set $\mathcal{V} \subseteq \overline{\mathcal{V}} \subset \mathbb{N}^{n+1}$ such that $\langle\overline{\mathcal{V}}\rangle_L = \langle\mathcal{V}\rangle$ is called a* (weak) involutive completion *of $\mathcal{V}$. An* obstruction to involution *for the set $\mathcal{V}$ is a multi index $\nu \in \langle\mathcal{V}\rangle \setminus \langle\mathcal{V}\rangle_L$.*

In Gröbner basis theory there exists the concept of an autoreduced set. Equivalently there is also such a term in involutive theory.

**Definition 1.26.** *A set $\mathcal{V} \subset \mathbb{N}^{n+1}$ is called* involutively autoreduced *with respect to the involutive division $L$ if there exist no two distinct multi indices $\mu, \nu \in \mathcal{V}$ such that $\mu \mid_{L,\mathcal{V}} \nu$.*

An obvious observation is that every (strong) involutive basis is involutively autoreduced. Furthermore, the definition of the Janet division implies that $\mathcal{C}_{J,\mathcal{V}}(\mu) \cap \mathcal{C}_{J,\mathcal{V}}(\nu) = \emptyset$ whenever $\mu \neq \nu$. Hence, for the Janet division any set is involutively autoreduced.

We want to use (strong) involutive bases. The following proposition shows that it is always possible to extract a (strong) involutive bases from a weak involutive basis in the monomial case.

**Proposition 1.27** ([43, Prop. 3.1.12])**.** *If $\mathcal{V}$ is a weak involutive basis, then there exists a subset $\mathcal{V}' \subseteq \mathcal{V}$ which is a (strong) involutive basis of $\langle\mathcal{V}\rangle$.*

The minimal involutive basis of a monoid ideal is unique, if it exists. For a globally defined division, we can even show that any involutive basis is unique.

**Proposition 1.28** ([43, Prop. 3.1.21])**.** *Let $L$ be a globally defined division and $J \subseteq \mathbb{N}^{n+1}$ a monoid ideal. If $J$ has a strong involutive basis for $L$, then it is unique and thus minimal.*

Note that a finite involutive basis of $\langle\mathcal{V}\rangle$ does not always exist, as we see in the following example.

**Example 1.29.** *We use the Pommaret division and $\mathcal{V} = \{[1,1]\}$. The class of $[1,1]$ is zero and hence $X_P([1,1]) = \{0\}$. So $\mathcal{C}_P([1,1]) \subsetneq \mathcal{C}([1,1])$, but also every other multi index in $\langle\mathcal{V}\rangle$ has class zero. Hence, there cannot be a finite involutive basis of $\langle\mathcal{V}\rangle$. We can generate it involutively only with the infinite set $\{[1,k] \mid k \in \mathbb{N} \setminus \{0\}\}$.*

**Definition 1.30.** *An involutive division $L$ is called* noetherian *if any finite subset $\mathcal{V} \subset \mathbb{N}^{n+1}$ possesses a finite involutive completion with respect to $L$.*

**Lemma 1.31** ([43, Prop. 3.1.19])**.** *The Janet division is noetherian.*

So far we have only defined involutive bases for monoid ideals. Now we extend the theory to finitely generated graded submodules of $\mathcal{P}^m_{\mathbf{d}}$, where $\mathcal{P}$ is a polynomial ring over a noetherian $\Bbbk$-algebra. In the following, we equip the free module with an arbitrary term order $\prec$.

In addition to the conventions at the beginning of this chapter, we define for an element $\mathbf{f} \in \mathcal{P}^m_{\mathbf{d}}$ the *leading (module) term* $\operatorname{lt}_\prec(\mathbf{f}) := x^\mu \mathbf{e}_i$ which is the biggest module term in $\operatorname{supp}(\mathbf{f})$ with respect to $\prec$. Note that we ignore the coefficient of $\mathbf{f}$ in $\operatorname{lt}_\prec(\mathbf{f})$. Furthermore, we define for a set $\mathcal{B} \subseteq \mathcal{P}^m_{\mathbf{d}}$ the *leading set* as $\operatorname{lt}_\prec(\mathcal{B}) := \{\operatorname{lt}_\prec(\mathbf{f}) \mid \mathbf{f} \in \mathcal{B}\}$. For a submodule $\mathcal{U} \in \mathcal{P}^m_{\mathbf{d}}$ we define the *leading module* as $\operatorname{lt}_\prec(\mathcal{U}) := \langle\{\operatorname{lt}_\prec(\mathbf{f}) \mid \mathbf{f} \in \mathcal{U}\}\rangle$. We call a module element $\mathbf{f} \in \mathcal{P}^m_{\mathbf{d}}$ *monic* if the leading term has coefficient $1_A$.

**Definition 1.32.** *Let $\mathcal{P}^m_{\mathbf{d}}$ be a finitely generated free $\mathcal{P}$-module with grading $\mathbf{d}$ and free generators $\mathbf{e}_1, \dots, \mathbf{e}_m$. Let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a finitely generated graded submodule of $\mathcal{P}^m_{\mathbf{d}}$ with $\operatorname{lt}_\prec(\mathcal{U}) = \bigoplus_{k=1}^m \mathcal{J}^{(k)}\mathbf{e}_k$ such that the $\mathcal{J}^{(k)}$ are monomial ideals.*

*A finite monic subset $\mathcal{B} \subset \mathcal{U}$ is a* weak involutive basis *of $\mathcal{U}$ for an involutive division $L$ on $\mathbb{N}^{n+1}$ if its leading module terms $\operatorname{lt}_\prec(\mathcal{B})$ form a weak involutive basis of the monomial module $\operatorname{lt}_\prec(\mathcal{U})$. That is, the sets $\mathcal{B}^{(k)} := \{x^\mu \mid x^\mu \mathbf{e}_k \in \operatorname{lt}_\prec(\mathcal{B})\}$ are weak involutive bases of $\mathcal{J}^{(k)}$ for all $k \in \{1, \dots, m\}$.*

*The set $\mathcal{B}$ is a* (strong) involutive basis *of $\mathcal{U}$ if the sets $\mathcal{B}^{(k)}$ are strong involutive bases for $\mathcal{J}^{(k)}$ for all $k \in \{1, \dots, m\}$ and no two distinct elements of $\mathcal{B}$ have the same leading module terms.*

*The set $\mathcal{B}$ is called* (weakly) involutive *if it is a (weak) involutive basis of $\langle\mathcal{B}\rangle$.*

In the next definition we translate some notions from the monomial case to the module case.

**Definition 1.33.** *Let $\mathcal{B} \subset \mathcal{P}^m_{\mathbf{d}} \setminus \{0\}$ be a finite monic set and $L$ an involutive division on $\mathbb{N}^{n+1}$. Assume that $\mathbf{f} \in \mathcal{B}$ and $\operatorname{lt}_\prec(\mathbf{f}) = x^\mu \mathbf{e}_k$. By considering again the set $\mathcal{B}^{(k)}$ from Definition 1.32 we define the multiplicative variables of $\mathbf{f}$ as*

$$X_{L,\mathcal{B}}(\mathbf{f}) = \{x_i \mid i \in X_{L,\mathcal{B}^{(k)}}(x^\mu)\}\,.$$

*The* involutive span *of $\mathcal{B}$ is then the set*

$$\langle\mathcal{B}\rangle_L := \sum_{\mathbf{f} \in \mathcal{B}} A[X_{L,\mathcal{B}}(\mathbf{f})] \cdot \mathbf{f} \subseteq \langle\mathcal{B}\rangle. \tag{1.11}$$

Standard representations are an important aspect of Gröbner bases. We can also prove that there exist involutive standard representations which have furthermore the nice property of uniqueness.

**Theorem 1.34** ([41, Thm. 5.4]). *Let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a nonzero finitely generated graded submodule, $\mathcal{B} \subset \mathcal{U} \setminus \{0\}$ a finite monic set and L an involutive division on $\mathbb{N}^{n+1}$. Then the following two statements are equivalent.*

- *The set $\mathcal{B}$ is a weak involutive basis of $\mathcal{U}$ with respect to L and $\prec$.*
- *Every module element $\mathbf{f} \in \mathcal{U}$ can be written in the form*

$$\mathbf{f} = \sum_{\mathbf{h} \in \mathcal{B}} p_{\mathbf{h}} \cdot \mathbf{h} \tag{1.12}$$

*where the coefficients $p_{\mathbf{h}} \in A[X_{L,\mathcal{B}}(\mathbf{h})]$ satisfy $\mathrm{lt}_{\prec}(p_{\mathbf{h}} \cdot \mathbf{h}) \preceq \mathrm{lt}_{\prec}(\mathbf{f})$ for all module elements $\mathbf{h} \in \mathcal{B}$.*

*$\mathcal{B}$ is a strong involutive basis if and only if the representation* (1.12) *is unique for all $\mathbf{f} \in \mathcal{U}$.*

**Remark 1.35.** *In the preceding theorem we considered* monic *sets $\mathcal{B}$. It must be monic because we consider a polynomial ring over a noetherian $\Bbbk$-algebra, hence in general the leading coefficient is not invertible. If we consider a polynomial ring over a field, then we do not need this property.*

**Corollary 1.36** ([41, Cor. 5.5]). *Let the set $\mathcal{B}$ be a weak involutive basis of the finitely generated graded submodule $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$. Then $\langle \mathcal{B} \rangle_L = \mathcal{U}$*

Note that the converse is in general not true.

**Example 1.37.** *Consider in the ordinary polynomial ring $\Bbbk[x_0, x_1]$ with `degrevlex` order $\prec$ the ideal $\mathcal{I}$ generated by two polynomials $\mathbf{f}_1 = x_1^2$ and $\mathbf{f}_2 = x_1^2 + x_0^2$. Then the set $\mathcal{B} = \{\mathbf{f}_1, \mathbf{f}_2\}$ trivially satisfies $\langle \mathcal{B} \rangle_J = \mathcal{I}$, as with respect to the Janet division all variables are multiplicative for each generator. However, $\mathrm{lt}_{\prec}(\mathcal{B}) = \{[0,2]\}$ obviously does not generate $\mathrm{lt}_{\prec}(\mathcal{I})$ because it is clear, that $\mathrm{lt}_{\prec}(\mathcal{I}) = \langle \{[2,0],[0,2]\} \rangle$. Thus, $\mathcal{B}$ is not a weak Janet basis.*

Now we see that we can extract a strong involutive basis out of every weak involutive basis, as in the monoid case.

**Proposition 1.38** ([41, Prop. 5.7]). *Let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a finitely generated graded submodule and $\mathcal{B} \subset \mathcal{P}^m_{\mathbf{d}}$ a weak involutive basis of $\mathcal{U}$ for the involutive division L. Then there exists a subset $\mathcal{B}' \subseteq \mathcal{B}$ which is a strong involutive basis of $\mathcal{U}$.*

In our case there is no need to discuss about weak involutive bases, therefore we will discuss in the following only about strong involutive bases. Nevertheless, weak involutive bases are useful in a more general setting.

In the next definition we introduce the notation of an involutive normal form and an involutive autoreduced set which are similar to the usual definitions concerning Gröbner bases.

**Definition 1.39.** *Let $\mathcal{B} \subset \mathcal{P}^m_{\mathbf{d}}$ be a finite monic set and L an involutive division. A module element $\mathbf{g} \in \mathcal{P}^m_{\mathbf{d}}$ is* involutively reducible *with respect to $\mathcal{B}$, if it contains a module term $x^\mu \mathbf{e}_i$ such that $\mathrm{lt}_\prec(\mathbf{f}) \mid_{L,\mathcal{B}} x^\mu \mathbf{e}_i$ for some $\mathbf{f} \in \mathcal{B}$. It is in* involutive normal form *with respect to $\mathcal{B}$, if it is not involutively reducible.*

*The set $\mathcal{B}$ is involutively autoreduced, if no module element $\mathbf{f} \in \mathcal{B}$ contains a module term $x^\mu \mathbf{e}_i$ such that another module element $\mathbf{f}' \in \mathcal{B} \setminus \{\mathbf{f}\}$ exists with $\mathrm{lt}_\prec(\mathbf{f}') \mid_{L,\mathcal{B}} x^\mu \mathbf{e}_i$.*

*An* obstruction to involution *is a module element $\mathbf{g} \in \langle \mathcal{B} \rangle \setminus \langle \mathcal{B} \rangle_L$ possessing a (necessarily non-involutive) standard representation with respect to $\mathcal{B}$.*

If $\mathcal{G}$ is a Gröbner basis of the finitely generated graded submodule $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$, then any element of $\mathcal{U}$ has a standard representation with respect to $\mathcal{G}$. But this does not imply that for a given division $L$ the submodule $\mathcal{U}$ is free of obstructions to involution with respect to $\mathcal{G}$. In order to obtain at least a weak involutive basis, we must usually add further elements of $\mathcal{U}$ to $\mathcal{G}$ until $\langle \mathrm{lt}_\prec(\mathcal{G}) \rangle_L = \mathrm{lt}_\prec(\mathcal{U})$.

Often we are only interested in the leading term of a module element $\mathbf{g}$. Therefore, we introduce the following notions.

**Definition 1.40.** *Let $\mathcal{B} \subset \mathcal{P}^m_{\mathbf{d}}$ be a finite monic set and L an involutive division. A module element $\mathbf{g} \in \mathcal{P}^m_{\mathbf{d}}$ is* involutively head reducible *if $\mathrm{lt}_\prec(\mathbf{f}) \mid_{L,\mathcal{B}} \mathrm{lt}_\prec(\mathbf{g})$ for some $\mathbf{f} \in \mathcal{B}$. The set $\mathcal{B}$ is* involutively head autoreduced *if the leading exponent of every element $\mathbf{f} \in \mathcal{B}$ is not involutively divisible with respect to L and $\mathcal{B}$ by the leading exponent of an element $\mathbf{f}' \in \mathcal{B} \setminus \{\mathbf{f}\}$.*

If the set $\mathcal{B}$ is a strong involutive basis this immediately implies that $\mathcal{B}$ is involutively head autoreduced.

The involutive reduction is a restriction of the ordinary reduction. Due to that involutive normal forms can be computed similar to the normal forms computed with respect to Gröbner bases. If $\mathbf{g}'$ is an involutive normal form of $\mathbf{g} \in \mathcal{P}^m_{\mathbf{d}}$ with respect to the set $\mathcal{B}$ for the division $L$, then we write $\mathbf{g}' = \mathrm{NF}_{L,\mathcal{B}}(\mathbf{g})$. The ordinary normal form is unique if and only if it is computed with respect to a Gröbner basis. For the involutive normal form the situation is a bit different.

**Lemma 1.41** ([41, Lem. 5.12])**.** *The sum in* (1.11) *is direct if and only if the finite monic set $\mathcal{B} \subset \mathcal{P}^m_{\mathbf{d}} \setminus \{0\}$ is involutively head autoreduced with respect to the involutive division L.*

**Proposition 1.42** ([41, Prop. 5.14])**.** *The ordinary and the involutive normal form of any module element $\mathbf{g} \in \mathcal{P}^m_{\mathbf{d}}$ with respect to a finite monic weakly involutive set $\mathcal{B} \subset \mathcal{P}^m_{\mathbf{d}} \setminus \{0\}$ are identical.*

With the preceding proposition we are now able to extend the definition of a minimal involutive basis from $\mathbb{N}^{n+1}$ to a free module $\mathcal{P}^m_{\mathbf{d}}$. It is done in the same way as in the ordinary Gröbner bases case.

**Definition 1.43.** *Let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a nonzero finitely generated graded submodule and L an involutive division. An involutive basis $\mathcal{B}$ of $\mathcal{U}$ with respect to L is* minimal *if $\mathcal{B}^{(i)}$ is the minimal involutive basis of $\mathcal{J}^{(i)}$ for all $i \in \{1, \dots, m\}$, where $\mathcal{B}^{(i)}$ and $\mathcal{J}^{(i)}$ are defined like in Definition 1.32.*

By Proposition 1.28 we see that for any globally defined division any involutive basis is minimal. But in general uniqueness requires two additional assumptions in the module case.

**Proposition 1.44** ([22, Thm. 5.2])**.** *Let $\mathcal{P}$ be a polynomial ring over an arbitrary field $\mathbb{k}$. Furthermore, let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a nonzero finitely generated graded submodule and L an involutive division. Then $\mathcal{U}$ possesses at most one monic, involutively autoreduced, minimal involutive basis for the division L.*

The main objective of this section is to introduce a class of resolving decompositions which based on term orders. It turns out that the definition of involutive bases is not enough. In addition to that we need *continuity*.

**Definition 1.45.** *Let L be an involutive division and $\mathcal{V} \subset \mathbb{N}^{n+1}$ a finite set. Furthermore, let $\nu^{(1)}, \dots, \nu^{(t)}$ be a finite sequence of elements of $\mathcal{V}$ where every multi index $\nu^{(k)}$ with $k < t$ has a non-multiplicative index $j_k \in \overline{X}_{L,\mathcal{V}}(\nu^{(k)})$ such that $\nu^{(k+1)} \mid_{L,\mathcal{V}} \nu^{(k)} + 1_{j_k}$. The division L is* continuous *if any such sequence consists only of distinct elements, i. e. $\nu^{(k)} \neq \nu^{(l)}$ for all $k \neq l$.*

**Proposition 1.46** ([21, Cor. 4.11])**.** *The Janet and Pommaret division are continuous.*

**Definition 1.47.** *Let $\mathcal{B} \subset \mathcal{P}^m_{\mathbf{d}}$ be an involutive basis for the involutive division L. An L*-graph *of the basis $\mathcal{B}$ is a graph associated to the involutive basis $\mathcal{B}$. Its vertices are given by the terms in $\operatorname{lt}_\prec(\mathcal{B})$. If $x_j \in \overline{X}_{L,\mathcal{B}}(\mathbf{h})$ for some generator $\mathbf{h} \in \mathcal{B}$, then, by definition of an involutive basis, $\mathcal{B}$ contains a unique generator $\overline{\mathbf{h}}$ such that $\operatorname{lt}_\prec(\overline{\mathbf{h}})$ is an involutive divisor of $\operatorname{lt}_\prec(x_j\mathbf{h})$. In this case we include a directed edge from $\operatorname{lt}_\prec(\mathbf{h})$ to $\operatorname{lt}_\prec(\overline{\mathbf{h}})$.*

**Lemma 1.48** ([42, Lem. 5.5])**.** *If the division L is continues then the L-graph of any involutive set $\mathcal{B} \subset \mathcal{P}^m_{\mathbf{d}}$ is acyclic.*

**Proposition 1.49.** *Let $\prec$ be a term order on the free module $\mathcal{P}^m_{\mathbf{d}}$, L a continuous involutive division and $\mathcal{B} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ a finite, L-involutively autoreduced set which is a strong L-involutive basis of a finitely generated graded submodule $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$. Then $\mathcal{B}$ induces a resolving decomposition with $\operatorname{hm}(\mathbf{h}_i) = \operatorname{lt}_\prec(\mathbf{h}_i)$ and $X_{\mathcal{B}}(\mathbf{h}_i) = X_{L,\mathcal{B}}(\mathbf{h}_i)$ for all $i \in \{1, \dots, s\}$. We take as $\prec_{\mathcal{B}}$ the Schreyer order induced by $\mathcal{B}$ and $\prec$.*

*Proof.* Condition (i) of Definition 1.1 follows from Corollary 1.36, condition (ii) is a consequence of the fact that $\mathcal{B}$ is involutively autoreduced and condition (iii) follows from Lemma 1.41. According to Proposition 1.42 every $\mathbf{f} \in \mathcal{P}^m_{\mathbf{d}}$ possesses a unique normal form. In Remark 1.3 we have seen that this is equivalent to the fourth condition in our definition. Finally, (v) is satisfied because of Lemma 1.48 and the existence of an $L$-ordering. Let $\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}$ denote again the standard basis of the free module $\mathcal{P}^s$. Then an $L$-ordering $\prec_{L,\mathcal{B}}$ on $\mathcal{P}^s$ must satisfy that $\mathbf{f}_\alpha \prec_{L,\mathcal{B}} \mathbf{f}_\beta$ whenever there is a path from $\mathbf{h}_\alpha$ to $\mathbf{h}_\beta$ in the corresponding $\mathcal{B}$-graph. Due to Lemma 1.48 it is obvious that for every continuous division $L$ such a term order exists. Hence, an autoreduced involutive basis always induces a resolving decomposition. □

**Remark 1.50.** *The resolving decomposition $(\mathcal{B}^{(1)}, \mathrm{hm}(\mathcal{B}^{(1)}), X_{\mathcal{B}^{(1)}}, \prec_{\mathcal{B}^{(1)}})$ constructed in Theorem 1.11 is always a Janet basis of the first syzygy module with respect to the term order $\prec_{\mathcal{B}^{(0)}}$. This is simply due to the fact that the choice of the multiplicative variables in the resolving decomposition of the syzygy module made in Theorem 1.11 is actually inspired by what happens for the Janet division. Hence, in the special case that the resolving decomposition is induced by a Janet basis, it is easy to see that also the resolving decompositions of the higher syzygy modules are actually induced by Janet bases for a Schreyer order constructed as in Theorem 1.11.*

*At this point, one can also see some advantages of our general framework. Previous results require that the used involutive division is of Schreyer type (see [42, Def. 5.8]). This assumption ensures that we obtain at each step again an L-involutive basis for the syzygy module with respect to a Schreyer order. With the new approach, we automatically obtain Janet basis, as we can choose the head terms and the multiplicative variables as we like. Consequently, we can now use an involutive basis $\mathcal{B}$ for an arbitrary involutive division L as starting point for the construction of a resolution, provided its L-graph is acyclic (which is always the case if L is continuous). The construction will not necessarily lead to L-involutive bases of the syzygy modules, but for most applications this fact is irrelevant.*

### 1.2.2 Pommaret Bases

In the previous section we have seen in Example 1.29 that the Pommaret division does not always possess a finite Pommaret basis for a module. Hence, the Pommaret division is not noetherian. In the following we restrict ourselves to such polynomial modules which always induce finite Pommaret bases. In [42] it is shown that we can perform on every polynomial module a deterministic change of coordinates to get a new module which possess a finite Pommaret basis.

We show that one can easily extract the regularity, depth and projective dimension out of a Pommaret basis. At the end of this section we shortly analyse the saturation of a module with a finite Pommaret basis.

**Definition 1.51** ([42, Def. 2.1]). *The variables* $\mathbf{x}$ *are* $\delta$-regular *for the finitely generated graded submodule* $\mathcal{U} \subseteq \mathcal{P}_{\mathbf{d}}^m$ *if* $\mathcal{U}$ *possesses a finite Pommaret basis.*

In Proposition 1.49 we proved that every involutive basis induces a resolving decomposition. Hence, a Pommaret basis induces a resolving decomposition. In Corollary 1.13 we get upper bounds for the projective dimension and regularity and a lower bound for the depth of a submodule. If the resolving decomposition is induced by a finite Pommaret basis for a suitable term order we get equality in every case.

**Theorem 1.52** ([42, Thm. 8.11]). *Let* $\mathcal{B}$ *be a Pommaret basis of the finitely generated graded submodule* $\mathcal{U} \subseteq \mathcal{P}_{\mathbf{d}}^m$ *for a class respecting term order and set* $d = \min(\{\operatorname{cls}(\mathbf{h}) \mid \mathbf{h} \in \mathcal{B}\}) + 1$*. Then* $\operatorname{pd}(\mathcal{U}) = n + 1 - d$*.*

With the theorem of Auslander-Buchsbaum we get equality for the depth as well.

**Corollary 1.53.** *Let* $\mathcal{B}$ *be a Pommaret basis of the finitely generated graded submodule* $\mathcal{U} \subseteq \mathcal{P}_{\mathbf{d}}^m$ *for a class respecting term order and set* $d = \min(\{\operatorname{cls}(\mathbf{h}) \mid \mathbf{h} \in \mathcal{B}\})$*. Then* $\operatorname{depth}(\mathcal{U}) = d$*.*

For the depth and the projective dimensions it is enough to assume that the Pommaret basis is computed with respect to a class respecting term order. For the regularity it is necessary to choose the degree reverse lexicographic order.

**Theorem 1.54** ([42, Thm. 9.2]). *Let* $\mathcal{U} \subseteq \mathcal{P}_{\mathbf{d}}^m$ *be a finitely generated graded submodule with a finite homogeneous Pommaret basis with respect to the degree reverse lexicographic order. The regularity of* $\mathcal{U}$ *is* $q$ *if and only if* $\mathcal{U}$ *has homogeneous Pommaret basis of degree* $q$ *with respect to the degree reverse lexicographic order.*

The following lemma analyses the behaviour of elements in a $\delta$-regular monomial module. This lemma will be fundamental for the last section in this chapter.

**Lemma 1.55** ([7, Lem. 3]). *Let* $\mathcal{V} \subseteq \mathcal{P}_{\mathbf{d}}^m$ *be a* $\delta$*-regular monomial module with Pommaret basis* $\mathcal{B}$*. Then:*

*(i) If* $x^\alpha \mathbf{e}_k \in \mathcal{V} \setminus \mathcal{B}$ *then* $\frac{x^\alpha}{x_{\operatorname{cls}(x^\alpha)}} \mathbf{e}_k \in \mathcal{V}$*.*

*(ii) If* $x^\alpha \mathbf{e}_k \notin \mathcal{V}$ *and* $x_i x^\alpha \mathbf{e}_k \in \mathcal{V}$*, then either* $x_i x^\alpha \mathbf{e}_k \in \mathcal{B}$ *or* $i > \operatorname{cls}(x^\alpha)$*.*

*(iii) If* $x^\alpha \mathbf{e}_k \notin \mathcal{V}$ *and* $x^\delta x^\alpha \mathbf{e}_k \in \mathcal{C}_P(x^\beta \mathbf{e}_k)$ *such that* $x^\beta \mathbf{e}_k \in \mathcal{B}$ *and* $x^\delta x^\alpha \mathbf{e}_k = x^{\delta'} x^\beta \mathbf{e}_k$*, then* $x^{\delta'} \prec_{lex} x^\delta$*.*

For the rest of this section we introduce the saturation and the satiety of a module. Furthermore, we show some interesting properties concerning Pommaret bases and the saturation of modules in $\delta$-regular modules.

**Definition 1.56.** *Let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ a graded module. Then the* saturation *of $\mathcal{U}$ is defined as*

$$\mathcal{U}^{\mathrm{sat}} = \mathcal{U} : \mathcal{P}_+^\infty = \{\mathbf{f} \in \mathcal{P}^m_{\mathbf{d}} \mid \exists k \in \mathbb{N} \ : \ \mathbf{f} \cdot \mathcal{P}_k \subseteq \mathcal{U}\}.$$

*A module $\mathcal{U}$ is called* saturated *if $\mathcal{U} = \mathcal{U}^{\mathrm{sat}}$. The smallest value $q_0$ such that $\mathcal{U}_q = \mathcal{U}^{\mathrm{sat}}_q$ for all $q \geq q_0$ is called the* satiety $\mathrm{sat}(\mathcal{U})$ *of the module $\mathcal{U}$.*

Due to the next proposition it is easy to determine the saturation of an $\delta$-regular module.

**Proposition 1.57.** *Let $\mathcal{B}$ be a Pommaret basis of the graded module $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ for a class respecting term order $\prec$. We introduce the sets $\mathcal{B}_0 = \{\mathbf{h} \in \mathcal{B} \mid \mathrm{cls}(\mathbf{h}) = 0\}$ and*

$$\overline{\mathcal{B}_0} = \left\{ \frac{\mathbf{h}}{x_0^{\deg_{x_0}(\mathrm{lt}_\prec(\mathbf{h}))}} \mid \mathbf{h} \in \mathcal{B}_0 \right\}.$$

*Then $\overline{\mathcal{B}} = (\mathcal{B} \setminus \mathcal{B}_0) \cup \overline{\mathcal{B}_0}$ is a weak Pommaret basis of the saturation $\mathcal{U}^{\mathrm{sat}}$.*

*Proof.* The proof is analogous to the proof of [42, Prop. 10.1]. □

An involutive head autoreduction of the set $\overline{\mathcal{B}}$ yields a strong Pommaret basis for $\mathcal{U}^{\mathrm{sat}}$. Another trivial consequence of this proposition is that for $\delta$-regular coordinates the following equality is satisfied:

$$\mathcal{U}^{\mathrm{sat}} = \mathcal{U} : \mathcal{P}_+^\infty = \mathcal{U} : x_0^\infty.$$

**Corollary 1.58.** *Let $\mathcal{B}$ be a Pommaret basis of the module $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$. Then $\mathcal{U}$ is saturated if and only if $\mathcal{B}_0 = \emptyset$. If $\mathcal{U}$ is not saturated, then $\mathrm{sat}(\mathcal{U}) = \deg(\mathcal{B}_0)$.*

*Proof.* The proof is analogous to the proof of [42, Cor. 10.2]. □

Furthermore, we have the following formula relating the regularity and saturation.

**Corollary 1.59.** *Let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a module. Then*

$$\mathrm{reg}(\mathcal{U}) = \max(\{\mathrm{sat}(\mathcal{U}), \mathrm{reg}(\mathcal{U}^{\mathrm{sat}})\}).$$

*Proof.* The proof is analogous to the proof of[42, Cor. 10.4]. □

## 1.3 Stable Positions

In this section we introduce several kinds of stability. Stability is a classical combinatorial concept that plays an important role in the theory of monomial ideals. We recall the basic theory of the several stability concepts and show the connection to Pommaret bases. If not stated otherwise we use in this section a polynomial ring over a noetherian $\mathbb{k}$-algebra $A$, again.

The first concept which we introduce is the concept of quasi-stability.

**Definition 1.60.**

- *Let $\mathcal{J} \subseteq \mathcal{P}$ be a monomial ideal and $q$ the maximal degree of a minimal generator of $\mathcal{J}$. Then the ideal $\mathcal{J}$ is* quasi-stable *if for every term $x^\mu \in \mathcal{J}$ and every index $j > c = \operatorname{cls}(x^\mu)$ the term $x_j^q \frac{x^\mu}{x_c^{\mu_c}}$ also lies in $\mathcal{J}$.*
- *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a monomial submodule. Then $\mathcal{V} = \bigoplus_{i=1}^m \mathcal{J}_i \mathbf{e}_i$, such that all $\mathcal{J}_i \subseteq \mathcal{P}$ are monomial ideals. We define $\mathcal{V}$ to be* quasi-stable *if all $\mathcal{J}_i$ are quasi-stable.*

There are many equivalent definitions of quasi-stability. We repeat some of them in the following proposition. An important observation is that for monomial modules the terms $\delta$-regular and quasi-stable are equivalent. Hence, a monomial module has a finite Pommaret basis if and only if it is quasi-stable.

**Proposition 1.61.** *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a monomial module, such that $\mathcal{V} = \bigoplus_{i=1}^m \mathcal{J}_i \mathbf{e}_i$, where $\mathcal{J}_i \subseteq \mathcal{P}^m_{\mathbf{d}}$ are monomial ideals. Then the following statements are equivalent:*

- *$\mathcal{V}$ is quasi-stable.*
- *For each module term $x^\alpha \mathbf{e}_k \in \mathcal{V}$ and for all integers $i, j, l$ such that $1 \leq i < j \leq n$ and $x_i^l$ divides $x^\alpha$, there exists $s \geq 0$ such that $x_j^s \frac{x^\alpha}{x_i^l} \mathbf{e}_k \in \mathcal{V}$.*
- *For each term $x^\alpha \mathbf{e}_k \in \mathcal{V}$ and for all integers $i, j$ such that $1 \leq i < j \leq n$, there exists $s \geq 0$ such that $x_j^s \frac{x^\alpha}{x_i^{\alpha_i}} \mathbf{e}_k \in \mathcal{V}$.*
- *For all $1 \leq j \leq n$ and for all $1 \leq i \leq m$ we have*
$$\mathcal{J}_i : x_j^\infty = \mathcal{J}_i : \langle x_j, \ldots, x_n \rangle^\infty.$$
- *For all $1 \leq i \leq m$ the variable $x_0$ is not a zero divisor for $\mathcal{P}/\mathcal{J}_i^{\mathsf{sat}}$ and for all $1 \leq j < d$ the variable $x_{j+1}$ is not a zero divisor for $\mathcal{P}/\langle \mathcal{J}_i, x_0, \ldots, x_j \rangle^{\mathsf{sat}}$.*
- *$\mathcal{V}$ is $\delta$-regular.*

*Proof.* Using the identity $\mathcal{V} = \bigoplus_{i=1}^{m} \mathcal{J}_i \mathbf{e}_i$ we can apply [7, Thm. 1] to every $\mathcal{J}_i$ which shows the claim. □

The next concept that we introduce is the concept of stability.

**Definition 1.62.**

- *Let $\mathcal{J} \subseteq \mathcal{P}$ be a monomial ideal. Then $\mathcal{J}$ is* stable, *if for every term $x^\mu \in \mathcal{J}$ and every index $j > c = \operatorname{cls}(x^\mu)$ the term $x_j \frac{x^\mu}{x_c}$ also lies in $\mathcal{J}$.*
- *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a monomial submodule. Then $\mathcal{V} = \bigoplus_{i=1}^{m} \mathcal{J}_i \mathbf{e}_i$, such that all $\mathcal{J}_i \subseteq \mathcal{P}$ are monomial ideals. We define $\mathcal{V}$ to be* stable *if all $\mathcal{J}_i$ are stable.*

The following proposition shows that for stable ideals the Pommaret basis and the minimal basis coincide. This implies immediately that every stable ideal is always quasi-stable, too.

**Proposition 1.63.** *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a monomial module. $\mathcal{V}$ is stable, if and only if its minimal basis $\mathcal{B}$ is simultaneously a Pommaret basis.*

*Proof.* This is a direct consequence of [42, Prop. 8.6] which proves the claim for the ideal case. □

**Proposition 1.64.**

- *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a quasi-stable module generated in degrees less than or equal to $s$. The monomial module $\mathcal{V}$ is $s$-regular if and only if $\mathcal{V}_{\geq s}$ is stable.*
- *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a quasi-stable module and consider $s \geq \operatorname{reg}(\mathcal{V})$. Then $\mathcal{V}_{\geq s}$ is stable and the set of terms $\mathcal{V}_s \cap \mathbb{T}^m$ is its Pommaret basis.*

*Proof.* This is a direct consequence of [7, Prop. 1] which proves the claim for the ideal case. □

As a last combinatorial definition we define strong stability. Again we see immediately that every ideal which is strongly stable is stable, too.

**Definition 1.65.**

- *Let $\mathcal{J} \subseteq \mathcal{P}$ be a monomial ideal. Then $\mathcal{J}$ is* strongly stable *if for every term $x^\mu \in \mathcal{J}$ and every index pair $i < j$ such that $x_i | x^\mu$ the term $x_j \frac{x^\mu}{x_i}$ also lies in $\mathcal{J}$.*
- *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a monomial submodule. Then $\mathcal{V} = \bigoplus_{i=1}^{m} \mathcal{J}_i \mathbf{e}_i$, such that all $\mathcal{J}_i \subseteq \mathcal{P}$ are monomial ideals. We define $\mathcal{V}$ to be* strongly stable *if all $\mathcal{J}_i$ are strongly stable.*

The definitions above are always independent of the characteristic of the basic field. In the following we introduce the concept of Borel fixed ideals and modules. For this we need the assumption that we act over a polynomial ring over an arbitrary field $\mathbb{k}$. We will see that these ideals and modules depend on the characteristic of the basic field $\mathbb{k}$.

**Definition 1.66.** *Let $GL(n+1,\mathbb{k})$ be the general linear group. That is the group of invertible $n+1 \times n+1$-matrices with entries in $\mathbb{k}$. Every $g = (g_{i,j})_{i,j\in\{0,\ldots n\}} \in GL(n+1,\mathbb{k})$ induces an automorphism*

$$\begin{aligned} g : \mathcal{P}^m_{\mathbf{d}} &\longrightarrow \mathcal{P}^m_{\mathbf{d}} \\ \mathbf{f}(x_0,\ldots,x_n) &\longmapsto \mathbf{f}\left(\sum_{j=0}^{n} g_{0,j}x_j,\ldots,\sum_{j=0}^{n} g_{n,j}x_j\right). \end{aligned}$$

*For every module $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$, we write $g(\mathcal{U})$ for $\langle g(\mathbf{f}(x_0,\ldots,x_n)) \mid \mathbf{f} \in \mathcal{U}\rangle$.*

*We denote by $\mathfrak{B}$ the* Borel subgroup *of $GL(n+1,\mathbb{k})$ consisting of upper triangular matrices.*

**Definition 1.67.** *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a monomial module. We say that $\mathcal{V}$ is* Borel fixed *if for every $g \in \mathfrak{B}$, $g(\mathcal{V}) = \mathcal{V}$.*

If we consider only ideals of $\mathcal{P}$ for the definition above, we do not need to demand that the ideal must be monomial, because in [16, Thm. 15.23] it was shown that every Borel fixed ideal must be monomial. For submodules of $\mathcal{P}^m_{\mathbf{d}}$ this is not the case. For example consider $\mathcal{U} \subseteq \mathbb{k}[x,y]^2$ such that $\mathcal{U} = \langle y\mathbf{e}_1 + y\mathbf{e}_2\rangle$. It is obviously closed under the operation of the Borel subgroup, but it is not a monomial module. Nevertheless, we demand that the Borel fixed modules are monomial so that they fit in the context of stable modules.

Computationally it is very difficult to check if a module is fixed under the operators of the Borel subgroup. Therefore, we develop in the following a concept which implies an easy computational way to check if a module is Borel- fixed or not.

**Definition 1.68.** *Let $p$ be a prime number and $a, b \in \mathbb{N}$. We define $a <_p b$ if and only if $\binom{a}{b} \neq 0 \mod p$. We extend this definition for $p = 0$ demanding that $a <_0 b$ if and only if $a \leq b$ in the usual sense.*

**Definition 1.69.** *Let $x^\alpha \in \mathbb{T}$ and choose $i < j$ and $s > 0$ such that $x_i^s | x^\alpha$. We define the $s$th* increasing move *on the term $x^\alpha \in \mathbb{T}$ as*

$$e^{+(s)}_{i,j}(x^\alpha) = \frac{x_j^s}{x_i^s} x^\alpha = x_0^{\alpha_0} \cdots x_i^{\alpha_i - s} \cdots x_j^{\alpha_j + s} \cdots x_n^{\alpha_n}.$$

*The increasing move $e^{+(s)}_{i,j}(x^\alpha)$ is $p$-*admissible *on the term $x^\alpha$ if $e^{+(s)}_{i,j}(x^\alpha) \in \mathbb{T}$ and $s <_p \alpha_i$.*

**Definition 1.70.** *For a fixed prime p, we define the following relation on the terms of* $\mathbb{T}$*:* $x^\alpha \prec_p x^\beta$ *if and only if there is a p-admissible increasing move* $e_{i,j}^{+(s)}$ *such that* $e_{i,j}^{+(s)}(x^\alpha) = x^\beta$*. The transitive closure of this relation gives a partial order on the set of monomials of any fixed degree, that we will keep on denoting by* $\prec_p$*.*

**Theorem 1.71.** *Let* $\mathrm{char}(\Bbbk) = p \geq 0$ *and* $\mathcal{V} \in \mathcal{P}_{\mathbf{d}}^m$ *be a monomial module with minimal generating set* $\mathcal{B}$*.* $\mathcal{V}$ *is Borel fixed if and only if for every* $x^\alpha \mathbf{e}_k \in \mathcal{B}$*, if* $x^\alpha \prec_p x^\beta$ *then* $x^\beta \mathbf{e}_k \in \mathcal{V}$*.*

*Proof.* Using the identity $\mathcal{V} = \bigoplus_{i=1}^m \mathcal{J}_i \mathbf{e}_i$ we can apply [7, Thm. 6] to every $\mathcal{J}_i$ which shows the claim. □

In the following a monomial module $\mathcal{V} \subseteq \mathcal{P}_{\mathbf{d}}^m$ is called *p-Borel fixed* if it is Borel fixed in $\mathcal{P}_{\mathbf{d}}^m$, with $\mathrm{char}(\Bbbk) = p$. The question if a monomial module is Borel fixed or not depends on the characteristic. It is easy to see that for $\mathrm{char}(\Bbbk) = 0$ the property Borel fixed is equivalent to strong stability. For arbitrary characteristic it is only true that strong stability always induces Borel fixedness. But a $p$-Borel fixed module is not always strongly stable, as the following example shows.

**Example 1.72.** *Consider* $\mathcal{P} = \Bbbk[x_0, \dots, x_4]$*,* $\mathcal{B} = \{x_4, x_3, x_2^2, x_1^2, x_0^2\}$ *and* $\mathcal{J} = \langle \mathcal{B} \rangle$*. The ideal is not p-Borel fixed for* $p \neq 2$ *because* $e_{2,3}^{+(1)}(x_2^2) = x_2 x_3$ *is a p-admissible move for all* $p \neq 2$*, but* $\mathcal{J}$ *does not contain* $x_2 x_3$*.*

*But the ideal is* 2*-Borel fixed, because the increasing moves* $e_{i,j}^{+(1)}(x_i^2)$ *are not* 2*-admissible for all* $i \in \{1, 2, 3\}$ *and* $j > i$*. All other increasing moves are obviously* 2*-admissible which implies that* $\mathcal{J}$ *is* 2*-Borel fixed.*

*In contrast to that the ideal* $\mathcal{J} = \langle x_4, x_3, x_2, x_1, x_0^8 \rangle$ *is obviously strongly stable and hence p-admissible for all possible p.*

The definitions above are always for monomial modules. For a given term order $\prec$ we can extend the definitions to an arbitrary module $\mathcal{U} \subseteq \mathcal{P}_{\mathbf{d}}^m$. We say that $\mathcal{U}$ is in *quasi-stable/stable/strongly stable/p-Borel fixed Position* if $\mathrm{lt}_\prec(\mathcal{U})$ is quasi-stable/stable/strongly stable/$p$-Borel fixed.

## 1.4 Marked Modules

Resolving decompositions are generating sets which give us an easy way to gain much information about the minimal free resolution of a finitely generated module. In contrast to other special generating sets of finitely generated modules, the head module of a resolving decomposition does not depend on a term order. In the following we assume that we have a given head module. For modules which "contain" this given head module we will see that we can construct a resolving decomposition for them. This work is based on [8, 12, 13, 33], where the notions of a marked polynomial and marked basis were introduced. Now we extend this to finitely generated modules and show that a marked basis induces a resolving decomposition.

In general, it is not possible to construct an algorithm which computes a resolving decomposition for a given finitely generated module. The main problem is that due to the lack of a term order it is hard to define a terminating normal form algorithm. But, such an algorithm is mandatory for computing new generating sets for a given finitely generated module. We will see that we can obtain a resolving decomposition if the given finitely generated module "contains" a quasi-stable monomial module.

For completeness, we first generalize some notions introduced in [8, 12, 13, 33]. We assume that $\mathcal{P}^m_{\mathbf{d}} = A[\mathbf{x}]^m_{\mathbf{d}}$ is a free module over $A[\mathbf{x}]$, where $A$ is a $\Bbbk$-Algebra over a field of arbitrary characteristic.

**Definition 1.73.** *A* marked homogeneous module element *is a homogeneous module element in $\mathcal{P}^m_{\mathbf{d}}$ with a fixed term in its support whose coefficient is $1_A$ and which is called* head term*. We denote a marked homogeneous module element by $\mathbf{f}^k_\alpha$ if the head term is* $\mathrm{Ht}(\mathbf{f}^k_\alpha) = x^\alpha \mathbf{e}_k$.

*We associate the following sets to a marked module element $\mathbf{f}^k_\alpha$:*

- *the* multiplicative variables of $\mathbf{f}^k_\alpha$*:* $X_{P_m}(\mathbf{f}^k_\alpha) := X_P(\mathrm{Ht}(\mathbf{f}^k_\alpha))$*;*
- *the* non-multiplicative variables of $\mathbf{f}$*:* $\overline{X}_{P_m}(\mathbf{f}) := \overline{X}_P(\mathrm{Ht}(\mathbf{f}))$.

The following definition is fundamental for this section. It is modelled after a well-known characteristic property of Gröbner bases.

**Definition 1.74.** *Let $\mathcal{C} \subset \mathbb{T}^m$ be a finite set and $\mathcal{V}$ the module generated by $\mathcal{C}$ in $\mathcal{P}^m_{\mathbf{d}}$. A $\mathcal{C}$-*marked set *is a finite set $\mathcal{B} \subset \mathcal{P}^m_{\mathbf{d}}$ of marked homogeneous module elements $\mathbf{f}^k_\alpha$ with* $\mathrm{Ht}(\mathbf{f}^k_\alpha) = x^\alpha \mathbf{e}^k \in \mathcal{C}$ *and* $\mathrm{supp}(\mathbf{f}^k_\alpha - x^\alpha \mathbf{e}_k) \subset \langle \mathcal{N}(\mathcal{V}) \rangle$ *(obviously, $|\mathcal{C}| = |\mathcal{B}|$).*

*The $\mathcal{C}$-marked set $\mathcal{B}$ is a $\mathcal{C}$-*marked basis *if $\mathcal{N}(\mathcal{V})_s$ is a basis of $(\mathcal{P}^m_{\mathbf{d}})_s / \langle \mathcal{B} \rangle_s$ as $A$-module, i. e. if $(\mathcal{P}^m_{\mathbf{d}})_s = \langle \mathcal{B} \rangle_s \oplus \langle \mathcal{N}(\mathcal{V})_s \rangle^A$ for all $s$.*

As we wrote in Remark 1.3 the defining property of a marked basis is equivalent to the existence of a normal form algorithm. The next lemma shows that we find a marked set on every degree $s$ such that at this degree $s$ the marked set fulfils this property.

**Lemma 1.75.** *Let $\mathcal{C} \subset \mathbb{T}^m$ be a finite set and $\mathcal{V}$ the module generated by it in $\mathcal{P}^m_{\mathbf{d}}$. Let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a finitely generated graded module such that for every $s$ the set $\mathcal{N}(\mathcal{V})_s$ generates the $A$-module $(\mathcal{P}^m_{\mathbf{d}})_s/\mathcal{U}_s$. Then for every degree $s$ there exists a $\mathcal{V}_s \cap \mathbb{T}^m$-marked set $\mathcal{B}_s$ contained in $\mathcal{U}_s$ such that*

$$(\mathcal{P}^m_{\mathbf{d}})_s = \langle \mathcal{B}_s \rangle^A \oplus \langle \mathcal{N}(\mathcal{V})_s \rangle^A.$$

*Proof.* Let $\pi$ be the usual projection morphism of $\mathcal{P}^m_{\mathbf{d}}$ onto the quotient $\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}$. For every $x^\alpha \mathbf{e}_k \in \mathcal{V}_s \cap \mathbb{T}^m$, we consider $\pi(x^\alpha \mathbf{e}_k)$ and choose a representation $\pi(x^\alpha \mathbf{e}_k) = \sum_{x^\eta \mathbf{e}_l \in \mathcal{N}(\mathcal{V})_s} c^{\alpha k}_{\eta l} x^\eta \mathbf{e}_l$, $c^{\alpha k}_{\eta l} \in A$ which exists as $\mathcal{N}(\mathcal{V})_s$ generates $(\mathcal{P}^m_{\mathbf{d}})_s/\mathcal{U}_s$ as an $A$-module. We consider the set of marked module elements $\mathcal{B}_s = \{\mathbf{f}^k_\alpha\}_{x^\alpha \mathbf{e}_k \in \mathcal{V}_s}$, where $\mathbf{f}^k_\alpha := x^\alpha \mathbf{e}_k - \pi(x^\alpha \mathbf{e}_k)$ and $\mathrm{Ht}(\mathbf{f}^k_\alpha) = x^\alpha \mathbf{e}_k$.

We now prove that $(\mathcal{P}^m_{\mathbf{d}})_s = \langle \mathcal{B}_s \rangle^A \oplus \langle \mathcal{N}(\mathcal{V})_s \rangle^A$. At first, we prove that every term in $\mathbb{T}^m_s$ belongs to $\langle \mathcal{B}_s \rangle^A \oplus \langle \mathcal{N}(\mathcal{V})_s \rangle^A$. If $x^\beta \mathbf{e}_l \in \mathcal{N}(\mathcal{V})_s$, there is nothing to prove. If $x^\beta \mathbf{e}_l \in \mathcal{V}_s$, then there is $\mathbf{f}^l_\beta \in \mathcal{B}_s$ such that $\mathrm{Ht}(\mathbf{f}^l_\beta) = x^\beta \mathbf{e}_l$, hence we can write $x^\beta \mathbf{e}_l = \mathbf{f}^l_\beta + (x^\beta \mathbf{e}_l - \mathbf{f}^l_\beta) = \mathbf{f}^l_\beta + \pi(x^\beta \mathbf{e}_l)$.

We conclude by proving that $\langle \mathcal{B}_s \rangle^A \cap \langle \mathcal{N}(\mathcal{V})_s \rangle^A = \{0^m_A\}$. Let $\mathbf{g} \in \mathcal{P}^m_{\mathbf{d}}$ be an element contained in $\langle \mathcal{B}_s \rangle^A \cap \langle \mathcal{N}(\mathcal{V})_s \rangle^A$

$$\mathbf{g} = \sum_{\mathbf{f}^k_\alpha \in \mathcal{B}_s} \lambda_{\alpha k} \mathbf{f}^k_\alpha \in \langle \mathcal{N}(\mathcal{V})_s \rangle .$$

Since the head terms of $\mathbf{f}^k_\alpha$ cannot cancel each other, $\lambda_{\alpha k} = 0$ for every $\alpha$ and $k$ and hence $\mathbf{g} = 0$. □

We specialize now to the case that $\mathcal{V}$ is a quasi-stable module and assume that $\mathbf{P}(\mathcal{V})$ is its Pommaret basis. We study a reduction relation naturally induced by any basis marked over such a set $\mathbf{P}(\mathcal{V})$.In particular, we show that it is confluent and noetherian just as the familiar reduction relation induced by a Gröbner basis. In the following the set $\mathcal{B}^{(s)}$, defined below, plays an important role. We show that this set is for every degree $s$ a marked set which fulfils the condition of the lemma above.

**Definition 1.76.** *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a quasi-stable module and $\mathcal{B}$ be a $\mathbf{P}(\mathcal{V})$-marked set in $\mathcal{P}^m_{\mathbf{d}}$. We introduce the following sets:*

- $\mathcal{B}^{(s)} := \left\{ x^\delta \mathbf{f}^k_\alpha \mid \mathbf{f}^k_\alpha \in \mathcal{B},\ x^\delta \in A[X_{P_m}(\mathbf{f}^k_\alpha)],\ \deg(x^\delta \mathbf{f}^k_\alpha) = s \right\}$,
- $\widehat{\mathcal{B}}^{(s)} := \left\{ x^\delta \mathbf{f}^k_\alpha \mid \mathbf{f}^k_\alpha \in \mathcal{B},\ x^\delta \notin A[X_{P_m}(\mathbf{f}^k_\alpha)],\ \deg(x^\delta \mathbf{f}^k_\alpha) = s \right\}$
  $= \left\{ x^\delta \mathbf{f}^k_\alpha \mid \mathbf{f}^k_\alpha \in \mathcal{B},\ x^\delta \mathbf{f}^k_\alpha \notin \mathcal{B}^{(s)} \right\}$,
- $\mathcal{N}(\mathcal{V}, \langle \mathcal{B} \rangle) := \langle \mathcal{B} \rangle \cap \langle \mathcal{N}(\mathcal{V}) \rangle$.

**Lemma 1.77.** *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a quasi-stable module such that $\mathcal{V} = \bigoplus_{k=1}^m \mathcal{J}^{(k)}$ and $\mathcal{B}$ a $\mathbf{P}(\mathcal{V})$-marked set. For every product $x^\delta \mathbf{f}^k_\alpha$ with $\mathbf{f}^k_\alpha \in \mathcal{B}$, each term in* $\operatorname{supp}(x^\delta x^\alpha \mathbf{e}_k - x^\delta \mathbf{f}^k_\alpha)$ *either belongs to $\mathcal{N}(\mathcal{V})$ or is of the form $x^\eta x^\nu \mathbf{e}_l \in \mathcal{C}_P(x^\nu \mathbf{e}_l)$ with $x^\nu \mathbf{e}_l \in \mathbf{P}(\mathcal{V})$ and $x^\eta <_{\mathtt{lex}} x^\delta$.*

*Proof.* It is sufficient to consider $x^\delta x^\beta \mathbf{e}_l \in \operatorname{supp}(x^\delta x^\alpha \mathbf{e}_k - x^\delta \mathbf{f}^k_\alpha) \cap \mathcal{V}$. Then $x^\delta x^\beta \in \mathcal{J}^{(l)}$. Therefore, there exists $x^\gamma \in \mathbf{P}(\mathcal{J}^{(l)})$ such that $x^\delta x^\beta \in \mathcal{C}_P(x^\gamma)$. More precisely, if $x^\eta := x^\delta \frac{x^\beta}{x^\gamma}$, then $x^\eta \prec_{\mathtt{lex}} x^\delta$ by Lemma 1.55. □

Note in the next definition the use of the set $\mathcal{B}^{(s)}$ which means that we use here a generalization of the involutive reduction relation associated with the Pommaret division and not of the standard reduction relation in the theory of Gröbner bases. This modification is the key for circumventing the restrictions imposed by the results of [39]. It also entails that if a term is reducible, then there is only one element in the marked basis which can be used for its reduction.

**Definition 1.78.** *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a quasi-stable module and $\mathcal{B}$ a $\mathbf{P}(\mathcal{V})$-marked set. We denote by $\xrightarrow{G^{(s)}}$ the transitive closure of the relation $\mathbf{h} \xrightarrow{G^{(s)}} \mathbf{h} - \lambda x^\eta \mathbf{f}^k_\alpha$ where $x^\eta x^\alpha \mathbf{e}_k$ is a term that appears in $\mathbf{h}$ with a nonzero coefficient $\lambda \in A$ and which satisfies* $\deg(x^\eta x^\alpha \mathbf{e}_k) = s$ *and $x^\eta \mathbf{f}^k_\alpha \in \mathcal{B}^{(s)}$.*

*We will write $\mathbf{h} \xrightarrow{G^{(s)}}_* \mathbf{g}$ or $\mathsf{NF}_{G^{(s)}}(\mathbf{h}) = \mathbf{g}$ if $\mathbf{h} \xrightarrow{G^{(s)}} \mathbf{g}$ and $\mathbf{g} \in \langle \mathcal{N}(\mathcal{V}) \rangle$ and call it the $\mathcal{V}$-*normal form modulo $\langle \mathcal{B} \rangle$ of $\mathbf{h}$. *Observe that if $\mathbf{h} \in (\mathcal{P}^m_{\mathbf{d}})_s$, then $\mathbf{h} \xrightarrow{G^{(s)}} \mathbf{g} \in (\mathcal{P}^m_{\mathbf{d}})_s$. If $\mathsf{NF}_{G^{(s)}}(\mathbf{h}) = \mathbf{h}$, then we call $\mathbf{h}$ $\mathcal{V}$-*reduced.

**Proposition 1.79.** *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a quasi-stable module and $\mathcal{B}$ a $\mathbf{P}(\mathcal{V})$-marked set. The reduction relation $\xrightarrow{G^{(s)}}$ is confluent and noetherian.*

*Proof.* It is sufficient to prove that for every term $x^\gamma \mathbf{e}_k$ in $\mathcal{V}_s$, there is a unique $\mathbf{g} \in \mathcal{P}^m_{\mathbf{d}}$ such that $x^\gamma \mathbf{e}_k \xrightarrow{G^{(s)}}_* \mathbf{g}$ and $\mathbf{g} \in \langle \mathcal{N}(\mathcal{V}) \rangle^A$.

Since $x^\gamma \mathbf{e}_k \in \mathcal{V}_s$, there exists a unique $x^\delta \mathbf{f}^k_\alpha \in \mathcal{B}^{(s)}$ such that $x^\delta \operatorname{Ht}(\mathbf{f}^k_\alpha) = x^\gamma \mathbf{e}_k$. Hence, $x^\gamma \mathbf{e}_k \xrightarrow{G^{(s)}} x^\gamma \mathbf{e}_k - x^\delta \mathbf{f}^k_\alpha$. If we could proceed in the reduction without obtaining an element in $\langle \mathcal{N}(\mathcal{V}) \rangle$, we would obtain by Lemma 1.77 an infinite `lex`-descending chain of terms in $\mathbb{T}$ which is impossible since $\prec_{\mathtt{lex}}$ is a well-ordering. Hence, $\xrightarrow{G^{(s)}}$ is noetherian. Confluence is immediate by the uniqueness of the element of $\mathcal{B}^{(s)}$ that is used at each step of reduction. □

**Proposition 1.80.** *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a quasi-stable module and $\mathcal{B}$ be a $\mathbf{P}(\mathcal{V})$-marked set. Every term $x^\beta \mathbf{e}_k \in \mathbb{T}^m_s$ of degree s can be uniquely expressed in the form*

$$x^\beta \mathbf{e}_l = \sum_i \lambda_{\delta_i \alpha_i k_i} x^{\delta_i} \mathbf{f}^{k_i}_{\alpha_i} + \mathbf{g},$$

*where $\lambda_{\delta_i \alpha_i k_i} \in A \setminus \{0_A\}$, $x^{\delta_i} f^{k_i}_{\alpha_i} \in \mathcal{B}^{(s)}$, $\mathbf{g} \in \langle \mathcal{N}(\mathcal{V}) \rangle^A$ and the terms $x^{\delta_i}$ form a sequence which is strictly descending with respect to $\prec_{lex}$.*

*Proof.* For terms in $\mathcal{N}(\mathcal{V})$, there is nothing to prove. For $x^\beta \mathbf{e}_l \in \mathcal{V}$, it is sufficient to consider $g \in \langle \mathcal{N}(\mathcal{V}) \rangle^A$ such that $x^\beta \mathbf{e}_l \xrightarrow{G^{(s)}}_* \mathbf{g}$. The polynomials $x^{\delta_i} \mathbf{f}^{k_i}_{\alpha_i} \in \mathcal{B}^{(s)}$ are exactly those used during the reduction $\xrightarrow{G^{(s)}}$. They fulfil the statement on the terms $x^{\delta_i}$ by Lemma 1.77. □

**Corollary 1.81.** *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a quasi-stable module and $\mathcal{B}$ be a $\mathbf{P}(\mathcal{V})$-marked set. Consider a homogeneous element $\mathbf{g} \in \mathcal{P}^m_{\mathbf{d}}$ such that $\mathbf{g} = \sum_{i=1}^m \lambda_i x^{\delta_i} \mathbf{f}^{k_i}_{\alpha_i}$, with $\lambda_i \in A \setminus \{0\}$ and $x^{\delta_i} \mathbf{f}^{k_i}_{\alpha_i} \in \mathcal{B}^{(s)}$ with $s = \deg(g)$ and $x^{\delta_i} \mathbf{f}^{k_i}_{\alpha_i}$ pairwise different. Then $\mathbf{g} \neq 0^m_A$ and $\mathbf{g} \notin \langle \mathcal{N}(\mathcal{V}) \rangle^A$.*

*Proof.* The statement follows from the definition of $\mathcal{B}^{(s)}$ and the properties of $\xrightarrow{G^{(s)}}$. □

The following theorem and corollary collect some basic properties of sets marked over a Pommaret basis. They generalize analogous statements in [33, Thms. 1.7, 1.10] which consider only ideals and marked bases where the head terms generate a strongly stable ideal.

**Theorem 1.82.** *Let $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a quasi-stable module and $\mathcal{B}$ a $\mathbf{P}(\mathcal{V})$-marked set. Then, we have for every degree s the following decompositions of A-modules:*

*(i)* $\langle \mathcal{B} \rangle_s = \left\langle \mathcal{B}^{(s)} \right\rangle^A + \left\langle \widehat{\mathcal{B}}^{(s)} \right\rangle^A$;

*(ii)* $(\mathcal{P}^m_{\mathbf{d}})_s = \langle \mathcal{B}^{(s)} \rangle^A \oplus \langle \mathcal{N}(\mathcal{V})_s \rangle^A$;

*(iii) The A-module $\left\langle \mathcal{B}^{(s)} \right\rangle^A$ is free of rank equal to $|\mathcal{B}^{(s)}| = \mathrm{rk}(\mathcal{V}_s)$ and it is generated (as an A-module) by a unique $\mathcal{V}_s \cap \mathbb{T}^m$-marked set $\widetilde{\mathcal{B}}^{(s)}$;*

*(iv)* $\langle \mathcal{B} \rangle_s = \left\langle \mathcal{B}^{(s)} \right\rangle^A \oplus \mathcal{N}(\mathcal{V}, \langle \mathcal{B} \rangle)_s$.

*Moreover, the following conditions are equivalent:*

*(v) $\mathcal{B}$ is a $\mathbf{P}(\mathcal{V})$-marked basis;*

*(vi) for all degrees s,* $\langle \mathcal{B} \rangle_s = \left\langle \mathcal{B}^{(s)} \right\rangle^A$;

*(vii)* $\mathcal{N}(\mathcal{V}, \langle\mathcal{B}\rangle) = \{0_A^m\}$*;*

*(viii) for all s,* $\bigwedge^{Q(s)+1}\langle\mathcal{B}\rangle_s = 0_A$*, where* $Q(s) := \mathrm{rk}(\mathcal{V}_s)$*.*

*Proof.* Item (i): Immediate.

Item (ii) is a consequence of Proposition 1.80 and Corollary 1.81.

Item (iii): We can repeat the arguments of [33, Theorem 1.7]: for every $s$, we may construct a $\mathcal{V}_s \cap \mathbb{T}^m$-marked set $\widetilde{\mathcal{B}}^{(s)}$ such that $(\mathcal{P}_{\mathbf{d}}^m)_s = \langle\widetilde{\mathcal{B}}^{(s)}\rangle^A \oplus \langle\mathcal{N}(\mathcal{V})_s\rangle^A$ by using Lemma 1.75. By item (ii), the $\mathcal{V}_s \cap \mathbb{T}^m$-marked set $\widetilde{\mathcal{B}}^{(s)}$ is unique and furthermore $\langle\widetilde{\mathcal{B}}^{(s)}\rangle^A = \langle\mathcal{B}^{(s)}\rangle^A$.

Item (iv): By items (i) and (iii), we have $\langle\mathcal{B}\rangle_s = \langle\widetilde{\mathcal{B}}^{(s)}\rangle^A + \langle\widehat{\mathcal{B}}^{(s)}\rangle^A$. Recalling that $\langle\widetilde{\mathcal{B}}^{(s)}\rangle^A \cap \langle\mathcal{N}(\mathcal{V})_s\rangle^A = \{0_A^m\}$ by Lemma 1.75, it is sufficient to show that every $\mathbf{g} \in \langle\widehat{\mathcal{B}}^{(s)}\rangle^A$ can be written as $\mathbf{g} = \mathbf{f} + \mathbf{h}$ with $\mathbf{f} \in \langle\widetilde{\mathcal{B}}^{(s)}\rangle^A$ and $\mathbf{h} \in \langle\mathcal{N}(\mathcal{V})_s\rangle^A$: we express every term $x^\beta \mathbf{e}_l \in \mathcal{V}_s$ appearing in $\mathbf{g}$ with nonzero coefficient in the form $x^\beta\mathbf{e}_l = \widetilde{\mathbf{f}}^l_\beta + (x^\beta\mathbf{e}_l - \widetilde{\mathbf{f}}^l_\beta)$ where $\widetilde{\mathbf{f}}^l_\beta$ is the unique polynomial in $\widetilde{\mathcal{B}}^{(s)}$ with $\mathrm{Ht}(\widetilde{\mathbf{f}}^l_\beta) = x^\beta\mathbf{e}_l$. By construction, $\mathbf{h} \in \mathcal{N}(\mathcal{V}, \langle\mathcal{B}\rangle)_s$. By item (ii), we obtain the assertion.

Items (v), (vi), (vii) are equivalent by the previous items, using again the same proof as in [33, Thm. 1.7].

With respect to [33], the only new item is (viii), which is obviously equivalent to (vi) and (vii). In fact, by (iii) and (iv) we find that $\langle\mathcal{B}\rangle_s = \left\langle\mathcal{B}^{(s)}\right\rangle^A \oplus \mathcal{N}(\mathcal{V}, \langle\mathcal{B}\rangle)_s$ and $\mathrm{rk}\left\langle\mathcal{B}^{(s)}\right\rangle^A = \mathrm{rk}(\mathcal{V}_s) = Q(s)$. □

**Corollary 1.83.** *Let* $\mathcal{V} \subseteq \mathcal{P}_{\mathbf{d}}^m$ *be a quasi-stable module and* $\mathcal{B}$ *be a* $\mathbf{P}(\mathcal{V})$*-marked set. The following conditions are equivalent:*

*(i)* $\mathcal{B}$ *is a* $\mathbf{P}(\mathcal{V})$*-marked basis,*

*(ii)* $\langle\mathcal{B}\rangle_s = \left\langle\mathcal{B}^{(s)}\right\rangle^A$ *for every* $s \le \mathrm{reg}(\mathcal{V}) + 1$*,*

*(iii)* $\mathcal{N}(\mathcal{V}, \langle\mathcal{B}\rangle)_s = \{0_A^m\}$ *for every* $s \le \mathrm{reg}(\mathcal{V}) + 1$*,*

*(iv)* $\bigwedge^{Q(s)+1}\langle\mathcal{B}\rangle_s = 0_A$ *for every* $s \le \mathrm{reg}(\mathcal{V}) + 1$*.*

*Proof.* By the second part of Theorem 1.82, item (i) implies item (ii) and items (ii), (iii), (iv) are equivalent. For the proof that item (ii) implies (i), it is sufficient to repeat the arguments of [33, Thm. 1.10]. □

**Corollary 1.84.** *Let* $\mathcal{V} \subseteq \mathcal{P}_{\mathbf{d}}^m$ *be a saturated quasi-stable module and* $\mathcal{B}$ *be a* $\mathbf{P}(\mathcal{V})$*-marked set. Then the following conditions are equivalent:*

*(i)* $\mathcal{B}$ *is a* $\mathbf{P}(\mathcal{V})$*-marked basis,*

*(ii)* $\langle \mathcal{B} \rangle_{\mathrm{reg}(\mathcal{B})+1} = \left\langle \mathcal{B}^{(\mathrm{reg}(\mathcal{V})+1)} \right\rangle^A$,

*(iii)* $\mathcal{N}(\mathcal{V}, \langle \mathcal{B} \rangle)_{\mathrm{reg}(\mathcal{V})+1} = \{0^m_A\}$,

*(iv)* $\bigwedge^{Q+1} \langle \mathcal{B} \rangle_{\mathrm{reg}(\mathcal{V})+1} = 0_A$*, where* $Q := \mathrm{rk}(\mathcal{V}_{\mathrm{reg}(\mathcal{V})+1})$.

*Proof.* The equivalence among items (ii), (iii) and (iv) is immediate by Theorem 1.82. We only prove that items (i) and (iii) are equivalent. If $\mathcal{B}$ is a $\mathbf{P}(\mathcal{V})$-marked basis, then we have $\mathcal{N}(\mathcal{V}, \langle \mathcal{B} \rangle)_{\mathrm{reg}(\mathcal{V})+1} = \{0^m_A\}$ by Theorem 1.82.

We now assume that $\mathcal{N}(\mathcal{V}, \langle \mathcal{B} \rangle)_{\mathrm{reg}(\mathcal{V})+1} = \{0^m_A\}$ and prove that $\mathcal{N}(\mathcal{V}, \langle \mathcal{B} \rangle) = \{0^m_A\}$. By Corollary 1.83, it is sufficient to prove that $\mathcal{N}(\mathcal{V}, \langle \mathcal{B} \rangle)_s = \{0^m_A\}$ for every $s \leq \mathrm{reg}(\mathcal{V})$. If $\mathbf{f} \in \mathcal{N}(\mathcal{V}, \langle \mathcal{B} \rangle)_s$, with $s \leq \mathrm{reg}(\mathcal{V})$, then $x_0^{\mathrm{reg}(\mathcal{V})+1-s}\mathbf{f} \in \mathcal{N}(\mathcal{V}, \langle \mathcal{B} \rangle)_{\mathrm{reg}(\mathcal{V})+1}$, by Corollary 1.58 and Lemma 1.55 applied to $\mathcal{V}$. Hence, $\mathbf{f} = 0^m_A$. □

**Corollary 1.85.** *Let* $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ *be a quasi-stable module and* $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ *be a finitely generated graded submodule such that* $(\mathcal{P}^m_{\mathbf{d}})_s = \mathcal{U}_s \oplus \langle \mathcal{N}(\mathcal{V})_s \rangle^A$ *for every s. Then* $\mathcal{U}$ *is generated by a* $\mathbf{P}(\mathcal{V})$*-marked basis.*

*Proof.* The statement is an easy consequence of Theorem 1.82 as soon as we define a $\mathbf{P}(\mathcal{V})$-marked set generating $\mathcal{U}$.

By the hypotheses, for every degree $s$ and every module term $x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V})$ there is a unique element $\mathbf{h}^k_\alpha \in \langle \mathcal{N}(\mathcal{V})_s \rangle^A$ such that $x^\alpha \mathbf{e}_k - \mathbf{h}^k_\alpha \in \mathcal{U}_s$.

The collection $\mathcal{B}$ of the elements $x^\alpha \mathbf{e}_k - \mathbf{h}^k_\alpha$ is obviously a $\mathbf{P}(\mathcal{V})$-marked set and generates a graded submodule of $\mathcal{U}$. Moreover, $(\mathcal{P}^m_{\mathbf{d}})_s = \mathcal{U}_s \oplus \langle \mathcal{N}(\mathcal{V})_s \rangle^A = \langle \mathcal{B}^{(s)} \rangle^A \oplus \langle \mathcal{N}(\mathcal{V})_s \rangle^A$. Therefore, $\mathcal{U}_s = \langle \mathcal{B}^{(s)} \rangle^A \subseteq \mathcal{B}_s \subseteq \mathcal{U}_s$, so that $\mathcal{B}$ generates $\mathcal{U}$ as a graded $\mathcal{P}$-module. □

Finally, we give an algorithmic method to check whether a marked set is a marked basis using the reduction process introduced in Definition 1.78.

**Theorem 1.86.** *Let* $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ *be a quasi-stable module and* $\mathcal{B}$ *be a* $\mathbf{P}(\mathcal{V})$*-marked set. The set* $\mathcal{B}$ *is a* $\mathbf{P}(\mathcal{V})$*-marked basis if and only if*

$$\forall \mathbf{f}^k_\alpha \in \mathcal{B}, \forall x_i \in \overline{X}_{P_m}(\mathbf{f}^k_\alpha) : x_i \mathbf{f}^k_\alpha \xrightarrow{\mathcal{B}^{(s)}} 0^m_A.$$

*Proof.* We can repeat the arguments used in [12, Thm. 5.13] for the ideal case. □

**Corollary 1.87.** *Let $\mathcal{V} \subseteq \mathcal{P}_{\mathbf{d}}^m$ be a quasi-stable module and $\mathcal{B}$ be a $\mathbf{P}(\mathcal{V})$-marked basis. Then every homogeneous module element $\mathbf{h} \in \langle \mathcal{B} \rangle_s$ for an arbitrary $s \in \mathbb{N}$ reduces to zero with respect to $\xrightarrow{\mathcal{B}^{(s)}}$.*

At the end of this section we show that a $\mathbf{P}(\mathcal{V})$-marked basis over a noetherian $\Bbbk$-algebra is a resolving decomposition. Hence, we can apply the whole syzygy theory which we have done in the first part of this chapter.

**Theorem 1.88.** *Let $\mathcal{V} \subseteq \mathcal{P}_{\mathbf{d}}^m$ be a quasi-stable module and $\mathcal{B} = \{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ be a $\mathbf{P}(\mathcal{V})$-marked basis. Define $X_{\mathcal{B}}(\mathbf{h}) = X_{P_m}(\mathbf{h})$ for all $\mathbf{h} \in \mathcal{B}$. Let $\prec_{\mathtt{lex}}$ be the TOP-lift of the lexicographic ordering to $A[\mathbf{x}]^s$. Then the quadruple $(\mathcal{B}, \mathrm{Ht}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathtt{lex}})$ is a resolving decomposition.*

*Proof.* We see immediately that the conditions (i), (ii) and (iv) of Definition 1.1 are satisfied. The first part of condition (iii) follows from the fact that $\mathrm{Ht}(\mathcal{B})$ is a Pommaret basis. The second part follows from the uniqueness of the reduction process which is a consequence of Proposition 1.80 and Theorem 1.82 (iv). That $\prec_{\mathtt{lex}}$ fulfils the last condition of a resolving decomposition follows from Lemma 1.77. □

# 2 The Hilbert polynomial and the Theorems of Gotzmann

In this chapter we introduce the Hilbert function and the Hilbert polynomial. They measure the growth of the dimension of homogeneous components of ideals over polynomial rings.

First, we recall the basic definitions and results concerning the Hilbert function and the Hilbert polynomial. Then we show how to compute the Hilbert function and the Hilbert polynomial via a given resolving decomposition and introduce lexsegment ideals which play an important role later in this section.

The persistence theorem and the regularity theorem of Gotzmann for ideals are well-known results concerning the connection between the Hilbert function respectively Hilbert polynomial and the regularity of an ideal. We give for both theorems two alternative proofs. In addition to that, we introduce the Gotzmann number. It is based on a decomposition of a Hilbert polynomial and will be important in the next chapters.

Furthermore, we recall the results of Gasharov and Dellaca which extended the classical theorems of Gotzmann to the module case.

We consider again finitely generated graded submodules of $\mathcal{P}^m_{\mathbf{d}}$ where $\mathcal{P} = \mathbb{k}[\mathbf{x}]$ for an arbitrary field $\mathbb{k}$.

**Definition 2.1.** *For a finitely generated graded submodule $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ the* Hilbert function of *$\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}$ is defined by*

$$\begin{aligned} \mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}} : \mathbb{Z} &\longrightarrow \mathbb{Z} \\ j &\longrightarrow \dim_{\mathbb{k}}((\mathcal{P}^m_{\mathbf{d}}/\mathcal{U})_j). \end{aligned}$$

*In contrast to many other works about the Hilbert function we distinguish between the Hilbert function of $\mathcal{U}$ and $\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}$. The* volume function of *$\mathcal{U}$ is defined by*

$$\begin{aligned} \mathsf{HF}_{\mathcal{U}} : \mathbb{Z} &\longrightarrow \mathbb{Z} \\ j &\longrightarrow \dim_{\mathbb{k}}((\mathcal{U})_j). \end{aligned}$$

The following important result goes back to Hilbert and Serre. It motivates the definition of the Hilbert polynomial.

**Theorem 2.2** ([28, Thm. I.7.5]). *For a finitely generated graded submodule $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ with Hilbert function $\mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}$ there is a univariate polynomial $\mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(t) \in \mathbb{Q}[t]$ such that*

$$\mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(j) = \mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(j) \quad \textit{for } j \gg 0\,.$$

For the volume function there is an analogous result which provides the existence of a polynomial $\mathsf{HP}_{\mathcal{U}}(t)$.

**Definition 2.3.** *The above defined polynomial $\mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(t)$ is called the* Hilbert polynomial of $\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}$. *The polynomial $\mathsf{HP}_{\mathcal{U}}(t)$ is called the* volume polynomial of $\mathcal{U}$.

In the following, we need some simple results concerning the definitions above. We summarize these in the next lemma.

**Lemma 2.4.** *Let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a finitely generated graded submodule and $j \in \mathbb{Z}$.*

- $\mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(j) = \mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}}(j) - \mathsf{HF}_{\mathcal{U}}(j)$ *and*
- $\mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(t) = \mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}}(t) - \mathsf{HP}_{\mathcal{U}}(t)$.

*If furthermore $\mathbf{d} = (0, \dots, 0)$, then*

- $\mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}}(j) = m \cdot \binom{n+j}{n}$ *and*
- $\mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}}(t) = m \cdot \binom{n+t}{n}$.

**Remark 2.5.** *To make statements about the Hilbert function or the Hilbert polynomial of a module one often assumes that the given modules are in a stable position which is often easier to analyse. This restriction is possible because a $\mathbb{k}$-linear coordinate change does not change the $\mathbb{k}$-vector space dimension of homogeneous components of a module. Hence, one can obtain a stable module via coordinate transformations with the same Hilbert function and the same Hilbert polynomial.*

**Remark 2.6.** *It is also possible to define a Hilbert function and a Hilbert polynomial for an $A[\mathbf{x}]$-submodule $\mathcal{U}$ of $A[\mathbf{x}]^m_{\mathbf{d}}$ where $A$ is a $\mathbb{k}$-algebra and for every $j \in \mathbb{Z}$ the component of degree $j$ of $A[\mathbf{x}]^m_{\mathbf{d}}/\mathcal{U}$ is a free $A$-module. Then the Hilbert function respectively the Hilbert polynomial is given by the ranks of the homogeneous components of $A[\mathbf{x}]^m_{\mathbf{d}}/\mathcal{U}$*

If we consider the Hilbert polynomial or the Hilbert function of a finitely generated graded module $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$, we see immediately that the Hilbert polynomial or the Hilbert function of $\mathcal{U}$ and $\mathrm{lt}_{\prec}(\mathcal{U})$ coincide for a given term order $\prec$. It is obvious that the same is true for a resolving decomposition $(\mathcal{B}, \mathrm{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$ of $\mathcal{U}$ because of the definition of a resolving decomposition. Therefore, it is enough to consider the Hilbert polynomial or the Hilbert function of $\langle \mathrm{hm}(\mathcal{U})\rangle$. Due to that, it is easy to get the Hilbert function and the Hilbert polynomial, respectively the volume function and the volume polynomial for a finitely generated graded module with a given resolving decomposition.

**Lemma 2.7.** *Let $(\mathcal{B}, \mathrm{hm}(\mathcal{B}), X_{\mathcal{B}}, \prec_{\mathcal{B}})$ be a resolving decomposition of the finitely generated graded module $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$. Then*

$$\begin{aligned}
\mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(j) &= m \cdot \binom{n+j}{n} - \sum_{x^\mu \mathbf{e}_k \in \mathrm{hm}(\mathcal{B}) \wedge \deg(x^\mu \mathbf{e}_k) \leq j} \binom{j - \deg(x^\mu \mathbf{e}_k) + |X_{\mathcal{B}}(x^\mu)| - 1}{j - \deg(x^\mu \mathbf{e}_k)}, \\
\mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(t) &= m \cdot \binom{n+t}{n} - \sum_{x^\mu \mathbf{e}_k \in \mathrm{hm}(\mathcal{B})} \binom{t - \deg(x^\mu \mathbf{e}_k) + |X_{\mathcal{B}}(x^\mu)| - 1}{t - \deg(x^\mu \mathbf{e}_k)}, \\
\mathsf{HF}_{\mathcal{U}}(j) &= \sum_{x^\mu \mathbf{e}_k \in \mathrm{hm}(\mathcal{B}) \wedge \deg(x^\mu \mathbf{e}_k) \leq j} \binom{j - \deg(x^\mu \mathbf{e}_k) + |X_{\mathcal{B}}(x^\mu)| - 1}{j - \deg(x^\mu \mathbf{e}_k)}, \\
\mathsf{HP}_{\mathcal{U}}(t) &= \sum_{x^\mu \mathbf{e}_k \in \mathrm{hm}(\mathcal{B})} \binom{t - \deg(x^\mu \mathbf{e}_k) + |X_{\mathcal{B}}(x^\mu)| - 1}{t - \deg(x^\mu \mathbf{e}_k)}.
\end{aligned}$$

## 2.1 Gotzmann's Persistence Theorem

In this and the following section we restrict ourselves to the case of homogeneous ideals $\mathcal{I} \subseteq \mathcal{P}$. We introduce the Persistence Theorem of Gotzmann [23]. This is a fundamental theorem about the growth of the Hilbert function of an ideal. We present here a new proof using the $\beta$-vector of monomial quasi-stable ideals.

Fundamental for the analysis of the growth of Hilbert functions are the so called lexsegments which we define in the following.

**Definition 2.8.** *Let $d \in \mathbb{N}$ and $x^\mu \in \mathbb{T}$ be a monomial of degree $d$.*

- *The set $\mathcal{L}(x^\mu) := \{x^\nu \in \mathbb{T} \mid \deg(x^\nu) = d,\ x^\mu \preceq_{\mathtt{lex}} x^\nu\}$ is called a* lexsegment. *The empty set is also considered as a lexsegment.*
- *The $\mathbb{k}$-subspace $V$ of $\mathcal{P}_d$ is called* lexsegment space *if $V \cap \mathbb{T}$ is first the $\mathbb{k}$-basis of $V$ and second a lexsegment.*
- *An ideal $\mathcal{L} \subseteq \mathcal{P}$ is called* lexsegment ideal *if $\mathcal{L}_d$ is a lexsegment space in $\mathcal{P}_d$ for all $d \in \mathbb{N}$.*
- *An ideal $\mathcal{L} \subseteq \mathcal{P}$ is called a* basic lexsegment ideal (with $s$ elements) *if $\mathcal{L}$ is a lexsegment ideal which has a lexsegment (with $s$ elements) as minimal generating set.*

- *Let $\mathcal{I} \subseteq$ be a homogeneous ideal. Consequently, the* corresponding lexsegment ideal *$\mathcal{L} \subseteq \mathcal{P}$ is the lexsegment ideal which has the same Hilbert function then $\mathcal{I}$.*

In the following we need several times the growth theorem of Macaulay which gives us a fundamental statement of the growth of ideals. Especially, it shows us that the growth of a lexsegment ideal is the slowest possible.

**Theorem 2.9** (Theorem of Macaulay [24, Thm. 3.3 and Prop. 3.7])**.** *Let be $\mathcal{I} \subseteq \mathcal{P}$ be a homogeneous ideal and $\mathcal{L}$ its corresponding lexsegment ideal. Then*

$$\dim_{\Bbbk}(\mathcal{I}_{d+1}) \geq \dim_{\Bbbk}(\mathcal{P}_1\mathcal{L}_d) \quad \text{for } d \geq 0.$$

To proof the theorems of Gotzmann we first need a statement about the $\beta$-vector and basic lexsegment ideals.

**Definition 2.10.** *Let $\mathcal{J} \subseteq \mathcal{P}$ be a monomial ideal. We define $\mathcal{B}_q(\mathcal{J}) := \mathcal{J}_q \cap \mathbb{T}$ and*

$$\beta_q^{(k)}(\mathcal{J}) := |\{x^\mu \in \mathcal{B}_q(\mathcal{J}) \mid \operatorname{cls}(x^\mu) = k\}|.$$

*Moreover we define the $\beta$-*vector *as*

$$\beta_q(\mathcal{J}) := (\beta_q^{(0)}(\mathcal{J}), \dots, \beta_q^{(n)}(\mathcal{J})).$$

**Lemma 2.11.** *Let $d \in \mathbb{N}$ and $\mathcal{B} = \{x^{\mu^{(1)}}, \dots, x^{\mu^{(s)}}\}$ be a monomial Pommaret basis such that* $\deg(x^{\mu_i}) = d$ *for all possible $i$. Then $\mathcal{J} = \langle \mathcal{B} \rangle$ is stable.*

*Proof.* Let $x^\mu \in \mathcal{B} = \mathcal{J}_d \cap \mathbb{T}$ such that $\operatorname{cls}(x^\mu) = j$. We claim for $k > j$ that the element $x_j^{\mu_j - 1} \cdots x_k^{\mu_k + 1} \cdots x^{\mu_n}$ is in $\mathcal{B}$, too. Consider the element $x_k x^\mu = x_j^{\mu_j} \cdots x_k^{\mu_k+1} \cdots x^{\mu_n}$. The set $\mathcal{B}$ is a Pommaret basis, hence there exists $x^{\mu'}$ which divides $x_k x^\mu$ involutively. Due to the fact that every element in $\mathcal{B}$ has the same degree there exists an $l$ such that $x^{\mu'} = \frac{x_k}{x_l} x^\mu$. Suppose that $l > j$, then $\operatorname{cls}(x^{\mu'}) = \operatorname{cls}(x^\mu) = j$ and hence $x_l$ is non-multiplicative for $x^{\mu'}$. But then $x^{\mu'}$ is not an involutive divisor of $x_k x^\mu$, because $\deg_l(x^{\mu'}) < \deg_l(x_k x^\mu)$. Hence, $l = j$ and $x^{\mu'} = x_j^{\mu_j - 1} \cdots x_k^{\mu_k+1}$ which proves the claim.

Now we choose an arbitrary monomial $x^\nu \in \mathcal{J} \setminus \mathcal{B}$. Thus, there is an $x^\mu \in \mathcal{B}$ such that $x^\nu = x^\delta x^\mu$ and $x^\delta \in \mathbb{T}_{X_P(x^\mu)}$. Assume that $k = \operatorname{cls}(x^\mu)$ and $j = \operatorname{cls}(x^\nu)$. In addition to that, we choose an $l > j$ and we want to show that $x^{\nu'} = \frac{x_l}{x_j} x^\nu \in \mathcal{J}$. If $l \leq k$ then $x^{\nu'}$ also has $x^\mu$ as an involutive divisor which proves the claim. Assume that $l > k$ and $x^{\mu'} = \frac{x_l}{x_k} x^\mu$. It is obvious that $x^{\mu'} \in \mathcal{B}$ and that $x^{\mu'}$ is an involutive divisor of $x^{\nu'}$ which completes the proof. □

Now we prove the persistence theorem of Gotzmann. In contrast to many other poofs our proof needs only some easy combinatorial results concerning the growth of the $\beta$-vectors of stable ideals.

If a $\beta$-vector of a stable ideal is given of degree higher than or equal to the maximal degree of a minimal generator of this ideal, one can use the $\beta$-vector to compute the volume function.

**Lemma 2.12.** *Let $d \in \mathbb{N}$ and $\mathcal{B} = \{x^{\mu^{(1)}}, \dots, x^{\mu^{(s)}}\} \subset \mathbb{T}_d$ be a monomial Pommaret basis of the ideal $\mathcal{J}$. In addition to that, let $\beta_d(\mathcal{J}) = (\beta_d^{(0)}(\mathcal{J}), \dots, \beta_d^{(n)}(\mathcal{J}))$. We get for the volume function the following result:*

$$\mathsf{HF}_{\mathcal{J}}(j) = \sum_{i=0}^{n} \binom{j-d+i}{j-d} \beta_d^{(i)}(\mathcal{J}).$$

*Proof.* We know from Lemma 2.7 that

$$\mathsf{HF}_{\mathcal{J}}(j) = \sum_{x^\mu \in \mathcal{B}} \binom{j - \deg(x^\mu) + \operatorname{cls}(x^\mu)}{j - \deg(x^\mu)}.$$

Using the fact that every element in the Pommaret basis has the same degree we can simplify the formula further:

$$\begin{aligned} \mathsf{HF}_{\mathcal{J}}(j) &= \sum_{x^\mu \in \mathcal{B}} \binom{j - \deg(x^\mu) + \operatorname{cls}(x^\mu)}{j - \deg(x^\mu)} \\ &= \sum_{x^\mu \in \mathcal{B}} \binom{j - d + \operatorname{cls}(x^\mu)}{j - d} \\ &= \sum_{i=0}^{n} \binom{j-d+i}{j-d} \beta_d^{(i)}(\mathcal{J}). \end{aligned}$$

The last transformation is a consequence of the fact that there are exactly $\beta_d^{(i)}(\mathcal{J})$ elements of class $i$ in $\mathcal{B} = \mathcal{J}_d \cap \mathbb{T}$. □

If we have a monomial ideal which is generated by a monomial Pommaret basis in degree $d$, then we can define a shifted version of the volume function which simplifies the formula to compute the volume function out of the $\beta$-vector:

$$\begin{aligned} \mathsf{HF}_{\mathcal{J}}(d+j) &= \sum_{i=0}^{n} \binom{d+j-d+i}{d+j-d} \beta_d^{(i)}(\mathcal{J}) \\ &= \sum_{i=0}^{n} \binom{j+i}{j} \beta_d^{(i)}(\mathcal{J}) =: \mathsf{HF}_{\mathcal{J}}^{d}(j) \end{aligned}$$

for $j \geq 0$.

One main step to prove the persistence theorem of Gotzmann is the next proposition. It shows that the $\beta$-vector of a lexsegment is the $\prec_{\texttt{lex}}$-smallest among all stable ideals.

**Proposition 2.13.** *Let $d \in \mathbb{N}$ and $\mathcal{B} = \{x^{\mu^{(1)}}, \ldots, x^{\mu^{(s)}}\} \subset \mathbb{T}_d$ be a monomial Pommaret basis of the ideal $\mathcal{J}$. Then the $\beta$-vector of $\mathcal{J}$ is lexicographically greater than or equal to the $\beta$-vector $\beta_d(\mathcal{L})$ where $\mathcal{L}$ is a basic lexsegment ideal which is generated by the lexsegment in degree $d$ which has $s$ elements.*

*Proof.* The theorem of Macaulay (Theorem 2.9) indicates that if $\mathsf{HF}_{\mathcal{J}}(j) = \mathsf{HF}_{\mathcal{L}}(j)$, so $\mathsf{HF}_{\mathcal{J}}(j+1) \geq \mathsf{HF}_{\mathcal{L}}(j+1)$ holds for all $j \in \mathbb{N}$. We know that $\mathsf{HF}_{\mathcal{J}}(d) = \mathsf{HF}_{\mathcal{L}}(d)$, hence $\mathsf{HF}_{\mathcal{J}}(j) \geq \mathsf{HF}_{\mathcal{L}}(j)$ for all $j \geq d$.

Assume $\mathcal{J}$ has a $\beta$-vector which is lexicographically smaller than the $\beta$-vector of $\mathcal{L}$. We take a look at the difference of the shifted volume functions

$$\mathsf{HF}^d_{\mathcal{J}}(j) - \mathsf{HF}^d_{\mathcal{L}}(j) = \sum_{i=0}^{n} \binom{j+i}{j} \left(\beta_d^{(i)}(\mathcal{J}) - \beta_d^{(i)}(\mathcal{L})\right).$$

We can interpret this difference as a polynomial in $j$. The leading term of this polynomial has the degree of the maximal $i \in \{0, \ldots, n\}$ such that $\beta_d^{(i)}(\mathcal{J}) - \beta_d^{(i)}(\mathcal{L}) \neq 0$. Due to the assumption we know that $\beta_d(\mathcal{J}) \prec_{\texttt{lex}} \beta_d(\mathcal{L})$ which implies that $\beta_d^{(i)}(\mathcal{J}) - \beta_d^{(i)}(\mathcal{L}) < 0$. Therefore, the leading term of the polynomial above has a negative leading coefficient. However, then we have $\mathsf{HF}^d_{\mathcal{J}}(s) - \mathsf{HF}^d_{\mathcal{L}}(s) < 0$ for $s \gg 0$ which is a contradiction to the theorem of Macaulay. □

The next two statements concern multi indices and show the connection between them if they have the same content[1].

**Definition 2.14.** *Let $\alpha \in \mathbb{N}^{n+1}$ be a multi index, $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, n-1\}$. We define an* elementary change *$e_{i,j}(\alpha) := \alpha'$ such that $\alpha'$ fulfils the following conditions*

$$\begin{aligned}
\alpha'_i &= \alpha_i - 1 \\
\alpha'_{i-1} &= \alpha_{i-1} + 1 \\
\alpha'_j &= \alpha_j - 1 \\
\alpha'_{j+1} &= \alpha_{j+1} + 1 \\
\alpha'_k &= \alpha_k \text{ for all } k \in \{1, \ldots, n\} \setminus \{i, i-1, j, j+1\}.
\end{aligned}$$

[1]The content of a multi index $\alpha \in \mathbb{N}^{n+1}$ is the sum of its entries.

**Lemma 2.15.** *Let $\alpha, \beta \in \mathbb{N}^{n+1}$ be two multi indices such that*

$$\sum_{i=0}^{n} \alpha_i = \sum_{i=0}^{n} \beta_i \quad \text{and} \quad \sum_{i=0}^{n} i\alpha_i = \sum_{i=0}^{n} i\beta_i \,.$$

*Then we can transform $\alpha$ to $\beta$ by a finite number of elementary changes.*

*Proof.* We assume that $\alpha \neq \beta$ and perform an induction over $n$. For $n = 1$ and $n = 2$ the statement is trivial. Let $n \geq 2$ and consider $\alpha$ and $\beta$ such that $\sum \alpha_i = \sum \beta_i$ and $\sum i\alpha_i = \sum i\beta_i$. If $\alpha_n = \beta_n$ we can apply the induction hypothesis to $(\alpha_0, \ldots, \alpha_{n-1})$ and $(\beta_0, \ldots, \beta_{n-1})$ and the claim is following. Now assume without loss of generality that $\alpha_n > \beta_n$. The repeated application of elementary changes on $\alpha$ gives us an $\alpha'$ such that

$$\alpha'_n = e_{n,k_{\alpha_n - \beta_n}} \circ \cdots \circ e_{n,k_1}(\alpha)$$

for $k_1, \ldots, k_{\alpha_n - \beta_n} \in \{0, \ldots, n-2\}$ and $(\alpha')_n = \beta_n$. We find such a suitable chain of elementary changes because if not then we would get an element $\alpha''$ of the form $(0, \ldots, 0, \alpha''_n)$ with $\alpha''_n \neq \beta_n$ and $\alpha''_n = \sum_{i=0}^{n} \beta_i$. This has the consequence that $\sum_{i=0}^{n} i\alpha''_i = n\alpha''_n > \sum_{i=0}^{n} i\beta_i$ which is a contradiction of the assumption. Because we only apply elementary changes it is still true that $\sum \alpha'_i = \sum \beta_i$ and $\sum i\alpha'_i = \sum i\beta_i$. Furthermore, $\sum_{i=0}^{n-1} \alpha'_i = \sum_{i=0}^{n-1} \beta_i$ and $\sum_{i=0}^{n-1} i\alpha'_i = \sum_{i=0}^{n-1} i\beta_i$. Now we can apply the induction hypothesis again and the claim is following. □

**Proposition 2.16.** *Let $\alpha, \beta \in \mathbb{N}^{n+1}$ be like in Lemma 2.15. Additionally, let $\alpha \prec_{\texttt{lex}} \beta$. Then there exists a $j \in \{0, \ldots, n\}$ such that*

$$\sum_{k=j}^{n} \alpha_k > \sum_{k=j}^{n} \beta_k.$$

*Proof.* Due to Lemma 2.15 we know that there is a finite number of elementary transformations from $\alpha$ to $\beta$. Among all of these transformations we take a minimal set $E = \{e_{i_1,j_1}, \ldots, e_{i_r,j_r}\}$ which transforms $\alpha$ to $\beta$. This set is not necessarily unique.

At first, we apply to $\alpha$ all elementary transformations $e_{i,j} \in E$ such that $i \leq j$ and $i$ is decreasing. There must be at least one such transformation because $\alpha \prec_{\texttt{lex}} \beta$.

If we apply the first elementary transformation $e_{i_1,j_1}$ to $\alpha$, we see that

$$\sum_{k=i_1}^{n} \alpha_k > \sum_{k=i_1}^{n} e_{i_1,j_1}(\alpha)_k.$$

Now we assume that for $\alpha \prec_{\texttt{lex}} \alpha' = e_{i_{s-1},j_{s-1}} \circ \cdots \circ e_{i_1,j_1}(\alpha)$ the claim above is true which means that there is an $i$ such that

$$\sum_{k=i}^{n} \alpha_k > \sum_{k=1}^{n} \alpha'_k.$$

Let $i$ be the minimal number such that the inequality above is satisfied.

Now we apply $e_{i_s,j_s}$ on $\alpha'$. Assume that $i_s \leq j_s$. If $i = i_s$ the inequality above remains true. The case $i < i_s$ cannot appear because we apply the elementary transformations $e_{i,j}$ such that $i$ is decreasing. The only remaining case is $i > i_s$. Here we have to distinguish between four different cases:

$j_s \geq i$**:** The inequality obviously remains.

$j_s < i-1$**:** The inequality obviously remains.

$j_s = i-1$**:** Due to the order how we apply the elementary transformations we can follow that at least $\sum_{k=i_s}^{n} \alpha_k > \sum_{k=i_s}^{n} e_{i_s,j_s}(\alpha')_k$ is true.

$j_s < i_s$**:**

If $j_s < i_s$ we can assume that all elementary transformations $e_{i,j} \in E$ with $i_s \leq j_s$ are done. Now we have to do distinguish some further cases:

$i_s \leq i$**:** The inequality obviously remains.

$i_s > i$**:** We have to distinguish three more cases:

$j_s \geq i$**:** The inequality obviously remains.

$j_s < i-1$**:** The inequality obviously remains.

$j_s = i-1$**:** There must be at least one elementary transformation $e_{i_q,j_q}$ such that $i_q \leq j_q$ and $i_q = i$. If not, no inequality existed before. Hence, we can replace the elementary transformations $e_{i_q,j_q}$ and $e_{i_s,j_s}$ by $e_{i_s,j_q}$ which is a contradiction to the assumption that $E$ is minimal.

□

**Lemma 2.17.** *Let $\mathcal{I} \subseteq \mathcal{P}$ be a homogeneous ideal and $\mathcal{L}$ be the corresponding lexsegment ideal. Further, let $\mathcal{H}_1$ be a generic linear form of $\mathcal{P}$, $\mathcal{H}_2$ be a generic linear form of $\mathcal{P}/\mathcal{H}_1$, $\mathcal{H}_3$ be a generic linear form of $\mathcal{P}/\langle \mathcal{H}_1, \mathcal{H}_2 \rangle$, etc., $q \in \mathbb{N}$ and $s \in \{1, \ldots, n\}$. Then*

$$\dim_{\mathbb{k}}((\mathcal{P}/\langle \mathcal{L}, \mathcal{H}_1, \ldots, \mathcal{H}_s \rangle)_q) \geq \dim_{\mathbb{k}}((\mathcal{P}/\langle \mathcal{I}, \mathcal{H}_1, \ldots, \mathcal{H}_s \rangle)_q).$$

*Proof.* This statement is a repeated application of the theorem of Green (for example see [37, Sec. II.44]) and hence obvious. □

We denote by $\mathcal{J}_{\{x_0,\ldots,x_j\}}$ the ideal that we obtain by applying the generic linear forms $\langle x_0\rangle, \ldots, \langle x_j\rangle$ on the stable ideal $\mathcal{J}$.

**Theorem 2.18.** *Let* $\operatorname{char}(\mathbb{k}) = 0$, $p \in \mathbb{N}$ *and* $\mathcal{J}$ *be a stable ideal with minimal generating set* $\mathcal{B}$ *such that* $\deg(\mathbf{h}) = p$ *for every* $\mathbf{h} \in \mathcal{B}$. *Furthermore, let* $\mathcal{L}$ *be the corresponding lexsegment ideal of* $\mathcal{J}$. *Moreover, suppose that* $d \geq p$ *is an integer such that*

$$\dim_{\mathbb{k}}((\mathcal{P}/\mathcal{J})_{d+1}) = \dim_{\mathbb{k}}(\mathcal{P}_1(\mathcal{P}/\mathcal{J})_d),$$

*then* $\mathcal{J}$ *is d-regular and* $\dim_{\mathbb{k}}((\mathcal{P}/\mathcal{J})_{k+1}) = \dim_{\mathbb{k}}(\mathcal{P}_1(\mathcal{P}/\mathcal{J})_k)$ *for all* $k \geq d$.

*Proof.* At the beginning of the proof we consider the Pommaret basis of $\mathcal{J}$. It has degree $p$ due to the fact that $\mathcal{J}$ is stable, hence the Pommaret basis is $\mathcal{B}$ which only consists of elements of degree $p$. Now assume that we cut $\mathcal{J}$ at degree $k \geq p$. Then $\mathcal{J}_{\geq k}$ is also stable and hence the Pommaret basis coincides with the minimal generating set of $\mathcal{J}_{\geq k}$ which is $\mathcal{J}_k \cap \mathbb{T}$.

Now consider $\mathcal{J}_{\geq d}$ and the corresponding lexsegment ideal $\mathcal{L}_{\geq d}$. It is true that

$$\sum_{i=0}^{n} \beta^{(i)}_{\mathcal{L}_{\geq d}}(d) = \sum_{i=0}^{n} \beta^{(i)}_{\mathcal{J}_{\geq d}}(d)$$

and

$$\sum_{i=0}^{n} i\beta^{(i)}_{\mathcal{L}_{\geq d}}(d) = \sum_{i=0}^{n} i\beta^{(i)}_{\mathcal{J}_{\geq d}}(d) = \dim_{\mathbb{k}}(\mathcal{J}_{d+1}).$$

We assume that $\beta_{\mathcal{L}_{\geq d}}(d) \neq \beta_{\mathcal{J}_{\geq d}}(d)$. Then $\beta_{\mathcal{L}_{\geq d}}(d) \prec_{\texttt{lex}} \beta_{\mathcal{J}_{\geq d}}(d)$ due to Proposition 2.13. In addition to that, Proposition 2.16 implies the existence of $j \in \{0, \ldots, n\}$ such that

$$\sum_{k=j}^{n} \beta^{(i)}_{\mathcal{L}_{\geq d}}(d) > \sum_{k=j}^{n} \beta^{(i)}_{\mathcal{J}_{\geq d}}(d).$$

As a result, we have in the $(j-1)$th hyperplane that

$$\dim_{\mathbb{k}}(\mathcal{J}_{\geq d_{\{x_0,\ldots,x_{j-1}\}}}) = \sum_{k=j}^{n} \beta^{(i)}_{\mathcal{J}_{\geq d}}(d) < \sum_{k=j}^{n} \beta^{(i)}_{\mathcal{L}_{\geq d}}(d) = \dim_{\mathbb{k}}(\mathcal{L}_{\geq d_{\{x_0,\ldots,x_{j-1}\}}}).$$

However, this is a contradiction to Lemma 2.17 which claims that

$$\dim_{\mathbb{k}}(\mathcal{J}_{\geq d_{\{x_0,\ldots,x_{j-1}\}}}) \geq \dim_{\mathbb{k}}(\mathcal{L}_{\geq d_{\{x_0,\ldots,x_{j-1}\}}}).$$

This contradiction implies that $\beta_{\mathcal{L}_{\geq d}}(d) = \beta_{\mathcal{J}_{\geq d}}(d)$. The Hilbert function of $\mathcal{J}_{\geq d}$ can be computed via $\beta_{\mathcal{J}_{\geq d}}(d)$ because $\mathcal{J}_{\geq d}$ has no minimal generator of degree greater $d$. From this follows that $\mathcal{L}_{\geq d}$ cannot have a minimal generator of degree greater than $d$ because the Hilbert functions would not be coincide anymore. This implies that the growth of $\mathcal{J}_{\geq d}$ is equal to the growth of $\mathcal{L}_{\geq d}$. Together with Theorem 2.9 it follows the first part of the claim.

The maximal degree of an element in the Pommaret basis of $\mathcal{I}$ is $p$ and $d \geq p$ and thus $\mathcal{I}$ is trivially $d$-regular. □

The last theorem proved the persistence theorem for a field with characteristic zero. The next corollary extends this to fields of arbitrary characteristics.

**Corollary 2.19** (Gotzmann's Persistence Theorem)**.** *Let* $\operatorname{char}(\Bbbk) \geq 0$ *and* $\mathcal{I} \subseteq \mathcal{P}$ *be a homogeneous ideal with minimal generating set* $\mathcal{B}$ *and* $\deg(\mathcal{B}) = p$*. Moreover, that let* $\mathcal{L}$ *be the lexsegment ideal of* $\mathcal{I}$*. Further, suppose that* $d \geq p$ *is an integer such that*

$$\mathsf{HF}_{\mathcal{P}/\mathcal{I}}(d+1) = \dim_{\Bbbk}(\mathcal{P}_1(\mathcal{P}/\mathcal{I})_d),$$

*then* $\mathcal{I}$ *is* $d$*-regular and* $\mathsf{HF}_{\mathcal{P}/\mathcal{I}}(k+1) = \dim_{\Bbbk}(\mathcal{P}_1(\mathcal{P}/\mathcal{I})_k)$ *for all* $k \geq d$*.*

*Proof.* Without loss of generality we assume that $\mathcal{I}$ is a monomial ideal. Furthermore, assume that $\operatorname{char}(\Bbbk) > 0$. Now, assume another field $\Bbbk'$ with $\operatorname{char}(\Bbbk') = 0$. If we consider $\mathcal{I}$ not over $\Bbbk$ but over $\Bbbk'$, the Hilbert function does not change and so the corresponding lexsegment ideal remains. Hence, it is sufficient to show the statement for a monomial ideal over a polynomial ring with a field of characteristic zero.

Let $\operatorname{char}(\Bbbk) = 0$ and $\mathcal{J} \subseteq \mathcal{P}$ be a homogeneous ideal. The Hilbert function is stable under coordinate transformations. Thus, we can assume that $\mathcal{J}$ is in a stable position. In addition to that, we can assume that $\mathcal{J}$ is monomial due to the theorem of Macaulay. For a stable ideal $\mathcal{J}$ we can apply Theorem 2.18 to $\mathcal{J}$ and the claim immediately follows.

□

**Definition 2.20.** *Let* $\mathcal{I} \subseteq \mathcal{P}$ *and* $p$ *be like in Corollary 2.19. Then the smallest* $d$ *which fulfils the condition of Corollary 2.19 is called the* persistence index.

## 2.2 Gotzmann's Regularity Theorem

In this section we prove Gotzmann's regularity theorem [23]. It gives us an upper bound for the regularity among all saturated ideals with the same Hilbert polynomial.

The regularity theorem is based on the Gotzmann representation which we define below. Our approach shows that one can easily determine the Gotzmann representation out of the saturated lexsegment ideal which is unique for every Hilbert polynomial.

**Lemma 2.21** ([47, Cor. B.5.1]). *Let $\mathcal{I} \subset \mathcal{P}$ be a homogeneous ideal. The Hilbert polynomial of $\mathcal{P}/\mathcal{I}$ has the form*

$$\mathsf{HP}_{\mathcal{P}/\mathcal{I}}(t) = \binom{t+a_1}{a_1} + \binom{t+a_2-1}{a_2} + \cdots + \binom{t+a_s-(s-1)}{a_s}$$

*for some $a_1 \geq \cdots \geq a_s \geq 0$.*

The representation above is called the *Gotzmann representation of* $\mathsf{HP}_{\mathcal{P}/\mathcal{I}}$, the number $s$ is called the *Gotzmann number of* $\mathsf{HP}_{\mathcal{P}/\mathcal{I}}$, $a_1, \ldots, a_s$ are called the *Gotzmann coefficients* and $\binom{t+a_i-i+1}{a_i}$ are called the Gotzmann summands. Please note, that the Gotzmann representation is only defined for ideals which are not the whole polynomial ring.

Firstly, we show the uniqueness of the saturated lexsegment ideal among all lexsegment ideals with the same Hilbert polynomial. Secondly we show how to obtain the Gotzmann representation out of this saturated lexsegment ideal. Finally, we use this to prove the regularity theorem of Gotzmann.

**Lemma 2.22.** *Let $M$ be the set of all lexsegment ideal with the same Hilbert polynomial* $\mathsf{HP}$. *Then there exists an $\mathcal{L} \in M$ such that $\mathcal{L}' \subseteq \mathcal{L}$ for every $\mathcal{L}' \in M$.*

*Proof.* Assume there is an $\mathcal{L}''$ such that neither $\mathcal{L}'' \subset \mathcal{L}$ nor $\mathcal{L} \subset \mathcal{L}''$. Hence, there exist $x^\mu \in \mathcal{L}''$ with $x^\mu \notin \mathcal{L}$ and $x^\nu \in \mathcal{L}$, such that $x^\nu \notin \mathcal{L}''$.

However, there is a $d \in \mathbb{N}$ such that $\mathsf{HF}_{\mathcal{L}}(d+i) = \mathsf{HF}_{\mathcal{L}''}(d+i)$ for all $i \in \mathbb{N}$, which implies $\mathcal{L}_{d+i} = \mathcal{L}''_{d+i}$. But then $\mathcal{L} + \mathcal{L}''$ has also Hilbert polynomial $\mathsf{HP}$ and $\mathcal{L} \subset \mathcal{L} + \mathcal{L}''$ and $\mathcal{L}'' \subset \mathcal{L} + \mathcal{L}''$. This is a contradiction to the assumption that $\mathcal{L}$ is maximal. □

**Lemma 2.23.** *Let $M$ be as above, then the maximal $\mathcal{L} \in M$ is saturated.*

*Proof.* Let $\mathcal{B}_{\mathcal{L}}$ be the minimal generating set of $\mathcal{L}$ and assume that $\mathcal{L}$ is not saturated. Thus, there exists an $x^\mu \in \mathcal{B}_{\mathcal{L}}$ such that $\deg_0(x^\mu) = i > 0$ because $\mathcal{B}_{\mathcal{L}}$ is a Pommaret basis of $\mathcal{L}$, too. Due to the fact that $\mathcal{L}$ is stable we know that $x^{\mu - 1_0 + 1_i} \in \mathcal{L}$ for all $1 \leq i \leq n$. This implies that $\mathcal{L} + \mathcal{L}(x^{\mu - 1_0})$ is also a lexsegment ideal which has Hilbert polynomial $\mathsf{HP}$. This is a contradiction to the maximality of $\mathcal{L}$, because $\mathcal{L} \subset \mathcal{L} + \mathcal{L}(x^{\mu - 1_0})$. □

**Lemma 2.24.** *Let $M$ be as above, then $M$ contains only one saturated lexsegment ideal.*

*Proof.* Assume there are at least two saturated lexsegment ideals with the same Hilbert polynomial. We choose the saturated lexsegment ideals $\mathcal{L}^1$ and $\mathcal{L}^2$. Since both have the same Hilbert polynomial there exists $d \in \mathbb{N}$ such that $\mathsf{HF}_{\mathcal{L}^1}(d+i) = \mathsf{HF}_{\mathcal{L}^2}(d+i)$ for all $i \in \mathbb{N}$. Let $d'$ be the greatest degree such that $\mathsf{HF}_{\mathcal{L}^1}(d') \neq \mathsf{HF}_{\mathcal{L}^2}(d')$. Then there exists without loss of generality an $x^\mu \in \mathcal{L}^2_{d'}$ such that $x^\mu \notin \mathcal{L}^1_{d'}$. But we know that $\mathcal{L}^1_{d'+1} = \mathcal{L}^2_{d'+1}$ which implies $x_0 x^\mu \in \mathcal{L}^1_{d'+1}$. Consequently, $x_0 x^\mu$ must be a minimal generator of $\mathcal{L}_1$. This is a contradiction to the assumption that $\mathcal{L}_1$ is a saturated ideal. □

**Definition 2.25.** *Let $\mathcal{C} \subset \mathbb{T}$ be a finite set. We divide $\mathcal{C}$ for every $0 \leq i \leq n$ into subgroups which are represented by non-negative integers $d_n, \ldots, d_i$:*

$$[d_i, \ldots, d_n]_{\mathcal{C}} := \{x^\mu \in \mathcal{C} \mid d_j = \deg_j(x^\mu), i \leq j \leq n\}.$$

**Lemma 2.26.** *Let $\mathcal{L}$ be a saturated lexsegment ideal with minimal generating set $\mathcal{B}$. If*

$$[d_i, \ldots, d_n]_{\mathcal{B}} \neq \emptyset, \text{ but } [d_i + 1, \ldots, d_n]_{\mathcal{B}} = \emptyset,$$

*then*

$$[d_i - j, \ldots, d_n]_{\mathcal{B}} = \emptyset \text{ for } i \in \{0, \ldots, n\} \text{ and } 2 \leq j \leq d_i.$$

*Proof.* First, we proof that the set $[d_i, \ldots, d_n]_{\mathcal{B}}$ contains only the element $x^\alpha = x_i^{d_i} \cdots x_n^{d_n}$.

Assume that there is another element $x^\beta$ in this set. $x^\beta$ has a class that is smaller than $i$ which implies that there is a non-multiplicative prolongation $x_i x^\beta$. This element must be involutive divisible by an element $x^\gamma \in \mathcal{B}$. The element $x^\gamma$ must be divisible by $x_i^{d_i} \cdots x_n^{d_n}$. If this is not the case, every element in the set $[d_i, \ldots, d_n]_{\mathcal{B}}$ would be involutively divisible by $x^\gamma$ and hence the set would be empty. In fact, $x^\gamma$ must be $x_i^{d_i} \cdots x_n^{d_n}$ because $\deg_i(x^\gamma) < d_i + 1$, and hence $i$ must be multiplicative for $x^\gamma$. Hence, the only possible solution is $\{x^\alpha\} = [d_i, \ldots, d_n]_{\mathcal{B}}$.

Now assume that $\mathcal{B}$ contains an element $x^\nu x_i^{d_i - j} \cdots x_n^{d_n}$ such that $j \geq 2$ and $x^\nu \in \mathbb{T}_{\{0,\ldots,i-1\}}$. $\mathcal{L}$ is a lexsegment ideal, hence $x^\beta = x_0^{\deg(x^\nu)-1} x_i^{d_i - j + 1} \cdots x_n^{d_n} \in \mathcal{L}$. Note that $x^\nu \neq 1$. If this would not the case, then $x_i^{d_i - j} \cdots x_n^{d_n}$ would be an involutive divisor of $x^\alpha$ which implies that $x^\alpha \notin \mathcal{B}$. Neither $x_i^{d_i - j + 1} \cdots x_n^{d_n}$ nor an involutive divisor of it can be an involutive divisor of $x^\beta$, because this would be an involutive divisor of $x^\alpha$, too. However, then $x^\beta \in \mathcal{B}$ which is a contradiction to the fact that $\mathcal{L}$ is saturated. □

**Corollary 2.27.** *Let $\mathcal{L}$ be a saturated lexsegment ideal with minimal generating set $\mathcal{B}$ with maximal degree $d$. Moreover, let $d_i$ be the maximal degree of $x_i$ and $m$ be the smallest class which occurs for an element in $\mathcal{B}$. Then $x^\mu = x_n^{d_n - 1} \cdots x_{m-1}^{d_{m-1} - 1} x_m^{d_m} \in \mathcal{B}$ and $\deg(x^\mu) = d$.*

**Lemma 2.28.** *Let $\mathcal{L}$ be a saturated lexsegment ideal with minimal generating set $\mathcal{B}$ with maximal degree $d$ and denote by $\overline{\mathcal{L}} = \mathbb{T} \setminus (\mathcal{L} + \mathbb{T})$ its complementary set. Then there exists a finite set $\overline{\mathcal{B}} \subset \overline{\mathcal{L}}$ and for each $x^\mu \in \overline{\mathcal{B}}$ a set $X_\mu \subseteq \{0, \dots, n\}$ defining the disjoint decomposition*

$$\biguplus_{x^\mu \in \overline{\mathcal{B}}} \left( x^\mu + \mathbb{T}_{X_\mu} \right).$$

*It is possible to choose a decomposition such that $\overline{\mathcal{B}}$ contains exactly one element of degree $i$ for $0 \leq i < d$.*

*Proof.* The proof is based on [43, Prop. 5.1.4], [43, Alg. 5.2] and Lemma 2.26. We proceed iteratively over the sets $[d_i, \dots, d_n]_{\mathcal{B}}$. Let $d_i$ be the maximal degree of $x_i$ which occurs in $\mathcal{B}$. Due to Lemma 2.26 we know that only $[d_n]_{\mathcal{B}}$ and $[d_n - 1]_{\mathcal{B}}$ are non-empty sets of length one. Hence, we can add the set $\{(x_n^i, \{0, \dots, n-1\}) \mid 0 \leq i < d_n - 1\}$ to $\overline{\mathcal{B}}$ ($(x_n^i, \{0, \dots, n-1\})$ means that we add the element $x^\mu = x_n^i$, with $X_\mu = \{0, \dots, n-1\}$ to $\overline{\mathcal{B}}$). For the sets of length two, we know that the only non-empty sets that can occur are $[d_n - 1, d_{n-1}]_{\mathcal{B}}$ and $[d_n - 1, d_{n-1} - 1]_{\mathcal{B}}$. We have a similar situation as above and we add $\{(x_n^{d_n-1} x_{n-1}^i, \{0, \dots, n-2\}) \mid 0 \leq i < d_{n-1} - 1\}$ to $\overline{\mathcal{B}}$. Now we apply this technique to all sets of length $3, 4, \dots, n - m + 1$. Here $m$ is the smallest class which occurs in $\mathcal{B}$. However, for sets of length $n - m$ the situation is similar but one has to note that the set $[d_n, \dots, d_m - 1]$ is empty. This leads to the complementary decomposition

$$\begin{aligned} \overline{\mathcal{B}} = & \biguplus_{i=n,\dots,m+1} \left\{ (x_i^j x_{i+1}^{d_{i+1}-1} \cdots x_n^{d_n-1}, \{0, \dots, i-1\}) \mid 0 \leq j < d_i - 1 \right\} \\ & \cup \left\{ (x_m^j x_{m+1}^{d_{m+1}-1} \cdots x_n^{d_n-1}, \{0, \dots, m-1\}) \mid 0 \leq j < d_m \right\}. \end{aligned} \tag{2.1}$$

Now it is obvious, that the claim is fulfilled. □

**Corollary 2.29.** *Let $\mathcal{L}$ be a saturated lexsegment ideal as above. Then the Hilbert polynomial is*

$$\begin{aligned} \mathsf{HP}_{\mathcal{P}/\mathcal{L}}(t) = & \sum_{i=0}^{d_n-2} \binom{t - i + n - 1}{t - i} \\ & + \sum_{i=0}^{d_{n-1}-2} \binom{t - (d_n - 1 + i) + (n-1) - 1}{t - (d_n - 1 + i)} \\ & + \cdots \\ & + \sum_{i=0}^{d_{m+1}-2} \binom{t - (d_n - 1 + \cdots d_{m+2} - 1 + i) + (m+1) - 1}{t - (d_n - 1 + \cdots d_{m+2} - 1 + i)} \\ & + \sum_{i=0}^{d_m-1} \binom{t - (d_n - 1 + \cdots d_{m+1} - 1 + i) + m - 1}{t - (d_n - 1 + \cdots d_{m+1} - 1 + i)} \end{aligned} \tag{2.2}$$

*with respect to the notations of* $\overline{\mathcal{B}}$ *defined in* (2.1).

*Proof.* It is a simple application of the formula to compute the Hilbert polynomial from a polynomial decomposition which is defined in [43, Errata Prop. 5.2.1] on $\overline{\mathcal{B}}$. □

**Proposition 2.30.** *Let* $\mathcal{L}$ *be a saturated lexsegment ideal, and* $\overline{\mathcal{B}}$ *its complementary decomposition as constructed in* (2.1)*. Then* $\overline{\mathcal{B}}$ *provides the Gotzmann representation of* $\mathsf{HP}_{\mathcal{P}/\mathcal{L}}$*. with Gotzmann number*

$$s = d_m + \sum_{i=m+1}^{n} (d_i - 1)\,. \tag{2.3}$$

*Furthermore the Gotzmann coefficients are*

$$\begin{aligned} a_1, \dots, a_{d_n - 1} &= n - 1, \\ a_{d_n}, \dots, a_{d_n - 1 + d_{n-1} - 1} &= n - 2, \\ a_{d_n - 1 + d_{n-1}}, \dots, a_{d_n - 1 + d_{n-1} - 1 + d_{n-2} - 1} &= n - 3, \\ &\dots, \\ a_{d_n - 1 + \cdots d_{m+2} - 1 + d_{m+1}}, \dots, a_{d_n - 1 + \cdots + d_{m+1} - 1 + d_m} &= m - 1\,. \end{aligned}$$

*Proof.* This is a direct consequence of the previous corollary with (2.2). □

Now we want to prove Gotzmann's regularity theorem. In order to prove it we need a result from the PhD thesis of A.A. Reeves.

**Theorem 2.31** ([38, Appendix A, Fact 2, p. 83])**.** *Let* $\mathcal{J} \subseteq \mathcal{P}$ *be a saturated monomial stable ideal. Then the corresponding lexsegment ideal is a saturated monomial stable ideal, too.*

**Corollary 2.32.** *Let* $\mathsf{HP}$ *be a Hilbert polynomial with the corresponding saturated lexsegment ideal* $\mathcal{L}$*. All saturated stable ideals with Hilbert polynomial* $\mathsf{HP}$ *have the same lexsegment ideal, namely* $\mathcal{L}$*.*

**Theorem 2.33** (Gotzmann's regularity theorem)**.** *Let* $\mathcal{I}$ *be an ideal and* $\mathsf{HP}_{\mathcal{P}/\mathcal{I}}$ *the Hilbert polynomial with Gotzmann representation*

$$\mathsf{HP}_{\mathcal{P}/\mathcal{I}}(t) = \binom{t + a_1}{a_1} + \binom{t + a_2 - 1}{a_2} + \cdots + \binom{t + a_s - (s-1)}{a_s}$$

*with* $a_1 \geq a_2 \geq \cdots \geq a_s \geq 0$*. Then the saturation of* $\mathcal{I}$ *is s-regular.*

*Proof.* Without loss of generality we assume that $\mathcal{I}$ is a saturated monomial ideal. This property remains preserved under coordinate transformations. In addition to that, the regularity does not change under coordinate transformations, too. Hence, we can assume that $\mathcal{I}$ is a stable ideal. We know from Theorem 2.31, that the corresponding lexsegment ideal is the saturated lexsegment ideal corresponding to $\mathsf{HP}_{\mathcal{P}/\mathcal{I}}$. From Corollary 2.19 we know that the degree of the minimal generating set of $\mathcal{I}$ is smaller than $s$. Hence, $\mathcal{I}$ is $s$-regular because $\mathcal{I}$ is stable. □

**Corollary 2.34.** *Let $\mathcal{L}$ be a saturated lexsegment ideal with minimal generating set $\mathcal{B}$ and $s$ the corresponding Gotzmann number. Then $s = \deg(\mathcal{B})$ and the regularity of $\mathcal{L}$ is equal to the Gotzmann number.*

*Proof.* This is an easy consequence of (2.3), Corollary 2.27 and the fact, that the saturated lexsegment ideal is stable. □

Finally, we show that the regularity of a quasi-stable ideal is either smaller than or equal to its Gotzmann number or is equal to the persistence index.

**Lemma 2.35.** *Let $\mathcal{L}$ be a lexsegment ideal with minimal generating set $\mathcal{B}$ and Gotzmann number $s$ such that $\deg(\mathcal{B}) > s$. If $\mathsf{HF}_{\mathcal{P}/\mathcal{L}}(d) = \mathsf{HP}_{\mathcal{P}/\mathcal{L}}(d)$, then $d \geq \deg(\mathcal{B})$.*

*Proof.* We assume that this is not true and there is a $d$ with $s \leq d < \deg(\mathcal{B})$ such that $\mathsf{HF}_{\mathcal{P}/\mathcal{L}}(d) = \mathsf{HP}_{\mathcal{P}/\mathcal{L}}(d)$. Let $d$ be the biggest possible. Now we choose another lexsegment ideal $\mathcal{L}^s$ such that $\mathsf{HP}_{\mathcal{P}/\mathcal{L}^s} = \mathsf{HP}_{\mathcal{P}/\mathcal{L}}$ and the degree of the minimal generating set of $\mathcal{L}^s$ is $s$. Then

$$\mathsf{HP}_{\mathcal{L}}(d+1) = \mathsf{HP}_{\mathcal{L}^s}(d+1) = \dim_{\Bbbk}((\mathcal{L}^s)_{d+1}) = \dim_{\Bbbk}(\mathcal{P}_1(\mathcal{L}^s)_d) = \dim_{\Bbbk}(\mathcal{P}_1(\mathcal{L})_d).$$

However, from the assumption we know that $\mathcal{B}$ has an element of degree $d+1$. This implies that $\mathsf{HF}_{\mathcal{L}}(d+1) > \dim_{\Bbbk}(\mathcal{P}_1\mathcal{L}_d) = \mathsf{HP}_{\mathcal{L}}(d+1)$ and respectively $\mathsf{HF}_{\mathcal{P}/\mathcal{L}}(d+1) < \mathsf{HP}_{\mathcal{P}/\mathcal{L}}(d+1)$. However, this is a contradiction because [43, Errata Prop. 5.2.1] implies that

$$\begin{aligned}\mathsf{HF}_{\mathcal{L}}(d+1) &= \sum_{t\in\mathcal{B}\wedge\deg(t)\leq d+1} \binom{d+1-\deg(t)+\operatorname{cls}(t)-1}{q-\deg(t)} \\ &\leq \sum_{t\in\mathcal{B}} \binom{d+1-\deg(t)+\operatorname{cls}(t)-1}{q-\deg(t)} \\ &= \mathsf{HP}_{\mathcal{L}}(d+1).\end{aligned}$$

□

**Proposition 2.36.** *Let $\mathcal{I} \subseteq \mathcal{P}$ be an ideal with finite Pommaret basis $\mathcal{B}$, $\mathcal{L}$ the corresponding lexsegment ideal with minimal generating set $\mathcal{B}_{\mathcal{L}}$ and $s$ the corresponding Gotzmann number such that* $\deg(\mathcal{B}) > s$. *Then,* $\deg(\mathcal{B}) = \deg(\mathcal{B}_{\mathcal{L}})$.

*Proof.* We know that $\mathsf{HF}_{\mathcal{P}/\mathcal{I}}(\deg(\mathcal{B})) = \mathsf{HP}_{\mathcal{P}/\mathcal{I}}(\deg(\mathcal{B}))$ using the formula to compute the Hilbert polynomial starting from a Pommaret basis. Then it is also true that $\mathsf{HF}_{\mathcal{P}/\mathcal{L}}(\deg(\mathcal{B})) = \mathsf{HP}_{\mathcal{P}/\mathcal{L}}(\deg(\mathcal{B}))$. Lemma 2.35 implies $\deg(\mathcal{B}_{\mathcal{L}}) \leq \deg(\mathcal{B})$. Additionally, the persistence theorem of Gotzmann (Corollary 2.19) says that $\deg(\mathcal{B}) = \operatorname{reg}(\mathcal{I}) \leq \deg(\mathcal{B}_{\mathcal{L}})$ which implies $\deg(\mathcal{B}) = \deg(\mathcal{B}_{\mathcal{L}})$. □

**Corollary 2.37.** *Let $\mathcal{I}$ be a quasi-stable ideal with Pommaret basis $\mathcal{B}$, Gotzmann number $s$ and persistence index $p$. Then* $\operatorname{reg}(\mathcal{I}) \in \{1, \dots, s, p\}$.

*Proof.* The persistence index is an upper bound for the regularity of $\mathcal{I}$. If $\deg(\mathcal{B}) > s$, Proposition 2.36 implies $\operatorname{reg}(\mathcal{I}) = \deg(\mathcal{B}) = p$. □

## 2.3 Gotzmann's Theorems for Modules

In this section we show that one can extend Gotzmann's theorems to the module case. In order to transform the regularity theorem to modules we need a special grading for $\mathcal{P}^m_{\mathbf{d}}$.

At first, we have to introduce the Macaulay representation of natural numbers. Let $a, d \in \mathbb{N}$, then the *dth Macaulay representation of a* is the unique expression

$$a = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_e}{e},$$

with $e \in \mathbb{Z}$, satisfying $k_d > \cdots > k_e \geq e > 0$ (see for example [31, Prop. 5.5.1]). With this representation, the *dth Macaulay transformation of a* is

$$a^{<d>} = \binom{k_d + 1}{d + 1} + \binom{k_{d-1} + 1}{d} + \cdots + \binom{k_e + 1}{e + 1}.$$

In 1997 Gasharov [20] already extended the persistence theorem to the module case.

**Theorem 2.38** ([20, Thm. 4.2])**.** *Let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a finitely generated graded module. Furthermore, let* $l = \max(\{d_i \mid 1 \leq i \leq m\})$. *For each pair $(p, d)$ such that $p \geq 0$ and $d \geq p + l + 1$ we have*

- $\mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(d+1) \leq \mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(d)^{\langle d-l-p \rangle}$;

- *If $\mathcal{U}$ is generated in degree at most $d$ and* $\mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(d+1) = \mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(d)^{\langle d-l-p\rangle}$*, then* $\mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(d+i) = \mathsf{HF}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(d+i-1)^{\langle d+i-1-l-p\rangle}$ *for all $i \geq 1$.*

Now we recall the work of Dellaca who recently showed that one can extend the regularity theorem as well. However, we have to notice, that there are Hilbert polynomials where we cannot even find a Gotzmann representation. The reason for this is the grading we use. We see this in the next example.

**Example 2.39** ([15, Ex. 2.10])**.** *Let us consider the polynomial ring $\mathcal{P}^m_{\mathbf{d}} = \Bbbk[x_0, x_1]^1_{(1)}$, e.g.* $\deg(\mathbf{e}_1) = 1$*. Then* $\mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\langle 0\rangle}(t) = t$ *and by reason of degree, the Gotzmann representation must have one as the first Gotzmann coefficient. The Gotzmann representation can only add more positive terms to the first term $\binom{t+1}{1} = t+1$, hence there cannot exist a Gotzmann representation for this Hilbert polynomial.*

For the rest of the chapter we assume that $\mathcal{P}^m_{\mathbf{d}}$ has a grading such that $\mathbf{d} = (d_1, \ldots, d_m)$ with $d_i \leq 0$ for all $1 \leq i \leq m$. The next proposition shows that we can always find a Gotzmann representation in this case.

**Proposition 2.40** ([15, Prop. 3.1])**.** *Let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a finitely generated graded module. Then* $\mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(t)$ *has a unique Gotzmann representation.*

With this proposition Dellaca is able to proof Gotzmann's regularity theorem for modules.

**Theorem 2.41** ([15, Prop. 4.1])**.** *Let $\mathcal{U} \subseteq \mathcal{P}^m_{\mathbf{d}}$ be a finitely generated graded module such that $\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}$ has the following Gotzmann representation*

$$\mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{U}}(t) = \binom{t+a_1}{a_1} + \binom{t+a_2-1}{a_2} + \cdots + \binom{t+a_s-(s-1)}{a_s}.$$

*Then the module $\mathcal{U}$ is $s$-regular.*

**Example 2.42.** *We consider the polynomial $\mathcal{P}^3_{\mathbf{d}} = \Bbbk[x_0, x_1, x_2, x_3]^3_{\mathbf{d}}$ with $\mathbf{d} = (0, -1, -2)$. That gives* $\deg(\mathbf{e}_1) = 0$*,* $\deg(\mathbf{e}_2) = -1$ *and* $\deg(\mathbf{e}_3) = -2$*. Moreover, we consider the monomial ideals $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \subseteq \mathcal{P}$ with*

$$\begin{aligned}
\mathcal{J}_1 &= \langle x_0, x_1, x_2^2, x_3^3\rangle,\\
\mathcal{J}_2 &= \langle x_2, x_3\rangle,\\
\mathcal{J}_3 &= \langle x_1x_3, x_2x_3, x_3^3\rangle.
\end{aligned}$$

*Obviously, the corresponding Hilbert polynomials are* $\mathsf{HP}_{\mathcal{P}/\mathcal{J}_1}(t) = 0$, $\mathsf{HP}_{\mathcal{P}/\mathcal{J}_2}(t) = t + 1$ *and* $\mathsf{HP}_{\mathcal{P}/\mathcal{J}_3}(t) = t^2 + 3t + 6$*. Out of this Hilbert polynomials we can easily construct the Hilbert polynomials of* $\mathcal{J}_i \cdot \mathbf{e}_i$*:*

$$\begin{aligned}
\mathsf{HP}_{\mathcal{P}^3_{\mathbf{d}}/\mathcal{J}_1\cdot\mathbf{e}_1}(t) &= \mathsf{HP}_{\mathcal{P}/\mathcal{J}_1}(t) = 0,\\
\mathsf{HP}_{\mathcal{P}^3_{\mathbf{d}}/\mathcal{J}_2\cdot\mathbf{e}_2}(t) &= \mathsf{HP}_{\mathcal{P}/\mathcal{J}_2}(t+1) = t + 2,\\
\mathsf{HP}_{\mathcal{P}^3_{\mathbf{d}}/\mathcal{J}_3\cdot\mathbf{e}_3}(t) &= \mathsf{HP}_{\mathcal{P}/\mathcal{J}_3}(t+2) = t^2 + 7t + 16.
\end{aligned}$$

*Then the Hilbert polynomial of* $\mathcal{U} = \mathcal{J}_1 \cdot \mathbf{e}_1 \oplus \mathcal{J}_2 \cdot \mathbf{e}_2 \oplus \mathcal{J}_3 \cdot \mathbf{e}_3$ *is obviously*

$$\mathsf{HP}_{\mathcal{P}^3_{\mathbf{d}}/\mathcal{U}}(t) = \mathsf{HP}_{\mathcal{P}^3_{\mathbf{d}}/\mathcal{J}_1\cdot\mathbf{e}_1}(t) + \mathsf{HP}_{\mathcal{P}^3_{\mathbf{d}}/\mathcal{J}_2\cdot\mathbf{e}_2}(t) + \mathsf{HP}_{\mathcal{P}^3_{\mathbf{d}}/\mathcal{J}_3\cdot\mathbf{e}_3}(t) = t^2 + 8t + 18\,.$$

*A computation shows that the Gotzmann number of* $\mathsf{HP}_{\mathcal{P}^3_{\mathbf{d}}/\mathcal{U}}(t)$ *is* 45*. The Gotzmann coefficients are*

$$\begin{aligned}
a_1 = a_2 &= 2\,,\\
a_3 = \cdots = a_7 &= 1\,,\\
a_8 = \cdots = a_{45} &= 0\,.
\end{aligned}$$

# 3 Quasi-Stable Covering of Quot Schemes

In this chapter we introduce the Hilbert, Quot and Grassmann functor. We will see that the Quot functor is a simple generalization of the Hilbert functor. Furthermore, we show that the Quot functor can be seen as a subfunctor of the Grassmann functor. To define the Hilbert functor and the Quot functor we are going to remind the construction of the Hilbert polynomial for coherent $\mathcal{O}_X$-modules over projective $\mathbb{k}$-schemes at the beginning of this chapter. Furthermore, we show how the Hilbert polynomial behaves if we consider flat families of schemes. We are also going to remind the definition of representable functors and their properties.

Grothendieck proved in [25] that the Quot functor is representable and hence we can talk about Quot schemes. In this chapter we present a new constructive proof of representability of the Quot functor. For that we construct an open quasi-stable covering of the Grassmann functor and we show that we can restrict this quasi-stable covering to the Quot functor. Then we introduce marked functors and show that these functors are represented by marked schemes. By showing that the marked functors correspond to the subfunctors of the quasi-stable covering of the Quot functor we can finally show that the Quot functor is representable.

The basic ideas we use in this chapter were introduced by Brachat et al. [11]. They considered a strongly stable covering of the Grassmanian to give a new proof for the representability of Hilbert functors over fields of characteristic zero. This chapter generalizes this idea in two directions. First of all, we extend this idea to the more general case of Quot functors and secondly we use a quasi-stable covering of the Quot functor in order to show the representability over fields of arbitrary characteristic.

## 3.1 Representability and Flat Families of Projective Schemes

### 3.1.1 The Hilbert Polynomial for Coherent Sheaves

We define the Hilbert polynomial for coherent sheaves. For the attentive reader this seems to be confusing as we already introduced the Hilbert polynomial for modules in Definition 2.3. At the end of this section we resolve this ambiguity by showing that these definitions are compatible.

In the following let $X$ be a projective $\Bbbk$-Scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. The *Euler characteristic of* $\mathcal{F}$ ([28, Ex. III.5.1]) is defined as

$$\chi(\mathcal{F}) := \sum_{i=0}^{\infty}(-1)^i \dim_{\Bbbk}(H^i(\mathcal{F})).$$

**Lemma 3.1** ([28, Ex. III.5.1]). *Let* $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ *be an exact sequence of* $\mathcal{O}_X$*-modules then*

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

Using this lemma we can show the following theorem.

**Theorem 3.2** ([28, Ex. III.5.2 (a)]). *There exists a polynomial* $\mathsf{HP}_{\mathcal{F}}(t) \in \mathbb{Q}[t]$ *such that* $\chi(\mathcal{F}(s)) = \mathsf{HP}_{\mathcal{F}}(s)$ *for all* $s \in \mathbb{Z}$*. This polynomial is called the* Hilbert polynomial *of* $\mathcal{F}$

The following theorem allows us to simplify the definition of the Hilbert polynomial.

**Theorem 3.3** (Serres vanishing theorem [28, Thm. III.5.2]). *Let $A$ be a noetherian ring and $X$ be a projective scheme over $A$. Let $\mathcal{F}$ be a coherent sheaf on $X$. Then we have the following properties:*

- *For any integer $p \geq 0$ the $A$-module $H^p(\mathcal{F})$ is finitely generated.*
- *There exists an integer $s_0$ (which depends on $\mathcal{F}$) such that for every $s \geq s_0$ and for every $p \geq 1$ we have $H^p(\mathcal{F}(s)) = 0$.*

As all higher homologies of $\mathcal{F}$ vanish for $s \gg 0$ we get for $s \gg 0$

$$\mathsf{HP}_{\mathcal{F}}(s) = \dim_{\Bbbk}(H^0(\mathcal{F}(s))). \tag{3.1}$$

**Remark 3.4.** *We defined the Hilbert polynomial only for coherent $\mathcal{O}_X$-modules, where $X$ is a projective $\Bbbk$-scheme. But for example in [36] the definition is extended to the more general case, where $X$ is a finite type scheme over $\Bbbk$, whose support is proper over $\Bbbk$.*

*Due to the fact that the support of $X$ is proper over $\Bbbk$ the morphism* $\operatorname{supp}(\mathcal{F}) \to \operatorname{Spec}(\Bbbk)$ *is proper. This ensures that the cohomology groups in the definition of the Euler characteristic are finite. In addition to that there is a generalization of Theorem 3.3, where $X$ can be a proper scheme over* $\operatorname{Spec}(A)$*, where $A$ is a noetherian ring (see see [34, Rem. 3.3, Prop. 3.6]). Hence, it is possible to define the Hilbert polynomial in the same way for this more general case.*

In the second chapter we have seen the classical definition of a Hilbert polynomial in commutative algebra. In this section we defined a Hilbert polynomial, again. The following example illustrates that both definitions are essentially the same.

**Example 3.5.** *Let $\mathcal{P}^m = \mathbb{k}[x_0, \dots, x_n]^m$ be a polynomial module over an arbitrary field. Furthermore, let $\mathcal{U} \subseteq \mathcal{P}^m$ be a homogeneous module of $\mathcal{P}^m$. Then $\mathcal{M} = \mathcal{P}^m/\mathcal{U}$ is a finitely presented $\mathbb{k}$-module. Now we take a look at the module sheaf $\mathcal{F} = \widetilde{\mathcal{M}}$ which is induced by $\mathcal{M}$. It is obvious that this module sheaf is a coherent sheaf over the projective $\mathbb{k}$-scheme $X = \mathbb{P}^n$.*

*Via* (3.1) *we can compute the Hilbert polynomial of $\mathcal{F}$. It is (for $s \gg 0$)*

$$\mathsf{HP}_{\mathcal{F}}(s) = \dim_k(H^0(\mathcal{F}(s))) = \dim_k(\mathcal{F}(s)(X)) = \dim_k(\mathcal{M}_s).$$

*But this is exactly the definition of the Hilbert polynomial which we deduced from Theorem 2.3. Hence, we see that $\mathsf{HP}_{\mathcal{F}}(s) = \mathsf{HP}_{\mathcal{M}}(s)$.*

### 3.1.2 Flatness

Now we introduce the concept of flatness for schemes. The property of flatness is in general not easy to understand. In fact, there is no easy geometric interpretation of this property. Nevertheless, we will see that flatness is really important in the following. One reason is the impact of flatness on families of projective schemes. In general the fibres of these families can vary a lot. But flatness ensures that these fibres only vary continuously and this fact allows to study these fibres systematically.

**Definition 3.6** ([28, III.9]). *Let $X, Y$ be schemes, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Furthermore, let $f : X \longrightarrow Y$. We can endow $\mathcal{F}_x$ for $x \in X$ with the structure of an $\mathcal{O}_{Y,f(x)}$-module via the canonical homomorphism $f_x : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$. We say that*

- $\mathcal{F}$ is flat *if $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module at every point $x \in X$,*
- $\mathcal{F}$ is flat over $Y$*, if $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,f(x)}$ at every point $x \in X$ and*
- $f$ is flat *if $f_x$ is flat at every point $x \in X$.*

The next propositions collect some important properties of flat module sheaves.

**Proposition 3.7** ([28, Prop. III.9.2]).

- *An open immersion is flat.*
- *Flatness is preserved by base change: Let $f : X \longrightarrow Y$ be a morphism of schemes, let $\mathcal{F}$ be an $\mathcal{O}_X$-module which is flat over $Y$, and let $g : Y' \longrightarrow Y$ be any morphism. Let $X' = X \times_Y Y'$, $p_1 : X' \longrightarrow X$ be the first projection and let $\mathcal{F}' = p_1^\star(\mathcal{F})$. Then $\mathcal{F}'$ is flat over $Y'$.*

- *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module which is flat over $Y$, and assume also that $Y$ is flat over $Z$. Then $\mathcal{F}$ is flat over $Z$.*
- *Let $A \longrightarrow B$ be a ring homomorphism, and let $M$ be a $B$-module. Let $f : X = \operatorname{Spec}(B) \longrightarrow Y = \operatorname{Spec}(A)$ be the corresponding morphism of affine schemes, and let $\mathcal{F} = \widetilde{M}$. Then $\mathcal{F}$ is flat over $Y$ if and only if $M$ is flat over $A$.*
- *Let $X$ be a noetherian scheme, and $\mathcal{F}$ a coherent $O_X$-module. Then $\mathcal{F}$ is flat over $X$ if and only if it is locally free.*

**Proposition 3.8** ([28, Cor. III.9.4])**.** *Let $f : X \longrightarrow Y$ be a separated morphism of finite type of noetherian schemes, $\mathcal{F}$ be a quasi-coherent sheaf on $X$ and $Y$ be affine. For any point $y \in Y$ let $X_y$ be the fibre over $y$ and $\mathcal{F}_y$ be the induced sheaf. Moreover, let $k(y)$ denote the constant sheaf $k(y)$ on the closure of $y$ in $Y$. Then for all $i \geq 0$ there are natural isomorphisms*

$$H^i(X_y, \mathcal{F}_y) \cong H_i(X, \mathcal{F} \otimes k(y)).$$

In the following part we want to investigate the Hilbert polynomial in the context of flatness. But first of all we recall the definitions of families of schemes.

A *family of projective varieties* is a projective morphism of schemes

$$f : X \longrightarrow S.$$

The members of the family are the fibres of $f$ and $S$ is the *parameter scheme*.

A *family of closed subschemes of* $\mathbb{P}^r$ is a family $f$ as above which is part of a commutative diagram of morphisms:

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^r \times S \\ & {\scriptstyle f}\searrow & \downarrow \\ & & S \end{array}$$

realizing $X$ as a closed subscheme of the product $\mathbb{P}^r \times S$ and where the vertical arrow is the second projection.

The family $f : X \longrightarrow S$ is called *flat*, if $f$ is flat as a morphism of schemes.

**Proposition 3.9** ([44, Prop. 4.2.1 (ii)])**.** *Let $S$ be a connected scheme and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^r \times S$. For each $s \in S$ let $P_s(t) \in \mathbb{Q}[t]$ be the Hilbert polynomial of $\mathcal{F}_s$. $P_s(t)$ is independent of $s \in S$ if $\mathcal{F}$ is flat over $S$. Conversely, if $S$ is integral and $P_s(t)$ is independent of $s$ for all $s \in S$, then $\mathcal{F}$ is flat over $S$.*

The proposition leads immediately to a simple corollary about flat families of closed subschemes.

**Corollary 3.10** ([44, Cor. 4.2.2])**.** *If $X \subset \mathbb{P}^r \times S$ is a flat family of closed subschemes of $\mathbb{P}^r$ with $S$ connected then all fibres $X_s$ have the same Hilbert polynomial.*

The next example nicely illustrates the last corollary.

**Example 3.11** ([44, Ex. 4.2.3 (ii)])**.** *We consider the projective space $\mathbb{P}^3$ with homogeneous coordinates $\mathbf{x} = (x_0, x_1, x_2, x_3)$ and choose the curve*

$$C_u = \operatorname{Proj}(\Bbbk[\mathbf{x}]/\langle x_2, x_3\rangle) \cup \operatorname{Proj}(\Bbbk[\mathbf{x}]/\langle x_1, x_3 - ux_0\rangle)$$

*for every $u \in \Bbbk$. If $u \neq 0$ then $C_u$ consists of two disjoint lines, while*

$$C_0 = \operatorname{Proj}(\Bbbk[\mathbf{x}]/\langle x_1x_2, x_3\rangle)$$

*is a reducible conic in the plane $x_3 = 0$. The Hilbert polynomials are*

$$\begin{aligned} P_u(t) &= 2t + 2 \quad u \neq 0 \text{ and} \\ P_0(t) &= 2t + 1. \end{aligned}$$

*From Corollary 3.10 it follows immediately that $\{C_u\}_{u\in\Bbbk}$ cannot be a set of fibres of a flat family of closed subschemes of $\mathbb{P}^3$.*

### 3.1.3 Representable Functors

We now introduce the concept of representable functors. We will see that a representable functor can be represented by a scheme. This allow us to analyse and understand these functors much better. To introduce the concept of representable functors we define the functor of points and state the Yoneda lemma which is a fundamental lemma from category theory. After introducing representable functors we will see how to check if a given functor is representable.

In the following let $\underline{Sch}_S$ be the category of $S$-schemes.

**Definition 3.12.** *Let $F, G$ be two functors from a category $\mathcal{C}$ into a category $\mathcal{D}$. A family of morphisms $\alpha(S) : F(S) \longrightarrow G(S)$ for every object $S$ of $\mathcal{C}$ is* functorial in $S$ *or a* morphism of functors *if for every morphism $f : T \longrightarrow S$ in $\mathcal{C}$ the following diagram commutes.*

$$\begin{array}{ccc} F(T) & \xrightarrow{\alpha(T)} & G(T) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(S) & \xrightarrow{\alpha(S)} & G(S)\,. \end{array}$$

**Definition 3.13.** *We define for every S-scheme X the contravariant functor*

$$\begin{aligned} h_X : \underline{Sch}_S^\circ &\longrightarrow \underline{Set} \\ Y &\longmapsto h_X(Y) := \mathrm{Hom}_S(Y, X), \\ (u : Y' \to Y) &\longmapsto (h_X(u) : h_X(Y) \to h_X(Y'), x \mapsto x \circ u). \end{aligned}$$

*The functor above is called* the functor of points*. Furthermore, we define for an S-morphism* $f : X \longrightarrow Y$ *and an S-Scheme Z a* functor of morphisms

$$\begin{aligned} h_f(Z) : \ h_X(Z) &\longrightarrow h_Y(Z), \\ g &\longmapsto f \circ g. \end{aligned}$$

We obtain a covariant functor $X \mapsto h_X$ from $\underline{Sch}_S$ to the category $\widehat{\underline{Sch}_S}$ of functors $\underline{Sch}_S^\circ \to \underline{Set}$. For the following let $F$ be a functor from $\underline{Sch}_S^\circ$ to $\underline{Set}$, $X$ an $S$-Scheme and $\alpha : h_X \longrightarrow F$ a morphism of functors. Note that then $\alpha(X)(id_X) \in F(X)$.

**Lemma 3.14** (Yoneda Lemma [26, Lem. 4.6])**.** *The map*

$$\begin{aligned} \mathrm{Hom}_{\widehat{\underline{Sch}_S}}(h_X, F) &\longrightarrow F(X) \\ \alpha &\longmapsto \alpha(X)(id_X) \end{aligned}$$

*is bijective and functorial in X.*

For the category of $S$-schemes the Yoneda lemma implies the following nice identification.

**Corollary 3.15** ([26, Cor. 4.7])**.** *Let X and Y be S-Schemes. Then it is equivalent to give the following data.*

- *An S-morphism of schemes from X to Y.*
- *For all S-schemes T a map* $f(T) : X_S(T) \longrightarrow Y_S(T)$ *of sets which is functorial in T.*
- *For all affine S-schemes* $T = \mathrm{Spec}(B)$ *a map* $f(T) : X_S(T) \longrightarrow Y_S(T)$ *of sets which is functorial in B.*

**Definition 3.16.** *A functor* $F : \underline{Sch}_S^\circ \longrightarrow \underline{Set}$ *is* representable *if there exists an S-scheme X and an isomorphism* $\xi : h_X \longrightarrow F$*. We say that F* is representable by $X$.

The pair $(X, \xi)$ is then uniquely determined up to isomorphisms of schemes, due to the Yoneda Lemma.

**Example 3.17.** *Let $n \geq 0$ be an integer. The functor*

$$\begin{aligned} F : \underline{Sch}_{\mathbb{Z}}^{\circ} &\longrightarrow \underline{Set} \\ S &\longmapsto \mathcal{O}_S(S)^n \end{aligned}$$

*is representable by* $\operatorname{Spec}(\mathbb{Z}[x_1, \dots, x_n])$. *To see this we take a look at the functor* $h_{\operatorname{Spec}(\mathbb{Z}[x_1,\dots,x_n])}$. *This functor maps a scheme $S$ to* $h_{\operatorname{Spec}(\mathbb{Z}[x_1,\dots,x_n])}(S) = \operatorname{Hom}_{\mathbb{Z}}(S, \operatorname{Spec}(\mathbb{Z}[x_1, \dots, x_n])) \cong \operatorname{Hom}_{\underline{Ring}}(\mathbb{Z}[x_1, \dots, x_n], \mathcal{O}_S(S)) = \mathcal{O}_S(S)^n$. *Now it is obviously that* $h_{\operatorname{Spec}(\mathbb{Z}[x_1,\dots,x_n])}$ *and $F$ are isomorphic and hence $F$ is representable by* $\operatorname{Spec}(\mathbb{Z}[x_1, \dots, x_n])$.

In the following we try to solve the problem of checking if a given functor $F : \underline{Sch}_S^{\circ} \longrightarrow \underline{Set}$ is representable. If $j : U \longrightarrow X$ is an open immersion of $S$-schemes and $\xi \in F(X)$ we write $\xi_{|U}$ instead of $F(j)(\xi)$.

**Definition 3.18.** *$F$ is a* sheaf for the Zariski topology *(or* Zariski sheaf*) on $\underline{Sch}_S$ if the usual sheaf axioms are satisfied that is for every $S$-scheme $X$ and for every open covering $X = \bigcup_{i \in I} U_i$ we have:*
*Given $\xi_i \in F(U_i)$ for all $i \in I$ such that $\xi_{i|(U_i \cap U_j)} = \xi_{j|(U_i \cap U_j)}$ for all $i, j \in I$ there exists a unique element $\xi \in F(U)$ such that $\xi_{|U_i} = \xi_i$ for all $i \in I$.*

This definition immediately leads to the following simple proposition:

**Proposition 3.19** ([26, Prop. 8.8])**.** *Every representable functor $F : \underline{Sch}_S^{\circ} \longrightarrow \underline{Set}$ is a sheaf for the Zariski topology.*

Now we want to prove that every Zariski sheaf that has a Zariski covering by representable functors is representable itself. This will be later on one of the key properties to prove that the Grassmann functor is representable.

**Definition 3.20.** *Let $F : \underline{Sch}_S^{\circ} \longrightarrow \underline{Set}$ be a contravariant functor. A subfunctor $F'$ is an* open subfunctor *if for the morphism $f : F' \longrightarrow F$, for every $S$-scheme $X$ and for every $S$-morphism $g : X \longrightarrow F$ the second projection $f_{(X)} : F' \times_F X \longrightarrow X$ is an open immersion and $F' \times_F X$ is representable.*

**Definition 3.21.** *A family $(f_i : F_i \longrightarrow F)_{i \in I}$ of open subfunctors is called a* Zariski open covering of $F$ *if for every $S$-scheme $X$ and every $S$-morphism $g : X \longrightarrow F$ the images of the $(f_i)_{(X)}$ form a covering of $X$.*

**Theorem 3.22** ([26, Thm. 8.9])**.** *Let $F : \underline{Sch}_S^{\circ} \longrightarrow \underline{Set}$ be a functor such that $F$ is a sheaf for the Zariski topology and has a Zariski open covering $(f_i : F_i \longrightarrow F)_{i \in I}$ by representable functors $F_i$. Then $F$ is representable.*

We work mainly with schemes over an algebraic closed field $\mathbb{k}$. The following proposition shows that we can restrict us in this case to the functor of points of affine schemes over $\mathbb{k}$.

**Proposition 3.23** ([18, Prop. VI-2]). *If $R$ is a commutative ring, a scheme over $R$ is determined by the restriction of its functor of points to affine schemes over $R$; in fact the functor $h$ from the category of $R$-schemes to the category of functors from the category of $R$-algebras to the category of sets is an equivalence of the category of $R$-schemes with a full subcategory of the category of functors.*

A contravariant functor from the category of affine $\mathbb{k}$-schemes to the category of sets is obviously the same then a covariant functor from the category of $\mathbb{k}$-algebras to the category of sets. Hence, we can consider the contravariant functors which are defined in the next section as functors of covariant functors of $\mathbb{k}$-Algebras.

## 3.2 Functors

In this section we introduce several functors, especially the Hilbert and the Quot functor. Furthermore, we define the Grassmann functor. This functor plays an important role in the following because we use the Grassmann functor to present a new proof of the representability of the Quot functor.

### 3.2.1 Hilbert functor

We first introduce the Hilbert functor in a very general case. We will see that we can simplify the definition when we only consider the Hilbert functor over the projective $\mathbb{k}$-scheme $\mathbb{P}^n$

Let $Y$ be a projective scheme over $\mathbb{k}$. Furthermore, fix a numerical polynomial in other words choose $\mathsf{HP}(t) \in \mathbb{Q}[t]$ of the form

$$\mathsf{HP}(t) = \sum_{i=0}^{r} a_i \binom{t+r}{i} \tag{3.2}$$

with $a_i \in \mathbb{N}$ for all $i$. For every $Y$-scheme $S$ we define

$$\mathrm{Hilb}^{Y}_{\mathsf{HP}(t)}(S) := \left\{ \begin{array}{l} \text{flat families } X \subset Y \times S \text{ of closed subschemes of } Y \text{ parametrized} \\ \text{by } S \text{ with fibres having Hilbert polynomial } \mathsf{HP}(t) \end{array} \right\}$$

and for a morphism of schemes $f : S \longrightarrow T$ we define

$$\begin{aligned}\mathrm{Hilb}^{Y}_{\mathsf{HP}(t)}(f) : \mathrm{Hilb}^{Y}_{\mathsf{HP}(t)}(T) &\longrightarrow \mathrm{Hilb}^{Y}_{\mathsf{HP}(t)}(S)\\ X &\longmapsto f^{*}(X).\end{aligned}$$

This functor is well defined because the property flatness is preserved under base change by Proposition 3.7. We call this functor the *Hilbert functor of $Y$ relative to* $\mathsf{HP}(t)$.

The Hilbert functor was introduced by Grothendieck [25], who also proved that the Hilbert functor is representable. This motivates the following definition:

**Definition 3.24.** *The scheme which represents the functor* $\mathrm{Hilb}^{Y}_{\mathsf{HP}(t)}$ *is called the* Hilbert scheme *and we denote it by* $\mathbf{Hilb}^{Y}_{\mathsf{HP}(t)}$.

The Hilbert scheme $\mathbf{Hilb}^{Y}_{\mathsf{HP}(t)}$ has a universal element. That is a flat family $Z \subset Y \times \mathbf{Hilb}^{Y}_{\mathsf{HP}(t)}$ of closed subschemes of $Y$ with Hilbert polynomial $\mathsf{HP}(t)$ which fulfils the following universal property:
For all schemes $S$ and for all $X \subset Y \times S$ of closed subschemes of $Y$ having Hilbert polynomial $\mathsf{HP}(t)$ there is a unique morphism $S \longrightarrow \mathbf{Hilb}^{Y}_{\mathsf{HP}(t)}$ such that

$$\begin{array}{ccc} X = Z \times_{\mathbf{Hilb}^{Y}_{\mathsf{HP}(t)}} S & \longrightarrow & Z \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathbf{Hilb}^{Y}_{\mathsf{HP}(t)} . \end{array}$$

If $Y = \mathbb{P}^n$ for an $n \in \mathbb{N} \setminus \{0\}$ we write in the following $\mathrm{Hilb}^{n}_{\mathsf{HP}(t)}$, respectively $\mathbf{Hilb}^{n}_{\mathsf{HP}(t)}$. Furthermore, we restrict ourselves to the case that we only consider schemes over an algebraic closed field $\Bbbk$. Hence, we can use Proposition 3.23 to consider $\mathrm{Hilb}^{n}_{\mathsf{HP}(t)}$ as a functor from the category of affine $\Bbbk$-schemes to the category sets with the following map:

$$\mathrm{Hilb}^{n}_{\mathsf{HP}(t)}(\operatorname{Spec} A) = \left\{ X \subset \mathbb{P}^n \times A \;\middle|\; \begin{array}{l} X \text{ flat over } \operatorname{Spec}(A) \text{ with Hilbert} \\ \text{polynomial } \mathsf{HP}(t) \end{array} \right\}.$$

Since we are now considering affine $\Bbbk$-schemes, we take the graded module

$$\mathcal{M} = \bigoplus_{t \geq 0} H^0(X, \mathcal{O}_X(t))$$

over $A[\mathbf{x}]$ such that $\widetilde{\mathcal{M}} = \mathcal{O}_X$. In the image of the functor we assume that $\mathcal{O}_X$ is flat, hence $\mathcal{M}$ must be also flat by Proposition 3.7. Furthermore, we know that flatness is preserved under localization and that every finitely generated module is flat if and only if it is free. We denote by $k(\mathfrak{p})$ the residue field of $A_{\mathfrak{p}}$, where $\mathfrak{p} \in \operatorname{Spec}(A)$. Then we can define the Hilbert polynomial $\mathsf{HP}(t)$ of $\mathcal{M}_{\mathfrak{p}}$ as the Hilbert polynomial of $\mathcal{M}_{\mathfrak{p}} \otimes k(\mathfrak{p})$ which is

$$\mathsf{HP}(t) = \dim_{k(\mathfrak{p})}((\mathcal{M}_{\mathfrak{p}})_t \otimes k(\mathfrak{p})) \qquad t \gg 0.$$

By Corollary 3.10 we conclude that $\mathsf{HP}(t)$ does not depend on $\mathfrak{p} \in \operatorname{Spec} A$. Hence, we can redefine the Hilbert functor as a functor from the category of $\Bbbk$-Algebras to the category of sets: For every $\Bbbk$-Algebra $A$ let

$$\mathrm{Hilb}^n_{\mathsf{HP}(t)}(A) = \begin{Bmatrix} \mathcal{M}\text{, graded module over } A[\mathbf{x}] \text{ flat over } A \text{ with} \\ \text{Hilbert polynomial } \mathsf{HP}(t) \end{Bmatrix}$$

and for any homomorphism of $\Bbbk$-Algebras $f : A \longrightarrow B$ let

$$\begin{aligned} \mathrm{Hilb}^n_{\mathsf{HP}(t)}(f) : \mathrm{Hilb}^n_{\mathsf{HP}(t)}(A) &\longrightarrow \mathrm{Hilb}^n_{\mathsf{HP}(t)}(B) \\ \mathcal{M} &\longmapsto \mathcal{M} \otimes_A B. \end{aligned}$$

It is easy to see that the functor is still well defined again because flatness is preserved when tensoring with $B$.

### 3.2.2 Quot Functor

In the previous section we introduced the Hilbert functor. Now we define a generalization of this functor called Quot functor. We first introduce the Quot functor for the general case and then specialize to Quot functors over the projective space $\mathbb{P}^n$ over the algebraically closed field $\Bbbk$ again.

Let $S$ be a noetherian scheme. Furthermore, let $X$ be an $S$-scheme, $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and assume that $\mathsf{HP}(t) \in \mathbb{Q}[t]$ fulfils (3.2). Then we define the *quotient functor of $\mathcal{F}$ relative to* $\mathsf{HP}(t)$ as a functor from the category $S$-schemes to the category of sets. For every $S$-scheme $T$ we define

$$\mathrm{Quot}^{\mathcal{F},\mathsf{HP}(t)}_{X,S}(T) := \begin{Bmatrix} \mathcal{Q} \text{ quotients of } \mathcal{F}_T \text{ which are quasi-coherent and flat over } T, \\ \text{such that the fibres have Hilbert polynomial } \mathsf{HP}(t) \end{Bmatrix}.$$

For a morphism of $S$-schemes $f : T \longrightarrow T'$ we define

$$\begin{aligned} \mathrm{Quot}^{\mathcal{F},\mathsf{HP}(t)}_{X,S}(f) : \mathrm{Quot}^{\mathcal{F},\mathsf{HP}(t)}_{X,S}(T) &\longrightarrow \mathrm{Quot}^{\mathcal{F},\mathsf{HP}(t)}_{X,S}(S) \\ X &\longmapsto f^*(X). \end{aligned}$$

We have to prove that the functor $\mathrm{Quot}_{X,S}^{\mathcal{F},\mathsf{HP}(t)}$ is representable again but for the moment we assume that this is true which leads to the following definition:

Grothendieck [25] proved that the Quot functor is representable. We will not assume this in the following, because we will give a new proof of the representability of this functor under some hypothesis, which we introduce below. Grothendieck's work motivates the following definition

**Definition 3.25.** *The scheme which represents the functor* $\mathrm{Quot}_{X,S}^{\mathcal{F},\mathsf{HP}(t)}$ *is called the* Quot scheme *and we denote it by* $\boldsymbol{Quot}_{X,S}^{\mathcal{F},\mathsf{HP}(t)}$.

If $S = \mathrm{Spec}(\Bbbk)$ we write $\mathrm{Quot}_X^{\mathcal{F},\mathsf{HP}(t)}$, respectively $\mathbf{Quot}_X^{\mathcal{F},\mathsf{HP}(t)}$. If furthermore $\mathcal{F} = \mathcal{O}_X^{m,\mathbf{d}} := \bigoplus_{i=1}^m \mathcal{O}_X(d_i)$ for an $m \in \mathbb{N} \setminus \{0\}$ and $\mathbf{d} = (d_1, \ldots, d_m)$ with $d_i \in \mathbb{Z}$ we write $\mathrm{Quot}_{X,\mathsf{HP}(t)}^{m,\mathbf{d}}$, respectively $\mathbf{Quot}_{X,\mathsf{HP}(t)}^{m,\mathbf{d}}$.

Now we can see, how the Hilbert scheme is a specialization of the Quot scheme. For $m = 1$ and $\mathbf{d} = (0)$ the Quot functor $\mathrm{Quot}_{X,\mathsf{HP}(t)}^{m,\mathbf{d}}$ is equivalent to the Hilbert functor $\mathrm{Hilb}_{\mathsf{HP}(t)}^X$ due to the fact that quotients of the Quot functor correspond to the flat families of the Hilbert functor.

We can simplify the Quot functor even further. If $X = \mathbb{P}^n$ for an $n \in \mathbb{N} \setminus \{0\}$ we write $\mathrm{Quot}_{n,\mathsf{HP}(t)}^{m,\mathbf{d}}$, respectively $\mathbf{Quot}_{n,\mathsf{HP}(t)}^{m,\mathbf{d}}$. In the following we only work with the Quot functors which are of this form. Grothendieck [25] proved that $\mathrm{Quot}_{n,\mathsf{HP}(t)}^{m,\mathbf{d}}$ is a Zariski sheaf. Due to this we can consider $\mathrm{Quot}_{n,\mathsf{HP}(t)}^{m,\mathbf{d}}$ as a functor from the category of $\Bbbk$-algebras to the category of sets via Proposition 3.23 that is for every finitely generated $\Bbbk$-Algebra $A$ we assign

$$\mathrm{Quot}_{n,\mathsf{HP}(t)}^{m,\mathbf{d}}(A) = \left\{ \begin{matrix} \mathcal{Q} \text{ quotients of } \mathcal{O}_{\mathbb{P}_A^n}^{m,\mathbf{d}} \text{ which are flat over } \mathrm{Spec}(A) \text{ with Hilbert} \\ \text{polynomial } \mathsf{HP}(t) \end{matrix} \right\}.$$

Note that we can now skip the condition of quasi-coherence because this is always fulfilled in this case. Furthermore, it holds for any $\Bbbk$-algebra homomorphism $f : A \longrightarrow B$ that

$$\begin{aligned} \mathrm{Quot}_{n,\mathsf{HP}(t)}^{m,\mathbf{d}}(f) : \mathrm{Quot}_{n,\mathsf{HP}(t)}^{m,\mathbf{d}}(A) &\longrightarrow \mathrm{Quot}_{n,\mathsf{HP}(t)}^{m,\mathbf{d}}(B) \\ \widetilde{Q} &\longmapsto \widetilde{Q \otimes_A B}, \end{aligned}$$

where $Q = H_*^0 \mathcal{Q}$, for $\mathcal{Q} \in \mathrm{Quot}_{n,\mathsf{HP}(t)}^{m,\mathbf{d}}(A)$.

This is equivalent to consider the functor from the category of $\Bbbk$-algebras to the category of sets that associates to every $\Bbbk$-algebra the set

$$\mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(A) = \left\{ A\text{-flat quotients } Q \text{ of } A[\mathbf{x}]^m_{\mathbf{d}} \text{ with Hilbert polynomial } \mathsf{HP}(t) \right\}$$

and that associates to every $\Bbbk$-algebra homomorphism $f : A \longrightarrow B$ the morphism

$$\begin{aligned} \mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(f) : \mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(A) &\longrightarrow \mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(B) \\ Q &\longmapsto Q \otimes_A B\,. \end{aligned}$$

This is equivalent to consider the functor from the category of $\Bbbk$-algebras to the category of sets that associates to every $\Bbbk$-algebra the set

$$\mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(A) = \left\{ \begin{matrix} \mathcal{M} \text{ saturated submodules of } A[\mathbf{x}]^m_{\mathbf{d}} \text{ with } A[\mathbf{x}]^m_{\mathbf{d}}/\mathcal{M} \text{ flat with} \\ \text{Hilbert polynomial } \mathsf{HP}(t) \end{matrix} \right\}$$

and that associates to every $\Bbbk$-algebra homomorphism $f : A \longrightarrow B$ the morphism

$$\begin{aligned} \mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(f) : \mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(A) &\longrightarrow \mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(B) \\ \mathcal{M} &\longmapsto \mathcal{M} \otimes_A B\,. \end{aligned}$$

### 3.2.3 Grassmann Functor

In the following we define the Grassmann functor and show that this functor is representable. We follow Görtz and Wedhorn [26, (8.4)].

Like the Hilbert functor and the Quot functor the Grassmann functor $\mathrm{Gr}_{k,l}$ is a functor from the category of schemes to the category of sets for two positive integers $k, l$. For a scheme $S$ we define

$$\mathrm{Gr}_{k,l}(S) := \left\{ U \subseteq \mathcal{O}^l_S \mid \mathcal{O}^l_S/U \text{ is locally free } \mathcal{O}_S\text{-module of rank } l-k \right\}$$

where $\mathcal{O}^l_S = \bigoplus_{i=1}^l \mathcal{O}_S$ and for a morphism of schemes $f : S \longrightarrow T$ we define

$$\begin{aligned} \mathrm{Gr}_{k,l}(f) : \mathrm{Gr}_{k,l}(T) &\longrightarrow \mathrm{Gr}_{k,l}(S) \\ U &\longmapsto f^*(U). \end{aligned}$$

Obviously this functor is well defined. Furthermore, we observe the following fact.

**Lemma 3.26.** *The functors* $\mathrm{Gr}_{k,l}$ *are Zariski sheafs.*

*Proof.* For the proof see [26, p. 211] □

We use Theorem 3.22 to prove the representability of $\mathrm{Gr}_{k,l}$. Therefore, we construct at first a Zariski open covering of representable functors $F_i$ which covers $\mathrm{Gr}_{k,l}$ and then apply the theorem.

Let $\mathcal{I}$ be the set of subsets of $\{1, \ldots, l\}$ of cardinality $l-k$. For $I \in \mathcal{I}$ let $s_I : \mathcal{O}_S^{l-k} \longrightarrow \mathcal{O}_S^l$ the direct sum of coprojections $\mathcal{O}_S \longrightarrow \mathcal{O}_S^l$ corresponding to the elements of $I$. That is if $I = \{i_1, \ldots, i_{l-k}\}$ then a coprojection $\mathcal{O}_S \longrightarrow \mathcal{O}_S^l$ corresponds to $i_j$ if $\mathcal{O}_S$ maps into the $i_j$th component of $\mathcal{O}_S^l$.

With this notation we are able to introduce the following subfunctors of the Grassmann functor. For a scheme $S$ we define

$$\mathrm{Gr}_{k,l}^I(S) = \{U \in \mathrm{Gr}_{k,l}(S) \mid \mathcal{O}_S^{l-k} \xrightarrow{s_I} \mathcal{O}_S^l \to \mathcal{O}_S^l/U \text{ is an isomorphism}\}$$

and for a morphism of schemes $f : S \longrightarrow T$ we define

$$\begin{aligned} \mathrm{Gr}_{k,l}^I(f) : \mathrm{Gr}_{k,l}^I(T) &\longrightarrow \mathrm{Gr}_{k,l}^I(S) \\ U &\longmapsto f^*(U). \end{aligned}$$

Furthermore let $f_I : \mathrm{Gr}_{k,l}^I \longrightarrow \mathrm{Gr}_{k,l}$ the natural embedding of the $\mathrm{Gr}_{k,l}^I$ into $\mathrm{Gr}_{k,l}$.

To apply Theorem 3.22 we have to show that all $\mathrm{Gr}_{k,l}^I$ are representable and all $f_I : \mathrm{Gr}_{k,l}^I \longrightarrow \mathrm{Gr}_{k,l}$ form a Zariski open covering of $\mathrm{Gr}_{k,l}$. First we show that the functors $\mathrm{Gr}_{k,l}^I$ are representable.

**Lemma 3.27.** *The functors $\mathrm{Gr}_{k,l}^I$ are representable.*

*Proof.* Let $S$ be a scheme and $U \in \mathrm{Gr}_{k,l}^I$. The morphism $w : \mathcal{O}_S^{l-k} \longrightarrow \mathcal{O}_S^l/U$ is obviously an isomorphism. If we define the map

$$u_U : \mathcal{O}_S^l \longrightarrow \mathcal{O}_S^l/U \xrightarrow{w^{-1}} \mathcal{O}_S^{l-k}$$

we see immediately that the kernel is $U$ and that $u_U \circ s_I = \mathrm{id}_{\mathcal{O}_S^{l-k}}$.

Conversely, given a homomorphism $u : \mathcal{O}_S^l \longrightarrow \mathcal{O}_S^{l-k}$ with $u \circ s_I = \mathrm{id}_{\mathcal{O}_S^{l-k}}$ we get $\ker(U) \in \mathrm{Gr}_{k,l}^I(S)$. Hence, we can define a map

$$\begin{aligned} F(S) = \{u \in \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S^l, \mathcal{O}_S^{l-k}) \mid u \circ s_I = \mathrm{id}\} &\longrightarrow \mathrm{Gr}_{k,l}^I(S) \\ u &\longrightarrow \ker(u). \end{aligned}$$

This map is clearly bijective and functorial in $S$. Hence, we get a natural isomorphism between $F$ and $\mathrm{Gr}^I_{k,l}$. Now we define the map

$$\begin{aligned} F(S) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S^k, \mathcal{O}_S^{l-k}) = \mathcal{O}_S(S)^{k(l-k)} \cong \mathbb{A}^{k(l-k)}(S) \\ u &\longmapsto u_{|\mathcal{O}_S^k}. \end{aligned}$$

This map is bijective and functorial in $S$. Therefore, $F$ is representable by $\mathbb{A}^{k(l-k)}$ and hence $F \cong \mathrm{Gr}^I_{k,l}$ is representable. □

**Lemma 3.28.** *A Zariski open covering of* $\mathrm{Gr}_{k,l}$ *is* $(f_I : \mathrm{Gr}^I_{k,l} \longrightarrow \mathrm{Gr}_{k,l})_{I\in\mathcal{I}}$.

*Proof.* In [26, Lem. 8.13 (i)] it is shown that $f_I$ is an open subfunctor. Thus, we only have to show that $(f_I)_{(X)}$ form a covering of $X$ for an arbitrary scheme $X$. Let $g : X \longrightarrow \mathrm{Gr}_{k,l}$ be a morphism of functors which corresponds via the Yoneda Lemma to $U \in \mathrm{Gr}_{k,l}$. In [26, Lem. 8.13 (i)] we see that $U_I = \mathrm{Gr}^I_{k,l} \times_{\mathrm{Gr}_{k,l}} X$ is representable by an open subscheme of $X$ which we again denote by $U_I$. We have to show that $f : \amalg_I U_I \longrightarrow X$ which is induced by open immersion $U_I \longrightarrow X$, is surjective.

It is enough to show that $f$ is surjective on $\Bbbk$-valued points, where $\Bbbk$ is an arbitrary field. Let $x : \mathrm{Spec}(\Bbbk) \longrightarrow X$ be a $\Bbbk$-valued point of $X$. By composition with $g$ we get a $\Bbbk$-valued point of $\mathrm{Gr}_{k,l}$. This point corresponds to a $k$-dimensional vectors space of $\Bbbk^l$. By the definition of $x$ we know that $x$ lies in the image of $U^I(\Bbbk) \longrightarrow X(\Bbbk)$ if and only if the complement of $U$ is an $l-k$-dimensional vector subspace of $\Bbbk^l$. We can complete any basis of $U$ by part of the standard basis to a basis of $\Bbbk^l$. Thus, there exists a complement of $U$ of dimension $l-k$. Now it is obvious that $f$ is surjective and we are done. □

Due to the lemmata above it is easy to prove the following statement.

**Corollary 3.29.** *The functor* $\mathrm{Gr}_{k,l}$ *is representable.*

*Proof.* Due to Lemma 3.28 and Lemma 3.27 we can apply Theorem 3.22. □

## 3.3 A new Proof for the Representability of the Quot Functor

To prove that the Quot functor is representable we will follow the approach which was developed by Bayer [6] for Hilbert functors. He showed that the Hilbert functor is a subfunctor of the Grassmann functor which allows to deduce the representability of the Hilbert functor from the representability of the Grassmann functor.

In the previous section we constructed a Zariski open covering for the Grassmann functor. Using this open covering we were able to show that the Grassmann functor is representable. In this section we introduce the "quasi-stable open covering" for the Grassmann functor. Using this covering we show that the Quot functor is a subfunctor of the Grassmann functor which allows us to show that the Quot functor is representable.

The advantage of the quasi-stable open covering is the possibility to show that every subfunctor in this open covering is represented by a marked scheme which we also introduce in this section.

The idea to consider this covering is based on [9] where it is proved that a Hilbert functor can be covered by a strongly stable covering if it is considered over a field of characteristic zero. This work here does not only extend this idea to Quot functors it also allows considering Quot functors (respectively Hilbert functors) over fields of arbitrary characteristic.

In this section we fix the positive integers $n, m$ and we consider a polynomial $\mathsf{HP}(t)$ and a sequence of $m$ integers $\mathbf{d}$. As in the previous section we restrict us to the Quot functor $\mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}$. Hence, it is enough to consider the Grassmann functor also as a functor from the category of $\mathbb{k}$-algebras to the category of sets such that

$$\mathrm{Gr}_{k,l}(A) := \left\{A\text{-submodule } M \subseteq A^l \mid A^l/M \text{ is locally free } A\text{-module of rank } l-k\right\}$$

for a $\mathbb{k}$-algebra $A$ and two positive integers $k$ and $l$.

Instead of considering $A$-submodules $M$ of $A^l$ it is also possible to consider $A$-submodules of $A[\mathbf{x}]^m_{\mathbf{d}}$. Let now $s, p \in \mathbb{N}$. Then we define for $A$

$$\mathrm{Gr}^{m,n,\mathbf{d}}_{s,p}(A) = \begin{Bmatrix} A\text{-submodule } M \subseteq (A[\mathbf{x}]^m_{\mathbf{d}})_s \text{ such that } (A[\mathbf{x}]^m_{\mathbf{d}})_s/M \text{ is} \\ \text{locally free of rank } p \end{Bmatrix}.$$

As $(A[\mathbf{x}]^m_{\mathbf{d}})_s$ is isomorphic as an $A$-module to $A^{N^{m,n}_{\mathbf{d}}(s)}$ where $N^{m,n}_{\mathbf{d}}(s) := \mathsf{HP}_{A[\mathbf{x}]^m_{\mathbf{d}}}(s)$ we immediately see that $\mathrm{Gr}^{m,n,\mathbf{d}}_{s,p}$ can be identified by $\mathrm{Gr}_{N^{m,n}_{\mathbf{d}}(s)-p, N^{m,n}_{\mathbf{d}}(s)}$. From now on we always work with the "Grassmann functor" $\mathrm{Gr}^{m,n,\mathbf{d}}_{s,p}$.

We set a basis $\{b_1, \dots, b_p\}$ for $A^p$. Consider the complete list $(\mathbb{T}^m_{\mathbf{d}})_s = \{\tau_\ell\}_{\ell=1,\dots,N^{m,n}_{\mathbf{d}}(s)}$ of terms $\tau = x^\alpha \mathbf{e}_i$, $\deg(x^\alpha \mathbf{e}_i) = s$, of $(A[\mathbf{x}]^m_{\mathbf{d}})_s$. $(\mathbb{T}^m_{\mathbf{d}})_s$ is the basis we consider for the $A$-module $(A[\mathbf{x}]^m_{\mathbf{d}})_s$.

Consider $I = \{a_1, \dots, a_p\} \subseteq \{1, \dots, N^{m,n}_{\mathbf{d}}(s)\}$, $|I| = p$ and consider the injective morphism $\Gamma_I : A^p \to (A[\mathbf{x}]^m_{\mathbf{d}})_s, \quad b_i \mapsto \tau_{a_i}$ and the subfunctor $\mathrm{Gr}^{m,n,\mathbf{d}}_{s,p,I}$ that associates to every $\Bbbk$-algebra $A$ the set

$$\mathrm{Gr}^{m,n,\mathbf{d}}_{s,p,I}(A) = \{M \in \mathrm{Gr}^{m,n,\mathbf{d}}_{s,p}(A) \mid \pi_M \circ \Gamma_I \text{ is bijective}\}$$

where $\pi_M : (A[\mathbf{x}]^m_{\mathbf{d}})_s \to (A[\mathbf{x}]^m_{\mathbf{d}})_s/M$ is the canonical projection. In the previous section we have shown that we cover the Grassmann functor $\mathrm{Gr}^{m,n,\mathbf{d}}_{s,p}$ if we consider every possible $I \subset \{1,\dots,N^{m,n}_{\mathbf{d}}(s)\}$ with $|I| = p$.

In Theorem 2.41 we have seen how Dellaca generalizes the notion of Gotzmann number to modules. If we consider the corresponding modules as sheaves we get the following statement:
Let $\mathbf{d} = (d_1,\dots,d_m)$ with $d_i \leq 0$ for all $1 \leq i \leq m$. If $\mathcal{Q} = \mathcal{O}^{m,\mathbf{d}}_{\mathbb{P}^n_A}/M$ with Hilbert polynomial $\mathsf{HP}(t)$ and the Gotzmann number of $\mathsf{HP}(t)$ is $r$ then $M$ is $r$-regular. Therefore, for every $s \geq r$ there is a graded $A[\mathbf{x}]$-module $\mathcal{M}$ generated by $\mathcal{M}_r$ such that $M = \widetilde{\mathcal{M}}$. In this way the Quot-Functor can be considered by [15, Lem. 5.2 and Thm. 5.1] as a subfunctor of the Grassmann functor $\mathrm{Gr}^{m,n,\mathbf{d}}_{s,\mathsf{HP}(s)}$.

Considering now that $\mathbf{d}$ has the form like above then the Quot functor can be seen as a closed subfunctor of the Grassmann functor $\mathrm{Gr}^{m,n,\mathbf{d}}_{s,\mathsf{HP}(s)}$ which associates to every $\Bbbk$-Algebra the set

$$\mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(A) = \left\{\begin{matrix} M \in \mathrm{Gr}^{m,n,\mathbf{d}}_{s,\mathsf{HP}(s)}(A) \text{ with } A[\mathbf{x}]^m_{\mathbf{d}}/\langle M\rangle \text{ flat and with Hilbert} \\ \text{polynomial } \mathsf{HP}(t) \end{matrix}\right\}.$$

and to every $\Bbbk$-algebra homomorphism $f\colon A \to B$ the function

$$\begin{aligned} \mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(f)\colon\ \mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(A) &\longrightarrow \mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}(B) \\ M &\longmapsto M \otimes_A B\,. \end{aligned}$$

Furthermore, we define the natural transformation of functors

$$\mathcal{H} : \mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)} \to \mathrm{Gr}^{m,n,\mathbf{d}}_{s,\mathsf{HP}(s)}\ .$$

Now we can define the following open subfunctor of $\mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)}$ as

$$\mathrm{Quot}^{m,\mathbf{d},I}_{n,\mathsf{HP}(t)}(A) := \mathrm{Quot}^{m,\mathbf{d}}_{n,\mathsf{HP}(t)} \cap \mathrm{Gr}^{m,n,\mathbf{d}}_{s,\mathsf{HP}(s),I}(A)$$

for every $I \subseteq \{1,\dots,N^{m,n}_{\mathbf{d}}(s)\}$ with $|I| = \mathsf{HP}(s)$.

### 3.3.1 Quasi-Stable Covering

In this section we consider a different open covering of the Grassmanian which is based on quasi-stable modules. We will use this covering to show that we also find a similar covering for the Quot functor. Hence, we are mainly interested to find such a covering for $\mathrm{Gr}^{m,n,\mathbf{d}}_{s,\mathsf{HP}(s)}$ where $s$ is greater or equal to the Gotzmann number of $\mathsf{HP}(t)$. To ensure that all admissible Hilbert polynomials have a Gotzmann number we have to restrict ourselves to the case that $\mathbf{d} = (d_1, \dots, d_m)$ with $d_i \leq 0$ for all $1 \leq i \leq m$.

To simplify the notations we assume for the rest of this chapter the standard grading that is $\mathbf{d} = (0, \dots, 0)$. Hence, we can always skip the index $\mathbf{d}$. It is straightforward to see that everything that we will do in the following is also applicable for a non-standard grading as long as we assume that the grading satisfies the condition above.

The complement of the polynomial $\mathsf{HP}(t)$ is defined as $\mathsf{VP}(t) := N^{m,n}(t) - \mathsf{HP}(t)$. Let $I \subseteq \{1, \dots, N^{m,n}_{\mathbf{d}}(s)\}$ then we define the complementary set $I^c := \{1, \dots, N^{m,n}_{\mathbf{d}}(s)\} \setminus I$. Furthermore, we define $\mathcal{E}^{I_c} := \{\tau_i\}_{i \in I^c} \subseteq \mathbb{T}^m_s$.

**Lemma 3.30.** *Let us assume that the monomial module $\mathcal{V} := \langle \mathcal{E}^{I_c} \rangle$ is quasi-stable.*

*(i) Then $M \in \mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s)}(A)$ if and only if it is generated as an $A$-module by an $\mathcal{E}^{I_c}$-marked set.*

*(ii) If $M$ belongs to $\mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s)}(A)$, then for every $s' \geq s$ the $A$-module $\langle M \rangle_{s'}$ contains a free submodule of rank greater or equal to $\mathsf{VP}(s')$ generated by a $\mathcal{V} \cap \mathbb{T}^m_{s'}$-marked set.*

*Proof.* (i): If $M$ belongs to $\mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s)}(A)$, since $\pi_M \circ \Gamma_I$ is surjective, $\mathcal{E}^I$ is a generating set for the module $A[\mathbf{x}]^m_s / M$. Then for every $\tau \in \mathcal{E}^{I_c}$ we consider the polynomial $\mathbf{f}_\tau = \tau - \pi_M(\Gamma_I(\tau))$. $\mathbf{f}_\tau$ is a homogeneous marked element of $A[\mathbf{x}]^m$ with $\mathrm{Ht}(\mathbf{f}_\tau) = \tau$ and $\mathbf{f}_\tau - \tau \in \langle \mathcal{N}(\mathcal{V}) \rangle^A$. Hence, $\{\mathbf{f}_\tau\}_{\tau \in \mathcal{E}^{I_c}}$ is a $\mathcal{E}^{I_c}$-marked set contained in $M$. Observe that $\langle \{\mathbf{f}_\tau\}_{\tau \in \mathcal{E}^{I_c}} \rangle^A \subseteq M$ and $\mathrm{rk}(M_s) = \mathrm{rk}(\langle \{\mathbf{f}_\tau\}_{\tau \in \mathcal{E}^{I_c}} \rangle)$, hence $M = \langle \{\mathbf{f}_\tau\}_{\tau \in \mathcal{E}^{I_c}} \rangle$.

Vice versa, let $\{\mathbf{f}_\tau\}_{\tau \in \mathcal{E}^{I_c}}$ be the $\mathcal{E}^{I_c}$-marked set generating $M$. Then every $\tau \in A[\mathbf{x}]^m$ can be written modulo $M$ as $\tau = \tau - \mathbf{f}_\tau = \sum_{\tau' \in \mathcal{V}} a' \tau', a' \in A$. Hence, the $A$-module $M$ generated by $\{\mathbf{f}_\tau\}_{\tau \in \mathcal{E}^{I_c}}$ belongs to $\mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s)}(A)$.

(ii): Due to the fact, that $\langle M^{(s')} \rangle^A \subset \langle M \rangle_{s'}$ this statement follows from Theorem 1.82 (iii, iv). ☐

The following example shows that the modules $A[\mathbf{x}]^m / \langle \mathcal{E}^{I_c} \rangle$ and $A[\mathbf{x}]^m / \langle M \rangle$, with $M \in \mathrm{Quot}^{m,\mathsf{HP}(t),I}_n(A)$, in general do not have the same Hilbert polynomial or function.

**Example 3.31.** *In $A[\mathbf{x}] = \mathbb{k}[x_0, x_1, x_2]$ let $\mathcal{E}^{I_c} = (x_1x_2, x_0^2)$ and $\mathcal{M}$ be the ideal of $A[\mathbf{x}]$ generated by $\mathbf{f}_1 = x_1x_2 + x_0x_1$, $\mathbf{f}_2 = x_0^2 + x_0x_2$ which form a $\mathcal{E}^{I_c}$-marked set. The Hilbert polynomial of $A[\mathbf{x}]/\langle\mathcal{E}^{I_c}\rangle$ is constant, while the Hilbert polynomial of $A[\mathbf{x}]/\langle\mathbf{f}_1, \mathbf{f}_2\rangle$ has degree one. Hence, they also do not have the same Hilbert function.*

**Definition 3.32.** *Let $\mathsf{HP}(t)$ be an admissible Hilbert polynomial, and $s$ and $q$ be two non-negative integers.*

- *$\mathbb{QS}^{s,q}$ is the set of the quasi-stable modules in $\mathbb{k}[\mathbf{x}]^m$ generated by $q$ terms of degree $s$.*
- *$\mathbb{QS}^{s,q}_{\mathsf{HP}(t)}$ is the set of quasi-stable modules in $\mathbb{QS}^{s,q}$ with Hilbert polynomial $\mathsf{HP}(t)$.*
- *For every element $\mathfrak{G} \in GL(n+1, \mathbb{k})$ $\widetilde{\mathfrak{G}}$ denotes the automorphism induced by $\mathfrak{G}$ on $A[\mathbf{x}]^m_r$, the Grassmann functor and the Quot functor and $\overline{\mathfrak{G}}$ denotes the corresponding action on an element or a set of elements.*

For any $I$ such that $\langle\mathcal{E}^{I_c}\rangle \in \mathbb{QS}^{s,q}_{\mathsf{HP}(t)}$ and any $\mathfrak{G} \in GL(n+1, \mathbb{k})$ we consider the following subfunctor of the Grassmann functor:

$$\mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s),\mathfrak{G}}(A) = \left\{ M \in \mathrm{Gr}^{n,m}_{s,\mathsf{HP}(s)}(A) \mid \pi_M \circ \widetilde{\mathfrak{G}} \circ \Gamma_I \text{ is bijective} \right\} .$$

**Lemma 3.33.** *Let $(A, \mathfrak{m}, K)$ be a local ring and $M \in \mathrm{Gr}^{n,m}_{s,\mathsf{HP}(s)}(A)$. Then $M \in \mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s)}(A)$ if and only if $M \otimes_A K \in \mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s)}(K)$.*

*Proof.* By the extensions of the scalars it is clear that $M \otimes_A K \in \mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s)}(K)$ if $M \in \mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s)}(A)$. Therefore, we only prove the other direction.

Assume that $M \otimes_A K \in \mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s)}(K)$ and let $\{\bar{\mathbf{f}}_\tau\}_{\tau\in\mathcal{E}^{I_c}}$ the $\mathcal{E}^{I_c}$-marked set generating $M \otimes_A K$. Let us consider a set of polynomials $\{\mathbf{f}_\tau\}_{\tau\in\mathcal{E}^{I_c}} \subset M$ such that the image of each $\mathbf{f}_\tau$ in $K[\mathbf{x}]^m_s$ is $\bar{\mathbf{f}}_\tau$.

We construct a matrix $\mathfrak{M}_M \in A^{\mathsf{VP}(s)\times N^{m,n}(s)}$ for $M$. We order (in any way) the terms of $\mathbb{T}^m_s$: $x^{\alpha_1}\mathbf{e}_{k_1}, \dots, x^{\alpha_{N^{m,n}(s)}}\mathbf{e}_{k_{N^{m,n}(s)}}$ and the elements $\mathbf{f}_\tau$. The $j$th column of $M$ corresponds to the term $x^{\alpha_j}\mathbf{e}_{k_j}$. The $i$th row of $\mathfrak{M}_M$ corresponds to the coefficients in the element $i$th element in $\{\mathbf{f}_\tau\}_{\tau\in\mathcal{E}^{I_c}}$.

Considering the images of the entries in $K$ we obtain the analogous matrix $M'$ for $\{\bar{\mathbf{f}}_\tau\}_{\tau\in\mathcal{E}^{I_c}}$. By hypothesis the minor of these last matrix corresponding to $\mathcal{E}^{I_c}$ is invertible. Then the corresponding minor in $M$ is also invertible because $A$ is local.

We cannot say that $\{\mathbf{f}_\tau\}_{\tau\in\mathcal{E}^{I_c}}$ is a $\mathcal{E}^{I_c}$-marked set. But we can obtain a $\mathcal{E}^{I_c}$-marked set by performing a row reduction of $M$ such that the minor from above gets the identity matrix. ☐

**Remark 3.34.** *Consider a non quasi-stable module $\mathcal{V}$ which is generated by*

$$\mathcal{B}_{\mathcal{V}} = \{x^{\mu^{(1)}}\mathbf{e}_{k_1}, \dots, x^{\mu^{(q)}}\mathbf{e}_{k_q}\},$$

*where the maximal degree of a generator is s. Due to the fact that $\mathcal{V}$ is not quasi-stable, there exists an obstruction to quasi-stability: $x^\mu \mathbf{e}_k \in \mathcal{B}_{\mathcal{V}}$ and $j > c := \operatorname{cls}(x^\mu)$, such that $x_j^s \frac{x^\mu}{x_c^{\mu_c}}\mathbf{e}_k \notin \mathcal{V}$. This implies that $x_j^{\mu_c} \frac{x^\mu}{x_c^{\mu_c}}\mathbf{e}_k \notin \mathcal{V}$. If we now consider the module $\widetilde{\mathcal{V}}$ generated by $\mathcal{B}_{\widetilde{\mathcal{V}}} = \{x_j^{\mu_c} \frac{x^\mu}{x_c^{\mu_c}}\mathbf{e}_k, x^{\mu^{(1)}}\mathbf{e}_{k_1}, \dots, x^{\mu^{(q)}}\mathbf{e}_{k_q}\}$ it is clear that $\widetilde{\mathcal{V}}$ is somehow nearer to quasi-stability than $\mathcal{V}$. In fact, it is obvious that we can find now fewer pairs $(x^\mu \mathbf{e}_k, j)$ which are an obstruction to quasi-stability. Replacing all elements in the way described above leads to a quasi-stable module $\widetilde{\mathcal{V}}$. In [40] it is explained for the ideal case how one can perform deterministic linear coordinate transformations such that we obtain such a quasi-stable module $\widetilde{\mathcal{V}}$.*

With the knowledge of Remark 3.34 we define an *elementary move $m_{l,t,a}$* which is a linear transformation of variables of the form $x_i \mapsto x_i$ if $i \neq l$ and $x_l \mapsto x_l + a \cdot x_t$ for suitable indices $l < t$ and $a \in \mathbb{k}^\times$. If we apply $m_{l,t,a}$ to a term $x^\mu$ we obtain a polynomial

$$m_{l,t,a}(x^\mu) = \sum_{i=0}^{\mu_l} \binom{\mu_l}{i} a^i x^{\mu - i_l + i_t}.$$

For any characteristic the polynomial $m_{l,t,a}(x^\mu)$ contains at least two terms: $x^\mu$ with coefficient 1 and $x^{\mu - \mu_l + (\mu_l)_t}$ with coefficient $a^{\mu_l}$.

It is clear that a monomial module is marked on itself. When we apply a coordinate transformation on a monomial module, the new module is not monomial anymore it has a non-monomial minimal generating set. The next proposition shows that we are able to construct a marked set out of the new module again.

**Proposition 3.35.** *Let $s \geq 0$, $\operatorname{char}(\mathbb{k}) > sq$ and $\mathcal{B}_{\mathcal{V}} = \{x^{\mu(1)}\mathbf{e}_{k_1}, \dots, x^{\mu(q)}\mathbf{e}_{k_q}\} \subseteq \mathbb{T}_s^m$, such that $\mathcal{V} := \langle \mathcal{B}_{\mathcal{V}} \rangle$. Furthermore, let $F = \{\mathbf{g}_1, \dots, \mathbf{g}_q\} \subset \mathbb{k}[\mathbf{x}]_s^m$ marked over $\mathcal{V}$.*

*Let $x^\mu \mathbf{e}_k := x^{\mu(1)}\mathbf{e}_{k_1}$ an obstruction to quasi-stability for $\mathcal{V}$ that is it exists a $j > c := \operatorname{cls}(x^\mu)$ such that $x_j^s \frac{x^\mu}{x_c^{\mu_c}}\mathbf{e}_k \notin \mathcal{V}$ and let $\widetilde{F} = m_{c,j,a}(F)$ for an $a \in \mathbb{k}$. We define $x^{\widetilde{\mu}}\mathbf{e}_k := x_j^{\mu_c} \frac{x^\mu}{x_c^{\mu_c}}\mathbf{e}_k$.*

*Then we can construct a set $F' \subseteq \langle \widetilde{F} \rangle \cap \mathbb{k}[\mathbf{x}]_s^m$ from $\widetilde{F}$ via linear combinations which is marked over $\mathcal{B}_{\widetilde{\mathcal{V}}} = \{x^{\widetilde{\mu}}\mathbf{e}_k, x^{\mu(2)}\mathbf{e}_{k_2}, \dots, x^{\mu(q)}\mathbf{e}_{k_q}\}$ and $\widetilde{\mathcal{V}} = \langle \mathcal{B}_{\widetilde{\mathcal{V}}} \rangle$ which can be constructed from $\widetilde{F}$ via linear combinations.*

*Proof.* The considered term $x^\mu \mathbf{e}_k$ is replaced as follows

$$m_{c,j,a}(x^\mu)\mathbf{e}_k = \sum_{i=0}^{\mu_c} \binom{\mu_i}{i} a^i x^{\mu - i_c + i_j}\mathbf{e}_k.$$

Without loss of generality we see that by our choice of the pair $(c, j)$ the term $x^{\widetilde{\mu}}\mathbf{e}_k$ appears on the right hand side with a nonzero coefficient for the index value $i = \mu_c$. Applying the transformation $m_{c,j,a}$ to all generators $\mathbf{g}_i$ yields new generators $\widetilde{\mathbf{g}}_i$ and each $\widetilde{\mathbf{g}}_i$ still contains the term $x^{\alpha^{(i)}}\mathbf{e}_{k_i}$ with a coefficient which is a polynomial in $a$ with constant term 1. It may happen that the term $x^{\alpha^{(i)}}\mathbf{e}_{k_i}$ also appears in other generators $\widetilde{\mathbf{g}}_l$ now but then the coefficient of $x^{\alpha^{(i)}}\mathbf{e}_{k_i}$ in $\widetilde{\mathbf{g}}_l$ is always a polynomial in $a$ without a constant term. Furthermore, the term $x^{\widetilde{\mu}}\mathbf{e}_k$ appears in $\widetilde{\mathbf{g}}_1$. Its coefficient in particular contains the term $a^{\mu_c}$ coming from the above transformation of $x^{\mu}$. If $x^{\widetilde{\mu}}\mathbf{e}_k$ also lies in the support of some other generator $\widetilde{\mathbf{g}}_l$ then its coefficient cannot contain the term $a^{\mu_c}$ as $x^{\mu}\mathbf{e}_k$ only appears in $\mathbf{g}_1$, because $F$ is marked over $\mathcal{V}$.

These observations imply that after taking suitable linear combinations again we get new generators $\widetilde{\mathbf{h}}_1, \ldots, \widetilde{\mathbf{h}}_q$ such that for $i = 2, \ldots, q$ each $\widetilde{\mathbf{h}}_i$ is solved for $x^{\alpha^{(i)}}\mathbf{e}_{k_i}$ (and $x^{\alpha^{(i)}}\mathbf{e}_{k_i}$ does not appear in any other generator) and $\widetilde{\mathbf{h}}_1$ is solved for $x^{\widetilde{\mu}}\mathbf{e}_k$ (again with the term not appearing in any other generator). It cannot happen that for $i = 2, \ldots, q$ a term $x^{\alpha^{(i)}}\mathbf{e}_{k_i}$ vanishes by performing linear combinations on $\widetilde{\mathbf{g}}_1, \ldots \widetilde{\mathbf{g}}_q$ because there is exactly one term $x^{\alpha^{(i)}}\mathbf{e}_{k_i}$ which has as a coefficient a polynomial in $a$ with constant term 1. With the same argument it is clear that the term $x^{\widetilde{\mu}}\mathbf{e}_k$ does not vanish by performing linear combinations as its coefficient $a^{\mu_c}$ in $\widetilde{\mathbf{g}}_1$ is unique. But this implies that the set $F'$ obtains from $F$ by the coordinate transformation and suitable linear combinations is marked over $\mathcal{B}_{\widetilde{\mathcal{V}}}$.

Furthermore, in each polynomial $\widetilde{\mathbf{h}}_i$ the coefficient of the head module term is a polynomial in $a$ of degree at most $s$. Since we have $q$ such coefficients, the assumption $|\mathbb{k}| > sq$ guarantees that there exists a value for $a$ such that none of these polynomials vanish. □

**Proposition 3.36.** *The collection of subfunctors*

$$\left\{ \mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s),\mathfrak{G}}(A) \mid \mathfrak{G} \in GL(n+1, \mathbb{k}), I \ s.t \ \langle \mathcal{E}^{I_c} \rangle \in \mathbb{QS}^{s,\mathsf{VP}(s)} \right\}$$

*covers the Grassmann functor* $\mathrm{Gr}^{m,n}_{s,\mathsf{HP}(s)}$.

*Proof.* We have to proof that for every $\mathbb{k}$-Algebra $A$ and every $M \in \mathrm{Gr}^{m,n}_{s,\mathsf{HP}(s)}(A)$ an $I$ with $\langle \mathcal{E}^{I_c} \rangle \in \mathbb{QS}^{s,\mathsf{VP}(s)}$ and $\mathfrak{G} \in GL(n+1, \mathbb{k})$ exists such that $M \in \mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s),\mathfrak{G}}(A)$ or equivalently such that $\overline{\mathfrak{G}}^{-1} M \in \mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s)}(A)$.

As the question is local it is sufficient to consider the case that the ring $A$ is local. By Lemma 3.33 we may assume that $A$ is in fact a field.

Let $M \in \mathrm{Gr}^{m,n}_{s,\mathsf{HP}(s)}(A)$ for a field $A$, $\mathcal{J}$ be the set of subsets of $\mathbb{T}^m_s$ of cardinality $\mathsf{VP}(s)$ and let $\mathcal{E}^{I_c} \in \mathcal{J}$. Let $\Delta_{\mathcal{E}^{I_c}}(\mathfrak{M}_M)$ be the corresponding minor of $\mathfrak{M}_M$ of $\mathcal{E}^{I_c} \in \mathcal{J}$. It is obvious that there is at least one $\mathcal{E}^{I_c} \in \mathcal{J}$ such that $\Delta_{\mathcal{E}^{I_c}}(\mathfrak{M}_M) \neq 0$.

If $\langle \mathcal{E}^{I_c} \rangle \in \mathbb{QS}^{s,\mathsf{VP}(s)}$ we are done. Assume that this is not the case. Then there exists a module term $x^\mu \mathbf{e}_k \in \mathcal{E}^{I_c}$ and $j > c := \mathrm{cls}(x^\mu)$, such that $x_j^s \frac{x^\mu}{x_c^{\mu_c}} \mathbf{e}_k \notin \langle \mathcal{E}^{I_c} \rangle$. We denote by $\widetilde{\mathcal{E}}^{I_c} \in \mathcal{J}$ the set obtained by replacing in $\mathcal{E}^{I_c}$ the obstruction to quasi-stability $x^\mu \mathbf{e}_k$ with $x^{\widetilde{\mu}} \mathbf{e}_k := x_j^{\mu_c} \frac{x^\mu}{x_c^{\mu_c}} \mathbf{e}_k$.

It is obvious that there is an autoreduction on $M$ such that $M$ is marked over $\mathcal{E}^{I_c}$ otherwise $\Delta_{\mathcal{E}^{I_c}}(\mathfrak{M}_M)$ would be zero. Hence, we assume without loss of generality, that $M$ is marked over $\mathcal{E}^{I_c}$. Proposition 3.35 guarantees that there is a linear coordinate transformation $\mathfrak{G} \in GL(n+1, \mathbb{k})$ with respect to the elementary move $m_{c,j,a}$ for an $a \in A$, such that $\widetilde{M} = \overline{\mathfrak{G}}^{-1} M$ and $\widetilde{M}$ is marked over $\widetilde{\mathcal{E}}^{I_c}$. This implies that $\Delta_{\widetilde{\mathcal{E}}^{I_c}}(M_{\widetilde{M}}) \neq 0$.

The claim of the proposition follows from a simple termination argument which shows that we finally get an $\mathcal{E}^{I_c} \in \mathbb{QS}^{s,\mathsf{VP}(s)}$ by using the elementary moves from above. We introduce an ordering on $\mathcal{J}$. Given two sets $J_1, J_2 \in \mathcal{J}$, we first sort them according to $\prec_{TOP_{\mathtt{revlex}}}$ (greatest term first) and then compare the two sets entry by entry again with respect to $\prec_{TOP_{\mathtt{revlex}}}$. The set whose entry is greater when we find a difference for the first time is defined to be the greater set. That is

$$J_1 < J_2 \iff x^{\alpha_i} \mathbf{e}_{k_i} = x^{\beta_i} \mathbf{e}_{l_i} \text{ for } i < j \text{ and } x^{\alpha_j} \mathbf{e}_{k_j} \prec_{TOP_{\mathtt{revlex}}} x^{\beta_j} \mathbf{e}_{l_j}$$

for $J_1, J_2 \in \mathcal{J}$ with $J_1 = \{x^{\alpha_1} \mathbf{e}_{k_1}, \dots, x^{\alpha_{\mathsf{VP}(s)}} \mathbf{e}_{k_{\mathsf{VP}(s)}}\}$ and $J_2 = \{x^{\beta_1} \mathbf{e}_{l_1}, \dots, x^{\beta_{\mathsf{VP}(s)}} \mathbf{e}_{l_{\mathsf{VP}(s)}}\}$ and $j \in \{1, \dots, \mathsf{VP}(s)\}$.

We can decompose every $J \in \mathcal{J}$ into a set of sets $J^{(1)}, \dots, J^{(m)} \subseteq \mathbb{T}_s$, such that $J = J^{(1)} \mathbf{e}_1 \cup \cdots \cup J^{(m)} \mathbf{e}_m$. For the following it is enough to know that $\langle J \rangle$ is quasi-stable when for every $i \in \{1, \dots, m\}$ there is no term in $\mathbb{T}_s \setminus J^{(i)}$ which is greater as any element in $J^{(i)}$ with respect to the `revlex` order.

Our construction yields a set $\widetilde{\mathcal{E}}^{I_c}$ that is always larger than $\mathcal{E}^{I_c}$ with respect to this ordering and iterating it leads to a strictly ascending chain of sets in $\mathcal{J}$. Since $\mathcal{J}$ is a finite set the chain must be finite, too. However, our construction only stops when it reaches a set contained in $\mathbb{QS}^{s,\mathsf{VP}(s)}$. Therefore, we obtain after finitely many transformations a set $\mathcal{E}^{I_c} \in \mathbb{QS}^{s,\mathsf{VP}(s)}$ defining a quasi-stable module because the elementary moves construct a set $J = J^{(1)} \mathbf{e}_1 \cup \cdots \cup J^{(m)} \mathbf{e}_m$ such that $J^{(i)}$ is maximal with respect to the `revlex` order. □

**Definition 3.37.** *We call* quasi-stable subfunctor *of* $\mathrm{Gr}^{m,n}_{s,\mathsf{HP}(s)}$ *any element of the collection of subfunctors of Proposition 3.36. Moreover, we denote by* $\mathrm{Quot}^{m,\mathsf{HP}(t),I}_{n,\mathfrak{G}}$ *the open subfunctor* $\mathcal{H}^{-1}(\mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s),\mathfrak{G}}(A)) \cap \mathrm{Quot}^{m,\mathsf{HP}(t)}_{n}$.

**Theorem 3.38.** *The collection of subfunctors*

$$\left\{\mathrm{Quot}^{m,\mathsf{HP}(t),I}_{n,\mathfrak{G}} \mid \mathfrak{G} \in GL(n+1,\Bbbk), I \text{ s.t } \langle \mathcal{E}^{I_c}\rangle \in \mathbb{QS}^{s,\mathsf{VP}(s)}_{\mathsf{HP}(t)}\right\} \tag{3.3}$$

*covers the Quot functor* $\mathrm{Quot}^{m,\mathsf{HP}(t)}_{n}$.

*Proof.* By Proposition 3.36 we can immediately deduce that the Quot functor is covered by

$$\left\{\mathrm{Quot}^{m,\mathsf{HP}(t),I}_{n,\mathfrak{G}} \mid \mathfrak{G} \in GL(n+1,\Bbbk), I \text{ s.t } \langle \mathcal{E}^{I_c}\rangle \in \mathbb{QS}^{s,\mathsf{VP}(s)}\right\} . \tag{3.4}$$

We obtain the statement proving that $\mathrm{Quot}^{m,\mathsf{HP}(t),I}_{n}(A) \neq \emptyset$ for a quasi-stable $\langle \mathcal{E}^{I_c}\rangle$ if and only if $\langle \mathcal{E}^{I_c}\rangle \in \mathbb{QS}^{s,\mathsf{VP}(s)}_{\mathsf{HP}(t)}$. In fact, this implies that for every $\mathfrak{G} \in GL(n+1,\Bbbk)$ we have $\mathrm{Quot}^{m,\mathsf{HP}(t),I}_{n,\mathfrak{G}}(A) \neq \emptyset$ if and only if $\langle \mathcal{E}^{I_c}\rangle \in \mathbb{QS}^{s,\mathsf{VP}(s)}_{\mathsf{HP}(t)}$. As this is a local and set-theoretical fact we may assume that $A$ is a field.

Assume now that $M \in \mathrm{Quot}^{m,\mathsf{HP}(t),I}_{n}(A)$ for $\langle \mathcal{E}^{I_c}\rangle \in \mathbb{QS}^{s,\mathsf{VP}(s)}$.

Due to Lemma 3.30 we know that $M$ is generated by an $\mathcal{E}^{I_c}$-marked set. By Theorem 1.82 we know that $\langle M\rangle_{s'}$ contains an $A$-vector space of the same dimension as $\langle \mathcal{E}^{I_c}\rangle_{s'}$ for every $s' \geq s$. This implies by Theorem 1.82 that $N^{m,n}(s') - \mathsf{HP}(s') = \dim(\langle M\rangle_{s'}) \geq \dim(\langle \mathcal{E}^{I_c}\rangle_{s'})$. Furthermore, the growth theorem of Macaulay ([29, Lem. 23]) implies that $\dim(\langle \mathcal{E}^{I_c}\rangle_{s'}) \geq N(s') - \mathsf{HP}(s')$, hence we have equality and the Hilbert polynomial of $\langle \mathcal{E}^{I_c}\rangle$ must be $\mathsf{HP}(t)$.

For the other direction note that $\langle \mathcal{E}^{I_c}\rangle \in \mathbb{QS}^{s,\mathsf{VP}(s)}_{\mathsf{HP}(t)}$ induces $\mathcal{E}^{I_c} \in \mathrm{Quot}^{m,\mathsf{HP}(t),I}_{n}(A)$. □

**Definition 3.39.** *The* quasi-stable covering *of* $\mathrm{Quot}^{m,\mathsf{HP}(t)}_{n}$ *is the collection of the open subfunctors* (3.3) *of Theorem 3.38.*

**Remark 3.40.** *We considered for* $\mathrm{Quot}^{m,\mathsf{HP}(t)}_{n}$ *a quasi-stable covering in degree* $s \geq r$*, where* $r$ *is the Gotzmann number of* $\mathsf{HP}(t)$*. This implies that the considered modules* $\langle \mathcal{E}^{I_c}\rangle \in \mathbb{QS}^{s,\mathsf{VP}(s)}_{\mathsf{HP}(t)}$ *have regularity* $s$ *since they are generated in degree* $s$*. But this implies that* $\langle \mathcal{E}^{I_c}\rangle$ *is in fact a stable module. Hence, we have actually a stable open covering of the Quot functor. Nevertheless, we call this covering a quasi-stable covering because it is deduced from a quasi-stable covering of the Grassmann functor.*

**Remark 3.41.** *We constructed a quasi-stable covering of* $\mathrm{Quot}_n^{m,\mathsf{HP}(t)}$ *by using a deterministic change of coordinates. There always exists a change of coordinates to reach a p-Borel fixed position. It is well known that p-Borel fixed implies quasi-stability. Therefore, there exists a p-Borel fixed covering of the Quot functor which is in general a more sparse covering of the Quot functor than the quasi-stable covering. However, we prefer to work with the quasi-stable covering because this covering is independent of the characteristic.*

*Furthermore, in the next section we show that it is possible to compute equations for the open subscheme of the Quot scheme corresponding to each quasi-stable open subfunctor. The set of p-Borel fixed open subfunctors is contained in the set of quasi-stable open subfunctors. The computational cost to get such equations for open neighbourhoods of a given point of the Quot scheme can be significantly different depending on the neighbourhood we choose. Hence, it is an advantage to have a relatively dense covering in order to choose the more convenient one.*

**Corollary 3.42.** *Let* $M$ *be any element of* $\mathrm{Gr}^{m,n,I}_{s,\mathsf{HP}(s)}(A)$ *with* $\langle \mathcal{E}^{I_c}\rangle$ *quasi-stable. Then* $M \in \mathrm{Quot}_n^{m,\mathsf{HP}(t),I}(A)$ *if and only if for every* $s' \geq s$ *the* $A$*-module* $\langle M\rangle_{s'}$ *is free of rank* $\mathsf{VP}(s')$ *and it is generated by* $\langle \mathcal{E}^{I_c}\rangle \cap \mathbb{T}^m_{s'}$*-marked basis. Furthermore,* $\mathrm{Quot}_n^{m,\mathsf{HP}(t),I}(A) \neq \emptyset$ *if and only if the Hilbert polynomial of* $A[\mathbf{x}]^m/\langle \mathcal{E}^{I_c}\rangle$ *is* $\mathsf{HP}(t)$.

*Proof.* As the question is local again we once more assume that $A$ is a local ring.

We first consider the special case with $A$ a field. Due the proof of Theorem 3.38 we know that $M^{(s)} \subseteq \langle M\rangle_s$. Furthermore, the proof shows that the dimension of both vector spaces is $\mathsf{VP}(s')$. Hence, they must be equal for every degree $s'$ and this implies via Theorem 1.82 that $M$ is a $\mathcal{E}^{I_c}$-marked basis of $\langle M\rangle$.

We generalize this result to the case $(A, \mathfrak{m}, K)$ a local ring by the lemma of Nakayama, since for every $s' \geq s$ the $A$-module $\langle M\rangle_{s'}$ contains the free submodule of rank $\mathsf{VP}(s')$ that in Theorem 1.82 is denoted by $M^{(s')}$ and the two $A/\mathfrak{m}$-vector spaces $\langle M\rangle_{s'} \otimes_A A/\mathfrak{m}$ and $M^{(s')} \otimes_A A/\mathfrak{m}$ coincides as they have the same dimensions.

The rest of the statement has already proven in the proof of Theorem 3.38. $\square$

### 3.3.2 Marked Schemes

In Theorem 3.38 we proved that there is a quasi-stable covering of the Quot functor. Due to the fact that this covering is a Zariski open covering it remains to show that the subfunctors of the Quot functor are representable. For that we exhibit a natural scheme structure on the set containing all modules generated by a $\mathbf{P}(\mathcal{V})$-marked basis with $\mathcal{V}$ a quasi-stable module. A part of this section is part of [3].

Let $\mathbf{P}(\mathcal{V}) \subset \mathbb{T}^m$ be the Pommaret basis of the quasi-stable module $\mathcal{V} \subseteq A[\mathbf{x}]^m$. We consider the functor of the marked bases on $\mathbf{P}(\mathcal{V})$ from the category of $\Bbbk$-algebras to the category of sets

$$\mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V})} : \underline{\Bbbk\text{–Alg}} \longrightarrow \underline{\text{Sets}}$$

that associates to any $\Bbbk$-algebra $A$ the set

$$\mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V})}(A) := \{G \subset A[\mathbf{x}]^m \mid G \text{ is a } \mathbf{P}(\mathcal{V})\text{-marked basis}\}$$

or equivalently by Corollary 1.85,

$$\mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V})}(A) := \{\mathcal{M} \subseteq A[\mathbf{x}]^m \mid \mathcal{M} \text{ is generated by a } \mathbf{P}(\mathcal{V})\text{-marked basis}\}$$

and to any morphism $\sigma : A \to B$ the map

$$\begin{array}{rccc} \mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V})}(\sigma) : & \mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V})}(A) & \longrightarrow & \mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V})}(B) \\ & G & \longmapsto & \sigma(G)\ . \end{array}$$

Note that the image $\sigma(G)$ under this map is indeed again a $\mathbf{P}(\mathcal{V})$-marked basis, as we are applying the functor $- \otimes_A B$ to the decomposition $A[\mathbf{x}]^m_s = \langle G^{(s)} \rangle^A \oplus \langle \mathcal{N}(\mathcal{V})_s \rangle^A$ for every degree $s$.

**Corollary 3.43.** *Let $\mathbf{P}(\mathcal{V}) \subset \mathbb{T}^m$ be the Pommaret basis of the quasi-stable module $\mathcal{V} \subseteq A[\mathbf{x}]^m$. Then every module $\mathcal{M} \in \mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V})}(A)$ has the same Hilbert function as $\mathcal{V}$.*

*Proof.* This is a simple reformulation of Lemma 2.7. □

The functor that was introduced above turns out to be representable by an affine scheme that can be explicitly constructed by the following procedure. We consider the $\Bbbk$-algebra $\Bbbk[C]$ where $C$ denotes the finite set of variables

$$\left\{ C_{\alpha\eta kl} \mid x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V}), x^\eta \mathbf{e}_l \in \mathcal{N}(\mathcal{V}), \deg(x^\eta \mathbf{e}_l) = \deg(x^\alpha \mathbf{e}_k) \right\}$$

and construct the $\mathbf{P}(\mathcal{V})$-marked set $\overline{G} \subset \Bbbk[C][\mathbf{x}]^m$ consisting of all elements

$$F^k_\alpha = \left( x^\alpha - \sum_{x^\eta \mathbf{e}_k \in \mathcal{N}(\mathcal{V})_{\deg(x^\alpha \mathbf{e}_k)}} C_{\alpha\eta kk} x^\eta \right) \mathbf{e}_k - \sum_{\substack{x^\eta \mathbf{e}_l \in \mathcal{N}(\mathcal{V})_{\deg(x^\alpha \mathbf{e}_k)} \\ l \neq k}} C_{\alpha\eta kl} x^\eta \mathbf{e}_l \tag{3.5}$$

with $x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V})$. Then we compute all the complete reductions $x_i F^k_\alpha \xrightarrow{\mathcal{G}^{(s)}}_* L$ for every term $x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V})$ and every non-multiplicative variable $x_i \in \overline{X}_P(F^k_\alpha)$ and collect the coefficients of the monomials $x^\eta \mathbf{e}_j \in \mathcal{N}(\mathcal{V})$ of all the reduced elements $L$ in a set $R \subset \Bbbk[C]$.

**Theorem 3.44.** *The functor* $\mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V})}$ *is represented by the scheme* $\boldsymbol{Mf}^{n,m}_{\mathbf{P}(\mathcal{V})} := \mathrm{Spec}(\Bbbk[C]/\langle R\rangle)$.

*Proof.* We observe that each element $\mathbf{f}^k_\alpha$ of a $\mathbf{P}(\mathcal{V})$-marked set $G$ in $A[\mathbf{x}]^m$ can be written in the following form:

$$\mathbf{f}^k_\alpha = \left( x^\alpha - \sum_{x^\eta \mathbf{e}_k \in \mathcal{N}(\mathcal{V})_{\deg(x^\alpha \mathbf{e}_k)}} c_{\alpha\eta kk} x^\eta \right) \mathbf{e}_k - \sum_{\substack{x^\eta \mathbf{e}_l \in \mathcal{N}(\mathcal{V})_{\deg(x^\alpha \mathbf{e}_k)} \\ l \neq k}} c_{\alpha\eta kl} x^\eta \mathbf{e}_l, \quad c_{\alpha\eta kl} \in A.$$

Therefore $G$ can be obtained by specializing in $\overline{G}$ the variables $C_{\alpha\eta kl}$ to the constants $c_{\alpha\eta kl} \in A$. Moreover, $G$ is a $\mathbf{P}(\mathcal{V})$-marked basis if and only $x_i \mathbf{f}^k_\alpha \xrightarrow{G^{(s)}}_* 0$ for every $x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V})$ and $x_i \in \overline{X}_P(\mathbf{f}^k_\alpha)$. Equivalently $G$ is a $\mathbf{P}(\mathcal{V})$-marked basis if and only if the evaluation morphism $\varphi : \Bbbk[C] \to A$, $\varphi(C_{\alpha\eta kl}) = c_{\alpha\eta kl}$ factors through $\Bbbk[C]/\langle R\rangle$, namely, if and only if the following diagram commutes

$$\begin{array}{ccc} \Bbbk[C] & \xrightarrow{\varphi} & A \\ \downarrow & \nearrow & \\ \Bbbk[C]/\langle R\rangle & & \end{array}$$

□

The next lemma shows that the subfunctor of the Quot functor is isomorphic to a marked functor hence the subfunctor is representable by an affine scheme.

**Lemma 3.45.**

*(i) The subfunctor* $\mathrm{Quot}^{m,\mathsf{HP}(t),I}_n$ *with* $\langle \mathcal{E}^{I_c} \rangle \in \mathbb{QS}^{s,q}_{\mathsf{HP}(t)}$ *is isomorphic to the marked functor* $\mathrm{Mf}^{n,m}_{\mathcal{E}^{I_c}}$.

*(ii) The subfunctor* $\mathrm{Quot}^{m,\mathsf{HP}(t),I}_n$ *is representable by an affine subscheme of the affine space* $\mathbb{A}^{\mathsf{HP}(s)\cdot\mathsf{VP}(s)}$.

*Proof.* Item (i) is a straightforward consequence of Corollary 3.42; item (ii) follows from (i) and Theorem 3.44. □

Now we are able to prove the main result of this chapter about the representability of the Quot functor.

**Theorem 3.46.** *The Quot functor* $\mathrm{Quot}_n^{m,\mathsf{HP}(t)}$ *is the functor of points of a closed subscheme* $\boldsymbol{Quot}_n^{m,\mathsf{HP}(t)}$ *of the Grassmanian* $\mathrm{Gr}_{s,\mathsf{HP}(s)}^{m,n}$.

*Proof.* By Theorem 3.22, it suffices to check the representability on an open covering of $\mathrm{Gr}_{s,\mathsf{HP}(s)}^{m,n}$ and $\mathrm{Quot}_n^{m,\mathsf{HP}(t)}$: we choose the quasi-stable covering (Proposition 3.36 and Theorem 3.38).

Therefore, we can conclude by Corollary 3.45 (ii). □

A generating set $F \in \mathrm{Mf}_{\mathbf{P}(\mathcal{V})}^{n,m}(A)$ defines a quotient $A[\mathbf{x}]^m/\langle F\rangle$ that is a free $A$-module such that the family $\widetilde{A[\mathbf{x}]^m/\langle F\rangle} \to \mathcal{O}_{\mathbb{P}_A^n}^m$ is flat and defines a morphism from $\mathcal{O}_{\mathbb{P}_A^n}^m$ to a suitable Quot scheme by the universal property of Quot schemes. Since Quot schemes parametrize flat families of $\mathcal{O}_{\mathbb{P}_A^n}^m$-quotients and the same quotient can be defined by infinitely many different modules we first need to investigate the function that associates to every generating set in $\mathrm{Mf}_{\mathbf{P}(\mathcal{V})}^{n,m}(A)$ the quotient of $\mathcal{O}_{\mathbb{P}_A^n}^m$ it defines. In general, this function can be non-injective.

**Example 3.47.** *Consider the quasi-stable module*

$$\mathcal{V} = \langle x_2\mathbf{e}_1, x_1^2\mathbf{e}_1, x_2^2\mathbf{e}_2, x_1^2\mathbf{e}_2, x_0\mathbf{e}_2\rangle \subset A[x_0, x_1, x_2]^2$$

*with* $\mathbf{P}(\mathcal{V}) = \{x_2\mathbf{e}_1, x_1^2\mathbf{e}_1, x_2^2\mathbf{e}_2, x_1^2\mathbf{e}_2, x_0\mathbf{e}_2\}$*. For* $a \in A$ *consider the* $\mathbf{P}(\mathcal{V})$*-marked set*

$$F_{\mathcal{V},a} = \{x_2\mathbf{e}_1, x_1^2\mathbf{e}_1 + ax_1x_2\mathbf{e}_2, x_2^2\mathbf{e}_2, x_1^2\mathbf{e}_2, x_0\mathbf{e}_2\}\,.$$

*This set is in fact a* $\mathbf{P}(\mathcal{V})$*-marked basis for every* $a \in A$ *since every non-multiplicative prolongation reduces to zero. Moreover, for every* $a \in A$ *the module* $(F_{\mathcal{V},a})_{\geq 3}$ *coincides with* $\mathcal{V}_{\geq 3}$*. Therefore, for all* $a \in A$ *the modules* $\langle F_{\mathcal{V},a}\rangle$ *define the same quotient.*

For a marked module element $\mathbf{f}_\alpha^k$ we define $T(\mathbf{f}_\alpha^k) := \mathbf{f}_\alpha^k - x^\alpha\mathbf{e}_k$.

**Proposition 3.48.** *Let* $\mathcal{V}$ *be a quasi-stable module and let* $t$ *be the minimum degree such that* $\mathcal{V}_t \neq 0$*. Assume that no monomial module element of degree larger than* $t$ *in the Pommaret basis* $\mathbf{P}(\mathcal{V})$ *is divisible by* $x_0$ *(or equivalently that* $x_0^q\mathcal{N}(\mathcal{V})_{\geq t} \subseteq \mathcal{N}(\mathcal{V})_{\geq t+q}$ *for every* $q$*). Then for any two different* $\mathbf{P}(\mathcal{V})$*-marked bases* $F_\mathcal{V}$ *and* $G_\mathcal{V}$ *in* $A[\mathbf{x}]^m$ *the module sheaves* $\widetilde{A[\mathbf{x}]^m/\langle F_\mathcal{V}\rangle}$ *and* $\widetilde{A[\mathbf{x}]^m/\langle G_\mathcal{V}\rangle}$ *are different.*

*Proof.* There exists a module term $x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V})$ such that the corresponding polynomials $\mathbf{f}_\alpha^k \in F_\mathcal{V}$ and $\mathbf{g}_\alpha^k \in G_\mathcal{V}$ are different. If $\widetilde{A[\mathbf{x}]^m/\langle F_\mathcal{V}\rangle} = \widetilde{A[\mathbf{x}]^m/\langle G_\mathcal{V}\rangle}$, then $\langle F_\mathcal{V}\rangle_{\geq s} = \langle G_\mathcal{V}\rangle_{\geq s}$ for a sufficiently large $s$. Therefore, for $s \gg 0$ $x_0^s\mathbf{f}_\alpha^k$ is contained in $\langle G_\mathcal{V}\rangle$ and $x_0^s\mathbf{f}_\alpha^k - x_0^s\mathbf{g}_\alpha^k = x_0^s(T(\mathbf{g}_\alpha^k) - T(\mathbf{f}_\alpha^k)) \in \langle G_\mathcal{V}\rangle$. By definition, the support of $T(\mathbf{g}_\alpha^k) - T(\mathbf{f}_\alpha^k)$ is contained in $\mathcal{N}(\mathcal{V})$ due to the hypothesis on $\mathcal{V}$. Finally, by Theorem 1.82 (ii)-(vi) this implies $x_0^s(T(\mathbf{g}_\alpha^k) - T(\mathbf{f}_\alpha^k)) \in \langle F_\mathcal{V}\rangle_{s+|\alpha|} \cap \langle \mathcal{N}(\mathcal{V})\rangle^A = \{0\}$, so that $T(\mathbf{g}_\alpha^k) = T(\mathbf{f}_\alpha^k)$ against the assumption $\mathbf{f}_\alpha^k \neq \mathbf{g}_\alpha^k$.

□

**Definition 3.49.** *We say that $\mathcal{V}$ is a $t$-*truncated module *if $\mathcal{V} = \mathcal{V}'_{\geq t}$ for $\mathcal{V}'$ a saturated quasi-stable module.*

Observe that the monomial module elements divisible by $x_0$ in the Pommaret basis of a $t$-truncated module $\mathcal{V}$ (if any) are of degree $t$. Therefore, by Proposition 3.48, different $t$-truncation modules define different quotients. We emphasize that a priori the truncation degree $t$ can be any positive integer.

We now describe the relations among marked functors (respectively schemes) corresponding to different truncations of the same saturated quasi-stable modules $V$. We will prove that for sufficiently large integers $t$ the $V_{\geq t}$-marked schemes are all isomorphic. However, the construction of $\mathbf{Mf}^{n,m}_{\mathbf{P}(V_{\geq t})}$ given in Theorem 3.44 depends on $t$ since we obtain it as a closed subscheme of an affine space whose dimension increases with $t$. From a computational point of view it will be convenient to choose among isomorphic marked schemes the one corresponding to the minimum value of $t$ while for other applications higher values of $t$ can be more convenient.

**Theorem 3.50.** *Let $\mathcal{V}$ be a saturated quasi-stable module. Then for every $t > 0$ and for any ring $A$, $\mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}_{\geq t-1})}(A) \subseteq \mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}_{\geq t})}(A)$. More precisely,*

*(i) if in $\mathbf{P}(\mathcal{V})$ are no elements of degree $t+1$ divisible by the variable $x_1$ or $\mathcal{V}_{\geq s-1} = \mathcal{V}_{\geq s}$, then $\mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}_{\geq t-1})} \cong \mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}_{\geq t})}$;*

*(ii) otherwise, $\mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}_{\geq t-1})}$ is a closed subfunctor of $\mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}_{\geq t})}$.*

*Proof.* To prove the inclusion $\mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}_{\geq t-1})}(A) \subseteq \mathrm{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}_{\geq t})}(A)$ let us consider a $\mathbf{P}(\mathcal{V}_{\geq t-1})$-marked basis $F$. Recall that according to Theorem 1.82 $\widetilde{F}^{(t)}$ is the unique $\mathbf{P}(\mathcal{V}_t \cap \mathbb{T}^m)$-marked set which generates $\langle F^{(t)}\rangle^A$ as an $A$-module. The set

$$G := \widetilde{F}^{(t)} \cup \left\{ \mathbf{f}_\alpha^k \in F \mid x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V}) \text{ and } |\alpha| > t \right\}$$

is by construction a $\mathbf{P}(\mathcal{V}_{\geq t})$-marked set. In fact, $G$ is a $\mathbf{P}(\mathcal{V}_{\geq t})$-marked basis since $\langle G^{(t)}\rangle^A = \langle F^{(t)}\rangle^A$ by Theorem 1.82 and the generators of degree larger than $t$ are the same in the two marked sets.

From now on in this proof we denote by $\mathcal{V}'$ the truncation of $\mathcal{V}$ in degree $t-1$, by $F_{\mathcal{V}'}$ the marked set analogous to the one given via (3.5) that we used to construct the module $\langle R\rangle \subseteq \Bbbk[C]$ of $\mathbf{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}')}$. We denote the corresponding module $\langle R\rangle$ for $\mathbf{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}')}$ from now on by $\mathcal{M}_{\mathcal{V}'}$. We also let $A' := \Bbbk[C']^m/\mathcal{M}_{\mathcal{V}'}$, $\phi_{F_{\mathcal{V}'}} : \Bbbk[C'] \to A'$ the canonical map on the quotient and $\phi_{F_{\mathcal{V}'}}[\mathbf{x}]$ the extension to $\Bbbk[C'][\mathbf{x}] \to A'$. Moreover, $\mathcal{V}''$ will be the truncation of $\mathcal{V}$ in degree $t$ and $F_{\mathcal{V}''}$, $\Bbbk[C'']$, $\mathcal{M}_{\mathcal{V}''}$, $A''$, $\phi_{F_{\mathcal{V}''}}$ are defined analogously. By the definition of $\mathcal{M}_{\mathcal{V}'}$ and $\mathcal{M}_{\mathcal{V}''}$, we observe that $\phi_{F_{\mathcal{V}'}}[\mathbf{x}](F_{\mathcal{V}'})$ is a $\mathbf{P}(\mathcal{V}')$-marked basis in $A'[\mathbf{x}]$ and $\phi_{F_{\mathcal{V}''}}[\mathbf{x}](F_{\mathcal{V}''})$ is a $\mathbf{P}(\mathcal{V}'')$-marked basis in $A''[\mathbf{x}]$

We first prove (ii). Let us consider the $\mathbf{P}(\mathcal{V}'')$-marked set

$$G := \widetilde{F}^{(t)}_{\mathcal{V}'} \cup \left\{ \mathbf{f}^k_\alpha \in F_{\mathcal{V}'} \mid x^\alpha \mathbf{e}_k \in \mathbf{P}(V),\ |\alpha| > t \right\}.$$

By Theorem 1.82, $\phi_{F_{\mathcal{V}'}}[\mathbf{x}](G)$ is a $\mathbf{P}(\mathcal{V}'')$-marked basis of $A'[\mathbf{x}]$, since

$$\mathcal{N}\left(\mathcal{V}'', \langle \phi_{F_{\mathcal{V}'}}[\mathbf{x}](G)\rangle\right) \subseteq \mathcal{N}\left(\mathcal{V}', \langle \phi_{F_{\mathcal{V}'}}[\mathbf{x}](F_{\mathcal{V}'})\rangle\right) = \{0\}.$$

Thus, the ring homomorphism

$$\begin{aligned} \psi : \Bbbk[C''] &\to \Bbbk[C'] \\ C''_{\alpha\beta kl} &\mapsto \text{coefficient of } x^\beta \mathbf{e}_l \text{ in } \mathbf{g}^k_\alpha \in G \end{aligned}$$

induces a homomorphism $\overline{\psi} : A'' \to A'$ such that $\phi_{F_{\mathcal{V}'}} \circ \psi = \overline{\psi} \circ \phi_{F_{\mathcal{V}''}}$. Moreover, $\phi_{F_{\mathcal{V}'}} \circ \psi$ is surjective, being the composition of two surjective homomorphisms. Indeed,

$$C'_{\alpha\beta kl} = \begin{cases} \psi(C''_{\alpha\beta kl}), & \text{if } x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V}),\ |\alpha| \geq t, \\ \psi(C''_{\eta\gamma kl}),\ x^\eta \mathbf{e}_k = x_0 x^\alpha \mathbf{e}_k,\ x^\gamma = x_0 x^\beta \mathbf{e}_l, & \text{otherwise.} \end{cases}$$

Under our assumptions, for every $\mathbf{f}^k_\alpha \in F_{\mathcal{V}'}$ of degree $t-1$, $x_0 T(\mathbf{f}^k_\alpha)$ is a $\mathcal{V}''$-remainder, so that $x_0 \mathbf{f}^k_\alpha \in G$.

Therefore, the epimorphism $\overline{\psi}$ induces an isomorphism between a closed subscheme of $\mathbf{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}'')} = \mathrm{Spec}(\Bbbk[C'']/\mathcal{M}_{\mathcal{V}''})$ and $\mathbf{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}')} = \mathrm{Spec}(\Bbbk[C']/\mathcal{M}_{\mathcal{V}'})$.

To prove (i) we observe that the new condition on $\mathcal{V}$ implies that for every $x^\gamma \mathbf{e}_k \in \mathcal{N}(\mathcal{V})_t$ either $x_1 x^\gamma \mathbf{e}_k \in \mathcal{N}(\mathcal{V})_{t+1}$ or $x_1 x^\gamma \mathbf{e}_k = x_0 x^\delta \mathbf{e}_k$ with $x^\delta \mathbf{e}_k \in \mathcal{V}_t$ holds.

Exploiting this property we first prove that $C''_{\eta\gamma kl} \in \mathcal{M}_{\mathcal{V}''}$ if $x^\eta \mathbf{e}_k \in \mathcal{V}_t$, $x_0 | x^\eta$ and $x_0 \nmid x^\gamma$. Let $x^\epsilon \mathbf{e}_k = x_1 \frac{x^\eta}{x_0} \mathbf{e}_k$ and consider the non-multiplicative prolongation $x_0 \mathbf{g}^k_\epsilon$. Obviously the involutive divisor of $x_0x^\epsilon\mathbf{e}_k$ is $x^\gamma\mathbf{e}_k$ such that $x_0x^\epsilon\mathbf{e}_k = x_1x^\gamma\mathbf{e}_k$. The $\mathcal{V}$-remainder of this module element given by $\xrightarrow{F^{(\cdot)}_{\mathcal{V}''}}$ is of type

$$\mathbf{g} = x_0T(\mathbf{g}^k_\epsilon) - x_1T(\mathbf{g}^k_\eta) + x_0\sum C''_{\eta\delta kl}\mathbf{g}^l_\beta\,,$$

where $\mathbf{g}^l_\beta \in F_{\mathcal{V}''}$ and the sum is over the multi-indices $\beta$ and the indices $l$ such that $x^\beta\mathbf{e}_l := x_1\frac{x^\delta}{x_0}\mathbf{e}_l \in \mathcal{V}_t$ with $x^\delta$ divisible by $x_0$ and contained in the support of $T(\mathbf{g}^k_\eta)$ and $C''_{\eta\delta kl}$ the coefficient of $x^\delta\mathbf{e}_l$ in $\mathbf{g}^k_\eta$. If $x^\gamma\mathbf{e}_l$ is a term in the support of $T(\mathbf{g}^k_\eta)$ such that $x_0 \nmid x^\gamma$ then $x_1x^\gamma\mathbf{e}_l \in \mathcal{N}(\mathcal{V})_{t+1}$ is contained in the support of $\mathbf{g}$. By definition, $\mathcal{M}_{\mathcal{V}''}$ contains the $\mathbf{x}$-coefficients of $\mathbf{g}$, thus in particular the coefficient $C''_{\eta\gamma kl}$ of $x_1x^\gamma\mathbf{e}_l$ in $\mathbf{g}$.

For every $x^\alpha\mathbf{e}_k \in \mathcal{V}_{t-1}$ and $x^\eta\mathbf{e}_k = x_0x^\alpha\mathbf{e}_k$ let us denote by $\mathbf{h}^k_\alpha$ the module element in $\Bbbk[C''][\mathbf{x}]$ such that $\mathbf{g}^k_\eta = x_0\mathbf{h}^k_\alpha + \sum C''_{\eta\gamma kl}x^\gamma\mathbf{e}_l$ with $x_0 \nmid x^\gamma$, so that $\phi_{F_{\mathcal{V}''}}[\mathbf{x}](\mathbf{g}^k_\eta) = \phi_{F_{\mathcal{V}''}}[\mathbf{x}](x_0\mathbf{h}^k_\alpha)$.

Using these polynomials we can define the $\mathbf{P}(\mathcal{V}')$-marked set

$$H = \{\mathbf{h}^k_\alpha \mid x^\alpha\mathbf{e}_k \in \mathcal{V}_{t-1}\} \cup \{\mathbf{g}^k_\eta \in F_{\mathcal{V}''} \mid x_\eta\mathbf{e}_k \in \mathbf{P}(\mathcal{V}),\ |\eta| \geq t\}\,.$$

By construction, $\phi_{F_{\mathcal{V}''}}[\mathbf{x}](x_0H) \subseteq \phi_{F_{\mathcal{V}''}}[\mathbf{x}](F_{\mathcal{V}''})$, therefore $\phi_{F_{\mathcal{V}''}}[\mathbf{x}](H)$ is a $\mathbf{P}(\mathcal{V}')$-marked basis by Theorem 1.82. In fact, if the support of an element $\mathbf{u}$ in the module $\langle\phi_{F_{\mathcal{V}''}}[\mathbf{x}](H)\rangle$ only contains module terms of $\mathcal{N}(\mathcal{V})$, then $x_0\mathbf{u}$ is in $\langle\phi_{F_{\mathcal{V}''}}[\mathbf{x}](F_{\mathcal{V}''})\rangle$ and has the same support, so that $\mathbf{u} = 0$ since $\mathcal{N}(\mathcal{V}, \langle\phi_{F_{\mathcal{V}''}}[\mathbf{x}](F_{\mathcal{V}''})\rangle) = \{0\}$.

Thus, the ring homomorphism

$$\begin{aligned}\varphi : \Bbbk[C'] &\to \Bbbk[C'']\\ C'_{\alpha\beta kl} &\mapsto \text{coefficient of } x^\beta\mathbf{e}_l \text{ in } \mathbf{h}^k_\alpha \text{ if } |\alpha| = s-1\\ C'_{\eta\gamma kl} &\mapsto \text{coefficient of } x^\gamma\mathbf{e}_l \text{ in } \mathbf{g}^l_\eta \text{ if } x^\eta\mathbf{e}_l \in \mathbf{P}(\mathcal{V}),\ |\eta| \geq t\end{aligned}$$

induces a homomorphism $\overline{\varphi} : A' \to A''$.

Finally, $\overline{\psi}$ and $\overline{\varphi}$ are inverses of each other. Indeed, if we apply to the $\mathbf{P}(\mathcal{V}')$-marked set $H$ the construction from the first part of the proof we obtain a $\mathbf{P}(\mathcal{V}'')$-marked set $G'$ such that $\phi_{F_{\mathcal{V}''}}[\mathbf{x}](G')$ is a $\mathbf{P}(\mathcal{V}'')$-marked basis and $\phi_{F_{\mathcal{V}''}}[\mathbf{x}](G') \subseteq \phi_{F_{\mathcal{V}''}}[\mathbf{x}](F_{\mathcal{V}''})$, hence $\phi_{F_{\mathcal{V}''}}[\mathbf{x}](G') = \phi_{F_{\mathcal{V}''}}[\mathbf{x}](F_{\mathcal{V}''})$. □

**Corollary 3.51.** *Let $\mathcal{V}$ be a saturated quasi-stable monomial module with Hilbert polynomial* $\mathsf{HP}(t)$ *and Gotzmann number $r$ and regularity $d$. Then $\mathbf{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}_{\geq r})}$ and $\mathbf{Mf}^{n,m}_{\mathbf{P}(\mathcal{V}_{\geq d-1})}$ are isomorphic.*

# 4 Computation

In the first chapters we developed a method to compute Quot-schemes over fields of arbitrary characteristic. For that we first developed marked bases for modules and then we used the marked bases to define marked families. With the marked families we constructed a new open covering for a Quot-scheme.

In this chapter we use the theory developed in the earlier chapters to construct algorithmic methods. This will allow us to define a first algorithm for computing Quot-schemes over fields of arbitrary characteristic. It turns out that we can optimize the algorithms a lot. Therefore, a large part of this chapter is devoted to this aspect.

Some of the algorithms, which we will develop in this chapter are implemented in the computer algebra system CoCoALib ([1]). Namely, the computation of Hilbert schemes for fields of arbitrary characteristic. At the end of this chapter we will use the algorithms to compute some examples.

## 4.1 Computation of Saturated Quasi-Stable Ideals

In the following we describe how to compute all saturated quasi-stable ideals which have a given Hilbert polynomial. As main reference we use [7].

As before we consider a field $\Bbbk$ of arbitrary characteristic. In this section we consider the polynomial ring $\mathcal{P}^{(l)} := \Bbbk[x_l, \dots, x_n]$ with $0 \leq l \leq n$. We denote by $\mathbb{T}(l)$ the set of terms of $\mathcal{P}^{(l)}$.

**Definition 4.1.** *Let* $\mathsf{HP}(t) \in \mathbb{Q}[t]$. *Then we define*

$$\Delta\mathsf{HP}(t) := \mathsf{HP}(t) - \mathsf{HP}(t-1).$$

*In addition to that we define recursively*

$$\Delta^k\mathsf{HP}(t) := \Delta^{k-1}\mathsf{HP}(t) - \Delta^{k-1}\mathsf{HP}(t-1)$$

*with* $\Delta^0\mathsf{HP}(t) = \mathsf{HP}(t)$.

If $\mathsf{HP}(t)$ has Gotzmann number $r$, then the Gotzmann number of the Hilbert polynomial $\Delta\mathsf{HP}(t)$ is $\leq r$.

Now let $\mathcal{J} \subseteq \mathcal{P}^{(l)}$ be a monomial ideal, hence $\mathcal{J}$ is generated by a unique minimal set of terms. This set is denoted by $\mathcal{B}^{\min}(\mathcal{J})$. If $\mathcal{J}$ is quasi-stable then we have seen that there exists a unique minimal Pommaret basis $\mathbf{P}(\mathcal{J})$.

**Definition 4.2.** *[7, Def. 3] Let $\mathcal{J}$ be a quasi-stable ideal in $\mathcal{P}^{(l)}$. We define the following sets of terms*

$$\mathbf{P}(\mathcal{J})(j) := \{x^\alpha \in \mathbf{P}(\mathcal{J}) \mid \operatorname{cls}(x^\alpha) = j\} \quad \textit{and}$$
$$\overline{\mathbf{P}(\mathcal{J})(j)} := \left\{ \frac{x^\alpha}{x_j^{\alpha_j}} \mid x^\alpha \in \mathbf{P}(\mathcal{J})(j) \right\}.$$

**Lemma 4.3.** *[7, Lem. 1] Let $\mathcal{J}$ be a quasi-stable ideal in $\mathcal{P}^{(l)}$ and consider $l \leq j \leq n$. The ideal $\mathcal{J} : \langle x_n, \dots, x_j \rangle^\infty$ has the weak Pommaret basis*

$$\overline{\mathbf{P}(\mathcal{J})(j)} \cup \bigcup_{i=j+1}^{n} \mathbf{P}(\mathcal{J})(i)\,.$$

*Furthermore no term in the Pommaret basis of $\mathcal{J} : \langle x_n, \dots, x_j \rangle^\infty$ is divisible by $x_m$ with $m \leq j$.*

**Lemma 4.4** ([7, Lem. 2])**.** *Let $\mathcal{J} \subseteq \mathcal{P}^{(l)}$ be a quasi-stable monomial ideal. Then the Pommaret basis $\mathbf{P}(\mathcal{J})$ is also a Pommaret basis for the ideal $\mathcal{J} \cdot \mathcal{P}^{(l-1)}$.*

**Definition 4.5** ([7, Def. 4])**.** *Let $\mathcal{J} \subseteq \mathcal{P}^{(l)}$ be a stable monomial ideal. The term $x^\alpha \in \mathcal{J}$ is* St-minimal *if $\frac{x_j x^\beta}{x_{\operatorname{cls}(x^\beta)}} \neq x^\alpha$ for every $x^\beta \in \mathcal{J}$ and for every $j > \operatorname{cls}(x^\beta)$.*

**Definition 4.6.** *Let $\mathcal{J} \subseteq \mathcal{P}^{(l)}$ be a stable monomial ideal generated in degree $s$, i.e. $\mathcal{B}^{\min}(\mathcal{J}) = \mathcal{J}_s \cap \mathbb{T}(l)$. The* set of St-minimal elements *of $\mathcal{J}$ is defined as the set of monomial generators of $\mathcal{J}$ which are St-minimal. This set is denoted by $\mathcal{S}_{\min}(\mathcal{J})$ or $\mathcal{S}_{\min}(\mathcal{J}_s)$.*

**Corollary 4.7** ([7, Cor. 1])**.** *Let $\mathcal{J}$ be a quasi-stable ideal in $\mathcal{P}^{(l)}$, consider $s \geq \operatorname{reg}(\mathcal{J})$ and $x^\alpha \in \mathbf{P}(\mathcal{J}_{\geq s}) = \mathcal{J}_s \cap \mathbb{T}(l)$. Then the ideal generated by the set $(\mathcal{J}_s \cap \mathbb{T}(l)) \setminus \{x^\alpha\}$ is stable if and only if $x^\alpha \in \mathcal{S}_{\min}(\mathcal{J}_{\geq s})$.*

**Definition 4.8.** *Given a quasi-stable ideal $\mathcal{J} \subseteq \mathcal{P}^{(l)}$, then the ideal $(\mathcal{J} : x_l^\infty) : x_{l+1}^\infty$ is called the* $x_{l+1}$-saturation *of $\mathcal{J}$ and is denoted it by $\mathcal{J}_{x_l x_{l+1}}$. The quasi-stable ideal $\mathcal{J}$ is $x_{l+1}$-saturated if $\mathcal{J} = \mathcal{J}_{x_l x_{l+1}}$.*

**Proposition 4.9** ([7, Prop. 2])**.** *Let $\mathcal{J} \subseteq \mathcal{P}^{(l)}$ be a saturated quasi-stable ideal, $r$ be the Gotzmann number of $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{J}}(t)$ and let the integer $s \geq r$. We define the ideal $\mathcal{I} := \mathcal{J}_{x_l x_{l+1}}$ and $q := \dim_{\Bbbk}(\mathcal{I}_s) - \dim_{\Bbbk}(\mathcal{J}_s)$. Then the Hilbert polynomial of $\mathcal{I}$ is $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{I}}(t) = \mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{J}}(t) - q$.*

**Lemma 4.10** ([7, Lem. 4]). *Let $\mathcal{J}$ be a quasi-stable ideal in $\mathcal{P}^{(l)}$ and let $r$ be the Gotzmann number of $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{J}}(t)$. For an arbitrary $s \geq r$, consider an St-minimal term $x^\beta \in \mathcal{J}_s$ with $\operatorname{cls}(x^\beta) = l$ and let $\mathcal{M} := (\mathcal{J}_s \cap \mathbb{T}(l)) \setminus \{x^\beta\}$. Then the ideal generated by $\mathcal{M}$ is stable and its Hilbert polynomial is $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{J}}(t) + 1$.*

**Lemma 4.11** ([7, Lem. 5]). *Let $\mathcal{I}$ and $\mathcal{J}$ be quasi-stable ideals in $\mathcal{P}^{(l)}$. If for every $s \gg 0$ we have $\mathcal{I}_s \subseteq \mathcal{J}_s$ and $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{I}}(t) = \mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{J}}(t) + a$ with $a \in \mathbb{N}$, then $\mathcal{I}$ and $\mathcal{J}$ have the same $x_{l+1}$-saturation and for every $s \gg 0$ there is a term $x^\alpha \in \mathcal{J}_s \setminus \mathcal{I}_s$, with $\operatorname{cls}(x^\alpha) = l$ which is St-minimal.*

**Algorithm 1** REMOVE($\mathcal{I}, l, n, s, q, x^\beta$)

**Input:** $\mathcal{I}$ a quasi-stable ideal
**Input:** $l$ first index of the variables in the polynomial ring
**Input:** $n$ last index of the variables of the polynomial ring
**Input:** $s$ upper bound for $\operatorname{reg}(\mathcal{I})$
**Input:** $q$ non-negative Integer
**Input:** $x^\beta$ monomial
**Output:** $\mathcal{L}$ set of saturated quasi-stable ideals $\mathcal{J}$ obtained by removing $q$ St-minimal terms divisible by $x_l$ from $\mathcal{I}_s \cap \mathbb{T}(l)$ and saturating

$\mathcal{L} \leftarrow \emptyset$
**if** $q = 0$ **then**
  **return** $\{\mathcal{I}^{\text{sat}}\}$
**else**
  $\mathcal{M} \leftarrow$ STMINIMAL($\mathcal{I}, l, n, s$)
  **for all** $x^\alpha \in \mathcal{M}$ **do**
    **if** $x^\alpha \succ x^\beta$ **then**
      $\mathcal{L} \leftarrow \mathcal{L} \cup$ REMOVE($\langle(\mathcal{I}_s \cap \mathbb{T}(l)) \setminus \{x^\alpha\}\rangle, l, n, s, q-1, x^\alpha$)
    **end if**
  **end for**
**end if**
**return** $\mathcal{L}$

The computation of all quasi-stable ideals with the same Hilbert polynomial $\mathsf{HP}(t)$ uses the $x_{l+1}$-saturation. For the computation we relate the Hilbert polynomial of a quasi-stable ideal $\mathcal{J}$ to that of $\langle \mathcal{J}, x_l\rangle / \langle x_l \rangle$.

In $\mathcal{P}^{(l)}$ consider a quasi-stable ideal $\mathcal{J}$ with Hilbert polynomial $\mathsf{HP}(t)$. Obviously $x_l$ is a non-zero divisor in $\mathcal{P}^{(l)}/\mathcal{J}^{\text{sat}}$. The ideal $\langle \mathcal{J}, x_l\rangle / \langle x_l \rangle \subseteq \mathcal{P}^{(l+1)}$ has the same Hilbert polynomial as $\langle \mathcal{J}, x_l \rangle \subseteq \mathcal{P}^{(l)}$ and $\mathcal{J}^{\text{sat}}$ has the same Hilbert polynomial as $\mathcal{J}$, because for $s \gg 0$ $\mathcal{J}_s^{\text{sat}} = \mathcal{J}_s$. Hence, we can consider the short exact sequence

$$0 \longrightarrow (\mathcal{P}^{(l)}/\mathcal{J}^{\text{sat}})_{s-1} \xrightarrow{\cdot x_l} (\mathcal{P}^{(l)}/\mathcal{J}^{\text{sat}})_s \longrightarrow (\mathcal{P}^{(l)}/\langle \mathcal{J}^{\text{sat}}, x_l\rangle)_s \longrightarrow 0$$

**Algorithm 2** QuasiStable($l, n, \mathsf{HP}(t), s$)

**Input:** $l$ first index of the variables in the polynomial ring
**Input:** $n$ last index of the variables of the polynomial ring
**Input:** $\mathsf{HP}(t)$ admissible Hilbert polynomial
**Input:** $s$ positive integer upper bound for the Gotzmann number of $\mathsf{HP}(t)$
**Output:** $\mathcal{F}$ set of quasi-stable ideals $\mathcal{J}$ in $\mathcal{P}^{(l)}$ having Hilbert polynomial $\mathsf{HP}(t)$

**if** $\mathsf{HP}(t) = 0$ **then**
  **return** $\{\langle 1 \rangle\}$
**else**
  $\mathcal{E} \leftarrow$ QuasiStable($l+1, n, \Delta\mathsf{HP}(t), s$)
  $\mathcal{F} \leftarrow \emptyset$
  **for all** $\mathcal{J} \in \mathcal{E}$ **do**
    $\mathcal{I} \leftarrow \mathcal{J} \cdot \mathcal{P}^{(l)}$
    $q \leftarrow \mathsf{HP}(s) - \binom{n-l+s}{s} + \dim_{\Bbbk}(\mathcal{I}_s)$
    **if** $q \geq 0$ **then**
      $\mathcal{F} \leftarrow \mathcal{F} \cup$ Remove($\mathcal{I}, l, n, s, q, 1$)
    **end if**
  **end for**
  **return** $\mathcal{F}$
**end if**

and we obtain that the Hilbert polynomial of $\langle \mathcal{J}^{\text{sat}}, x_l \rangle$ is $\Delta\mathsf{HP}(t)$ in $\mathcal{P}^{(l)}$. This is also the Hilbert polynomial of $\langle \mathcal{J}, x_l \rangle$, since $\langle \mathcal{J}^{\text{sat}}, x_l \rangle_s = \langle \mathcal{J}_s^{\text{sat}}, x_l \rangle_s = \langle \mathcal{J}_s, x_l \rangle_s = \langle \mathcal{J}, x_l \rangle_s$, for every $s \geq r$, where $r$ is the Gotzmann number of $\mathsf{HP}(t)$. Furthermore, observe that $\langle \mathcal{J}, x_l \rangle / \langle x_l \rangle \subseteq \mathcal{P}^{(l+1)}$ is also quasi-stable and $\Big(\langle \mathcal{J}, x_l \rangle / \langle x_l \rangle\Big)^{\text{sat}}$ is generated by $\mathcal{J}_{x_l x_{l+1}} \cap \mathbb{T}(l+1)$ in $\mathcal{P}^{(l+1)}$ due to Proposition 1.61.

To compute all saturated quasi-stable ideals in $\mathcal{P}^{(l)}$ with a given Hilbert polynomial $\mathsf{HP}(t)$ we use recursion on the number of variables. Assume that we have a complete list of saturated quasi-stable ideals in $\mathcal{P}^{(l+1)}$ generated in degree less than or equal to the Gotzmann number of $\mathsf{HP}(t)$ with Hilbert polynomial $\Delta\mathsf{HP}(t)$. Then we embed all the ideals of this list in $\mathcal{P}^{(l)}$. In $\mathcal{P}^{(l)}$ these ideals have Hilbert polynomial $\mathsf{HP}(t) + q$, where $q$ is an integer. Turning Lemma 4.10 into an algorithm finishes the construction and leads to the Algorithms 1 and 2.

To get a better understanding why the Algorithms 1 and 2 work we recall the proofs of [7].

**Theorem 4.12** ([7, Thm. 4])**.** *Algorithm 1, Remove($\mathcal{I}, l, n, s, q, x^{\beta}$), returns the set of all saturated quasi-stable ideals in the polynomial ring $\mathcal{P}^{(l)}$ contained in $\mathcal{I}_s$, having the same $x_{l+1}$-saturation as $\mathcal{I}$ and having Hilbert polynomial $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{I}}(t) + q$.*

*Proof.* For $q = 0$ the algorithm terminates obviously with the correct output in line (3). If $q > 0$, the algorithm computes in line (5) the St-minimal terms $x^\alpha$ having class $l$. By Lemma 4.10 the set of terms in $(\mathcal{I}_s \cap \mathbb{T}(l)) \setminus \{x^\alpha\}$ generates a stable ideal with Hilbert polynomial $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{I}}(t) + 1$.

The termination of the algorithm is obvious, because at each recursive call at line (8) the number of terms to remove decreases. The if-condition in line (7) avoids the repeated computation of the same result several times: If $x^\alpha$ and $x^\beta$ are two St-minimal elements of class $l$ of $\mathcal{I}$ then the algorithm without the if-condition would generate both the ideal

$$\left\langle ((\mathcal{I}_s \cap \mathbb{T}(l)) \setminus \{x^\alpha\}) \setminus \{x^\beta\} \right\rangle^{\text{sat}}$$

and the ideal

$$\left\langle ((\mathcal{I}_s \cap \mathbb{T}(l)) \setminus \{x^\beta\}) \setminus \{x^\alpha\} \right\rangle^{\text{sat}}$$

which are obviously the same. By using an arbitrary term order $\prec$ in line (7) the algorithm avoids these duplicate computations.

Applying Algorithm 1 to a quasi-stable ideal $\mathcal{I}$ we obtain as output saturated ideals having the same $x_{l+1}$-saturation as $\mathcal{I}$ by Lemma 4.11 and having the Hilbert polynomial $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{I}}(t) + q$. $\square$

**Theorem 4.13** ([7, Thm. 5])**.** *Algorithm 2, QUASISTABLE($l, n, \mathsf{HP}(t), s$), returns the set of all quasi-stable saturated ideals in the polynomial ring $\mathcal{P}^{(l)}$ with Hilbert polynomial $\mathsf{HP}(t)$.*

*Proof.* We perform an induction on $\Delta^m \mathsf{HP}(t)$. It is sufficient to take $s$ to be an upper bound for the Gotzmann numbers of $\Delta^m \mathsf{HP}(t)$ for all $m \geq 0$.

If $\mathsf{HP}(t) = 0$, then the ideal $\langle 1 \rangle$ is the only quasi-stable saturated ideal with this Hilbert polynomial. This shows that the algorithm is correct in this case.

Now we assume that $\mathsf{HP}(t) \neq 0$ and that Algorithm 2 returns the correct set for $\Delta\mathsf{HP}(t)$. Line (4) returns the complete list of $x_{l+1}$-saturation of the saturations of the ideals we look for. Consider $\mathcal{J} \subseteq \mathcal{P}^{(l+1)}$ belonging to the output of QUASISTABLE ($l + 1, n, \Delta\mathsf{HP}(t), s$)). Then the ideal $\mathcal{I} = \mathcal{J} \cdot \mathcal{P}^{(l)}$ is quasi-stable by Lemma 4.4 and Proposition 1.64. Furthermore, the Hilbert polynomial of $\mathcal{I}$ is $\mathsf{HP}(t) + q$, where $q$ is defined at line (8). Now there are three possibilities:

- if $q < 0$, there exist no saturated quasi-stable ideals $\mathcal{I} \subseteq \mathcal{P}^{(l)}$ with Hilbert polynomial $\mathsf{HP}(t)$ and such that $\langle \mathcal{I}, x_l \rangle / \langle x_l \rangle = \mathcal{J}$, hence $\mathcal{J}$ has to be discarded, by Proposition 4.9;
- if $q = 0$, then $\mathcal{J} \cdot \mathcal{P}^{(l)}$ is one of the ideals which we want to obtain;

- if $q > 0$, we apply Algorithm 1 to obtain the saturated quasi-stable ideals $\mathcal{I} \subseteq \mathcal{P}^{(l)}$ with Hilbert polynomial $\mathsf{HP}(t)$ and such that $\langle \mathcal{I}, x_l \rangle / \langle x_l \rangle = \mathcal{J}$.

□

### 4.1.1 An Efficient Algorithm to Compute Saturated Quasi-Stable Ideals

Our goal is a fast implementation of the algorithms mentioned in the section above. Therefore, we investigate some effort to obtain this goal now.

The key point of a good implementation is a good balance between a memory efficient representation of the ideals and the fast computation of the different operations of the algorithms. In Algorithm 1 and Algorithm 2 we can identify the following costly operations:

(i) The computation of the St-minimal elements of an ideal in Algorithm 1 in line (5);

(ii) The computation of $\mathcal{J} \cdot \mathcal{P}^{(l)}$ in Algorithm 2 in line (7);

(iii) The computation of the dimension of $\mathcal{I}_s$ in Algorithm 2 in line (8).

Our idea is now to represent the ideals in Algorithm 1 and 2 by two different sets. We represent an ideal $\mathcal{I}$ by its Pommaret basis and by the St-minimal set of $\mathcal{I}_s$.

In the following let $\mathcal{I}$ be a quasi-stable saturated ideal and $s$ be greater than or equal to the Gotzmann number of the Hilbert polynomial of $\mathcal{I}$. As a first step we will show how to efficiently check whether an element belongs to $\mathcal{I}_s$, only knowing $\mathcal{S}_{\min}(\mathcal{I}_s)$.

The following ordering is based on ideas of [32] where the author wants to compute Borel fixed ideals in characteristic zero, e. g. to compute strongly stable ideals.

**Definition 4.14.** *Let $x^\alpha, x^\beta \in \mathbb{T}(l)_s$. Additionally, let $d_i(x^\alpha, x^\beta) := \deg_i(x^\alpha) - \deg_i(x^\beta)$ for all $i \in \{l, \dots, n\}$. Then $x^\alpha \prec_{\mathrm{St}} x^\beta$ if and only if*

- $d_i(x^\alpha, x^\beta) \geq 0$ *for all* $i \in \{l, \dots, \mathrm{cls}(x^\beta) - 1\}$,
- $d_i(x^\alpha, x^\beta) > 0$ *for at least one* $i \in \{l, \dots, \mathrm{cls}(x^\beta) - 1\}$,
- $d_i(x^\alpha, x^\beta) \leq 0$ *for all* $i \in \{\mathrm{cls}(x^\beta) + 1, \dots n\}$ *and*
- $d_i(x^\alpha, x^\beta) < 0$ *for at least one* $i \in \{\mathrm{cls}(x^\beta) + 1, \dots n\}$.

**Lemma 4.15.** *Let $\mathcal{J} \subseteq \mathcal{P}^{(l)}$ be a stable monomial ideal generated in degree s. Let $x^\alpha \in \mathcal{J}_s$ and $x^\beta \in \mathbb{T}(l)_s$ such that $x^\alpha \preceq_{\mathrm{St}} x^\beta$. Then $x^\beta \in \mathcal{J}_s$.*

*Proof.* Beside equality of $x^\alpha$ and $x^\beta$ we distinguish between the cases $d_{\mathrm{cls}(x^\beta)}(x^\alpha, x^\beta) \geq 0$ and $d_{\mathrm{cls}(x^\beta)}(x^\alpha, x^\beta) < 0$.

For the first case we prove the statement by an induction over the sum

$$d(x^\alpha, x^\beta) := \sum_{i=l}^{\mathrm{cls}(x^\beta)} d_i(x^\alpha, x^\beta).$$

Suppose $d(x^\alpha, x^\beta) = 1$, hence

$$\sum_{i=\mathrm{cls}(x^\beta)+1}^{n} d_i(x^\alpha, x^\beta) = -1$$

and so we must have be $d_k(x^\alpha, x^\beta) = 1$ and $d_j(x^\alpha, x^\beta) = -1$ for some $k \leq \mathrm{cls}(x^\beta)$ and $j > \mathrm{cls}(x^\beta)$. Furthermore, $d_i(x^\alpha, x^\beta) = 0$ for all $i \in \{l, \ldots, n\} \setminus \{k, j\}$. Hence, $\frac{x_j}{x_k} x^\alpha = x^\beta$ and therefore $x^\beta \in \mathcal{J}_s$ because $\mathcal{J}$ is stable and $\mathrm{cls}(x^\alpha) = k$.

When $d(x^\alpha, x^\beta) > 1$ we consider an element $x^\gamma = \frac{x_j}{x_k} x^\alpha$ with $d_k(x^\alpha, x^\beta) > 0$ and $d_j(x^\alpha, x^\beta) < 0$ such that $k = \mathrm{cls}(x^\alpha)$. It is obvious that $x^\alpha \prec_{\mathrm{St}} x^\gamma$, hence $x^\gamma \in \mathcal{J}$ by the beginning of the induction. It is also clear that $x^\gamma \prec_{\mathrm{St}} x^\beta$ and that $d(x^\gamma, x^\beta) = d(x^\alpha, x^\beta) - 1$. Hence, the claim follows.

The proof for $d_{\mathrm{cls}(x^\beta)}(x^\alpha, x^\beta) < 0$ is essentially the same. But now we perform an induction over the sum

$$d(x^\alpha, x^\beta) := \sum_{i=l}^{\mathrm{cls}(x^\beta)-1} d_i(x^\alpha, x^\beta).$$

. □

**Corollary 4.16.** *Let $\mathcal{J} \subseteq \mathcal{P}^{(l)}$ be a stable monomial ideal. Let $x^\alpha \in \mathcal{J}_s$ be an St-minimal element. Then there does not exist any $x^\beta \in \mathcal{J}_s$ such that $x^\beta \prec_{\mathrm{St}} x^\alpha$. Moreover, there exists for any non St-minimal element in $\mathcal{J}_s$ an St-minimal element in $\mathcal{J}_s$ which is smaller with respect to $\succ_{\mathrm{St}}$.*

*Proof.* Let $x^\alpha$ be an St-minimal element of $\mathcal{J}_s$. Assume that there exists an element $x^\beta \in \mathcal{J}_s$, such that $x^\beta \prec_{\mathrm{St}} x^\alpha$. Then $d_i(x^\beta, x^\alpha) \geq 0$ for all $i \in \{l, \ldots, \mathrm{cls}(x^\alpha) - 1\}$. There is a strict inequality for at least one $i$. Additionally, $d_i(x^\beta, x^\alpha) \leq 0$ for all $i \in \{\mathrm{cls}(x^\alpha) + 1, \ldots, n\}$. Hence,

$$\frac{x_l^{d_l(x^\beta,x^\alpha)} \cdots x_{\mathrm{cls}(x^\alpha)-1}^{d_{\mathrm{cls}(x^\alpha)-1}(x^\beta,x^\alpha)}}{x_{\mathrm{cls}(x^\alpha)+1}^{-d_{\mathrm{cls}(x^\alpha)+1}(x^\beta,x^\alpha)} \cdots x_n^{-d_n(x^\beta,x^\alpha)}} x^\beta = x^\alpha\,,$$

which shows, that $x^\alpha$ cannot be St-minimal in this situation.

The second part of the corollary is obvious. □

In the following we assume that $\mathcal{J}$ and $\mathcal{I}$ in the Algorithms 1 and 2 are represented by their St-minimal sets in degree $s$. Now we show how to perform the costly computations mentioned above.

The computation of the St-minimal elements in Algorithm 1 in line (5) is trivial because $\mathcal{I}$ is represented exactly by the set of St-minimal elements of $\mathcal{I}_s$.

The next problem which occurs is the computation of the St-minimal set of $\mathcal{J} := \langle(\mathcal{I}_s \cap \mathbb{T}(l)) \setminus \{x^\alpha\}\rangle$ where $x^\alpha \in \mathcal{S}_{\min}(\mathcal{I}_s)$. It is obvious that $\mathcal{S} := \mathcal{S}_{\min}(\mathcal{I}_s) \setminus \{x^\alpha\} \subseteq \mathcal{S}_{\min}(\mathcal{J})$. Now we have to find a set $\mathcal{T}$, such that $\mathcal{S} \cup \mathcal{T} = \mathcal{S}_{\min}(\mathcal{J})$. For this we compute all possible increasing moves starting from $x^\alpha$. It is clear that $\mathcal{S}_{\min}(\mathcal{J}) \subseteq \mathcal{S} \cup \mathcal{T}''$, where

$$\mathcal{T}'' = \left\{ \frac{x_i}{x_{\mathrm{cls}(x^\alpha)}} x^\alpha \mid i \in \{\mathrm{cls}(x^\alpha)+1, \dots, n\} \right\}$$

is the set of all increasing moves starting from $x^\alpha$. Then we check for every $x^\gamma \in \mathcal{T}''$ if there is an $x^\beta \in \mathcal{S}_{\min}(\mathcal{I}_s) \setminus \{x^\alpha\}$ such that $x^\beta$ is smaller than $x^\gamma$ with respect to $\succ_{\mathrm{St}}$. If this is not the case the element could be a new St-minimal element of $\mathcal{J}$ and we add it to a new set $\mathcal{T}'$.

Now we have to check that the remaining elements in $\mathcal{T}'$ are St-minimal for $\mathcal{J}$. We already know, that $\mathcal{S}_{\min}(\mathcal{I}_s \setminus \{x^\alpha\}) \subseteq \mathcal{S} \cup \mathcal{T}'$ and that there exists no $x^\beta \in \mathcal{S}$ which is smaller than an element in $\mathcal{T}'$ with respect to $\prec_{\mathrm{St}}$. Hence, the only case which can happen now, is that there are elements $x^\beta, x^\gamma \in \mathcal{T}'$, such that $x^\beta \prec_{\mathrm{St}} x^\gamma$. To find these elements we can use the special structure of elements in $\mathcal{T}'$ and the next lemma.

**Lemma 4.17.** *Let $x^\alpha \in \mathbb{T}(l)_s$, $x^\beta = \frac{x_i}{x_{\mathrm{cls}(x^\alpha)}} x^\alpha$ and $x^\gamma = \frac{x_j}{x_{\mathrm{cls}(x^\alpha)}} x^\alpha$ with $i > j$. Then it never holds that $x^\beta \prec_{\mathrm{St}} x^\gamma$.*

*Proof.* We take a look at $d_k(x^\beta, x^\gamma)$ for all $k \in \{l, \dots, n\}$. It is clear that $d_k(x^\beta, x^\gamma) = 0$ for all $k \notin \{i, j\}$, $d_i(x^\beta, x^\gamma) = 1$ and $d_j(x^\beta, x^\gamma) = -1$. But then it can never happen that $x^\beta \prec_{\mathrm{St}} x^\gamma$ according to Definition 4.14 because $i > j$ and $\mathrm{cls}(x^\alpha) \leq i$. □

To construct the set $\mathcal{T}$ out of $\mathcal{T}'$ we proceed as follows: We move at first the element $\frac{x_i}{x_{\mathrm{cls}(x^\alpha)}} x^\alpha$ with the smallest index $i$ from $\mathcal{T}'$ to $\mathcal{T}$. Then we remove all elements from $\mathcal{T}'$ which are bigger than the chosen element with respect to $\prec_{\mathrm{St}}$. After that we choose again the element $\frac{x_j}{x_{\mathrm{cls}(x^\alpha)}} x^\alpha$ with the smallest index $j$ from the new set $\mathcal{T}'$, and move it into $\mathcal{T}$, and repeat. Lemma 4.17 guarantees that the first element which we move from $\mathcal{T}'$ to $\mathcal{T}$ is not smaller than the second element which we move from $\mathcal{T}'$ to $\mathcal{T}$ with respect to $\prec_{\mathrm{St}}$. Finally, we get a set $\mathcal{T}$, such that $\mathcal{S} \cup \mathcal{T} = \mathcal{S}_{\min}(\mathcal{J})$.

The process described above is formalized in Algorithm 3.

**Algorithm 3** StMinimalElements($l, n, \mathcal{S}_{\min}(\mathcal{I}_s), x^\alpha$)

**Input:** $l$ first index of the variables in the polynomial ring
**Input:** $n$ last index of the variables of the polynomial ring
**Input:** $\mathcal{S}_{\min}(\mathcal{I}_s)$ St-minimal elements of $\mathcal{I}_s$
**Input:** $x^\alpha \in \mathcal{S}_{\min}(\mathcal{I}_s)$
**Output:** The set $\mathcal{S}_{\min}(\mathcal{I}_s \setminus \{x^\alpha\})$

```
S ← S_min(I_s) \ {x^α}
i ← cls(x^α) + 1
while i ≤ n do
    x^δ ← (x_i / x_cls(x^α)) x^α
    if there does not exist x^γ ∈ S such that x^γ ≺_St x^δ then
        S ← S ∪ {x^δ}
    end if
    i ← i + 1
end while
return S
```

Now we determine how to compute the St-minimal set of $\mathcal{I}_s$, where $\mathcal{I} = \mathcal{J} \cdot \mathcal{P}^{(l)}$ and $\mathcal{J} \subseteq \mathcal{P}^{(l+1)}$ is a saturated quasi-stable ideal and $s \geq \operatorname{reg}(\mathcal{J})$.

**Lemma 4.18.** *Let $\mathcal{J} \subseteq \mathcal{P}^{(l)}$ be a saturated quasi-stable monomial ideal and let $s \geq \operatorname{reg}(\mathcal{J})$. Then*

$$\mathcal{S}_{\min}(\mathcal{J}_s) \subseteq \left\{ x^\alpha x_l^{s-|\alpha|} \mid x^\alpha \in \mathcal{B}^{\min}(\mathcal{J}) \right\}.$$

*Proof.* Let $x^\gamma \in \mathcal{S}_{\min}(\mathcal{J}_s)$. Then $x^\gamma = x^\delta x^\beta$ for some $x^\beta \in \mathbf{P}(\mathcal{J})$ and $x^\delta \in \Bbbk[X_{\mathcal{P}}(x^\beta)]$.

It is clear that $x^\delta = x_l^{\gamma_l}$ because $x^\beta$ is saturated and if $x^\delta \neq x_l^{\gamma_l}$ then the element $x_l^{\deg(x^\delta)} x^\beta$ is smaller with respect to the St-minimal order.

If $x^\beta \in \mathcal{B}^{\min}(\mathcal{J}) \cap \mathbf{P}(\mathcal{J})$ we are done. If not, there must be an $x^\alpha \in \mathcal{B}^{\min}(\mathcal{J})$ and an $x^{\delta'} \in \Bbbk[x_l, \ldots, x_n] \setminus \Bbbk[X_{\mathcal{P}}(x^\alpha)]$ with $x^\beta = x^{\delta'} x^\alpha$, hence $x^\gamma = x^\delta x^{\delta'} x^\alpha$. But with the same arguments as above we show that $x^\alpha x^\delta x_l^{\deg(x^{\delta'})}$ is more St-minimal than $x^\gamma$ which is again a contradiction. □

**Definition 4.19.** *Let $\mathcal{J} \subseteq \mathcal{P}^{(l)}$ be a stable monomial ideal generated in degree s. We define*

$$\mathcal{S}_{\min_l}(\mathcal{J}_s) := \left\{ x^\alpha \in \mathcal{S}_{\min}(\mathcal{J}_s) \mid \operatorname{cls}(x^\alpha) = l \right\}.$$

**Lemma 4.20.** *Let $\mathcal{J} \subseteq \mathcal{P}^{(l)}$ be a saturated quasi-stable monomial ideal and let $s \geq \operatorname{reg}(\mathcal{J})$. Then*
$$\mathcal{S}_{\min_l}(\mathcal{J}_s) = \left\{ x^\alpha x_l^{s-\deg(x^\alpha)} \mid x^\alpha \in \mathcal{B}^{\min}(\mathcal{J}) \text{ and } \deg(x^\alpha) < s \right\}.$$

*Proof.* By Lemma 4.18 it is obvious that

$$\mathcal{S}_{\min_l}(\mathcal{J}_s) \subseteq \left\{ x^\alpha x_l^{s-\deg(x^\alpha)} \mid x^\alpha \in \mathcal{B}^{\min}(\mathcal{J}) \text{ and } \deg(x^\alpha) < s \right\}.$$

For the other direction assume that

$$x^\alpha x_l^{s-\deg(x^\alpha)} \in \left\{ x^\alpha x_l^{s-\deg(x^\alpha)} \mid x^\alpha \in \mathcal{B}^{\min}(\mathcal{J}) \text{ and } \deg(x^\alpha) < s \right\}$$

is not St-minimal. Therefore, there must be a $x^\gamma \in \mathcal{J}_s$ such that $\frac{x^\gamma x_j}{x_{\operatorname{cls}(x^\gamma)}} = x^\alpha x_l^{s-\deg(x^\alpha)}$. But this means that $\operatorname{cls}(x^\gamma) = l$, hence $x^\gamma x_j = x^\alpha x_l^{s-\deg(x^\alpha)+1}$ or equivalent $x^\gamma = \frac{x^\alpha}{x_j} x_l^{s-\deg(x^\alpha)+1}$. Let $x^\gamma = x^\delta x^\beta$ with $x^\beta \in \mathbf{P}(\mathcal{J})$ and $x^\delta \in \Bbbk[X_\mathcal{P}(x^\beta)]$. Then $x^\delta x^\beta = \frac{x^\alpha}{x_j} x_l^{s-\deg(x^\alpha)+1}$. $\mathcal{J}$ is saturated, hence $\deg_l(\beta) = 0$ and we can write $x^{\delta'} x^\beta = \frac{x^\alpha}{x_j}$ or equivalent $x_j x^{\delta'} x^\beta = x^\alpha$. Hence, $x^\alpha$ is not a minimal generator of $\mathcal{J}$ which is a contradiction by assumption. □

With the lemma above it turns out to be easy to compute the St-minimal elements of $\mathcal{I}_s$.

**Lemma 4.21.** *Let $\mathcal{J} \subseteq \mathcal{P}^{(l+1)}$ be a saturated quasi-stable monomial ideal, $s \geq \operatorname{reg}(\mathcal{J})$ and $\mathcal{I} = \mathcal{J} \cdot \mathcal{P}^{(l)}$. Then*

$$\mathcal{S}_{\min}(\mathcal{I}_s) = \left\{ x^\alpha \in \mathcal{S}_{\min}(\mathcal{J}_s) \mid \operatorname{cls}(x^\alpha) > l+1 \right\} \cup \left\{ \frac{x_l^{\alpha_{l+1}}}{x_{l+1}^{\alpha_{l+1}}} x^\alpha \mid x^\alpha \in \mathcal{S}_{\min_{l+1}}(\mathcal{J}_s) \right\}.$$

*Proof.* Let
$$\mathcal{B}_1 = \left\{ \frac{x^\alpha}{x_{l+1}^{\alpha_{l+1}}} \mid x^\alpha \in \mathcal{S}_{\min_{l+1}}(\mathcal{J}_s) \right\}$$

and $\mathcal{B}_2$ be the set of all terms which can be reached by increasing moves starting from elements in $\mathcal{S}_{\min}(\mathcal{J}_s) \setminus S_{\min_{l+1}}(\mathcal{J}_s)$. It is obvious that $\langle \mathcal{B}_1 \rangle + \langle \mathcal{B}_2 \rangle = \mathcal{J} \subseteq \mathcal{P}^{(l+1)}$. By construction $\langle \mathcal{B}_2 \rangle$ is saturated and $\mathcal{B}_2$ is a Pommaret basis of $\langle \mathcal{B}_2 \rangle$. Hence, $\mathcal{B}_2$ is also a Pommaret basis for $\langle \mathcal{B}_2 \rangle \cdot \mathcal{P}^{(l)}$ by Lemma 4.4. This implies that $\mathcal{S}_{\min}(\mathcal{J}_s) \setminus S_{\min_{l+1}}(\mathcal{J}_s)$ is the St-minimal set of $\langle \mathcal{B}_2 \rangle \cdot \mathcal{P}^{(l)}$.

We know that $\mathcal{B}_1$ is a minimal generating set for $\langle \mathcal{B}_1 \rangle$. Hence, it is also a minimal generating set of $\langle \mathcal{B}_1 \rangle \cdot \mathcal{P}^{(l)}$. Therefore, we know that

$$\mathcal{S}_{\min_l}\left((\langle \mathcal{B}_1 \rangle \cdot \mathcal{P}^{(l)})_{\geq s}\right) = \left\{ x^\alpha x_l^{s-|\alpha|} \mid x^\alpha \in \mathcal{B}_1 \right\} ,$$

by Lemma 4.20.

It is obvious that $\mathcal{I} = \langle \mathcal{B}_1 \rangle \cdot \mathcal{P}^{(l)} + \langle \mathcal{B}_2 \rangle \cdot \mathcal{P}^{(l)}$. This implies that

$$\mathcal{S}_{\min}(\mathcal{I}_s) \subseteq \mathcal{S}_{\min_l}\left((\langle \mathcal{B}_1 \rangle \cdot \mathcal{P}^{(l)})_{\geq s}\right) \cup \mathcal{S}_{\min}(\mathcal{J}_s) \setminus S_{\min_{l+1}}(\mathcal{J}_s) .$$

Now it is left to prove that

$$\mathcal{S}_{\min}(\mathcal{I}_s) = \mathcal{S}_{\min_l}\left((\langle \mathcal{B}_1 \rangle \cdot \mathcal{P}^{(l)})_{\geq s}\right) \cup \mathcal{S}_{\min}(\mathcal{J}_s) \setminus S_{\min_{l+1}}(\mathcal{J}_s) .$$

It is clear that there are no increasing moves from an element of $\mathcal{S}_{\min_l}((\langle \mathcal{B}_1 \rangle \cdot \mathcal{P}^{(l)})_{\geq s})$ to $\mathcal{S}_{\min}(\mathcal{J}_s) \setminus S_{\min_{l+1}}(\mathcal{J}_s)$ and vice versa. It is also clear that there cannot be an element which can be obtained by increasing moves from an element of the set

$$\mathcal{S}_{\min_l}((\langle \mathcal{B}_1 \rangle \cdot \mathcal{P}^{(l)})_{\geq s}) \cup \mathcal{S}_{\min}(\mathcal{J}_s) \setminus S_{\min_{l+1}}(\mathcal{J}_s)$$

which lies outside $\mathcal{I}_s$ and vice versa. Hence, the statement is true. □

The last task on the list at the beginning of this section is the computation of $q$ in line (8) in Algorithm 2. Specifically the dimension of $\mathcal{I}_s$, where $\mathcal{I}$ is a quasi-stable ideal with a regularity which is less than or equal to $s$. For this computation we simply use the Pommaret basis of $\mathcal{I}$, because it is easy to compute the Hilbert function in degree $s$ starting from of the Pommaret basis which gives us immediately the dimension of $\mathcal{I}_s$.

$$\dim(\mathcal{I}_s) = \binom{s+n-l}{s} - \mathsf{HF}_{\mathcal{P}^{(l)}/\mathcal{I}}(s).$$

Now we have to determine how the Pommaret basis changes when we remove an St-minimal element from $\mathcal{I}_s$.

Assume that we remove an St-minimal element $x^\alpha$ from $\mathcal{S}_{\min}(\mathcal{I}_s)$. Let $\mathcal{J} := \langle \mathcal{I}_s \setminus \{x^\alpha\} \rangle^{\mathrm{sat}}$. We have to determine how to change $\mathbf{P}(\mathcal{I})$ to get $\mathbf{P}(\mathcal{J})$. It turns out, that the modifications are quite easy. It is clear that $x^{\alpha_{\mathrm{sat}}} := \frac{x^\alpha}{x_l^{\alpha_l}} \notin \mathcal{J}$, but every $x_i x^{\alpha_{\mathrm{sat}}} \in \mathcal{J}$ for $i \in \{l+1, \ldots, n\}$. We know that $x_i$ is non-multiplicative for $x^{\alpha_{\mathrm{sat}}}$ for all $i > \mathrm{cls}(x^{\alpha_{\mathrm{sat}}})$, hence these elements have a divisor which is already in the Pommaret basis and we do not need to treat them. The only possible elements which we may have to add to the Pommaret basis of $\mathcal{J}$ are the elements $x_i x^{\alpha_{\mathrm{sat}}}$ with $i \in \{l+1, \ldots, \min(x^{\alpha_{\mathrm{sat}}})\}$. It turns out that we have to add all of them to the Pommaret basis of $\mathcal{I}$ to get the Pommaret basis of $\mathcal{J}$ because these elements have the involutive divisor $x^{\alpha_{\text{sat}}}$ in $\mathcal{I}$ and therefore do not occur in the Pommaret basis of $\mathcal{I}$. Hence, the Pommaret basis of $\mathcal{J}$ is

$$\mathbf{P}(\mathcal{J}) = \mathbf{P}(\mathcal{I}) \cup \{x_i x^{\alpha_{\text{sat}}} \mid i \in \{l+1, \dots, \text{cls}(x^{\alpha_{\text{sat}}})\}\} \setminus \{x^{\alpha_{\text{sat}}}\}.$$

The last thing which we have to determine is the behaviour of the Pommaret basis in Algorithm 2 line (7), that is we have a Pommaret basis of $\mathbf{P}(\mathcal{J})$ of a saturated quasi-stable $\mathcal{J} \subseteq \mathcal{P}^{(l+1)}$ and we want to compute the Pommaret basis $\mathbf{P}(\mathcal{I})$ of $\mathcal{I} := \mathcal{J} \cdot \mathcal{P}^{(l)}$ Thanks to Lemma 4.4 the solution is easy because it says that the Pommaret bases for $\mathcal{J}$ and $\mathcal{I}$ are equivalent.

Now we are able to state new effective versions for the Algorithms 1 and 2 which call the Algorithms 4 and 5. In Algorithm 4 it is possible that we have sometimes to compute $\text{cls}(1)$. In this case we always consider $\text{cls}(1) = n$.

**Algorithm 4** REMOVE$((\mathcal{B}^{\text{PB}}, \mathcal{B}^{\text{St}}), l, n, s, q, x^\beta)$

**Input:** $(\mathcal{B}^{\text{PB}}, \mathcal{B}^{\text{St}})$ such that $\mathcal{B}^{\text{PB}} = \mathbf{P}(\mathcal{I})$ and $\mathcal{B}^{\text{St}}$ is the St-minimal set of a quasi-stable ideal $\mathcal{I}$
**Input:** $l$ first index of the variables in the polynomial ring
**Input:** $n$ last index of the variables of the polynomial ring
**Input:** $s$ upper bound for $\text{reg}(\mathcal{I})$
**Input:** $q$ non-negative Integer
**Input:** $x^\beta$ monomial
**Output:** $\mathcal{L}$, the set of pairs $(\mathcal{B}_i^{\text{PB}}, \mathcal{B}_i^{\text{St}})$ obtained by removing $q$ St-minimal terms divisible by $x_l$ from $\mathcal{I}_s \cap \mathbb{T}(l)$ and saturating

$\mathcal{L} \leftarrow \emptyset$
**if** $q = 0$ **then**
  **return** $\{(\mathcal{B}^{\text{PB}}, \mathcal{B}^{\text{St}})\}$
**else**
  **for all** $x^\alpha \in \mathcal{B}^{\text{St}}$ **do**
    **if** $x^\alpha \succ x^\beta$ **then**
      $\mathcal{B}_\alpha^{\text{PB}} \leftarrow \mathcal{B}^{\text{PB}} \cup \left\{ x_i \frac{x^\alpha}{x_l^{\alpha_l}} \mid i \in \{l+1, \dots, \text{cls}(\frac{x^\alpha}{x_l^{\alpha_l}})\} \right\} \setminus \left\{ \frac{x^\alpha}{x_l^{\alpha_l}} \right\}$
      $\mathcal{B}_\alpha^{\text{St}} \leftarrow$ STMINIMALELEMENTS$(l, n, \mathcal{B}^{\text{St}}, x^\alpha)$
      $\mathcal{L} \leftarrow \mathcal{L} \cup$ REMOVE$((\mathcal{B}_\alpha^{\text{PB}}, \mathcal{B}_\alpha^{\text{St}}), l, n, s, q-1, x^\alpha)$
    **end if**
  **end for**
**end if**
**return** $\mathcal{L}$

**Algorithm 5** $\textsc{QuasiStable}(l, n, \mathsf{HP}(t), s))$

**Input:** $l$ first index of the variables in the polynomial ring
**Input:** $n$ last index of the variables of the polynomial ring
**Input:** $\mathsf{HP}(t)$ admissible Hilbert polynomial
**Input:** $s$ positive integer upper bounding the Gotzmann number of $\mathsf{HP}(t)$
**Output:** $\mathcal{F}$, the set of all pairs $(\mathcal{B}^{\mathrm{PB}}, \mathcal{B}^{\mathrm{St}})$, such that $\mathcal{B}^{\mathrm{PB}} = \mathbf{P}(\mathcal{J})$ for a saturated quasi-stable ideal $\mathcal{J}$ in $\mathcal{P}^{(l)}$ having Hilbert polynomial $\mathsf{HP}(t)$

```
if HP(t) = 0 then
    return {(1, x_l^s)}
else
    E ← QuasiStable(l + 1, n, ΔHP(t), s)
    F ← ∅
    for all (B^PB, B^St) ∈ E do
        B ← {x^α ∈ B^St | cls(x^α) > l + 1} ∪ {(x_l^{α_{l+1}} / x_{l+1}^{α_{l+1}}) x^α | x^α ∈ B^St and cls(x^α) = l + 1}
        q ← HP(s) − binom(n−l+s, s) + dim_k(((⟨B^PB⟩ · P^(l))_s)
        if q ≥ 0 then
            F ← F ∪ Remove((B^PB, B), l, n, s, q, 1)
        end if
    end for
    return F
end if
```

where line 7 reads $\mathcal{B} \leftarrow \left\{x^\alpha \in \mathcal{B}^{\mathrm{St}} \mid \mathrm{cls}(x^\alpha) > l+1\right\} \cup \left\{ \frac{x_l^{\alpha_{l+1}}}{x_{l+1}^{\alpha_{l+1}}} x^\alpha \mid x^\alpha \in \mathcal{B}^{\mathrm{St}} \text{ and } \mathrm{cls}(x^\alpha) = l+1 \right\}$ and line 8 reads $q \leftarrow \mathsf{HP}(s) - \binom{n-l+s}{s} + \dim_{\Bbbk}((\langle \mathcal{B}^{\mathrm{PB}} \rangle \cdot \mathcal{P}^{(l)})_s)$.

**Example 4.22.** *We want to compute all saturated quasi-stable ideals in $\Bbbk[x_0, x_1, x_2]$ with Hilbert polynomial $\mathsf{HP}(t) = 4$. Figure 4.1 sketches the computation which is done by the Algorithms 3, 4 and 5.*

*As a result we get five different saturated quasi-stable ideals $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4, \mathcal{J}_5 \subset \Bbbk[x_0, x_1, x_2]$ with*

$$\begin{aligned}
\mathcal{B}^{\min}(\mathcal{J}_1) &= \{x_1, x_2^4\}\,, \\
\mathcal{B}^{\min}(\mathcal{J}_2) &= \{x_1^3, x_2x_1, x_2^2\}\,, \\
\mathcal{B}^{\min}(\mathcal{J}_3) &= \{x_1^2, x_2x_1, x_2^3\}\,, \\
\mathcal{B}^{\min}(\mathcal{J}_4) &= \{x_1^2, x_2^2\}\,, \\
\mathcal{B}^{\min}(\mathcal{J}_5) &= \{x_1^4, x_2\}\,.
\end{aligned}$$

Algorithm 3 Algorithm 5 and Algorithm 4 are implemented in the computer algebra system CoCoALib ([1]). We applied the implementation to some bigger examples. Table 4.1 exhibits the computation times of these examples, and Table 4.2 shows the number of saturated quasi-stable ideals for the given input. The first column gives the

QuasiStable$(0,2,4,4)$

- **4:** $\mathcal{E} \leftarrow$ QuasiStable$(1,2,0,4)$
  - **2: return** $\{(\{1\},\{x_1^4\})\}$
- **5:** $\mathcal{F} \leftarrow \emptyset$
- **6: for** $(\{1\},\{x_1^4\}) \in \mathcal{E}$
  - **7:** $\mathcal{B} \leftarrow \{x_0^4\}$
  - **8:** $q \leftarrow 4 - \binom{2-0+4}{4} + \binom{2-0+4}{4} = 4$
  - **10:** $\mathcal{F} \leftarrow$ Remove$((\{1\},\{x_0^4\}),0,2,4,4,1)$
    - **1:** $\mathcal{L} \leftarrow \emptyset$
    - **5: for** $x_0^4 \in \{x_0^4\}$
      - **7:** $\mathcal{B}^{\text{PB}}_{x_0^4} \leftarrow \{1\} \cup \{x_1,x_2\} \setminus \{1\}$
      - **8:** $\mathcal{B}^{\text{St}}_{x_0^4} \leftarrow$ StMinimalElements$(0,2,\{x_0^4\},x_0^4)$
        - **1:** $\mathcal{S} \leftarrow \emptyset$
        - **2:** $i \leftarrow 1$
        - **3: while** $i = 1 \leq 2$
          - **4:** $x^\delta \leftarrow \frac{x_1}{x_0}x_0^4$
          - **6:** $\mathcal{S} \leftarrow \mathcal{S} \cup \{x_1x_0^3\} = \{x_1x_0^3\}$
          - **8:** $i \leftarrow 2$
        - **3: while** $i = 2 \leq 2$
          - **4:** $x^\delta \leftarrow \frac{x_2}{x_0}x_0^4$
          - **6:** $\mathcal{S} \leftarrow \mathcal{S} \cup \{x_2x_0^3\} = \{x_2x_0^3, x_1x_0^3\}$
          - **8:** $i \leftarrow 3$
        - **10: return** $\mathcal{S} = \{x_2x_0^3, x_1x_0^3\}$
      - **9:** $\mathcal{L} \leftarrow \mathcal{L} \cup$ Remove$((\mathcal{B}^{\text{PB}}_{x_0^4}, \mathcal{B}^{\text{St}}_{x_0^4}),0,2,4,3,x_0)$
        - **1:** $\mathcal{L} \leftarrow \emptyset$
        - **5: for** $x_2x_0^3 \in \{x_2x_0^3, x_1x_0^3\}$
          - **7:** $\mathcal{B}^{\text{PB}}_{x_2x_0^3} \leftarrow \{x_1,x_2\} \cup \{x_2^2x_0^2, x_2x_1x_0^2\} \setminus \{x_2\} = \{x_2^2x_0^2, x_2x_1x_0^2, x_2\}$
          - **8:** $\mathcal{B}^{\text{St}}_{x_2x_0^3} \leftarrow$ StMinimalElements$(0,2,\{x_2x_0^3, x_1x_0^3\}, x_2x_0^3) = \{x_1x_0^3\}$
          - **9:** $\mathcal{L} \leftarrow \mathcal{L} \cup$ Remove$(\mathcal{B}^{\text{PB}}_{x_2x_0^3}, \mathcal{B}^{\text{St}}_{x_2x_0^3},0,2,4,2,x_2x_0^3) = \{(\{x_1x_0^3\},\{x_1,x_2^4\})\}$
        - **5: for** $x_1x_0^3 \in \{x_2x_0^3, x_1x_0^3\}$
          - **7:** $\mathcal{B}^{\text{PB}}_{x_1x_0^3} \leftarrow \{x_1,x_2\} \cup \{x_2x_1x_0^2, x_1^2x_0^2\} \setminus \{x_1\}$
          - **8:** $\mathcal{B}^{\text{St}}_{x_1x_0^3} \leftarrow$ StMinimalElements$(0,2,\{x_2x_0^3, x_1x_0^3\}, x_1x_0^3) = \{x_2x_1x_0^2, x_2x_0^3\}$
          - **9:** $\mathcal{L} \leftarrow \mathcal{L} \cup$ Remove$(\mathcal{B}^{\text{PB}}_{x_1x_0^3}, \mathcal{B}^{\text{St}}_{x_1x_0^3},0,2,4,2,x_1x_0^3)$
            $= \{(\{x_2^2x_0^2, x_2x_1x_0^2, x_1^3x_0\},\{x_2x_1, x_2^2, x_1^3\}), (\{x_1^2x_0^2, x_2x_1x_0^2, x_2^3x_0\},\{x_1^2, x_2x_1, x_2^3\}),$
            $(\{x_1^2x_0^2, x_2^2x_0^2\},\{x_1^2, x_2^2\}), (\{x_2x_0^3, x_1^4\},\{x_2, x_1^4\})\}$
        - **13: return** $\mathcal{L}$
    - **13: return** $\mathcal{L}$
- **13: return** $\mathcal{F} = \{(\{x_1x_0^3\},\{x_1,x_2^4\}), (\{x_2^2x_0^2, x_2x_1x_0^2, x_1^3x_0\},\{x_2x_1, x_2^2, x_1^3\}),$
  $(\{x_1^2x_0^2, x_2x_1x_0^2, x_2^3x_0\},\{x_1^2, x_2x_1, x_2^3\}), (\{x_1^2x_0^2, x_2^2x_0^2\},\{x_1^2, x_2^2\}), (\{x_2x_0^3, x_1^4\},\{x_2, x_1^4\})\}$

Figure 4.1: Computation of quasi-stable ideals

| HP$(t)$ | $G$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ | $n=7$ | $n=8$ |
|---|---|---|---|---|---|---|---|
| 5 | 5 | 0.000 | 0.000 | 0.004 | 0.008 | 0.016 | 0.024 |
| 10 | 10 | 0.012 | 0.120 | 0.684 | 2.972 | 10.000 | 29.152 |
| 15 | 15 | 0.288 | 20.607 | 79.904 | - | - | - |
| $5t+1$ | 11 | 0.008 | 0.072 | 0.392 | 1.536 | 42.586 | 12.452 |
| $5t+7$ | 17 | 0.732 | 29.124 | - | - | - | - |
| $2t^2+8t-46$ | 16 | 0.000 | 0.096 | 1.488 | 17.838 | 55.424 | 205.424 |
| $2t^2+8t-42$ | 20 | 0.032 | 41.640 | 12.024 | 73.632 | - | - |
| $4t^2-12t+10$ | 20 | 0.040 | 1.272 | 12.392 | 71.032 | - | - |
| $4t^2-12t+14$ | 24 | 0.796 | 63.748 | - | - | - | - |

Table 4.1: Time for computing all saturated quasi-stable ideals for a given Hilbert polynomial in a polynomial ring $\Bbbk[x_0,\dots,x_n]$

given Hilbert polynomial and the second column gives the corresponding Gotzmann number. The first row says the number of variables which we consider. If $n=3$ we compute the example in the polynomial ring $\Bbbk[x_0,\dots,x_3]$, hence we consider $n+1$ variables. That is the entry in the second row and third column is the computation time in seconds (respectively the number of saturated quasi-stable ideals) for the input $n=3$ and HP$(t)=5$.

For the computation we have used a computer with 8 GB main memory and Intel i7-5500U processor. The operating system was Ubuntu 16.04.1 LTS. We compiled CoCoALib with gcc 5.4.0. We restrict our program to use only 7.8 GB main memory.

For some examples we were not able to compute the result. This is denoted in both tables by "-". The problem was always that we ran out of memory. We see that the computation time is reasonably fast. Even the big examples could be computed within four minutes. The results are usually very big which explains why we ran so often out of memory. There is even an example which has as result more than a million different saturated quasi-stable ideals. It is obvious that the Gotzmann number and the number of variables are two values which influence the complexity most strongly.

The big results also show that it is not useful to compute all marked scheme according to a quasi-stable cover for a Hilbert scheme. There are simply too many marked schemes, which we would need to compute.

| HP($t$) | $G$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ | $n=7$ | $n=8$ |
|---|---|---|---|---|---|---|---|
| 5 | 5 | 24 | 59 | 120 | 216 | 357 | 554 |
| 10 | 10 | 500 | 3 122 | 13 220 | 43 352 | 118 874 | 285 784 |
| 15 | 15 | 6 879 | 108 802 | 982 615 | - | - | - |
| $5t+1$ | 11 | 370 | 2 562 | 10 039 | 29 165 | 70 317 | 149 149 |
| $5t+7$ | 17 | 21 648 | 579 747 | - | - | - | - |
| $2t^2+8t-46$ | 16 | 66 | 468 | 1 614 | 4 048 | 8 460 | 15 696 |
| $2t^2+8t-42$ | 20 | 1 200 | 24 052 | 170 526 | 741 684 | - | - |
| $4t^2-12t+10$ | 20 | 1 811 | 31 236 | 204 855 | 851 736 | - | - |
| $4t^2-12t+14$ | 24 | 23 153 | 1 068 586 | - | - | - | - |

Table 4.2: Number of all saturated quasi-stable ideals for a given Hilbert polynomial in a polynomial ring $\Bbbk[x_0, \ldots, x_n]$

## 4.2 Computation of Saturated $p$-Borel Fixed Ideals

In [7] an algorithm was presented to compute all Borel fixed ideals for a given Hilbert polynomial over a field $\Bbbk$ with characteristic $p$. In this section we briefly recall the construction of this algorithm. The structure of the algorithm is similar to that of the algorithm to compute quasi-stable ideals. But instead of removing St-minimal elements we remove $p$-minimal elements which are introduced below.

**Definition 4.23** ([7, Def. 10]). *Let $\mathcal{J}$ be a p-Borel fixed ideal and consider $x^\alpha \in \mathcal{J}$. $x^\alpha$ is* $p$-minimal (with respect to $\mathcal{J}$) *if there is no other term in $\mathcal{J} \cap \mathbb{T}(l)_{\deg(\alpha)}$ smaller than $x^\alpha$, with respect to $\prec_p$.*

**Definition 4.24.** *Let $\mathcal{J}$ be a p-Borel fixed ideal and s an integer. Then we define the set*

$$\operatorname{Min}_p(\mathcal{J}_s) := \{x^\alpha \mid x^\alpha \text{ is p-minimal in } \mathcal{J} \cap \mathbb{T}(l)_s\}.$$

**Lemma 4.25** ([7, Lem. 6]). *Let $\mathcal{J}$ be a p-Borel fixed ideal in $\mathcal{P}^{(l)}$ and let r be the Gotzmann number of $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{J}}(t)$. Let $s \geq r$, let $x^\alpha \in \mathcal{J}_s$ be a p-minimal and St-minimal term in $\mathcal{J}_s$ with* $\operatorname{cls}(x^\alpha) = l$. *Let $\mathcal{M} := (\mathcal{J}_s \cap \mathbb{T}(l)) \setminus \{x^\alpha\}$. Then $\langle \mathcal{M} \rangle$ is p-Borel fixed and $\mathsf{HP}_{\mathcal{P}^{(l)}/\langle \mathcal{M} \rangle}(t) = \mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{J}}(t) + 1$.*

**Lemma 4.26** ([7, Lem. 7]). *Let $\mathcal{I}$ and $\mathcal{J}$ be p-Borel fixed ideals in $\mathcal{P}^{(l)}$. If, for every $s \gg 0$, we have $\mathcal{I}_s \subseteq \mathcal{J}_s$ and $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{I}}(t) = \mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{J}}(t) + a$, with $a \in \mathbb{N}_0$, then $\mathcal{I}$ and $\mathcal{J}$ have the same $x_{l+1}$-saturation and for every $s \gg 0$ there is an $x^\alpha \in \mathcal{J}_s \setminus \mathcal{I}_s$, with $\operatorname{cls}(x^\alpha) = l$ which is St-minimal and p-minimal.*

The idea of computing all $p$-Borel fixed ideals with the same Hilbert polynomial, is the same as for computing all quasi-stable ideals with the same Hilbert polynomial.

Instead of removing arbitrary St-minimal elements during the removal procedure we only remove St-minimal elements of class $l$ which are in addition $p$-minimal according to Lemma 4.25, now. In the following we call the set of these elements $p$-St-minimal elements. This leads to Algorithms 6 and 7.

**Algorithm 6** $p$-REMOVE$(\mathcal{I}, l, n, s, q, p)$

**Input:** $\mathcal{I}$ a $p$-Borel fixed ideal
**Input:** $l$ first index of the variables in the polynomial ring
**Input:** $n$ last index of the variables of the polynomial ring
**Input:** $s$ upper bound for $\operatorname{reg}(\mathcal{I})$
**Input:** $q$ non-negative integer
**Input:** $x^\beta$ monomial
**Input:** $p$ characteristic of the coefficient field
**Output:** $\mathcal{L}$, the set of the saturated $p$-Borel fixed ideals $\mathcal{J}$ obtained by removing $q$ terms divisible by $x_l$ from $\mathcal{I}_s$

```
L ← ∅
if q = 0 then
    return {I^sat}
else
    M ← p-StMinimal(I, l, n, p, s)
    for all x^α ∈ M do
        L ← L ∪ p-Remove(⟨(I_s ∩ T(l)) \ {x^α}⟩, l, n, s, q − 1, p)
    end for
end if
return L
```

**Theorem 4.27** ([7, Thm. 7])**.** *Algorithm 6, $p$-REMOVE$(\mathcal{I}, l, n, s, q, p)$, returns the set of all saturated $p$-Borel fixed ideals in the polynomial ring $\mathcal{P}^{(l)}$ contained in $\mathcal{I}_s$, having the same $x_{l+1}$-saturation as $\mathcal{I}$ and having Hilbert polynomial $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{I}}(t) + q$.*

**Theorem 4.28** ([7, Thm. 8])**.** *Algorithm 7, BOREL$(l, n, \mathsf{HP}(t), s, p)$, returns the set of all saturated $p$-Borel fixed ideals in the polynomial ring $\mathcal{P}^{(l)}$ with Hilbert polynomial $\mathsf{HP}(t)$.*

### 4.2.1 An Efficient Algorithm to Compute Saturated $p$-Borel Fixed Ideals

As in Section 4.1.1 we will develop a datastructure which allow us to compute saturated $p$-Borel fixed ideals quickly. In Algorithms 6 and 7 we mainly work with the stable ideal $\mathcal{I}_{\geq s}$, and with its minimal generating set $\mathcal{I}_s$. In the optimized version we consider the Pommaret basis of $\mathcal{I}$. In Lemma 4.25 we have seen that we have to consider St-minimal elements of class $l$ which are $p$-minimal for $\mathcal{I}_s$. Lemma 4.20 shows the close connection

**Algorithm 7** BOREL($l, n, \mathsf{HP}(t), s, p$)

**Input:** $l$ first index of the variables in the polynomial ring
**Input:** $n$ last index of the variables of the polynomial ring
**Input:** $\mathsf{HP}(t)$ admissible Hilbert polynomial
**Input:** $s$ positive integer upper bounding the Gotzmann number of $\mathsf{HP}(t)$
**Input:** $p$ characteristic of the coefficient field
**Output:** $\mathcal{F}$, the set of all saturated $p$-Borel fixed ideals $\mathcal{J}$ in the polynomial ring $\mathcal{P}^{(l)}$ having Hilbert polynomial $\mathsf{HP}(t)$

1: **if** $\mathsf{HP}(t) = 0$ **then**
2:     **return** $\{\langle 1 \rangle\}$
3: **else**
4:     $\mathcal{E} \leftarrow$ BOREL($l+1, n, \Delta\mathsf{HP}(t), s, p$)
5:     $\mathcal{F} \leftarrow \emptyset$
6:     **for all** $\mathcal{J} \in \mathcal{E}$ **do**
7:         $\mathcal{I} \leftarrow \mathcal{J} \cdot \mathcal{P}^{(l)}$
8:         $q \leftarrow \mathsf{HP}(s) - \binom{n-l+s}{s} + \dim_{\Bbbk}(\mathcal{I}_s)$
9:         **if** $q \geq 0$ **then**
10:             $\mathcal{F} \leftarrow \mathcal{F} \cup p$-REMOVE($\mathcal{I}, l, n, s, q, p$)
11:         **end if**
12:     **end for**
13:     **return** $\mathcal{F}$
14: **end if**

between St-minimal elements of class $l$ and the minimal generating system of $\mathcal{I}$. Hence, it is enough to consider only the ideal $\mathcal{I}$ with its Pommaret basis. The reason why we consider the Pommaret basis of $\mathcal{I}$ and not only the minimal generating set of $\mathcal{I}$ is, that it is much easier to determine $\dim_{\Bbbk}(\mathcal{I}_s)$ from the Pommaret basis than from the minimal generating system of $\mathcal{I}$.

Adding a variable to the polynomial ring as in Algorithm 7, line (7) is also easy, because the Pommaret basis does not change, by Lemma 4.4.

In Algorithm 6 we remove a $p$-St-minimal element $x^\alpha$ from $\mathcal{I}_s \cap \mathbb{T}(l)$. In particular $x^\alpha$ is St-minimal, and we have already seen in Section 4.1.1 how to compute $\mathcal{J}$ such that $\mathcal{J}_{\geq s} = \langle (\mathcal{I}_s \cap \mathbb{T}(l)) \setminus \{x^\alpha\} \rangle$. If $x^\alpha$ is St-minimal, then there exists an $x^\beta \in \mathcal{B}^{\min}(\mathcal{I}) \subset \mathbf{P}(\mathcal{I})$, such that $x^\alpha = x^\beta x_l^{s-|\beta|}$ and $\deg(x^\beta) < s$. Then we have seen that $\mathcal{J} = \langle \mathcal{I}_s \setminus \{x^\alpha\} \rangle^{\mathrm{sat}}$ has as Pommaret basis

$$\mathcal{J} = \langle \mathcal{I}_s \setminus \{x^\alpha\} \rangle^{\mathrm{sat}} = \left\langle \mathcal{B}^{\min}(\mathcal{J}) \setminus \{x^\alpha\} \cup \{x^\alpha x_i \mid i \in \{l+1, \ldots, n\}\} \right\rangle.$$

The only problem left is to determine if $x^\alpha$ is $p$-minimal as well. It turns out that we do not have to consider $x^\alpha$, it is enough to consider only $x^\beta$. This is done in the next proposition which is in fact a reformulation of Lemma 4.25.

**Proposition 4.29.** *Let $\mathcal{I} \subseteq \mathcal{P}^{(l)}$ be a saturated p-Borel fixed ideal and $s \geq \operatorname{reg}(\mathcal{I})$. Furthermore, let $x^\alpha \in \mathcal{B}^{\min}(\mathcal{I})$ with $\deg(x^\alpha) \leq s$. Then*

$$\mathcal{J} = \left\langle \mathcal{B}^{\min}(\mathcal{I}) \setminus \{x^\alpha\} \cup \left\{ x^\alpha x_i \mid i \in \{l+1, \ldots, n\} \right\} \right\rangle$$

*is a saturated p-Borel fixed ideal with $\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{J}}(t) = \mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{I}}(t) + 1$ if and only if $x^\alpha$ is p-minimal with respect to the set $\mathcal{B}^{\min}(\mathcal{I})_{\deg(x^\alpha)}$.*

*Proof.* If $x^\alpha$ is not $p$-minimal with respect to the set $\mathcal{B}^{\min}(\mathcal{I})_{\deg(x^\alpha)}$ it is clear that $\mathcal{J}$ is not $p$-Borel fixed.

For the other direction we show first that $x^\alpha$ being $p$-minimal with respect to the set $\mathcal{B}^{\min}(\mathcal{I})_{\deg(x^\alpha)}$ implies that $x^\alpha$ is $p$-minimal with respect to $\mathcal{I}_{\deg(x^\alpha)} \cap \mathbb{T}(l)$. Suppose that this is not the case. Then there exists an $x^\beta \in \mathcal{B}^{\min}(\mathcal{I})_{\leq \deg(x^\alpha)}$ such that there is a chain of $p$-admissible moves from $x^\beta x_l^{\deg(x^\alpha)-\deg(x^\beta)}$ to $x^\alpha$. We can arrange the moves in such a way that every 0-admissible move $e_{i,j}^{+(s)}$ with $i \neq l$ is also applicable at $x^\beta$. But then we would have found an element $x^\gamma \in \mathcal{I}$ such that $x^\gamma x^\delta = x^\alpha$ which contradicts to $x^\alpha \in \mathcal{B}^{\min}(\mathcal{I})$.

As all $p$-admissible moves are a subset of the 0-admissible moves the statement holds for any suitable $p$. Using the same arguments above again, we can show that $x^\alpha x_l^i$ is $p$-minimal with respect to $\mathcal{I}_{\deg(x^\alpha)+i} \cap \mathbb{T}(l)$ for every $i$.

Consider now $\mathcal{J}_{\geq s}$. It is clear that

$$\mathcal{J}_s \cap \mathbb{T}(l) = \left(\mathcal{I}_s \cap \mathbb{T}(l)\right) \setminus \left\{ x^\alpha x_l^{s-\deg(x^\alpha)} \right\}$$

and $\mathcal{J}_{\geq s} = \langle \mathcal{J}_s \rangle$. This concludes the proof because

$$\mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{J}}(t) = \mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{J}_{\geq s}} = \mathsf{HP}_{\mathcal{P}^{(l)}/\mathcal{I}}(t) + 1\,,$$

by Lemma 4.25. □

Therefore, we have to check only if there is an element $x^\beta \in \mathcal{B}^{\min}(\mathcal{I})_{\deg(x^\alpha)}$ with $x^\beta \prec_p x^\alpha$. If this is not the case, $\mathcal{J}$ is $p$-Borel fixed and $x^\alpha$ is $p$-St-minimal which leads to Algorithm 8.

Algorithms 9 and 10 consider all the changes which we suggested above. In Algorithm 9, line (7) the same optimization as in Algorithm 1, line (7) is used, which avoids computing duplicate results.

**Algorithm 8** $p$-StMinimal($\mathbf{P}(\mathcal{I}), l, n, p, s$)

**Input:** $\mathbf{P}(\mathcal{I})$ Pommaret basis of a saturated $p$-Borel fixed ideal
**Input:** $l$ first index of the variables in the polynomial ring
**Input:** $n$ last index of the variables of the polynomial ring
**Input:** $p$ characteristic of the coefficient field
**Input:** $s$ positive integer upper bounding the Gotzmann number of $\mathsf{HP}(t)$
**Output:** $\mathcal{S}$, the set of terms such that for every term $x^\alpha \in \mathcal{S}$ the element $x^\alpha x_l^{s-\deg(x^\alpha)}$ is $p$-St-minimal element of $\mathcal{I}_s \cap \mathbb{T}(l)$.

$\mathcal{B}^{\min} \leftarrow \{x^\alpha \in \mathbf{P}(\mathcal{I}) \mid \nexists x^\beta \in \mathbf{P}(\mathcal{I}) \text{ s.t. } x^\beta | x^\alpha\}$
$\mathcal{S} \leftarrow \emptyset$
**for all** $x^\alpha \in \mathcal{B}^{\min}$ **do**
  **if** $\deg(x^\alpha) < s$ **then**
    $\mathcal{B}^{\min}_\alpha \leftarrow \{x^\beta \in \mathcal{B}^{\min} \mid \deg(x^\beta) = \deg(x^\alpha)\}$
    IsPMinimal $\leftarrow$ true
    **for all** $x^\beta \in \mathcal{B}^{\min}_\alpha$ **do**
      **if** $x^\beta \prec_p x^\alpha$ **then**
        IsPMinimal $\leftarrow$ false
      **end if**
    **end for**
    **if** IsPMinimal $=$ true **then**
      $\mathcal{S} \leftarrow \mathcal{S} \cup \{x^\alpha\}$
    **end if**
  **end if**
**end for**
**return** $\mathcal{S}$

**Example 4.30.** *In Example 4.22 we computed all saturated quasi-stable ideals in $\Bbbk[x_0, x_1, x_2]$ for $\mathsf{HP}(t) = 4$. We do not sketch the computation of $p$-Borel fixed ideals here, because the computation is very similar to the computation of saturated quasi-stable ideals and notable differences only occur if we do the computation in all detail.*

*Nevertheless, we want to take a look at all saturated $p$-Borel fixed ideals for $p = 0$ and $p = 2$. For $p = 0$ we get the following two ideals $\mathcal{J}_1, \mathcal{J}_2$ with*

$$\mathcal{B}^{\min}(\mathcal{J}_1) = \{x_1^3, x_2x_1, x_2^2\}\,,$$
$$\mathcal{B}^{\min}(\mathcal{J}_2) = \{x_1^4, x_2\}\,.$$

*For $p = 2$ we get additionally the ideal $\mathcal{J}_3$ with*

$$\mathcal{B}^{\min}(\mathcal{J}_3) = \{x_1^2, x_2^2\}\,.$$

**Algorithm 9** $p$-REMOVE($\mathbf{P}(\mathcal{I}), l, n, q, x^\beta, p$)

**Input:** $\mathbf{P}(\mathcal{I})$ Pommaret basis of a saturated $p$-Borel fixed ideal
**Input:** $l$ first index of the variables in the polynomial ring
**Input:** $n$ last index of the variables of the polynomial ring
**Input:** $q$ non-negative Integer
**Input:** $x^\beta$ monomial
**Input:** $p$ characteristic of the coefficient field
**Output:** $\mathcal{L}$ set of Pommaret bases of saturated $p$-Borel fixed ideals obtained by removing $q$ $p$-St-minimal terms divisible by $x_l$ from $\mathcal{I}_s$

```
ℒ ← ∅
if q = 0 then
    return P(ℐ)
else
    ℳ ← p-StMinimal(P(ℐ), l, n, p, s)
    for all x^α ∈ ℳ do
        if x^α x_l^{s−deg(x^α)} > x^β x_l^{s−deg(x^β)} then
            ℒ ← ℒ ∪
      p-Remove(P(ℐ) \ {x^α} ∪ {x^α x_i | i ∈ {l+1, ..., cls(x^α)}}, l, n, q−1, x^α, p)
        end if
    end for
end if
return ℒ
```

*We see that $\mathcal{J}_3$ is obviously not 0-Borel fixed because $x_1x_2$ does not belong to $\mathcal{J}_3$. If we compare the results with all saturated quasi-stable ideals with Hilbert polynomial $\mathsf{HP}(t) = 4$ from Example 4.22 we see that there are two quasi-stable ideals ($\mathcal{I}_1 = \langle x_1, x_2^4\rangle$ and $\mathcal{I}_2 = \langle x_1^2, x_2x_1, x_2^3\rangle$) which are neither 0-Borel fixed nor 2-Borel fixed. In fact, for both ideals there is no $p$ such that they are Borel fixed because $e^{+(1)}_{1,2}(x_1) = x_2 \notin \mathcal{I}_1$ and $e^{+(1)}_{1,2}(x_2x_1) = x_2^2 \notin \mathcal{I}_2$ and these are $p$-admissible moves for every possible $p$.*

Algorithm 9 and Algorithm 10 are implemented in the computer algebra system CoCoALib. As for the quasi-stable case we applied the implementation to some bigger examples. Table 4.3 exhibits the computation time of these examples and Table 4.4 shows the number of saturated quasi-stable ideals for the given input. The structure of the tables is similar to the tables in the quasi-stable case. But $p$-Borel fixedness is dependent on the characteristic. Hence, we divide every row, which correspond to an example with one specific Hilbert polynomial into four small rows. The upper row corresponds to computations of saturated 0-Borel fixed ideals, the second to computations of saturated 2-Borel fixed ideals, and so on.

**Algorithm 10** BOREL($l, n, \mathsf{HP}(t), s, p)$)

**Input:** $l$ first index of the variables in the polynomial ring
**Input:** $n$ last index of the variables of the polynomial ring
**Input:** $\mathsf{HP}(t)$ admissible Hilbert polynomial
**Input:** $s$ positive integer upper bounding the Gotzmann number of $\mathsf{HP}(t)$
**Input:** $p$ characteristic of the coefficient field
**Output:** $\mathcal{F}$, the set of Pommaret bases of $p$-Borel fixed ideals in the polynomial ring $\mathcal{P}^{(l)}$ having Hilbert polynomial $\mathsf{HP}(t)$

**if** $\mathsf{HP}(t) = 0$ **then**
    **return** $\{\{1\}\}$
**else**
    $\mathcal{E} \leftarrow$ BOREL$(l+1, n, \Delta\mathsf{HP}(t), s, p)$
    $\mathcal{F} \leftarrow \emptyset$
    **for all** $\mathbf{P}(\mathcal{J}) \in \mathcal{E}$ **do**
        $\mathbf{P}(\mathcal{I}) \leftarrow \mathbf{P}(\mathcal{J}) \cdot \mathcal{P}^{(l)}$
        $q \leftarrow \mathsf{HP}(s) - \binom{n-l+s}{s} + \dim_{\Bbbk}(\mathcal{I}_s)$
        **if** $q \geq 0$ **then**
            $\mathcal{F} \leftarrow \mathcal{F} \cup p$-REMOVE$(\mathbf{P}(\mathcal{I}), l, n, q, 1, p)$
        **end if**
    **end for**
    **return** $\mathcal{F}$
**end if**

In contrast to the structure of Table 4.1 and 4.2 we introduced a third column where we denote the number $p$. That is, if $p = 0$ we refer to 0-Borel fixed ideals.

Again we used for the computation a computer with Intel i7-5500U processor and 8 GB main memory. The operating system was Ubuntu 16.04.1 LTS. We compiled CoCoALib with gcc 5.4.0. We restricted our program to use only 7.8 GB main memory.

The Hilbert polynomials which we consider in these examples are a superset of the Hilbert polynomials considered in Table 4.1 and Table 4.2. But instead of considering the examples only in polynomial rings $\Bbbk[x_0, \ldots, x_n]$ with $n \in \{3, 4, 5, 6, 7, 8\}$ we now consider polynomial rings with $n \in \{5, 10, 15, 20\}$. The new Hilbert polynomials have in general greater Gotzmann numbers.

Despite the fact that we consider larger examples the computations times are much better. Every example which we compute here finished within 20 seconds. The reason for that is, that there are many more saturated quasi-stable ideals than $p$-Borel fixed ideals. For example, take the Hilbert polynomial $4t^2 - 12t + 10$ and $n = 5$. There are 204 855 different quasi-stable ideals but only 1 135 2-Borel fixed, which is only about one twentieth. The smaller result implies that we have to consider far fewer sets during the computation, and this explains why these algorithms are much faster than the algorithms for computing saturated quasi-stable ideals.

For every example we computed once all saturated $p$-Borel fixed ideals for the primes $p \in \{0, 2, 5, 19\}$. We see in Table 4.4 that usually there are many more 2-Borel fixed ideals than 0-Borel fixed ideals. In our examples there are at most three times as much. If we now compare for a specific example all saturated $p$-Borel fixed ideals with $p \in \{2, 5, 19\}$ we see that the number of ideals quickly descends to the number of 0-Borel fixed ideals as the prime $p$ increases. In fact for $p = 19$ we get always the same result as for $p = 0$ in our set of examples.

This means for the computation of Hilbert schemes that the computation over fields with small finite characteristic is more costly because we need usually a larger covering. But it is possible that we can save the extra costs because we work over a finite field, so the arithmetic is usually faster. For fields with larger finite characteristic the result is almost the same.

| $\mathsf{HP}(t)$ | $G$ | $p$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ |
|---|---|---|---|---|---|---|
| 5 | 5 | 0 | 0.000 | 0.000 | 0.004 | 0.004 |
| | | 2 | 0.000 | 0.000 | 0.000 | 0.000 |
| | | 5 | 0.000 | 0.000 | 0.004 | 0.004 |
| | | 19 | 0.000 | 0.000 | 0.000 | 0.000 |
| 10 | 10 | 0 | 0.000 | 0.004 | 0.004 | 0.008 |
| | | 2 | 0.004 | 0.004 | 0.004 | 0.012 |
| | | 5 | 0.004 | 0.004 | 0.004 | 0.008 |
| | | 19 | 0.000 | 0.004 | 0.004 | 0.008 |
| 15 | 15 | 0 | 0.008 | 0.028 | 0.044 | 0.060 |
| | | 2 | 0.012 | 0.052 | 0.084 | 0.116 |
| | | 5 | 0.008 | 0.028 | 0.048 | 0.060 |
| | | 19 | 0.008 | 0.028 | 0.048 | 0.064 |
| 20 | 20 | 0 | 0.052 | 0.212 | 0.372 | 0.480 |
| | | 2 | 0.120 | 0.488 | 0.844 | 1.080 |
| | | 5 | 0.052 | 0.216 | 0.388 | 0.484 |
| | | 19 | 0.056 | 0.212 | 0.392 | 0.504 |
| 25 | 25 | 0 | 0.320 | 1.520 | 2.540 | 3.388 |
| | | 2 | 0.856 | 4.264 | 6.956 | 9.192 |
| | | 5 | 0.320 | 1.536 | 2.580 | 3.492 |
| | | 19 | 0.360 | 1.612 | 2.636 | 3.580 |

| HP($t$) | $G$ | $p$ | $n = 5$ | $n = 10$ | $n = 15$ | $n = 20$ |
|---|---|---|---|---|---|---|
| $5t + 1$ | 11 | 0 | 0.000 | 0.004 | 0.004 | 0.008 |
| | | 2 | 0.004 | 0.000 | 0.008 | 0.008 |
| | | 5 | 0.004 | 0.000 | 0.008 | 0.008 |
| | | 19 | 0.004 | 0.004 | 0.004 | 0.008 |
| $5t + 7$ | 17 | 0 | 0.052 | 0.160 | 0.236 | 0.288 |
| | | 2 | 0.076 | 0.228 | 0.312 | 0.388 |
| | | 5 | 0.052 | 0.164 | 0.232 | 0.296 |
| | | 19 | 0.052 | 0.164 | 0.232 | 0.288 |
| $5t + 13$ | 23 | 0 | 1.296 | 4.988 | 7.408 | 8.988 |
| | | 2 | 2.600 | 9.220 | 13.068 | 16.364 |
| | | 5 | 1.292 | 4.996 | 7.416 | 9.128 |
| | | 19 | 1.316 | 4.928 | 7.288 | 9.100 |
| $8t - 6$ | 22 | 0 | 0.076 | 0.244 | 0.356 | 0.456 |
| | | 2 | 0.116 | 0.336 | 0.476 | 0.596 |
| | | 5 | 0.080 | 0.248 | 0.368 | 0.448 |
| | | 19 | 0.076 | 0.244 | 0.372 | 0.448 |
| $8t - 3$ | 25 | 0 | 0.368 | 1.228 | 1.792 | 2.228 |
| | | 2 | 0.620 | 1.916 | 2.708 | 3.360 |
| | | 5 | 0.368 | 1.224 | 1.804 | 2.252 |
| | | 19 | 0.372 | 1.236 | 1.808 | 2.260 |
| $8t$ | 28 | 0 | 1.580 | 5.928 | 8.732 | 10.688 |
| | | 2 | 3.036 | 10.616 | 15.208 | 18.556 |
| | | 5 | 1.560 | 5.892 | 8.800 | 10.788 |
| | | 19 | 1.600 | 6.068 | 8.752 | 11.048 |
| $2t^2 + 8t - 46$ | 16 | 0 | 0.008 | 0.028 | 0.036 | 0.048 |
| | | 2 | 0.016 | 0.040 | 0.052 | 0.072 |
| | | 5 | 0.008 | 0.032 | 0.036 | 0.052 |
| | | 19 | 0.008 | 0.028 | 0.040 | 0.048 |
| $2t^2 + 8t - 42$ | 20 | 0 | 0.016 | 0.044 | 0.060 | 0.080 |
| | | 2 | 0.024 | 0.064 | 0.080 | 0.104 |
| | | 5 | 0.016 | 0.044 | 0.064 | 0.080 |
| | | 19 | 0.016 | 0.048 | 0.060 | 0.080 |
| $2t^2 + 8t - 38$ | 24 | 0 | 0.088 | 0.280 | 0.392 | 0.488 |
| | | 2 | 0.124 | 0.372 | 0.512 | 0.624 |
| | | 5 | 0.092 | 0.288 | 0.396 | 0.492 |
| | | 19 | 0.092 | 0.284 | 0.396 | 0.608 |

| $\mathrm{HP}(t)$ | $G$ | $p$ | $n=5$ | $n=10$ | $n=15$ | $n=20$ |
|---|---|---|---|---|---|---|
| $4t^2-12t+10$ | 20 | 0 | 0.012 | 0.024 | 0.032 | 0.048 |
| | | 2 | 0.016 | 0.032 | 0.044 | 0.056 |
| | | 5 | 0.012 | 0.024 | 0.036 | 0.048 |
| | | 19 | 0.012 | 0.024 | 0.036 | 0.048 |
| $4t^2-12t+14$ | 24 | 0 | 0.112 | 0.344 | 0.464 | 0.576 |
| | | 2 | 0.176 | 0.496 | 0.672 | 0.848 |
| | | 5 | 0.116 | 0.336 | 0.480 | 0.584 |
| | | 19 | 0.112 | 0.340 | 0.472 | 0.588 |
| $4t^2-12t+18$ | 28 | 0 | 1.088 | 3.976 | 5.480 | 6.888 |
| | | 2 | 1.996 | 6.692 | 9.068 | 11.436 |
| | | 5 | 1.072 | 3.964 | 5.552 | 6.964 |
| | | 19 | 1.092 | 4.024 | 5.604 | 6.996 |

Table 4.3: Time for computing all saturated $p$-Borel fixed ideals for a given Hilbert polynomial in a polynomial ring $\Bbbk[x_0,\dots,x_n]$ and a given characteristic of $\Bbbk$

| $\mathrm{HP}(t)$ | $G$ | $p$ | $n=5$ | $n=10$ | $n=15$ | $n=20$ |
|---|---|---|---|---|---|---|
| 5 | 5 | 0 | 5 | 5 | 5 | 5 |
| | | 2 | 6 | 6 | 6 | 6 |
| | | 5 | 5 | 5 | 5 | 5 |
| | | 19 | 5 | 5 | 5 | 5 |
| 10 | 10 | 0 | 42 | 50 | 50 | 50 |
| | | 2 | 63 | 75 | 75 | 75 |
| | | 5 | 43 | 51 | 51 | 51 |
| | | 19 | 42 | 50 | 50 | 50 |
| 15 | 15 | 0 | 287 | 417 | 425 | 425 |
| | | 2 | 591 | 851 | 863 | 863 |
| | | 5 | 312 | 445 | 453 | 453 |
| | | 19 | 287 | 417 | 425 | 425 |
| 20 | 20 | 0 | 1 732 | 3 130 | 3 263 | 3 271 |
| | | 2 | 4 303 | 8 073 | 8 342 | 8 354 |
| | | 5 | 2 017 | 3 540 | 3 676 | 3 684 |
| | | 19 | 1 732 | 3 130 | 3 263 | 3 271 |
| 25 | 25 | 0 | 9 501 | 21 616 | 23 158 | 23 291 |
| | | 2 | 27 795 | 67 600 | 71 817 | 72 086 |
| | | 5 | 11 696 | 25 801 | 27 471 | 27 607 |
| | | 19 | 9 501 | 21 616 | 23 158 | 23 291 |

| HP$(t)$ | $G$ | $p$ | $n=5$ | $n=10$ | $n=15$ | $n=20$ |
|---|---|---|---|---|---|---|
| $5t+1$ | 11 | 0 | 89 | 98 | 98 | 98 |
| | | 2 | 101 | 110 | 110 | 110 |
| | | 5 | 89 | 98 | 98 | 98 |
| | | 19 | 89 | 98 | 98 | 98 |
| $5t+7$ | 17 | 0 | 3 028 | 4 560 | 4 587 | 4 587 |
| | | 2 | 4 541 | 6 648 | 6 677 | 6 677 |
| | | 5 | 3 043 | 4 575 | 4 602 | 4 602 |
| | | 19 | 3 028 | 4 560 | 4 587 | 4 587 |
| $5t+13$ | 23 | 0 | 58 124 | 123 689 | 126 962 | 127 030 |
| | | 2 | 121 343 | 245 079 | 249 804 | 249 882 |
| | | 5 | 59 550 | 125 513 | 128 787 | 128 855 |
| | | 19 | 58 124 | 123 689 | 126 962 | 127 030 |
| $8t-6$ | 22 | 0 | 4 171 | 6 741 | 6 837 | 6 838 |
| | | 2 | 6 438 | 9 956 | 10 066 | 10 067 |
| | | 5 | 4 248 | 6 824 | 6 920 | 6 921 |
| | | 19 | 4 171 | 6 741 | 6 837 | 6 838 |
| $8t-3$ | 25 | 0 | 17 334 | 32 073 | 32 848 | 32 868 |
| | | 2 | 30 835 | 54 024 | 55 003 | 55 024 |
| | | 5 | 17 932 | 32 817 | 33 592 | 33 612 |
| | | 19 | 17 334 | 32 073 | 32 848 | 32 868 |
| $8t$ | 28 | 0 | 68 291 | 144 660 | 149 777 | 149 976 |
| | | 2 | 137 987 | 277 237 | 284 502 | 284 737 |
| | | 5 | 71 777 | 149 758 | 154 895 | 155 094 |
| | | 19 | 68 291 | 144 660 | 149 777 | 149 976 |
| $2t^2+8t-46$ | 16 | 0 | 34 | 38 | 38 | 38 |
| | | 2 | 37 | 41 | 41 | 41 |
| | | 5 | 34 | 38 | 38 | 38 |
| | | 19 | 34 | 38 | 38 | 38 |
| $2t^2+8t-42$ | 20 | 0 | 481 | 670 | 671 | 671 |
| | | 2 | 593 | 806 | 807 | 807 |
| | | 5 | 481 | 670 | 671 | 671 |
| | | 19 | 481 | 670 | 671 | 671 |
| $2t^2+8t-38$ | 24 | 0 | 4 774 | 8 393 | 8 476 | 8 476 |
| | | 2 | 6 738 | 11 346 | 11 436 | 11 436 |
| | | 5 | 4 808 | 8 428 | 8 511 | 8 511 |
| | | 19 | 4 774 | 8 393 | 8 476 | 8 476 |

| HP($t$) | $G$ | $p$ | $n=5$ | $n=10$ | $n=15$ | $n=20$ |
|---|---|---|---|---|---|---|
| $4t^2-12t+10$ | 20 | 0 | 631 | 856 | 857 | 857 |
| | | 2 | 861 | 1 135 | 1 136 | 1 136 |
| | | 5 | 631 | 856 | 857 | 857 |
| | | 19 | 631 | 856 | 857 | 857 |
| $4t^2-12t+14$ | 24 | 0 | 6 394 | 10 986 | 11 082 | 11 082 |
| | | 2 | 10 087 | 16 511 | 16 623 | 16 623 |
| | | 5 | 6 430 | 11 023 | 11 119 | 11 119 |
| | | 19 | 6 394 | 10 986 | 11 082 | 11 082 |
| $4t^2-12t+18$ | 28 | 0 | 51 527 | 112 852 | 115 295 | 115 332 |
| | | 2 | 96 668 | 196 002 | 199 285 | 199 326 |
| | | 5 | 52 606 | 114 265 | 116 708 | 116 745 |
| | | 19 | 51 527 | 112 852 | 115 295 | 115 332 |

Table 4.4: Number of all saturated $p$-Borel fixed ideals for a given Hilbert polynomial in a polynomial ring $\Bbbk[x_0,\ldots,x_n]$ and a given characteristic of $\Bbbk$

## 4.3 Computation of Saturated Monomial Modules in Certain Stability Positions

In this section we determine how to compute all saturated monomial modules in a specific stability position for a given Hilbert polynomial. It turns out that the main point in the development of an algorithm is the fact that one can identify a monomial module $\mathcal{V} \subseteq \mathcal{P}^m_{\mathbf{d}}$ with $\mathcal{V} = \bigoplus_{i=1}^m \mathcal{J}_i \mathbf{e}_i$, where $\mathcal{J}_i$ are monomial ideals in $\mathcal{P}$.

Let us assume that the Hilbert polynomial of $\mathcal{V}$ is $\mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{V}}(t) \in \mathbb{Q}[t]$. Then we have seen in Section 2.3 that

$$\mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{V}}(t) = \mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{I}_1\cdot\mathbf{e}_1} + \cdots + \mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{I}_m\cdot\mathbf{e}_m}.$$

The Hilbert polynomial of $\mathcal{I}_i$ is

$$\mathsf{HP}_{\mathcal{P}/\mathcal{I}_i}(t) = \mathsf{HP}_{\mathcal{P}^m_{\mathbf{d}}/\mathcal{I}_i\cdot\mathbf{e}_i}(t+\deg(\mathbf{e}_i)) \tag{4.1}$$

for $i \in \{1,\ldots,m\}$.

This observation suggests a way of computing to compute all saturated monomial modules in a certain stability position for a given Hilbert polynomial. We first divide the given admissible Hilbert polynomial into $m$ admissible Hilbert polynomials for the modules $\mathcal{I}_i \cdot \mathbf{e}_i$ such that the sum of the $m$ polynomials is equal to the original polynomial. The polynomial corresponding to $\mathcal{I}_i \cdot \mathbf{e}_i$ must take into account the degree of

$\mathbf{e}_i$. Hence we shift the polynomial as in (4.1) to get a polynomial corresponding to $\mathcal{I}_i$. If this polynomial is an admissible Hilbert polynomial it is possible to compute all saturated monomial ideals which are quasi-stable (Algorithm 5) or $p$-Borel fixed fixed (Algorithm 10).

Hence the main purpose of this section is to develop an algorithm which can divide a given admissible Hilbert polynomial $\mathsf{HP}(t) \in \mathbb{Q}[t]$ into $m$ admissible polynomials $\mathsf{HP}_1(t), \ldots, \mathsf{HP}_m(t) \in \mathbb{Q}[t]$, that is into polynomials which have a Gotzmann representation, such that $\mathsf{HP}(t) = \sum_{i=1}^{m} \mathsf{HP}_i(t)$.

Assume that we have polynomials as in the paragraph before. Furthermore we assume that the Gotzmann coefficients of $\mathsf{HP}(t)$ are $(a_1, \ldots, a_s)$ and that the Gotzmann coefficients of $\mathsf{HP}_i(t)$ are $(a_1^{(i)}, \ldots, a_{s_i}^{(i)})$ for every $i \in \{1, \ldots, m\}$. Then the first observation which we make is that the Gotzmann coefficients $a_i^{(j)}$ are bounded by $a_1$:

**Lemma 4.31.** *The Gotzmann coefficient $a_1^{(i)}$ is smaller than or equal to $a_1$ for every $i \in \{1, \ldots, m\}$.*

*Proof.* Assume that this is not the case. Then choose $i$, such that $a_1^{(i)}$ is maximal. Then $\deg(\mathsf{HP}_i(t)) = a_1^{(i)} > a_1$ and $\mathrm{lc}(\mathsf{HP}_i(t)) > 0$. Since the leading coefficients for all indices $j$ such that $\deg(\mathsf{HP}_j(t)) = a_1^{(i)}$ are positive we get that

$$\deg(\mathsf{HP}(t)) = \deg(\sum_{k=1}^{m} \mathsf{HP}_k(t)) = a_1^{(i)} > a_1 \,.$$

But this is a contradiction. □

For a given polynomial $p$ Algorithm 11 computes all possible tuples of admissible polynomials $(p_1, \ldots, p_m)$ whose sum is exactly $p$.

The idea of the algorithm is to compute for a given admissible polynomial $p$ firstly the number of Gotzmann coefficients with the highest value among all Gotzmann coefficients. Among all Gotzmann coefficients of the polynomials $(p_1, \ldots, p_m)$, there must be exactly the same number of Gotzmann coefficients with this value. Then we subtract the polynomials $p_1, \ldots, p_m$ from $p$ and obtain a lower degree polynomial. Then we call the recursively the algorithm but now we add the computed polynomials $p'_1, \ldots, p'_m$ to the already obtained polynomials $p_1, \ldots, p_m$. For the computation of $p'_1, \ldots, p'_m$ we take into account that these Gotzmann coefficients are not the highest Gotzmann coefficients which leads to different Gotzmann summands. We call the algorithm recursively until we reach as input polynomial 0.

**Algorithm 11** DistributePolynomial$((p, (\mathcal{G}_1, \ldots, \mathcal{G}_m), m))$

**Input:** $p \in \mathbb{Q}[t]$ such that $\mathrm{lc}(p) \cdot (\deg(p))! \in \mathbb{N}$
**Input:** $(\mathcal{G}_1, \ldots, \mathcal{G}_m)$ sequence of sequences of natural numbers
**Input:** $m$ positive integer which corresponds to the rank of $\mathcal{P}^m_{\mathbf{d}}$
**Output:** $\mathcal{M}$, the set of module elements of $\mathbb{Q}[t]^m$

1: $\mathcal{M} \leftarrow \emptyset$
2: **if** $p = 0$ **then**
3:     **return** $\{$GotzmannCoeffToPol$(m, (\mathcal{G}_1, \ldots, \mathcal{G}_m))\}$
4: **end if**
5: $d \leftarrow \deg(p)$
6: $c \leftarrow \mathrm{lc}(p) \cdot d!$
7: **for all** $(\mathcal{S}_1, \ldots \mathcal{S}_m) \in$ Multiset(d,c,m) **do**
8:     $q \leftarrow p$
9:     **for all** $j = 1$ to $m$ **do**
10:         $a \leftarrow |\mathcal{G}_j|$
11:         **for** $k = 1$ to $|\mathcal{S}_j|$ **do**
12:             $q \leftarrow q - \binom{t+d-k-a+1}{d}$
13:         **end for**
14:         $\mathcal{G}_j \leftarrow \mathcal{G}_j + \mathcal{S}_j$
15:     **end for**
16:     **if** $\mathrm{lc}(q) \cdot (\deg(q))! \notin \mathbb{N}$ **then**
17:         **continue**
18:     **end if**
19:     $\mathcal{M} \leftarrow \mathcal{M} \cup$ DistributePolynomial$(q, (\mathcal{G}_1, \ldots \mathcal{G}_m), m)$
20: **end for**
21: **return** $\mathcal{M}$

In line (3) the algorithm calls the method GotzmannCoeffToPol$(m, (\mathcal{G}_1, \ldots, \mathcal{G}_m))$. This method assumes that $\mathcal{G}_i$ is a sequence of Gotzmann coefficients and it computes the corresponding polynomial $p_i(t) \in \mathbb{Q}[t]$. It returns the sequence of polynomials $(p_1(t), \ldots, p_m(t))$.

In line (7) the algorithm calls the method Multiset$(d, c, m)$. This method computes all $(\mathcal{S}_1, \ldots, \mathcal{S}_m)$ such that there are exactly $c$ copies of $d$ among all sequences $\mathcal{S}_i$.

The sum operation in line (14) is meant as the concatenation of two sequences of natural numbers.

**Theorem 4.32.** *Let $p \in \mathbb{Q}[t]$ such that there exists a Gotzmann representation for this polynomial and $m \in \mathbb{N} \setminus \{0\}$. Then Algorithm 11 returns all sequences of polynomials $(p_1, \ldots, p_m)$, such that $p = \sum_{i=1}^m p_i$ and there exists a Gotzmann representation for every $p_i$.*

*Proof.* First of all we note that Algorithm 11 terminates. Let $p$ be the input of Algorithm 11 and $q$ the result in line (19). We define $p_{\text{supp}} := p - \text{lc}(p) \cdot \text{lt}(p)$ and $d = \deg(p)$. Then

$$\begin{aligned} q &= p - \sum_{i=1}^{lc(p)\cdot d!} \binom{t+d-a_i}{d} \\ &= p - \sum_{i=1}^{lc(p)\cdot d!} \frac{(t+d-a_i)\cdot \dots \cdot (t-a_i+1)}{d!} \\ &= \text{lc}(p)p^d + p_{\text{supp}} - \sum_{i=1}^{lc(p)\cdot d!} \frac{t^d}{d!} - \sum_{i=1}^{lc(p)\cdot d!} \frac{q_i}{d!} \\ &= p_{\text{supp}} - \sum_{i=1}^{lc(p)\cdot d!} \frac{q_i}{d!}, \end{aligned}$$

where $q_i = (t+d-a_i)\cdot \dots \cdot (t-a_i+1) - t^d$. It is obvious that the degree of $q$ is less then the degree of $p$ and that the leading term of $p$ vanishes during the algorithm which proves termination.

If $a$ is a Gotzmann coefficient, then note that the degree of the corresponding Gotzmann summand is always $a$ and the leading coefficient of the Gotzmann summand is $\frac{1}{a!}$, no matter at which position this Gotzmann coefficient appears in the Gotzmann representation.

Now we come to the proof of the correctness of Algorithm 11. It is clear that every sequence of polynomials which are in the output of Algorithm 11 satisfy the demanded properties. Hence we only have to show, that the algorithm provides all possible sequences of polynomials which satisfy there properties.

Let $p \in \mathbb{Q}[t]$ be a polynomial which has a Gotzmann representation. Furthermore let $(p_1, \dots, p_m)$ be a sequence of rational polynomials such that there exists a Gotzmann representation for every polynomial and $p = \sum_{i=1}^{m} p_i$. Let $(\mathcal{S}_1, \dots, \mathcal{S}_m)$ be the corresponding Gotzmann coefficients of $(p_1, \dots, p_m)$. From the observation above their are $\text{lc}(p) \cdot (\deg(p))!$ Gotzmann coefficients with value $\deg(p)$. We distribute them over the sequences $\mathcal{S}'_1, \dots, \mathcal{S}'_m$ so that in all the $\mathcal{S}'_i$ together there are exactly the same number of Gotzmann coefficients with value $\deg(p)$ as in $\mathcal{S}_1$. Let $p'_i$ the result of subtracting from $p_i$ all Gotzmann summands with Gotzmann coefficient $\deg(p)$. It is obvious that the sum over all $p'_i$ is equal to $q$ at line (19). Now we call the algorithm again and we add to the sequences of Gotzmann coefficients $\mathcal{S}'_1, \dots, \mathcal{S}'_m$ the Gotzmann coefficients $\deg(q)$. Let $q'$ be the polynomial which we gave as argument in line (19). It is again obvious that $q'$ is equal to the sum of all $p_i$ after subtracting all Gotzmann summands with Gotzmann coefficients $\deg(p)$ and $\deg(q)$.

We proceed recursively. At the last recursion we have as input $p = 0$ and $(\mathcal{S}_1, \ldots, \mathcal{S}_m)$ which obviously leads to the polynomials $(p_1, \ldots, p_m)$. $\square$

**Example 4.33.** *Let $p = 2t + 3 \in \mathbb{Q}[t]$. Then Algorithm 11 provides eight different results:*

$$\begin{aligned}
(2t+3, 0) &\quad \textit{with Gotzmann coefficients} \quad ((1,1,0,0),())\,,\\
(2t+2, 1) &\quad \textit{with Gotzmann coefficients} \quad ((1,1,0),(0))\,,\\
(2t+1, 2) &\quad \textit{with Gotzmann coefficients} \quad ((1,1),(0,0))\,,\\
(t+2, t+1) &\quad \textit{with Gotzmann coefficients} \quad ((1,0),(1))\,,\\
(t+1, t+2) &\quad \textit{with Gotzmann coefficients} \quad ((1),(1,0))\,,\\
(2, 2t+1) &\quad \textit{with Gotzmann coefficients} \quad ((0,0),(1,1))\,,\\
(1, 2t+2) &\quad \textit{with Gotzmann coefficients} \quad ((0),(1,1,0))\,,\\
(0, 2t+3) &\quad \textit{with Gotzmann coefficients} \quad ((),(1,1,0,0))\,.
\end{aligned}$$

*This example shows, that the Gotzmann coefficients of p are not simply the collected Gotzmann coefficients of the polynomials $(p_1, p_2)$, because the Gotzmann coefficients of p are $(1, 1, 0, 0)$.*

With Algorithm 11 we are now able to define an algorithm to compute all saturated monomial modules which satisfy a certain stability condition.

The method StabilityAlgorithm in Algorithm 12 at line (9) computes all ideals in $\mathcal{P}$ which satisfy the stability condition stated in the variable StabilityPosition. For us the interesting stability position are quasi-stability and $p$-Borel fixed fixedness. Then we call the corresponding algorithms which we have developed before.

The correctness and the termination of Algorithm 12 is obvious for the stability positions quasi-stable and $p$-Borel fixed. For other stability conditions we first need to develop the corresponding algorithm in the ideal case and we need to clarify if the stability positions of $\mathcal{J}_1, \ldots, \mathcal{J}_m$ have as a consequence the stability position of the module $\mathcal{V} = \bigoplus_{i=1}^m \mathcal{J}_i \mathbf{e}_i$

**Example 4.34.** *We consider $\mathcal{P}^2_{\mathbf{d}}$ with $\mathbf{d} = (0, -1)$ and $\mathcal{P} = \Bbbk[x_0, x_1, x_2, x_3]$. We want to compute all saturated 0-Borel monomial modules for the Hilbert polynomial $\mathsf{HP}(t) = 2t + 3$. In Example 4.33 we have already computed all pairs $(p_1, p_2)$. We sketch now the rest of Algorithm 12 for the pairs $(2t + 1, 2)$ and $(t + 2, t + 1)$.*

*Let us begin with $(t + 2, t + 1)$. First we have to compute the shifting in the second component. This leads to computing all quasi-stable ideals in $\mathcal{P}$ for $p_1 = t + 2$ and $p_2 = t$. But it turns out that for $p_2$ there exists no quasi-stable ideal with $p_2$ as Hilbert polynomial, because $p_2$ does not have a Gotzmann representation. Hence there are no monomial modules $\mathcal{J}_1\mathbf{e}_1 \oplus \mathcal{J}_2\mathbf{e}_2$ such that $\mathsf{HP}_{\mathcal{P}/\mathcal{J}_1}(t) = p_1$ and $\mathsf{HP}_{\mathcal{P}/\mathcal{J}_2}(t) = p_2$.*

**Algorithm 12** MONOMIALMODULES(HP, $\mathcal{P}^m_{\mathbf{d}}$, STABILITYPOSITION)

**Input:** $\mathsf{HP} \in \mathbb{Q}[t]$ admissible Hilbert polynomial
**Input:** $\mathcal{P}^m_{\mathbf{d}}$, a graded free module with $\mathbf{d} = (d_1, \dots, d_m)$, such that $d_i \leq 0$ for all $1 \leq i \leq m$
**Input:** STABILITYPOSITION, a flag which indicates which stability condition the saturated monomial modules should fulfil
**Output:** $\mathcal{M}$, the set of saturated monomial modules which fulfil the stability condition STABILITYPOSITION

$\mathcal{M} \leftarrow \emptyset$
**for all** $(p_1(t), \dots, p_m(t)) \in$ DISTRIBUTEPOLYNOMIAL($\mathsf{HP}(t), ((), \dots, ()), m$) **do**
  $\mathcal{M}_1, \dots, \mathcal{M}_m \leftarrow \emptyset$
  **for** $i = 1$ to $m$ **do**
    $p_i(t) \leftarrow p_i(t + \deg(\mathbf{e}_i))$
    **if** $p_i(t)$ is not an admissible Hilbert polynomial **then**
      **break**
    **end if**
    $\mathcal{M}_i \leftarrow$ STABILITYALGORITHM(STABILITYPOSITION, $p_i(t), \mathcal{P}^m_{\mathbf{d}}$)
  **end for**
  **for all** $\mathcal{J}_1 \in \mathcal{M}_i, \dots, \mathcal{J}_m \in \mathcal{M}_m$ **do**
    $\mathcal{M} \leftarrow \mathcal{M} \cup \{\mathcal{J}_1\mathbf{e}_1 \oplus \cdots \oplus \mathcal{J}_m\mathbf{e}_m\}$
  **end for**
**end for**
**return** $\mathcal{M}$

*For $(2t+1, 2)$ we begin again with the shifting in the second component. Therefore it turns out, that we have to compute all saturated 0-Borel ideals in $\mathcal{P}$ for $p_1 = 2t+1$ and $p_2 = 1$. For $p_1$ there are three different saturated 0-Borel ideals in $\mathcal{P}$:*

$$\mathcal{J}_1^{(1)} = \langle x_3^2, x_3x_2, x_3x_1, x_2^3, x_2^2x_1 \rangle\,,$$
$$\mathcal{J}_1^{(2)} = \langle x_2^2, x_3^2, x_3x_2, x_3x_1^2 \rangle\,,$$
$$\mathcal{J}_1^{(3)} = \langle x_3, x_2^3, x_2^2x_1^2 \rangle\,.$$

*For $p_2 = 1$, there is only one 0-Borel ideal in $\mathcal{P}$:*

$$\mathcal{J}_2 = \langle x_3, x_2, x_1 \rangle\,.$$

*Hence we get three different saturated monomial modules which are 0-Borel with Hilbert polynomial $\mathsf{HP}(t)$.*

$$\mathcal{V}_1 = \mathcal{J}_1^{(1)}\mathbf{e}_1 \oplus \mathcal{J}_2\mathbf{e}_2\,, \qquad \mathcal{V}_2 = \mathcal{J}_1^{(2)}\mathbf{e}_1 \oplus \mathcal{J}_2\mathbf{e}_2\,, \qquad \mathcal{V}_3 = \mathcal{J}_1^{(3)}\mathbf{e}_1 \oplus \mathcal{J}_2\mathbf{e}_2\,.$$

*For the original problem we get a total of eight different saturated monomial modules.*

## 4.4 Computation of Marked Families

In order to compute an open covering for Quot schemes we have seen in the previous section how to compute all saturated quasi-stable modules for a given Hilbert polynomial. If we consider the truncation of these modules in degree equal to the regularity or higher and compute the corresponding marked families we get an open covering of the Quot scheme as we have seen in Theorem 3.46 and Theorem 3.50.

### 4.4.1 Computing Marked Families Using Superminimal Generators

In Section 3.3.2 we have seen a first method to compute a marked family based on Theorem 1.86. There we embed the marked scheme for a quasi-stable module $\mathcal{V}$ with $\mathbf{P}(\mathcal{V}) \subseteq \mathcal{V}_r$ in an affine space of dimension $|\mathbf{P}(\mathcal{V})| \cdot |\mathcal{N}(\mathcal{V})_r|$. We will see that we can eliminate a significant number of variables of the equations of the marked scheme which allows us to embed the marked scheme of $\mathcal{V}$ in an affine space of much lower dimension than $|\mathbf{P}(\mathcal{V})| \cdot |\mathcal{N}(\mathcal{V})_r|$. Usually the new equations are of higher degree, but from a computational point of view fewer variables are often better. The ideas used here are based of [8]. There they consider the marked schemes only for ideals which are strongly stable. Now, we extend this ideas to marked schemes over quasi-stable modules.

For simplicity, we consider for the rest of this chapter only the standard grading $\mathbf{d} = (0, \dots, 0)$ for computing the marked schemes. Nevertheless, it is simple to adept the techniques to arbitrary gradings. In this section we always consider a polynomial module $A[\mathbf{x}]^m$, where $A$ is a $\Bbbk$-algebra. Furthermore, we consider $t$-truncated modules $\mathcal{V} \subseteq A[\mathbf{x}]^m$ such that $\mathcal{V} = \mathcal{V}^{\text{sat}}_{\geq t}$. It is obvious that we have in this case $\mathbf{P}(\mathcal{V})_{\geq t} \subseteq \mathbf{P}(\mathcal{V}^{\text{sat}})$. For the following we define $\underline{\mathcal{V}} := \mathcal{V}^{\text{sat}}$. Instead of writing $\mathbf{Mf}^{n,m}_{\mathbf{P}(\mathcal{V})}(A)$ when we consider the marked scheme over the quasi-stable module $\mathcal{V} \subseteq A[\mathbf{x}]^m$ we will write for simplicity for the rest of this chapter always $\mathbf{Mf}(\mathcal{V}) := \mathbf{Mf}^{n,m}_{\mathbf{P}(\mathcal{V})}(A)$.

At first, we state some basic facts we need in the following.

**Lemma 4.35.** *Let $\mathcal{V}$ be a quasi-stable $t$-truncation. Then:*

*(i)* $\mathbf{P}(\mathcal{V}) \cap \mathbf{P}(\underline{\mathcal{V}}) = \mathbf{P}(\underline{\mathcal{V}})_{\geq t}$

*(ii) Let $x^\beta \mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}}) \setminus \mathbf{P}(\mathcal{V})$, then $x_0^{t-\deg(x^\beta \mathbf{e}_k)} x^\beta \mathbf{e}_k \in \mathbf{P}(\mathcal{V})$*

*(iii) Let $x^\gamma \mathbf{e}_k \in A[\mathbf{x}]^m_{\geq t}$. For all $i \in \mathbb{N}$ $x_0^i x^\gamma \mathbf{e}_k \in \mathcal{V}$ if and only if $x^\gamma \mathbf{e}_k \in \mathcal{V}$*

*(iv)* $\mathcal{N}(\mathcal{V})_{\geq t} = \mathcal{N}(\underline{\mathcal{V}})_{\geq t}$

*(v) For all $\mathbf{g} \in A[\mathbf{x}]^m_{\geq t}$: $\mathbf{g}$ is $\mathcal{V}$-reduced if and only if $\mathbf{g}$ is $\underline{\mathcal{V}}$-reduced.*

*(vi) If $\mathcal{U}$ belongs to **Mf**$(\mathcal{V})$ with a $\mathbf{P}(\mathcal{V})$-marked basis $G$, then every homogeneous module element $\mathbf{g} \in \mathcal{U}_s$ with $s \geq t$ satisfies that $\mathsf{NF}_{G^{(s+i)}}(x_0^i\mathbf{g}) = x_0^i\mathsf{NF}_{G^{(s)}}(\mathbf{g})$.*

*Proof.* The first two items are a straightforward consequence of the definition of $t$-truncation. For the third item we consider only the non-trivial part. If $x_0^i x^\gamma \mathbf{e}_k \in \mathcal{V}$, then $x^\gamma \mathbf{e}_k$ belongs to $\underline{\mathcal{V}}$. Since $\mathcal{V}$ is a $t$-truncation and $x^\gamma \mathbf{e}_k \in A[\mathbf{x}]^m_{\geq t}$, then $x^\gamma \mathbf{e}_k \in \mathcal{V}$, too. The fourth and fifth statements are obviously equivalent to the third statement. For the sixth statement one has to note that the normal form with respect to $G$ is unique. Furthermore, $\mathsf{NF}_{G^{(s+i)}}(x_0^i\mathbf{g})$ and $x_0^i\mathsf{NF}_{G^{(s)}}(\mathbf{g})$ are $\mathcal{V}$-reduced forms of $x_0^i\mathbf{g}$ and then they must coincide. □

**Lemma 4.36.** *Let $\mathcal{V} \subseteq A[\mathbf{x}]^m$ be a quasi-stable module. If $x^\epsilon \mathbf{e}_k$ belongs to $\mathcal{N}(\mathcal{V})$ and $x^\delta x^\epsilon \mathbf{e}_k$ belongs to $\mathcal{V}$ for some $x^\delta$, then $x^\delta x^\epsilon \mathbf{e}_k = x^\eta x^\alpha \mathbf{e}_k$, such that $x^\alpha \mathbf{e}_k$ is a Pommaret divisor of $x^\delta x^\epsilon \mathbf{e}_k$ with $x^\eta \prec_{\mathtt{lex}} x^\delta$. Furthermore, $x^{\eta_{\mathrm{sat}}} \prec_{\mathtt{lex}} x^{\delta_{\mathrm{sat}}}$.*

*Proof.* We can assume that $x^\delta$ and $x^\eta$ are coprime. If $x^\eta = 1$, the statement is obvious. If $x^\eta \neq 1$, then $x_{\mathrm{cls}(x^\delta)} | x^\alpha$ because $x^\delta$ and $x^\eta$ are coprime, hence $\mathrm{cls}(x^\delta) \geq \mathrm{cls}(x^\alpha)$. Furthermore, every $i$ such that $\deg_i(x^\eta) \neq 0$ is smaller than or equal to $\mathrm{cls}(x^\alpha)$, so it is smaller than $\mathrm{cls}(x^\delta)$ because $x^\delta$ and $x^\eta$ are coprime. This inequality implies as already known that $x^\eta \prec_{\mathtt{lex}} x^\delta$ and newly that $x^{\eta_{\mathrm{sat}}} \prec_{\mathtt{lex}} x^{\delta_{\mathrm{sat}}}$. □

To compute the embedding in the affine space of marked scheme we have seen before that we have to compute all reductions of the non-multiplicative prolongations of the set $\overline{G}$, which we defined in Section 3.3.2. For a $t$-truncated module $\mathcal{V}$ the set $\overline{G}$ has $|\mathbf{P}(\mathcal{V})|$ elements. Now, we define the set of superminimal generators which has in general fewer elements. For the next definition, recall that $x^{\alpha_{\mathrm{sat}}} := \frac{x^\alpha}{x_0^{\deg_0(x^\alpha)}}$.

**Definition 4.37.** *The* set of superminimal generators *of a quasi-stable module $\mathcal{V}$ is*

$$\mathbf{sP}(\mathcal{V}) := \{x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V}) \mid x^{\alpha_{\mathrm{sat}}} \mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})\}.$$

It is easy to determine the number of elements in the set of superminimal generators for a quasi-stable module $\mathcal{V}$. The number is exactly the number of elements in the Pommaret basis of $\underline{\mathcal{V}}$.

**Example 4.38.** *Consider*

$$\begin{aligned} V_1 &= \{x_2^3, x_2x_1, x_2x_1x_0, x_2x_0^2\} \subset \Bbbk[x_0, x_1, x_2] \\ V_2 &= \{x_2^4, x_2^3x_1, x_2^2x_1, x_2x_1^2, x_2x_1x_0, x_1^3, x_1^2x_0, x_1x_0^2\} \subset \Bbbk[x_0, x_1, x_2] \\ \mathcal{V} &= V_1\mathbf{e}_1 \oplus V_2\mathbf{e}_2 \subset \Bbbk[x_0, x_1, x_2]^2 . \end{aligned}$$

*Then* $\underline{\mathcal{V}} = \langle x_2\mathbf{e}_1, x_2^4\mathbf{e}_2, x_1\mathbf{e}_2\rangle$ *with* $\mathcal{V} = \underline{\mathcal{V}}_{\geq 3}$. *We immediately see that* $\mathbf{P}(\mathcal{V}) = V_1\mathbf{e}_1 \cup V_2\mathbf{e}_2$ *and* $\mathbf{P}(\underline{\mathcal{V}}) = \{x_2\mathbf{e}_1, x_2^4\mathbf{e}_2, x_2^3x_1\mathbf{e}_2, x_2^2x_1\mathbf{e}_2, x_2x_1\mathbf{e}_2, x_1\mathbf{e}_2\}$. *Then*

$$\mathbf{sP}(\mathcal{V}) = \{x_2x_0^2\mathbf{e}_1, x_2^4\mathbf{e}_2, x_2^3x_1\mathbf{e}_2, x_2^2x_1\mathbf{e}_2, x_2x_1x_0\mathbf{e}_2, x_1x_0^2\mathbf{e}_2\}\,.$$

**Definition 4.39.** *Let* $\mathcal{V}$ *be a quasi-stable module. An* $\mathbf{sP}(\mathcal{V})$-marked superminimal set *is a finite set of marked module elements* $\mathbf{f}_\alpha^k = x^\alpha\mathbf{e}_k + \sum c_{\alpha,\beta,k,l}x^\beta\mathbf{e}_l$ *such that the head module terms form the set of superminimal generators* $\mathbf{sP}(\mathcal{V})$ *of* $\mathcal{V}$ *and they are pairwise different. Furthermore, all elements* $x^\beta\mathbf{e}_l \in \operatorname{supp}(\mathbf{f}_\alpha^k - x^\alpha\mathbf{e}_k)$ *are in* $\mathcal{N}(\mathcal{V})$.

*Every* $\mathbf{P}(\mathcal{V})$*-marked set G contains a subset*

$$sG := \{\mathbf{f}_\alpha^k \in G \mid x^\alpha\mathbf{e}_k \in \mathbf{sP}(\mathcal{V})\}\,,$$

*which is an* $\mathbf{sP}(\mathcal{V})$*-marked superminimal set. It is called* the set of superminimals of $G$. *If G is a* $\mathcal{V}$*-marked basis it is called* $\mathbf{sP}(\mathcal{V})$-superminimal basis.

**Definition 4.40.** *Let* $\mathcal{V}$ *be a quasi-stable module, G be a* $\mathbf{P}(\mathcal{V})$*-marked set and two module elements* $\mathbf{g}$ *and* $\mathbf{g}_1$*. We say that* $\mathbf{g}$ *is in sG-relation with* $\mathbf{g}_1$ *if there is a module term* $x^\gamma\mathbf{e}_k \in \operatorname{supp}(\mathbf{g}) \cap \mathcal{V}$*, such that* $x^\gamma\mathbf{e}_k$ *is divisible by a superminimal generator* $x^\alpha\mathbf{e}_k \in \mathbf{sP}(\mathcal{V})$ *with* $x^\gamma\mathbf{e}_k = x^\epsilon x^\alpha\mathbf{e}_k$ *such that* $x^{\alpha_{\text{sat}}}$ *is a Pommaret divisor of* $x^\gamma$ *and* $\mathbf{g}_1 = \mathbf{g} - cx^\epsilon\mathbf{f}_\alpha^k$.

*We call a* superminimal reduction *the transitive closure of the above relation and denote it by* $\xrightarrow{sG}$. *Moreover, we say that:*

- $\mathbf{g}$ can be reduced to $\mathbf{g}_1$ by $\xrightarrow{sG}$ *if* $\mathbf{g} \xrightarrow{sG} \mathbf{g}_1$;
- $\mathbf{g}$ is reduced with respect to *sG if no module term in* $\operatorname{supp}(\mathbf{g})$ *is divisible by a module term of* $\mathbf{sP}(\mathcal{V})$*;*
- $\mathbf{g}$ is strongly reduced *if for every i,* $x_0^i\mathbf{g}$ *is reduced with respect to sG.*

**Remark 4.41.**

*(i) A homogeneous module element* $\mathbf{h}$ *is strongly reduced if and only if no module terms in* $\operatorname{supp}(\mathbf{h})$ *are divisible by a module term of* $\mathbf{P}(\underline{\mathcal{V}})$*, that is* $\mathbf{h}$ *is* $\underline{\mathcal{V}}$*-reduced. In fact, if* $x^\gamma\mathbf{e}_k \in \operatorname{supp}(\mathbf{h}) \cap \underline{\mathcal{V}}$ *then* $x^\gamma\mathbf{e}_k = x^\eta x^{\alpha_{\text{sat}}}\mathbf{e}_k$ *with* $x^{\alpha_{\text{sat}}}\mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})$ *is a Pommaret divisor of* $x^\gamma\mathbf{e}_k$ *such that* $x^\alpha\mathbf{e}_k = x_0^q x^{\alpha_{\text{sat}}}\mathbf{e}_k \in \mathbf{P}(\mathcal{V})$*. Thus* $x_0^q\mathbf{h}$ *can be reduced by* $\xrightarrow{sG}$ *using the module element* $\mathbf{f}_\alpha^k$.

*(ii) The module elements* $x^\epsilon\mathbf{f}_\alpha^k$*, that we use for the reduction procedure* $\xrightarrow{sG}$ *have pairwise different head module terms. Moreover, if* $x^\delta\mathbf{f}_{\alpha'}^{k'}$ *is used in the* $\xrightarrow{sG}$ *reduction of* $x^\epsilon T(\mathbf{f}_\alpha^k)$ *then* $x^{\delta_{\text{sat}}} \prec_{lex} x^{\epsilon_{\text{sat}}}$.

If we consider a quasi-stable module $\mathcal{V}$ without further hypothesis we cannot generalize the properties of reduction $\xrightarrow{G^{(s)}}$ to $\xrightarrow{sG}$ as shown in the following example:

**Example 4.42** ([8, Ex. 3.13]). *Consider* $\Bbbk[x_0, x_1, x_2]$ *and*

$$\mathcal{J} = \langle x_2^3, x_2^2x_1, x_2^2x_0, x_2x_1x_0, x_1^4, x_1^3x_0, x_1^2x_0^2 \rangle$$

*with* $\underline{\mathcal{J}} = \langle x_2^2, x_2x_1, x_1^2 \rangle$. *The set of superminimals of* $\mathcal{J}$ *is* $\mathbf{sP}(\mathcal{J}) = \{x_2^2x_0, x_2x_1x_0, x_1^2x_0^2\}$. *Now we consider the* $\mathbf{sP}(\mathcal{J})$*-marked superminimal set*

$$sG = \{\mathbf{f}_1 = x_2^2x_0, \mathbf{f}_2 = x_2x_1x_0 - x_1^3, \mathbf{f}_3 = x_1^2x_0^2 - x_2x_0^3\}.$$

*The superminimal reduction with respect to sG is not noetherian:*

$$x_1^3x_0^2 \xrightarrow{sG} x_1^3x_0^2 - x_1\mathbf{f}_3 = x_2x_1x_0^3 \xrightarrow{sG} x_2x_1x_0^3 - x_0^2\mathbf{f}_2 = x_1^3x_0^2.$$

However, if we assume that the quasi-stable module $\mathcal{V}$ is also a $t$-truncated module, then the reduction $\xrightarrow{sG}$ turns out to be noetherian and satisfies several good properties, similar to the ones of $\xrightarrow{G^{(s)}}$ as the following theorem shows. In the following we use for a marked polynomial $\mathbf{f}_\alpha^k$ again the tail of $\mathbf{f}_\alpha^k$ which is defined as $T(\mathbf{f}_\alpha^k) := \mathbf{f}_\alpha^k - x^\alpha\mathbf{e}_k$.

**Theorem 4.43.** *Let* $\mathcal{V}$ *be a t-truncated quasi-stable module and sG be an* $\mathbf{sP}(\mathcal{V})$*-marked superminimal set. Then*

*(i)* $\xrightarrow{sG}$ *is noetherian.*

*(ii) For every homogeneous module element* $\mathbf{h}$ *there exists an integer q and a unique strongly reduced module element* $\mathbf{h}_q$ *such that* $x_0^q\mathbf{h} \xrightarrow{sG} \mathbf{h}_q$. *If* $\overline{q}$ *is the minimum one and* $\overline{\mathbf{h}} := \mathbf{h}_{\overline{q}}$, *then* $\mathbf{h}_q = x_0^{q-\overline{q}}\overline{\mathbf{h}}$ *for every* $q \geq \overline{q}$. *There is an effective procedure to compute* $\overline{q}$ *and* $\overline{\mathbf{h}}$.

*Proof.* (i) Since $\mathcal{V}$ is a quasi-stable $t$-truncation $\mathcal{N}(\mathcal{V})_{\geq t} = \mathcal{N}(\underline{\mathcal{V}})_{\geq t}$ by Lemma 4.35 (iv). If $\xrightarrow{sG}$ is not noetherian we would be able to find an infinite descending chain of module terms with respect to $\prec_{\texttt{lex}}$, by Lemma 4.36.

(ii) It is sufficient to prove the statement for module terms $\mathbf{h} = x^\gamma\mathbf{e}_k \in \mathcal{V}$. Let $x^\gamma\mathbf{e}_k = x^\eta x^{\alpha_{\text{sat}}}\mathbf{e}_k$, such that $x^{\alpha_{\text{sat}}}\mathbf{e}_k \in \mathbf{P}(\mathcal{V})$ is a Pommaret divisor of $x^\gamma\mathbf{e}_k$. If $x^\eta = 1$, then $x^\alpha = x_0^{q_\alpha}x^{\alpha_{\text{sat}}} \in \mathbf{sP}(\mathcal{V})$, $\mathbf{f}_\alpha^k$ belongs to $sG$ and $x_0^{q_\alpha}x^{\alpha_{\text{sat}}}\mathbf{e}_k \xrightarrow{sG} T(\mathbf{f}_\alpha^k)$, where $\operatorname{supp}(T(\mathbf{f}_\alpha^k)) \subseteq \mathcal{N}(\mathcal{V})$. In this case $\overline{\mathbf{h}} = T(\mathbf{f}_\alpha^k)$ and $\overline{q} = q_\alpha$.

If $x^\eta \neq 1$ we can assume that the statement holds for any module monomial $x^\epsilon\mathbf{e}_k = x^{\eta'}x^{\beta_{\text{sat}}}\mathbf{e}_k$ such that $x^{\eta'} \prec_{\texttt{lex}} x^\eta$ and $x^{\beta_{\text{sat}}}\mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})$ is a Pommaret divisor of $x^\epsilon\mathbf{e}_k$. We perform a first reduction $x_0^{q_\alpha}x^\gamma\mathbf{e}_k \xrightarrow{sG} x^\eta T(\mathbf{f}_\alpha^k)$. If $x^\eta T(\mathbf{f}_\alpha^k)$ is strongly reduced we are done. Otherwise, we have $x^\eta \neq x_0^{|\eta|}$. For every module term $x^\epsilon \mathbf{e}_k \in \mathrm{supp}(x^\eta T(f_\alpha^k)) \cap \underline{\mathcal{V}}$ we have $x^\epsilon \mathbf{e}_k = x^{\eta'} x^{\beta_{\mathrm{sat}}} \mathbf{e}_k$ such that $x^{\eta_{\mathrm{sat}}} \mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})$ is a Pommaret divisor of $x^\epsilon \mathbf{e}_k$ with $x^{\eta'} \prec_{\mathtt{lex}} x^\eta$ by Lemma 4.36. So, we also have $x_0^q x^{\eta'} \prec_{\mathtt{lex}} x^\eta$ for every $q$. By the inductive hypothesis we can find a suitable power $q$ of $x_0$ such that every module term in $x_0^q x^\eta T(\mathbf{f}_\alpha^k)$ can be reduced $\xrightarrow{sG}$ to a strongly reduced module monomial.

It remains to prove the uniqueness of the strongly reduced module term $\mathbf{h}_q$. Let us consider two different strongly reduced $\xrightarrow{sG}$ reductions of $x_0^q \mathbf{h}$: their difference is again strongly reduced and can be written as $\sum a_i x^{\eta_i} \mathbf{f}_{\alpha_i}^{k_i}$ with $a_i \in A \setminus \{0\}$ and $x_i^\eta \mathbf{f}_{\alpha_i}^{k_i}$ pairwise different. Let $x^{\eta_1} \mathbf{f}_{\alpha_1}^{k_1}$ be such that for every $i \geq 2$, either $x^{\eta_1} \succ_{\mathtt{lex}} x^{\eta_i}$ or $k_1 \neq k_i$. Then $x^{\eta_1} x^{\alpha_1} \mathbf{e}_{k_1}$ should cancel with a module term in $\mathrm{supp}(x^{\eta_i} T(\mathbf{f}_{\alpha_i}^{k_i}))$ for some $i$, but this is impossible as observed in Remark 4.41 (ii).

Observe that though for a fixed $x^\gamma \mathbf{e}_k = x^\eta x^{\alpha_{\mathrm{sat}}} \mathbf{e}_k$, such that $x^{\alpha_{\mathrm{sat}}} \mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})$ is a Pommaret divisor of $x^\gamma \mathbf{e}_k$ there are infinitely many module terms $x^\epsilon \mathbf{e}_k = x^{\eta'} x^{\beta_{\mathrm{sat}}} \mathbf{e}_k$, such that $x_{\mathrm{sat}}^\beta \mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})$ is a Pommaret divisor of $x^\epsilon \mathbf{e}_k$ and $x^{\eta'} \mathbf{e}_k \prec_{\mathtt{lex}} x^\eta \mathbf{e}_k$. We use the inductive hypothesis only with respect to the finite number of them that appear on the support of $x^\eta T(\mathbf{f}_\alpha^k)$. For this reason our procedure is effective. □

**Theorem 4.44.** *Let $\mathcal{V}$ be a t-truncated quasi-stable module and sG be an $\mathbf{sP}(\mathcal{V})$-marked superminimal set which is the superminimal basis of a module $\mathcal{U}$ of **Mf**$(\mathcal{V})$, then*

*(i) $\xrightarrow{sG}$ computes the $\mathcal{V}$-normal forms modulo $\mathcal{U}$. More precisely, for every homogeneous module element:*

$$\mathsf{NF}_{sG}(\mathbf{h}) = \begin{cases} \mathbf{h}, & if\ \deg(\mathbf{h}) < t\,, \\ \frac{\overline{\mathbf{h}}}{x_0^{\bar{q}}}, & if\ \deg(\mathbf{h}) \geq t \ and\ x_0^{\bar{q}}\mathbf{h} \xrightarrow{sG} \overline{\mathbf{h}}\,. \end{cases}$$

*(ii) It is true for every homogeneous module element $\mathbf{h}$, that*

$$\mathbf{h} \in \mathcal{U} \quad \Longleftrightarrow \quad \deg(\mathbf{h}) \geq t \ and\ x_0^{\bar{q}}\mathbf{h} \xrightarrow{sG} 0\,.$$

*(iii) There is a one-to-one correspondence between a module in **Mf**$(\mathcal{V})$ and $\mathbf{sP}(\mathcal{V})$-superminimal bases.*

*Proof.* If $\mathbf{h}$ is a homogeneous module element and $\mathbf{h} \xrightarrow{sG} \mathbf{h}_1$ with $\mathbf{h}_1$ strongly reduced, then by uniqueness of $\mathcal{V}$-normal forms modulo $\mathcal{U}$ we have $\mathbf{h}_1 = \mathsf{NF}_{sG}(\mathbf{h})$.

(i) If $\deg(\mathbf{h}) < t$ we are done. Otherwise from Theorem 4.43 (ii) we have that $x_0^{\bar{q}}\mathbf{h} \xrightarrow{sG} \overline{\mathbf{h}}$ and $\overline{\mathbf{h}}$ is a $\mathcal{V}$-reduced form modulo $\mathcal{U}$. thus $x_0^q \mathsf{NF}_{Gs}(\mathbf{h})$ is also $\mathcal{V}$-reduced, by Lemma

4.35 (iii). Hence, we get the desired equality by uniqueness of $\mathcal{V}$-normal forms modulo $\mathcal{U}$.

(ii) This is a consequence of (i) and of Corollary 1.87.

(iii) This is a straightforward consequence of (ii). □

Whenever $\mathcal{V}$ is a quasi-stable $t$-truncated module and $G$ is the $\mathbf{P}(\mathcal{V})$-marked basis and $sG$ is the $\mathbf{sP}(\mathcal{V})$-marked superminimal basis of a module $\mathcal{U} \in \mathbf{Mf}(\mathcal{V})$, then $sG$ is a subset of $G$. Nevertheless, it is interesting to notice that not every step of reduction by $\xrightarrow{sG}$ is also a step of reduction by $\xrightarrow{G^{(s)}}$, as shown in the following example.

**Example 4.45** ([8, Ex. 3.15]). *Consider in $\Bbbk[x_0, x_1, x_2]$ the ideal $\mathcal{J} = \langle x_1^2, x_0x_2, x_1x_2, x_2^2 \rangle$ which is a quasi-stable ideal and a 2-truncation of $\underline{\mathcal{J}} = \langle x_2, x_1^2 \rangle$. Let G be a $\mathbf{P}(\mathcal{J})$-marked set.*

- *The term $x_2x_1^2$ is non-reducible with respect to sG, because the only term of $\mathbf{sP}(\mathcal{J}) = \{x_0x_2, x_1^2\}$ dividing it is $x_1^2$, but $x_1^2$ is not the Pommaret divisor of $x_2x_1^2$ . On the other hand, $x_2x_1^2$ is Pommaret divisible by $x_2x_1$, so $x_2x_1^2 \xrightarrow{G^{(3)}} x_1T(\mathbf{f})$ where $\mathbf{f} \in G^{(2)}$ with $\mathrm{Ht}(\mathbf{f}) = x_2x_1$.*

- *The only way to reduce $x_0x_2^2$ via $\xrightarrow{G^{(3)}}$ leads to $x_0T(\mathbf{f}')$, where $\mathbf{f}'$ is the unique polynomial of $G^{(2)}$ such that $\mathrm{Ht}(\mathbf{f}') = x_2^2$. Moreover, $x_0T(\mathbf{f}')$ is not further reducible, because all the monomials of its support belong to $\mathcal{N}(\mathcal{J})$ according to Lemma 4.35. On the other hand, according to Definition 4.39, a first step of reduction of the term $x_0x_2^2$ via $\xrightarrow{sG}$ is $x_0x_2^2 \xrightarrow{sG} x_2T(\mathbf{f}'')$, where $\mathbf{f}''$ is the polynomial in sG with $\mathrm{Ht}(\mathbf{f}'') = x_0x_2$. Since $x_2$ is a term of $\mathbf{P}(\underline{\mathcal{J}})$, every term appearing in $\mathrm{supp}(x_2T(\mathbf{f}''))$ belongs to $\mathcal{J}$ and so we will need further steps of reduction via $\xrightarrow{sG}$ to compute a polynomial non-reducible with respect to sG.*

**Lemma 4.46.** *Let $\mathcal{V}$ be a quasi-stable t-truncated module, G be a $\mathbf{P}(\mathcal{V})$-marked set and $\mathbf{h}$ be a homogeneous module element of degree $q \geq t$. Then $\mathbf{h} \in \langle G^{(q)} \rangle$ if and only if $x_0\mathbf{h} \in \langle G^{(q+1)} \rangle$.*

*Proof.* If $\mathbf{h} \in \langle G^{(q)} \rangle$ then it is obvious that $x_0\mathbf{h} \in \langle G^{(q+1)} \rangle$

Vica versa, assume that $x_0\mathbf{h} \in \langle G^{(q+1)} \rangle$. This is equivalent to $x_0\mathbf{h} \xrightarrow{G^{(q+1)}} 0$. Every module term in $\mathrm{supp}(x_0\mathbf{h})$ can be written as $x_0x^\epsilon\mathbf{e}_k$. Observe that $x_0x^\epsilon\mathbf{e}_k \notin \mathbf{P}(\mathcal{V})$ because $\deg(x_0x^\epsilon\mathbf{e}_k) > t$ by Lemma 4.35 (i). Then, if $x_0x^\epsilon\mathbf{e}_k$ belongs to $\mathcal{V}$ we can find a Pommaret divisor $x^\alpha\mathbf{e}_k \in \mathbf{P}(\mathcal{V})$ such that $x_0x^\epsilon\mathbf{e}_k = x^\eta x^\alpha\mathbf{e}_k$ and $x^\eta \neq 1$. By the properties of the Pommaret division $x^\eta$ must be divisible by $x_0$, hence $x^\eta = x_0x^{\eta'}$.

Summing up, in order to reduce the module term $x_0x^{\epsilon}\mathbf{e}_k$ of $\mathrm{supp}(x_0\mathbf{h})$ using $G^{(q+1)}$, we use the module element $x_0x^{\eta'}\mathbf{f}^k_\alpha \in G^{(q+1)}$. If the coefficient of $x_0x^{\epsilon}\mathbf{e}_k$ in $x_0\mathbf{h}$ is $a$ we obtain

$$x_0\mathbf{h} \xrightarrow{G^{(q+1)}} x_0(\mathbf{h} - ax^{\eta'}\mathbf{f}^k_\alpha)\,.$$

At every step of reduction we obtain a module element which is divisible by $x_0$. In particular

$$x_0\mathbf{h} \in \langle G^{(q+1)}\rangle \quad\Longrightarrow\quad x_0\mathbf{h} = x_0\sum a_i x^{\eta_i}\mathbf{f}^{k_i}_{\alpha_i},\ \text{where } x_0x^{\eta_i}\mathbf{f}^{k_i}_{\alpha_i} \in G^{(q+1)}.$$

Then we have that $\mathbf{h} = \sum a_i x^{\eta_i}\mathbf{f}^{k_i}_{\alpha_i}$ and $x^{\eta_i}\mathbf{f}^{k_i}_{\alpha_i} \in G^{(q)}$, that is $\mathbf{h} \in \langle G^{(q)}\rangle$. □

The next lemma is an analogous of Proposition 1.80 and is gives us an effective procedure to check if a given module is contained in a marked scheme.

**Lemma 4.47.** *Let $\mathcal{V}$ be a quasi-stable t-truncated module, G be a $\mathbf{P}(\mathcal{V})$-marked set $\mathbf{f}^k_\alpha \in G$ and $i > \mathrm{cls}(x^\alpha)$. If $x_0^q x_i \mathbf{f}^k_\alpha \xrightarrow{sG} \mathbf{h}$, then*

$$x_0^q x_i \mathbf{f}^k_\alpha - \mathbf{h} = \sum a_j x^{\eta_j}\mathbf{f}^{k_j}_{\beta_j}$$

*with $\mathbf{f}^{k_j}_{\beta_j} \in sG$, $x^{\eta_j} \prec_{\mathtt{lex}} x_i$ and $x^{\eta_j^{\mathrm{sat}}} \prec_{\mathtt{lex}} x_i$.*

*Proof.* For every module term $x_0^q x_i x^{\epsilon}\mathbf{e}_k \in \mathrm{supp}(x_0^q x_i \mathbf{f}^k_\alpha - \mathbf{h})$ there is a Pommaret divisor $x^{\beta_{\mathrm{sat}}}\mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})$ such that $x_0^q x_i x^{\epsilon}\mathbf{e}_k = x^{\eta}x^{\beta_{\mathrm{sat}}}\mathbf{e}_k$ with $x^{\eta} \prec_{\mathtt{lex}} x_0^q x_i$ and $x^{\eta_{\mathrm{sat}}} \prec_{\mathtt{lex}} x_i$ by Lemma 4.35 (iii) and Lemma 4.36. The same holds for any further reduction and the same argument applies to module terms appearing in $\mathrm{supp}(x_0^q x^{\gamma'} T(\mathbf{f}^{k'}_{\alpha'}))$. □

With the lemma above we are now able to state and proof an analogous of Theorem 1.86.

**Theorem 4.48.** *Let $\mathcal{V}$ be a quasi-stable t-truncated module, G be a $\mathbf{P}(\mathcal{V})$-marked set and $\mathcal{U}$ be the graded module generated by G. The following statements are equivalent:*

*(i)* $\mathcal{U} \in \boldsymbol{Mf}(\mathcal{V})$.

*(ii) For every $\mathbf{f}^k_\alpha \in G$ and $i > \mathrm{cls}(x^\alpha)$ there exists a non-negative integer q such that $x_0^q x_i \mathbf{f}^k_\alpha \xrightarrow{sG} 0$.*

*(iii) For every $\mathbf{f}^k_\alpha \in G$ and $i > \mathrm{cls}(x^\alpha)$ there exists a non-negative integer q such that $x_0^q x_i \mathbf{f}^k_\alpha = \sum a_j x^{\eta_j}\mathbf{f}^{k_j}_{\beta_j}$ with $x^{\eta_j} \prec_{\mathtt{lex}} x_i$ and $\mathbf{f}^{k_j}_{\beta_j} \in sG$.*

*Proof.* If $\mathcal{U} \in \mathbf{Mf}(\mathcal{V})$ we can apply Theorem 4.44 (ii) because any non-multiplicative prolongation of elements in $G$ belongs to $\mathcal{U}$.

If statement (ii) holds, then we get (iii) by Lemma 4.47.

We now assume that statement (iii) holds. We define $G^* := \bigcup_{q\in\mathbb{N}} G^{(q)}$. By Theorem 1.82 it is sufficient to prove that $\langle G\rangle = \langle G^*\rangle^A$. It is enough to show that $x^\eta G^* \subseteq \langle G^*\rangle^A$. The statement is obviously true for $x^\eta = 1$. For the following we assume that the statement holds for any term $x^{\eta'}$ such that $x^{\eta'} \prec_{\texttt{lex}} x^\eta$

If $\deg(x^\eta) > 1$ we can consider any product $x^\eta = x^{\eta_1}x^{\eta_2}$ such that $\deg(x^{\eta_1}) > 0$ and $\deg(x^{\eta_2}) > 0$. Since $x^{\eta_1} \prec_{\texttt{lex}} x^\eta$ and $x^{\eta_2} \prec_{\texttt{lex}} x^\eta$ we immediately obtain by induction

$$x^\eta G^* = x^{\eta_1}(x^{\eta_2}G^*) \subseteq x^{\eta_1}\langle G^*\rangle^A \subseteq \langle G^*\rangle^A .$$

If $\deg(x^\eta) = 1$ we need to prove that $x_iG^* \subseteq \langle G^*\rangle$. Lemma 4.46 shows that $x_0G^* \subseteq \langle G^*\rangle^A$. Therefore, we only have to prove that $x_iG^* \subseteq \langle G^*\rangle^A$ for $i \geq 1$, assuming that the statement holds for every $x^{\eta'} \prec_{\texttt{lex}} x_i$. We consider $\mathbf{g}^k_\beta = x^\delta\mathbf{f}^k_\alpha \in G^*$ where $x^\delta$ is multiplicative for $x^\alpha$ and $x^\beta = x^\delta x^\alpha$. If $x_i\mathbf{g}^k_\beta$ does not belong to $G^*$ then $x_ix^\beta$ is non-multiplicative for $x^\alpha$, so $i > \operatorname{cls}(x^\alpha)$. So $x_i \succ_{\texttt{lex}} x^\delta$ and it is sufficient to prove the statement for $x_i\mathbf{f}^k_\alpha$.

By hypothesis there is a $q$ such that $x_0^qx_i\mathbf{f}^k_\alpha = \sum a_jx^{\eta_j}\mathbf{f}^{k_j}_{\beta_j}$ where all $x^{\eta_j}$ are lower than $x_i$ with respect to $\prec_{\texttt{lex}}$ and $\mathbf{f}^{k_j}_{\beta_j} \in sG$. Then all $x^{\eta_j}\mathbf{f}^{k_j}_{\beta_j}$ belong to $\langle G^*\rangle^A$ by induction and we conclude that $x_i\mathbf{f}^k_\alpha \in \langle G^*\rangle^A$ by Lemma 4.47. □

For the theorem above we have to consider all non-multiplicative prolongations for every element in $G$. The next theorem shows that we only have to consider all non-multiplicative prolongations for elements in $sG$ and and some other prolongations of elements in $G$.

**Theorem 4.49.** *Let $\mathcal{V} \subseteq A[\mathbf{x}]^m$ be a quasi-stable t-truncated module, $G$ be a $\mathbf{P}(\mathcal{V})$-marked set and $\mathcal{U}$ be the graded module generated by $G$. We define the following two sets*

$$L_1 := \left\{ x_i\mathbf{f}^k_\alpha \mid \mathbf{f}^k_\alpha \in sG \text{ and } i > \operatorname{cls}(x^{\alpha_{\mathrm{sat}}}) \right\},$$
$$L_2 := \left\{ x_i\mathbf{f}^k_\alpha \;\middle|\; \begin{array}{l} \mathbf{f}^k_\alpha \in G \text{ with } \deg(\mathbf{f}^k_\alpha) = t \text{ such that } x_ix^\alpha = x_0x^\beta \\ \text{for } \mathbf{f}^k_\beta \in G \text{ and } i = \operatorname{cls}(x^{\alpha_{\mathrm{sat}}}) \end{array} \right\}.$$

*Then $\mathcal{U} \in \mathbf{Mf}(\mathcal{V})$ if and only if for all $x_i\mathbf{f}^k_\alpha \in L_1 \cup L_2$ there exists a $q$ such that $x_0^qx_i\mathbf{f}^k_\alpha \xrightarrow{sG} 0$.*

*Proof.* If $\mathcal{U} \in \mathbf{Mf}(\mathcal{V})$ we can apply Theorem 4.48 and the statement follows immediately.

Vica versa, by Theorem 1.82 it is sufficient to prove that $\langle G \rangle = \langle G^* \rangle^A$. It is enough to show that $x_i G^* \subseteq \langle G^* \rangle^A$. We proceed by induction on the variables. By construction we have $x_0 G^* = \langle G^* \rangle^A$. We now assume that $\langle x_0, \dots, x_{i-1} \rangle G^* \subseteq \langle G^* \rangle^A$ and we prove that $x_i G^* \subseteq \langle G^* \rangle^A$. Consider $x^\delta \mathbf{f}^k_\beta \in G^*$. The statement is that $x_i x^\delta \mathbf{f}^k_\beta$ is contained in $\langle G^* \rangle^A$. If $x_i x^\delta \mathbf{f}^k_\beta$ does not belong to $G^*$, then $x_i x^\delta$ is not multiplicative for $x^\beta$, so $i > \operatorname{cls}(x^\beta)$ because $x^\delta$ is multiplicative for $x^\beta$. Hence it is sufficient to prove the statement for $x_i \mathbf{f}^k_\beta$, because by induction then we have $x^\delta x_i \mathbf{f}^k_\beta \in \langle G^* \rangle^A$. In the following we consider, that $x^\beta \mathbf{e}_k = x^\eta x^{\alpha_{\text{sat}}} \mathbf{e}_k$, such that $x^\eta$ is multiplicative with respect to $x^{\alpha_{\text{sat}}} \mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})$.

We have a first case when $x^\eta = 1$. Then $x^\beta \mathbf{e}_k = x^{\alpha_{\text{sat}}} \mathbf{e}_k$ and $\mathbf{f}^k_\beta \in sG$. We consider $x_i x^{\alpha_{\text{sat}}} \mathbf{e}_k = x^{\eta'} x^{\alpha'_{\text{sat}}} \mathbf{e}_k$ such that $x^{\eta'}$ is multiplicative for $x^{\alpha'_{\text{sat}}} \mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})$. Since $i > \operatorname{cls}(x^{\alpha_{\text{sat}}})$ we have that $x_i \mathbf{f}^k_\beta \in L_1$. Hence by hypothesis and by Lemma 4.47 there is a $q$ such that

$$x_0^q x_i \mathbf{f}^k_\beta = \sum a_j x^{\eta_j} \mathbf{f}^{k_j}_{\alpha_j},$$

with $x^{\eta_j} \prec_{\texttt{lex}} x_i$ and $\mathbf{f}^{k_j}_{\alpha_j} \in sG$. Hence $x^{\eta_j} \mathbf{f}^{k_j}_{\alpha_j} \in \langle G^* \rangle^A$ by induction on the variables and so $x_i \mathbf{f}^k_\beta$ belongs to $\langle G^* \rangle^A$ by Lemma 4.46.

We have a second case when $x^\eta = x_0^q \neq 1$. Then $\deg(x^\beta \mathbf{e}_k) = t$ and $\mathbf{f}^k_\beta$ belongs to $sG$. Let $x_i x^\beta \mathbf{e}_k = x^{\eta'} x^{\alpha'_{\text{sat}}} \mathbf{e}_k$, such that $x^{\eta'}$ is multiplicative for $x^{\alpha'_{\text{sat}}} \mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})$. If $i > \operatorname{cls}(x^{\alpha'_{\text{sat}}})$ then $x^{\eta'}$ is not divisible by $x_i$ and we repeat the argument above. Otherwise, $i \leq \operatorname{cls}(x^{\alpha'_{\text{sat}}})$ and $x_i$ does not divide $x^{\eta'}$. If $x_i$ would divide $x^{\eta'}$ we would have find a Pommaret divisor of $x^{\alpha_{\text{sat}}} \mathbf{e}_k$ in $\mathbf{P}(\underline{\mathcal{V}})$. So $i = \operatorname{cls}(x^{\alpha'_{\text{sat}}})$ and $x^{\eta'} \prec_{\texttt{lex}} x_i$. Then $x_i \mathbf{f}^k_\beta$ belongs to $L_2$ and we repeat the same reasoning above.

We now assume the statement holds for every $\mathbf{f}^k_{\beta'}$ with $x^{\beta'} \mathbf{e}_k = x^{\eta'} x^{\alpha'_{\text{sat}}} \mathbf{e}_k$ such that $x^{\eta'}$ is multiplicative for $x^{\alpha'_{\text{sat}}} \mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})$ and $x^{\eta'} \prec_{\texttt{lex}} x^\eta$. By the base of the induction we can suppose that $x^\eta \succeq_{\texttt{lex}} x_1$. Hence $\mathbf{f}^k_\beta$ does not belong to $sG$ and it has degree $t$. Let $j = \operatorname{cls}(x^{\beta_{\text{sat}}})$.

We first suppose that $x_i \preceq_{\texttt{lex}} x_j$. Then $j \geq i > \operatorname{cls}(x^\beta)$ and $x_0$ divides $x^\beta$. Then there exists an element $\mathbf{f}^k_{\beta'}$ with $x^{\beta'} \mathbf{e}_k = x_i \frac{x^\beta}{x_0} \mathbf{e}_k$ such that $x_i x^\beta = x_0 x^{\beta'}$ which shows that $x_i \mathbf{f}^k_\beta$ belongs to $L_2$ and we can repeat the argument of the previous case.

We now assume that $x_i > x_j$ and choose $\mathbf{f}^k_{\beta'}$ such that $x^{\beta'} \mathbf{e}_k = x_0 \frac{x^\beta}{x_j} \mathbf{e}_k$. Therefore $x_j \mathbf{f}^k_{\beta'}$ belongs to $L_2$ and by the hypothesis and by Lemma 4.47 there exists an integer $q$ such that

$$x_0^q x_j \mathbf{f}^k_{\beta'} = \sum a_l x^{\eta_l} \mathbf{f}^{k_l}_{\alpha_l}$$

with $x^{\eta_l} \prec_{\texttt{lex}} x_j$ and $\mathbf{f}^{k_l}_{\alpha_l} \in sG$. We now multiply the equation by $x_i$. We see that $x_i\mathbf{f}^{k_l}_{\alpha_l}$ belongs to $\langle G^* \rangle^A$ because $\mathbf{f}^{k_l}_{\alpha_l} \in sG$ and by the first two cases. Finally $x_i\mathbf{f}^k_\beta$ belongs to $\langle G^* \rangle^A$ because of Lemma 4.46. □

Now we see how we obtain $\mathbf{Mf}(\mathcal{V})$ by using the theory of $\mathbf{sP}(\mathcal{V})$-marked sets. As in Section 3.3.2 we start with a $\mathbf{P}(\mathcal{V})$-marked set $\overline{G} = \{F^k_\alpha \in \Bbbk[C][\mathbf{x}]^m \mid x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V})\}$ consisting of elements as defined in (3.5).

**Definition 4.50.** *If $\overline{G}$ is the set of marked module elements for $\mathcal{V}$ like defined above then we define the* set of superminimals *and denote it by $s\overline{G}$ as the subset of $\overline{G}$ such that $F^k_\alpha \in s\overline{G}$ if $F^k_\alpha \in \overline{G}$ with $x^\alpha \mathbf{e}_k \in \mathbf{sP}(\mathcal{V})$. We denoted by $C$ the set of variables appearing in the tails of the module elements in $\overline{G}$. Now we denote by $\widetilde{C}$ the set of variables appearing in the tails of the module elements in $s\overline{G}$.*

In Section 3.3.2 we have seen that we obtain a $\mathbf{P}(\mathcal{V})$-marked basis for a module $\mathcal{U} \in \mathbf{Mf}(\mathcal{V})$ by specializing in a suitable way the variables $C$ in $\overline{G}$. The set of superminimals $sG$ of $\mathcal{U}$ is obtained in the same way by $s\overline{G}$ through the same specialization of the variables $\widetilde{C}$.

**Definition 4.51.** *Let $x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V})$ and $q$ be an integer such that $x_0^q x^\alpha \mathbf{e}_k \xrightarrow{s\overline{G}} H^k_\alpha \in \Bbbk[\widetilde{C}][\mathbf{x}]^m$ with $H^k_\alpha$ strongly reduced (the integer $q$ exists by Theorem 4.43). We can write $H^k_\alpha = \overline{H}^k_\alpha + x_0^q \widetilde{H}^k_\alpha$, where no module term appearing in $\overline{H}^k_\alpha$ is divisible by $x_0^q$. We will denote by:*

- $B = \left\{ C_{\alpha\gamma kl} - \operatorname{coeff}_{\widetilde{H}^k_\alpha}(x_\gamma \mathbf{e}_l) \mid x_\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V}) \setminus \mathbf{sP}(\mathcal{V}),\ x^\gamma \mathbf{e}_l \in \mathcal{N}(\mathcal{V})_{\deg(x^\alpha \mathbf{e}_k)} \right\}$ *the set of the coefficients of $T(F^k_\alpha) - \widetilde{H}^k_\alpha$ for every $x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V}) \setminus \mathbf{sP}(\mathcal{V})$.*
- $D_1 \subset \Bbbk[\widetilde{C}]$ *the set of coefficients of $\overline{H}^k_\alpha$ for every $x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V}) \setminus \mathbf{sP}(\mathcal{V})$.*
- $D_2$ *the set of coefficients of the strongly reduced module elements in $\langle s\overline{G} \rangle$.*

Observe that $B, D_1, D_2$ are well-defined because of the uniqueness of $H^k_\alpha$, by Theorem 4.43 (ii). We have seen in Theorem 3.44 that the marked scheme $\mathbf{Mf}(\mathcal{V})$ can be defined through an ideal $\mathcal{R} := \langle R \rangle$. The next theorem shows that we can intersect $\mathcal{R}$ with the polynomial ring $\Bbbk[\widetilde{C}]$ and the newly obtained ideal still represents $\mathbf{Mf}(\mathcal{V})$.

**Theorem 4.52.** *The marked scheme **Mf**$(\mathcal{V})$ is defined by the ideal $\widetilde{\mathcal{R}} := \mathcal{R} \cap \Bbbk[\widetilde{C}]$ as a subscheme of the the affine space $\mathbb{A}^{|\widetilde{C}|}$, where $|\widetilde{C}| = \sum_{x_\alpha \mathbf{e}_k \in \mathbf{sP}(\mathcal{V})} |\mathcal{N}(\mathcal{V})_{\deg(x_\alpha \mathbf{e}_k)}|$. Moreover $\mathcal{R} = \langle B \cup D_1 \cup D_2 \rangle^{\Bbbk[C]}$ and $\widetilde{\mathcal{R}} = \langle D_1 \cup D_2 \rangle^{\Bbbk[\widetilde{C}]}$.*

*Proof.* For the first part it is sufficient to prove that $\mathcal{R}$ contains $B$ and so it contains an element of type $C_{\alpha\gamma kl} - E_{\alpha\gamma kl}$ for every $C_{\alpha\gamma kl} \in C \setminus \widetilde{C}$ where $E_{\alpha\gamma kl} \in \Bbbk[\widetilde{C}]$. Because this allows the elimination of the variables $C_{\alpha\gamma kl} \in C \setminus \widetilde{C}$.

By the construction in Definition 4.51, it is clear that $H^k_\alpha$ belongs to $\Bbbk[\widetilde{C}][\mathbf{x}]^m$ and that $x_0^q T(F^k_\alpha)$ and $H^k_\alpha$ are strongly reduced. Thus their difference $x_0^q T(F^k_\alpha) - H^k_\alpha$ is strongly reduced and moreover it belongs to $\mathcal{R}$ , because $x_0^q T(F^k_\alpha) - H^k_\alpha = x_0^q F^k_\alpha - x_0^q x_\alpha \mathbf{e}_k - H^k_\alpha$. Hence its coefficients belong to $\mathcal{R}$ and in particular the coefficient of $x_0^q x^\gamma \mathbf{e}_l$ is of the type $C_{\alpha\gamma kl} - E_{\alpha\gamma kl}$ with $E_{\alpha\gamma kl} \in \Bbbk[\widetilde{C}]$. Then $B \subseteq \mathcal{R}$ and $\mathcal{R}$ is generated by $B \cup \widetilde{\mathcal{R}}$.

To prove the second part it is sufficient to show that $\mathcal{R} \cap \Bbbk[\widetilde{C}] = \langle D_1 \cup D_2 \rangle^{\Bbbk[\widetilde{C}]}$.

"$\supseteq$": Taking the coefficients of $x_0^q T(F^k_\alpha) - H^k_\alpha$ of module terms that are not divisible by $x_0^q$, we see that $\mathcal{R}$ contains the coefficients of $\overline{H}^k_\alpha$. Then $D_1 \subseteq \mathcal{R} \cap \Bbbk[\widetilde{C}]$ because $\overline{H}^k_\alpha \in \Bbbk[\widetilde{C}][\mathbf{x}]^m$.

Moreover, we recall that $\mathcal{R}$ is made by all the coefficients in the module elements of $\langle \overline{G} \rangle$ that are strongly reduced. Indeed, $\mathcal{R}$ is made by all the coefficients of the module elements of $\langle \overline{G} \rangle$ that are $\mathcal{V}$-reduced. This is true because the degree of the module terms in the variables $\mathbf{x}$ of every module element in $\langle \overline{G} \rangle$ is greater than or equal to $t$. Then $\mathcal{V}$-reduced is equivalent to $\underline{\mathcal{V}}$-reduced, that is strongly reduced by Lemma 4.35 (iv). Then $D_2 \subseteq \mathcal{R} \cap \Bbbk[\widetilde{C}]$, because $\langle s\overline{G} \rangle^{\Bbbk[\widetilde{C}][\mathbf{x}]^m} \subset \langle \overline{G} \rangle^{\Bbbk[C][\mathbf{x}]^m}$.

"$\subseteq$": For every module element $F \in \Bbbk[C][\mathbf{x}]^m$ let us denote by $F^E$ the module element in $\Bbbk[\widetilde{C}][\mathbf{x}]^m$ obtained by substituting every $C_{\alpha\beta kl} \in C \setminus \widetilde{C}$ by $E_{\alpha\beta kl}$. If $F$ is strongly reduced then $F^E$ is strongly reduced, too. Observe that for every $x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V})$ we have $F^{k,E}_\alpha = x^\alpha \mathbf{e}_k - \widetilde{H}^k_\alpha$ and moreover $x_0^q(x^\alpha \mathbf{e}_k - \widetilde{H}^k_\alpha) - \overline{H}^k_\alpha \in \langle s\overline{G} \rangle^{\Bbbk[\widetilde{C}][\mathbf{x}]}$. In particular, $x_0^q F^{k,E}_\alpha$ and $x_0^q(x^\alpha \mathbf{e}_k - \widetilde{H}^k_\alpha) - \overline{H}^k_\alpha$ are equal modulo $D_1$

It remains to prove that every element $K \in \mathcal{R} \cap \Bbbk[\widetilde{C}]$ can be obtained modulo $D_1$ as an coefficient in some strongly reduced module element of the module $\langle s\overline{G} \rangle \subseteq \Bbbk[\widetilde{C}]$. We know that $K$ is a coefficient in a strongly reduced module element $D \in \langle \overline{G} \rangle$.

If $D = \sum D^k_\alpha F^k_\alpha \in \langle \overline{G} \rangle$, then for a suitable $q$,

$$x_0^q D^E = \sum D^{k,E}_\alpha (x_0^q(x^\alpha \mathbf{e}_k - \widetilde{H}^k_\alpha) - \overline{H}^k_\alpha) \mod D_1 \tag{4.2}$$

and the module element on the right-hand side of the equality is strongly reduced and it belongs to $\langle s\overline{G} \rangle^{\Bbbk[\widetilde{C}][\mathbf{x}]^m}$. Therefore $K$ is still one of the coefficients of $D^E$ since it does not contain any variable in $C \setminus \widetilde{C}$ and it remains unchanged. Then $K \in \langle D_1 \cup D_2 \rangle^{\Bbbk[\widetilde{C}]}$. □

**Proposition 4.53.** *Let $\widetilde{\mathcal{R}}$ be as in Theorem 4.52 and let $\mathcal{Q}$ be any ideal in $\Bbbk[\widetilde{C}]$. Assume that $\mathcal{Q} \subseteq \widetilde{\mathcal{R}}$ and that the following conditions hold:*

*(i) For every module term $x^\beta \mathbf{e}_k \in \mathbf{P}(\mathcal{V}) \setminus \mathbf{sP}(\mathcal{V})$ such that $x^\beta \mathbf{e}_k = x^\delta x^{\alpha_{\text{sat}}}\mathbf{e}_k$ with $x^\delta$ is multiplicative for $x^{\alpha_{\text{sat}}}\mathbf{e}_k \in \mathbf{P}(\underline{\mathcal{V}})$ there exists $q$ such that we have a formula of type*

$$x_0^q x^\beta \mathbf{e}_k = \sum b_i x^{\eta_i} F_{\alpha_i}^{k_i} + H_\beta^k,$$

*with $b_i \in \Bbbk[\widetilde{C}]$, $F_{\alpha_i}^{k_i} \in s\overline{G}$, $x^{\eta_i} \preceq_{lex} x^\delta$, $x^{(\eta_j)_{\text{sat}}}$ is multiplicative with respect to $x^{\alpha_j}\mathbf{e}_{k_j} \in \mathbf{P}(\underline{\mathcal{V}})$ and $H_\beta^k = \overline{H}_\beta^k + x_0^q \widetilde{H}_\beta^k$ with $H_\beta^k$ strongly reduced, $x_0^q$ does not divide any module term in $\operatorname{supp}(\overline{H}_\beta^k)$ and $\operatorname{coeff}(\overline{H}_\beta^k) \subseteq \mathcal{Q}$.*

*(ii) For every module element $F_\alpha^k \in s\overline{G}$ and for every $i > \operatorname{cls}(x^{\alpha_{\text{sat}}})$ there exists a $q$ such that we have a formula of type*

$$x_0^q x_i F_\alpha^k = \sum b_j x^{\eta_j} F_{\alpha_j}^{k_j} + H_{i,\alpha}^k$$

*where $b_j \in \Bbbk[\widetilde{C}]$, $F_{\alpha_j}^{k_j} \in s\overline{G}$, $x^{\eta_j} \prec_{lex} x_i$, $x^{(\eta_j)_{\text{sat}}}$ is multiplicative with respect to $x^{\alpha_j}\mathbf{e}_{k_j} \in \mathbf{P}(\underline{\mathcal{V}})$ and $H_{i,\alpha}^k$ is strongly reduced with $\operatorname{coeff}(H_{i,\alpha}^k) \subseteq \mathcal{Q}$.*

*Then $\mathcal{Q} = \langle D_1 \cup D_2 \rangle = \widetilde{\mathcal{R}}$.*

*Proof.* Due to (i) we immediately observe that $D_1 \subseteq \mathcal{Q}$.

For the inclusion $D_2 \subseteq \mathcal{Q}$ we show that if (i) and (ii) hold for $\mathcal{Q}$ we obtain that for every $F_\alpha \in s\overline{G}$ and for every $x^\delta$ there exists a $q$ such that

$$x_0^q x^\delta F_\alpha^k = \sum b_j x^{\eta_j} F_{\alpha_j}^{k_j} + H \tag{4.3}$$

with $b_j \in \Bbbk[\widetilde{C}]$, $F_{\alpha_j}^{k_j} \in s\overline{G}$, $x^{\eta_j} \prec_{\texttt{lex}} x^\delta$, $x^{(\eta_j)_{\text{sat}}}$ is multiplicative with respect to $x^{\alpha_j}\mathbf{e}_{k_j} \in \mathbf{P}(\underline{\mathcal{V}})$ and $H$ is strongly reduced with $\operatorname{coeff}(H) \subseteq \mathcal{Q}$.

For $\deg(x^\delta) = 1$ the statement is obviously be true, due to (ii). Assume that $\deg(x^\delta) > 1$ and that the thesis holds for every $x^{\delta'} \prec_{\texttt{lex}} x^\delta$. Let $i = \operatorname{cls}(x^\delta)$ and $x^{\delta'} = \frac{x^\delta}{x_i}$.

By the inductive hypothesis we have an integer $q$ such that

$$x_0^q x^{\delta'} F_\alpha^k = \sum b'_j x^{\eta'_j} F_{\alpha_j}^{k_j} + H_{\delta',\alpha}^k,$$

with $x^{\eta_j} \prec_{\texttt{lex}} x^{\delta'}$. So, multiplying by $x_i$ we obtain

$$x_0^q x^\delta F_\alpha^k = \sum b'_j x_i x^{\eta'_j} F_{\alpha_j}^{k_j} + x_i H_{\delta',\alpha}^k$$

and the thesis holds for every module element $x_i x^{\eta'_j} F^{k_j}_{\alpha_j}$ because $x_i x^{\eta'_j} \prec_{\texttt{lex}} x_i x^{\delta'} = x^{\delta}$. Then, we replace such module elements by formula (4.3) and obtain for $q' \geq q$

$$x_0^{q'} x^{\delta} F^k_{\alpha} = \sum b_s x^{\eta_s} F^{k_s}_{\alpha_s} + H' + x_i H^k_{\delta',\alpha}$$

where the first sum satisfies the conditions of (4.3) and $H'$ is strongly reduced with $\mathrm{supp}(H') \subset \mathcal{N}(\mathcal{V})$ and $\mathrm{coeff}(H) \subseteq \mathcal{Q}$.

Note that $\mathrm{coeff}(x_i H^k_{\delta',\alpha}) = \mathrm{coeff}(H^k_{\delta',\alpha}) \subseteq \mathcal{Q}$, but we do not know if $\mathrm{supp}(x_i H^k_{\delta',\alpha}) \subseteq \mathcal{N}(\mathcal{V})$. If the coefficient of $x^{\beta'} \mathbf{e}_l \in \mathrm{supp}(H^k_{\delta',\alpha})$ is $b$ and $x^{\beta} \mathbf{e}_k = x_i x^{\beta'} \mathbf{e}_k \in \mathcal{V}$ then we can use (i) obtaining an integer $q''$ such that $x_0^{q''} b x^{\beta} \mathbf{e}_k = \sum b a_l x^{\gamma_i} F^{k_l}_{\alpha_l} + b H^k_{\beta}$. Moreover if $x^{\beta} \mathbf{e}_k = x^{\epsilon} x^{\alpha} \mathbf{e}_k$ such that $x^{\epsilon}$ is multiplicative for $x^{\alpha} \mathbf{e}_k \in \mathbf{P}(\mathcal{V})$ then $x^{\gamma_i} \preceq_{\texttt{lex}} x^{\epsilon} \prec_{\texttt{lex}} x_i \prec_{\texttt{lex}} x^{\delta}$, where the second inequality is due to the fact that $x^{\beta'} \in \mathcal{N}(\mathcal{V})$ and to Lemma 1.55. All coefficients of $H^k_{\beta}$ belong to $Q$ because they are divisible by $b$. Replacing all such module terms $x^{\beta} \mathbf{e}_k$ we obtain the statement and $H$ is strongly reduced with coefficients in $\mathcal{Q}$, because it is the sum of strongly reduced module elements with coefficients in $\mathcal{Q}$.

We can also prove the uniqueness of such a rewriting: thanks to the uniqueness of the decompositions of the Pommaret division with respect to $\underline{\mathcal{V}}$, the module elements $x^{\eta_j} F^{k_j}_{\alpha_j}$ that can appear in (4.3) have pairwise different head terms. Hence we can express every module element in $\mathcal{Q}$ in a unique way like in (4.3). And this implies the equality of $\mathcal{Q}$ and $\widetilde{\mathcal{R}}$. □

Proposition 4.53 is very important from the computational point of view. Indeed, its condition (i) allows to explicitly construct the set of module elements $B$, namely to write a $\mathcal{V}$-marked set $\overline{G}'$ in $\Bbbk[\widetilde{C}][\mathbf{x}]$ whose superminimal set is $s\overline{G}$. using such a $\mathcal{V}$-marked set in $\Bbbk[\widetilde{C}][\mathbf{x}]$ we can either use Theorem 4.48 or Theorem 4.49 to obtain a set of generators for $\widetilde{\mathcal{R}}$.

Now we give a pseudo-code description of an algorithm, which computes the affine representation of $\mathbf{Mf}(\mathcal{V})$, where $\mathcal{V}$ is a $t$-truncation. The algorithm is based on Theorem 4.49 and Proposition 4.53. But first of all we define some auxiliary functions, which we need to state the algorithm:

- SUPERMINIMALREDUCTION($\mathbf{f}, sG$): Given an $\mathbf{sP}(\mathcal{V})$-marked superminimal set $sG$ and a module element $\mathbf{f}$, it returns a pair $(q, \mathbf{h})$ where $q$ is the minimal power of $x_0$ such that there is a superminimal reduction of $x_0^q \mathbf{f}$ to a strongly reduced module element and $\mathbf{h}$ is such a module element, namely $x_0^q \mathbf{f} \xrightarrow{sG} \mathbf{h}$.
- QUOTIENTANDREMAINDER($\mathbf{f}, q$): Given a module element $\mathbf{f}$ and a non-negative integer $q$, it returns the pair of module elements $(\overline{H}, \widetilde{H})$ such that $\mathbf{f} = \overline{H} + x_0^q \widetilde{H}$.

- COEFF($\mathbf{f}, x^\alpha \mathbf{e}_k$): It returns the coefficient of the module term $x^\alpha \mathbf{e}_k$ in the module element $\mathbf{f}$. If $x^\alpha \mathbf{e}_k \notin \operatorname{supp}(\mathbf{f})$ the method returns zero.

**Algorithm 13** MARKEDSCHEME($\mathcal{V}$)

**Input:** $\mathcal{V}$ quasi-stable $t$-truncated module
**Output:** An ideal defining the marked scheme $\mathbf{Mf}(\mathcal{V})$

$s\overline{G} \leftarrow \emptyset$
**for all** $x^\alpha \mathbf{e}_k \in \mathbf{sP}(\mathcal{V})$ **do**
  $F_\alpha^k \leftarrow x^\alpha \mathbf{e}_k$
  **for all** $x^\beta \mathbf{e}_l \in \mathcal{N}(\mathcal{V})_{\deg(x^\alpha \mathbf{e}_k)}$ **do**
    $F_\alpha^k \leftarrow C_{\alpha\beta kl} x^\beta \mathbf{e}_l$
  **end for**
  $s\overline{G} \leftarrow s\overline{G} \cup \{F_\alpha^k\}$
**end for**
$\mathcal{E} \leftarrow \emptyset$
$\overline{\mathbf{P}}(\mathcal{V}) \leftarrow \mathbf{P}(\mathcal{V}) \setminus \mathbf{sP}(\mathcal{V})$
**for all** $x^\alpha \mathbf{e}_k \in \overline{\mathbf{P}}(\mathcal{V})$ **do**
  $(q, H) \leftarrow$ SUPERMINIMALREDUCTION($x^\alpha \mathbf{e}_k, s\overline{G}$)
  $(\overline{H}, \widetilde{H}) \leftarrow$ QUOTIENTANDREMAINDER($H, q$)
  **for all** $x^\eta \mathbf{e}_l \in \operatorname{supp}(\overline{H})$ **do**
    $\mathcal{E} \leftarrow \mathcal{E} \cup \{$ COEFF($\overline{H}, x^\eta \mathbf{e}_l$) $\}$
  **end for**
**end for**
$L_1 \leftarrow \{x_i H_\alpha^k \mid H_\alpha^k \in s\overline{G}$ and $i > \operatorname{cls}(x^{\alpha_{\text{sat}}})\}$
**for all** $x_i H_\alpha^k \in L_1$ **do**
  $(q, H) \leftarrow$ SUPERMINIMALREDUCTION($x_i H_\alpha^K, s\overline{G}$)
  **for all** $x^\eta \mathbf{e}_l \in \operatorname{supp}(\overline{H})$ **do**
    $\mathcal{E} \leftarrow \mathcal{E} \cup \{$ COEFF($\overline{H}, x^\eta \mathbf{e}_l$) $\}$
  **end for**
**end for**
**return** $\mathcal{E}$

**Theorem 4.54.** *Algorithm 13 terminates and is correct.*

*Proof.* To prove that the algorithm terminates it is sufficient to recall Theorem 4.43 (i) where it is proved that the superminimal reduction is noetherian.

Hence we only have to show that the algorithm is correct, that is it returns a set of generators for the ideal defining $\mathbf{Mf}(\mathcal{V})$. The starting point is the $\mathcal{V}$-marked superminimal set $s\overline{G}$, having parameters in $\widetilde{C}$ as coefficients of every module term in the tails and get a set $\mathcal{E}$ of polynomials in $\Bbbk[\widetilde{C}]$. We will show that the ideal generated by $\mathcal{E}$ coincides with the ideal $\mathcal{Q}$ defined in Theorem 4.52.

Indeed in the first part (lines (11)-(17)), the algorithm computes the superminimal reduction $H$ of each module term $x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathcal{V}) \setminus \mathbf{sP}(\mathcal{V})$ and it imposes the conditions required by Proposition 4.53 (i). This means that the algorithm computes the set $D_1$ defined in Definition 4.51.

In the second part (lines (18)-(24)), the algorithm considers non-multiplicative prolongations $x_i H^k_\alpha$ of superminimal generators $H^k_\alpha$, such that $x_i x^\alpha \mathbf{e}_k = x^\eta x^{\alpha_{\text{sat}}} \mathbf{e}_k$. Then $x^\eta \prec_{\texttt{lex}} x_i$ by Lemma 4.36. These prolongations correspond to the set $L_1$ defined in Theorem 4.49.

At line (20) of the algorithm we compute the superminimal reduction of the associated non-multiplicative prolongations

$$x_0^q x_i H^k_\alpha \xrightarrow{sG} H\,,$$

that is applying Lemma 4.47:

$$x_0^q x_i H^k_\alpha = H + \sum b_j x^{\eta_j} F^{k_j}_{\beta_j}$$

with $b_j \in \Bbbk[\widetilde{C}]$, $F^{k_j}_{\beta_j} \in s\overline{G}$, $x^{\eta_j} \prec_{\texttt{lex}} x_i x_0^t$ and $x^{(\eta_j)_{\text{sat}}} \prec_{\texttt{lex}} x_i$.

The module element $H$ is strongly reduced and it belongs to the module $\langle s\overline{G}\rangle \subseteq \Bbbk[\widetilde{C}][\mathbf{x}]$, hence its coefficients belong to $D_2 \subseteq \mathcal{Q}$.

Then by construction, $\mathcal{Q}$ is contained in $\widetilde{\mathcal{R}}$ and it satisfies the condition required by Proposition 4.53 (ii), hence $\mathcal{Q} = \widetilde{\mathcal{R}}$. □

### 4.4.2 Computing Marked Families Using Dehomogenization

The main problem of Algorithm 13 is the computation of superminimal reductions. Assume that we compute the superminimal reduction of an element $\mathbf{f}$. Then we have to determine a non-negative $q$, such that there is a reduction from $x_0^q \mathbf{f}$ to a strongly reduced module element. The value of the integer $q$ cannot be predicted before the superminimal reduction and due to that $q$ could be arbitrary large. As we have to multiply several times with $x_0^{q'}$ with $q' \leq q$ during the superminimal reduction this is one bottleneck for the superminimal reduction.

In the following we use another approach which computes exactly the same result than Algorithm 13 but without using the costly superminimal reduction. It is the straightforward generalization of an idea from [10].

Let us consider the quasi-stable module $\mathfrak{V} \subseteq A[x_1, \dots, x_n]^m$ and a non-negative integer $t$. We know that the module $\mathcal{V} = \mathcal{B}^{\min}(\mathfrak{V}) \cdot A[x_0, \dots, x_n]^m$ is a saturated quasi-stable module in $A[x_0, \dots, x_n]^m$. We define $\mathbf{h}^h$ as the homogenization of $\mathbf{h} \in A[x_1, \dots, x_n]$ with the variable $x_0$ and $\mathfrak{V}^h$ as the homogenization of $\mathfrak{V}$ with the variable $x_0$. Then we immediately see that $\mathcal{V}_{\geq t} = (\mathfrak{V}^h)_{\geq t}$. In the following we denote as usual $A[x_0, \dots, x_n]$ as $A[\mathbf{x}]$ and $A[x_1, \dots, x_n]$ as $A[\mathbf{x}']$. Originally we defined a marked module element as a homogeneous module element. In the following do not demand that a marked module element in $A[\mathbf{x}']^m$ must be homogeneous, anymore.

**Definition 4.55.** *Let $\mathfrak{V} \subseteq A[\mathbf{x}']^m$ be a quasi-stable module and $\mathcal{V} = \mathfrak{V}^h$.*

- *A* $[\mathbf{P}(\mathfrak{V}), t]$-marked set $\mathfrak{B}$ *is a finite set of monic marked module elements $\mathbf{f}_\alpha^k \in A[\mathbf{x}']$ such that the head module terms $\operatorname{Ht}(\mathbf{f}_\alpha^k) = x^\alpha \mathbf{e}_k$ are pairwise different and form the Pommaret basis $\mathbf{P}(\mathfrak{V})$ and $\operatorname{supp}(\mathbf{f}_\alpha^k - x^\alpha \mathbf{e}_k) \subseteq \mathcal{N}(\mathfrak{V})_{\leq s}$ with $s = \max(\{t, \deg(x^\alpha \mathbf{e}_k)\})$.*
- *A $[\mathbf{P}(\mathfrak{V}), t]$-marked set $\mathfrak{B}$ is a* $[\mathbf{P}(\mathfrak{V}), t]$-marked basis *if there is a $\mathbf{P}(\mathcal{V}_{\geq t})$-marked basis $\mathcal{B}$ and a non-negative integer $q_\alpha$ for every $x^\alpha \mathbf{e}_k \in \mathbf{P}(\mathfrak{V})$ such that $x_0^{q_\alpha} \mathbf{f}_\alpha^k$ belongs to $\mathcal{B}$.*

**Definition 4.56.** *Let $\mathfrak{B}$ be a $[\mathbf{P}(\mathfrak{V}), t]$-marked set. We define*

$$\widetilde{\mathfrak{B}} := \left\{ x^\delta \mathbf{f}_\alpha^k \mid \mathbf{f}_\alpha^k \in \mathfrak{B} \text{ and } x^\delta \in A[X_P(x^\alpha \mathbf{e}_k)] \right\} .$$

*We will denote by $\xrightarrow{\widetilde{\mathfrak{B}}}$ the transitive closure of the following reduction relation on $A[\mathbf{x}']$: $\mathbf{f}$ is in relation with $\mathbf{f}'$ if $\mathbf{f}' = \mathbf{f} - c x^\delta \mathbf{f}_\alpha^k$ where $x^\delta \mathbf{f}_\alpha^k \in \widetilde{\mathfrak{B}}$ and $c$ is the coefficient of the module term $x^\alpha x^\delta \mathbf{e}_k$ in $\mathbf{f}$. We will write $\mathbf{f} \xrightarrow{\widetilde{\mathfrak{B}}}_* \mathbf{h}$ if $\mathbf{f} \xrightarrow{\widetilde{\mathfrak{B}}} \mathbf{h}$ and $\mathbf{h} \in \langle \mathcal{N}(\mathfrak{V}) \rangle$.*

**Proposition 4.57.** *Let $\mathfrak{B}$ be a $[\mathbf{P}(\mathfrak{V}), t]$-marked set.*

*(i) The reduction relation $\xrightarrow{\widetilde{\mathfrak{B}}}$ is noetherian and confluent.*

*(ii) For every module element $\mathbf{f} \in A[\mathbf{x}']^m$ there is $\mathbf{h} \in \langle \mathcal{N}(\mathfrak{V}) \rangle$ such that $\mathbf{f} \xrightarrow{\widetilde{\mathfrak{B}}}_* \mathbf{h}$. The module element $\mathbf{h}$ is a $\mathfrak{V}$-reduced form of $\mathbf{f}$ modulo the module $\langle \mathfrak{B} \rangle$.*

*(iii) $A[\mathbf{x}']^m = \langle \widetilde{\mathfrak{B}} \rangle \oplus \langle \mathcal{N}(\mathfrak{V}) \rangle$.*

*Proof.* The proof is analogous to the proof of [10, Prop. 4.3]. □

Even if the definition of $[\mathbf{P}(\mathfrak{V}), t]$-marked basis relies on the existence of a homogeneous marked basis in $A[\mathbf{x}]^m$ the following theorem shows that it is equivalent to consider a condition which involves only $\mathfrak{V}$-reduced forms modulo $\langle \mathfrak{B} \rangle$ and their degrees.

**Theorem 4.58.** *Let $\mathfrak{B}$ be a $[\mathbf{P}(\mathfrak{V}), t]$-marked set. It is a $[\mathbf{P}(\mathfrak{V}), t]$-marked basis if and only if every module element $\mathbf{f} \in A[\mathbf{x}']^m$ has a $\mathfrak{V}$-normal form $\mathbf{h}$ modulo $\langle \mathfrak{B} \rangle$ with $\deg(\mathbf{h}) \leq \max(\{\deg(\mathbf{f}), t\})$.*

*Proof.* One direction of the proof is analogous to the proof of [10, Thm. 4.4] and the other direction is analogous to the proof of [10, Thm. 4.5] $\square$

Let $\mathfrak{B}$ be a $[\mathbf{P}(\mathfrak{V}), t]$-marked basis. By Theorem 4.58 we can consider for every $x^\beta \mathbf{e}_k \in \mathfrak{V}_{\leq t} \setminus \mathbf{P}(\mathfrak{V})$ the module element $\mathbf{f}^k_\beta := x^\beta \mathbf{e}_k - \mathbf{h}$ where $\mathbf{h}$ is the $\mathfrak{V}$-normal form of $x^\beta \mathbf{e}_k$ modulo $\langle \mathfrak{B} \rangle$ which belongs to $\langle \mathcal{N}(\mathfrak{V})_{\leq t} \rangle$. We denote by $\overline{\mathfrak{B}}$ the set $\{\mathbf{f}^k_\beta \mid x^\beta \mathbf{e}_k \in \mathfrak{V}_{\leq t} \setminus \mathbf{P}(\mathfrak{V})\} \subseteq \langle \mathfrak{B} \rangle$.

**Corollary 4.59.** *There is a bijective correspondence between the set of $[\mathbf{P}(\mathfrak{V}), t]$-marked bases and the set of $\mathbf{P}(\mathcal{V}_{\geq t})$-marked bases.*

*Proof.* The proof is analogous to the proof of [10, Cor. 4.8]. $\square$

Finally, we show that $[\mathbf{P}(\mathfrak{V}), t]$-marked bases have the expected good behaviour with respect to the homogenization. We define $\mathfrak{B}^h := \{(\mathbf{f}^k_\alpha)^h \mid \mathbf{f}^k_\alpha \in \mathfrak{B}\}$ and $\widetilde{\mathfrak{B}}^h := \{(\mathbf{f}^k_\alpha)^h \mid \mathbf{f}^k_\alpha \in \widetilde{\mathfrak{B}}\}$.

**Theorem 4.60.** *If $\mathfrak{B}$ is a $[\mathbf{P}(\mathfrak{V}), t]$-marked basis and $\mathcal{B}$ is its corresponding $\mathbf{P}(\mathcal{V}_{\geq t})$-marked basis then $\langle \mathcal{B} \rangle = \langle \mathfrak{B} \rangle^h_{\geq t} = \langle \mathfrak{B}^h \cup \widetilde{\mathfrak{B}}^h \rangle_{\geq t}$ and $\langle \mathfrak{B} \rangle = \langle \mathfrak{B}^h \cup \widetilde{\mathfrak{B}}^h \rangle^{\mathrm{sat}} = \langle \mathcal{B} \rangle^{\mathrm{sat}}$.*

*Proof.* The proof is analogous to the proof of [10, Thm. 4.10]. $\square$

The next theorem gives us an effective criterion for $[\mathbf{P}(\mathfrak{V}), t]$-marked bases.

**Theorem 4.61.** *A $[\mathbf{P}(\mathfrak{V}), t]$-marked set $\mathfrak{B}$ is a $[\mathbf{P}(\mathfrak{V}), t]$-marked basis if and only if the two following statements hold:*

*(i) For every $x^\beta \mathbf{e}_k \in \mathfrak{V}_{\leq t} \setminus \mathbf{P}(\mathfrak{V})$, $x^\beta \mathbf{e}_k \xrightarrow{\widetilde{\mathfrak{B}}}{}^* \mathbf{h}^k_\alpha$ with $\mathbf{h}^k_\alpha \in \langle \mathcal{N}(\mathfrak{V})_{\leq t} \rangle^A$.*

*(ii) For every $\mathbf{f}^k_\alpha \in \mathfrak{B}$ and $i > \mathrm{cls}(x^\alpha)$, $x_i \mathbf{f}^k_\alpha \xrightarrow{\widetilde{\mathfrak{B}}}{}^* 0$.*

*Proof.* The proof is analogous to the proof of [10, Thm. 5.1]. $\square$

**Example 4.62.** *Consider $\mathfrak{V} \subset \Bbbk[x_1, x_2]$ which is generated by the Pommaret basis*

$$\mathbf{P}(\mathfrak{V}) = \{x_2^2\mathbf{e}_1, x_2x_1^2\mathbf{e}_1, x_1^2\mathbf{e}_1, x_2^3\mathbf{e}_2, x_2^2x_1\mathbf{e}_2\}\,.$$

*As a first example we consider the set*

$$\mathfrak{B}_1 = \{x_2^2\mathbf{e}_1 + x_1^4\mathbf{e}_2, x_2x_1^2\mathbf{e}_1, x_1^2\mathbf{e}_1, x_2^3\mathbf{e}_2, x_2^2x_1\mathbf{e}_2 + x_1x_2\mathbf{e}_1\}\,.$$

*This set is obviously a $[\mathbf{P}(\mathfrak{V}), 4]$-marked set. But it is not a $[\mathbf{P}(\mathfrak{V}), 4]$-marked basis because $x_2(x_2^2x_1\mathbf{e}_2 + x_1x_2\mathbf{e}_1) \xrightarrow{\widetilde{\mathfrak{B}_1}}{}^* -x_1^5\mathbf{e}_1$, which violates the second condition of Theorem 4.61.*
*As a second example we consider the set*

$$\mathfrak{B}_2 = \{x_2^2\mathbf{e}_1 + x_1^4\mathbf{e}_2, x_2x_1^2\mathbf{e}_1, x_1^2\mathbf{e}_1, x_2^3\mathbf{e}_2, x_2^2x_1\mathbf{e}_2\}\,.$$

*This set is obviously a $[\mathbf{P}(\mathfrak{V}), t]$-marked set for every $t \geq 4$, but it is not a $[\mathbf{P}(\mathfrak{V}), 4]$-marked basis. Due to the first condition of Theorem 4.61 $x_2^3\mathbf{e}_1$ must reduces to $\mathbf{h} \in \langle \mathcal{N}(\mathfrak{V})_{\leq 4}\rangle^{\Bbbk}$. But $x_2^3\mathbf{e}_1 \xrightarrow{\widetilde{\mathfrak{B}_2}}{}^* -x_2x_1^4\mathbf{e}_2 \notin \langle \mathcal{N}(\mathfrak{V})\rangle_{\leq 4}$.*
*However, one can easily check that it is a $[\mathbf{P}(\mathfrak{V}), t]$-marked basis for $t \geq 5$.*

In the last part of the example we have seen an example of a given $[\mathbf{P}(\mathfrak{V}), t]$-marked set which is always a $[\mathbf{P}(\mathfrak{V}), t]$-marked basis if $t$ is large enough. The next theorem shows that if a $[\mathbf{P}(\mathfrak{V}), t]$-marked set is a $[\mathbf{P}(\mathfrak{V}), t]$-marked basis for a large $t$, then it is always a $[\mathbf{P}(\mathfrak{V}), t']$ for $t'$ large enough. In addition, it gives a precise bound, how large $t$ must be.

**Theorem 4.63.** *Let $\mathfrak{B}$ be a $[\mathbf{P}(\mathfrak{V}), t]$-marked set, with $t \geq \operatorname{sat}(\mathfrak{V})$. $\mathfrak{B}$ is a $[\mathbf{P}(\mathfrak{V}), t-1]$-marked basis if and only if $\mathfrak{B}$ is a $[\mathbf{P}(\mathfrak{V}), t]$-marked basis.*

*Proof.* The proof is analogous to the proof of [10, Thm. 5.4]. □

Using the theorem above, we can improve Theorem 4.61 for large $t$.

**Corollary 4.64.** *Let $\mathfrak{B}$ be a $[\mathbf{P}(\mathfrak{V}), t]$-marked basis with $t \geq \operatorname{sat}(\mathfrak{V})$.*

- *$\mathfrak{B}$ is a $[\mathbf{P}(\mathfrak{V}), t]$-marked basis if and only if it is a $[\mathbf{P}(\mathfrak{V}), t-1]$-marked basis.*
- *$\mathfrak{B}$ is a $[\mathbf{P}(\mathfrak{V}), \operatorname{sat}(\mathfrak{V}) - 1]$-marked basis if and only if*
  - *(i) For every $x^\beta\mathbf{e}_k \in \mathfrak{V}_{\leq \operatorname{sat}(\mathfrak{V})-1} \setminus \mathbf{P}(\mathfrak{V})$ there is a reduction $x^\beta\mathbf{e}_k \xrightarrow{\widetilde{\mathfrak{B}}}{}^* \mathbf{h}_\alpha^k$ with $\mathbf{h}_\alpha^k \in \langle \mathcal{N}(\mathfrak{V})_{\leq \operatorname{sat}(\mathfrak{V})-1}\rangle^A$.*
  - *(ii) For every $\mathbf{f}_\alpha^k \in \mathfrak{B}$ and $i > \operatorname{cls}(x^\alpha)$, $x_i\mathbf{f}_\alpha^k \xrightarrow{\widetilde{\mathfrak{B}}}{}^* 0$.*

It turns out that we can simplify the check for a $[\mathbf{P}(\mathfrak{V}), \mathrm{sat}(\mathfrak{V})-1]$-marked basis even more, if $A[\mathbf{x}']^m/\mathfrak{V}$ is an artinian module, that is $\mathcal{N}(\mathfrak{V})$ is a finite set of terms and we have that $\mathcal{N}(\mathfrak{V})_{\mathrm{reg}(\mathfrak{V})-1} = \mathcal{N}(\mathfrak{V})_{\mathrm{reg}(\mathfrak{V})+r}$ for a non-negative integer $r$. Furthermore, it is equivalent to the fact, that $\mathsf{HP}_{A[\mathbf{x}']/\mathfrak{V}}(t) = 0$.

**Proposition 4.65.** *Let $\mathfrak{B}$ be a $[\mathbf{P}(\mathfrak{V}), t]$-marked set with $t \geq \mathrm{sat}(\mathfrak{V})$. If $\mathcal{N}(\mathfrak{V})$ is finite then $\mathfrak{B}$ is a $[\mathbf{P}(\mathfrak{V}), \mathrm{sat}(\mathfrak{V})-1]$-marked basis if and only if $x_i\mathbf{f}^k_\alpha \xrightarrow{\widetilde{\mathfrak{B}}}_* 0$ for every $\mathbf{f}^k_\alpha \in \mathfrak{B}$ and for every $i > \mathrm{cls}(x^\alpha)$.*

*Proof.* The proof is analogous to the proof of [10, Prop. 5.6]. □

Now we describe a new algorithm for computing the marked scheme corresponding to the saturated quasi-stable ideal $\mathcal{V} \subseteq \Bbbk[\mathbf{x}]^m$. Let $\mathfrak{V} \subseteq \Bbbk[\mathbf{x}']^m$ the corresponding dehomogenization of $\mathcal{V}$, that is $\mathfrak{V} := \langle \mathcal{B}^{\min}(\mathcal{V})\rangle \cdot \Bbbk[\mathbf{x}']^m$.

For every $x^\alpha\mathbf{e}_k \in \mathbf{P}(\mathfrak{V})$ let $q_\alpha := \max(\{t, \deg(x^\alpha\mathbf{e}_k)\})$. We define the following set of parameters

$$\mathfrak{C} := \{\mathfrak{C}_{\alpha\eta kl} \mid x^\alpha\mathbf{e}_k \in \mathbf{P}(\mathfrak{V}),\ x^\eta\mathbf{e}_l \in \mathcal{N}(\mathfrak{V})_{\leq q_\alpha}\}$$

and construct the $\mathbf{P}(\mathfrak{V})$-marked set $\mathfrak{G} \subset \Bbbk[\mathfrak{C}][\mathbf{x}']^m$ consisting of all elements

$$\mathfrak{F}^k_\alpha := \left(x^\alpha - \sum_{x^\eta\mathbf{e}_k \in \mathcal{N}(\mathfrak{V})_{\leq q_\alpha}} \mathfrak{C}_{\alpha\eta kk}x^\eta\right)\mathbf{e}_k - \sum_{\substack{x^\eta\mathbf{e}_l \in \mathcal{N}(\mathcal{V})_{\leq q_\alpha} \\ l \neq k}} \mathfrak{C}_{\alpha\eta kl}x^\eta\mathbf{e}_l$$

with $x^\alpha\mathbf{e}_k \in \mathbf{P}(\mathfrak{V})$.

For every $x^\beta\mathbf{e}_k \in \mathfrak{V}_{\leq t} \setminus \mathbf{P}(\mathfrak{V})$, let $\mathbf{h}^k_\beta$ the unique module element in $\langle \mathcal{N}(\mathfrak{V})\rangle^{\Bbbk[\mathfrak{C}]}$ such that $x^\beta\mathbf{e}_k \xrightarrow{\widetilde{\mathfrak{G}}}_* \mathbf{h}^k_\beta$. We write $\mathbf{h}^k_\beta = (\mathbf{h}^k_\beta)^{(\leq t)} + (\mathbf{h}^k_\beta)^{(>t)}$ with $(\mathbf{h}^k_\beta)^{(\leq t)} \in \langle \mathcal{N}(\mathfrak{V})_{\leq t}\rangle^{\Bbbk[\mathfrak{C}]}$ and $(\mathbf{h}^k_\beta)^{(>t)} \in \langle \mathcal{N}(\mathfrak{V})_{\leq \deg(\mathbf{h}^k_\beta)} \setminus \mathcal{N}(\mathfrak{V})_{\leq t}\rangle^{\Bbbk[\mathfrak{C}]}$.

For every $\mathfrak{F}^k_\alpha \in \mathfrak{G}$ and for every $i > \mathrm{cls}(x^\alpha)$, let $\mathbf{h}^k_{\alpha,i}$ be the unique module element in $\langle \mathcal{N}(\mathfrak{V})\rangle \subseteq \Bbbk[\mathfrak{C}][\mathbf{x}']$, such that $x_i\mathfrak{F}^k_\alpha \xrightarrow{\widetilde{\mathfrak{G}}}_* \mathbf{h}^k_{\alpha,i}$.

Let $\mathfrak{R} \subseteq \Bbbk[\mathfrak{C}]$ be the ideal generated by the coefficients in $\Bbbk[\mathfrak{C}]$ of the module elements $(\mathbf{h}^k_\beta)^{(>t)}$ for every $x^\beta\mathbf{e}_k \in \mathfrak{V}_{\leq t} \setminus \mathbf{P}(\mathfrak{V})$ and those of the module elements $\mathbf{h}^k_{\alpha,i}$ for every $\mathfrak{F}^k_\alpha \in \mathfrak{G}$ and every $i > \mathrm{cls}(x^\alpha)$.

**Theorem 4.66.** *The marked scheme $\boldsymbol{Mf}(\mathcal{V}_{\geq t})$ is defined by the ideal $\langle \mathfrak{R}\rangle$ as a subscheme of the affine space $\mathbb{A}^{|\mathfrak{C}|}$.*

*Proof.* Using the same arguments as in Theorem 3.44 we can show that the set of all $[\mathbf{P}(\mathfrak{V}), t]$-marked bases over noetherian $\Bbbk$-algebras can be represented by the affine scheme $\operatorname{Spec}(\Bbbk[\mathfrak{C}]/\langle \mathfrak{R} \rangle)$. We have seen in Corollary 4.59 that there is a bijective correspondence between $[\mathbf{P}(\mathfrak{V}), t]$-marked bases and $\mathbf{P}(\mathcal{V}_{\geq t})$-marked bases. Therefore, $\operatorname{Spec}(\Bbbk[\mathfrak{C}]/\langle \mathfrak{R} \rangle)$ is also the representation of all $\mathbf{P}(\mathcal{V}_{\geq t})$-marked bases over noetherian $\Bbbk$-algebras and so of $\mathbf{Mf}(\mathcal{V}_{\geq t})$. □

**Remark 4.67.** *The ideal $\mathfrak{R}$ constructed above is exactly the ideal $\widetilde{\mathcal{R}}$ constructed by* super-minimal reduction *of homogeneous module elements in Theorem 4.52 for a marked scheme $\mathbf{Mf}(\mathcal{V}_{\geq t})$.*

Now we describe another algorithm to compute marked schemes $\mathbf{Mf}(\mathcal{V}_{\geq t})$, where $\mathcal{V} \subseteq \Bbbk[\mathbf{x}]^m$ is a saturated quasi-stable module and $t$ is an integer. For every $\mathcal{V}_{\geq t}$ there exists a quasi-stable module $\mathfrak{V} \subseteq \Bbbk[\mathbf{x}']$, such that $\mathcal{B}^{\min}(\mathfrak{V}) \cdot \Bbbk[\mathbf{x}] = \mathcal{V}$ and $\mathcal{V}_{\geq t} = \mathfrak{V}^h_{\geq t}$.

For simplicity, we choose as input of Algorithm 16 $\mathfrak{V} \subseteq \Bbbk[\mathbf{x}']$ and the non-negative integer $t$. The output of the algorithm is a set of equations which defines the ideal $\langle \mathfrak{R} \rangle$, which corresponds to the marked scheme $\mathbf{Mf}(\mathcal{V}_{\geq t})$.

At first, we define some additionally auxiliary functions, which we need for stating the algorithm:

- LOWERPART$(\mathbf{g}, q)$: Given a module element $\mathbf{g}$ and a non-negative integer $q$, it returns the pair of module elements $(\mathbf{g}_1, \mathbf{g}_2)$ such that $\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2$, $\deg(\mathbf{g}_1) \leq q$ and for every $x^\gamma \mathbf{e}_k \in \operatorname{supp}(\mathbf{g}_2)$, $\deg(x^\gamma \mathbf{e}_k) \geq q + 1$.
- REDUCTION$(\mathbf{g}, \mathfrak{G})$: Given a $[\mathbf{P}(\mathfrak{V}), t]$-marked set $\mathfrak{G}$ and a module element $\mathbf{g}$, it returns the module element $\mathbf{h}$ such that $\mathbf{g} \xrightarrow{\widetilde{\mathfrak{G}}}_* \mathbf{h}$ with $\mathbf{h} \in \langle \mathcal{N}(\mathfrak{V}) \rangle$ according to Definition 4.56.
- LOWTERMS$(\mathfrak{V}, t)$: Given a quasi-stable module $\mathfrak{V}$ it determines the set of module terms in $\mathfrak{V}_{\leq t}$.

Algorithm 14 corresponds to the first condition of Theorem 4.61 and Algorithm 15 corresponds to the second condition of Theorem 4.61. We have seen that we can skip many reduction in the case, when $A[\mathbf{x}']^m/\mathfrak{V}$ is an artinian module. We have taken this into account at Algorithm 16 line (11). The improvement of Theorem 4.63 is taken into account at Algorithm 16 line (2).

**Algorithm 14** REDUCTIONONE($\mathfrak{V}, t, \mathfrak{G}$)

**Input:** $\mathfrak{V} \subseteq \Bbbk[\mathbf{x}']^m$, a quasi-stable module
**Input:** $t$ a non-negative integer
**Input:** $\mathfrak{G}$ a $[\mathbf{P}(\mathfrak{V}), t]$-marked set
**Output:** Conditions to impose on the module elements in $\mathfrak{G}$ to fulfil Theorem 4.61 (i)

```
ℰ ← ∅
ℬ ← LowTerms(𝔙, t) \ P(𝔙)
for all x^β e_k ∈ ℬ do
    h ← Reduction(x^β e_k, 𝔊)
    (h_1, h_2) ← LowerPart(h, t)
    for all x^γ e_l ∈ supp(h_2) do
        ℰ ← ℰ ∪ {Coeff(h_2, x^γ e_l)}
    end for
end for
return ℰ
```

## 4.5 Computation of Concrete Hilbert Schemes

Now we want to use the tool set, which we developed earlier in this chapter, to compute several examples. Algorithms 5, 10 and 16 with their corresponding subalgorithms are implemented experimentally in the computer algebra system COCOALIB([1]). At the moment the implementation of Algorithm 16 can only compute Hilbert schemes. In this chapter we illustrated only some interesting examples. We also tested our experimental implementation with some larger and we were able to compute the Hilbert scheme of points in $\mathbb{P}^3$ up to 14 points.

### 4.5.1 Reducedness of $\mathrm{Hilb}_4^3$

It is still an open question if the Hilbert scheme for $n = 3$ with Hilbert polynomial $\mathsf{HP}(t) = 4$ is reduced or not over a field of characteristic zero. First of all we shortly repeat the definition and some properties of a reduced scheme.

**Definition 4.68.** *Let $X$ be a scheme. It is* reduced *if every local ring $\mathcal{O}_{X,x}$ is reduced, that is if $r \in \mathcal{O}_{X,x}$ with $r^n = 0$ for some $n > 0$, then $r = 0$. Or in other words: the ring $\mathcal{O}_{X,x}$ does not contain any nilpotent elements.*

**Lemma 4.69.** *A scheme $X$ is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for all open subsets $U \subseteq X$.*

**Algorithm 15** REDUCTIONTWO($\mathfrak{V}, t, \mathfrak{G}$)

**Input:** $\mathfrak{V} \subseteq \mathbb{k}[\mathbf{x}']^m$, a quasi-stable module
**Input:** $t$ a non-negative integer
**Input:** $\mathfrak{G}$ a $[\mathbf{P}(\mathfrak{V}), t]$-marked set
**Output:** Conditions to impose on the module elements in $\mathfrak{G}$ to fulfil Theorem 4.61 (ii)

```
ℰ ← ∅
for all f_α^k ∈ 𝔊, i > cls(x^α) do
    h ← REDUCTION(x_i f_α^k, 𝔊)
    for all x^γ e_l ∈ supp(h) do
        ℰ ← ℰ ∪ {COEFF(h_2, x^γ e_l)}
    end for
end for
return ℰ
```

*Proof.* At first we suppose that $X$ is reduced and $U$ is an open subset of $X$. Let $s$ be a section of $\mathcal{O}_X(U)$, such that $s^n = 0$ for some $n > 0$. For each $u \in U$ let $\rho_u : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,u}$ be the map which sends a section $s$ over $U$ to the corresponding stalk at $u$. Then $\rho_u(s^n) = 0$ for all $u \in U$. This implies that $s = 0$ because the map

$$\begin{aligned} \mathcal{O}_X(U) &\longrightarrow \prod_{x \in U} \mathcal{O}_{X,x} \\ s &\longmapsto \{\rho_x(s)\} \end{aligned}$$

is injective. This shows that $\mathcal{O}_X(U)$ is reduced.

Conversely, suppose that $\mathcal{O}_X(U)$ is a reduced ring for every open subset $U \subseteq X$ and let $x \in U$. We pick an element with a representation $(V, f)$ from $\mathcal{O}_{X,x}$. Suppose $f$ is nonzero. Then $f$ is not nilpotent in $\mathcal{O}_X(V)$. Hence, $(V, f)$ is not equivalent to a nilpotent element in $\mathcal{O}_{X,x}$. Hence, $\mathcal{O}_{X,x}$ is reduced. □

**Proposition 4.70.** *Let $X$ be a scheme. Then the following is equivalent:*

*(i) $X$ is reduced.*

*(ii) There exists an affine open covering $X = \bigcup_i U_i$, such that $\mathcal{O}_X(U_i)$ is reduced.*

*(iii) $\mathcal{O}_X(U)$ is reduced for every affine open $U \subseteq X$.*

*Proof.* (i) $\Leftrightarrow$ (ii): $X$ can be covered by affine open sets. Let $\{U_i\}$ be an affine open covering of $X$. We know $\mathcal{O}_X(U_i)$ is reduced for each $i$ by Lemma 4.69. Conversely, assume that $X$ is covered by an affine open covering $\{U_i\}$ such that $U_i = \operatorname{Spec}(A_i)$ with $A_i$ reduced. Consider $x \in U_i$, then $\mathcal{O}_{X,x} = (A_i)_x$. Since $A_i$ is reduced and reducedness does not vanish when localizing we get that $\mathcal{O}_{X,x}$ is reduced.

**Algorithm 16** MARKEDSCHEME($\mathfrak{V}, t$)

**Input:** $\mathfrak{V} \subseteq \Bbbk[\mathbf{x}']^m$, a quasi-stable module
**Input:** $t$ a non-negative integer
**Output:** A set of equations defining the ideal which defines the marked scheme $\mathbf{Mf}(\mathcal{V}_{\geq t})$

```
E ← ∅
t_0 ← min({t, sat(V) − 1})
G ← ∅
for all x^α e_k ∈ P(V) do
    F^k_α ← x^α e_k
    for all x^η e_l ∈ N(V_{≤max({t_0, deg(x^α e_k)})}) do
        F^k_α ← F^k_α + C_{αηkl} x^η e_l
    end for
end for
E ← REDUCTIONTWO(V, t_0, G)
if t_0 = reg(V) − 1 and N(V_{≤reg(V)−1}) = N(V_{≤reg(V)}) then
    return E
end if
E ← E ∪ REDUCTIONONE(V, t_0, G)
return E
```

(i) ⇔ (iii): One direction is clear by Lemma 4.69. Conversely, suppose $\mathcal{O}_X(U)$ is reduced for every affine open $U \subset X$. We can choose an affine open covering of $X$. Hence, by the equivalence of (ii) and (i) the claim follows. □

To check if $\mathbf{Hilb}^3_4$ is reduced we compute an open covering by marked schemes and check if the corresponding rings, which are associated to the marked schemes, are reduced. The reducedness of marked scheme then implies that $\mathbf{Hilb}^3_4$ is reduced.

For $\mathbf{Hilb}^3_4$ we compute the Borel fixed open covering. At first, we have to compute all saturated 0-Borel fixed ideals with Hilbert polynomial $\mathsf{HP}(t) = 4$ in a polynomial ring with four variables. Using our implementation in the computer algebra system CoCoALib, we get the following three saturated 0-Borel fixed ideals.

$$\mathcal{I}_1 = \langle x_1^2, x_2^2, x_2x_1, x_3^2, x_3x_2, x_3x_1 \rangle\,,$$
$$\mathcal{I}_2 = \langle x_3, x_2^2, x_2x_1, x_1^3 \rangle\,,$$
$$\mathcal{I}_3 = \langle x_3, x_2, x_1^4 \rangle\,.$$

For these three ideals we computed the corresponding marked schemes, using CoCoALib and our implementation of Algorithm 16. It turns out that we can embed

$\mathrm{Mf}(\mathcal{I}_1)$ in the affine space $\mathbb{A}^{24}$, $\mathrm{Mf}(\mathcal{I}_2)$ in $\mathbb{A}^{16}$ and $\mathrm{Mf}(\mathcal{I}_3)$ in $\mathbb{A}^{12}$.

For $\mathrm{Mf}(\mathcal{I}_1)$ the algorithm produces 32 equations, for $\mathrm{Mf}(\mathcal{I}_2)$ the algorithm produces 16 equations and for $\mathrm{Mf}(\mathcal{I}_3)$ it only produces one equation, namely 0.

The marked scheme $\mathrm{Mf}(\mathcal{I}_3)$ is a really simple space, because it is equal to $\mathbb{A}^{12}$. Hence, we can immediately deduce that the ring corresponding to $\mathrm{Mf}(\mathcal{I}_3)$ is a simple polynomial ring over $\Bbbk$ with 12 variables which is obviously reduced.

Let $\mathcal{R}_1$ be the ideal generated by the equations corresponding to $\mathrm{Mf}(\mathcal{I}_1)$ and $\mathcal{R}_2$ the ideal generated by the equations corresponding to $\mathrm{Mf}(\mathcal{I}_2)$. We assume that $\mathcal{R}_1 \subseteq \Bbbk[C_1,\dots,C_{32}] := \Bbbk[C']$ and $\mathcal{R}_2 \subseteq \Bbbk[C_1,\dots,C_{16}] := \Bbbk[C'']$. The ring $\Bbbk[C']/\mathcal{R}_1$, respectively $\Bbbk[C'']/\mathcal{R}_2$ is reduced if and only if $\mathcal{R}_1$, respectively $\mathcal{R}_2$ is a radical ideal. To check this we computed with the computer algebra system SINGULAR ([14]) the radicals of $\mathcal{R}_1$ and $\mathcal{R}_2$. It turns out that both ideals are radical ideals.

Finally, this leads to the following theorem:

**Theorem 4.71.** *The Hilbert scheme* $\mathbf{Hilb}^3_4$ *is reduced.*

### 4.5.2 $\mathbf{Quot}^{2,(0,0)}_{2,t+2}$

In the following, we compute an open covering of the Quot-scheme for the Hilbert polynomial $\mathsf{HP}(t) = t+2$ over the polynomial module $\Bbbk[x_0,x_1,x_2]^2_{(0,0)}$, where $\Bbbk$ is a field of characteristic zero. Currently, there is no implementation of Algorithm 16 for computing Quot schemes. Hence, we will do the computations manually.

At first, we have to compute all saturated 0-Borel fixed modules in $\Bbbk[x_0,x_1,x_2]^2_{(0,0)}$. Using the algorithm MONOMIALMODULES ($t+2, \Bbbk[x_0,x_1,x_2]^2_{(0,0)}$, 0-Borel-fixed) (Algorithm 12) we have to compute at first all results of the algorithm DISTRIBUTEPOLYNOMIAL($t+2,((),()),2$) (Algorithm 11). There we get the following distributions:

$$\begin{aligned} &\mathsf{HP}_1 = (1, t+1)\,, \quad \mathsf{HP}_2 = (0, t+2)\,, \\ &\mathsf{HP}_3 = (t+1, 1)\,, \quad \mathsf{HP}_4 = (t+2, 0)\,. \end{aligned} \tag{4.4}$$

This intermediate result leads to the following four saturated 0-Borel fixed modules:

$$\begin{aligned} \mathcal{V}_1 &= \langle x_2\mathbf{e}_1, x_1\mathbf{e}_1, x_2\mathbf{e}_2\rangle\,, & \mathcal{V}_2 &= \langle \mathbf{e}_1, x_2^2\mathbf{e}_2, x_2x_1\mathbf{e}_2\rangle\,, \\ \mathcal{V}_3 &= \langle x_2\mathbf{e}_1, x_2\mathbf{e}_2, x_1\mathbf{e}_2\rangle\,, & \mathcal{V}_4 &= \langle x_2^2\mathbf{e}_1, x_2x_1\mathbf{e}_1, \mathbf{e}_2\rangle\,. \end{aligned}$$

To compute the open covering we have to compute the marked schemes $\mathbf{Mf}((\mathcal{V}_1)_{\geq 2})$, $\mathbf{Mf}((\mathcal{V}_2)_{\geq 2})$, $\mathbf{Mf}((\mathcal{V}_3)_{\geq 2})$ and $\mathbf{Mf}((\mathcal{V}_4)_{\geq 2})$. We have to consider the 2-truncations because the Gotzmann number of $\mathsf{HP}(t)$ is two. It is clear that $\mathbf{Mf}((\mathcal{V}_1)_{\geq 2}) \cong \mathbf{Mf}((\mathcal{V}_3)_{\geq 2})$

and $\mathbf{Mf}((\mathcal{V}_2)_{\geq 2}) \cong \mathbf{Mf}((\mathcal{V}_4)_{\geq 2})$. Hence, we only compute $\mathbf{Mf}((\mathcal{V}_1)_{\geq 2})$ and $\mathbf{Mf}((\mathcal{V}_2)_{\geq 2})$. Furthermore, it is enough to compute $\mathbf{Mf}((\mathcal{V}_1)_{\geq 1})$, because $\mathbf{Mf}((\mathcal{V}_1)_{\geq 1}) \cong \mathbf{Mf}((\mathcal{V}_1)_{\geq 2})$, due to Corollary 3.51.

We use Algorithm 16 to compute the marked schemes. For that we have to define $\mathfrak{V}_1 \subseteq \Bbbk[x_1, x_2]^2$. But, as we have seen in the previous section, this is trivial and $\mathbf{P}(\mathfrak{V}_1) = \{x_2\mathbf{e}_1, x_1\mathbf{e}_1, x_2\mathbf{e}_2\}$. To compute $\mathbf{Mf}((\mathcal{V}_1)_{\geq 1})$, we have to call MARKEDSCHEME($\mathfrak{V}_1, 1$). Due to the fact, that $\operatorname{sat}(\mathfrak{V}_1) = 1$ this is equivalent to call MARKEDSCHEME($\mathfrak{V}_1, 0$), due to Corollary 4.64. In the first part of the algorithm we compute the set $\mathfrak{G} \subset \Bbbk[\mathfrak{C}^1][x_1, x_2]^2$, which contains the following three elements:

$$\begin{aligned}
\mathcal{F}_1 &= x_2\mathbf{e}_1 + \mathfrak{C}_1\mathbf{e}_1 + \mathfrak{C}_2\mathbf{e}_2 + \mathfrak{C}_3 x_1\mathbf{e}_2\,,\\
\mathcal{F}_2 &= x_1\mathbf{e}_1 + \mathfrak{C}_4\mathbf{e}_1 + \mathfrak{C}_5\mathbf{e}_2 + \mathfrak{C}_6 x_1\mathbf{e}_2\,,\\
\mathcal{F}_3 &= x_2\mathbf{e}_2 + \mathfrak{C}_7\mathbf{e}_1 + \mathfrak{C}_8\mathbf{e}_2 + \mathfrak{C}_9 x_1\mathbf{e}_2\,.
\end{aligned}$$

For simplicity, we enumerate the variables $\mathfrak{C}^1 = (\mathfrak{C}_1, \ldots, \mathfrak{C}_9)$ and the elements in $\mathfrak{G}$ sequentially and not like in Algorithm 16.

In the second part of the algorithm we call the subalgorithm REDUCTIONTWO($\mathfrak{V}, 0, \mathfrak{G}$) (Algorithm 15). In this algorithm we have to compute all possible non-multiplicative prolongations of elements in the $[\mathbf{P}(\mathfrak{V}_1), 0]$-marked set. There is only one prolongation:

$$\begin{aligned}
x_2\mathfrak{F}_2 \xrightarrow{\mathfrak{G}}{}^{*} &(-\mathfrak{C}_6\mathfrak{C}_9 - \mathfrak{C}_3)x_1^2\mathbf{e}_2 + (\mathfrak{C}_6^2\mathfrak{C}_7 - \mathfrak{C}_3\mathfrak{C}_4 + \mathfrak{C}_1\mathfrak{C}_6 - \mathfrak{C}_6\mathfrak{C}_8 - \mathfrak{C}_5\mathfrak{C}_9 - \mathfrak{C}_2)x_1\mathbf{e}_2\\
&+ (\mathfrak{C}_5\mathfrak{C}_6\mathfrak{C}_7 - \mathfrak{C}_2\mathfrak{C}_4 + \mathfrak{C}_1\mathfrak{C}_5 - \mathfrak{C}_5\mathfrak{C}_8)\mathbf{e}_2 + (\mathfrak{C}_4\mathfrak{C}_6\mathfrak{C}_7 - \mathfrak{C}_5\mathfrak{C}_7)\mathbf{e}_1\,.
\end{aligned}$$

Out of this normal form we can extract the following set of polynomials:

$$\begin{aligned}
\mathcal{E}_1 = \{ &-\mathfrak{C}_6\mathfrak{C}_9 - \mathfrak{C}_3, \mathfrak{C}_6^2\mathfrak{C}_7 - \mathfrak{C}_3\mathfrak{C}_4 + \mathfrak{C}_1\mathfrak{C}_6 - \mathfrak{C}_6\mathfrak{C}_8 - \mathfrak{C}_5\mathfrak{C}_9 - \mathfrak{C}_2,\\
&\mathfrak{C}_5\mathfrak{C}_6\mathfrak{C}_7 - \mathfrak{C}_2\mathfrak{C}_4 + \mathfrak{C}_1\mathfrak{C}_5 - \mathfrak{C}_5\mathfrak{C}_8, \mathfrak{C}_4\mathfrak{C}_6\mathfrak{C}_7 - \mathfrak{C}_5\mathfrak{C}_7\}\,.
\end{aligned}$$

It is clear that $\mathcal{N}((\mathfrak{V}_1)_{\leq \operatorname{reg}(\mathfrak{V}_1)-1}) \neq \mathcal{N}((\mathfrak{V}_1)_{\leq \operatorname{reg}(\mathfrak{V}_1)})$. Therefore, we have to call REDUCTIONONE($\mathfrak{V}_1, 0, \mathfrak{G}$) (Algorithm 14), too. But we have to compute nothing in this algorithm, because there are no terms in $(\mathfrak{V}_1 \setminus \mathbf{P}(\mathfrak{V}_1) \cap \mathbb{T}^2)_{\leq 0}$. As a result we get that $\mathbf{Mf}((\mathcal{V}_1)_{\geq 2})$ is a subspace of $\mathbb{A}^9$, more exactly

$$\mathbf{Mf}((\mathcal{V}_1)_{\geq 2}) \cong \Bbbk[\mathfrak{C}^1]/\langle \mathcal{E}_1 \rangle\,.$$

Now it remains to compute $\mathbf{Mf}((\mathcal{V}_2)_{\geq 2})$. This is equivalent to compute for $\mathbf{P}(\mathfrak{V}_2) = \{\mathbf{e}_1, x_2^2\mathbf{e}_2, x_2x_1\mathbf{e}_2\} \subset \Bbbk[x_1, x_2]^2$ the algorithm MARKEDSCHEME($\mathfrak{V}_2, 1$). Again, we compute at first the set $\mathfrak{G} \subset \Bbbk[\mathfrak{C}^2][x_1, x_2]^2$, which contains the three following elements:

$$\begin{aligned}
\mathfrak{F}_1 &= \mathbf{e}_1 + \mathfrak{C}_1\mathbf{e}_2 + \mathfrak{C}_2 x_1\mathbf{e}_2 + \mathfrak{C}_3 x_2\mathbf{e}_2\,,\\
\mathfrak{F}_2 &= x_2^2\mathbf{e}_2 + \mathfrak{C}_4\mathbf{e}_2 + \mathfrak{C}_5 x_1\mathbf{e}_2 + \mathfrak{C}_6 x_2\mathbf{e}_2 + \mathfrak{C}_7 x_1^2\mathbf{e}_2\,,\\
\mathfrak{F}_3 &= x_1x_2\mathbf{e}_2 + \mathfrak{C}_8\mathbf{e}_2 + \mathfrak{C}_9 x_1\mathbf{e}_2 + \mathfrak{C}_{10} x_2\mathbf{e}_2 + \mathfrak{C}_{11} x_1^2\mathbf{e}_2\,.
\end{aligned}$$

Again, we enumerate the variables $\mathfrak{C}^2 = (\mathfrak{C}_1, \ldots, \mathfrak{C}_{11})$ and the elements in $\mathfrak{G}$ sequentially.

Then we have to call the subalgorithm REDUCTIONTWO($\mathfrak{V}_2, 1, \mathfrak{G}$), where we compute the normal forms of all non-multiplicative prolongations. We only have to compute one prolongation, again:

$$\begin{aligned} x_2\mathfrak{F}_3 \xrightarrow{\mathfrak{G}}{}^* & (-\mathfrak{C}_{11}^2 - \mathfrak{C}_7)x_1^3\mathbf{e}_2 + (\mathfrak{C}_{10}\mathfrak{C}_{11}^2 - \mathfrak{C}_7\mathfrak{C}_{10} + \mathfrak{C}_6\mathfrak{C}_{11} - 2\mathfrak{C}_9\mathfrak{C}_{11} - \mathfrak{C}_5)x_1^2\mathbf{e}_2 \\ & + (\mathfrak{C}_9\mathfrak{C}_{10}\mathfrak{C}_{11} + \mathfrak{C}_6\mathfrak{C}_9 - \mathfrak{C}_9^2 - \mathfrak{C}_5\mathfrak{C}_{10} - \mathfrak{C}_8\mathfrak{C}_{11} - \mathfrak{C}_4)x_1\mathbf{e}_2 \\ & + (\mathfrak{C}_{10}^2\mathfrak{C}_{11} - \mathfrak{C}_9\mathfrak{C}_{10} + \mathfrak{C}_8)x_2\mathbf{e}_2 + (\mathfrak{C}_8\mathfrak{C}_{10}\mathfrak{C}_{11} + \mathfrak{C}_6\mathfrak{C}_8 - \mathfrak{C}_8\mathfrak{C}_9 - \mathfrak{C}_4\mathfrak{C}10)\mathbf{e}_2\,. \end{aligned}$$

Out of this normal form we can extract the following set of polynomials:

$$\begin{aligned} \mathcal{E}_2 = \{ & -\mathfrak{C}_{11}^2 - \mathfrak{C}_7, \mathfrak{C}_{10}\mathfrak{C}_{11}^2 - \mathfrak{C}_7\mathfrak{C}_{10} + \mathfrak{C}_6\mathfrak{C}_{11} - 2\mathfrak{C}_9\mathfrak{C}_{11} - \mathfrak{C}_5, \\ & \mathfrak{C}_9\mathfrak{C}_{10}\mathfrak{C}_{11} + \mathfrak{C}_6\mathfrak{C}_9 - \mathfrak{C}_9^2 - \mathfrak{C}_5\mathfrak{C}_{10} - \mathfrak{C}_8\mathfrak{C}_{11} - \mathfrak{C}_4, \mathfrak{C}_{10}^2\mathfrak{C}_{11} - \mathfrak{C}_9\mathfrak{C}_{10} + \mathfrak{C}_8, \\ & \mathfrak{C}_8\mathfrak{C}_{10}\mathfrak{C}_{11} + \mathfrak{C}_6\mathfrak{C}_8 - \mathfrak{C}_8\mathfrak{C}_9 - \mathfrak{C}_4\mathfrak{C}_{10}\}\,. \end{aligned}$$

It remains to call REDUCTIONONE($\mathfrak{V}, 1, \mathfrak{G}$). In contrast to the first case we have to compute two normals forms:

$$\begin{aligned} x_1\mathbf{e}_1 & \xrightarrow{\mathfrak{G}}{}^* (\mathfrak{C}_3\mathfrak{C}_{11} - \mathfrak{C}_2)x_1^2\mathbf{e}_2 + (\mathfrak{C}_3\mathfrak{C}_9 - \mathfrak{C}_1)x_1\mathbf{e}_2 + (\mathfrak{C}_3\mathfrak{C}_{10})x_2\mathbf{e}_2 + (\mathfrak{C}_3\mathfrak{C}_8)\mathbf{e}_2\,, \\ x_2\mathbf{e}_1 & \xrightarrow{\mathfrak{G}}{}^* (\mathfrak{C}_3\mathfrak{C}_7 + \mathfrak{C}_2\mathfrak{C}_{11})x_1^2\mathbf{e}_2 + (\mathfrak{C}_3\mathfrak{C}_5 + \mathfrak{C}_2\mathfrak{C}_9)x_1\mathbf{e}_2 + (\mathfrak{C}_3\mathfrak{C}_6 + \mathfrak{C}_2\mathfrak{C}_{10} - \mathfrak{C}_1)x_2\mathbf{e}_2 \\ & \quad + (\mathfrak{C}_3\mathfrak{C}_4 + \mathfrak{C}_2\mathfrak{C}_8)\mathbf{e}_2\,. \end{aligned}$$

Hence, we have to add the following set of polynomials to $\mathcal{E}_2$:

$$\begin{aligned} \mathcal{E}_2 = & \mathcal{E}_2 \cup \{\mathfrak{C}_3\mathfrak{C}_{11} - \mathfrak{C}_2, \mathfrak{C}_3\mathfrak{C}_9 - \mathfrak{C}_1, \mathfrak{C}_3\mathfrak{C}_{10}, \mathfrak{C}_3\mathfrak{C}_8, \\ & \mathfrak{C}_3\mathfrak{C}_7 + \mathfrak{C}_2\mathfrak{C}_{11}, \mathfrak{C}_3\mathfrak{C}_5 + \mathfrak{C}_2\mathfrak{C}_9, \mathfrak{C}_3\mathfrak{C}_6 + \mathfrak{C}_2\mathfrak{C}_{10} - \mathfrak{C}_1, \mathfrak{C}_3\mathfrak{C}_4 + \mathfrak{C}_2\mathfrak{C}_8, \}\,. \end{aligned}$$

As a result we get that $\mathbf{Mf}((\mathcal{V}_2)_{\geq 2})$ is a subspace of $\mathbb{A}^{11}$, more exactly

$$\mathbf{Mf}((\mathcal{V}_2)_{\geq 2}) \cong \Bbbk[\mathfrak{C}^2]/\langle\mathcal{E}_2\rangle\,. \tag{4.5}$$

To summarize, we have seen that we can cover $\mathbf{Quot}_{2,t+2}^{2,(0,0)}$ by four marked schemes. Two schemes, which are isomorphic are a subspace of $\mathbb{A}^9$. The other two schemes, which are also isomorphic, are a subspace of $\mathbb{A}^{11}$.

### 4.5.3 $\mathbf{Quot}_{2,t+2}^{2,(-1,0)}$

Now we compute the Quot scheme for the Hilbert polynomial $\mathsf{HP}(t) = t + 2$ over the polynomial module $\Bbbk[x_0, x_1, x_2]^2_{(-1,0)}$, where $\Bbbk$ is a field of characteristic zero. This is nearly the same Quot scheme as in the section above. We do not use the standard grading now, but we shift the degree of the first free generator by $-1$. We will see that we obtain a completely different open covering. The algorithms defined in Section 4.4 are formulated only for the standard grading. Nevertheless, they are also applicable for a non-standard grading. Therefore, we can use these algorithms for this example, too.

Firstly, we call MONOMIALMODULES($(t+2, \Bbbk[x_0,x_1,x_2]^2_{(-1,0)}$, 0-Borel-fixed). For that algorithm we have to call DISTRIBUTEPOLYNOMIAL($t+2$,((), ()),2) which produces the pairs of polynomials from (4.4). In the next step we have to shift the polynomials according to the algorithm:

$$\begin{aligned} \mathsf{HP}_1 &= (1, t+1)\,, & \mathsf{HP}_2 &= (0, t+2)\,, \\ \mathsf{HP}_3 &= (t, 1)\,, & \mathsf{HP}_4 &= (t+1, 0)\,. \end{aligned}$$

It turns out that for $\mathsf{HP}_3$ there is no saturated 0-Borel fixed module. For the other three pairs of polynomials we get in total three 0-Borel fixed modules:

$$\begin{aligned} \mathcal{V}_1 &= \langle x_2\mathbf{e}_1, x_1\mathbf{e}_1, x_2\mathbf{e}_2 \rangle\,, & \mathcal{V}_2 &= \langle \mathbf{e}_1, x_2^2\mathbf{e}_2, x_2x_1\mathbf{e}_2 \rangle\,, \\ \mathcal{V}_3 &= \langle x_2\mathbf{e}_1, \mathbf{e}_2 \rangle\,. \end{aligned}$$

Even, if the modules $\mathcal{V}_1$ and $\mathcal{V}_2$ also occur in the example of the section above, we will see that the corresponding marked schemes are only in one case equal.

At first, we compute $\mathbf{Mf}((\mathcal{V}_1)_{\geq 2})$, which is equivalent to compute for

$$\mathbf{P}(\mathfrak{V}_1) = \{x_2\mathbf{e}_1, x_1\mathbf{e}_1, x_2\mathbf{e}_2\}$$

the algorithm MARKEDSCHEME($\mathfrak{V}_1, -1$). The satiety of $\mathfrak{V}_1$ is zero which explains the second argument. The set $\mathfrak{G} \subset \Bbbk[\mathfrak{C}^1][x_1,x_2]^2_{(-1,0)}$ contains the following three elements:

$$\begin{aligned} \mathfrak{F}_1 &= x_2\mathbf{e}_1 + \mathfrak{C}_1\mathbf{e}_1 + \mathfrak{C}_2\mathbf{e}_2\,, \\ \mathfrak{F}_2 &= x_1\mathbf{e}_1 + \mathfrak{C}_3\mathbf{e}_1 + \mathfrak{C}_4\mathbf{e}_2\,, \\ \mathfrak{F}_3 &= x_2\mathbf{e}_2 + \mathfrak{C}_5\mathbf{e}_1 + \mathfrak{C}_6\mathbf{e}_2 + \mathfrak{C}_7x_1\mathbf{e}_2\,. \end{aligned}$$

As usual we enumerate the variables $\mathfrak{C}^1 = (\mathfrak{C}_1, \ldots, \mathfrak{C}_7)$ and the elements in $\mathfrak{G}$ sequentially.

After initializing $\mathfrak{G}$ we have to compute in REDUCTIONTWO($\mathfrak{V}_1, -1, \mathfrak{G}$) the normal forms of the non-multiplicative prolongation. Again we have to compute only one prolongation:

$$x_2\mathfrak{F}_2 \xrightarrow{\mathfrak{G}}{}^{*} (-\mathfrak{C}_4\mathfrak{C}_7 - \mathfrak{C}_2)x_1\mathbf{e}_2 + (-\mathfrak{C}_2\mathfrak{C}_3 + \mathfrak{C}_1\mathfrak{C}_4 - \mathfrak{C}_4\mathfrak{C}_6)\mathbf{e}_2 + (-\mathfrak{C}_4\mathfrak{C}_5)\mathbf{e}_1\,.$$

Out of this we can extract the following set of polynomials:

$$\mathcal{E}_1 = \{-\mathfrak{C}_4\mathfrak{C}_7 - \mathfrak{C}_2, -\mathfrak{C}_2\mathfrak{C}_3 + \mathfrak{C}_1\mathfrak{C}_4 - \mathfrak{C}_4\mathfrak{C}_6, -\mathfrak{C}_4\mathfrak{C}_5\}\,.$$

There are no normal form computations which we have to perform in algorithm REDUCTIONONE($\mathfrak{V}_1, -1, \mathfrak{G}$). Hence, $\mathbf{Mf}((\mathcal{V}_1)_{\geq 2} \cong \Bbbk[\mathfrak{C}^1]/\langle\mathcal{E}_1\rangle$.

Now we compute $\mathbf{Mf}((\mathcal{V}_2)_{\geq 2})$, which is equivalent to compute for

$$\mathbf{P}(\mathfrak{V}_2) = \{\mathbf{e}_1, x_2^2\mathbf{e}_2, x_2x_1\mathbf{e}_2\}$$

the algorithm MARKEDSCHEME($\mathfrak{V}_2, 1$). The satiety of $\mathfrak{V}_2$ is two, hence we have chosen one as second argument. The set $\mathfrak{G} \subset \Bbbk[\mathfrak{C}^2][x_1, x_2]^2_{(-1,0)}$ contains the following three elements:

$$\begin{aligned}
\mathfrak{F}_1 &= \mathbf{e}_1 + \mathfrak{C}_1\mathbf{e}_2 + \mathfrak{C}_2x_1\mathbf{e}_2 + \mathfrak{C}_3x_2\mathbf{e}_2\,,\\
\mathfrak{F}_2 &= x_2^2\mathbf{e}_2 + \mathfrak{C}_4\mathbf{e}_2 + \mathfrak{C}_5x_1\mathbf{e}_2 + \mathfrak{C}_6x_2\mathbf{e}_2 + \mathfrak{C}_7x_1^2\mathbf{e}_2\,,\\
\mathfrak{F}_3 &= x_1x_2\mathbf{e}_2 + \mathfrak{C}_8\mathbf{e}_2 + \mathfrak{C}_9x_1\mathbf{e}_2 + \mathfrak{C}_{10}x_2\mathbf{e}_2 + \mathfrak{C}_{11}x_1^2\mathbf{e}_2\,.
\end{aligned}$$

The variables in $\mathfrak{C}^2$ are $(\mathfrak{C}_1, \dots, \mathfrak{C}_{11})$.

Now we call in REDUCTIONTWO($\mathfrak{V}_2, 1, \mathfrak{G}$) the normal forms of the non-multiplicative prolongations. Again we have to compute only one prolongation:

$$\begin{aligned}
x_2\mathfrak{F}_3 \xrightarrow{\mathfrak{G}}{}^{*} &(-\mathfrak{C}_{11}^2 - \mathfrak{C}_7)x_1^3\mathbf{e}_2 + (\mathfrak{C}_{10}\mathfrak{C}_{11}^2 - \mathfrak{C}_7\mathfrak{C}_{10} + \mathfrak{C}_6\mathfrak{C}_{11} - 2\mathfrak{C}_9\mathfrak{C}_{11} - \mathfrak{C}_5)x_1^2\mathbf{e}_2\\
&+ (\mathfrak{C}_9\mathfrak{C}_{10}\mathfrak{C}_{11} + \mathfrak{C}_6\mathfrak{C}_9 - \mathfrak{C}_9^2 - \mathfrak{C}_5\mathfrak{C}_{10} - \mathfrak{C}_8\mathfrak{C}_{11} - \mathfrak{C}_4)x_1\mathbf{e}_2\\
&+ (\mathfrak{C}_{10}^2\mathfrak{C}_{11} - \mathfrak{C}_9\mathfrak{C}_{10} + \mathfrak{C}_8)x_2\mathbf{e}_2 + (\mathfrak{C}_8\mathfrak{C}_{10}\mathfrak{C}_{11} + \mathfrak{C}_6\mathfrak{C}_8 - \mathfrak{C}_8\mathfrak{C}_9 - \mathfrak{C}_4\mathfrak{C}_{10})\mathbf{e}_2\,.
\end{aligned}$$

Out of this we can extract the following set of polynomials:

$$\begin{aligned}
\mathcal{E}_2 = \{ &-\mathfrak{C}_{11}^2 - \mathfrak{C}_7, \mathfrak{C}_{10}\mathfrak{C}_{11}^2 - \mathfrak{C}_7\mathfrak{C}_{10} + \mathfrak{C}_6\mathfrak{C}_{11} - 2\mathfrak{C}_9\mathfrak{C}_{11} - \mathfrak{C}_5,\\
&\mathfrak{C}_9\mathfrak{C}_{10}\mathfrak{C}_{11} + \mathfrak{C}_6\mathfrak{C}_9 - \mathfrak{C}_9^2 - \mathfrak{C}_5\mathfrak{C}_{10} - \mathfrak{C}_8\mathfrak{C}_{11} - \mathfrak{C}_4, \mathfrak{C}_{10}^2\mathfrak{C}_{11} - \mathfrak{C}_9\mathfrak{C}_{10} + \mathfrak{C}_8,\\
&\mathfrak{C}_8\mathfrak{C}_{10}\mathfrak{C}_{11} + \mathfrak{C}_6\mathfrak{C}_8 - \mathfrak{C}_8\mathfrak{C}_9 - \mathfrak{C}_4\mathfrak{C}_{10}\}\,.
\end{aligned}$$

In the algorithm REDUCTIONONE($\mathfrak{V}_2, 1, \mathfrak{G}$) we have to perform several normal form computations:

$$
\begin{aligned}
x_1\mathbf{e}_1 &\xrightarrow{\mathfrak{G}}{}^{*}(\mathfrak{C}_3\mathfrak{C}_{11}-\mathfrak{C}_2)x_1^2\mathbf{e}_2+(\mathfrak{C}_3\mathfrak{C}_9-\mathfrak{C}_1)x_1\mathbf{e}_2+(\mathfrak{C}_3\mathfrak{C}_{10})x_2\mathbf{e}_2+(\mathfrak{C}_3\mathfrak{C}_8)\mathbf{e}_2\,,\\
x_2\mathbf{e}_1 &\xrightarrow{\mathfrak{G}}{}^{*}(\mathfrak{C}_3\mathfrak{C}_7+\mathfrak{C}_2\mathfrak{C}_{11})x_1^2\mathbf{e}_2+(\mathfrak{C}_3\mathfrak{C}_5+\mathfrak{C}_2\mathfrak{C}_9)x_1\mathbf{e}_2+(\mathfrak{C}_3\mathfrak{C}_6+\mathfrak{C}_2\mathfrak{C}_{10}-\mathfrak{C}_1)x_2\mathbf{e}_2\\
&\quad+(\mathfrak{C}_3\mathfrak{C}_4+\mathfrak{C}_2\mathfrak{C}_8)\mathbf{e}_2\,,\\
x_1^2\mathbf{e}_1 &\xrightarrow{\mathfrak{G}}{}^{*}(\mathfrak{C}_3\mathfrak{C}_{11}-\mathfrak{C}_2)x_1^3\mathbf{e}_2+(-\mathfrak{C}_3\mathfrak{C}_{10}\mathfrak{C}_{11}+\mathfrak{C}_3\mathfrak{C}_9-\mathfrak{C}_1)x_1^2\mathbf{e}_2\\
&\quad+(-\mathfrak{C}_3\mathfrak{C}_9\mathfrak{C}_{10}+\mathfrak{C}_3\mathfrak{C}_8)x_1\mathbf{e}_2+(-\mathfrak{C}_3\mathfrak{C}_{10}^2)x_2\mathbf{e}_2+(-\mathfrak{C}_3\mathfrak{C}_8\mathfrak{C}_{10})\mathbf{e}_2\,,\\
x_1x_2\mathbf{e}_1 &\xrightarrow{\mathfrak{G}}{}^{*}(\mathfrak{C}_3\mathfrak{C}_7+\mathfrak{C}_2\mathfrak{C}_{11})x_1^3\mathbf{e}_2+(-\mathfrak{C}_3\mathfrak{C}_6\mathfrak{C}_{11}-\mathfrak{C}_2\mathfrak{C}_{10}\mathfrak{C}_{11}+\mathfrak{C}_3\mathfrak{C}_5+\mathfrak{C}_2\mathfrak{C}_9+\mathfrak{C}_1\mathfrak{C}_{11})x_1^2\mathbf{e}_2\\
&\quad+(-\mathfrak{C}_3\mathfrak{C}_6\mathfrak{C}_9-\mathfrak{C}_2\mathfrak{C}_9\mathfrak{C}_{10}+\mathfrak{C}_3\mathfrak{C}_4+\mathfrak{C}_2\mathfrak{C}_8+\mathfrak{C}_1\mathfrak{C}_9)x_1\mathbf{e}_2\\
&\quad+(-\mathfrak{C}_3\mathfrak{C}_6\mathfrak{C}_{10}-\mathfrak{C}_2\mathfrak{C}_{10}^2+\mathfrak{C}_1\mathfrak{C}_{10})x_2\mathbf{e}_2+(-\mathfrak{C}_3\mathfrak{C}_6\mathfrak{C}_8-\mathfrak{C}_2\mathfrak{C}_8\mathfrak{C}_{10}+\mathfrak{C}_1\mathfrak{C}_8)\mathbf{e}_2\,,\\
x_2^2\mathbf{e}_1 &\xrightarrow{\mathfrak{G}}{}^{*}(-\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_{11}+\mathfrak{C}_2\mathfrak{C}_7)x_1^3\mathbf{e}_2\\
&\quad+(\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_{10}\mathfrak{C}_{11}-\mathfrak{C}_3\mathfrak{C}_6\mathfrak{C}_7-\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_9-\mathfrak{C}_3\mathfrak{C}_5\mathfrak{C}_{11}-\mathfrak{C}_2\mathfrak{C}_6\mathfrak{C}_{11}+\mathfrak{C}_2\mathfrak{C}_5\\
&\quad+\mathfrak{C}_1\mathfrak{C}_7)x_1^2\mathbf{e}_2\\
&\quad+(\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_9\mathfrak{C}_{10}-\mathfrak{C}_3\mathfrak{C}_5\mathfrak{C}_6-\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_8-\mathfrak{C}_3\mathfrak{C}_5\mathfrak{C}_9-\mathfrak{C}_2\mathfrak{C}_6\mathfrak{C}_9+\mathfrak{C}_2\mathfrak{C}_4+\mathfrak{C}_1\mathfrak{C}_5)x_1\mathbf{e}_2\\
&\quad+(\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_{10}^2-\mathfrak{C}_3\mathfrak{C}_6^2-\mathfrak{C}_3\mathfrak{C}_5\mathfrak{C}_{10}-\mathfrak{C}_2\mathfrak{C}_6\mathfrak{C}_{10}+\mathfrak{C}_3\mathfrak{C}_4+\mathfrak{C}_1\mathfrak{C}_6)x_2\mathbf{e}_2\\
&\quad+(\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_8\mathfrak{C}_{10}-\mathfrak{C}_3\mathfrak{C}_4\mathfrak{C}_6-\mathfrak{C}_3\mathfrak{C}_5\mathfrak{C}_8-\mathfrak{C}_2\mathfrak{C}_6\mathfrak{C}_8+\mathfrak{C}_1\mathfrak{C}_4)\mathbf{e}_2\,,
\end{aligned}
$$

Hence, we have to add the following set of polynomials to $\mathcal{E}_2$:

$$
\begin{aligned}
\mathcal{E}_2 =&\mathcal{E}_2\cup\{\mathfrak{C}_3\mathfrak{C}_{11}-\mathfrak{C}_2,\mathfrak{C}_3\mathfrak{C}_9-\mathfrak{C}_1,\mathfrak{C}_3\mathfrak{C}_{10},\mathfrak{C}_3\mathfrak{C}_8,\mathfrak{C}_3\mathfrak{C}_7+\mathfrak{C}_2\mathfrak{C}_{11},\mathfrak{C}_3\mathfrak{C}_5+\mathfrak{C}_2\mathfrak{C}_9,\\
&\mathfrak{C}_3\mathfrak{C}_6+\mathfrak{C}_2\mathfrak{C}_{10}-\mathfrak{C}_1,\mathfrak{C}_3\mathfrak{C}_4+\mathfrak{C}_2\mathfrak{C}_8,\mathfrak{C}_3\mathfrak{C}_{11}-\mathfrak{C}_2,-\mathfrak{C}_3\mathfrak{C}_{10}\mathfrak{C}_{11}+\mathfrak{C}_3\mathfrak{C}_9-\mathfrak{C}_1,\\
&-\mathfrak{C}_3\mathfrak{C}_9\mathfrak{C}_{10}+\mathfrak{C}_3\mathfrak{C}_8,-\mathfrak{C}_3\mathfrak{C}_{10}^2,-\mathfrak{C}_3\mathfrak{C}_8\mathfrak{C}_{10},\mathfrak{C}_3\mathfrak{C}_7+\mathfrak{C}_2\mathfrak{C}_{11},\\
&-\mathfrak{C}_3\mathfrak{C}_6\mathfrak{C}_{11}-\mathfrak{C}_2\mathfrak{C}_{10}\mathfrak{C}_{11}+\mathfrak{C}_3\mathfrak{C}_5+\mathfrak{C}_2\mathfrak{C}_9+\mathfrak{C}_1\mathfrak{C}_{11},\\
&-\mathfrak{C}_3\mathfrak{C}_6\mathfrak{C}_9-\mathfrak{C}_2\mathfrak{C}_9\mathfrak{C}_{10}+\mathfrak{C}_3\mathfrak{C}_4+\mathfrak{C}_2\mathfrak{C}_8+\mathfrak{C}_1\mathfrak{C}_9,-\mathfrak{C}_3\mathfrak{C}_6\mathfrak{C}_{10}-\mathfrak{C}_2\mathfrak{C}_{10}^2+\mathfrak{C}_1\mathfrak{C}_{10},\\
&-\mathfrak{C}_3\mathfrak{C}_6\mathfrak{C}_8-\mathfrak{C}_2\mathfrak{C}_8\mathfrak{C}_{10}+\mathfrak{C}_1\mathfrak{C}_8,-\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_{11}+\mathfrak{C}_2\mathfrak{C}_7,\\
&\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_{10}\mathfrak{C}_{11}-\mathfrak{C}_3\mathfrak{C}_6\mathfrak{C}_7-\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_9-\mathfrak{C}_3\mathfrak{C}_5\mathfrak{C}_{11}-\mathfrak{C}_2\mathfrak{C}_6\mathfrak{C}_{11}+\mathfrak{C}_2\mathfrak{C}_5+\mathfrak{C}_1\mathfrak{C}_7,\\
&\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_9\mathfrak{C}_{10}-\mathfrak{C}_3\mathfrak{C}_5\mathfrak{C}_6-\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_8-\mathfrak{C}_3\mathfrak{C}_5\mathfrak{C}_9-\mathfrak{C}_2\mathfrak{C}_6\mathfrak{C}_9+\mathfrak{C}_2\mathfrak{C}_4+\mathfrak{C}_1\mathfrak{C}_5,\\
&\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_{10}^2-\mathfrak{C}_3\mathfrak{C}_6^2-\mathfrak{C}_3\mathfrak{C}_5\mathfrak{C}_{10}-\mathfrak{C}_2\mathfrak{C}_6\mathfrak{C}_{10}+\mathfrak{C}_3\mathfrak{C}_4+\mathfrak{C}_1\mathfrak{C}_6,\\
&\mathfrak{C}_3\mathfrak{C}_7\mathfrak{C}_8\mathfrak{C}_{10}-\mathfrak{C}_3\mathfrak{C}_4\mathfrak{C}_6-\mathfrak{C}_3\mathfrak{C}_5\mathfrak{C}_8-\mathfrak{C}_2\mathfrak{C}_6\mathfrak{C}_8+\mathfrak{C}_1\mathfrak{C}_4\}\,.
\end{aligned}
$$

As a result we get that $\mathbf{Mf}((\mathcal{V}_2)_{\geq 2})$ is a subspace of $\mathbb{A}^{11}$, more exactly

$$\mathbf{Mf}((\mathcal{V}_2)_{\geq 2}) \cong \Bbbk[\mathfrak{C}^2]/\langle\mathcal{E}_2\rangle\,.$$

A simple computation shows that this scheme is exactly the same scheme as defined in (4.5).

At last, we have to compute $\mathbf{Mf}((\mathcal{V}_3)_{\geq 2})$, which is equivalent to compute for $\mathbf{P}(\mathfrak{V}_3) = \{x_2\mathbf{e}_1, \mathbf{e}_2\}$ the algorithm MARKEDSCHEME($\mathfrak{V}_3, -2$). The module $\mathfrak{V}_3$ is saturated, hence the saturation of $\mathfrak{V}_3$ is equal to $\mathfrak{V}_3$ in every degree. Therefore, the satiety is the lowest possible degree, which is $-1$. Hence, the second argument of the algorithm is $-2$. The set $\mathfrak{G} \subset \Bbbk[\mathfrak{C}^3][x_1, x_2]^2_{(-1,0)}$ contains the following two elements:

$$\mathfrak{F}_1 = x_2\mathbf{e}_1 + \mathfrak{C}_1\mathbf{e}_1 + \mathfrak{C}_2 x_1\mathbf{e}_1\,,$$
$$\mathfrak{F}_2 = \mathbf{e}_2 + \mathfrak{C}_3\mathbf{e}_1 + \mathfrak{C}_4 x_1\mathbf{e}_1\,.$$

The variables in $\mathfrak{C}^4$ are $(\mathfrak{C}_1, \dots, \mathfrak{C}_4)$.

The algorithm REDUCTIONTWO($\mathfrak{V}_3, -1, \mathfrak{G}$) computes the normal forms of non-multiplicative prolongations. But in $\mathfrak{V}_3$, there are no non-multiplicative prolongations. Hence, the set $\mathcal{E}_3$ remains empty. For the algorithm REDUCTIONONE($\mathfrak{V}_3, -1, \mathfrak{G}$) we have the same situation. But this implies that $\mathbf{Mf}((\mathcal{V}_3)_{\geq 2}) \cong \mathbb{A}^4$.

To summarize, we have seen that we can cover $\mathbf{Quot}^{2,(-1,0)}_{2,t+2}$ by three marked schemes. One scheme is isomorphic to $\mathbb{A}^4$, one scheme is isomorphic to a subspace of $\mathbb{A}^7$ and one is isomorphic to the subspace of $\mathbb{A}^{11}$.

If we compare this Quot scheme with the Quot scheme for the same Hilbert polynomial but with the standard grading, we see that this tiny change produces a totally different result.

### 4.5.4 The Hilbert Polynomial $\mathrm{HP}(t) = 4$ in Polynomial Rings with Four Variables over Fields with Different Characteristic

A main advantage of the approach developed in this thesis is that we can compute Quot schemes over fields with arbitrary characteristic. As an example we compute $\mathbf{Hilb}^3_4$ once over the field $\Bbbk_1 = \mathbb{C}$ which is a field of characteristic zero and $\Bbbk_2 = \mathbb{Z}/2\mathbb{Z}$ which is a field of characteristic two.

At first, we compute as usual the set of saturated 0-Borel fixed ideals, respectively the set of 2-Borel fixed modules. There are three saturated 0-Borel ideals $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ with

$$\mathbf{P}(\mathcal{V}_1) = \{x_1^2, x_2^2, x_2x_1, x_3^2, x_3x_2, x_3x_1\}\,, \qquad \mathbf{P}(\mathcal{V}_2) = \{x_3, x_2^2, x_2x_1, x_1^3\}\,,$$
$$\mathbf{P}(\mathcal{V}_3) = \{x_3, x_2, x_1^4\}\,.$$

The set of saturated 2-Borel fixed ideals contains an additional ideal $\mathcal{V}_4$ with

$$\mathbf{P}(\mathcal{V}_4) = \{x_3, x_1^2, x_2^2\} .$$

For the Hilbert scheme over $\Bbbk_1$ we have to compute the marked schemes $\mathbf{Mf}_{\Bbbk_1}((\mathcal{V}_1)_{\geq 4})$, $\mathbf{Mf}_{\Bbbk_1}((\mathcal{V}_2)_{\geq 4})$, $\mathbf{Mf}_{\Bbbk_1}((\mathcal{V}_3)_{\geq 4})$. We have done this with our experimental implementation in CoCoALib. The marked scheme $\mathbf{Mf}_{\Bbbk_1}((\mathcal{V}_1)_{\geq 4})$ is a subspace of $\mathbb{A}^{24}_{\Bbbk_1}$, the marked scheme $\mathbf{Mf}_{\Bbbk_1}((\mathcal{V}_2)_{\geq 4})$ is a subspace of $\mathbb{A}^{16}_{\Bbbk_1}$ and the marked scheme $\mathbf{Mf}_{\Bbbk_1}((\mathcal{V}_3)_{\geq 4})$ is the affine space $\mathbb{A}^{12}_{\Bbbk_1}$.

For the Hilbert scheme over $\Bbbk_2$ we have a similar situation: $\mathbf{Mf}_{\Bbbk_2}((\mathcal{V}_1)_{\geq 4})$ is a subspace of $\mathbb{A}^{24}_{\Bbbk_2}$, $\mathbf{Mf}_{\Bbbk_2}((\mathcal{V}_2)_{\geq 4})$ is a subspace of $\mathbb{A}^{16}_{\Bbbk_2}$ and $\mathbf{Mf}_{\Bbbk_2}((\mathcal{V}_3)_{\geq 4})$ is the affine space $\mathbb{A}^{12}_{\Bbbk_2}$. In addition, $\mathbf{Mf}_{\Bbbk_2}((\mathcal{V}_4)_{\geq 3})$ is a subspace of $\mathbb{A}^{16}_{\Bbbk_2}$.

The scheme $\mathbf{Hilb}^3_4$ considered over the field $\Bbbk_1$ has a different open covering then over the field $\Bbbk_2$. But this does not mean that both Hilbert schemes behave different or have different invariants. This needs a deeper analysis of these results, which is not the goal of this thesis.

# 5 Conclusion

In the first chapter we introduced resolving decompositions (Definition 1.1). We have introduced continuous involutive divisions as an example. We investigated the computation of free resolutions and Betti numbers induced from resolving decompositions. In [19] it was shown how to compute free resolutions and Betti numbers by involutive bases. By using resolving decompositions it could be possible to get other generating systems which behave computationally better than involutive bases. Furthermore, we introduced in the first chapter marked bases over modules which extended the previous work of Cioffi and Roggero [13].

In the second chapter we gave alternative proofs for the Gotzmann regularity and persistence theorem. These are important ingredients in an alternative proof of the representability of the Quot functor. For Hilbert functions the theorems of Macaulay and Green are very important. We are convinced that one can use Pommaret bases to get a simple proof for both theorems. In addition to that, our hope is that this leads to a deeper understanding of the combinatorial structure of Pommaret bases.

In this thesis we have seen a new method to prove the representability of the Quot functor (Theorem 3.46). Using this we were able to give algorithms to compute the corresponding Quot schemes.

It seems to be worth trying to use the same techniques to show the representability of the multigraded Hilbert functors introduced by Haiman and Sturmfels [27]. Another interesting aspect is to analyse, with the techniques introduced in this thesis, special loci of Quot schemes. For example, the subscheme of a specific Quot scheme which contains all modules with a bounded regularity or projective dimension. For the specific case of the subscheme of Quot schemes with bounded regularity or projective dimension most of the work should already be done in the first chapter. Marked bases over modules induce a resolving decomposition. By using this theory it should be easy to construct marked schemes which take into account bounded regularity or projective dimension.

In the fourth chapter we developed algorithms for computing Quot schemes. In the first two sections we developed efficient algorithms for computing saturated quasi-stable and $p$-Borel fixed ideals. By using these algorithms we showed in the third section how to compute saturated quasi-stable and saturated $p$-Borel fixed modules. The fourth section was devoted to the computation of marked schemes. Combining the algorithms for the computation of saturated quasi-stable or $p$-Borel fixed modules and the computation of marked schemes we are finally able to compute Quot schemes. Finally, we saw in Section 4.5 some example applications of computing Quot schemes. Specifically we have computed the first concrete representation of a Quot scheme which is not a Hilbert scheme, ever. Formerly it was a hard task to compute even small Quot schemes. We have seen in this section that our approach is a much better approach which allows to compute many Quot schemes.

In Section 4.5 we saw that not only the computation of an open covering of a Quot scheme is a challenging problem. For analysing a Quot scheme it is necessary to have effective algorithms to analyse the marked schemes. But, in general, marked schemes are represented by ideals in polynomial rings with very many variables, even if the algorithms presented in Section 4.4 are better than the standard approach which we introduced at first. Furthermore, the ideals are generated by very dense polynomials. Both facts cause big problems for algorithmic tools to analyse the marked schemes. Most of the algorithmic tools are based on Gröbner bases. The computation of Gröbner bases in polynomial rings with many variables and dense generating systems is still very challenging. Hence, for the algorithmic analysis of Quot schemes we need efficient techniques to deal with dense generating systems in polynomial rings with many variables. Another problem is that the tools to analyse Quot schemes are not fully developed. For example, we checked in Section 4.5 if a Hilbert scheme is reduced. For that we computed the radicals of ideals which represent the marked schemes. But it would be enough to have an algorithm which only checks if the ideal is radical or not.

As we have stated, we are able to compute an open covering of the Quot scheme. In general, it is possible to remove some marked schemes from the open covering and the remaining set of marked schemes is still an open covering of the Quot scheme. We know that the quasi-stable covering of a Quot scheme is in general a covering where we can remove many elements and still have an open covering. An idea to get a better covering would be to use this covering and find a subcover which still covers the whole Quot scheme. There are two main problems which we need to solve for this approach. The first problem is to check if the subcover still covers the whole Quot scheme. The second problem is to verify that the subcover is better than the usual covering.

Nevertheless, the techniques developed in this thesis are a good starting point for a deeper computational analysis of Quot schemes, which are currently not well explored. This especially includes the Hilbert schemes over fields of arbitrary characteristic.

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# Eidesstattliche Erklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbstständig, ohne unerlaubte Hilfe Dritter angefertigt und andere als die in der Dissertation angegebenen Hilfsmittel nicht benutzt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen sind, habe ich als solche kenntlich gemacht. Dritte waren an der inhaltlich-materiellen Erstellung der Dissertation nicht beteiligt; insbesondere habe ich hierfür nicht die Hilfe eines Promotionsberaters in Anspruch genommen. Kein Teil dieser Arbeit ist in einem anderen Promotions- oder Habilitationsverfahren verwendet worden.

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