# A free boundary approach to the Rosensweig instability of ferrofluids

E. Parini* and A. Stylianou†

April 18, 2017

**Abstract**

We establish the existence of saddle points for a free boundary problem describing the two-dimensional free surface of a ferrofluid undergoing normal field instability. The starting point are the ferro-hydrostatic equations for the magnetic potentials in the ferrofluid and air, and the function describing their interface. These constitute the strong form for the Euler-Lagrange equations of a convex-concave functional, which we extend to include interfaces that are not necessarily graphs of functions. Saddle points are then found by iterating the direct method of the calculus of variations and applying classical results of convex analysis. For the existence part we assume a general nonlinear magnetization law; for a linear law we also show, via convex duality, that the saddle point is a constrained minimizer of the relevant energy functional.

# 1 Introduction

## 1.1 The ferro-hydrostatic equations

Let $\Omega \subset \mathbb{R}^2$ be open, connected and bounded with a Lipschitz continuous boundary, and $b, \tau > 0$. Moreover, let $\mu \in C_b^1(\mathbb{R})$, and for a function $\eta : \mathbb{R}^2 \longrightarrow \mathbb{R}$ define the sets

$$
\begin{aligned}
D_\eta^+ &\coloneqq \left\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \Omega \ \text{ and } \ z \in \big(\eta(x,y), 1\big)\right\}, \\
D_\eta^- &\coloneqq \left\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \Omega \ \text{ and } \ z \in \big(-1, \eta(x,y)\big)\right\}, \\
\partial_{cyl} D &\coloneqq \left\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \partial\Omega \ \text{ and } \ z \in (-1,1)\right\} \\
\partial_{top} D &\coloneqq \left\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \Omega \ \text{ and } \ z = 1\right\}, \\
\partial_{bot} D &\coloneqq \left\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \Omega \ \text{ and } \ z = -1\right\},
\end{aligned}
\tag{1}
$$

and the function

$$
M(s) \coloneqq \int_0^s t \cdot \mu(t)\, dt. \tag{2}
$$

*Aix Marseille Univ, CNRS, Centrale Marseille, I2M, 39 Rue Frederic Joliot Curie, 13453 Marseille, France

†Institut für Mathematik, Universität Kassel, 34132 Kassel, Germany

[1]**Keywords:** ferrofluids; free boundary problem; convex-concave functional

[2]**2010 AMS Subject Classification:** 35R35; 49J35; 35Q61; 35Q35

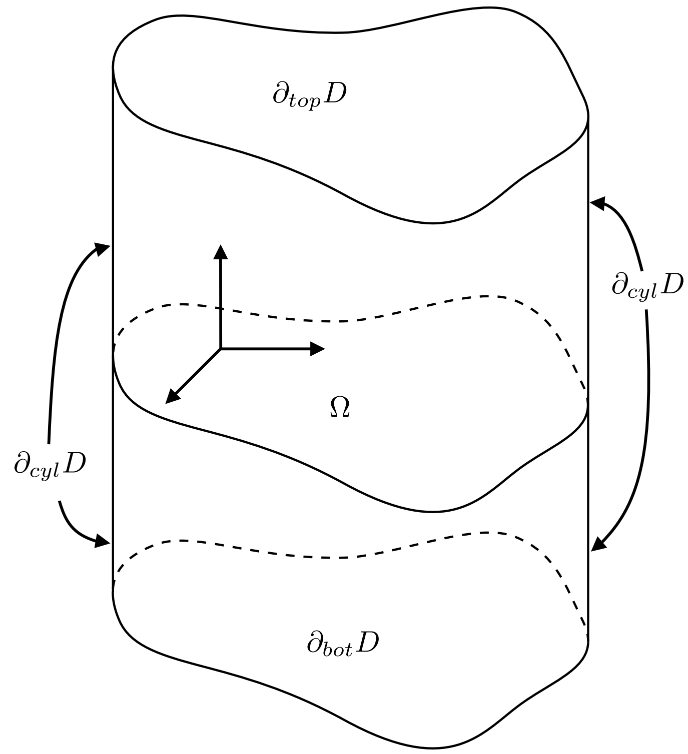


*Figure 1:* The cylindrical domain in which the ferrofluid is contained. The domain $\Omega$ lies at $z = 0$ and corresponds to the undisturbed surface of the ferrofluid before the magnet is turned on.

Consider the following problem: Find sufficiently smooth functions $\phi, \psi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $\eta : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with $|\eta| < 1$ that satisfy the boundary value problem

$$\begin{aligned} \Delta\psi &= 0 && \text{in } D_\eta^+, \\ \operatorname{div}\left(\mu\left(|\nabla\phi|\right)\nabla\phi\right) &= 0 && \text{in } D_\eta^-, \\ \psi_z &= \mu(1) && \text{on } \partial_{top}D, \\ \mu\left(|\nabla\phi|\right)\phi_z &= \mu(1) && \text{on } \partial_{bot}D, \\ \psi &= 0 && \text{on } \partial_{cyl}D, \\ \phi &= 0 && \text{on } \partial_{cyl}D, \end{aligned} \tag{3}$$

together with the compatibility conditions

$$\begin{aligned} \phi &= \psi && \text{on } \Omega \times \{z = \eta(x,y)\} \\ \mu\left(|\nabla\phi|\right)\phi_{\mathbf{n}} &= \psi_{\mathbf{n}} && \text{on } \Omega \times \{z = \eta(x,y)\}, \end{aligned} \tag{4}$$

and the free surface equation

$$\begin{aligned} M\left(|\nabla\phi|\right) - \frac{1}{2}|\nabla\psi|^2 + \sqrt{1+|\nabla\eta|^2}\left(\psi_z\psi_{\mathbf{n}} - \mu(|\nabla\phi|)\phi_z\phi_{\mathbf{n}}\right) \\ +\tau\operatorname{div}\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}} - b\eta - p_0 = 0, \end{aligned} \tag{5}$$

on $\Omega \times \{z = \eta(x,y)\}$. Here,

$$\mathbf{n} := \frac{1}{\sqrt{1+|\nabla\eta|^2}}(-\eta_x, -\eta_y, 1),$$

is the unit normal of the surface $\Omega \times \{z = \eta(x, y)\}$ in the direction of positive $z$, and the constant $p_0 := M(1) + \mu(1)\left(\frac{1}{2}\mu(1) - 1\right)$.

Finally we would like to point out the following notational convention: the dimension of all operators appearing in the paper conform to the dimension of the domain of definition of their arguments, that is,

$$\nabla\phi = (\phi_x, \phi_y, \phi_z),\ \operatorname{div}\eta = \eta_x + \eta_y,\ |\nabla\phi| = (\phi_x^2 + \phi_y^2 + \phi_z^2)^{1/2},\ |\nabla\eta| = (\eta_x^2 + \eta_y^2)^{1/2},\ \text{etc.}$$

## 1.2 Physical attributes and modelling of the normal field instability of a ferrofluid

The system of partial differential equations described in the previous subsection arises as the mathematical model of an incompressible ferrofluid undergoing the so-called *normal field instability* or *Rosensweig instability* (refer for example to Rosensweig's monograph [20]): In an experiment, a vertical magnetic field is applied to a static ferrofluid layer, and various patterns (typically regular cellular hexagons) emerge on the fluid surface as the field strength is increased through a critical value.

Note that the strength of the applied field does not seem to appear in the system; it is rescaled to 1. The function $\eta$ defines the (rescaled) interface between the ferrofluid and air (or another fluid conforming to a linear magnetization law), that is, they occupy the regions $D_\eta^-$ and $D_\eta^+$ respectively and are subjected to a parallel vertical magnetic field; for more details on extracting the ferrofluid system from Maxwell's equations and the appropriate rescaling of the initial physical laws see [9] and references therein. The real functions $M$ and $\mu$ describe the magnetization law for the ferrofluid and the unknown functions $\phi, \psi$ are the magnetic potentials in the ferrofluid and air respectively. A typical case consists in ferrofluids following a nonlinear Langevin law (see [15, 19]), that is,

$$\mu(s) = \begin{cases} 1 + \dfrac{M_s}{s}\left(\coth(\gamma s) - \dfrac{1}{\gamma s}\right), & \text{for } s \neq 0, \\ 1 + \dfrac{M_s \gamma}{3}, & \text{for } s = 0, \end{cases} \tag{6}$$

where $M_s$ is the saturation magnetization and $\gamma$ is the Langevin parameter (they are both positive constants). Lastly, the parameters $b$ and $\tau$ are respectively the rescaled gravity acceleration and the coefficient of surface tension for the ferrofluid.

Since the invention of the ferrofluids ([18]) there has been a number of works studying surface instabilities using formal analysis (see [9] and references therein). A first rigorous treatment of regular patterns assuming a linear magnetization law was given by Twombly and Thomas [22]. For small amplitude localized patterns and nonlinear laws, Groves *et al* produced a rigorous theory in [9], using a technique known as *Kirchgässner reduction.*

This work lies between the latter two papers in the following sense: we allow for general nonlinear laws and simultaneously pose no assumption on the smallness of solutions. This is achieved through the study of the problem as a free boundary problem, that is, the originally unknown function $\eta$ that models the free interface of the ferrofluid is replaced by the characteristic function of the set occupied by the ferrofluid. The new set of unknowns (the magnetic potentials of the two fluids and the characteristic function of the ferrofluid) is then found as a critical point of an appropriate functional.

## 1.3 The variational structure of the ferrofluid system

The equations (3)-(5) describing the ferrofluid system have a variational structure: they correspond to the strong Euler-Lagrange equations of the functional

$$\begin{aligned} F(u,\eta) = &\int_\Omega \left( \int_{-1}^{\eta(x,y)} M(|\nabla u|)\, dz + \int_{\eta(x,y)}^{1} \frac{1}{2}|\nabla u|^2\, dz \right) dxdy \\ &+ \mu(1) \int_\Omega (u|_{z=-1} - u|_{z=1})\, dxdy \\ &- \int_\Omega \left( \frac{b}{2}\eta^2 + p_0\eta \right) dxdy - \tau \int_\Omega \sqrt{1+|\nabla\eta|^2}\, dxdy. \end{aligned} \tag{7}$$

This means that, assuming that $(u,\eta)$ is a critical point of $F$ and allowing

$$\psi = u|_{D_\eta^+} \in W^{2,2}\left(D_\eta^+\right) \quad \text{and} \quad \phi = u|_{D_\eta^-} \in W^{2,2}\left(D_\eta^-\right),$$

and for some smoothness of the interface, for example $\eta \in C^1$ and $|\eta| < 1$, one obtains a strong solution to the ferrofluid system of equations.

Note that $F$ is not a "pure" energy functional: it is convex with respect to $u$ and "almost" concave with respect to $\eta$ (the variable integral is not affine but is still bounded). This implies that it does not possess minimizers: taking a highly oscillating interface will produce "energies" that tend to $-\infty$ as the oscillations increase.

**Remark 1.1.** The physical parameters $b$ and $\tau$ are of the order of $H^{-2}$, where $H$ denotes the strength of the applied field (see [9]). Thus, dividing (7) by $-H^2$, letting $H \longrightarrow 0$, and assuming incompressibility of the fluid, we are led to the problem of minimizing the functional

$$\eta \mapsto \frac{\tilde{b}}{2} \int_\Omega \eta^2\, dxdy + \tilde{\tau} \int_\Omega \sqrt{1+|\nabla\eta|^2}\, dxdy.$$

Due to the absence of side conditions, the minimizer is the function $\eta = 0$. This justifies physical intuition that the surface of the ferrofluid remains flat in the absence of external field.

The next observation concerns the mathematical properties of the nonlinear magnetization law. The following lemma can be proven with elementary analytical arguments.

**Lemma 1.2.** *Let $\mu, M : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by (2) and (6).*

1. *$\mu$ is an even function.*
2. *$\mu \in C(\mathbb{R})$ and $1 < \mu(s) \leq 1 + \frac{M_s\gamma}{3}$ with equality if and only if $s = 0$.*
3. *For all $s \geq 0$ holds that*

   $$\frac{1}{2}s^2 \leq M(s) \leq \frac{1}{2}\left(\frac{\gamma}{3}M_s + 1\right)s^2,$$

   *with equality if and only if $s = 0$.*
4. *$M$ is a convex function.*

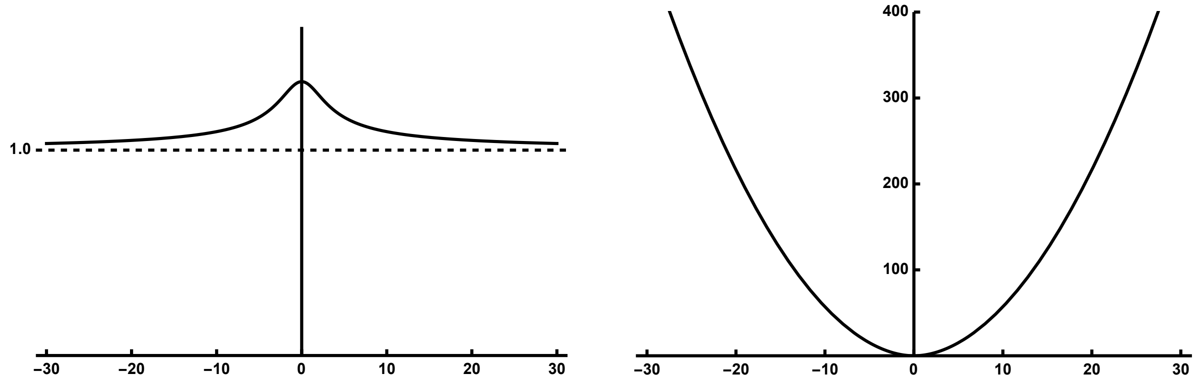


*Figure 2:* A plot of the function $\mu$ defined as in (6) with $\gamma = M_s = 1$ *(left)* and of its primitive *(right)*.

**Sketch of proof.** 1. and 2. follow from the explicit expression of $\mu$. 3. follows from 2., and 4. from the fact that a primitive is explicitly computable for a Langevin law: $M(s) = s^2/2 + M_s\left(\ln\left(\sinh(\gamma s)\right) - \ln s - \ln\gamma\right)/\gamma$. ■

This has two direct consequences: First, we have that $p_0 > 0$ since the expression $\mu(1)\left(\frac{1}{2}\mu(1) - 1\right)$ has an infimum value of $-\frac{1}{2}$ for $\mu(1) \longrightarrow 1$ (which is not attained since $\mu(1) > 1$) and $M(1) > \frac{1}{2}$. Second, we can define the domain of (7); the functional is well defined for $u \in H^1(D)$ and $\eta \in BV(\Omega)$.

# 2 The mathematical setting

## 2.1 A more general framework

From now on, and in order to shorten the formulas, we will use the notation $d\mathbf{x}$ for the Lebesgue measure in $\mathbb{R}^3$. We will drop the assumption that the interface can be described by the graph of a function $\eta$ and use only the minimal assumptions needed for the magnetization of the ferrofluid. Motivated by the properties of generic Langevin laws, we pose the following:

**Assumptions 2.1.** *1. The physical constants $b, \tau, \mu, p_0 \in \mathbb{R}$ are positive.*

*2. The function $M : \mathbb{R} \longrightarrow \mathbb{R}$ is convex and there exists $C_M > 1$ such that*

$$\frac{1}{2}s^2 \leq M(s) \leq \frac{C_M}{2}s^2 \quad \text{for all } s \in \mathbb{R}. \tag{8}$$

Concerning the magnetic potential, define the space

$$H^1_{cyl}(D) \coloneqq \left\{u \in H^1(D) : u|_{\partial_{cyl} D} = 0\right\} \tag{9}$$

equipped with the norm $\|u\|_{cyl} \coloneqq \|\nabla u\|_{L^2}$ which, due to Poincaré's inequality, is equivalent to the standard Sobolev norm. In contrast to [9], we address the problem in a weaker form, namely as a free boundary problem. To that end, define the set of characteristic functions

$$X(D) \coloneqq \left\{\chi \in BV(D) : \chi \in \{0,1\} \text{ in } D,\ \int_D \chi \, d\mathbf{x} = |\Omega|\right\} \tag{10}$$

and note that it is a weakly-$*$ closed subset of $BV(D)$ (every $L^1$ convergent sequence has an a.e. convergent subsequence and thus the limit function will be a.e. either 0 or 1. The integrals then converge to the correct value due to the dominated convergence theorem). The volume condition is due to the fact that the ferrofluid is assumed to be incompressible.

We consider the functional $J : H^1_{cyl}(D) \times X(D) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} J(u,\chi) \coloneqq & \int_D \left( \chi\, M(|\nabla u|) + \frac{1}{2}(1-\chi)\,|\nabla u|^2 \right) d\mathbf{x} + \mu \int_\Omega (u|_{z=-1} - u|_{z=1})\, dxdy \\ & - \int_D (b\, z\, \chi + p_0\, \chi)\, d\mathbf{x} - \tau \int_D |\nabla \chi|, \end{aligned} \tag{11}$$

where $\int_D |\nabla\chi|$ denotes the total variation of $\chi$, and note that

$$J(u,\chi) = J_1(u,\chi) - J_2(\chi) - \mu \int_D u_z\, d\mathbf{x}, \tag{12}$$

where

$$J_1(u,\chi) \coloneqq \int_D \left( \chi\, M(|\nabla u|) + \frac{1}{2}(1-\chi)\,|\nabla u|^2 \right) d\mathbf{x}, \tag{13}$$

$$J_2(\chi) \coloneqq \int_D (b\, z\, \chi + p_0\, \chi)\, d\mathbf{x} + \tau \int_D |\nabla \chi|. \tag{14}$$

The reason to consider the above functional lies in the following: as already mentioned, critical points of (7) are weak solutions of the ferrofluid problem (3)-(5). The functional $J$ defined above is an appropriate extension of (7) where we have dropped the assumption that the interface can be described by the graph of $\eta$. The function $\eta$ is replaced by a function $\chi$ which, as a characteristic function, yields the set that is occupied by the ferrofluid. Precisely, for a function $\eta \in BV(\Omega)$ holds (see eg. [8, Theorem 16.4, p.163])

$$J(u, \chi_{\{z<\eta\}}) = F(u,\eta) - \frac{b}{2}|\Omega|.$$

All in all, critical points $(u,\chi)$ with $\chi$ being the characteristic function of some open $D_F \subset D$ with appropriately smooth boundary will satisfy problem (3)-(5) locally, in the sense that the part of $\partial D_F$ that lies inside $D$ is locally the graph of $\eta$, and

$$\psi \coloneqq u|_{D\setminus D_F} \quad \text{and} \quad \phi \coloneqq u|_{D_F}$$

satisfy the differential equations in a weak sense. These critical points are sought as saddle points of $J$ since the functional is now convex-concave: $J_2$ is convex, $J_1(\cdot,\chi)$ is convex and $J_1(u,\cdot)$ is affine.

## 2.2 The framework of abstract minimax theory of convex-concave functions

Before we proceed with the exposition we give some definitions, following the analysis of *saddle* or *convex-concave* functions given in [3] and [17].

**Definition 2.2.** *Let* $\mathrm{X}$ *and* $\mathrm{Y}$ *be two nonempty sets and* $\Phi : \mathrm{X}\times\mathrm{Y} \longrightarrow \mathbb{R}\cup\{+\infty\}$ *a function. We say that* $(\mathrm{x}_0, \mathrm{y}_0)$ *is a* saddle point *of* $\Phi$*, if we have*

$$\Phi(\mathrm{x}_0, \mathrm{y}) \leq \Phi(\mathrm{x}_0, \mathrm{y}_0) \leq \Phi(\mathrm{x}, \mathrm{y}_0) \tag{15}$$

*for all* $(\mathrm{x}, \mathrm{y}) \in \mathrm{X} \times \mathrm{Y}$.

**Definition 2.3.** *Let* $\mathrm{X}$ *and* $\mathrm{Y}$ *be two nonempty sets and* $\Phi : \mathrm{X}\times\mathrm{Y} \longrightarrow \mathbb{R}\cup\{+\infty\}$ *a function. We say that* $\Phi$ *has a* saddle value $c$ *if*

$$\sup_{\mathrm{y}\in\mathrm{Y}} \inf_{\mathrm{x}\in\mathrm{X}} \Phi(\mathrm{x}, \mathrm{y}) = \inf_{\mathrm{x}\in\mathrm{X}} \sup_{\mathrm{y}\in\mathrm{Y}} \Phi(\mathrm{x}, \mathrm{y}) =: c. \tag{16}$$

**Definition 2.4.** *Let* $\mathrm{X}$ *and* $\mathrm{Y}$ *be two nonempty sets and* $\Phi : \mathrm{X}\times\mathrm{Y} \longrightarrow \mathbb{R}\cup\{+\infty\}$ *a function. We say that* $\Phi$ satisfies a minimax equality *at* $(\mathrm{x}_0, \mathrm{y}_0) \in \mathrm{X} \times \mathrm{Y}$ *if:*

1. *The function* $\Phi$ *has a saddle value.*
2. *There exists* $\mathrm{x}_0 \in \mathrm{X}$ *such that* $\sup_{\mathrm{y}\in\mathrm{Y}} \Phi(\mathrm{x}_0, \mathrm{y}) = \inf_{\mathrm{x}\in\mathrm{X}} \sup_{\mathrm{y}\in\mathrm{Y}} \Phi(\mathrm{x}, \mathrm{y})$.
3. *There exists* $\mathrm{y}_0 \in \mathrm{Y}$ *such that* $\inf_{\mathrm{x}\in\mathrm{X}} \Phi(\mathrm{x}, \mathrm{y}_0) = \sup_{\mathrm{y}\in\mathrm{Y}} \inf_{\mathrm{x}\in\mathrm{X}} \Phi(\mathrm{x}, \mathrm{y})$.

One can then directly prove the following fundamental result.

**Proposition 2.5** ([3, Proposition 2.105],[17, Proposition 2.3.5])**.** *Let* $\mathrm{X}$ *and* $\mathrm{Y}$ *be two nonempty sets and* $\Phi : \mathrm{X} \times \mathrm{Y} \longrightarrow \mathbb{R} \cup \{+\infty\}$*. The function* $\Phi$ *satisfies a minimax equality at* $(\mathrm{x}_0, \mathrm{y}_0)$ *if and only if* $(\mathrm{x}_0, \mathrm{y}_0)$ *is a saddle point of* $\Phi$ *in* $\mathrm{X} \times \mathrm{Y}$.

We outline the strategy for proving existence of saddle points: The special type of coupling between $u$ and $\chi$ that occurs only in $J_1$ allows us to iteratively apply the direct method of the calculus of variations to solve the min-max and the max-min problem. The next step is to show that the functional has a saddle value, that is, the max-min and the min-max problems are solved at the same value. To that end, we will use a tool from [17], a "coincidence theorem", which is a corollary of a classical result of Knaster, Kuratowski and Mazurkiewicz [11]; we provide it here without a proof.

**Proposition 2.6** ([17, Proposition 2.3.10])**.** *Assume that* $\mathbf{X}$ *and* $\mathbf{Y}$ *are Hausdorff topological vector spaces,* $\mathrm{X} \subseteq \mathbf{X}$ *and* $\mathrm{Y} \subseteq \mathbf{Y}$ *are nonempty, compact and convex sets, and* $F, G : \mathrm{X} \longrightarrow 2^{\mathrm{Y}}$ *are two set-valued maps that satisfy:*

1. *For all* $\mathrm{x} \in \mathrm{X}$*, the set* $F(\mathrm{x})$ *is open in* $\mathrm{Y}$ *and the set* $G(\mathrm{x})$ *is not empty and convex.*
2. *For all* $\mathrm{y} \in \mathrm{Y}$*, the set* $G^{-1}(\mathrm{y}) := \{\mathrm{x} \in \mathrm{X} : \mathrm{y} \in G(\mathrm{x})\}$ *is open in* $\mathrm{X}$ *and the set* $F^{-1}(\mathrm{y}) := \{\mathrm{x} \in \mathrm{X} : y \in F(\mathrm{x})\}$ *is not empty and convex.*

*Then there exists* $\mathrm{x}_0 \in \mathrm{X}$ *such that* $F(\mathrm{x}_0) \cap G(\mathrm{x}_0) \neq \emptyset$.

With the help of the latter and applying Proposition 2.5, we will prove the existence of saddle points for $J$ in the next section.

# 3 The main results

## 3.1 Existence of saddle points

We begin the section with our main existence theorem and its proof.

**Theorem 3.1.** *The functional $J$, defined by* (11)*, possesses a non-trivial saddle point*

$$(u_0, \chi_0) \in H^1_{cyl}(D) \times X(D),$$

*that is,* $u_0 \not\equiv 0$ *and*

$$J(u_0, \chi_0) = \min_{u \in H^1_{cyl}(D)} \max_{\chi \in X(D)} J(u, \chi).$$

**Proof.** We are looking for two pairs of functions $(u_{\rm Mm}, \chi_{\rm Mm}), (u_{\rm mM}, \chi_{\rm mM}) \in H^1_{cyl}(D) \times X(D)$, such that

$$J(u_{\rm Mm}, \chi_{\rm Mm}) = \max_{\chi \in X(D)} \min_{u \in H^1_{cyl}(D)} J(u, \chi), \tag{17}$$

$$J(u_{\rm mM}, \chi_{\rm mM}) = \min_{u \in H^1_{cyl}(D)} \max_{\chi \in X(D)} J(u, \chi). \tag{18}$$

We first deal with the max-min case (17): Fix $\chi \in X(D)$ and note that

$$\int_D u_z \, d\mathbf{x} \le \int_D |u_z| \, d\mathbf{x} \le \|u_z\|_{L^2(D)} \|1\|_{L^2(D)} \le \sqrt{|D|} \, \|u\|_{cyl} \, . \tag{19}$$

This, together with the growth condition on $M$ and (12) we calculate

$$\begin{aligned} J(u, \chi) &\ge \int_D \left( \chi \, \frac{1}{2} |\nabla u|^2 + \frac{1}{2}(1 - \chi) \, |\nabla u|^2 \right) d\mathbf{x} - \mu \int_D u_z \, d\mathbf{x} - J_2(\chi) \\ &\ge \frac{1}{2} \, \|u\|^2_{cyl} - \mu \, \sqrt{|D|} \, \|u\|_{cyl} - J_2(\chi), \end{aligned} \tag{20}$$

which implies that the functional $J(\cdot, \chi)$ is bounded below and, in turn, the existence of a minimizing sequence $\{u_{k,\chi}\}_{k \in \mathbb{N}} \subset H^1_{cyl}(D)$. Relation (20) implies that this sequence is bounded and thus possesses a weak limit $u_\chi \in H^1_{cyl}(D)$. Moreover, the functional $J(\cdot, \chi)$ is weakly lower semi-continuous: the real function $h : \mathbb{R}^3 \times D \longrightarrow \mathbb{R}$ with

$$h(\xi, x, y, z) = \chi(x, y, z) \, M(|\xi|) + \frac{1}{2}\Big(1 - \chi(x, y, z)\Big) \, |\xi|^2 - \mu \, \xi_3$$

is Carathéodory, satisfies $h(\xi, x, y, z) \ge \frac{1}{2} |\xi|^2 - \mu \, \xi_3 \in L^1(D)$ for almost all $\xi$, and it is strictly convex in $\xi$ a.e. in $D$ (since $M$ is convex and $\xi \mapsto |\xi|^2$ is strictly convex); see for example [21, Theorem 1.6]. Moreover, the boundary integral term is weakly continuous, since the trace operator is compact from $H^1(D)$ into $L^2(\partial D)$ (see for example [16, Theorem 6.2, p.103]). Thus, $u_\chi \in H^1_{cyl}(D)$ is a unique minimizer of $J(\cdot, \chi)$ in $H^1_{cyl}(D)$.

The set

$$\mathbf{X} := \Big\{ J(u_\chi, \chi) = \min_{u \in H^1_{cyl}(D)} J(u, \chi) : \chi \in X(D) \Big\} \tag{21}$$

is bounded above:

$$J(u_\chi, \chi) \le J(0, \chi) = -J_2(\chi) \le \frac{b}{2} \, |\Omega|. \tag{22}$$

This implies that there exists a sequence $\{\chi_k\}_{k\in\mathbb{N}} \subseteq X(D)$ such that, if we denote $u_k \coloneqq u_{\chi_k}$,

$$\lim_{k\to\infty} J(u_k, \chi_k) = \lim_{k\to\infty} \min_{u\in H^1_{cyl}(D)} J(u, \chi_k) = \sup \mathbf{X}.$$

Suppose that $J_2(\chi_k) \to +\infty$. Then, we have by (22) $J(u_k, \chi_k) \to -\infty$, a contradiction, since $\sup \mathbf{X} > -\infty$. Therefore, $J_2(\chi_k)$ is uniformly bounded, and therefore there exists a function $\chi_{\text{Mm}} \in X(D)$ such that $\chi_k \overset{*}{\rightharpoonup} \chi_{\text{Mm}}$ in $BV(D)$ up to a subsequence. We claim that the functional $J(u, \cdot)$ is weakly-$*$ upper semi-continuous, so that

$$J(u, \chi_{\text{Mm}}) \geq \limsup_{k\to\infty} J(u, \chi_k) \geq \lim_{k\to\infty} \min_{u\in H^1_{cyl}(D)} J(u, \chi_k) = \sup \mathbf{X},$$

for all $u \in H^1_{cyl}(D)$, and thus

$$\min_{u\in H^1_{cyl}(D)} J(u, \chi_{\text{Mm}}) = J(u_{\chi_{\text{Mm}}}, \chi_{\text{Mm}}) \geq \sup \mathbf{X}.$$

However, $J(u_{\chi_{\text{Mm}}}, \chi_{\text{Mm}}) \in \mathbf{X}$ and thus $J(u_{\chi_{\text{Mm}}}, \chi_{\text{Mm}}) = \sup \mathbf{X}$ so that we have solved (17) with $u_{\text{Mm}} \coloneqq u_{\chi_{\text{Mm}}}$.

*Proof of the claim.* Fix $u \in H^1_{cyl}(D)$ and take a sequence $\{\chi_k\}_{k\in\mathbb{N}} \subset X(D)$ and $\chi \in X(D)$ such that $\chi_k \overset{*}{\rightharpoonup} \chi$. Since the total variation is weakly-$*$ lower semi-continuous in $BV(D)$ and $\chi_k \overset{*}{\rightharpoonup} \chi$ implies $\chi_k \longrightarrow \chi$ in $L^1(D)$ we get that $J_2$ is lower semi-continuous. To finish the proof of the claim we need to prove that $J_1(u, \cdot)$ is continuous with respect to the $L^1$-topology: consider the sequence of real numbers $\left\{J_1(u, \chi_k)\right\}_{k\in\mathbb{N}}$ and pick an arbitrary subsequence $\left\{J_1(u, \chi_{k_l})\right\}_{l\in\mathbb{N}}$. For the sequence of functions $\{\chi_{k_l}\}_{l\in\mathbb{N}}$ it still holds that $\chi_{k_l} \longrightarrow \chi$ in $L^1(D)$, and thus we can extract a subsequence $\{\chi_{k_{l_m}}\}_{m\in\mathbb{N}}$ converging to $\chi$ almost everywhere. Moreover, since $C_M > 1$

$$\chi_{k_{l_m}} M(|\nabla u|) + \frac{1}{2}(1 - \chi_{k_{l_m}}) |\nabla u|^2 \leq \frac{1}{2} |\nabla u|^2 + \frac{1}{2} \chi_{k_{l_m}} (C_M - 1) |\nabla u|^2 \leq \frac{C_M}{2} |\nabla u|^2$$

so that, using Lebesgue's dominated convergence theorem, we get

$$J_1(u, \chi_{k_{l_m}}) \longrightarrow J_1(u, \chi).$$

All in all, we have shown that each subsequence $\left\{J_1(u, \chi_{k_l})\right\}_{l\in\mathbb{N}}$ possesses a further subsequence that converges to the same limit, namely to $J_1(u, \chi)$. Thus

$$J_1(u, \chi_{n,k}) \longrightarrow J_1(u, \chi_n)$$

which finishes the proof of the claim.

Next, we solve the min-max problem (18) in a similar manner: using (19) and the growth condition of $M$ we get

$$J(u, \chi) \leq \frac{C_M}{2} \|u\|^2_{cyl} + \mu \sqrt{|D|}\, \|u\|_{cyl} + \frac{b}{2} |\Omega| - \min\{p_0, \tau\}\, \|\chi\|_{BV},$$

since

$$\begin{aligned}
J_2(\chi) &= b \int_\Omega \left( \int_{-1}^0 z\,\chi\, dz + \int_0^1 z\,\chi\, dz \right) dxdy + p_0 \int_D |\chi|\, d\mathbf{x} + \tau \int_D |\nabla \chi| \\
&\geq b \int_\Omega \int_{-1}^0 z\,\chi\, dzdxdy + p_0 \int_D |\chi|\, d\mathbf{x} + \tau \int_D |\nabla \chi| \\
&\geq -\frac{b}{2} |\Omega| + p_0 \int_D |\chi|\, d\mathbf{x} + \tau \int_D |\nabla \chi| \\
&\geq -\frac{b}{2} |\Omega| + \min\{p_0, \tau\} \left( \int_D |\chi|\, d\mathbf{x} + \int_D |\nabla \chi| \right),
\end{aligned}$$

so that there exists a bounded maximizing sequence $\{\chi_{k,u}\}_{k\in\mathbb{N}} \subset X(D)$. Thus, there exists $\chi_u \in X(D)$ which, by the weak-$*$ upper semi-continuity of $J(u,\cdot)$, is a maximizer of $J(u,\cdot)$. The set

$$\mathbf{Y} \coloneqq \left\{ J(u,\chi_u) = \max_{\chi\in X(D)} J(u,\chi) : u \in H^1_{cyl}(D) \right\} \tag{23}$$

is bounded below:

$$\begin{aligned} J(u,\chi_u) &\geq J(u,\chi_{\{z>0\}}) \\ &= \int_\Omega \int_0^1 M(|\nabla u|)\, dzdxdy + \frac{1}{2}\int_\Omega \int_{-1}^0 |\nabla u|^2\, dzdxdy - \mu \int_D u_z\, d\mathbf{x} \\ &\quad - \int_\Omega \int_0^1 (b\,z + p_0)\, dzdxdy - \tau \int_D |\nabla \chi_{\{z>0\}}| \\ &\geq \frac{1}{2}\, \|u\|^2_{cyl} - \mu\sqrt{|D|}\, \|u\|_{cyl} - (b + p_0 + \tau)\, |\Omega| \\ &\geq -\,(b + p_0 + \tau + \mu^2)\, |\Omega| \end{aligned} \tag{24}$$

for all $u \in H^1_{cyl}(D)$. Thus there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} J(u_k,\chi_k) = \lim_{k\to\infty} \max_{\chi\in X(D)} J(u_k,\chi) = \inf \mathbf{Y},$$

where again $\chi_k \coloneqq \chi_{u_k}$. Estimate (24) implies that $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $H^1_{cyl}(D)$ and thus there exists $u_{\mathrm{mM}} \in H^1_{cyl}(D)$ such that $u_k \rightharpoonup u_{\mathrm{mM}}$. As already shown, $J(\cdot,\chi)$ is weakly lower semi-continuous so that

$$J(u_{mM},\chi) \leq \liminf_{k\to\infty} J(u_k,\chi) \leq \lim_{k\to\infty} \max_{\chi\in X(D)} J(u_k,\chi) = \inf \mathbf{Y},$$

for all $\chi \in X(D)$. Taking the maximum over $\chi$ we get $J(u_{mM},\chi_{u_{mM}}) \leq \inf \mathbf{Y}$ and, since $J(u_{mM},\chi_{u_{mM}}) \in \mathbf{Y}$, we get $J(u_{\mathrm{mM}},\chi_{\mathrm{mM}}) = \inf \mathbf{Y}$ where $\chi_{\mathrm{mM}} \coloneqq \chi_{u_{\mathrm{mM}}}$.

The last step is to prove that the functional $J$ has a saddle value, that is,

$$J(u_{\mathrm{Mm}},\chi_{\mathrm{Mm}}) = J(u_{\mathrm{mM}},\chi_{\mathrm{mM}})$$

To that end, first note that we directly obtain that

$$\max_{\chi\in X(D)} \min_{u\in H^1_{cyl}(D)} J(u,\chi) \leq \min_{u\in H^1_{cyl}(D)} \max_{\chi\in X(D)} J(u,\chi),$$

that is,

$$J(u_{\mathrm{Mm}},\chi_{\mathrm{Mm}}) \leq J(u_{\mathrm{mM}},\chi_{\mathrm{mM}}).$$

We will need the closed convex hull of $X(D)$, namely

$$\overline{\mathrm{conv}}\, X(D) = \left\{ \rho \in BV(D) : 0 \leq \rho \leq 1 \text{ a.e. in } D,\ \int_D \rho\, d\mathbf{x} = |\Omega| \right\}.$$

Note that $\overline{\mathrm{conv}}\, X(D)$ is a weakly-$*$ closed and convex subset of $BV(D)$ and that all partial semi-continuity properties of $J$ still hold in it. Assume there exists $c \in \mathbb{R}$ such that

$$\max_{\chi\in X(D)} \min_{u\in H^1_{cyl}(D)} J(u,\chi) < c < \min_{u\in H^1_{cyl}(D)} \max_{\chi\in X(D)} J(u,\chi), \tag{25}$$

and define the set-valued maps $F, G : H^1_{cyl}(D) \longrightarrow 2^{\overline{\text{conv}}\, X(D)}$ by

$$F(u) := \big\{\rho \in \overline{\text{conv}}\, X(D) : J(u,\rho) < c\big\} \quad \text{and} \quad G(u) := \big\{\rho \in \overline{\text{conv}}\, X(D) : J(u,\rho) > c\big\}.$$

Since $J(u,\cdot)$ is weakly-$*$ upper semi-continuous we get that $F(u)$ is open for each $u \in H^1_{cyl}(D)$. For each $u \in H^1_{cyl}(D)$ the set $G(u)$ is not empty, due to (25), and convex: let $\rho_1, \rho_2 \in \overline{\text{conv}}\, X(D)$ such that $J(u,\rho_1), J(u,\rho_2) > c$ and $t \in [0,1]$ and calculate

$$J\big(u, t\,\rho_1 + (1-t)\,\rho_2\big) \geq t\, J(u,\rho_1) + (1-t)\, J(u,\rho_2) > t\, c + (1-t)\, c = c, \tag{26}$$

since the functional $J(u,\cdot)$ is concave. Moreover, $G^{-1}(\rho) = \big\{u \in H^1_{cyl}(D) : J(u,\rho) > c\big\}$ is open for each $\rho \in \overline{\text{conv}}\, X(D)$, since $J(\cdot,\rho)$ is lower semi-continuous, and $F^{-1}(\rho) = \big\{u \in H^1_{cyl}(D) : J(u,\rho) < c\big\}$ is not empty (due to (25)) and convex for every $\rho \in \overline{\text{conv}}\, X(D)$, since $J(\cdot,\rho)$ is convex (argue just like (26)). Thus, applying Proposition 2.6 ($BV(D)$ is isomorphic to the dual of a separable Banach space–see for example [2, Remark 3.12, p. 124]–and the weak-$*$ topology is always Hausdorff in the dual of a Banach space) to obtain $(\tilde u, \tilde\rho) \in H^1_{cyl}(D) \times \overline{\text{conv}}\, X(D)$ such that $\tilde\rho \in F(\tilde u) \cap G(\tilde u)$, i.e., $c < J(\tilde u, \tilde\rho) < c$, a contradiction.

Thus, Proposition 2.5 implies that the functional $J$ possesses a saddle point $(u_0, \chi_0) \in H^1_{cyl}(D) \times X(D)$, i.e.,

$$J(u_0,\chi) \leq J(u_0,\chi_0) \leq J(u,\chi_0) \quad \text{for all} \quad (u,\chi) \in H^1_{cyl}(D) \times X(D), \tag{27}$$

given by $(u_0,\chi_0) = (u_{\mathrm{mM}}, \chi_{\mathrm{Mm}})$.

Finally we illustrate the non-triviality of the saddle point: Let $\varphi \in C_0^\infty(\Omega)$ satisfy $\varphi \geq 0$ and set $\tilde u(x,y,z) := z\,\varphi(x,y)$, so that $\tilde u \in H^1_{cyl}(D)$. It holds that

$$\int_D \tilde u_z \, d\mathbf{x} = \int_D \varphi \, d\mathbf{x} = 2 \int_\Omega \varphi \, dxdy > 0.$$

For any $\chi \in X(D)$ and $\varepsilon > 0$ we obtain using (8) that

$$\begin{aligned} J(\varepsilon\,\tilde u, \chi) &\leq \varepsilon^2 \int_D \left( \frac{C_M\,\chi}{2} |\nabla \tilde u|^2 + \frac{1-\chi}{2} |\nabla \tilde u|^2 \right) d\mathbf{x} - \varepsilon\,\mu \int_D \tilde u_z \, d\mathbf{x} - J_2(\chi) \\ &= \varepsilon^2 \int_D \left( \frac{C_M\,\chi}{2} |\nabla \tilde u|^2 + \frac{1-\chi}{2} |\nabla \tilde u|^2 \right) d\mathbf{x} - \varepsilon\,\mu \int_\Omega \varphi \, dxdy + J(0,\chi). \end{aligned}$$

Thus, for $\varepsilon$ small we get $J(\varepsilon\,\tilde u, \chi) < J(0,\chi)$, in particular for $\chi = \chi_0$. ■

## 3.2 Qualitative properties of the optimal configuration

Let $(u_0,\chi_0)$ be a saddle point of $\mathcal{E}$. Define the set that the ferrofluid occupies by

$$D_F := \{x \in D \,|\, \chi_0(x) = 1\}. \tag{28}$$

For all $(u,\chi) \in H^1_{cyl}(D) \times X(D)$ holds

$$J(u_0,\chi) \leq J(u,\chi_0).$$

Moreover

$$J(u_0,\chi) \geq \frac{1}{2} \int_D |\nabla u_0|^2 \, d\mathbf{x} - \mu \int_D (u_0)_z \, d\mathbf{x} - J_2(\chi)$$

and

$$J(u,\chi_0) \le \frac{C_M}{2}\int_D |\nabla u|^2\, d\mathbf{x} - \mu \int_D (u)_z\, d\mathbf{x} - J_2(\chi_0),$$

so that altogether

$$\frac{1}{2}\left( \|u_0\|_{cyl}^2 - C_M\, \|u\|_{cyl}^2 \right) - \mu \int_D \big((u_0)_z - u_z\big)\, d\mathbf{x} \le J_2(\chi) - J_2(\chi_0). \tag{29}$$

This enables us to prove an estimate on the norm of the optimal solution.

**Proposition 3.2.** *It holds that* $\|u_0\|_{cyl} \le 2\,\mu\,\sqrt{|D|}$.

**Proof.** For $u = 0$ and $\chi = \chi_0$ in (29) we get that

$$\frac{1}{2}\,\|u_0\|_{cyl}^2 \le \mu \int_D (u_0)_z\, d\mathbf{x} \le \mu\,\sqrt{|D|}\,\|u_0\|_{cyl}\,.$$

The last inequality is due to (19). ∎

The gravity term in $J_2$ allows us to prove an estimate that justifies the physical intuition that a heavy ferrofluid (with $b$ large) will not float in the air.

**Proposition 3.3.** *Let* $D_F$ *be as in* (28) *and* $\partial_{bot}D$*, the bottom part of the boundary, as in* (1)*. Then*

$$\operatorname{dist}(D_F, \partial_{bot}D) \le \frac{\mu^2}{b}\left(1 - \frac{1}{C_M}\right).$$

**Proof.** If $d := \operatorname{dist}(D_F, \partial_{bot}D) = 0$ there is nothing to prove. Suppose that $d > 0$. For any $\delta \in [0,d)$, define $A_\delta =: D_F - (0,0,\delta)$, and note that $\operatorname{dist}(A_\delta, \partial_{bot}D) > 0$. Let $\alpha \in [0, C_M^{-\frac{1}{2}})$ (to be chosen appropriately) and set $u = \alpha\, u_0$ in inequality (29) to obtain

$$\begin{aligned} J_2(\chi) - J_2(\chi_0) &\ge \frac{1}{2}\big(1 - C_M\,\alpha^2\big)\,\|u_0\|_{cyl}^2 - \mu\,\big(1-\alpha\big)\int_D (u_0)_z\, d\mathbf{x} \\ &\ge \frac{1}{2}\big(1 - C_M\,\alpha^2\big)\,\|u_0\|_{cyl}^2 - \mu\,\big(1-\alpha\big)\,\sqrt{|D|}\,\|u_0\|_{cyl} \\ &\ge -\,\frac{\mu^2\,|D|\,(1-\alpha)^2}{2(1 - C_M\,\alpha^2)} \end{aligned}$$

Setting $\chi = \chi_{A_\delta}$, the left-hand side becomes

$$J_2(\chi_{A_\delta}) - J_2(\chi_0) = b \int_D z\,(\chi_{A_\delta} - \chi_0)\, d\mathbf{x},$$

since a rigid motion of $D_F$ away from the boundary does not change its perimeter. It holds that

$$\int_{D_F} z\, d\mathbf{x} = \int_{A_\delta} (z+\delta)\, d\mathbf{x} = \int_{A_\delta} z\, d\mathbf{x} + \delta\,|\Omega|,$$

so that, altogether, we get

$$-\frac{\mu^2\,(1-\alpha)^2}{(1 - C_M\,\alpha^2)} \le -b\,\delta,$$

or, equivalently,

$$\delta \leq \frac{\mu^2 (1-\alpha)^2}{b(1 - C_M \alpha^2)}.$$

The function on the right hand side is minimized for $\alpha = \alpha_* \coloneqq C_M^{-1} < C_M^{-\frac{1}{2}}$. Choosing $\alpha = \alpha_*$ we obtain

$$\delta \leq \frac{\mu^2}{b}\left(1 - \frac{1}{C_M}\right),$$

and taking the supremum on all admissible $\delta$ we finish the proof. ■

**Remark 3.4.** The same argumentation provides with a proof that there will be no disconnected ferrofluid bubbles floating far from the rest of the ferrofluid mass. A complementary argument can be used to show that there cannot be any air bubbles in the ferrofluid too close to the interface.

**Remark 3.5.** One could have chosen to eliminate the quadratic part of (3.2) by choosing $u = C_M^{-1/2} u_0$ to obtain a bound directly. The argumentation in the proof above aims to illustrate that the result of Proposition 3.3 is optimal, in the sense that a different rescaling of the optimal solution will not provide with a better estimate. This, of course, does not mean that the proposition is optimal per se: We expect that an explicit relation between the parameters exists, that acts as a necessary and sufficient condition for asserting that $\mathrm{dist}(D_F, \partial_{bot} D) = 0$. This follows physical intuition, since a light ferrofluid in a strong magnetic field will completely leave the bottom and stick to the upper part of the container. However, finding such a relation needs sharper density estimates for the optimal solution, whose extraction and manipulation lies beyond the scope of this paper.

## 3.3 Duality theory for linear magnetization laws

In this section we focus in the linear case, that is, we assume that

$$M(s) = \frac{\mu}{2} s^2$$

for a fixed $\mu > 1$. We prove that the saddle point that we found is a minimizer of the energy functional of the system. In fact, we show that the energy functional is conjugate to our convex-concave functional $J$. The main impact of this section is that, after obtaining Theorem 3.7, we can apply regularity results that have been developed for minimizers of free discontinuity problems to the saddle point from Theorem 3.1, in order to obtain Theorem 3.8 and Corollary 3.9.

Among the first works studying properties of optimal configurations that are minimizers of a corresponding energy we should mention those by Ambrosio and Buttazzo [1] and by Lin [13]. Apart from other technical results, Ambrosio and Buttazzo also proved Hölder continuity of the optimal solution and openness of the optimal set, whereas Lin worked in a space of currents. There has been a number of works following; Larsen [12] showed $C^1$ regularity away from the boundary for the components of the optimal set; Fusco and Julin [7] dealt with the so-called *Taylor cones* – conical points on the free surface of a fluid inside an electric field – and with refined regularity assertions on the minimizers; De Philippis and Figalli [5] studied the dimension of the set of singularities of the boundary of the optimal set; Carozza *et al* [4] deal also with energies with general potentials. Many other works on this subject can be found in the references in these citations; this list is by far not exhaustive.

The main result of this section is given in its end, following the necessary discussion on duality theory. We follow the notation and exposition of Ekeland-Temam [6, Chapter III, 4].

Fix $\chi \in X(D)$ and define $V := H^1_{cyl}(D)$ and $Y := L^2(D, \mathbb{R}^3) \cong Y^*$. Moreover, define

$$J_\chi(\nabla u) := J(u, \chi) = \int_D f_\chi(\nabla u)\, d\mathbf{x} - J_2(\chi), \tag{30}$$

where $f_\chi : D \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$f_\chi(\xi) := \frac{\mu\,\chi}{2}\,|\xi|^2 + \frac{1-\chi}{2}\,|\xi|^2 - (0,0,\mu)\cdot\xi. \tag{31}$$

Moreover, define the family of perturbations $\Phi_\chi : V \times Y \longrightarrow \mathbb{R}$ by

$$\Phi_\chi(u,p) := J_\chi(\nabla u - p).$$

In the previous section we have shown that there exists a unique solution $u_\chi \in V$ to the minimization problem

$$\min_{u\in V} J_\chi(\nabla u) = \min_{u\in V} \Phi_\chi(u,0). \tag{32}$$

The dual problem is

$$\sup_{p^*\in Y} \big\{ -\Phi^*_\chi(0,p^*)\big\} \tag{33}$$

and it holds that

$$\begin{aligned}
\Phi^*_\chi(0,p^*) &= \sup\left\{ \int_D p^*\cdot p\, d\mathbf{x} - J_\chi(\nabla u - p) : u \in V, p \in Y\right\}\\
&= \sup\left\{\sup\left\{ \int_D p^*\cdot p\, d\mathbf{x} - J_\chi(\nabla u - p) : p \in Y\right\} : u\in V\right\}\\
&= \sup\left\{\sup\left\{ \int_D p^*\cdot \nabla u\, d\mathbf{x} - \int_D p^*\cdot q\, d\mathbf{x} - J_\chi(q) : \nabla u - q \in Y\right\} : u\in V\right\}\\
&= \sup\left\{\sup\left\{ \int_D p^*\cdot \nabla u\, d\mathbf{x} - \int_D p^*\cdot q\, d\mathbf{x} - J_\chi(q) : q \in Y\right\} : u\in V\right\}\\
&= \begin{cases} \sup\left\{ -\int_D p^*\cdot q\, d\mathbf{x} - J_\chi(q) : q\in Y\right\}, & \text{when } p^*\in Y_d,\\ +\infty, & \text{otherwise}\end{cases}\\
&= \begin{cases} J^*_\chi(-p^*), & \text{when } p^*\in Y_d,\\ +\infty, & \text{otherwise,}\end{cases}
\end{aligned}$$

where

$$Y_d := \left\{ p^* \in Y : \int_D p^*\cdot\nabla u\, d\mathbf{x} = 0 \text{ for all } u \in V\right\}.$$

Thus, (33) is equivalent to

$$\sup_{p^*\in Y_d}\big\{-J^*_\chi(-p^*)\big\} = -\inf_{p^*\in Y_d} J^*_\chi(-p^*). \tag{34}$$

Next, we calculate the conjugate functional with the help of [6, Proposition 1.2, p.78]

$$J^*_\chi(p^*) = \int_D f^*_\chi(p^*)\, d\mathbf{x} + J_2(\chi), \tag{35}$$

since $J_2(\chi)$ is a constant and $f_\chi^*(p^*)$ is calculated pointwise in $D$, so that,

$$f_\chi^*(p^*) = \begin{cases} \left(\xi \mapsto \frac{1}{2}|\xi|^2 - (0,0,\mu)\cdot\xi\right)^*(p^*) & \text{for } \chi = 0, \\ \left(\xi \mapsto \frac{\mu}{2}|\xi|^2 - (0,0,\mu)\cdot\xi\right)^*(p^*) & \text{for } \chi = 1, \end{cases}$$

$$= \frac{\chi}{2\,\mu}\big|p^* + (0,0,\mu)\big|^2 + \frac{1-\chi}{2}\big|p^* + (0,0,\mu)\big|^2,$$

where we used classical properties of the convex conjugate (ex. [10, Proposition 1.3.1, p.42]). Thus, the dual problem (34) becomes

$$-\inf_{p^*\in Y_d} \tilde{\mathcal{E}}_\chi(p^*), \tag{36}$$

where

$$\tilde{\mathcal{E}}_\chi(p^*) \coloneqq \int_D \left\{ \frac{\chi}{2\,\mu}\left|p^* - (0,0,\mu)\right|^2 + \frac{1-\chi}{2}\left|p^* - (0,0,\mu)\right|^2 \right\} d\mathbf{x} + J_2(\chi). \tag{37}$$

Next, we calculate the derivative of the perturbations

$$\begin{aligned} &\langle \Phi'_\chi(u,p),(v,q)\rangle \\ &= \int_D \Big\{ \chi\,\mu\,(\nabla u - p)\cdot(\nabla v - q) + (1-\chi)\,(\nabla u - p)\cdot(\nabla v - q) - (0,0,\mu)\cdot(\nabla v - q) \Big\}\, d\mathbf{x}, \end{aligned}$$

so that the differential satisfies

$$\mathrm{d}\Phi_\chi(u,p) = \begin{pmatrix} \chi\,\mu\,(\nabla u - p) + (1-\chi)\,(\nabla u - p) - (0,0,\mu) \\ -\chi\,\mu\,(\nabla u - p) - (1-\chi)\,(\nabla u - p) + (0,0,\mu) \end{pmatrix}^{\top} \in Y \times Y \subset V^* \times Y.$$

According to [6, Proposition 5.1, p.21],

$$\Phi_\chi(u,0) + \Phi_\chi^*(0,p^*) = 0$$

is equivalent to

$$(0,p^*) \in \partial\Phi_\chi(u,0), \tag{38}$$

the latter denoting the subdifferential of $\Phi_\chi$. But $\Phi_\chi$ is differentiable, so that $\partial\Phi_\chi(u,0) = \{\mathrm{d}\Phi_\chi(u,0)\}$. Thus, (38) is equivalent to

$$0 = \chi\,\mu\,\nabla u + (1-\chi)\,\nabla u - (0,0,\mu) \qquad \text{(in the sense of distributions)} \tag{39}$$

$$p^* = -\chi\,\mu\,\nabla u - (1-\chi)\,\nabla u + (0,0,\mu), \tag{40}$$

where (39) translates to

$$\int_D \Big( \chi\,\mu\,\nabla u + (1-\chi)\,\nabla u - (0,0,\mu) \Big)\cdot \nabla v\, d\mathbf{x} = 0 \text{ for all } v \in V. \tag{41}$$

A solution to the equation (41) is $u = u_\chi$, the minimizer of the primal problem (32). Set

$$p_\chi^* \coloneqq -\chi\,\mu\,\nabla u_\chi - (1-\chi)\,\nabla u_\chi + (0,0,\mu).$$

Then, from [6, Proposition 2.4, p.53] we get that $p_\chi^*$ is a solution to the dual problem (36). Define $\mathcal{E}_\chi : Y \longrightarrow \mathbb{R}$ by

$$\mathcal{E}_\chi(q) \coloneqq \int_D \left( \frac{\chi\,\mu}{2}\,|q|^2 + \frac{1-\chi}{2}\,|q|^2 \right) d\mathbf{x} + J_2(\chi). \tag{42}$$

We have the following:

**Lemma 3.6.** *The function $u_\chi$ satisfies*

$$\mathcal{E}_\chi(\nabla u_\chi) = \min\left\{\mathcal{E}_\chi(\nabla u) : u \in V_\chi\right\} = \min_{p^* \in Y_d} \tilde{\mathcal{E}}_\chi(p^*), \tag{43}$$

*where the space $V_\chi$ is defined by*

$$V_\chi := \left\{ v \in V : \int_D \big( -\chi\,\mu\,\nabla v - (1-\chi)\,\nabla v + (0,0,\mu)\big) \cdot \nabla u \, d\mathbf{x} = 0 \text{ for all } u \in V \right\}. \tag{44}$$

*In particular, it holds*

$$\mathcal{E}_\chi(\nabla u_\chi) = \min\left\{\mathcal{E}_\chi(\nabla u) : u \in V_\chi \text{ and } u = u_\chi \text{ on } \partial D\right\}. \tag{45}$$

**Proof.** First note that

$$\begin{aligned}
&\min_{p^* \in Y_d} \tilde{\mathcal{E}}_\chi(p^*) = \tilde{\mathcal{E}}_\chi(p^*_\chi) \\
&\quad = \int_D \left( \frac{\chi}{2\,\mu} \left|\chi\,\mu\,\nabla u_\chi + (1-\chi)\,\nabla u_\chi\right|^2 + \frac{1-\chi}{2} \left|\chi\,\mu\,\nabla u_\chi + (1-\chi)\,\nabla u_\chi\right|^2 \right) d\mathbf{x} + J_2(\chi) \\
&\quad = \int_D \left( \frac{\chi\,\mu}{2} |\nabla u_\chi|^2 + \frac{1-\chi}{2} |\nabla u_\chi|^2 \right) d\mathbf{x} + J_2(\chi).
\end{aligned}$$

Moreover

$$\begin{aligned}
\mathcal{E}_\chi(\nabla u_\chi) &= \min\left\{ \tilde{\mathcal{E}}_\chi(p^*) : p^* \in Y_d \right\} \\
&= \min\left\{ \tilde{\mathcal{E}}_\chi(p^*) : p^* \in Y \text{ and } \int_D p^* \cdot \nabla u \, d\mathbf{x} = 0 \text{ for all } u \in V \right\} \\
&= \min\Bigg\{ \tilde{\mathcal{E}}_\chi\big( -\chi\,\mu\,q - (1-\chi)\,q + (0,0,\mu)\big) : q \in Y \text{ and } \\
&\qquad \int_D \big( -\chi\,\mu\,q - (1-\chi)\,q + (0,0,\mu)\big) \cdot \nabla u \, d\mathbf{x} = 0 \text{ for all } u \in V \Bigg\} \\
&= \min\Bigg\{ \mathcal{E}_\chi(q) : q \in Y \text{ and } \\
&\qquad \int_D \big( -\chi\,\mu\,q - (1-\chi)\,q + (0,0,\mu)\big) \cdot \nabla u \, d\mathbf{x} = 0 \text{ for all } u \in V \Bigg\} \\
&\le \inf\left\{ \mathcal{E}_\chi(\nabla v) : v \in V_\chi \right\}.
\end{aligned}$$

Since from its definition $u_\chi \in V_\chi$, we obtain that

$$\mathcal{E}_\chi(\nabla u_\chi) = \min_{u \in V_\chi} \mathcal{E}_\chi(\nabla u).$$

Since the minimization problem does not change when we consider it in the class of functions that satisfy the "correct" (i.e., $u = u_\chi$ on $\partial D$) boundary condition, we obtain

$$u_\chi = \arg\min \left\{ \mathcal{E}_\chi(\nabla v) : v \in V_\chi \text{ and } v = u_\chi \text{ on } \partial D \right\}.$$

■

We can now prove the following:

**Theorem 3.7.** *Define the energy functional*

$$\mathcal{E}(u,\chi) \coloneqq \mathcal{E}_\chi(\nabla u) - \int_D (0,0,\mu)\cdot\nabla u \, d\mathbf{x},$$

*where* $\mathcal{E}_\chi$ *is given by* (42)*, and let* $(u_0,\chi_0) \in H^1_{cyl}(D)\times X(D)$ *satisfy*

$$J(u_0,\chi_0) = \min_{u\in H^1_{cyl}(D)} \max_{\chi\in X(D)} J(u,\chi).$$

*Then* $(u_0,\chi_0)$ *satisfies*

$$\mathcal{E}(u_0,\chi_0) = \min\Big\{\mathcal{E}(u,\chi) : (u,\chi)\in H^1_{cyl}(D)\times X(D) \text{ with } u=u_0 \text{ on } \partial D\Big\}.$$

**Proof.** From the discussion above we get that for $\chi \in X(D)$, the dual problem (36) is equivalent to the problem

$$\max\Big\{ -\mathcal{E}_\chi(\nabla v) : v\in V_\chi \text{ and } v = u_\chi \text{ on } \partial D\Big\}, \tag{46}$$

where $V_\chi$ is defined in (44). From [6, Proposition 2.4, p.53] we get that

$$\max\Big\{ -\mathcal{E}_\chi(\nabla v) : v\in V_\chi \text{ and } v = u_\chi \text{ on } \partial D\Big\} = \min\Big\{J(v,\chi) : v\in H^1_{cyl}(D)\Big\} \tag{47}$$

and, from Lemma 3.6, that

$$u_\chi = \arg\min\Big\{\mathcal{E}_\chi(\nabla v) : v\in V_\chi \text{ and } v = u_\chi \text{ on } \partial D\Big\} \tag{48}$$

for all $\chi\in X(D)$. Since the function

$$\chi \mapsto \min\Big\{J(v,\chi) : v\in H^1_{cyl}(D)\Big\}$$

(minimizers are unique so the mapping is well-defined as a real function) is maximized for $\chi = \chi_0$, we get that $\chi_0$ maximizes

$$\begin{aligned}\chi \mapsto\ &\max\Big\{ -\mathcal{E}_\chi(\nabla v) : v\in V_\chi \text{ and } v = u_\chi \text{ on } \partial D\Big\}\\ &= -\min\Big\{\mathcal{E}_\chi(\nabla v) : v\in V_\chi \text{ and } v = u_\chi \text{ on } \partial D\Big\}\\ &= -\mathcal{E}_\chi(u_\chi),\end{aligned}$$

where, the last equality is due to (48). Using equation (48) again, we get that

$$\mathcal{E}_{\chi_0}(\nabla u_0) = \min\Big\{\mathcal{E}_\chi(\nabla u) : (u,\chi)\in V_0\times X(D) \text{ with } u = u_0 \text{ on } \partial D\Big\},$$

where $V_0 \coloneqq V_{\chi_0}$. Because of

$$\int_D (0,0,\mu) \cdot \nabla u_0 \, d\mathbf{x} = \int_D (u_0)_z \, d\mathbf{x} = \int_\Omega (u_0|_{z=1} - u_0|_{z=-1}) \, dxdy,$$

we can add the missing term on both sides to obtain

$$\begin{aligned}
&\mathcal{E}_{\chi_0}(u_0) - \int_D (0,0,\mu) \cdot \nabla u_0 \, d\mathbf{x} \\
&\quad = \min \left\{ \mathcal{E}_\chi(\nabla u) - \int_D (0,0,\mu) \cdot \nabla u_0 \, d\mathbf{x} : (u,\chi) \in V_0 \times X(D) \text{ with } u = u_0 \text{ on } \partial D \right\} \\
&\quad = \min \left\{ \mathcal{E}_\chi(\nabla u) - \int_D (0,0,\mu) \cdot \nabla u \, d\mathbf{x} : (u,\chi) \in V_0 \times X(D) \text{ with } u = u_0 \text{ on } \partial D \right\},
\end{aligned}$$

that is,

$$\mathcal{E}(u_0,\chi_0) = \min \left\{ \mathcal{E}(u,\chi) : (u,\chi) \in V_0 \times X(D) \text{ with } u = u_0 \text{ on } \partial D \right\}. \tag{49}$$

In order to finish the proof, note that the minimizer of $\mathcal{E}$ in $H^1_{cyl}(D) \times X(D)$ belongs to $V_0 \times X(D)$, since the side condition in $V_0$ is nothing else than the partial Euler-Lagrange equation of $\mathcal{E}$. ■

Since $(u_0,\chi_0)$ minimizes an energy functional, it is possible to apply the theory developed in the references to obtain regularity results. More precisely, we have the following proposition.

**Theorem 3.8.** *Let $(u_0,\chi_0)$ be a minimizer of $\mathcal{E}$ and $D_F$ be given by* (28)*, that is, the set occupied by the ferrofluid. Then $u \in C^{0,\frac{1}{2}}_{loc}(D)$, and $\partial D_F \cap D$ is locally a $C^{1,\alpha}$-submanifold of $\mathbb{R}^3$ for some $\alpha \in (0,1)$, that is, up to a relatively closed singular set $\Sigma$ which satisfies $\mathcal{H}^2(\Sigma) = 0$. Here $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure in $\mathbb{R}^3$, restricted on $\partial D_F \cap D$.*

**Proof.** Because of Theorem 3.7, The regularity results of [14, Theorem 1.1] and [14, Theorem 1.2] apply and provide the claim. In the notation of that paper, we need to set $F(x,u,p) = \frac{1}{2}|p|^2$, and $G(x,u,p) = \frac{\mu-1}{2}|p|^2 + bz + p_0$ (note that $\mu > 1$ by assumption). ■

As a direct consequence of the above we obtain that the optimal set is equivalent to a relatively open set.

**Corollary 3.9.** *Let $(u_0,\chi_0)$ be a minimizer of $\mathcal{E}$ and $D_F$ as in Theorem 3.8. Let $\tilde{\chi}$ be the characteristic function of the set $D_F \setminus (\partial D_F \cap D)$. Then $\mathcal{E}(u_0,\tilde{\chi}) \le \mathcal{E}(u_0,\chi_0)$.*

**Proof.** Theorem 3.8 implies that $\mathcal{H}^3(\partial D_F \cap D) = 0$ since the set $(\partial D_F \cap D) \setminus \Sigma$ is a 2-dimensional submanifold. So we get from the definition of the perimeter that $\int_D |\nabla \tilde{\chi}| = \int_D |\nabla \chi_0|$ which implies the corollary. ■

Thus in the linear case one can obtain a solution as a minimizer instead of a saddle point. That, in turn, allows for the application of the deep theory which was developed in the references listed in the beginning of Section 3.3 for minimizers of free discontinuity problems.

# References

[1] L. Ambrosio and G. Buttazzo. An optimal design problem with perimeter penalization. *Calc. Var. Partial Differential Equations*, 1(1):55–69, 1993.

[2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[3] V. Barbu and T. Precupanu. *Convexity and optimization in Banach spaces*. Springer Monographs in Mathematics. Springer, Dordrecht, fourth edition, 2012.

[4] M. Carozza, I. Fonseca, and A. Passarelli di Napoli. Regularity results for an optimal design problem with a volume constraint. *ESAIM Control Optim. Calc. Var.*, 20(2):460–487, 2014.

[5] G. De Philippis and A. Figalli. A note on the dimension of the singular set in free interface problems. *Differential Integral Equations*, 28(5/6):523–536, 05 2015.

[6] I. Ekeland and R. Témam. *Convex analysis and variational problems*, volume 28 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, english edition, 1999. Translated from the French.

[7] N. Fusco and V. Julin. On the regularity of critical and minimal sets of a free interface problem. *Interfaces Free Bound.*, 17(1):117–142, 2015.

[8] E. Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[9] M. D. Groves, D. J. B. Lloyd, and A. Stylianou. Pattern formation on the free surface of a ferrofluid: Spatial dynamics and homoclinic bifurcation. *Physica D: Nonlinear Phenomena*, 350:1–12, 2017.

[10] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex analysis and minimization algorithms. II*, volume 306 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1993. Advanced theory and bundle methods.

[11] B. Knaster, C. Kuratowski, and S. Mazurkiewicz. Ein Beweis des Fixpunktsatzes für $n$-dimensionale Simplexe. *Fundam. Math.*, 14:132–137, 1929.

[12] C. J. Larsen. Regularity of components in optimal design problems with perimeter penalization. *Calc. Var. Partial Differential Equations*, 16(1):17–29, 2003.

[13] F.-H. Lin. Variational problems with free interfaces. *Calc. Var. Partial Differential Equations*, 1(2):149–168, 1993.

[14] F. H. Lin and R. V. Kohn. Partial regularity for optimal design problems involving both bulk and surface energies. *Chinese Ann. Math. Ser. B*, 20(2):137–158, 1999. A Chinese summary appears in Chinese Ann. Math. Ser. A **2**0 (1999), no. 2, 267.

[15] D. J. B. Lloyd, C. Gollwitzer, I. Rehberg, and R. Richter. Homoclinic snaking near the surface instability of a polarisable fluid. *Journal of Fluid Mechanics*, 783:283–305, 11 2015.

[16] J. Nečas. *Direct methods in the theory of elliptic equations*. Springer Monographs in Mathematics. Springer, Heidelberg, 2012. Translated from the 1967 French original by Gerard Tronel and Alois Kufner, Editorial coordination and preface by Šárka Nečasová and a contribution by Christian G. Simader.

[17] N. S. Papageorgiou and S. Th. Kyritsi-Yiallourou. *Handbook of applied analysis*, volume 19 of *Advances in Mechanics and Mathematics*. Springer, New York, 2009.

[18] S. S. Papell. Low viscosity magnetic fluid obtained by the colloidal suspension of magnetic particles, 11 1965. US Patent 3,215,572.

[19] R. Richter and A. Lange. Surface instabilities of ferrofluids. In S. Odenbach, editor, *Colloidal Magnetic Fluids*, volume 763 of *Lecture Notes in Physics*, pages 1–91. Springer Berlin Heidelberg, 2009.

[20] R. E. Rosensweig. *Ferrohydrodynamics*. Cambridge Monographs on Mechanics. Cambridge University Press, 1985.

[21] M. Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* Springer-Verlag, Berlin, fourth edition, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems.

[22] E. E. Twombly and J. W. Thomas. Bifurcating instability of the free surface of a ferrofluid. *SIAM J. Math. Anal.*, 14(4):736–766, 1983.